Predictive Reliabilities for Electronic Components

by

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(ABSTRACT)

A reliability model to study the behavior of an electronic component subject to several failure mechanisms is developed. The mechanisms considered for the analysis are of degradation type where the number of defects for a mechanism increases with time, eventually causing the failure of the component. The failure pattern of the component subject to a single mechanism with given initial and final number of defects is modelled as a pure birth process. Failure time for this mechanism is expressed as the first passage time of the birth process to state k from initial state i. First passage time distribution is derived for different forms of transition rates. When the initial and final states of the process are considered as random, the failure time is expressed as the mixture distribution obtained from the conditional first passage time distributions. The mixture distributions are well represented by a Weibull distribution. A computer program is developed to compute the parameters of the Weibull distribution iteratively by the method of matching moments. The approximation results are statistically validated.

The results for a single mechanism are extended to the case of multiple mechanisms. Extreme-value theory and competing risk theory are applied to analyze the simultaneous effects of multiple mechanisms. It is shown that the aggregate failure time distribution has a Weibull form for both the theories.

The model explains the influence of physical and chemical properties of the component and the operating conditions on the failure times. It can be used for accelerated testing and for incorporating reliability at product design stage.

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CHAPTER 1

Introduction and Background

1.1 Introduction

The popular method in estimating the reliability of a device is a data analysis method in which the data on failure times is collected and analyzed statistically. While this method is often reasonably accurate, it is a black-box approach. It fails to explain the reliability in terms of physical and chemical properties of the materials used in the device. Any changes in the materials or in the design require collection of lifetime data again to estimate the reliability function. Even for the same device, a change in a single operating condition such as temperature may require that the process of estimating the reliability from accumulated lifetime data be repeated.

An improvement on the above method is a model that can explain the reliability in terms of the properties of the materials used in the component and the operating

conditions. The model should capture the dependence of failures upon changes in the chemical and physical states of the component during operation. One possible approach to constructing such a model is to study the reliability behavior in terms of the failure mechanisms operating within the device. These mechanisms reflect the impact of chemical and physical properties of the device, the operating conditions and the duration of the operation on the failure times. A model that is intended to represent the causal relationship between the failure behavior of a component and the failure mechanisms active in the component is developed here.

A component, while in operation, is subject to many forces that may induce failure by causing certain changes in the structure of the component. These changes are called failure mechanisms. Some examples of the failure mechanisms are creep, crack growth and fatigue. In general the mechanisms of interest may be physical or chemical in nature. Once the mechanisms are classified, it may be possible to formulate probability based mathematical models for each of the classes. The models should reflect the influence of environmental variables and the constituents of the components upon the mechanisms. Then, in each class, individual mechanisms that are most prevalent should be examined. It may then be possible to study the behavior of the component failure with respect to the aggregate effect of many such individual failure mechanisms.

One class of failure mechanisms is the degradation process. As the name implies the process is a gradual one. There is a gradual change of the physical or chemical structure of the component that slowly reduces the strength or the resistance to failure of the component, ultimately resulting in failure.

The present study deals with degradation processes in which changes occur in discrete units. Conceptually, this process can be thought of as a chemical degradation process, in which reactant A is converted into reaction product B. As the reaction continues, there is an accumulation of the product B that is assumed to inhibit the function of the component. Completion of the reaction may not be required for a failure. Instead, failure is defined as the accumulation of a sufficient mass of reaction product to degrade performance to an unacceptable level.

Failures of the chemical degradation type are common in many types of components, particularly in electronic components. An example is oxidation of copper "lands" in integrated circuits. This is a chemical process. Another example is electromigration in integrated circuits. The momentum of electrons is transmitted to the metal atoms resulting in a movement of these metal atoms to different parts of the component. This eventually results in failure due to an open circuit. This is a physical mechanism.

For a given type of a component, there are many random variables that influence the failure behavior of the component. At the aggregate level, the time to failure depends upon the various failure mechanisms acting on the component simultaneously. The proposed model considers only those failure mechanisms that possess a common behavior, namely, degradation type. For each individual mechanism of this type, there are certain variables that are random. Some examples of the random variables are the initial and final number of molecules of the reaction product B and the time taken for the amount of reaction product B to increase by one molecule.

To capture the randomness of the degradation, the accumulation of the reaction product B is modelled as a stochastic process. In particular, the accumulation is

modelled as Markov process. The assumption is that the future course of reaction is independent of the past history of the reaction, provided the present extent of the reaction is known.

The model is constructed under the assumption that the reaction is an irreversible reaction. That is, the degradation is non-decreasing with time. In practical applications, there may be some reaction in the reverse direction but it is considered to be negligible for the present analysis.

1.2 Description of the Approach

Let X(t) represent the state variable at any time t so that X(t) = j represents the number of molecules of reaction product B present at time t. It is reasonable to assume that initially there is a certain mass of the reaction product B. Thus, there exists an initial state, X(0) = i. The state variable X(t) changes only in unit steps. As already stated, the reverse reaction rate is assumed to be zero. Since the process is a Markov process and because of the additional assumption that the reverse reaction does not occur, the random process is a pure birth process. Failure occurs when a sufficient mass of the product B is accumulated (i.e., arrival of the process to a particular state k). For the pure birth process, the failure time is the first passage time of the process from the initial state i to the final state k. In general, the initial state may be a random variable. This may be due to the variability in the component caused by the manufacturing process. For the present study it is indeed considered as a random variable with known probability distribution function. The assumptions about its probability distribution should be reinforced by results from defect models

and available defect data. It is also conceivable that the failure threshold for a degradation reaction is a random variable. This might be a consequence of the geometry of the component or of the random dispersion of the reaction product within the material. If either the initial state or the final state or both are random variables, then the first passage time distribution for a failure mechanism is a mixture distribution. For a given failure mechanism, the mixture distribution can be obtained from its conditional distributions, each conditioned on given initial and final states.

For the chemical degradation process, the reaction rate is the fraction of product B produced per unit time. This reaction rate represents the transition rate for the birth process. The transition rate, $\lambda_j(t)$, of the birth process is the rate of change of the transition probability, when the system is in state j at time t. The transition rate may depend on the state X(t) and time t. For the current model, the transition rate is assumed to be a function of time t, state X(t) and the environment. This will not only capture the essentials of the factors that control the reaction but also will reflect the effects of certain changes in the operating conditions on the reaction rate.

Once a single failure mechanism is studied, the results can be extended to multiple mechanisms. A component may be subject to several failure mechanisms simultaneously. It is assumed that all the mechanisms display similar behavior. The component may fail due to any of the mechanisms. Each of the mechanisms would individually result in a particular lifetime value for the component. In the aggregate, the lifetime of the component is the minimum of all the individual lifetimes. To obtain the distribution of the minimum of the lifetimes, extreme-value theory or competing risk theory may be applied. Extreme-value theory assumes that all the individual lifetimes are identically distributed and that the number of mechanisms acting on the

component is quite large. In competing risk theory both these assumptions can be relaxed. That is, the individual lifetimes may be different in their probability distributions and the number of mechanisms considered may be finite.

1.3 Problem Analysis

The objective of the present dissertation is to develop a methodology to study the failure behavior of a component in terms of the failure mechanisms acting on the component. Analysis of the failure time of a single mechanism constitutes the initial step in the study. A stochastic process model for the mechanism is constructed and the first passage time distribution for the process to visit a specific final state is obtained to describe this failure time. Results obtained for a single mechanism are expanded to include the presence of multiple mechanisms using extreme-value and competing risk theories.

Parts of the model have been used before to study reliability, but not in such a comprehensive and sequential manner. The transition rate function used in the model is a general one and takes into account the concentration of the reaction product, time, and the operating conditions. Almost all of the first passage problems have been considered for a given initial and final states only. For the present analysis, these two states are assumed as random variables. Reliability models for electronic components usually deal with a single failure mechanism, the one that dominates the failure of the component. The current model considers the presence of multiple mechanisms and analyzes the aggregate effect of these mechanisms.

In this dissertation, six different forms of transition rates are considered for failure mechanisms. These forms represent patterns that may correspond to the various mechanisms common to electronic components. The first passage time distributions for all the six cases are shown to be Generalized Gamma distributions. These distributions are used to obtain the numerical approximations for mixture distributions when the initial and final states of the mechanisms are assumed to be random. Results show that the mixtures are well represented by a Weibull distribution.

Based on the results of single failure mechanism, the failure time of the component subject to multiple mechanisms is analyzed. It is shown that in both the cases of extreme-value theory and competing risk theory, the aggregate failure time of the component has a Weibull distribution. Weibull distribution is the most frequently assumed distribution for the lifetimes of electronic components. The results of the current model support this assumption.

Reliability is an important factor in the present day competitive, cost conscious world of manufacturing. Electronic component reliability is all the more important because of the ubiquity of electronic devices. A continuous effort is being made to develop models that can predict the reliability of electronic components with higher and higher degrees of precision. It is hoped that this model will contribute to that endeavor.

CHAPTER 2

Literature Review

Estimation of the reliability of electronic components has become very important because of the extensive use of electronic equipment in modern times. The estimation is generally based on either empirical procedures or stochastic modelling procedures. In an empirical procedure, the reliability or the failure rate of the component is estimated using observed failure times of copies of the component. MIL-HDBK-217, the most important reference manual for reliability of electronic components, lists the failure rates for various components. These rates are based on either laboratory data or field data. Using suitable conversion factors, failures at different operating conditions may be obtained from the failure rates at standard conditions. In a stochastic modelling approach, a mathematical model for a component failure is developed that will yield a probability distribution function for the failure times. Some of the parameters of the functions have to be obtained from the physical properties of the the components and the operating conditions.

The most extensively used formula in failure rate models is an Arrhenius relationship (O'Connor (1983)):

$$\lambda(T) = \lambda(T_R) \exp\left[\frac{E_a}{K}\left[\frac{1}{T_R} - \frac{1}{T}\right]\right] \tag{2.1}$$

where,

$\lambda(T)$ failure rate at temperature T

T Temperature

T_R reference temperature

E_a Activation energy for the particular failure mechanism

K Boltzmann's constant.

Once a failure rate is empirically estimated at a reference temperature, failure rate at any desired temperature can be calculated using the above formula.

Frost and Poole (1987) develop a stochastic model for estimating the reliability of an integrated circuit subject to electromigration. An integrated circuit is regarded as a linear array of very small elements. Failure of any element causes the failure of the component as a whole. For each element, the electromigration failure time distribution is assumed to be lognormal with density function of the form:

$$P_E(t) = \frac{1}{\sigma\sqrt{2\pi}\, t} \exp\left[-1/2\left(\frac{\ln t - \ln t_{50}}{\sigma}\right)^2\right] \tag{2.2}$$

where,

t_{50} = median time to failure

σ = standard deviation of the logarithm of time.

The median of the distribution, t_{50}, is calculated by correlating it to the properties of the element. Using the median values, the failure rate is calculated for each element. The component failure rate is then obtained by summing up the failure rates of all the elements. Only one failure mechanism is considered for the analysis.

In a degradation model, it is assumed that one of the physical entities (or a state variable) of the component is gradually decreasing, and as a result, the component is becoming more susceptible to failure. The state variable may be random dependent or random-independent (Kapur (1974)). For a random-dependent case, the initial value of the variable is random but once a certain value is realized, it changes over time in a deterministic way. Much of the mechanical reliability literature reflects this approach (Bratt, Reethof and Weber (1964), Shooman (1968)). A state variable is assumed random-independent if at any time t, only its probability distribution may be known. In this case, the process can be modelled as a stochastic process. Failure occurs when the process reaches a certain level for the first time. Analyses of this type address the "level crossing" problem. Only simple cases have been solved in this area. Folks and Chhikara (1978) analyze the problem with the state variable being a Brownian motion and failure occurring when the process reaches a fixed level. Basu and Ebrahimi (1983) consider the case when both the state variable and the threshold level are Brownian motions.

For a discrete-space continuous-time Markov process, the first passage time can be obtained by solving the Chapman-Kolmogorov forward differential equations. This may involve direct solution techniques or transform techniques such as Laplace transform techniques. Several papers and text books deal with the pure birth processes and absorption probabilities for different forms of birth rates. Karlin (1975),

Bartlett (1966) and Cox (1965) are some of the textbooks that are particularly helpful. Srinivasan (1978) gives the solution for transition probabilities of a pure birth process with a general form for for the transition rate, $\lambda_i(t)$. The Laplace transform technique is used to solve for the transition probabilities. The Laplace transform function is then resolved into its partial fractions from which the inverse is obtained to yield the transition probabilities.

When the parameters in a probability distribution function are random variables, one can obtain the unconditional distribution by the weighted sum of all the distribution functions using the probabilities on the random parameters as the weights. This is called a mixture. In the present dissertation, the initial and final states of the stochastic process pertaining to a single failure mechanism are assumed to be random. The failure time distribution for the mechanism is then expressed as the mixture of first passage time distributions conditioned on initial and final states. Mixtures of distributions are first investigated by Lundburg (1909) in connection with insurance and risk theory. These are some of the first studies in the area of stochastic processes and they parallel the development of the models of Brownian Motion and related processes. The model addresses insurance claims where the accumulated claim value is a stochastic process dependent on number of claims and the number of claims is Poisson distributed. The additional assumption is that the claim amounts are independent and identically distributed.This is a part of a theory called the collective risk theory. A major portion of research in mixtures is still carried out by risk analysts developing different types of models and better approximation methods (Buhlmann (1970)).

When the mixing distribution is Poisson, the mixing process is called Poisson mixing. The distribution that is being mixed can be either a discrete or a continuous distribution provided it has a single parameter. Haight (1968) gives a brief review of Poisson mixing. In almost every work on Poisson mixing, the parameter being mixed is the number of convolutions of a distribution with itself. This is called a compound Poisson process. Feller (1943) and Gurland (1958) deal with different types of mixture distributions assuming various probability distributions for the parameter that is being mixed. The reliability model that is developed here involves Poisson mixing of Generalized Gamma distributions which has not been solved in closed form. Neuts (1981) describes some of the properties of this mixture and presents the computational techniques to obtain its probabilities.

Tran (1987) studies the unimolecular degradation process. It is assumed that the transition rates are independent of both time and state. The mixture of the first passage time is analyzed under the assumption that both the initial and final states are random variables with Poisson probability distributions. Only numerical methods are used for the mixture and the numerical approximations fit a three parameter Weibull distribution well.

When a component is subject to multiple failure mechanisms, extreme value theory or competing risk theory may be used for modelling. Extreme value theory analysis is based upon the assumption that the component is subject to a large number of failure mechanisms, all having a common distribution. The system can be viewed as a chain with n links each of which represents a failure mechanism. The system fails when the weakest link in the system fails. This is also called a series structure. The distribution of the lifetime of the chain is equal to the distribution of the lifetime

of the weakest link. If there is no specific link that is the weakest link, the lifetime of the chain has the same distribution as the minimum of lifetimes of all the links. For a large n, Gumbel (1958) shows that the minimum displays a limiting distribution. This is the minimum value or the extreme value approach. Gumbel further shows that there are only three types of extreme value distributions. The extreme value distribution occurs in many practical situations. As an example, the strength of a material may have a Weibull distribution which is an extreme value distribution. This happens because the strength is a result of several forces in the material. The extreme value distribution has been investigated by Fisher and Tippet (1928), Gnedenko (1943) and others. Gumbel (1958) contains a bibliography and discussion of applications.

Competing risk analysis models may be used to evaluate the effect of several failure mechanisms acting simultaneously on the component with each of them generating a distinct lifetime distribution of. The number of mechanisms may be small. The resulting lifetime distribution is the minimum of these distributions. Assuming that all mechanisms are independent and their number is small, Chiang (1964) explores the problem in terms of disease processes. The hazard rates of individual failure mechanisms are said to be proportional if the ratio of hazard rates of any two mechanisms is constant over time. Elandt-Johnson (1976) considers the case when there are proportional hazard rates and the independence assumption is relaxed.

In summary, many relevant results have been developed in different areas. A subset of these results is used here. Other needed results, particularly in the area of mixture distributions and multiple mechanisms, do not exist. They will be developed in this dissertation.

CHAPTER 3

Development of a Component Failure Model

3.1 Introduction

In this chapter, the structure of a general model for a component reliability is developed. The model represents the failure behavior of a component in terms of its failure mechanisms. Each mechanism can be represented as a chemical degradation process in which the molecules of reactant A are continuously changing into reaction product B. The actual process may be a chemical or a physical process depending on the nature of its degradation. An example is the oxidation of gallium arsenide in liquid crystal displays (Kool (1968)). While the LCD is in operation, oxygen may react with the gallium arsenide present in the crystal thereby reducing the amount of arsenide and resulting in a loss of brilliance of the crystal. This is a chemical process. Another example is electromigration in integrated circuits which is a physical

process. Here the continuous impact of electrons on the metal grains causes the grains to move in the direction of the electron flow. This transfer of metal atoms occurs when there are defective metalization layers caused by flaws in the pattern mask or errors in the manufacturing process (Amarasekhara (1988)).

Every component is subject to certain mechanisms that induce failure in the component. Failure is the inability of the component to perform according to a set standard. In the case of degradation mechanisms, completion of the reaction is not always necessary for failure. The component fails if there is sufficient amount of reaction product B to reduce the performance of the component to an unacceptable level.

In the model, the behavior of a single failure mechanism is studied first. The results are then used to analyze the effect of several mechanisms of the same class acting on the component simultaneously.

3.2 Assumptions on Component Failure

For the model, the following assumptions are made concerning component failure.

1. The component is subject to multiple failure mechanisms. These mechanisms are acting on the component simultaneously.

2. All failure mechanisms portray similar behavior.

3. Each failure mechanism exerts a lifetime distribution on the component. It is assumed that the lifetime distributions generated by individual failure mechanisms are independent of each other. Each mechanism exerts the same lifetime distribution whether it is acting alone or in the presence of other failure mechanisms.

3.3 Assumptions on the failure mechanisms

The main assumptions about individual mechanisms are the following.

1. Each failure mechanism is of degradation type. The state changes in unit steps. The failure mechanism involves changes in the physical or chemical structure of the component which ultimately result in the failure of the component. This physical or chemical structure is defined as the state of the process. For the degradation process, the state changes gradually. There is no instantaneous change of large magnitude in the value of the state. It is also assumed that when there is any change in the state, it occurs only in discrete unit steps.

2. The degradation process is irreversible.

   The change in the state of the failure process is unidirectional. In real-life situations, there may be changes in the reverse direction but for all practical purposes, they are considered to be negligible. In the terminology of the chemical degradation process, the reaction is irreversible.

3. The future state of the process depends only upon the present state and not upon the entire past history of the process.

   This assumption does not imply that the past course of history has no bearing on the future. It states that given the present state, the future is independent of the past.

4. Reaction rate $\lambda_j(t)$ may depend upon the state of the process, time and operating conditions. The effect of the state and time are independent of each other.

   The reaction rate, $\lambda_j(t)$, is the rate at which the process changes its state from j to j+1 at time t. In a chemical degradation, it is the rate at which an additional molecule of B is formed per unit time given that j molecules of B have already been created.

   Depending upon the degradation process, the reaction rate may be a function of either one or both of the current state and time. Any changes in the physical or chemical structure of the component or in the environment should be reflected in the reaction rate function of the model and hence in the equation for failure times of the component.

   The effects of the state and time on the reaction rate are assumed to be independent of each other. The reaction rate $\lambda_j(t)$ can be expressed as a product of two separate functions, one each of state and time. That is, $\lambda_j(t) = g(j) \cdot h(t)$, where g and h are functions of state j and time t, respectively. Note that h(t) also includes the pertinent equations for the kinetics of the reaction.

5. The process has an initial state i and a final state k. A component fails when the process reaches the final state k.

It is a reasonable assumption that the component has a certain number of defects (molecules of reaction product B) at the beginning of operation. Consider this number to be the initial state, i. As the operation continues, there is an increase in the number of molecules of reaction product. Accumulation of these defects impedes the proper functioning of the component. Eventually, at a particular level of defects, the component fails. This failure state is the final state k. For any process, i and k may have fixed or random values.

3.4 Randomness in the Failure Process

The reliability model captures much of the essence of a real life situation. Most of the characteristics and parameters of the model are considered to be random rather than deterministic. The randomness of the failure process is described in three levels. At the bottom level, a single failure mechanism which has fixed and known initial and final states is considered. In this situation, the failure time of the component or the required time for the process to reach the final state from the initial state is random because the reaction rate or the time the process spends in a particular state is random. At the next level, this randomness is compounded by considering the randomness in the initial and final states of the process. At the third level, the overall failure process of the component is studied. This failure process is a result of multiple failure mechanisms, each behaving randomly. To capture all this randomness in a mathematical model, a stochastic model is constructed.

3.5 Mathematical Model

Assume that the initial and final states are fixed and are known. By assumption, the future course of the process is independent of the past, provided that the the present state is known. For the representative chemical degradation process, X(t) represents the number of molecules of product B present at time t. The initial state X(0) = i is the initial number of product B molecules present at the start of operation. Failure occurs when the process reaches state k. Here, k is the absorbing state for the pure birth process. The corresponding transition diagram is shown in Figure 1.

To study the dynamic behavior of the process, one needs the transition rates of the process. These transition rates are derived from the reaction rates of the degradation process. For the degradation process, the reaction rate $\lambda_j(t)$ is defined as the fractional change in product B per unit time at time t. So,

$$\lambda_j(t) = \frac{\Delta n}{\Delta t} \frac{1}{n_j}. \tag{3.1}$$

Rearranging terms,

$$\frac{\Delta n}{n_j} = \lambda_j(t).\Delta t \tag{3.2}$$

But, $\frac{\Delta n}{n_j}$ is the probability of change in the state during Δt. So the transition rates of the pure birth process are the same as the reaction rates of the degradation process. For a given initial state i and final state k, all the transition rates can thus be obtained.

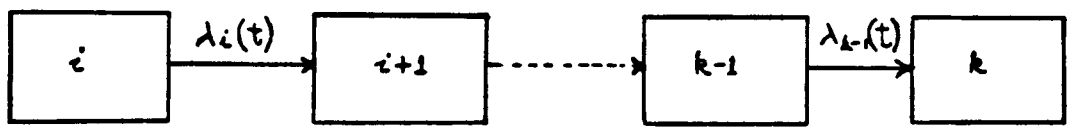

Figure 1. Transition diagram

$$A = \begin{bmatrix} -\lambda_l(t) & \lambda_l(t) & 0 & \dots & \dots & 0 & 0 \\ 0 & -\lambda_{l+1}(t) & \lambda_{l+1}(t) & \dots & \dots & 0 & 0 \\ 0 & 0 & -\lambda_{l+2}(t) & \dots & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \dots & -\lambda_{k-1}(t) & \lambda_{k-1}(t) \\ 0 & 0 & 0 & \dots & \dots & 0 & 0 \end{bmatrix}$$

Figure 2. Transition Matrix A

The general state transition for a Markov process is defined as the solution to the Chapman-Kolmogorov forward differential equations:

$$P' = PA. \tag{3.3}$$

Here P is the matrix of transition probabilities and P' the derivative of P with respect to time (Cinlar (1975)). Any entry (i,j) of the matrix P is defined as P(i,j,t), the probability of the process being in state j at time t when the initial state is i. A is the matrix of transition rates. It is also called the generator matrix of the process. The entries of A are given by,

$$A_{jj} = -\lambda_j(t)$$

$$A_{j,j+1} = \lambda_j(t), \quad for \quad l \le j \le k-1 \tag{3.4}$$

and 0 elsewhere. The matrix A is shown in Figure 2.

The matrix A has a special structure as it represents a pure birth process. The process may only go from state j to state j + 1 at any instant. Thus, the entries on the right super-diagonal are positive. Once the process enters the failure state k, it cannot go to any other state. This makes k an absorbing state and the entries of A corresponding to row k are equal to zero. All the states other than k are transient. The probability of the process remaining in a transient state decreases as time increases. Thus, the transition rates or the entries on the diagonal are negative. The rest of the elements are all equal to zero since no other transition can take place.

In the general case, the non-zero components of the generator matrix A depend upon the state, time and operating conditions. Specific assumptions concerning the reaction rate functions lead to different models and correspondingly different results.

Lifetimes as First Passage Times

The lifetime of a component is equal to the time at which the process reaches the failure state k. Since the process is a pure birth process, the failure time is the first passage time from state i to state k. The probability distribution of first passage time, p(i,k,t), thus gives the lifetime distribution. This is the conditional distribution for given initial and final states and is denoted by $F(t|i,k)$.

One or both of the extreme states may be random in a given mechanism. The randomness in these two states is incorporated by considering them as random variables and assuming suitable probability distribution functions for these two random variables. For the initial state i, the possible distributions have to be discrete distributions. The choice may depend upon the actual physical entity the state represents. An obvious choice is the Poisson distribution. Extensive experience with counting processes has shown that individual, convoluted and compound counting processes usually follow or converge to a Poisson distribution (Bortkewitsch (1898)). Another choice is the negative binomial distribution. This distribution can occur in the case of mixing a Poisson distribution whose mean is gamma distributed (Nachlas, Ricapito, and Wiesel (1983)). The binomial distribution may be used if the state of the process represents the number of defects in the component and each particle in the component has the same probability of being defective. Similarly, the probability distribution for the final state k depends on the physical nature of the device. To account for the randomness in the two extreme states for a failure mechanism, mixture distributions are computed using the conditional first passage time distributions and the probability distributions of the two states. If P(i) and P(k) are the probability

density functions for I and k, then the unconditional or the marginal distribution F(t) is given by,

$$F(t) = \sum_k \sum_l F(t|l,k)\ P(l)P(k). \qquad (3.5)$$

Sometimes it may be possible to assume a specific form of a probability distribution on the difference j between the initial and final states. In this case F(t) is given by,

$$F(t) = \sum_j F(t|j)\ P(j). \qquad (3.6)$$

Once a single mechanism is analyzed, the results can be extended to multiple mechanisms. It is assumed that the mechanisms are of similar behavior and that they are independent of each other. If n mechanisms are active in a component, then they generate n random failure times. For any given failure, only one of these failure times manifests itself and this is the actual observed failure time of the component. The probability distribution of the failure time of the component is given by the distribution of the minimum of the n individual failure times. A specific procedure to compute the distribution of the minimum depends on additional assumptions. Two procedures are applied for the present system; one is extreme-value theory and the other is competing risk theory.

3.6 Variations of the model

Variations of the model can be constructed on the basis of the nature of the transition rate, $\lambda_j(t)$, and the nature of the randomness in the initial and final number of defects. The present analysis considers the variations in the transition rate only.

It is assumed that the transition rate is a separable function of the state occupied and the time. The combined function will also reflect the operating conditions. The rate $\lambda_j(t)$ is expressed in the form $\lambda_j(t) = g(j)h(t)$.

Six realizations of the above equation might be considered. They are, in decreasing order of complexity:

1. $g(j) = j^n$ *and* $h(t) = m\alpha t^{m-1}$
2. $g(j) = j^n$ *and* $h(t) = \alpha$
3. $g(j) = j$ *and* $h(t) = m\alpha t^{m-1}$
4. $g(j) = j$ *and* $h(t) = \alpha$
5. $g(j) = 1$ *and* $h(t) = m\alpha t^{m-1}$
6. $g(j) = 1$ *and* $h(t) = \alpha$

The different forms of transition rates characterize various failure mechanisms common to electronic components. The choice of a particular form depends on the kinetics or the dynamics of the mechanism and on the results available from historical data. If the mechanism is of chemical nature, then the transition rate may be obtained from factors that influence the kinetics of the process such as the reaction energy, change in enthalpy, the concentration of the products and temperature. In the case of a physical mechanism, the rate may depend on the grain size, the momentum of electrons, temperature, or bond energy. In either case, the transition rate may further be influenced by the geometry of the component.

Failure mechanisms suitable for the present model are to be identified first. There are several candidate degradation mechanisms for the current model. Examples are oxidation, electromigration, and threshold-voltage shifting effects in MOS (Metal

Oxide Semiconductor) devices. It is possible to derive transition rates for some of these mechanisms from existing reliability models. In these models, a probability distribution function is assumed for the failure times of a mechanism. The corresponding variables of the function such as parameters and failure rates are expressed in terms of physiochemical properties of the component and operating conditions. Examination of these variables may indicate the rate of progress of the degradation process and possible forms of the transition rate. A few of the relevant works are discussed below.

Often, failure rate is modeled by an Arrhenius relationship (equation 2.1). Here, the failure rate is independent of time and the extent of degradation. This indicates that perhaps the transition rate for that failure mechanism also is time and state independent.

In the model developed by Frost and Poole (1987), the failure time of a component due to electromigration follows a lognormal distribution (equation 2.2). The median time to failure t_{50} is expressed as a function of current density, temperature, and conductor width:

$$t_{50} = A(W)J^{-N}\exp(E_a/kT),$$

where, J is the current density and A(W) is a material constant that is function of width of the component. The above form suggests that the transition rate may depend on the state of the process. The transition rate functions for cases 2 and 4 are worth considering for electromigration.

The six transition rate functions are constructed to account for the dependency of the transition rate on the state of the process and time. With a proper choice of constants, they can be applied to many of the degradation processes. Once the transition rate is selected, defect distributions for the initial and final states are to be identified. A transition rate function together with the defect distributions describe a failure process due to a single mechanism completely. Solution of the model in presence of several such mechanisms gives the reliability of a given component.

In the next chapter, a general solution procedure is presented to compute the reliability of the component for all the six variations of the transition rate function. Specific distributions are assumed to describe the randomness in the initial and final values of the degradation process. The failure time distribution is approximated when it is not feasible to obtain a closed form function. The solution procedures are illustrated with numerical examples.

CHAPTER 4

Solution Procedure

4.1 Introduction

Solution of the model defined above gives the overall failure behavior of a component. As in the model development, the solution procedure follows in three stages. Analysis of a single stochastic process with fixed initial and final states forms the first stage. The failure time for this process is given by the first passage time for the process from the initial state i to the final state k. The second stage deals with a process where the initial and final states are random. The failure time probability of the process is constructed as a mixture of the first passage time distributions obtained in the first stage. The mixture distribution represents a singe failure mechanism. Finally, the results of the second stage are expanded to cover the aggregate effect of several mechanisms simultaneously active in the component.

For a failure mechanism, an appropriate transition rate function must be selected. Once a particular form of transition rate is chosen, the first passage time distribution for the corresponding pure birth process is derived. This is carried out by solving the set of Chapman-Kolmogorov differential equations. Any of the standard techniques to solve ordinary differential equations can be used to obtain the distribution. In the present work, the solution is obtained by Laplace transform method. As an example, the solution procedure for one of the six cases is shown in detail in the current chapter.

The procedure to derive the mixture distribution depends on the form of the transition rate and the probability distributions of the initial and final states of the process. The conditional first passage time distribution used in the mixtures can be either a probability distribution function or its Laplace transform. The choice depends on the ease of algebraic manipulation of the mixture distribution. When it is not possible to obtain the mixture distribution in a closed form, the distribution is numerically approximated to a probability function of a known form. The selection of the approximating function is decided by the physical model or the norms set by the reliability practitioners. The approximation must be validated by appropriate statistical methods.

For multiple mechanisms, the failure time of the component can be expressed as the minimum of the individual failure times. To derive the distribution of the minimum, failure time functions of single mechanisms or their transforms are used. For some individual functions, finding the distribution of the minimum is a difficult task because the form of the function may not be simple. In that case, various approaches must be employed to obtain a solution. If all the mechanisms generate identical failure time distributions and the number of mechanisms in a component is large, then

extreme-value theory may be applied. Here, the minimum distribution is expressed as a limiting distribution. A detailed description of the procedure of the extreme-value theory is given in this chapter. If finding the distribution of the minimum is not possible and if the individual failure times are not identical, then an approximate solution must be derived to express the minimum distribution.

4.2 First Passage Time of a Birth Process

The first passage time distribution function is derived here for a single failure process with known transition rates and given initial and final states. This is carried out by solving the system of equations given by (3.3). These are the Chapman-Kolmogorov forward differential equations and they are:

$$P'(i,i,t) = -\lambda_i(t)P(i,i,t) \tag{4.1}$$

$$P'(i,j,t) = \lambda_{j-1}(t)P(i,j-1,t) - \lambda_j(t)P(i,j,t), \quad \text{for } i+1 \le j \le k-1 \tag{4.2}$$

$$P'(i,k,t) = \lambda_{k-1}(t)P(i,k-1,t). \tag{4.3}$$

The initial and final states are represented by i and k respectively.

When the transition rates of a process are independent of time, the process is time homogeneous and it is easy to follow the Laplace transform method to solve the Chapman-Kolmogorov equations. If the transition rates depend on time, then the process is time non-homogeneous and the set of equations will not be linear with

respect to time. Hence Laplace transform methods cannot be used directly. In this case, the process is converted to a time homogeneous process by time scale transformation (Cinlar (1975)) and the equations can be solved by transform methods. The solution procedure using the transform method is shown below:

Taking the Laplace transform of both sides, equation (4.1) yields

$$s.P(i,i,s) - P(i,i,0) = -\lambda_i P(i,i,s),$$

where P(i,i,0) = 1.0 from the initial condition.

Solving for P(i,i,s),

$$P(i,i,s) = \frac{1}{(s+\lambda_i(t))} \tag{4.4}$$

For j = 1, equation (4.2) is

$$P'(i,i+1,t) + \lambda_{i+1}(t)P(i,i+1,t) = \lambda_i(t)P(i,i,t)$$

Taking Laplace Transformations on both sides,

$$s.P(i,i+1,s) - P(i,i+1,0) + \lambda_{i+1}(t)P(i,i+1,s) = \lambda_i(t)P(i,i,s)$$

P(i,i+1,0) = 0. So,

$$P(i,i+1,s) = \frac{\lambda_i(t)}{(s+\lambda_{i+1}(t))} P(i,i,s)$$

$$= \lambda_i(t) \frac{1}{(s+\lambda_{i+1}(t))(s+\lambda_i(t))}$$

Solving iteratively, we find

$$P(i,j,s) = \lambda_{j-1}(t) \ldots \lambda_i(t) \frac{1}{(s+\lambda_j(t)) \ldots (s+\lambda_i(t))}, \quad \text{for } i<j<k. \tag{4.5}$$

Using Laplace Transforms on both sides of equation (4.3),

$$s.P(i,k,s) - 0 = \lambda_{k-1}(t) \ldots \lambda_i(t) \frac{1}{(s+\lambda_{k-1}(t)) \ldots (s+\lambda_i(t))}$$

from which,

$$P(i,k,s) = \lambda_{k-1}(t). \ldots \lambda_i(t) \frac{1}{s(s+\lambda_{k-1}(t)) \ldots (s+\lambda_i(t))} \tag{4.6}$$

The inverse Laplace transform of the above equation is obtained by first resolving it into its partial fractions. This can be written as,

$$P(i,k,s) = \frac{A_0}{s} - \sum_{j=i}^{k-1} \frac{A_j}{(s+\lambda_j(t))} \tag{4.7}$$

The coefficients A_0 and A_j's are given by

$$A_0 = 1$$

$$A_j = \lambda_{k-1}(t) \ldots \lambda_i(t) / \lambda_j(t) \prod_{\substack{r=i \\ r \neq j}}^{k-1} (\lambda_r(t) - \lambda_j(t)) \tag{4.8}$$

These values are substituted into equation (4.7) and the inverse of the Laplace transform is constructed term by term.

The explicit solution is shown below for case 2, where $\lambda_j(t) = \alpha j^n$

The first term of P(i,k,s) is 1/s and the inverse transform is 1.

The second term is

$$\left\{\alpha(k-1)^n \ldots \alpha i^n/\alpha i^n.\prod_{\substack{r=1\\ r\neq i}}^{k-1}(\alpha r^n - \alpha i^n)\right\}\frac{1}{(s+\alpha i^n)}$$

The inverse transform is $\left\{\alpha(k-1)^n \ldots \alpha i^n/\alpha i^n.\prod_{\substack{r=1\\ r\neq i}}^{k-1}(\alpha r^n - \alpha i^n)\right\}\exp(-\alpha i^n t)$

$$= \left\{(k-1)^n \ldots i^n/i^n.\prod_{\substack{r=1\\ r\neq i}}^{k-1}(r^n - i^n)\right\}\exp(-\alpha i^n t)$$

Similarly the other terms are evaluated and P(i,k,t) is given by

$$P(i,k,t) = 1 - \left(\prod_{j=i}^{k-1} j^n\right).\sum_{r=i}^{k-1}\{\exp[-\alpha r^n t]\}/r^n \prod_{\substack{j=i\\ j\neq r}}^{k-1}\{j^n - r^n\} \qquad (4.9)$$

Solutions to all the other cases are the following:

Case (1). $g(j) = j^n$ *and* $h(t) = m\alpha t^{m-1}$

$$P(i,k,t) = 1 - \left(\prod_{j=i}^{k-1} j^n\right) \cdot \sum_{r=i}^{k-1} \{\exp[-\alpha r^n t^m]\}/r^n \prod_{\substack{j=i \\ j \neq r}}^{k-1} \{j^n - r^n\} \tag{4.10}$$

Case (3). $g(j) = j$ *and* $h(t) = m\alpha t^{m-1}$

$$P(i,k,t) = 1 - \left(\prod_{j=i}^{k-1} j\right) \cdot \sum_{r=i}^{k-1} \{\exp[-\alpha r t^m]\}/r \prod_{\substack{j=i \\ j \neq r}}^{k-1} \{j - r\} \tag{4.11}$$

Case (4). $g(j) = j$ *and* $h(t) = \alpha$

$$P(i,k,t) = 1 - \left(\prod_{j=i}^{k-1} j\right) \cdot \sum_{r=i}^{k-1} \{\exp[-\alpha r t]\}/r \prod_{\substack{j=i \\ j \neq r}}^{k-1} \{j - r\} \tag{4.12}$$

Case (5). $g(j) = 1$ *and* $h(t) = m\alpha t^{m-1}$

$$P(i,k,t) = 1 - \sum_{r=0}^{k-i-1} [(\alpha t^m)^r \cdot \exp - \{\alpha t^m\}]/r! \tag{4.13}$$

Case (6). $g(j) = 1$ *and* $h(t) = \alpha$

$$P(i,k,t) = 1 - \sum_{r=0}^{k-i-1} [(\alpha t)^r . \exp - \{\alpha t\}]/r! \qquad (4.14)$$

Cases 2, 4 and 6 are special cases of 1, 3, and 5 respectively. All are realizations of the Generalized Gamma distributions (Stacy (1962)).

The term P(i,k,t) gives the probability of being in state k at time t when the initial state is i. Since the process is a pure birth process and k is the absorbing state, P(i,k,t) is equivalent to the distribution function for the first passage time to k from i. This is $F(t|i,k)$, the failure time distribution for the single mechanism with initial state i and final state k.

4.3 Mixture Distributions

If the initial and final states are random variables, then the first passage time for the process is given by a mixture distribution. For example, if i and j are random and their probability distributions are given by P(i) and P(k) respectively, then

$$F(t) = \sum_k \sum_i F(t|i,k)\ P(i)P(k)$$

Suppose the distance j between i and k is considered as random and the probability distribution on j is given by P(j). Then

$$F(t) = \sum_j F(t|j)\ P(j)$$

The density function, f(t), is used instead of the cumulative distribution function, F(t), whenever f(t) has a more manageable form. Since the expressions and the solution procedures depend on the form of $\lambda_i(t)$, the solutions are treated separately.

4.3.1 Case 6.

This is the simplest of all the cases. The transition rate $\lambda_i(t)$ is independent of both state and time. The first passage time for specific i and k is given by equation (4.14),

$$F(t|j) = 1 - \sum_{r=0}^{j-1}[(\alpha t)^r . \exp -(\alpha t)]/r! \ ,$$

where j is the distance between i and k. The probability density function is given by

$$f(t|j) = \alpha e^{-\alpha t}\frac{(\alpha t)^{j-1}}{(j-1)!} \tag{4.15}$$

If j has a Poisson distribution with mean μ,

$$f(t) = \frac{1}{(1-e^{-\mu})}\sum_{j=1}^{\infty}\alpha e^{-\alpha t}\frac{(\alpha t)^{j-1}}{(j-1)!}e^{-\mu}\frac{\mu^j}{j!} \tag{4.16}$$

In the model, instantaneous failures at time zero are not allowed. The value 0 is excluded from the range of values of the Poisson variable j. So the term $1/(1-e^{-\mu})$ appears in the above expression as the normalizing factor.

Computing the failure probabilities using the above expression is complicated. Moreover, if further analysis is needed where several failure mechanisms are involved, then a more manageable form of the density function is desired. Therefore, the function is approximated to a probability density function of a known form. Plots

of similar types of functions suggest that the model exhibits Weibull distribution behavior (Tran (1987)). Because the Weibull distribution is an extreme value distribution, it is well suited for the present model.

Since instantaneous failures at time zero are not allowed and the initial state i is assumed to be strictly less than the final value k, the approximating Weibull distribution is taken to be a function of shape and scale parameter only. The location parameter is set at zero.

The approximation is carried out by the method of matching moments. The moments of the original distribution and the approximating distribution should be equal. The parameters of the Weibull distribution are to be chosen in such a way that the first and second moments of the distribution are equal to the first and second moments of the mixture distribution respectively. Since the moments of the mixture distribution are functions of the physical characteristics of the failure model, this procedure of approximating will preserve the information about the physical properties of the model.

The procedure of approximation is carried out in the following steps.

(1) Compute the first and second moments of the mixture distribution for specific numerical values of its parameters.

(2) Estimate the parameters of the Weibull distribution by matching its first two moments with those of the mixture distribution.

(3) Check the goodness of the approximation by comparing the third and fourth moments of the mixture distribution with those of the Weibull distribution.

(4) Perform a goodness-of-fit test to validate the approximation.

The moments of the mixture distribution are derived from the Laplace transforms.

The n th moment is given by $\mu_n = (-1)^n \frac{d^n f(s)}{ds^n}$

where f(s) is the Laplace transform of the failure density distribution.

The Laplace transform of the mixture distribution (equation (4.16)) is given by

$$f(s) = \frac{1}{(1-e^{-\mu}).} \sum_{j=1}^{\infty} \left(\frac{\alpha}{\alpha+s}\right)^j e^{-\mu} \frac{\mu^j}{j!}$$

$$= \frac{1}{(1-e^{-\mu})} \sum_{j=0}^{\infty} \left(\frac{\alpha}{\alpha+s}\right)^j e^{-\mu} \frac{\mu^j}{j!} \quad - e^{-\mu}$$

$$= \frac{e^{-\mu s/\alpha+s} - e^{-\mu}}{(1-e^{-\mu})} \tag{4.17}$$

The first four moments are now computed using f(s).

First moment μ_1:

$$\frac{df(s)}{ds} = \frac{1}{(1-e^{-\mu})} \left[\frac{d}{ds}\left(-\frac{\mu s}{(\alpha+s)}\right)\right] e^{-\mu s/\alpha+s}$$

$$= \frac{1}{(1-e^{-\mu})} \left[-\frac{\alpha\mu}{(\alpha+s)^2}\right] e^{-\mu s/\alpha+s}$$

$$-\frac{df(s)}{ds}\Big|_{s=0} = \frac{1}{(1-e^{-\mu})}\left(\frac{\alpha\mu}{\alpha^2}\right) = \frac{1}{(1-e^{-\mu})}\left(\frac{\mu}{\alpha}\right)$$

So,

$$\mu_1 = \frac{1}{(1 - e^{-\mu})} \left(\frac{\mu}{\alpha}\right) \tag{4.18}$$

Second moment μ_2:

$$\frac{d\,f(s)}{ds} = \frac{d}{ds} \frac{1}{(1 - e^{-\mu})} \left[- \frac{\alpha\mu}{(\alpha + s)^2} \right] e^{-\mu s/\alpha + s}$$

$$\frac{d^2 f(s)}{ds^2} = \frac{1}{(1 - e^{-\mu})} \left\{ \left[\frac{2\alpha\mu}{(\alpha + s)^3} \right] e^{-\mu s/\alpha + s} + \frac{-\alpha\mu}{(\alpha + s)^2} \frac{-\alpha\mu}{(\alpha + s)^2} e^{-\mu s/\alpha + s} \right\}$$

$$= \frac{1}{(1 - e^{-\mu})} \frac{\alpha\mu}{(\alpha + s)^3} e^{-\mu s/\alpha + s} \left[2 + \frac{\alpha\mu}{(\alpha + s)} \right]$$

$$\frac{d^2 f(s)}{ds^2} \Big|_{s=0} = \mu_2 = \frac{1}{(1 - e^{-\mu})} \frac{\alpha\mu}{\alpha^2} \left[2 + \frac{\alpha\mu}{\alpha} \right]$$

$$= \mu_1 \frac{1}{\alpha} (2 + \mu) \tag{4.19}$$

Third moment μ_3:

$$\frac{d^3 f(s)}{ds^3} = \frac{1}{(1 - e^{-\mu})} \left\{ \frac{d}{ds} \left[2 \frac{\alpha\mu}{(\alpha + s)^3} + \frac{(\alpha\mu)^2}{(\alpha + s)^4} \right] e^{-\mu s/\alpha + s} \right.$$

$$\left. + \left[\frac{2\alpha\mu}{(\alpha + s)^3} + \frac{(\alpha\mu)^2}{(\alpha + s)^4} \right] \frac{d}{ds} \left(e^{-\mu s/\alpha + s} \right) \right\}$$

$$= \frac{1}{(1-e^{-\mu})}\left\{\left[-\frac{6\alpha\mu}{(\alpha+s)^4}-\frac{4(\alpha\mu)^2}{(\alpha+s)^5}\right]e^{-\mu s/\alpha+s}\right.$$

$$\left.+\left[\frac{2\alpha\mu}{(\alpha+s)^3}+\frac{(\alpha\mu)^2}{(\alpha+s)^4}\right]\frac{(-\alpha\mu)}{(\alpha+s)^2}e^{-\mu s/\alpha+s}\right\}$$

$$= \frac{1}{(1-e^{-\mu})}\left[-\frac{6\alpha\mu}{(\alpha+s)^4}-\frac{4(\alpha\mu)^2}{(\alpha+s)^5}-\frac{2(\alpha\mu)^2}{(\alpha+s)^5}-\frac{(\alpha\mu)^3}{(\alpha+s)^6}\right]e^{-\mu s/\alpha+s}$$

$$= -\frac{1}{(1-e^{-\mu})}\left[\frac{6\alpha\mu}{(\alpha+s)^4}+\frac{6(\alpha\mu)^2}{(\alpha+s)^5}+\frac{(\alpha\mu)^3}{(\alpha+s)^6}\right]e^{-\mu s/\alpha+s}$$

$$\frac{d^3f(s)}{ds^3}\Big|_{s=0} = \frac{-1}{(1-e^{-\mu})}\left[\frac{6\alpha\mu}{\alpha^4}+\frac{6(\alpha\mu)^2}{\alpha^5}+\frac{(\alpha\mu)^3}{\alpha^6}\right]$$

From which,

$$\mu_3 = \mu_1\frac{1}{\alpha^2}[6+6\mu+\mu^2] \tag{4.20}$$

Fourth moment μ_4 :

$$\frac{d^4f(s)}{ds^4} = \frac{-1}{(1-e^{-\mu})}\left\{\frac{d}{ds}\left[\frac{6\alpha\mu}{(\alpha+s)^4}+\frac{6(\alpha\mu)^2}{(\alpha+s)^5}+\frac{(\alpha\mu)^3}{(\alpha+s)^6}\right]e^{-\mu s/\alpha+s}\right.$$

$$\left.+\left[\frac{6\alpha\mu}{(\alpha+s)^4}+\frac{6(\alpha\mu)^2}{(\alpha+s)^5}+\frac{(\alpha\mu)^3}{(\alpha+s)^6}\right]\frac{d}{ds}[e^{-\mu s/\alpha+s}]\right\}$$

$$= \frac{1}{(1-e^{-\mu})} \left\{ \left[\frac{24\alpha\mu}{(\alpha+s)^5} + \frac{30(\alpha\mu)^2}{(\alpha+s)^6} + \frac{6(\alpha\mu)^3}{(\alpha+s)^7} \right] e^{-\mu s/\alpha+s} \right.$$

$$\left. + \left[\frac{6\alpha\mu}{(\alpha+s)^4} + \frac{(6\alpha\mu)^2}{(\alpha+s)^5} + \frac{(\alpha\mu)^3}{(\alpha+s)^6} \right] \frac{\alpha\mu}{(\alpha+s)^2} e^{-\mu s/\alpha+s} \right\}$$

$$= \frac{1}{(1-e^{-\mu})} \left\{ \frac{24\alpha\mu}{(\alpha+s)^5} + \frac{30(\alpha\mu)^2}{(\alpha+s)^6} + \frac{6(\alpha\mu)^3}{(\alpha+s)^7} \right.$$

$$\left. + \frac{6(\alpha\mu)^2}{(\alpha+s)^6} + \frac{6(\alpha\mu)^3}{(\alpha+s)^7} + \frac{(\alpha\mu)^4}{(\alpha+s)^8} \right\} e^{-\mu s/\alpha+s}$$

$$= \frac{1}{(1-e^{-\mu})} \left[\frac{24\alpha\mu}{(\alpha+s)^5} + \frac{36(\alpha\mu)^2}{(\alpha+s)^6} + \frac{12(\alpha\mu)^3}{(\alpha+s)^7} + \frac{(\alpha\mu)^4}{(\alpha+s)^8} \right]$$

$$\frac{d^4 f(s)}{ds^4}\Big|_{s=0} = \frac{1}{(1-e^{-\mu})} \left(\frac{24\mu}{\alpha^4} + \frac{36\mu^2}{\alpha^4} + \frac{12\mu^3}{\alpha^4} + \frac{\mu^4}{\alpha^4} \right)$$

$$= \frac{1}{(1-e^{-\mu})} \frac{\mu}{\alpha} \frac{1}{\alpha^3} (24 + 36\mu + 12\mu^2 + \mu^3)$$

So,

$$\mu_4 = \mu_1 \frac{1}{\alpha^3} (24 + 36\mu + 12\mu^2 + \mu^3) \tag{4.21}$$

Step 2: Estimation of Weibull parameters:

The first and second moments of the Weibull distribution are set equal to those of the mixture distribution. The scale parameter θ and the shape parameter β are estimated by the method of matching moments(Lloyd and Lipow (1977)). Since the moments of the Weibull distribution do not have closed form expressions in terms of the scale and shape parameters, the parameters are computed by using an iterative procedure.

The Weibull distribution that is used in the analysis is of the form

$$f(t) = \theta\beta t^{\beta-1}\exp(-\theta t^{\beta}). \qquad \theta, \beta > 0, \ t > 0$$

θ is called the scale parameter and β, the shape parameter.

The first moment μ_1 is given by

$$\mu_1 = \int_0^{\infty} t\,\theta\beta t^{\beta-1}\exp(-\theta t^{\beta})dt.$$

Let $T = \theta t^{\beta}$.

Then $t = (T/\theta)^{1/\beta}$, and $\theta\beta t^{\beta-1}dt = dT$.

Therefore,

$$\mu_1 = (1/\theta)^{1/\beta}\int_0^{\infty} T^{1/\beta}\exp(-T) = (1/\theta)^{1/\beta}\Gamma(\frac{1}{\beta}+1), \tag{4.22}$$

where $\Gamma(x)$ is the Gamma Function defined as

$$\Gamma(x) = \int_0^{\infty} T^{x-1} \exp(-T) dT.$$

Similarly, the second moment μ_2 is given by

$$\mu_2 = (1/\theta)^{2/\beta} \int_0^{\infty} T^{2/\beta} \exp(-T) dT = (1/\theta)^{2/\beta} \Gamma(\frac{2}{\beta} + 1) \tag{4.23}$$

From equation (4.22),

$$\theta = \left[\frac{1}{\mu_1} \Gamma(\frac{1}{\beta} + 1) \right]^{\beta} \tag{4.24}$$

And, from equation (4.23), $\theta^{2/\beta} = \frac{1}{\mu_2} \Gamma(\frac{2}{\beta} + 1)$

from which,

$$\beta = (2 \log \theta)/[-\log \mu_2 + \log \Gamma(\frac{2}{\beta} + 1)] \tag{4.25}$$

Using the values of μ_1 and μ_2 of the mixture distribution obtained in equations (4.18) and (4.19), θ and β are calculated from equations (4.24) and (4.25). A computer program is written in FORTRAN to solve for the parameters iteratively. The program is provided with a starting solution and at each iteration, the values of the parameters are updated until they converge to specific values to simultaneously yield the first

and second moments (Lloyd and Lipow (1974, pp. 186-189)). In the program, the Gamma function $\Gamma(x)$ is computed by polynomial approximation (Abromowitz and Stegun (1970, pp.257)).

Step 3. Comparison of higher moments for validation:

The third and fourth moments of the Weibull distribution are given by,

$$\mu_3 = (1/\theta)^{3/\beta}\Gamma(\frac{3}{\beta} + 1) \tag{4.26}$$

and,

$$\mu_4 = (1/\theta)^{4/\beta}\Gamma(\frac{4}{\beta} + 1). \tag{4.27}$$

The third and fourth moments of the approximating Weibull distribution are compared with the corresponding moments of the mixture distribution given by equations (4.20) and (4.21). These comparisons indicate the accuracy of the approximation. The first two moments of the Weibull distribution are computed from the estimated parameters to check the convergence of the iterative procedure.

Step 4. Kolmogorov-Smirnov Test for Goodness of Fit:

The Kolmogorov-Smirnov test for goodness of fit(Kraft and Van Eeden (1968)) is applied to see how well the Weibull distribution approximates the mixture distribution. This is to confirm the conclusions reached from the comparison of higher moments in the previous step. Although the K-S test of goodness of fit is used to check the match of a distribution to random samples, it can be used to test the goodness of the

approximation with slight modifications (Sculli and Wong (1985)). For the present context, the values of the actual cumulative function are treated as random values from an undetermined distribution. Then the K-S one sample test can be applied. In the test, the approximation is checked by measuring the error D_n , given by

$$D_n = max.|F(y_i) - E(y_i)|, \quad i = 1,..n \tag{4.28}$$

where,

F( ) is the approximating Weibull distribution

E( ) is the mixture distribution

n is the number of classes in the interval.

From the value of D_n, the K-S test statistic is calculated. The K-S test statistic is given by

$$K = n \times D_n \tag{4.29}$$

The critical values of K for acceptance or rejection of the approximation are obtained from tables (Kraft and Van Eeden (1968)). For a level of significance of 0.04, the critical value is 3.0.

The number of intervals, n, is usually taken between 30 and 40. The K-S test statistic is computed by a FORTRAN program which is included in Appendix B.

4.3.2 Case 5.

The mixture distribution is analyzed for the case where the transition function is dependent upon time t, i.e., for $\lambda_j(t) = m\alpha t^{m-1}$, where m is a constant. The conditional density function for first passage time is given by

$$f(t|j) = m\alpha t^{m-1} \frac{(\alpha t^m)^{j-1}}{(j-1)!} \exp(-\alpha t^m), \tag{4.30}$$

where j is the difference between the final and initial states. If j is assumed to be Poisson distributed with mean μ, then the mixture distribution gives the failure distribution for a mechanism. As in the previous case, the mixture distribution does not yield a closed form expression. The distribution is again approximated as a Weibull distribution. The procedure follows the same four steps enumerated for the previous case.

As the first step, the first two moments of the mixture and the Weibull are matched. Since it is not possible to construct the Laplace transform of the mixture distribution, a moment for the distribution is calculated by obtaining the conditional moment and then finding the mixture of the conditional moment.

From equation (4.30), the conditional mean for a given j is

$$\mu_{1|j} = \int_0^\infty t\, m\alpha t^{m-1} \frac{(\alpha t^m)^{j-1}}{(j-1)!} \exp(-\alpha t^m)\, dt$$

Let $T = \alpha t^m$. Then, $m\alpha t^{m-1}\, dt = dT$ and $t = \left(\frac{T}{\alpha}\right)^{1/m}$

So,

$$\mu_{1|j} = \int_0^\infty (T/\alpha)^{1/m} \frac{T^{j-1}}{(j-1)!} \exp(-T)\, dT$$

$$= (1/\alpha)^{1/m} \frac{(j + \frac{1}{m} - 1)!}{(j-1)!} \int_0^\infty \frac{(T)^{j+\frac{1}{m}-1}}{(j+\frac{1}{m}-1)!} \exp(-T) dT$$

Since the integrand is a gamma probability density function, the value of the integral is equal to 1. Thus,

$$\mu_{1|j} = (\alpha)^{-1/m} \frac{(j + \frac{1}{m} - 1)!}{(j-1)!} \tag{4.31}$$

Similarly, it can be shown that the conditional second moment is

$$\mu_{2|j} = (\alpha)^{-2/m} \frac{(j + \frac{2}{m} - 1)!}{(j-1)!} \tag{4.32}$$

From equations (4.31) and (4.32), the moments for the mixture distribution are given as

$$\mu_1 = \frac{1}{(1-e^{-\mu})} \sum_{j=1}^{\infty} (\alpha)^{-1/m} \frac{(j+\frac{1}{m}-1)!}{(j-1)!} e^{-\mu} \frac{\mu^j}{j!} \tag{4.33}$$

and

$$\mu_2 = \frac{1}{(1-e^{-\mu})} \sum_{j=1}^{\infty} (\alpha)^{-2/m} \frac{(j+\frac{2}{m}-1)!}{(j-1)!} e^{-\mu} \frac{\mu^j}{j!} \tag{4.34}$$

The moments of the mixture distribution are estimated numerically and matched with the moments of the Weibull distribution. The estimation of Weibull parameters and the validation of the approximation follow exactly the same procedure as in the previous case.

The approximation is carried out for two values of m; m = 2 and m = 0.5. Since case 6 corresponds to having a value of m equal to one, the inclusion of the above two m values will better describe the effect of time dependency upon the failure process.

The results of the approximation for case 6 and case 5 are presented in Tables 1, 2, and 3. In these tables, α is the constant in the transition rate, and μ is the mean of distance between the initial and final states of the birth process. The estimated shape and scale parameters are given by β' and θ' respectively. The first four moments of the mixture and Weibull distributions are also presented along with the ratios of corresponding third and fourth moments of the Weibull and the mixture

distributions. From the tables, it can be seen that the higher moments of the Weibull distribution are quite close to those of the exact mixture distribution. This shows that Weibull distribution serves well as an approximating function for this case. The K values of the K-S test confirm this conclusion.

The Weibull distribution is taken as a valid representation of the mixture distributions. Once the parameters of the Weibull distribution are estimated, the distribution function can be used for computing the failure probabilities for a mechanism or for further analysis to include multiple mechanisms.

4.4 Multiple Mechanisms

The results of the analysis of a single failure mechanism are now expanded to multiple mechanisms. The mechanisms are assumed to be active in the component simultaneously and are independent of each other. The component behavior can be studied using either the extreme value theory or the competing risk theory. In either case, the distribution of the aggregate failure time is given by the distribution of the minimum of individual failure times.

Extreme value theory is applied if the individual failure distributions are independent and identically distributed and their number is large (Cramer (1946)). The procedure for obtaining the aggregate failure distribution function is shown below. It is assumed that each individual distribution is a Weibull function.

Let X be the minimum of n random variables $X_1, X_2, \ldots, X_n$. Let F be the distribution function for X and G be the distribution function for X_j, j = 1 to n. Then

$$F(t) = P(X \le t) = 1 - P(X > t)$$
$$= 1 - \{P(X_1 > t)\, P(X_2 > t) \ldots P(x_n > t)\}$$
$$= 1 - (1 - G(t))^n$$

Define a random variable η_n as

$$\eta_n = nG(X)$$

Let $\Gamma_n(t) = P(\eta_n \le t), \quad 0 \le t \le n$

So, $\Gamma_n(t) = P[X \le G^{-1}(t/n)] = F[G^{-1}(t/n)]$

$$= 1 - [1 - G(G^{-1}(t/n))]^n$$
$$= 1 - (1 - t/n)^n$$

Let

$$\Gamma(t) = \lim_{t \to \infty} \Gamma_n(t) = \lim_{t \to \infty} [1 - (1 - t/n)^n] = 1 - e^{-t} \tag{4.35}$$

So, the sequence of random variables η_n converges to a random variable, say η.

If G(t) is a Weibull distribution,

$$\eta_n = nG(X) = n(1 - e^{-\theta x^\beta})$$

Or,

$$X^\beta = \frac{1}{\theta}\left[\log\left(\frac{1}{1 - \frac{\eta_n}{n}}\right)\right]$$

$$= \frac{1}{\theta}\left[\frac{\eta_n}{n} + \left(\frac{\eta_n}{n}\right)^2 + \left(\frac{\eta_n}{n}\right)^3 + \cdots\right]$$

Ignoring all the terms with powers greater than 1,

$$X^{\beta} = \frac{1}{\theta}(\eta/n)$$

Or,

$$X = (\eta/n\theta)^{1/\beta}$$

The distribution of X is now calculated. By extreme value theory (equation (4.35)),

$$P[\eta \leq t\ ] = \int_0^t e^{-x}dX = 1 - e^{-t}$$

Now,

$$P(X \leq t) = P(\ [\eta/n\theta]^{1/\beta} \leq t)$$
$$= P(\eta \leq n\theta t^{\beta}\ ) = 1 - e^{-n\theta t^{\beta}}$$

Hence X has a Weibull function with β as the shape parameter and $n\theta$ as the scale parameter. So if all the individual failure mechanisms have identical Weibull distributions for failure with shape parameter β and scale parameter θ, then the overall failure distribution is again a Weibull distribution with the shape parameter equal to that of the individual distributions and the scale parameter equal to the sum of their scale parameters.

In competing risk theory, it is assumed that the component is subject to multiple failure mechanisms and the number of mechanisms, n, is finite. All the lifetimes generated by individual failure mechanisms have Weibull form of distribution. The distribution of the component lifetime is given by the distribution of the minimum of individual distributions. Three situations are considered for the individual distributions: (1) all the distributions are identical, (2) the shape parameter β is the same for all the distributions, and (3) the shape and scale parameters are different for each distribution. The three situations are analyzed below.

(1) Let $X_1, X_2, \ldots, X_n$ be the n random variables representing the lifetimes corresponding to the n individual failure mechanisms. All n variables are identically distributed, each having a Weibull distribution with θ and β as the parameters.

Let $X = min.\{X_1, X_2, \ldots, X_n\}$.

The distribution of X is given by

$$
\begin{aligned}
P(X \leq t) &= 1 - P(X > t) \\
&= 1 - P(X_1 > t)\ P(X_2 > t)\ \ldots\ldots, P(X_n > t) \\
&= 1 - \left[\exp(-\theta t^{\beta})\right]\left[\exp(-\theta t^{\beta})\right]\ldots\left[\exp(-\theta t^{\beta})\right] \\
&= 1 - \exp(-\theta t^{\beta} - \theta t^{\beta} \ldots\ldots - \theta t^{\beta}) \\
&= 1 - \exp(-n\theta t^{\beta})
\end{aligned}
$$

So the minimum is also a Weibull distribution with shape parameter β and scale parameter $n\theta$.

(2) This is a situation in which the individual distributions have a common shape parameter β. Their scale parameters are different from each other. Let the scale parameters be $\theta_1, \theta_2, \ldots\ldots, \theta_n$, respectively. The probability distribution of X = min. $\{X_1, X_2, \ldots, X_n\}$ is given by

$$P(X \leq t) = 1 - \exp[-(\theta_1 + \theta_2 + \ldots\ldots + \theta_n)t^{\beta}],$$

which is again a Weibull distribution with scale parameter $(\theta_1 + \theta_2 + \ldots. + \theta_n)$ and shape parameter β.

(3) Here, the n random variables have different scale parameters ($\theta_1, \theta_2, \ldots\ldots, \theta_n$) and different shape parameters $(\beta_1, \beta_2, \ldots\ldots, \beta_n)$. In this case,

$$p(X \leq t) = 1 - \exp[-(\theta_1 t^{\beta 1} + \theta_2 t^{\beta 2} + \ldots\ldots + \theta_n t^{\beta n})]$$

This expression does not conform to any known probability distribution. However, it can be directly used to compute the lifetime distribution of the component.

The above distribution may have a Weibull type of extreme-value form since it is the distribution of the minimum of multiple random variables. To test the validity of this assumption, the distribution of the minimum of two Weibull random variables is approximated as a Weibull distribution. The goodness of the approximation is checked by the comparison of the corresponding higher moments and the K-S test. The procedure is described below.

Weibull approximation to the minimum of two Weibull distributions:
Let X_i, $i = 1,2$ be a random variable with Weibull distribution and θ_i and β_i as scale and shape parameters respectively. It is assumed that the minimum of two Weibull random variables is also a Weibull. The first two moments of the original distribution are computed first.

let $X = min.\{X_1, X_2\}$

The distribution of X is given by

$$\begin{aligned} P(X \leq t) &= 1 - P(X > t) \\ &= 1 - P(X_1 > t)\, P(X_2 > t) \\ &= 1 - \left[\exp(-\theta_1 t^{\beta_1})\right]\left[\exp(-\theta_2 t^{\beta_2})\right] \\ &= 1 - \exp(-\theta_1 t^{\beta_1} - \theta_2 t^{\beta_2}) \end{aligned} \tag{4.36}$$

The density function of X is given by

$$f(t) = (\beta_1 \theta_1 t^{\beta_1 - 1} + \beta_2 \theta_2 t^{\beta_2 - 1}) \exp(-\theta_1 t \beta_1 - \theta_2 t^{\beta_2})$$

The first moment $\mu_1 = \int_0^\infty t\,(\beta_1 \theta_1 t^{\beta_1 - 1} + \beta_2 \theta_2 t^{\beta_2 - 1}) \exp(-\theta_1 t^{\beta_1} - \theta_2 t^{\beta_2}) dt$,

and the second moment $\mu_2 = \int_0^\infty t^2(\beta_1 \theta_1 t^{\beta_1 - 1} + \beta_2 \theta_2 t^{\beta_2 - 1}) \exp(-\theta_1 t^{\beta_1} - \theta_2 t^{\beta_2}) dt$.

The two moments are numerically estimated. The first two moments of the approximating Weibull distribution are matched with these two values and the parameters θ' and β' of the approximating distribution are estimated. The procedure is similar to the one described in section 4.3. Using the estimated parameters, the third and fourth moments are calculated and compared with the corresponding moments of the original distribution (equation 4.36)). K-S Test is performed for further validation. The results are presented in Table 4 for different combinations of parameter values.

The results show that the third and fourth moments of the approximating distribution are very close to those of the original distribution. This indicates that the Weibull assumption for the minimum is a valid assumption. The values of the K-S test statistic further support this assumption.

If the minimum of two Weibull distributions is approximately a Weibull distribution, then by recursive argument, the minimum of n Weibull distributions is again another Weibull. Therefore, even for the situation in which the individual failure distributions are different from each other both in their scale and shape parameters, the aggregate distribution is still a Weibull distribution.

Table 1. Results for Weibull Approximation

$\lambda_f(t) = \alpha$

	moment#	mixture(m)	Weibull(W)	Ratio(W/m)
$\alpha = 2.0$ $\mu = 60.0$ $\theta' = 0.22E-9$ $\beta' = 6.40$	1 2 3 4	30.00 930.00 297.45E1 980.19E3	29.99 929.77 296.22E1 960.08E3	1.00 1.00 0.96 0.98 K-S = 0.18
$\alpha = 2.0$ $\mu = 80.0$ $\theta' = .67E-12$ $\beta' = 7.47$	1 2 3 4	40.00 164.00 688.60E2 295.86E4	40.00 164.00 686.80E2 292.95E7	1.00 1.00 0.99 0.99 K-S = 0.23
$\alpha = 2.0$ $\mu = 90.0$ $\theta' = 4.37E-13$ $\beta' = 7.96$	1 2 3 4	45.00 207.00E1 972.68E2 466.57E4	44.99 207.00E1 970.38E2 462.54E4	1.00 1.00 0.99 0.99 K-S = 0.87
$\alpha = 5.0$ $\mu = 60.0$ $\theta' = .78E-7$ $\beta' = 6.4$	1 2 3 4	12.00 148.80 190.40E1 250.93E2	12.00 148.80 189.70E1 247.40E28	1.00 1.00 0.99 0.98 K-S = 0.65
$\alpha = 5.0$ $\mu = 80.0$ $\theta' = 0.63E-9$ $\beta' = 7.47$	1 2 3 4	16.00 262.40 440.70E1 757.38E2	16.00 262.40 439.50E1 749.90	1.00 1.00 0.97 0.99 K-S = 0.71
$\alpha = 5.0$ $\mu = 90.00$ $\theta' = 0.64E-10$ $\beta' = 7.96$	1 2 3 4	18.00 331.20 622.50E1 119.40E3	17.99 331.30 620.90E1 118.30E3	1.00 1.00 0.99 0.99 K-S = 0.94

Table 1. (continued)

	moment#	mixture(m)	Weibull(W)	Ratio(W/m)
$\alpha = 8.0$ $\mu = 60.0$ $\theta' = 0.16E-5$ $\beta' = 6.40$	1 2 3 4	7.50 58.12 464.80 382.90E1	7.50 58.12 463.00 377.80E1	1.00 1.00 0.99 0.98 K-S = 0.09
$\alpha = 8.0$ $\mu = 80.0$ $\theta' = 0.21E-7$ $\beta' = 7.47$	1 2 3 4	10.00 102.50 107.60E1 115.57E2	10.00 102.50 107.30E1 114.44E2	1.00 1.00 0.99 0.99 K-S = 0.12
$\alpha = 8.0$ $\mu = 90.0$ $\theta' = 0.27E-8$ $\beta' = 7.95$	1 2 3 4	11.25 129.40 151.98E1 182.26E2	11.25 129.40 151.65E1 180.72E2	1.00 1.00 0.99 0.99 K-S = 0.46

Table 2. Results for Weibull Approximation

$\lambda_f(t) = m\alpha t^{m-1}$

m = 2.0

	moment#	mixture(m)	Weibull(W)	Ratio(W/m)
$\alpha = 2.0$ $\mu = 60.0$ $\theta' = 0.97E-10$ $\beta' = 13.28$	1 2 3 4	5.45 30.00 166.37 930.00	5.45 30.00 166.27 928.03	1.00 1.00 0.99 0.99 K-S = 0.23
$\alpha = 2.0$ $\mu = 80.0$ $\theta' = 0.25E-12$ $\beta' = 15.48$	1 2 3 4	6.30 40.30 255.36 164.0E1	6.30 40.00 255.30 163.8E1	1.00 1.00 0.99 0.99 K-S = 0.11
$\alpha = 2.0$ $\mu = 90.0$ $\theta' = 0.15E-13$ $\beta' = 15.49$	1 2 3 4	6.69 45.00 304.4 207.0E1	6.69 45.03 304.6 207.0E1	1.00 1.00 1.00 1.00 K-S = 0.07
$\alpha = 5.0$ $\mu = 60.0$ $\theta' = 0.43E-7$ $\beta' = 13.28$	1 2 3 4	3.45 12.00 42.09 148.8	3.45 12.00 42.06 148.4	1.00 1.00 1.00 1.00 K-S = 0.07
$\alpha = 5.0$ $\mu = 80.0$ $\theta' = 0.34E-9$ $\beta' = 15.38$	1 2 3 4	3.99 16.00 64.60 262.40	3.99 15.99 64.57 262.0	1.00 1.00 1.00 1.00 K-S = 0.10
$\alpha = 5.0$ $\mu = 90.0$ $\theta' = 0.29E-10$ $\beta' = 16.46$	1 2 3 4	4.23 18.00 77.00 331.20	4.23 18.00 76.97 330.75	1.00 1.00 0.99 0.99 K-S = 0.09

Table 2. (continued)

	moment#	mixture(m)	Weibull(W)	Ratio(W/m)
$\alpha = 8.0$ $\mu = 60.0$ $\theta' = 0.97E-6$ $\beta' = 13.28$	1 2 3 4	2.72 7.50 20.79 58.12	2.72 7.50 20.78 58.00	1.00 1.00 1.00 0.99 K-S = 0.45
$\alpha = 8.0$ $\mu = 80.0$ $\theta' = 0.11E-7$ $\beta' = 15.47$	1 2 3 4	3.15 10.00 31.92 102.50	3.12 10.00 31.90 102.30	1.00 1.00 1.00 1.00 K-S = 0.07
$\alpha = 8.0$ $\mu = 90.0$ $\theta' = 0.36E-8$ $\beta' = 15.65$	1 2 3 4	3.35 11.25 38.05 129.37	3.35 11.26 38.16 129.90	1.00 1.00 1.00 1.00 K-S = 0.09

Table 3. Results for Weibull Approximation

$\lambda_f(t) = m\alpha t^{m-1}$

m = 0.5

	moment#	mixture(m)	Weibull(W)	Ratio(W/m)
$\alpha = 2.0$ $\mu = 60.0$ $\theta' = 0.99E-9$ $\beta' = 2.98$	1 2 3 4	930.00 980.19E3 115.84E6 152.16E10	930.00 980.19E3 113.20E6 140.80E10	1.00 1.00 0.97 0.92 K-S = 1.7
$\alpha = 2.0$ $\mu = 80.0$ $\theta' = 0.37E-11$ $\beta' = 3.50$	1 2 3 4	164.00E1 292.85E4 583.38E7 125.06E11	164.00E1 292.85E4 573.67E7 117.88E11	1.00 1.00 0.98 0.94 K-S = 0.98
$\alpha = 2.0$ $\mu = 90.0$ $\theta' = 0.27E-12$ $\beta' = 3.74$	1 2 3 4	207.00E1 466.57E4 113.93E8 300.06E11	206.98 466.49E24 112.22E8 284.64E11	1.00 1.00 0.98 0.94 K-S = 2.5
$\alpha = 5.0$ $\mu = 60.0$ $\theta' = 0.23E-6$ $\beta' = 2.98$	1 2 3 4	148.80 250.93E2 474.48E4 997.20E6	148.80 250.93E2 464.03E4 922.78E6	1.00 1.00 0.98 0.93 K-S = 2.12
$\alpha = 5.0$ $\mu = 80.0$ $\theta' = 0.23E-8$ $\beta' = 3.5$	1 2 3 4	262.40 757.38E2 238.95E5 819.56E7	262.41 757.44E2 235.00E5 772.70E7	1.00 1.00 0.98 0.94 K-S = 1.81
$\alpha = 5.0$ $\mu = 90.0$ $\theta' = 0.25E-9$ $\beta' = 3.74$	1 2 3 4	331.20 119.44E3 466.66E5 196.60E8	331.20 119.44E3 459.76E5 186.60E8	1.00 1.00 0.98 0.94 K-S = 1.45

Table 3. (continued)

	moment#	mixture(m)	Weibull(W)	Ratio(W/m)
$\alpha = 8.0$ $\mu = 60.0$ $\theta' = 0.39E-5$ $\beta' = 2.98$	1 2 3 4	58.12 382.8E1 282.80E3 232.18E5	58.12 382.8E1 276.50E3 214.85E5	1.00 1.00 0.97 0.92 K-S = 0.54
$\alpha = 8.0$ $\mu = 80.0$ $\theta' = 0.62E-7$ $\beta' = 3.50$	1 2 3 4	102.50 115.57E2 142.40E4 190.82E6	102.50 115.57E2 140.00E4 179.88E6	1.00 1.00 0.98 0.94 K-S = 0.12
$\alpha = 8.0$ $\mu = 90.0$ $\theta' = 0.85E-8$ $\beta' = 3.74$	1 2 3 4	129.37 182.26E2 278.15E4 457.85E6	129.37 182.26E2 274.04E4 434.47E6	1.00 1.00 0.98 0.94 K-S = 1.4

Table 4. Weibull Approximation for the min. of two variables.

	moment#	minimum(m)	Weibull(W)	Ratio(W/m)
$\theta_1 = 0.22E-9$ $\beta_1 = 6.40$ $\theta_2 = 0.22E-9$ $\beta_2 = 6.40$ $\theta' = 0.43E-09$ $\beta' = 6.40$	1 2 3 4	26.99 753.00 215.80E2 633.30E6	26.98 753.0 215.70E2 632.90E6	1.00 1.00 1.00 0.99 K-S = 1.27
$\theta_1 = 0.219E-09$ $\beta_1 = 6.40$ $\theta_2 = 0.67E-12$ $\beta_2 = 7.47$ $\theta' = 0.17E-9$ $\beta' = 6.51$	1 2 3 4	29.55 90.18 282.50E2 905.90E3	29.55 90.18 282.50E2 905.80E3	1.00 1.00 1.00 0.99 K-S = 0.87
$\theta_1 = 0.44E-13$ $\beta_1 = 7.96$ $\theta_2 = 0.77E-7$ $\beta_2 = 6.40$ $\theta' = 0.77E-7$ $\beta' = 6.4$	1 2 3 4	12.02 149.3 190.60E1 249.10E2	12.02 149.3 190.60E1 249.10E2	1.00 1.00 1.00 1.00 K-S = 0.72
$\theta_1 = 0.62E-7$ $\beta_1 = 3.50$ $\theta_2 = 0.85E-9$ $\beta_2 = 3.74$ $\theta' = 0.63E-7$ $\beta' = 3.57$	1 2 3 4	93.45 957.90E1 105.00E4 122.53E6	93.45 957.90E1 105.00E4 122.52E6	1.00 1.00 1.00 1.00 K-S = 1.04
$\theta_1 = 0.44E-13$ $\beta_1 = 7.96$ $\theta_2 = 0.16E-5$ $\beta_2 = 6.40$ $\theta' = 0.16E-5$ $\beta' = 6.40$	1 2 3 4	7.51 58.33 465.50 380.32E1	7.51 58.33 465.50 380.32E1	1.00 1.00 1.00 1.00 K-S = 0.61
$\theta_1 = 0.99E-09$ $\beta_1 = 2.98$ $\theta_2 = 0.234E-6$ $\beta_2 = 2.98$ $\theta' = 0.24E-6$ $\beta' = 2.97$	1 2 3 4	14.96 254.00E2 472.80E4 946.50E6	14.96 254.00E2 472.80E4 946.60E6	1.00 1.00 1.00 0.99 K-S = 0.90

Table 4. (continued)

	moment#	minimum(m)	Weibull(W)	Ratio(W/m)
$\theta_1 = 0.22E-9$ $\beta_1 = 6.40$ $\theta_2 = 0.23E-6$ $\beta_2 = 2.98$ $\theta' = 0.26E-9$ $\beta' = 6.35$	1 2 3 4	30.00 932.00 297.40E2 972.00E3	30.00 932.00 297.50E2 972.30E3	1.00 1.00 1.00 1.00 K-S = 1.17
$\theta_1 = 0.22E-9$ $\beta_1 = 6.40$ $\theta_2 = 0.97E-10$ $\beta_2 = 13.30$ $\theta' = 0.52E-11$ $\beta' = 14.89$	1 2 3 4	5.53 30.82 173.20 982.70	5.53 30.82 172.70 973.70	1.00 1.00 0.99 0.99 K-S = 1.83
$\theta_1 = 0.25E-12$ $\beta_1 = 15.50$ $\theta_2 = 0.97E-10$ $\beta_2 = 13.30$ $\theta' = 0.65E-4$ $\beta' = 5.53$	1 2 3 4	5.28 29.12 162.20 911.30	5.28 29.12 166.20 917.60	1.00 1.00 1.02 0.98 K-S = 2.44

CHAPTER 5

Conclusions and Discussion

5.1 Conclusions

A reliability model which captures the dynamic behavior of failure of a component in terms of the properties of the material and operating conditions is developed. The model analyzes the effect of degradation mechanisms on component failure time. A component subject to a single mechanism is assumed to have an initial number i, and a final number k of defects. During operation, the component fails when the number of defects equals the final number. For the case of a single mechanism with fixed i and k, the component failure distribution is given as the first passage time distribution of the representative pure birth process to the final state. First passage time distributions can be derived using direct methods or Laplace transform methods for various forms of transition rates. All these distributions have the form of generalized Gamma

distribution. For the simplest case, where the transition rate is independent of state and temperature, the first passage time distribution is a Gamma distribution.

When the initial and final states of a mechanisms are random, the failure time of the component is given by mixture distribution derived from the conditional first passage time distributions. The mixture distribution is analyzed for two types of transition rates. In both the cases, the mixture distribution does not have a closed form when the distance between the final and initial states is taken to be Poisson distributed with mean μ. From the results of approximation, it can be concluded that the mixture follows a Weibull form of distribution. The mean time to failure is directly proportional to the mean of the difference between the initial and final number of defects. This is because the time to failure is greater if the process has to traverse a larger distance from the initial to the final state. As the transition rate increases, the process changes states faster and hence component fails earlier. The mean failure time is inversely proportional to α , the constant in the transition rate. Since α includes the reaction kinetics, it dictates the reaction rate and hence the failure rate and the mean failure time. The shape parameter β of the Weibull distribution remains almost the same for a fixed value of μ. The shape parameter is quite insensitive to the changes in the transition rates. The variable m, the factor describing the effect of time on the transition rate function, has a great impact on the failure times. As the value of m increases, the mean of the failure time decreases exponentially.

Multiple mechanisms are incorporated into the model by using either the extreme-value theory or the competing risk theory. In both the cases, the overall distribution is a Weibull distribution. The results conform with the general practice of assuming a Weibull distribution for the failure time of electronic components.

5.2 Extensions

The model by its very structure creates an exhaustive range of possibilities for future research. It has a two dimensional expansion potential, practical and analytical.

Experimental work needs to be done to identify the forms of the transition rate for failure mechanisms. Another major area of research that is needed is in the study of defect distributions. Although considerable work has been carried out in the area of defect distributions, much is needed to identify the distributions of the defects at which failures occur.

On the theoretical front, analysis of the model is interesting if some of the present assumptions are relaxed. For example the degradation process may be allowed to have a reaction in the reverse direction . The process will then no longer be a pure birth process and the first passage time distributions will be quite complicated to obtain. One advantage of this analysis would be that more failure mechanisms could be analyzed. Trapping of hot electrons is an example. Once the reaction is allowed to be reversible, the model may be used with some modifications to analyze mechanical systems as well.

In summary, a component failure model that explains the causal relationship between the failure of a component and factors like the properties of the material of the component and the operating conditions is developed. This relationship can be used for estimating the reliability of a component at different operating conditions without resorting to the accelerated testing methodology. Another advantage is that reliability can be built into a component from the design stage itself using this model.

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Appendix A.

Program to Compute Weibull Parameters

```
C   GIVEN THE FIRST AND SECOND MOMENTS, THIS PROGRAM
C   COMPUTES ALPHA AND BETA OF WEIBULL DIST. BY AN ITERATIVE
C   PROCEDURE.
C
      DOUBLE PRECISION A(2,2), B(2,2), X0(2), X1(2), XNEW(2), DEN
      DOUBLE PRECISION  RMU1, RMU2 , RMU3, RMU4, ALPHA , RMU
      COMMON RMU1, RMU2,RMU3 , RMU4
      READ(5,10) ALPHA, RMU
C  ALPHA IS THE PARAMETER FOR EXPO. DIST. OF THE BIRTH PROCESS
C  RMU IS THE MEAN OF POISSON DIST. OF THE DISTANCE
   10 FORMAT(D10.5,2X,D10.5)
      WRITE(6,20)ALPHA, RMU
```

```
 20  FORMAT(2X,'ALPHA = ', D10.5,'  MU = ', D10.2)
C     COMPUTE THE FIRST MOMENT RMU1 AND THE SECOND MOMENT RMU2
C
      DEN = 1.D00
      IF(RMU.LT.20.D0)  DEN = 1.D0-(DEXP(-RMU))
        RMU1 = (RMU / ALPHA)*(1.D0/DEN)
       RMU2 = RMU1 *((RMU/ALPHA) +(2.D0/ALPHA))
      RMU3=RMU1*(6.D0+6.D0*RMU + (RMU**2))*(1.D0/ALPHA**2)
      RMU4 = RMU1*(24.D0 + 36.D0*RMU + 12.D0*(RMU**2) + (RMU**3))*
     1        (1.D0/(ALPHA**3))
     WRITE(6,35)RMU1,RMU2 ,DEN
 35  FORMAT(2X,'RMU1=',D12.5, 3X, 'RMU2=',D12.5,5X,'DEN',D12.5)
     CALL SR1
      STOP
      END
C     SUBROUTINE TO ITERATIVELY FIND ALPHA AND BETA OF THE
C  WEIBULL DISTRIBUTION.
C
      SUBROUTINE SR1
      DOUBLE PRECISION  RJ(2,2), RIJ(2,2), B(2,2), A(2,2)
      DOUBLE PRECISION ALPHA0, BETA0, ALPHA1, BETA1,ALPHANU, BETANU
     DOUBLE PRECISION RMU1, RMU2,FUNC1,FUNC2,RMEAN,DET,RMEAN2,RMEAN3,
     1        RMEAN4 ,RMU3, RMU4
      COMMON RMU1, RMU2 , RMU3, RMU4
C
C     INIITIALIZE ALPHA AND BETA  AND THE COUNTER
      K = 1
```

```
      READ(5,10)BETA0
   10 FORMAT(D10.5)
      ALPHA0 = FUNC1(BETA0)+0.005
  100 WRITE(6,15) ALPHA0, BETA0
   15 FORMAT(2X, 'ALPHA0= ', D12.5,3X,'BETA0=', D12.5)
      ALPHA1 = FUNC1(BETA0)
      BETA1 = FUNC2(ALPHA0, BETA0)
      WRITE(6,55)ALPHA1, BETA1
   55 FORMAT(2X, 'ALPHA1', D12.5, ' BETA1', D12.5)
C        COMPUTE APPROXIMATE VALUES OF PARTIAL DIFFERENTIAL FUNCTIONS
C
      RJ(1,1) = 0.D0
      RJ(1,2) = (FUNC1(BETA1)-ALPHA1)/ (BETA1 - BETA0)
      RJ(2,1) = (FUNC2(ALPHA1,BETA0) - BETA1) / (ALPHA1 -ALPHA0)
      RJ(2,2) = (FUNC2(ALPHA0, BETA1) - BETA1) / (BETA1 - BETA0)
C   CALCULATE THE I-J MATRIX
      RIJ(1,1) = 1.D0 - RJ(1,1)
      RIJ(1,2) = -RJ(1,2)
      RIJ(2,1) = -RJ(2,1)
      RIJ(2,2) = 1.D0 -RJ(2,2)
C
C   FIND B, THE INVERSE OF I-J.
C
      DET = RIJ(1,1)*RIJ(2,2) - RIJ(1,2)*RIJ(2,1)
     B(1,1) = RIJ(2,2)/DET
     B(1,2) = -RIJ(1,2)/DET
     B(2,1) = -RIJ(2,1) / DET
```

```
      B(2,2) = RIJ(1,1)/DET
C
C   COMPUTE A = I-B
C
      A(1,1) = 1.D0-B(1,1)
      A(1,2) = -B(1,2)
      A(2,1) = -B(2,1)
      A(2,2) = 1.D0 - B(2,2)
C   CALCULATE THE NEXT SET OF VALUES
C
      ALPHANU = A(1,1)*ALPHA0 + A(1,2)*BETA0 + B(1,1)*ALPHA1 +
     1  B(1,2)*BETA1
      BETANU = A(2,1)*ALPHA0 + A(2,2)*BETA0 + B(2,1)*ALPHA1 +
     1  B(2,2)*BETA1
C
C  CHECK FOR CONVERGENCE
      IF((DABS(ALPHANU-ALPHA0)).LE.1.D-14.OR.K.GT.6)GOTO 40
      ALPHA0 = ALPHANU
      BETA0 = BETANU
        K = K+1
      GOTO 100
 40   WRITE(6,50)ALPHANU, BETANU,K
 50   FORMAT(2X, 'ALPHANU', D15.06, 2X, 'BETANU', 3X, D12.6,4X,I5)
      RMEAN = ((1.D0/ALPHANU)**(1.D0/BETANU)) * GAMMA1(1.D0/BETANU)
      RMEAN2 = ((1.D0/ALPHANU)**(2.D0/BETANU))* GAMMA1(2.D0/BETANU)
      RMEAN3 = ((1.D0/ALPHANU)**(3.D0/BETANU))* GAMMA1(3.D0/BETANU)
      RMEAN4 = ((1.D0/ALPHANU)**(4.D0/BETANU))* GAMMA1(4.D0/BETANU)
      WRITE(6,30)RMU1, RMU2, RMU3, RMU4
```

```
 30   FORMAT(2X,'RMU1=  ', D12.5, 2X, 'RMU2=  ', D12.5,'RMU3= ',D12.5,
     1 'RMU4 =  ',D12.5)
      WRITE(6,44)RMEAN, RMEAN2, RMEAN3,RMEAN4
 44   FORMAT(2X, 'RMEAN = ', D12.5,3X, D12.5,2X,'RMEAN3= ',D12.5,
     1 'RMEAN4 = ',D12.5)
C
      RETURN
      END
C
C     FUNCTION FOR THE ALPHA VALUE
C
      FUNCTION FUNC1( BETA)
      DOUBLE PRECISION ALPHA, BETA, A, VALUE, GAMMA1, FUNC1
       DOUBLE PRECISION RMU1, RMU2
      COMMON RMU1, RMU2, RMU3,RMU4
     A = 1.D0/BETA
     WRITE(6,30)BETA
 30   FORMAT(2X, 'BETA IN FUNC1', D15.7)
     VALUE = GAMMA1(A)
     FUNC1 = (VALUE/RMU1)**BETA
      RETURN
      END
C
C     FUNCTION FOR THE BETA VALUE
      FUNCTION FUNC2(ALPHA, BETA)
      DOUBLE PRECISION ALPHA, BETA, A, VALUE, GAMMA1, FUNC2
       DOUBLE PRECISION RMU1, RMU2
      COMMON RMU1, RMU2 , RMU3,RMU4
```

```
      A =2.D0/BETA
      VALUE = GAMMA1(A)
      FUNC2 = (2.D0*DLOG(ALPHA)) / DLOG(VALUE/RMU2)
      WRITE(6,66)FUNC2, ALPHA, VALUE
   66 FORMAT(2X,'FUNC2 ALPHA VALUE', 4X,3D12.7)
      RETURN
      END
C
C     NUMERICAL APPROXIMATION FOR GAMMA FUNCTION

C      THIS FUNCTION CALCULATES GAMMA FUNCTION BY APPROXIMATION
      FUNCTION GAMMA1(A)
      DOUBLE PRECISION GAMMA1, A, B, PROD,VALUE
      REAL*8 A1, A2, A3, A4, A5
     DATA A1,A2,A3,A4,A5/-5748.646D-4,9512.363D-4,-6998.588D-4,
    1 4245.549D-4,-1010.678D-4/
      B = A
      PROD = 1.D0
   30 IF(A.LE.1.D0) GO TO 40
         PROD = PROD*A
         A = A-1.D0
         GO TO 30
   40 VALUE = 1.D0+A1*A + A2*(A**2) + A3*(A**3) + A4*(A**4) +A5*(A**5)
     GAMMA1 = PROD*VALUE
      WRITE(6,20)GAMMA1,B
   20 FORMAT(2X, 'GAMMA', 5X, D10.5, ' FOR ARG.', 2X,D12.5)
      RETURN
      END
```

Appendix B.

Computation of K-S Test Statistic

```
C     THIS PROGRAM COMPUTES THE K-S TEST STATISTIC BY
C     COMPARING THE CDF OF THEORETICAL AND
C     FITTED WEIBULL.
      DIMENSION F(20),THEO(20), YBUL(20)
      INTEGER T, T1,T2
      DOUBLE PRECISION ALPHA, RMU,RM,F,THEO, YBUL, FT
      DOUBLE PRECISION ALPHANU, BETANU
      DOUBLE PRECISION  DMAX, DIFF, FMIX, PDF
      COMMON ALPHA, RMU, RM
C      DO 100 IC = 1,9
C     READ TRANSITION RATE, MEAN OF POISSON DISTANCE AND POWER IN LAMBDA
      READ(5,10)ALPHA, RMU ,RM
   10  FORMAT(D15.3,D15.3,D15.3)
      READ(5,15)ALPHANU,BETANU
```

```
 15   FORMAT(D15.3, D15.3)
      WRITE(6,*)ALPHANU, BETANU
C   FREQUENCIES ARE TABULATED FOR T =1 TO T =20
      T1 = 1
      T2 = 20
      THEO(1) = FMIX(1.D0)/1.D0

C      CALCULATE THE CDF FOR WEIBULL DISTRIBUTION
      YBUL(1) = 1.D0 - DEXP(-ALPHANU*(FT**BETANU))
C      DMAX IS THE MAXIMUM  ABSOLUTE DIFFERENCE
      DMAX = DABS(THEO(1)-YBUL(1))
      DO 40 T = T1+1, T2
      FT = DFLOAT(T)/1.D00
      YBUL(T) = 1.D0 - DEXP(-ALPHANU*(FT**BETANU))
      PDF = FMIX(FT)/1.D00
      THEO(T) =  THEO(T-1) + PDF
      DIFF = DABS(THEO(T) - YBUL(T))
      IF(DMAX.LT.DIFF)  DMAX = DIFF
      WRITE(6,35)DIFF,PDF, THEO(T), YBUL(T), T
 35   FORMAT(2X,'DIFF,PDF,THEO.CDF, YBUL CDF =', 4(D14.5, 2X), I5)
 40   CONTINUE
      WRITE(6,50)DMAX
 50   FORMAT(//2X, ' DMAX = ', D14.5)
C100   CONTINUE
      STOP
      END

      FUNCTION FMIX(FT)
```

```
      DOUBLE PRECISION ALPHA,RMU,RM,FT,FJ, SUM,POW
      DOUBLE PRECISION TERM(200) , FMIX
      COMMON ALPHA, RMU, RM
      SUM = 0.D0
C
      WRITE(6,35)ALPHA, RMU, RM, FT
   35  FORMAT(2X,4(D14.3, 2X))
      POW = ALPHA * (FT ** RM)
      TERM(1) = (DEXP(- POW/2.D0))*ALPHA * RM *(DEXP(- POW/2.D0))*
     1  (FT ** (RM-1.D0)) *  (DEXP(-RMU)) *RMU
C       WRITE(6,40)TERM(1)
      DO 20 J = 2,100
      FJ = DFLOAT(J)
      TERM(J) = (TERM(J-1) * POW * RMU)/((FJ-1.D0)*FJ)
C
C     STOP IF TERM VALUE IS TOO LOW
      IF(FJ.GT.RMU.AND.TERM(J).LE.1.D-20) GOTO 30
        WRITE(6,40)TERM(J)
   40  FORMAT(2X, 'TERM   = ', D15.6)
      SUM = SUM + TERM(J)
   20  CONTINUE
   30  FMIX = SUM
      RETURN
       END
```

Appendix C.

Approximation for Minimum of Two Weibull Random Variables

```
C    WEIBULL APPROXIMATION FOR MINIMUM OF TWO WEIBULL DIST.


C    GIVEN THE FIRST AND SECOND MOMENTS, THIS PROGRAM
C    COMPUTES ALPHA AND BETA OF WEIBULL DIST. BY AN ITERATIVE
C    PROCEDURE.
C
      DOUBLE PRECISION THETA1, THETA2, BETA1, BETA2
      DOUBLE PRECISION  RMU1, RMU2 , RMU3, RMU4
      DOUBLE PRECISION DT, DF(4), SUM(4), PROD1
      INTEGER T,TT
      COMMON RMU1, RMU2,RMU3 , RMU4
        READ(5,5)TT
   5  FORMAT(I5)
CC     DO 200 KK=1,7
```

```
      READ(5,10) THETA1, BETA1, THETA2, BETA2
      WRITE(6,*)THETA1, BETA1, THETA2, BETA2
C  THETA IS THE SCALE PARAMETER ABD BETA IS THE SHAPE PARAMETER
 10   FORMAT(4D15.3)
C     SUM IS THE SUMMATION OF MIN.DENSITY VALUES. SUM = MOMENT
     DO 20 I=1,4
 20   SUM(I) = 0.D0
     DO 100 T = 1,TT

C     PROD1 = DENSITY FUN. OF MIN. WITHOUT THE EXP. TERMS
C     DF = VARIABLE TO EXPRESS THE LOG. OF THE DENSITY
      DT = DFLOAT(T)
      PROD1 = (BETA1 * THETA1 * (DT ** (BETA1 - 1.D0)))
    1      + (BETA2 * THETA2 * (DT ** (BETA2 - 1.D0)))
      DO 80 K = 1,4
      DF(K) = DFLOAT(K) * DLOG(DT) + DLOG(PROD1) -
    1 (THETA1 * (DT ** (BETA1)) + THETA2 * (DT ** ( BETA2)) )
      SUM(K) = SUM(K) + DEXP(DF(K))
 80   CONTINUE
 100  CONTINUE
     RMU1 = SUM(1)
      RMU2 = SUM(2)
      RMU3 = SUM(3)
      RMU4 = SUM(4)
     WRITE(6,35)RMU1,RMU2 ,RMU3,RMU4
 35   FORMAT(2X,'RMU1=',D12.5, 3X, 'RMU2=',D12.5,5X,'RMU3=',D12.5,
    1 5X, 'RMU4 = ', D12.5)
       CALL SR1
```

```
      CONTINUE
      STOP
      END



C     SUBROUTINE TO ITERATIVELY FIND ALPHA AND BETA OF THE
C     WEIBULL DISTRIBUTION.

      SUBROUTINE SR1
      DOUBLE PRECISION  RJ(2,2), RIJ(2,2), B(2,2), A(2,2)
      DOUBLE PRECISION ALPHA0, BETA0, ALPHA1, BETA1,ALPHANU, BETANU
     DOUBLE PRECISION RMU1, RMU2,FUNC1,FUNC2,RMEAN,DET,RMEAN2,RMEAN3,
    1         RMEAN4 ,RMU3, RMU4
      COMMON RMU1, RMU2 , RMU3, RMU4
C
C     INIITIALIZE ALPHA AND BETA  AND THE COUNTER
      K = 1
      READ(5,10)BETA0
   10  FORMAT(D10.5)
      ALPHA0 = FUNC1(BETA0)+5.D-03
  100  WRITE(6,15) ALPHA0, BETA0
   15  FORMAT(2X, 'ALPHA0= ', D12.5,3X,'BETA0=', D12.5)
      ALPHA1 = FUNC1(BETA0)
      BETA1 = FUNC2(ALPHA0, BETA0)

      RJ(1,1) = 0.D0
      RJ(1,2) = (FUNC1(BETA1)-ALPHA1)/ (BETA1 - BETA0)
```

```
      RJ(2,1) = (FUNC2(ALPHA1,BETA0) - BETA1) / (ALPHA1 -ALPHA0)
      RJ(2,2) = (FUNC2(ALPHA0, BETA1) - BETA1) / (BETA1 - BETA0)
C     CALCULATE THE I-J MATRIX
        RIJ(1,1) = 1.D0 - RJ(1,1)
        RIJ(1,2) = -RJ(1,2)
        RIJ(2,1) =  -RJ(2,1)
        RIJ(2,2) = 1.D0 -RJ(2,2)
C
C     FIND B, THE INVERSE OF I-J.
C
        DET = RIJ(1,1)*RIJ(2,2) - RIJ(1,2)*RIJ(2,1)
      B(1,1) = RIJ(2,2)/DET
      B(1,2) = -RIJ(1,2)/DET
      B(2,1) = -RIJ(2,1) / DET
      B(2,2) = RIJ(1,1)/DET
C
C     COMPUTE A = I-B
C
      A(1,1) = 1.D0-B(1,1)
      A(1,2) = -B(1,2)
      A(2,1) = -B(2,1)
      A(2,2) = 1.D0 - B(2,2)
C     CALCULATE THE NEXT SET OF VALUES
C
       ALPHANU = A(1,1)*ALPHA0 + A(1,2)*BETA0 + B(1,1)*ALPHA1 +
     1  B(1,2)*BETA1
       BETANU = A(2,1)*ALPHA0 + A(2,2)*BETA0 + B(2,1)*ALPHA1 +
```

```
     1  B(2,2)*BETA1
C
C  CHECK FOR CONVERGENCE
      IF((DABS(ALPHANU-ALPHA0)).LE.1.D-14.OR.K.GT.7)GOTO 40
       ALPHA0 = ALPHANU
       BETA0 = BETANU
         K = K+1
       GOTO 100
   40  WRITE(6,50)ALPHANU, BETANU,K
   50  FORMAT(2X, 'ALPHANU', D15.06, 2X, 'BETANU', 3X, D12.6,4X,I5)
       RMEAN = ((1.D0/ALPHANU)**(1.D0/BETANU)) * GAMMA1(1.D0/BETANU)
       RMEAN2 = ((1.D0/ALPHANU)**(2.D0/BETANU))* GAMMA1(2.D0/BETANU)
       RMEAN3 = ((1.D0/ALPHANU)**(3.D0/BETANU))* GAMMA1(3.D0/BETANU)
       RMEAN4 = ((1.D0/ALPHANU)**(4.D0/BETANU))* GAMMA1(4.D0/BETANU)
       WRITE(6,30)RMU1, RMU2, RMU3, RMU4
   30  FORMAT(2X,'RMU1=  ', D12.5, 2X, 'RMU2=  ', D12.5,'RMU3= ',D12.5,
      1 'RMU4 =  ',D12.5)
       WRITE(6,44)RMEAN, RMEAN2, RMEAN3,RMEAN4
   44  FORMAT(2X, 'RMEAN =  ', D12.5,3X, D12.5,2X,'RMEAN3= ',D12.5,
      1 'RMEAN4 = ',D12.5)
C
       RETURN
       END
C
C      FUNCTION FOR THE ALPHA VALUE
C
       FUNCTION FUNC1( BETA)
       DOUBLE PRECISION ALPHA, BETA, A, VALUE, GAMMA1, FUNC1
```

```
      DOUBLE PRECISION RMU1, RMU2
      COMMON RMU1, RMU2, RMU3,RMU4
     A = 1.D0/BETA
     WRITE(6,30)BETA
  30 FORMAT(2X, 'BETA IN FUNC1', D15.7)
     VALUE = GAMMA1(A)
     FUNC1 = (VALUE/RMU1)**BETA
      RETURN
      END
C
C     FUNCTION FOR THE BETA VALUE
      FUNCTION FUNC2(ALPHA, BETA)
      DOUBLE PRECISION ALPHA, BETA, A, VALUE, GAMMA1, FUNC2
      DOUBLE PRECISION RMU1, RMU2
      COMMON RMU1, RMU2 , RMU3,RMU4
      A =2.D0/BETA
      VALUE = GAMMA1(A)
      FUNC2 = (2.D0*DLOG(ALPHA)) / DLOG(VALUE/RMU2)
CC     WRITE(6,66)FUNC2, ALPHA, VALUE
CC66   FORMAT(2X,'FUNC2 ALPHA VALUE', 4X,3D12.7)
      RETURN
      END
C
C     NUMERICAL APPROXIMATION FOR GAMMA FUNCTION

C     THIS FUNCTION CALCULATES GAMMA FUNCTION BY APPROXIMATION
      FUNCTION GAMMA1(A)
      DOUBLE PRECISION GAMMA1, A, B, PROD,VALUE
```

```
      REAL*8 A1, A2, A3, A4, A5
     DATA A1,A2,A3,A4,A5/-5748.646D-4,9512.363D-4,-6998.588D-4,
    1 4245.549D-4,-1010.678D-4/
      B = A
      PROD = 1.D0
30    IF(B.LE.1.D0)  GO TO 40
          PROD = PROD*B
          B = B-1.D0
          GO TO 30
40    VALUE = 1.D0+A1*B + A2*(B**2) + A3*(B**3) + A4*(B**4) +A5*(B**5)
     GAMMA1 =  PROD*VALUE
       WRITE(6,20)GAMMA1,B
20   FORMAT(2X, 'GAMMA', 5X, D10.5, ' FOR ARG.', 2X,D12.5)
        RETURN
        END
```

The vita has been removed from the scanned document