ZERO-POLE INTERPOLATION
OF NONREGULAR RATIONAL MATRIX FUNCTIONS

by

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(ABSTRACT)

In this thesis the right and left pole structure of a not necessarily regular rational matrix function W is described in terms of pairs of matrices–right and left pole pairs. The concept of orthogonality in $\mathcal{R}^n$ is investigated. Using this concept, the right and left zero structure of a rational matrix function W is desribed in terms of pairs and triples of matrices–right and left null pairs and right and left kernel triples. The definition of a spectral triple of a regular rational matrix function over a subset σ of $\mathbf{C}$ is extended to the nonregular case. Given a rational matrix function W and a subset σ of $\mathbf{C}$, the left null-pole subspace of W over σ is described in terms of a left kernel triple and a left σ-spectral triple for W. A sufficient condition for the minimality of McMillan degree of a rational matrix function H which is right equivalent to W on σ, that is a rational matrix function H of the same size and with the same left null-pole subspace over σ as W, is developed. An algorithm for constructing a rational matrix function W with a left kernel triple $(A_\kappa, B_\kappa, D_\kappa)$ and left null and right pole pairs over $\sigma \subset \mathbf{C}$ (A_ζ, B_ζ) and (C_π, A_π), repectively, from a regular rational matrix function with left null and right pole pairs over σ (A_ζ, B_ζ) and (C_π, A_π) is described. Finally, a necessary and sufficient condition for existence of a rational matrix function W with a given left kernel triple and a given left spectral triple over a subset σ of $\mathbf{C}$ is established.

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Już mówił:

– Właśnie zdałem przyśpieszoną maturę. Czekam na przyszłość, jak na pociąg na małej stacyjce. Może będzie to zagraniczny ekspres, a może zwykły osobowy gruchot, który skręci na jakąś lokalną linję małych zagmatwań. –

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Introduction

The central notion of this thesis is that of a matrix over the field $\mathcal{R}$ of scalar rational functions. Such matrices can be also viewed as meromorphic functions from the extended complex plane $\mathbf{C}_\infty$ into the space of matrices over $\mathbf{C}$. As a consequence, matrices over $\mathcal{R}$ can be approached from the point of view of an algebraist or an analyst. Although the distinction is not always clear-cut, the author identifies himself more with the latter point of view. This is reflected in the adopted name for matrices over $\mathcal{R}$: matrices over $\mathcal{R}$ are called rational matrix functions in this thesis. Matrices of any size over the ring of polynomials in z will be called matrix polynomials. The set of all $m \times n$ rational matrix functions, that is rational matrix functions of size $m \times n$, will be denoted by $\mathcal{R}^{m\times n}$.

Extensive research on rational matrix functions carried out recently is motivated to a large extent by many applications in various branches of engineering. Transfer functions and system matrices in Systems Theory are rational matrix functions (see e.g. [K,R]). The concept of a rational matrix function is fundamental for the H^∞ control theory (see e.g. [Fr]).

A rational matrix function W is said to be <u>regular</u> if W is square and det W is not identically equal to 0. Regular rational matrix functions of size $n \times n$, n a positive integer, form an algebra $\mathcal{A}$ over $\mathcal{R}$ such that each nonzero element of $\mathcal{A}$ has an inverse in $\mathcal{A}$. In fact, explicit spectral data formulae for the inverse of a regular rational matrix function W are available (see [BGK, Chapter I] if $W(\infty)$ is an invertible matrix and [BGR3, Chapter 5] otherwise). The theory of regular rational matrix functions makes extensive use of the existence of, and formulas for, W^{-1}. Consequently, many methods and techniques developed for rational matrix functions cannot be transfered directly to the study of nonregular rational matrix functions, that is rational matrix functions W such that W is not square or the determinant of W is identically equal to 0. This thesis generalizes certain results on regular rational matrix functions to the nonregular case.

The basic tool in the study of the properties of a rational matrix function W is provided by the Smith-McMillan form and a Smith-McMillan factorization of W which we now describe. Let q be a monic polynomial which is the least common multiple of the denominators of all entries of W. Then $\tilde{W}(z) \equiv q(z)W(z)$ is a matrix over the Euclidean domain of polynomials in z. Let $\tilde{D}$ be a Smith's normal form of $\tilde{W}$ (see e.g. Theorem 26.2 in [McD] or Theorem 3.8 in [J]). Thus,

$$\tilde{D} = \begin{bmatrix} p_1 & & & & & & 0 \\ & p_2 & & & & & 0 \\ & & \ddots & & & & \vdots \\ & & & p_k & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \\ & & & & & & \vdots \\ 0 & 0 & \dots & & & & 0 \end{bmatrix}$$

where $p_1, p_2, \dots, p_k$ are monic polynomials such that $p_i | p_{i+1}$ $(i = 1, 2, \dots, k-1)$ and there exist units E, F in the rings of square matrices of appropriate sizes over polynomials in z such that $\tilde{W} = E\tilde{D}F$. So $W = EDF$ where

$$D = \begin{bmatrix} p_1/q & & & & & & 0 \\ & p_2/q & & & & & 0 \\ & & \ddots & & & & \vdots \\ & & & p_k/q & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \\ & & & & & & \vdots \\ 0 & 0 & \dots & & & & 0 \end{bmatrix} \tag{1}$$

and E, F are matrix polynomials with constant nonzero determinants. The rational matrix function (1) is uniquely determined by W and is called the Smith-McMillan form of W (see e.g. [K]). The factorization EDF of W is called a Smith-McMillan factorization of W. We note that matrix polynomials E, F in a Smith-McMillan factorization EDF of W are not at all unique.

While the concept of a Smith form of a matrix polynomial goes back to 19th century (see [McD]), the concept of a Smith-McMillan form of a rational matrix function is fairly recent. The Smith-McMillan form of a square rational matrix function W has been introduced in the study of properties of electrical circuits in [McM]. Using the notation in [McM], let $\delta(W, \lambda)$ denote the sum of multiplicities of poles at $\lambda \in \mathbf{C}$ of the nonzero entries in the Smith-McMillan form of W. Let $\delta(W, \infty)$ denote $\delta(W \circ T, T^{-1}(\infty))$ where T is a Möbius transformation such that $T^{-1}(\infty) \in \mathbf{C}$. Then $\delta(W, \infty)$ does not depend on the choice of T and the number

$$\delta(W) = \sum_{\lambda \in \mathbf{C}_\infty} \delta(W, \lambda) \tag{2}$$

has been called in [McM] the degree of a rational matrix function W. The number $\delta(W)$ defined as in (2) for a not necessarily square rational matrix function W is called the McMillan degree of W.

If W is a nonregular $m \times n$ rational matrix function then after considering a Smith-McMillan factorization of W we see that the columns of W are contained in a proper subspace of the $\mathcal{R}$-vector space $\mathcal{R}^{m \times 1}$ or the rows of W are contained in a proper subspace of the $\mathcal{R}$-vector space $\mathcal{R}^{1 \times n}$. Forney investigated in [F] bases for a subspace over $\mathcal{R}$ of $\mathcal{R}^{1 \times n}$. Let the degree of a row vector polynomial $g = [\, g_1 \quad g_2 \quad \cdots \quad g_n \,]$ be

$$\deg g = \max \{\deg g_1, \deg g_2, \ldots, \deg g_n\}.$$

Forney defined a minimal polynomial basis for a subspace Λ of $\mathcal{R}^{1 \times n}$ to be a polynomial basis $\{v_1, v_2, \ldots, v_k\}$ for Λ such that $\sum_{i=1}^{k} \deg v_i$ is minimal. He showed that the degrees of the row vector polynomials in any minimal polynomial basis for Λ are invariant and depend only on Λ. Forney proved that a polynomial basis $\{v_1, v_2, ..., v_k\}$ for Λ is a minimal polynomial basis if and only if the coefficients of the highest degree terms in $v_1, v_2, ..., v_k$ are linearly independent and $v_1(\lambda), v_2(\lambda), ..., v_k(\lambda)$ are linearly independent for every $\lambda \in \mathbf{C}$. This characterization of a minimal polynomial basis underlies the algorithm in [F] of reduction of a basis for Λ to a minimal polynomial basis.

It has been also shown in [F] that the sum of degrees of the row vector polynomials in a minimal polynomial basis for a subspace Λ of $\mathcal{R}^{1\times n}$ is equal to the sum of the degrees of vector polynomials in a minimal polynomial basis for the right annihilator of Λ in $\mathcal{R}^{n\times 1}$. The degrees of the row (resp. column) vector polynomials in a minimal polynomial basis for a left (resp. right) annihilator of W are called in the literature the left (resp. right) Forney indices of W (see [BCRo]).

Let W be an $m \times n$ rational matrix function and choose a Smith-McMillan factorization EDF of W. Let

$$D = \begin{bmatrix} D_{11} & 0 \\ 0 & 0 \end{bmatrix}$$

with D_{11} regular and partition E, F conformably. Then

$$\begin{aligned} W &= EDF \\ &= [E_1 \quad E_2] \begin{bmatrix} D_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \\ &= E_1 D_{11} F_1. \end{aligned}$$

The columns of the matrix polynomial E_1 form a basis (over $\mathcal{R}$) for the column space of W. Choose a matrix polynomial $\tilde{E}_1$ whose columns form a minimal polynomial basis for the column space of W. Then $E_1 = \tilde{E}_1 Q_1$ for some regular rational matrix function Q_1. Choose another matrix polynomial $\tilde{F}_1$ whose rows form a minimal polynomial basis for the row space of W. Then $F_1 = Q_2 \tilde{F}_1$ for some regular rational matrix function Q_2. (In fact, it can be shown that Q_1 and Q_2 are matrix polynomials with constant nonzero determinants.) So

$$W = \tilde{E}_1 Q_1 D_{11} Q_2 \tilde{F}_1$$
$$\equiv \tilde{E}_1 \tilde{D} \tilde{F}_1,$$

with $\tilde{D}$ a regular rational matrix function . Now for each $\lambda \in \mathbf{C}$ let $\delta_z(W, \lambda)$ denote the sum of multiplicities of zeros at λ of the nonzero entries in the Smith-McMillan form of W and let $\delta_z(W, \infty)$ denote $\delta_z(W(z^{-1}), 0)$. Let $\delta_z(W) = \sum_{\lambda \in \mathbf{C}_\infty} \delta_z(W, \lambda)$. Then (see [VK])

$$\delta(W) - \delta_z(W) = \delta(\tilde{E}_1, \infty) + \delta(\tilde{F}_1, \infty).$$

Thus, the McMillan degree of W differs from $\delta_z(W)$ by the sum of left and right Forney indices of W.

In the early literature on rational matrix functions, the terms "zero and pole structure of a rational matrix function W" referred to the zeros and poles in $\mathbf{C}$ of the nonzero entries in the Smith-McMillan form of W and zeros and poles at 0 of the nonzero entries in the Smith-McMillan form of $W(z^{-1})$. Today more and more authors take the attitude that the zero and pole structure of a rational matrix function W should reflect more complete properties of W (see [CPW, WSCP, GLR, GKLR, BGR1]). We extend the approach of Ball-Gohberg-Rodman to the nonregular case. Suppose we multiply W on the right by a rational vector function ϕ which is analytic and nonzero at $\lambda \in \mathbf{C}_\infty$. After considering a Smith-McMillan factorization of W, we see that $W\phi$ can be analytic at λ, can have a pole at λ, or can vanish at λ identically or to a certain order. The right pole structure of W at $\lambda \in \mathbf{C}_\infty$ is related to a possible singular part of $W\phi$ at λ where ϕ is a rational vector function analytic at λ. The right zero structure of W is related to the functions ϕ such that $W\phi$ vanishes at λ. Similarly, suppose we multiply W on the left by a rational vector function ϕ which is analytic and nonzero at $\lambda \in \mathbf{C}_\infty$. The left pole structure of W at λ is related to the possible singular part at λ of ϕW. The left zero structure of W at λ is related to the functions ϕ such that ϕW vanishes at λ.

Suppose σ is a subset of $\mathbf{C}_\infty$, W and H are two $m \times n$ rational matrix functions and there exists an $n \times n$ regular rational matrix function Q with no zeros nor poles in σ such that $W = HQ$. Then W and H are said to be right equivalent on σ. Since the singular part of a function $W\phi$ at $\lambda \in \sigma$ coincides with the singular part at λ of $H(Q\phi)$, rational matrix functions which are right equivalent on σ have the same right pole structure on σ. Similarly, if W and H are right equivalent on σ then W and H have the same left zero structure on σ. However, even in the regular case, the converse statement is not true (see [GKLR]): two $m \times n$ rational matrix functions with the same right pole and left zero structure on $\sigma \subset \mathbf{C}_\infty$ may fail to be right equivalent on σ. Thus, the assertion that W and H are right equivalent on $\sigma \subset \mathbf{C}_\infty$ is stronger than the assertion that W and H have the same right pole and left zero structure on σ. To capture exactly this notion of right equivalence, another invariant, namely null-pole coupling matrix, has been introduced in the regular case (see [GKLR] and [BRan1]). The extension of the null-pole coupling matrix to the nonregular case is defined in Chapter III.

The thesis is organized as follows. In Chapter I we show how the right and left pole structure of an $m \times n$ rational matrix function W can be described in terms of pairs of matrices–left and right pole pairs for W. We also show how W can be represented in terms of its right and left pole pairs. In Chapter II we describe left and right zero structure of a rational matrix function W in terms of pairs and triples of matrices–left and right null pairs and left and right kernel triples. The regular case is worked out in detail in [BGR1]. While the generalization of the definition of left and right pole pairs to the nonregular case was straightforward, the analogous generalization of the definition of left and right null pairs was not obvious. Our approach differs from the approach in [BCRo] as we utilize the concept of orthogonality in non-Archimedean normed spaces, introduced in the study of rational matrix functions by Forney in [F] and later pursued by Kailath-Verghese (see [VK]).

Chapter III characterizes right equivalence of rational matrix functions on a subset σ of **C** in terms of left spectral triples containing data as defined in Chapters I and II. In Chapter III we also develop a sufficient condition for a rational matrix function H which is right equivalent to a given rational matrix function W on $\sigma \subset \mathbf{C}$ to have a minimal McMillan degree.

In Chapter IV we solve the local inverse spectral problem, that is we answer the question "what are the conditions for existence of a rational matrix function with a given right pole and left zero structure on $\sigma \subset \mathbf{C}$". We also solve the refined version of the local inverse spectral problem, that is, we indicate a necessary and sufficient condition for existence of a rational matrix function with a given left spectral triple over $\sigma \subset \mathbf{C}$ and a given left kernel triple.

Chapter I

Pole Structure

We say that a rational matrix function W has a pole at $\lambda \in \mathbf{C}_\infty$ (or λ is a pole of W) if some entry of W has a pole at λ.

Let $W \in \mathcal{R}^{m\times n}$ and let $\lambda \in \mathbf{C}$ be a pole of W. Suppose that the first l diagonal entries in the Smith-McMillan form D of W have a pole at λ and all other entries of D are analytic at λ. Then we say that the geometric multiplicity of the pole of W at λ equals l. The multiplicities of poles at λ of the diagonal entries of D are called the partial multiplicities of the pole of W at λ. The geometric multiplicity and the partial multiplicities of a pole of W at infinity are defined to be the geometric multiplicity and the partial multiplicities of the pole of the rational matrix function $H(z) = W(z^{-1})$ at 0. The sum of partial multiplicities of the pole of W at $\lambda \in \mathbf{C}_\infty$ is called the (total) multiplicity of the pole of W at λ. Thus, the McMillan degree of a rational matrix function W is equal to the sum of multiplicities of all poles of W. The multiplicity of a pole of a rational matrix function W at $\lambda \in \mathbf{C}_\infty$ is also called in the literature the local degree of W at λ or the pole multiplicity of W at λ.

We note that if a rational (row or column) vector function W has a pole at $\lambda \in \mathbf{C}_\infty$ then the geometric multiplicity of the pole of W at λ equals 1 and the total multiplicity of the pole of W at λ equals n where n is such that

$$W(z) = \sum_{i=-n}^{\infty} (z-\lambda)^i W_i$$

with all W_i's constant vectors and $W_{-n} \neq 0$. Thus, the multiplicity of a pole of a rational vector function W at λ equals the largest multiplicity of a pole at λ of some entry of W.

In this chapter we define pole functions and pole pairs for a not necessarily regular rational matrix function W. Pole pairs have been introduced and extensively used in the study of regular rational matrix functions (see [GKLR, BGR1, GK1]). The concept of pole functions appears in [BGR1]. Sections 1.1 and 1.2 contain the basic definitions. In Section 1.3 we discuss realization of a rational matrix function in terms of its right and left pole pairs.

1.1 Right pole pairs

Let W be an $m \times n$ rational matrix function and let $\lambda \in \mathbf{C}_\infty$. A function $\psi \in \mathcal{R}^{m\times 1}$ is called a right pole function for W at λ if

(i) ψ is analytic at λ and $\psi(\lambda) \neq 0$,

(ii) there is a positive integer k and a function $\phi \in \mathcal{R}^{n\times 1}$ such that ϕ is analytic at λ and

$$\psi(z) = \begin{cases} (z-\lambda)^k W(z)\phi(z), & \text{if } \lambda \in \mathbf{C} \\ z^{-k} W(z)\phi(z), & \text{if } \lambda = \infty. \end{cases} \tag{1}$$

The maximal integer k such that (1) holds for an appropriate ϕ is called the order of the right pole function ψ.

We note that $W \in \mathrm{R}_{m\times n}$ has a right pole function at λ if and only if W has a pole at λ.

Suppose $W \in \mathcal{R}^{m\times n}$ has a pole at $\lambda \in \mathbf{C}_\infty$. Then the values at λ of right pole functions for W at λ form, together with the 0 vector, a subspace V of $\mathbf{C}^m$. It can be seen from a Smith-McMillan factorization of W that dim V is equal to the geometric multiplicity of the pole of W at λ. We say that right pole functions $\psi_1, \psi_2, ..., \psi_\eta$ for W at λ of orders $k_1, k_2, ..., k_\eta$ respectively form a canonical set of right pole functions for W at λ if

(i) $\{\psi_1(\lambda), \psi_2(\lambda), ..., \psi_k(\lambda)\}$ is a basis for V,

(ii) $\sum_{i=1}^{\eta} k_i$ is maximal subject to condition (i).

It follows from a Smith-McMillan factorization of W that the orders of the pole functions in any canonical set of right pole functions for W at λ coincide with the partial pole multiplicities for W at λ.

A canonical set of right pole functions for an $m \times n$ rational matrix function W at $\lambda \in \mathbf{C}_\infty$ can be found similarly as in the regular case (cf. [BGR1]). Choose a right pole function ψ_1 for W at λ of a maximal possible order. Given the right pole functions $\psi_1, \psi_2, ..., \psi_i$ for W at λ, choose from right pole functions for W at λ a function ψ_{i+1} of possibly maximal order such that $\psi_{i+1}(\lambda) \notin$ span $\{\psi_1(\lambda), \psi_2(\lambda), ..., \psi_i(\lambda)\}$. Continue until the span of the values at λ of the right pole functions for W at λ has been exausted.

Let $W \in \mathcal{R}^{m \times n}$ and let $\lambda \in \mathbf{C}_\infty$ be a pole of W. Choose a canonical set of right pole functions for W at λ, $\{\psi_1, \psi_2, ..., \psi_\eta\}$. Suppose the order of ψ_i $(i = 1, 2, ..., \eta)$ as a right pole function for W at λ is k_i and let $\psi_{i,j} \in \mathbf{C}^{m \times 1}$ $(1 \le i \le \eta, j \ge 0)$ be such that

$$\psi_i(z) = \begin{cases} \displaystyle\sum_{j=0}^{\infty} (z - \lambda)^j \psi_{i,j}, & \text{if } \lambda \in \mathbf{C} \\ \displaystyle\sum_{j=0}^{\infty} z^{-j} \psi_{i,j}, & \text{if } \lambda = \infty \end{cases} \tag{2}$$

Any ordered pair of matrices (C, A), where C equals

$$\left[\psi_{1,0} \quad \psi_{1,1} \quad \cdots \quad \psi_{1,k_1-1} \quad \psi_{2,0} \quad \psi_{2,1} \quad \cdots \quad \psi_{2,k_2-1} \quad \cdots \quad \psi_{\eta,0} \quad \psi_{\eta,1} \quad \cdots \quad \psi_{\eta,k_\eta-1}\right] S,$$

and

$$A = \begin{cases} S^{-1}\begin{bmatrix} J_{k_1}(\lambda) & & & \\ & J_{k_2}(\lambda) & & \\ & & \ddots & \\ & & & J_{k_q}(\lambda) \end{bmatrix} S, & \text{if } \lambda \in \mathbf{C} \\ \\ S^{-1}\begin{bmatrix} J_{k_1}(0) & & & \\ & J_{k_2}(0) & & \\ & & \ddots & \\ & & & J_{k_q}(0) \end{bmatrix} S, & \text{if } \lambda = \infty \end{cases} \tag{3}$$

for some invertible matrix S of appropriate size, is called a right pole pair for W at λ (see [BGR1, GK1]).

Let W be an $m \times n$ rational matrix function and let $\sigma \subset \mathbf{C}$. Let $\lambda_1, \lambda_2, ..., \lambda_p$ be the poles of W in σ and let (C_i, A_i) be a right pole pair for W at λ_i $(i = 1, 2, ..., p)$. Any ordered pair of matrices (C, A), where

$$C = \begin{bmatrix} C_1 & C_2 & ... & C_p \end{bmatrix} S,$$

and

$$A = S^{-1}\begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_p \end{bmatrix} S$$

for some invertible matrix S of an appropriate size, is called a right pole pair for W over σ (or a σ-right pole pair for W). We note that if W is analytic on σ, the right pole pair for W over σ is vacuous. Also, if λ is the only pole of W in σ, the right pole pair for W at λ is the right pole pair for W over σ.

If (C, A) is a right pole pair for a rational matrix function W over $\sigma \subset \mathbf{C}$ and W is analytic on $\mathbf{C}_\infty \backslash \sigma$, (C, A) is called a global right pole pair for W.

Let σ be a subset of the complex plane and let $W \in \mathcal{R}^{m \times n}$. Then any two right pole pairs $(C_1, A_1), (C_2, A_2)$ for W over σ are right-similar, that is

$$C_2 = C_1 S \quad \text{and} \quad A_2 = S^{-1} A_1 S$$

for some invertible matrix S (see Theorem 3.2 in [BGR1]). Moreover, the similarity matrix S is unique. Similarly, any two right pole pairs for W at infinity are right-similar. Also, if (C, A) is a right pole pair for W at $\lambda \in \mathbf{C}_\infty$ and A is in Jordan form then the columns of C coincide with appropriate coefficients in the Taylor expansions of the functions in some canonical set of right pole functions for W at λ.

It is shown in [BGR1] (Theorems 3.1 and 3.3) that a right pole pair (C, A) for $W \in \mathcal{R}^{m\times n}$ over $\sigma \in \mathbf{C}$ is observable, that is

$$\ker \begin{bmatrix} C \\ CA \\ \vdots \\ CA^l \end{bmatrix} = (0)$$

for sufficiently large integers l. Observable pairs are also called null-kernel pairs in the literature.

1.2 Left pole pairs

Let W be an $m\times n$ rational matrix function and let $\lambda \in \mathbf{C}_\infty$. A function $\psi \in \mathcal{R}^{1\times m}$ is called a left pole function for W at λ if

(i) ψ is analytic at λ and $\psi(\lambda) \neq 0$,

(ii) there is a positive integer k and a function $\phi \in \mathcal{R}^{1\times m}$ such that ϕ is analytic at λ and

$$\psi(z) = \begin{cases} (z-\lambda)^k \phi(z) W(z), & \text{if } \lambda \neq \infty \\ z^{-k}\phi(z)W(z), & \text{if } \lambda = \infty. \end{cases} \tag{4}$$

The maximal integer k such that (4) holds for an appropriate ϕ is called the order of the left pole function ψ.

The left pole functions $\psi_1, \psi_2, ..., \psi_\eta$ for W at λ of orders $k_1, k_2, ..., k_\eta$ respectively form a canonical set of left pole functions for W at λ if

(i) $\psi_1(\lambda), \psi_2(\lambda), ..., \psi_k(\lambda)$ are linearly independent,

(ii) $\sum_{i=1}^{\eta} k_i$ is maximal subject to condition (i).

We note that the number of functions in any canonical set of left pole functions for W at λ is equal to the geometric multiplicity of the pole of W at λ. Also, the orders of the functions in a canonical set of left pole functions for W are equal to the partial multiplicities of the pole of W at λ.

Let $\{\psi_1, \psi_2, ..., \psi_\eta\}$ be a canonical set of left pole functions for $W \in \mathcal{R}^{m\times n}$ at $\lambda \in \mathbf{C}_\infty$ of orders $k_1, k_2, ..., k_\eta$ respectively and let $\psi_{i,j}$ be such that (2) holds. Let S be an invertible matrix of the same size as A. An ordered pair of matrices (A, B), where

$$A = \begin{cases} S \begin{bmatrix} J_{k_1}(\lambda) & & & \\ & J_{k_2}(\lambda) & & \\ & & \ddots & \\ & & & J_{k_\eta}(\lambda) \end{bmatrix} S^{-1}, & \text{if } \lambda \in \mathbf{C} \\ S \begin{bmatrix} J_{k_1}(0) & & & \\ & J_{k_2}(0) & & \\ & & \ddots & \\ & & & J_{k_\eta}(0) \end{bmatrix} S^{-1}, & \text{if } \lambda = \infty \end{cases}$$

and

$$B = S \begin{bmatrix} \psi_{1,k_1-1} \\ \psi_{1,k_1-2} \\ \vdots \\ \psi_{1,0} \\ \psi_{2,k_2-1} \\ \psi_{2,k_2-2} \\ \vdots \\ \psi_{2,0} \\ \vdots \\ \psi_{\eta,k_\eta-1} \\ \psi_{\eta,k_\eta-2} \\ \vdots \\ \psi_{\eta,0} \end{bmatrix},$$

is called a left pole pair for W at λ.

Let $W \in \mathcal{R}^{m\times n}$ and let $\sigma \subset \mathbf{C}$. Let $\lambda_1, \lambda_2, ..., \lambda_p$ be the poles of W in σ and let (A_i, B_i) be a left pole pair for W at λ_i $(i = 1, 2, ..., p)$. Any ordered pair of matrices

(A, B), where

$$A = S \begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_p \end{bmatrix} S^{-1}, \qquad B = S \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_p \end{bmatrix}$$

for some invertible matrix S of an appropriate size, is called a <u>left pole pair</u> for W over σ (or a σ-left pole pair for W).

If (A, B) is a left pole pair for a rational matrix function W over $\sigma \subset \mathbf{C}$ and W is analytic on $\mathbf{C}_\infty \backslash \sigma$, (A, B) is called the <u>global left pole pair</u> for W.

We note that similarly as in the regular case any two σ-left pole pairs (A_1, B_1) and (A_2, B_2) for a rational matrix function W over $\sigma \subset \mathbf{C}$ (resp. at infinity) are <u>left-similar</u>, that is

$$A_2 = SA_1S^{-1} \qquad \text{and} \qquad B_2 = SB_1$$

for some matrix S (see [BGR1]). Also, the similarity matrix S is unique. It is shown in [BGR1] (Theorem 3.4) that a left pole pair (A, B) for W over σ is <u>controllable</u>, that is for sufficiently large integers l the matrix

$$\begin{bmatrix} B & AB & \dots & A^l B \end{bmatrix}$$

has full row rank. Controllable pairs are also called full-rank pairs in the literature.

1.3 Realization theory

The realization theory for rational matrix functions in the context of right and left pole pairs can be based on the following lemma.

Lemma 1.1 *Let (C, A) be a right pole pair for $W \in \mathcal{R}^{m \times n}$ at $\lambda \in \mathbf{C}_\infty$. Then there exists a unique matrix B such that the rational matrix function*

$$\begin{cases} W(z) - C(z - A)^{-1}B, & \text{if } \lambda \in \mathbf{C} \\ W(z) - C(z^{-1} - A)^{-1}B, & \text{if } \lambda = \infty \end{cases} \tag{5}$$

is analytic at λ. Moreover, (A, B) is a left pole pair for W at λ.

Lemma 1.1 can be proved in the same way as Theorem 5.1 in [BGR1].

We note that the left version of Lemma 1.1 also holds; that is, if the left pole pair (A, B) for W at λ is given then there exists a unique matrix C such that the rational matrix function (5) is analytic at λ. Moreover, (C, A) is a right pole pair for W at λ.

In view of Liouville's Theorem, Lemma 1.1 implies that each rational matrix function has the following representation.

Theorem 1.2 *Let W be a rational matrix function. Let (C, A) and (C_∞, A_∞) be right pole pairs for W over $\mathbf{C}$ and at infinity. Then there exist unique matrices B, B_∞ and D such that*

$$W(z) = C(z - A)^{-1}B + D + C_\infty(z^{-1} - A_\infty)^{-1}B_\infty. \tag{6}$$

Any representation of a rational matrix function W of the form (6) is called a <u>realization</u> of W, and is usually written down as (A, B, C, D, E, F, G) where $E = C_\infty$, $F = B_\infty$ and $G = A_\infty$. The realization (6) of W in which $\sigma(A_\infty) \subset \{0\}$ has been called in [C] normal. It is possible to include in the last term in (6) the singular part of W at other poles of W besides infinity. This approach has been adopted in [BCR] (see also Chapter 5 in [BGR3]), where the term $E(z^{-1} - G)^{-1}F$ in (A, B, C, D, E, F, G) realizes the singular part of W outside a region $\Omega \subset \mathbf{C}$ which contains 0. We will not follow, however, this more general approach. In the sequel we will use realizations as in Theorem 1.2.

We note that if $W(z)$ is analytic at infinity, the right pole pair (C_∞, A_∞) for W at infinity is vacuous and the last term in (6) does not occur. In this case the realization

$D + C(z - A)^{-1}B$ is written down in the form (A, B, C, D). If W is analytic on $\mathbf{C}$, that is, if W is a matrix polynomial, the first term on the right hand side of (6) does not occur. If W has no poles in $\mathbf{C}_\infty$, $W(z) = D$.

It is well known that the sum of sizes of matrices A and A_∞ in (6) is at least equal to the McMillan degree of W. A realization $C(\phi(z) - A)^{-1}B + D + E(\psi(z) - G)^{-1}F$ of W, where ϕ and ψ are scalar rational functions such that the sizes of matrices A and G add up to the McMillan degree of W, is said to be minimal. Since the orders of pole functions in a canonical set of right pole functions for a rational matrix function W at $\lambda \in \mathbf{C}_\infty$ are equal to partial multiplicities of the pole of W at λ, the sum of sizes of matrices A and A_∞ in (6) is equal to the McMillan degree of W. Thus, the realization (6) of W is minimal. Minimal realizations of the form (6) have, in fact, the following property.

Theorem 1.3 *Let (A, B, C, D, E, F, G) be a realization of a rational matrix function W with $\sigma(G) \subset \{0\}$. Then the following are equivalent:*

(i) (C, A) is a right pole pair for W over $\mathbf{C}$ and (E, G) is a right pole pair for W at infinity;

(ii) (A, B) is a left pole pair for W over $\mathbf{C}$ and (G, F) is a left pole pair for W at infinity;

(iii) (A, B, C, D, E, F, G) is a minimal realization of W.

Theorem 1.3 follows immediately from Theorem 1.2 and the fact that if

$$(A_1, B_1, C_1, D_1, E_1, F_1, G_1) \quad \text{and} \quad (A_2, B_2, C_2, D_2, E_2, F_2, G_2)$$

are two minimal realizations of a rational matrix function W such that $\sigma(G_1) =$

$\sigma(G_2) \subset \{0\}$, then

$$A_2 = SA_1S^{-1}, \quad B_2 = SB_1, \quad C_2 = C_1S$$

$$G_2 = TG_1T^{-1}, \quad F_2 = TF_1, \quad E_2 = E_1T$$

for some unique invertible matrices S and T (see Theorem I.7 in [C] or Chapter 5 in [BGR3]).

Chapter II

Zero Structure

We say that a point $\lambda \in \mathbf{C}$ is a zero of a rational matrix function W (or, equivalently, that W has a zero at λ) if λ is a zero of some nonzero diagonal entry in the Smith-McMillan form of W (see eg. [R]). W has a zero at infinity if $H(z) = W(z^{-1})$ has a zero at 0. The number of nonzero diagonal entries in the Smith-McMillan form D of W which vanish at $\lambda \in \mathbf{C}$ is called the geometric multiplicity of the zero of W at λ. The orders of zeros at λ of the nonzero diagonal entries in the Smith-McMillan form of W are called the partial multiplicities of the zero of W at λ. The geometric multiplicity and the partial multiplicities of the zero of W at infinity are defined to be the geometric multiplicity and the partial multiplicities of the zero of $H(z) = W(z^{-1})$ at 0. The sum of partial multiplicities of a zero of W at $\lambda \in \mathbf{C}_\infty$ is called the (total) multiplicity of the zero of W at λ. Partial multiplicities and the total multiplicity of a zero of a rational matrix function W at $\lambda \in \mathbf{C}_\infty$ are also called in the literature the partial and total zero multiplicities of W at λ.

If W is a rational (row or column) vector function which has a zero at $\lambda \in \mathbf{C}_\infty$, the geometric multiplicity of the zero of W at λ equals 1. In this case the multiplicity of the zero of W at λ is an integer n such that the rational vector function

$$\begin{cases} (z-\lambda)^{-n} W(z), & \text{if } \lambda \in \mathbf{C} \\ z^n W(z), & \text{if } \lambda = \infty \end{cases}$$

is analytic and nonzero at λ.

In the study of zero structure of a rational matrix function the concept of null functions (see Sections 2.3 and 2.4 below) proves useful. Null functions are also called root functions in the literature. They were first introduced in [KT] in the context of analytic operator functions of several variables. The definition of a null function has been extended in [GS] to the case of a meromorphic operator function. Null functions were used in the context of nonregular rational matrix functions in [BCRo]. We note that our definition of null functions in Sections 2.3 and 2.4 below differs from the respective definitions in [KT, GS, BCRo] in that we do not allow null functions to have infinite order.

Let $\sigma \subset \mathbf{C}_\infty$ and let W be a rational matrix function which is analytic in σ. If W is regular, then it can be seen from a Smith-McMillan factorization of W that the set of points $\lambda \in \sigma$ such that the matrix $W(\lambda)$ is singular consists of isolated points. In a more general case, where W is not square or det W vanishes identically, there may exist rational matrix functions ϕ such that ϕ is analytic in σ, $\phi(\lambda) \neq 0$ and $\phi(\lambda)W(\lambda) = 0$ for all $\lambda \in \sigma$. For this reason the spectrum of a rectangular analytic matrix function is said in [Ro] to be continuous. We shall call the zero structure of a rational matrix function W related to the left (or right) annihilator of W the <u>continuous</u> <u>left</u> (or <u>right</u>) <u>zero</u> <u>structure</u> of W. Thus, the continuous zero structure of a rational matrix function corresponds to infinite order null functions as defined in [GS, BCRo]. We shall call the zero structure of W related to the (finite order) null functions the <u>discrete</u> <u>zero</u> <u>structure</u> of W.

The zero structure of a regular rational matrix function W can be described by pairs of matrices–left and right null pairs. Null pairs are also called eigenpairs or standard pairs in the literature. Null pairs were introduced originally in [GR1, GR2, Ro] (see also [GLR]) for analytic matrix and operator functions. They evolved from Jordan chains at $\lambda_0 \in \mathbf{C}_\infty$ for an analytic matrix or operator function L, that is, chains of vectors $x_0, x_1, ..., x_k$ such that

$$\sum_{i=0}^{j} \frac{1}{i!} L^{(i)} x_{j-i}(\lambda_0) = 0$$

for $j = 0, 1, ..., k$. The definition of a Jordan chain for L above extends the definition of a Jordan chain of a matrix. Indeed, if A is a square matrix and $L(z) = z - A$ then $\{x_0, x_1, ...x_k\}$ is a Jordan chain for L at λ_0 if and only if λ_0 is an eigenvalue of A and $\{x_0, x_1, ...x_k\}$ is a corrersponding Jordan chain (see [GLR]).

Null pairs are used to describe the discrete zero structure of a rectangular rational matrix function in [BCRo]. We note that our definition of null pairs below differs from the respective defintion in [BCRo].

In Chapter II we describe the right and left zero structure of a rational matrix function W. We begin with the description of the left annihilator of W (Section 2.1). Then we analyze the concept of orthogonality in $\mathcal{R}^n$ (Section 2.2). In Section 2.3 we define left null pairs for W. Section 2.4 contains definitions and facts referring to the right zero structure of W. Since the proofs of all assertions made in Section 2.4 have analogues in Section 2.1 and 2.3, they are omitted.

2.1 Continuous left zero structure

Let W be an $m \times n$ rational matrix function. We will denote by W^{ol} the left annihilator of W. Thus,

$$W^{ol} = \{r \in \mathcal{R}^{1\times m} : \quad rW = 0\}.$$

We note that if W is right invertible, then $W^{ol} = (0)$. If $W^{ol} \neq (0)$, a matrix polynomial whose rows form a minimal polynomial basis for W^{ol} is called a left kernel polynomial for W (see [BCRo]). We will assume that the rows in any left kernel polynomial for W are ordered according to decreasing degrees.

If Λ is a subspace of $\mathcal{R}^n$ and $\lambda \in \mathbf{C}_\infty$, we will denote by $\Lambda(\lambda)$ the set of values at λ of those functions in Λ which are analytic at λ. Plainly, $\Lambda(\lambda)$ is a subspace of $\mathbf{C}^n$. The space $\Lambda(\lambda)$ can be characterized equivalently as the linear span over $\mathbf{C}$ of the leading coefficients in the expansions at λ of the functions in Λ. In particular, if W is an $m \times n$ rational matrix function, then $W^{ol}(\lambda)$ is a subspace of $\mathbf{C}^{1\times n}$ formed by the values at λ of functions $\phi \in \mathcal{R}^{1\times m}$ such that $\phi W = 0$ and ϕ is analytic at λ.

Let Λ be a subspace of $\mathcal{R}^{1\times n}$ and let P be a matrix polynomial such that the rows $P_1, P_2, ..., P_k$ of P form a minimal polynomial basis for Λ. Suppose first $\lambda \in \mathbf{C}$. Plainly,

$$\text{span}\,\{P_1(\lambda), P_2(\lambda), ..., P_k(\lambda)\} \subset \Lambda(\lambda). \tag{1}$$

Since $P(\lambda)$ has full row rank, $P_1(\lambda), P_2(\lambda), ..., P_k(\lambda)$ are linearly independent. Hence, if ψ is a linear combination over scalar rational functions of $P_1, P_2, ..., P_k$, then the leading coefficient in the Laurent expansion of ψ at λ is contained in span $\{P_1(\lambda), P_2(\lambda), ..., P_k(\lambda)\}$. It follows that inclusion (1) is an equality. Thus, $\{P_1(\lambda), P_2(\lambda), ..., P_k(\lambda)\}$ is a basis for $\Lambda(\lambda)$ for every $\lambda \in \mathbf{C}$. Suppose now $\lambda = \infty$ and let $\tilde{P}_i(z) = z^{-\alpha_i} P_i(z)$ where α_i is such that $\tilde{P}_i$ is analytic and nonzero at infinity $(i = 1, 2, ..., k)$. Since the leading coefficients in the Laurent expansions of $P_1, P_2, ..., P_k$ at infinity are linearly independent, $\tilde{P}_1(\infty), \tilde{P}_2(\infty), ..., \tilde{P}_k(\infty)$ are linearly independent. Hence, by the same reasoning as above, $\Lambda(\infty) = \text{span}\{\tilde{P}_1(\infty), \tilde{P}_2(\infty), ..., \tilde{P}_k(\infty)\}$. So the leading coefficients of the rows of P form a basis for $\Lambda(\infty)$. In particular, if P and Λ are a left kernel polynomial and the left annihilator of an $m \times n$ rational matrix function W, then, for each $\lambda \in \mathbf{C}$, $\{P_1(\lambda), P_2(\lambda), ..., P_k(\lambda)\}$ is a basis for $W^{ol}(\lambda)$ and $\{\tilde{P}_1(\infty), \tilde{P}_2(\infty), ..., \tilde{P}_k(\infty)\}$ is a basis for $W^{ol}(\infty)$.

In view of Theorem 1.2, any matrix polynomial P has a realization of the form

$$D + E(z^{-1} - G)^{-1}F$$

where $D = P(0)$ and $(E, G), (G, F)$ are right and left pole pairs for P at infinity. We now describe one canonical form for such a realization of a left kernel polynomial.

Proposition 2.1 *Let $P \in \mathcal{R}^{k\times n}$ be a matrix polynomial with the i'th row*

$$P_i(z) = P_{i,\eta_i} + zP_{i,\eta_i-1} + \dots + z^{\eta_i}P_{i,0}$$

($P_{i,0} \neq 0$ if $P_i \neq 0$). Suppose that the first l rows of P have a pole at infinity and the last $k-l$ rows of P are constant. Then

$$P(z) = P(0) \; + \; \begin{bmatrix} E_1 & E_2 & \dots & E_l \end{bmatrix} \cdot$$

$$\cdot \left(z^{-1} - \begin{bmatrix} J_{\eta_1}(0) & & & \\ & J_{\eta_2}(0) & & \\ & & \ddots & \\ & & & J_{\eta_l}(0) \end{bmatrix} \right)^{-1} \begin{bmatrix} P_{1,\eta_1-1} \\ P_{1,\eta_1-2} \\ \vdots \\ P_{1,0} \\ P_{2,\eta_2-1} \\ P_{2,\eta_2-2} \\ \vdots \\ P_{2,0} \\ \vdots \\ P_{l,\eta_l-1} \\ P_{l,\eta_l-2} \\ \vdots \\ P_{l,0} \end{bmatrix} \tag{2}$$

where E_i is the $k \times \eta_i$ matrix with 1 at the position $(i,1)$ and 0's elsewhere.

Proof Since

$$\left(z^{-1} - \begin{bmatrix} J_{\eta_1}(0) & & & \\ & J_{\eta_2}(0) & & \\ & & \ddots & \\ & & & J_{\eta_l}(0) \end{bmatrix} \right)^{-1} =$$

$$= \begin{bmatrix} (z^{-1} - J_{\eta_1}(0))^{-1} & & & \\ & (z^{-1} - J_{\eta_2}(0))^{-1} & & \\ & & \ddots & \\ & & & (z^{-1} - J_{\eta_l}(0))^{-1} \end{bmatrix}$$

and

$$\left(z^{-1} - J_{\eta_i}(0) \right)^{-1} = \begin{bmatrix} z & z^2 & \dots & z^{\eta_i} \\ 0 & z & \dots & z^{\eta_i-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & z \end{bmatrix},$$

the second term on the right hand side in (2) equals

$$\sum_{i=1}^{l} E_i \begin{bmatrix} zP_{i,\eta_i-1} + z^2 P_{i,\eta_i-2} + \dots + z^{\eta_i} P_{i,0} \\ * \\ \vdots \\ * \end{bmatrix}.$$

So equality (2) follows.

□

If a rational vector function ϕ has a pole at $\lambda \in \mathbf{C}_\infty$ of order k, we will call the coefficient of

$$\begin{cases} (z-\lambda)^{-k}, & \text{if } \lambda \in \mathbf{C} \\ z^k, & \text{if } \lambda = \infty \end{cases}$$

in the Laurent expansion of ϕ at λ the leading coefficient of ϕ at λ. It turns out that the realization (2) of P is minimal if and only if the leading coefficients at infinity of the rows of P are linearly independent.

Proposition 2.2 *The realization* (2) *of a matrix polynomial P is minimal if and only if the leading coefficients at infinity of the columns of P which have a pole at infinity are linearly independent.*

Proof Since

$$\left(\begin{bmatrix} J_{\eta_1}(0) & & & \\ & J_{\eta_2}(0) & & \\ & & \ddots & \\ & & & J_{\eta_l}(0) \end{bmatrix}, \begin{bmatrix} P_{1,\eta_1-1} \\ P_{1,\eta_1-2} \\ \vdots \\ P_{1,0} \\ P_{2,\eta_2-1} \\ P_{2,\eta_2-1} \\ \vdots \\ P_{2,0} \\ \vdots \\ P_{l,\eta_l-1} \\ P_{l,\eta_l-2} \\ \vdots \\ P_{l,0} \end{bmatrix} \right)$$

is a left pole pair at infinity for P if and only if $P_{1,0}, P_{2,0}, ..., P_{l,0}$ are linearly independent, the assertion follows by Theorem 1.3.

□

Corollary 2.3 *If P is a left kernel polynomial for some rational matrix function then the realization* (2) *of P is minimal.*

The matrices $E_1, E_2, ..., E_l$ in the realization (2) of P are determined by the matrices $P(0)$ and $J_{\eta_1}(0), J_{\eta_2}(0), ..., J_{\eta_l}(0)$. Indeed, the number of rows of any E_i equals the number of rows of $P(0)$, and the number of columns of E_i equals the number of columns of $J_{\eta_i}(0)$. Consequently, the ordered triple of matrices

$$\left(P(0), \quad \begin{bmatrix} J_{\eta_1}(0) & & & \\ & J_{\eta_2}(0) & & \\ & & \ddots & \\ & & & J_{\eta_l}(0) \end{bmatrix}, \quad \begin{bmatrix} P_{1,\eta_1-1} \\ P_{1,\eta_1-2} \\ \vdots \\ P_{1,0} \\ P_{2,\eta_2-1} \\ P_{2,\eta_2-1} \\ \vdots \\ P_{2,0} \\ \vdots \\ P_{l,\eta_l-1} \\ P_{l,\eta_l-2} \\ \vdots \\ P_{l,0} \end{bmatrix} \right) \tag{3}$$

determines a matrix polynomial P uniquely. A triple of matrices as in (3) which describes a left kernel polynomial for a rational matrix function W will be called a <u>left kernel triple</u> for W.

We note that a left kernel triple for a rational matrix function W can be easily read off from a left kernel polynomial P for W. Also, a triple of matrices $(A_\kappa, B_\kappa, D_\kappa)$ is a left kernel triple for some rational matrix function W if A_κ is in the Jordan form, $\sigma(A_\kappa) = \{0\}$, and the rows of the corresponding matrix polynomial form a minimal polynomial basis for some subspace of $\mathcal{R}^{1\times m}$.

Finally, we note that if a left kernel triple $(A_\kappa, B_\kappa, D_\kappa)$ is given, we can easily find the corresponding left kernel polynomial for W. A basis for $W^{ol}(\lambda)$ at $\lambda \in \mathbf{C}_\infty$ can be computed from $(A_\kappa, B_\kappa, D_\kappa)$ using nested multiplication. For future reference, we state this in the following proposition.

Proposition 2.4 *Let*

$$\left(\begin{bmatrix} P_{1,n_1} \\ P_{2,n_2} \\ \vdots \\ P_{l,n_l} \\ P_{l+1,0} \\ P_{l+2,0} \\ \vdots \\ P_{k,0} \end{bmatrix}, \begin{bmatrix} J_{n_1}(0) & & & \\ & J_{n_2}(0) & & \\ & & \ddots & \\ & & & J_{n_l}(0) \end{bmatrix}, \begin{bmatrix} P_{1,n_1-1} \\ P_{1,n_1-2} \\ \vdots \\ P_{1,0} \\ P_{2,n_2-1} \\ P_{2,n_2-1} \\ \vdots \\ P_{2,0} \\ \vdots P_{l,n_l-1} \\ P_{l,n_l-2} \\ \vdots \\ P_{l,0} \end{bmatrix} \right)$$

be a left kernel triple for an $m \times n$ rational matrix function W. Then $\{P_{1,0}, P_{2,0}, \ldots, P_{k,0}\}$ is a basis for $W^{ol}(\infty)$ and, if $\lambda \in \mathbf{C}$, the set of vectors $\{v_1, v_2, \ldots, v_k\}$ such that

$$v_i = \begin{cases} P_{i,n_i} + \lambda P_{i,n_i-1} + \ldots + \lambda^{n_i} P_{i,0}, & \text{if } i \le l \\ P_{i,0}, & \text{if } i > l \end{cases}$$

is a basis for $W^{ol}(\lambda)$.

2.2 Orthogonality in $\mathcal{R}^n$

We describe now the concept of orthogonality in $\mathcal{R}^n$ induced by a standard valuation on $\mathcal{R}$. This concept has been used in [F] (see also [K]). It has been introduced originally by Monna in the study of non-Archimedean normed and, more generally, locally convex spaces (see [M1, M2]).

Section 2.2 is organized as follows. In Subsection 2.2.1 we define orthogonality of subspaces of $\mathcal{R}^n$. In Subsection 2.2.2 we connect orthogonality of algebraic complements Λ and Ω of $\mathcal{R}^n$ with orthogonality of their annihilators Λ° and Ω° in the dual space of $\mathcal{R}^n$ where by the dual space of $\mathcal{R}^n$ we understand the $\mathcal{R}$-vector space of the $\mathcal{R}$-valued linear functionals on $\mathcal{R}^n$. In Subsection 2.2.3 we specialize orthogonality of subspaces of $\mathcal{R}^n$ to orthogonality of functions in $\mathcal{R}^n$ and define orthogonal and orthonormal bases for subspaces of $\mathcal{R}^n$. In Subsection 2.2.4 we connect orthogonality of rational column vector functions $v_1, v_2, ..., v_k$ with the spectral points of a rational matrix function $[v_1 \quad v_2 \quad \dots \quad v_k]$ where by spectral points of a rational matrix function we understand its poles and zeros.

2.2.1 Orthogonal subspaces in $\mathcal{R}^n$

Let λ be a point on the Riemann sphere $\mathbf{C}_\infty$. We define a function $|\cdot|_{z=\lambda}$ from $\mathcal{R}$ into the set of real numbers by putting

$$|r|_{z=\lambda} = \begin{cases} 0, & \text{if } r = 0 \\ e^{-\eta}, & \text{if } r \neq 0 \end{cases}$$

where η is the unique integer such that

$$r(z) = \begin{cases} (z-\lambda)^\eta \, \tilde{r}(z), & \text{if } \lambda \in \mathbf{C} \\ z^{-\eta} \, \tilde{r}(z), & \text{if } \lambda = \infty \end{cases}$$

with $\tilde{r}$ analytic and nonzero at λ. $|\cdot|_{z=\lambda}$ is a real valuation of R. Since $|n|_{z=\lambda} \leq 1$ for every integer n, the valuation $|\cdot|_{z=\lambda}$ is non-Archimedean and the stronger triangle inequality

$$|r_1 + r_2|_{z=\lambda} \leq \max \{|r_1|_{z=\lambda}, |r_2|_{z=\lambda}\}$$

holds for all $r_1, r_2 \in \mathcal{R}$ (see [VdW]).

Let n be a positive integer and let $\lambda \in \mathbf{C}_\infty$. We define a function $\|\cdot\|_{z=\lambda}$ on the product of n copies of $\mathcal{R}$ by putting

$$\|(r_1, r_2, ..., r_n)\|_{z=\lambda} = \max \{|r_1|_{z=\lambda}, |r_2|_{z=\lambda}, ..., |r_n|_{z=\lambda}\}.$$

In this way $\mathcal{R}^n$ becomes a normed vector space over the real valued field $(\mathcal{R}, |\cdot|_{z=\lambda})$. We note that $(\mathcal{R}^n, \|\cdot\|_{z=\lambda})$ is not a Banach space. Indeed, exp (z) is an example of a non-rational function which is in the completion of $(\mathcal{R}, \|\cdot\|_{z=\lambda})$ for any $\lambda \in \mathbf{C}$.

More generally, let Λ be a subspace of $\mathcal{R}^n$ and let $\sigma \subset \mathbf{C}_\infty$. The family of norms $\{\|\cdot\|_{z=\lambda} : \lambda \in \sigma\}$ and valuations $\{|\cdot|_{z=\lambda} : \lambda \in \sigma\}$ endow Λ and $\mathcal{R}$ with topologies such that the addition $+ : \Lambda \times \Lambda \to \Lambda$ and multiplication $\cdot : \mathcal{R} \times \Lambda \to \Lambda$ are continuous. The resulting topological vector space will be denoted by (Λ, σ).

(Λ, σ) is a family of normed vector spaces with one underlying field of scalars, $\mathcal{R}$, and the same set of elements, Λ. Since the topology τ_σ of Λ is generated by a family of seminorms (in fact, norms), it seems natural to call this topology a locally convex one. The difficulty lies in the fact that the norms $\|\cdot\|_{z=\lambda}$ are related to the same underlying field $\mathcal{R}$ with different valuations. Consequently, the definition of a convex set in the theory of non-Archimedean locally convex spaces (see [M2]) does not apply to (Λ, σ) in the nontrivial case when σ contains more than one point.

Following the definition of orthogonality in a non-Archimedean normed space, we shall say that two subspaces Λ and Ω of $\mathcal{R}^n$ are <u>orthogonal</u> at $\lambda \in \mathbf{C}_\infty$ if

$$\|x + y\|_{z=\lambda} = \max\{\|x\|_{z=\lambda}, \|y\|_{z=\lambda}\} \tag{4}$$

for each $x \in \Lambda$, $y \in \Omega$. We shall say that Λ and Ω are orthogonal on $\sigma \subset \mathbf{C}_\infty$ if they are orthogonal at every point of σ.

In the study of orthogonality in $\mathcal{R}^n$ below, we shall need the following lemma.

Lemma 2.5 *Let Λ be a subspace of $\mathcal{R}^n$ and let $\lambda \in \mathbf{C}_\infty$. Then there exists a basis $\{v_1, v_2, ..., v_k\}$ for Λ such that the rational matrix function $[v_1 \quad v_2 \quad \dots \quad v_k]$ has neither a pole nor a zero at λ.*

Proof Choose an algebraic basis $\{w_1, w_2, ..., w_k\}$ for Λ and let

$$W = \begin{bmatrix} w_1 & w_2 & \dots & w_k \end{bmatrix}.$$

Considering, if necessary, $H(z) = W(z^{-1})$ we may assume $\lambda \in \mathbf{C}$. Choose a Smith-McMillan factorization EDF of W and let $v_1, v_2, ..., v_k$ be the first k columns of E. Clearly $\{v_1, v_2, ..., v_k\}$ is a basis for Λ and the rational matrix function $V = \begin{bmatrix} v_1 & v_2 & \dots & v_k \end{bmatrix}$ is analytic at λ. Since

$$E \begin{bmatrix} I \\ 0 \end{bmatrix} I$$

is a Smith-McMillan factorization of V, V does not have a zero at λ.

□

We can now characterize the orthogonality of two subspaces of $\mathcal{R}^n$ at a point $\lambda \in \mathbf{C}_\infty$ in terms of linear algebra. Recall that if Λ is a subspace of $\mathcal{R}^n$ and $\lambda \in \mathbf{C}_\infty$ then $\Lambda(\lambda)$ denotes the subspace of $\mathbf{C}^n$ formed by the values at λ of those functions in Λ which are analytic at λ.

Proposition 2.6 *Let Λ and Ω be two subspaces of $\mathcal{R}^n$ and let $\lambda \in \mathbf{C}_\infty$. Then Λ and Ω are orthogonal at λ if and only if $\Lambda(\lambda) \cap \Omega(\lambda) = (0)$.*

Proof Suppose Λ and Ω are orthogonal at λ and let $\{v_1, v_2, ..., v_k\}$ and $\{w_1, w_2, ..., w_l\}$ be algebraic bases for Λ and Ω respectively. In view of Lemma 2.5, we may assume that $v_1, v_2, ..., v_k, w_1, w_2, ..., w_l$ are analytic at λ and $\{v_1(\lambda), v_2(\lambda), ..., v_k(\lambda)\}$ and $\{w_1(\lambda), w_2(\lambda), ..., w_l(\lambda)\}$ are linearly independent sets. Suppose that

$$\sum_{i=1}^{k} \alpha_i v_i(\lambda) + \sum_{j=1}^{l} \beta_j w_j(\lambda) = 0$$

for some numbers α_i, β_j. Then $\| \sum_{i=1}^{k} \alpha_i v_i + \sum_{j=1}^{l} \beta_j w_j \|_{z=\lambda} < 1$. So, by the orthogonality of Λ and Ω at λ, $\| \sum_{i=1}^{k} \alpha_i v_i \|_{z=\lambda} < 1$ and $\| \sum_{j=1}^{l} \beta_j w_j \|_{z=\lambda} < 1$. Hence

$\sum_{i=1}^{k} \alpha_i v_i(\lambda) = 0$ and $\sum_{j=1}^{l} \beta_j w_j(\lambda) = 0$. Since $\{v_1(\lambda), v_2(\lambda), ..., v_k(\lambda)\}$ and $\{w_1(\lambda), w_2(\lambda), ..., w_l(\lambda)\}$ are linearly independent sets, $\alpha_1 = \alpha_2 = ... = \alpha_k = \beta_1 = \beta_2 = ... = \beta_l = 0$. Thus, $v_1(\lambda), v_2(\lambda), ..., v_k(\lambda), w_1(\lambda), w_2(\lambda), ..., w_l(\lambda)$ are linearly independent and $\Lambda(\lambda) \cap \Omega(\lambda) = (0)$.

Conversely, suppose $\Lambda(\lambda) \cap \Omega(\lambda) = (0)$. Choose a basis $\{v_1, v_2, ..., v_k\}$ for Λ and a basis $\{w_1, w_2, ..., w_l\}$ for Ω such that $v_1, v_2, ..., v_k, w_1, w_2, ..., w_l$ are analytic at λ and $\{v_1(\lambda), v_2(\lambda), ..., v_k(\lambda)\}$ and $\{w_1(\lambda), w_2(\lambda), w_l(\lambda)\}$ are linearly independent sets. Then $v_1(\lambda), v_2(\lambda), ..., v_k(\lambda), w_1(\lambda), w_2(\lambda), ..., w_l(\lambda)$ are linearly independent. Let $r_1, r_2, ..., r_{k+l}$ be scalar rational functions, not all equal to 0. Choose κ such that

$$|r_\kappa|_{z=\lambda} = \max\left\{|r_1|_{z=\lambda}, |r_2|_{z=\lambda}, ..., |r_{k+l}|_{z=\lambda}\right\}.$$

We assume without loss of generality $1 \le \kappa \le k$. The linear independence of $v_1(\lambda)$, $v_2(\lambda), ..., v_k(\lambda), w_1(\lambda), w_2(\lambda), ..., w_l(\lambda)$ implies then the equalities

$$\| \sum_{i=1}^{k} r_i v_i + \sum_{j=1}^{l} r_{k+j} w_j \|_{z=\lambda} = |r_\kappa|_{z=\lambda}$$

and

$$\| \sum_{i=1}^{k} r_i v_i \|_{z=\lambda} = |r_\kappa|_{z=\lambda}.$$

Since $\| \sum_{j=1}^{l} r_{k+j} w_j \|_{z=\lambda} \le |r_\kappa|_{z=\lambda}$,

$$\| \sum_{i=1}^{k} r_i v_i + \sum_{j=1}^{l} r_{k+j} w_j \|_{z=\lambda} = \max\left\{\| \sum_{i=1}^{k} r_i v_i \|_{z=\lambda}, \| \sum_{j=1}^{l} r_{k+j} w_j \|_{z=\lambda}\right\}.$$

Thus, Λ and Ω are orthogonal at λ.

$\square$

It follows from the definition of orthogonality (see equality (4)) that two subspaces of $\mathcal{R}^n$ orthogonal at a single point $\lambda \in \mathbf{C}_\infty$ necessarily have the trivial intersection. Let Λ, Ω and Σ be subspaces of $\mathcal{R}^n$. We say that the subspace Ω is an orthogonal complement of the subspace Λ in (Σ, σ) if Λ and Ω are orthogonal on σ and $\Lambda + \Omega = \Sigma$. The existence of orthogonal complements of a subspace Λ of $(\mathcal{R}^n, \sigma)$ follows from the next proposition.

Proposition 2.7 *Let Λ and Ω be subspaces of $\mathcal{R}^n$ which are orthogonal on a proper subset σ of $\mathbf{C}_\infty$. Then Ω has an extension to an orthogonal complement of Λ in $(\mathcal{R}^n, \sigma)$.*

Proof We identify $\mathcal{R}^n$ with $\mathcal{R}^{n \times 1}$. After applying a suitable Möbius transformation, we may assume $\sigma \subset \mathbf{C}$. Find a basis $\{w_1, w_2, ..., w_k\}$ for $\Lambda + \Omega$ and let EDF be a Smith-McMillan factorization of the rational matrix function $[\, w_1 \quad w_2 \quad \dots \quad w_k \,]$. We show that the subspace V spanned by the last $n - k$ columns of E is orthogonal to $\Lambda + \Omega$ on σ. Then the subspace $\Omega + V$ is an orthogonal complement of Λ in $(\mathcal{R}^n, \sigma)$. In fact, it suffices to show that the spaces spanned by the first k and the last $n - k$ columns of E are orthogonal on σ. But this follows from Proposition 2.6 and the properties of a Smith-McMillan factorization of a rational matrix function.

$\square$

Corollary 2.8 *If σ is a proper subset of $\mathbf{C}_\infty$ then every subspace Λ of $\mathcal{R}^n$ has an orthogonal complement in $(\mathcal{R}^n, \sigma)$.*

We note that unlike in a Hilbert space, an orthogonal complement of Λ in $(\mathcal{R}^n, \sigma)$ is generally not unique. We also note that there exist subspaces of $\mathcal{R}^n$ which are orthogonal on the whole extended complex plane. Indeed, if $S = \{c_1, c_2, ..., c_n\}$ is a basis for $\mathbf{C}^n$ and $S_1 \cup S_2$ is a partition of S, the subspaces of $\mathcal{R}^n$ spanned by the (constant) rational vector functions contained in S_1 and S_2, respectively, are orthogonal on $\mathbf{C}_\infty$. In general, however, as can be seen from the following example, the hypothesis $\sigma \neq \mathbf{C}_\infty$ in Proposition 2.7 and Corollary 2.8 is necessary.

Example 2.9 Let Λ be a subspace of $\mathcal{R}^2$ spanned by $[1 \quad z]^T$ and let Ω be a subspace of $\mathcal{R}^2$ which is orthogonal to Λ on $\mathbf{C}_\infty$. Then $\dim \Omega \leq 1$. Suppose $\dim \Omega = 1$ and let $[p_1 \quad p_2]^T$ be a minimal polynomial basis for Ω. In view of Proposition 2.6, orthogonality of Λ and Ω on $\mathbf{C}$ implies that

$$\det \begin{bmatrix} 1 & p_1(z) \\ z & p_2(z) \end{bmatrix}$$

does not vanish in $\mathbf{C}$. By the fundamental theorem of algebra, $zp_1(z) - p_2(z) = c$ for some constant c. If $p_1 \neq 0$ then

$$\begin{aligned} \| p_1(z)[1 \quad z] - [p_1(z) \quad p_2(z)] \|_{z=\lambda} &= \| [0 \quad c] \|_{z=\infty} = 1 \\ &< e \leq \| p_1(z)[1 \quad z] \|_{z=\infty} \\ &\leq \max \{ \| p_1(z)[1 \quad z] \|_{z=\infty}, \| [p_1 \quad p_2] \|_{z=\infty} \}, \end{aligned}$$

a contradiction. Thus, $p_1 = 0$ and Ω is spanned by $[0 \quad p_2]^T$. If $p_2 \neq 0$,

$$\begin{aligned} \| [1 \quad z] - (z/p_2(z))[0 \quad p_2(z)] \|_{z=\infty} &= 1 \\ &< e = \max \{ \| [1 \quad z] \|_{z=\infty}, \| (z/p_2(z))[0 \quad p_2(z)] \|_{z=\infty} \} \end{aligned}$$

which is again a contradiction. Thus, the only subspace of $\mathcal{R}^2$ orthogonal to Λ on $\mathbf{C}_\infty$ is (0).

2.2.2 $\mathcal{R}^n$ as a subspace of its dual

There is a natural identification of $\mathcal{R}^n$ with its dual space, given by

$$x^*(y) = \sum_{i=1}^{n} x_i(z) y_i(z)$$

for all $x = (x_1, x_2, ..., x_n), y = (y_1, y_2, ..., y_n) \in \mathcal{R}^n$. We shall denote the image of the embedding map $x \to x^*$ by $\mathcal{R}^{n*}$. If $\mathcal{R}^n$ is identified with $\mathcal{R}^{n\times 1}$ (resp. $\mathcal{R}^{1\times n}$),

$\mathcal{R}^{n*} = \mathcal{R}^{1\times n}$ (resp. $\mathcal{R}^{n*} = \mathcal{R}^{n\times 1}$). The members of $\mathcal{R}^{n*}$ are continuous maps from $(\mathcal{R}^n, \sigma)$ to $(\mathcal{R}, \sigma)$.

Proposition 2.10 *Let $\sigma \subset \mathbf{C}_\infty$ and let $x \in \mathcal{R}^n$. Then x^* is continuous as a map $(\mathcal{R}^n, \sigma) \to (\mathcal{R}, \sigma)$.*

Proof It suffices to show that if V, a neighborhood of 0 in $(\mathcal{R}, \sigma)$, is given, then there exists V_n, a neighborhood of 0 in $(\mathcal{R}^n, \sigma)$, such that $x^*(V_n) \subset V$. Let $x = (x_1, x_2, ..., x_n)$. We may assume $x \neq 0$. If V contains the set $\{r \in \mathcal{R} : \ |r|_{s=\lambda_i} < \epsilon \ (i = 1, 2, ..., k)\}$, let $\tilde{\epsilon} = \epsilon / \max\{|x_j|_{s=\lambda_i} : \ 1 \leq i \leq k, \ 1 \leq j \leq n\}$. Then $V_n = \{r \in \mathcal{R}^n : \|r\|_{s=\lambda_i} < \tilde{\epsilon} \ \ (i = 1, 2, ..., k)\}$ is a neighborhood of 0 in $(\mathcal{R}^n, \sigma)$ and $x^*(v_n) \subset V$.

□

Let Λ be an m-dimensional subspace of $\mathcal{R}^n$. By an algebraic argument, the elements of $\mathcal{R}^{n*}$ which annihilate Λ form an $n - m$ dimensional subspace of $\mathcal{R}^{n*}$. Using the polar notation, we shall denote this subspace by Λ°. Similarly, given a subspace Ω of $\mathcal{R}^{n*}$, there exists a unique subspace Ω° of $\mathcal{R}^n$ such that Ω annihilates Ω°. Clearly, $(\Lambda^\circ)^\circ = \Lambda$. It turns out that the map which sends each subspace Λ of $\mathcal{R}^n$ to Λ° preserves orthogonality in the following sense.

Theorem 2.11 *Let Λ, Ω be algebraic complements in $\mathcal{R}^n$ and let $\sigma \subset \mathbf{C}_\infty$. Then Λ and Ω are orthogonal on σ if and only if Λ° and Ω° are orthogonal on σ.*

Proof We identify $\mathcal{R}^n$ with $\mathcal{R}^{n\times 1}$ and show that Λ° and Ω° are orthogonal on σ whenever Λ and Ω are. Choose any $\lambda \in \sigma$. In view of Lemma 2.5 we can find bases $\{v_1, v_2, ..., v_k\}$ and $\{w_1, w_2, ..., w_l\}$ for Λ and Ω such that the rational matrix functions $V = [v_1 \ v_2 \ ... \ v_k]$ and $W = [w_1 \ w_2 \ ... \ w_l]$ have neither a pole nor a zero at λ. Then the diagonal entries in the Smith-McMillan forms of V and of W have neither poles nor zeros at λ. So, as can be seen from Smith-McMillan factorizations of V and W, $V\phi_1, W\phi_2$ are analytic at λ whenever ϕ_1, ϕ_2 are analytic at λ, and $V\phi_1, W\phi_2$ do not vanish at λ whenever ϕ_1, ϕ_2 do not have a zero at λ. Hence if $\phi = \begin{bmatrix} \phi_1^T & \phi_2^T \end{bmatrix}^T \in \mathcal{R}^{n\times 1}$ is analytic and nonzero at λ,

$$\begin{aligned} \| \, [V \quad W] \, \phi \, \|_{z=\lambda} &= \max \{ \|V\phi_1\|_{z=\lambda}, \; \|W\phi_2\|_{z=\lambda} \} \\ &= 1. \end{aligned}$$

In particular the columns of $[V(\lambda) \quad W(\lambda)]$ are linearly independent. It follows that $[V \quad W]$ is a regular rational matrix function that is analytic at λ and does not have a zero at λ. Hence $[V \quad W]^{-1}$ is analytic at λ and does not have a zero at λ. Consequently, $\psi [V \quad W]^{-1}$ is analytic and does not have a zero at λ whenever $\psi \in \mathcal{R}^{1\times m}$ is analytic and nonzero at λ. So $\|\psi\|_{z=\lambda} = 1$ implies $\|\psi [V \quad W]^{-1}\|_{z=\lambda} = 1$. Hence $\|\psi [V \quad W]^{-1}\|_{z=\lambda} = \|\psi\|_{z=\lambda}$ for all $\psi \in \mathcal{R}^{1\times n}$. Now let $\psi_1 \in \mathcal{R}^{1\times n}$ have the last l components 0 and let $\psi_2 \in \mathcal{R}^{1\times n}$ have the first k components 0. Then $\|\psi_1 + \psi_2\|_{z=\lambda} = \max \{\|\psi_1\|_{z=\lambda}, \|\psi_2\|_{z=\lambda}\}$ and

$$\begin{aligned} \|(\psi_1 + \psi_2) [V \quad W]^{-1}\|_{z=\lambda} &= \|\psi_1 + \psi_2\|_{z=\lambda} \\ &= \max \{\|\psi_1\|_{z=\lambda}, \; \|\psi_2\|_{z=\lambda}\} \\ &= \max \{\|\psi_1 [V \quad W]^{-1}\|_{z=\lambda}, \|\psi_2 [V \quad W]^{-1}\|_{z=\lambda}\}. \end{aligned}$$

It follows that the spaces spanned by the first k and last l rows of $[V \quad W]^{-1}$ are orthogonal at λ. Since the former space can be identified with Ω° and the latter space with Λ°, Λ° and Ω° are orthogonal at λ.

□

Stated in other words, Theorem 2.11 says that Λ and Ω are orthogonal complements in $(\mathcal{R}^n, \sigma)$ if and only if Λ° and Ω° are orthogonal complements in $(\mathcal{R}^{n*}, \sigma)$.

2.2.3 Orthogonal and orthonormal bases

An algebraic basis $\{v_1, v_2, ..., v_k\}$ for a subspace Λ of $\mathcal{R}^n$ is said to be an orthogonal basis for (Λ, σ), σ a subset of $\mathbf{C}_\infty$, if v_j $(j = 1, 2, ..., k)$ is orthogonal to span $\{v_i : 1 \leq i \leq k, i \neq j\}$ on σ. An orthogonal basis $\{v_1, v_2, ..., v_k\}$ for (Λ, σ) such that $\|v_j\|_{s=\lambda} = 1$ for every $j \in \{1, 2, ..., k\}$ and each $\lambda \in \sigma$ is called an orthonormal basis for (V, σ).

It follows by induction from the definition of orthogonality that an algebraic basis $\{v_1, v_2, ..., v_k\}$ for Λ is an orthogonal basis for (V, σ) if and only if

$$\|\sum_{i=1}^{k} r_i v_i\|_{s=\lambda} = \max\ \{\|r_i v_i\|_{s=\lambda} :\ \ 1 \leq i \leq k\} \tag{5}$$

for all $\lambda \in \sigma$ and all rational scalar functions $r_1, r_2, ..., r_k$. Elements $v_1, v_2, ..., v_k$ of $\mathcal{R}^n$ such that equality (5) holds for all scalar rational functions $r_1, r_2, ..., r_k$ and all points $\lambda \in \sigma$ are said to be orthogonal on σ. Thus, an algebraic basis $\{v_1, v_2, ..., v_k\}$ for a subspace Λ of $\mathcal{R}^n$ is an orthogonal basis for (Λ, σ) if and only if $v_1, v_2, ..., v_k$ are orthogonal on σ.

As a consequence of characterization (5), orthogonal bases have the following property.

Proposition 2.12 *Let Λ be a subspace of $\mathcal{R}^n$, let $\sigma \subset \mathbf{C}_\infty$, and let $r_1, r_2, ..., r_k$ be nonzero scalar rational functions. Then $\{v_1, v_2, ..., v_k\}$ is an orthogonal basis for (Λ, σ) if and only if $\{r_1 v_1, r_2 v_2, ..., r_k v_k\}$ is an orthogonal basis for (Λ, σ).*

Orthogonal bases can be also characterized more directly.

Proposition 2.13 *The set $\{v_1, v_2, ..., v_k\} \subset \Lambda \subset \mathcal{R}^n$ is an orthogonal basis for (Λ, σ) if and only if*

(i) the functions $v_1, v_2, ..., v_k$ span Λ,

(ii) for every $\lambda \in \sigma$ the leading coefficients at λ of $v_1, v_2, ..., v_k$ are linearly independent.

Proof Suppose $\{v_1, v_2, ..., v_k\}$ is an orthogonal basis for (Λ, σ). Then (i) holds and all v_i's are distinct from 0. If (ii) does not hold, there exists $\lambda \in \sigma$ and scalar rational functions $r_1, r_2, ..., r_k$ such that $r_1v_1, r_2v_2, ..., r_kv_k$ are analytic at λ, some r_iv_i are nonzero at λ and the constant term in the Laurent expansion of $\sum_{i=1}^{k} r_iv_i$ is zero. Then

$$\|\sum_{i=1}^{k} r_iv_i\|_{z=\lambda} < 1 = \max\{\|r_iv_i\|_{z=\lambda} : \quad i = 1, 2, ..., k\},$$

a contradiction.

Suppose now that $\{v_1, v_2, ..., v_k\} \subset \Lambda$ is such that (i) and (ii) hold. Then $\{v_1, v_2, ..., v_k\}$ is an algebraic basis for Λ. Let scalar rational functions $r_1, r_2, ..., r_k$ be arbitrary. Fix $\lambda \in \sigma$ and choose $j \in \{1, 2, ..., k\}$ such that

$$\|r_jv_j\|_{z=\lambda} = \max\{\|r_iv_i\|_{z=\lambda} : \quad 1 \le i \le k\}.$$

Then (ii) implies that

$$\|\sum_{i=1}^{k} r_iv_i\|_{z=\lambda} = \|r_jv_j\|_{z=\lambda}.$$

So (5) holds and it follows that $\{v_1, v_2, ..., v_k\}$ is an orthogonal basis for (Λ, σ).

$\square$

It follows from the results in [F] that a minimal polynomial basis for a subspace Λ of $\mathcal{R}^n$ is an orthogonal basis for $(\Lambda, \mathrm{C}_\infty)$. We state this in the next theorem.

Theorem 2.14 *If Λ is a subspace of $\mathcal{R}^n$, $(\Lambda, \mathrm{C}_\infty)$ has an orthogonal basis.*

In particular, $(\mathcal{R}^n, \mathrm{C}_\infty)$ has an orthogonal basis. In fact it follows from Proposition 2.13 that if $\{c_1, c_2, ..., c_n\}$ is a basis for C^n (over the base field C) and $r_1, r_2, ..., r_n$ are nonzero scalar rational functions, then $\{r_1(z)c_1, r_2(z)c_2, ..., r_n(z)c_n\}$ is an orthogonal basis for $(\mathcal{R}^n, \mathrm{C}_\infty)$.

In view of Proposition 2.12, Theorem 2.14 has the following corollary.

Corollary 2.15 *Let Λ be a subspace of $\mathcal{R}^n$ and let σ be a proper subset of $\mathbf{C}_\infty$. Then (Λ, σ) has an orthonormal basis.*

2.2.4 Orthogonality and spectral points

An orthonormal basis $\{v_1, v_2, ..., v_k\}$ for (Λ, σ) can be characterized in terms of poles and zeros of the rational matrix function $[\,v_1 \quad v_2 \quad \ldots \quad v_k\,]$ as follows.

Theorem 2.16 *Let $\{v_1, v_2, ..., v_k\}$ be a basis for a subspace Λ of $\mathcal{R}^{m\times 1}$ and let $\sigma \subset \mathbf{C}_\infty$. Then $\{v_1, v_2, ..., v_k\}$ is an orthonormal basis for (Λ, σ) if and only if the rational matrix function $[v_1\ v_2\ ...\ v_k]$ has no poles nor zeros in σ.*

Proof We may assume $\sigma = \{\lambda\}$ with $\lambda \in \mathbf{C}_\infty$. We note that since $\{v_1, v_2, ..., v_n\}$ are linearly independent over $\mathcal{R}$, the rational matrix function $W = [v_1\ v_2\ ...\ v_k] \in \mathcal{R}^{m\times n}$ is left invertible.

Suppose first that $\{v_1, v_2, ..., v_k\}$ is an orthonormal basis for $(\Lambda, \{\lambda\})$. Then W is analytic at λ. Also for any $\phi = [\phi_1\ \phi_2\ ...\ \phi_k]^T \in \mathcal{R}^{k\times 1}$ with $\|\phi\|_{z=\lambda} = 1$

$$\begin{aligned} \|W\phi\|_{z=\lambda} &= \max\left\{\|\phi_i v_i\|_{z=\lambda} : \quad 1 \le i \le k\right\} \\ &= \max\left\{|\phi_i|_{z=\lambda} : \quad 1 \le i \le k\right\} \\ &= \|\phi\|_{z=\lambda} \\ &= 1. \end{aligned}$$

Hence no diagonal entry in the Smith-McMillan form of W (the Smith-McMillan form of $W(z^{-1})$ if $\lambda = \infty$) has a zero at λ (at 0 if $\lambda = \infty$).

Suppose now that W has neither a zero nor a pole at λ. If $\|v_i\|_{z=\lambda} > 1$ for some i, v_i and hence also W has a pole at λ. If $\|v_i\|_{z=\lambda} < 1$ for some i, there is a nonzero constant function $\phi \in \mathcal{R}^{k\times 1}$ such that $W\phi = v_i$ has a zero at λ. Then some diagonal entry in the middle factor in a Smith-McMillan factorization EDF of W

(resp. $W(z^{-1})$ if $\lambda = \infty$) has a zero at λ (resp. at 0 if $\lambda = \infty$), a contradiction. It follows that $\|v_i\|_{z=\lambda} = 1$ for all $i = 1, 2, \ldots, k$. Suppose there are scalar rational functions $r_1, r_2, \ldots, r_k$ such that $\|\sum_{i=1}^{k} r_i v_i\|_{z=\lambda} < \max\{\|r_i v_i\|_{z=\lambda} : 1 \le i \le k\}$. Then not all $r_1, r_2, \ldots, r_k$ are zero and we may assume $\|(r_1, r_2, \ldots, r_k)\|_{z=\lambda} = 1$. Then $\max\{\|r_i v_i\|_{z=\lambda} : 1 \le i \le k\} = \max\{|r_i|_{z=\lambda} : 1 \le i \le k\} = 1$ and Wr has a zero at λ where $r = [r_1\ r_2\ \ldots\ r_k]^T$ is a function analytic and nonzero at λ. Consequently, some diagonal entry in the Smith-McMillan form of W (resp. of $W(z^{-1})$ if $\lambda = \infty$) has a zero at λ (resp. at 0 if $\lambda = \infty$), a contradiction. Thus, $\{v_1, v_2, \ldots, v_k\}$ is an orthonormal basis for $(\Lambda, \{\lambda\})$.

□

It follows from Theorem 2.16 that if W is a left invertible rational matrix function with no poles nor zeros in $\sigma \subset \mathbf{C}_\infty$, then the columns of W form an orthonormal basis for (Λ, σ) where Λ is the column space of W. Thus, if the columns of W fail to be orthogonal at λ, λ is a pole or a zero of W. The next proposition shows that the study of the right pole structure at λ of a rational matrix function W can be simplified if the span of some columns of W is orthogonal at λ to the span of the other columns of W. We note that in Proposition 2.17 we do not assume linear independence of the columns of the involved rational matrix functions.

Proposition 2.17 *Let W_1 and W_2 be $m \times n_1$ and $m \times n_2$ rational matrix functions with the canonical sets of right pole functions at $\lambda \in \mathbf{C}_\infty$ Ψ_1 and Ψ_2, respectively, and let $W = [\,W_1 \quad W_2\,]$. If the columns of W_1 and W_2 are contained in subspaces of $\mathcal{R}^m$ Λ_1 and Λ_2, respectively, and Λ_1, Λ_2 are orthogonal at λ, then each right pole function ψ for W_1 (resp. W_2) at λ of order k is a right pole function for W at λ of order k and $\Psi = \Psi_1 \cup \Psi_2$ is a canonical set of right pole functions for W at λ.*

Proof The first assertion follows from the definition of a right pole function and Proposition 2.6. Also by Proposition 2.6 the leading coefficients in the Taylor expansions at λ of functions in Ψ_1 and Ψ_2 are linearly independent. Let ψ be a right pole function for W at λ of order k. Then $\psi(z) = \psi_1(z) + \psi_2(z)$ where

$$\psi_i(z) = \begin{cases} (z-\lambda)^k W_i(z)\phi_i(z), & \text{if } \lambda \in \mathbf{C} \\ \\ z^{-k} W_i(z)\phi_i(z), & \text{if } \lambda = \infty. \end{cases}$$

for some rational vector function ϕ_i which is analytic at λ ($i = 1, 2$). So $\psi(\lambda)$ is contained in the span of the values at λ of the functions in Ψ of order at least equal to k. Thus, Ψ is a canonical set of right pole functions for W at λ.

□

Since the orders of pole functions in a canonical set of right pole functions for a rational matrix function W at $\lambda \in \mathbf{C}_\infty$ coincide with the partial multiplicities of the pole of W at λ, Proposition 2.17 has the following corollary.

Corollary 2.18 *Let W_1 and W_2 be $m \times n_1$ and $m \times n_2$ rational matrix functions, let $\lambda \in \mathbf{C}_\infty$, and let $k_1, k_2, ..., k_{m_1}$ and $l_1, l_2, ..., l_{m_2}$ be partial multiplicities of the pole at λ of W_1 and W_2. If the column spaces of W_1 and W_2 are orthogonal at λ, then $k_1, k_2, ..., k_{m_1}, l_1, l_2, ..., l_{m_2}$ are the partial multiplicities at λ of the pole of the rational matrix function $[\,W_1 \quad W_2\,]$.*

Proposition 2.17 shows that if a rational matrix function W can be split into a block matrix function $W = [\,W_1 \quad W_2\,]$ such that the column space of W_1 is orthogonal to the column space of W_2 at $\lambda \in \mathbf{C}_\infty$, then the right pole structure of W at λ can be investigated by considering the right pole structure at λ of W_1 and W_2 separately. Proposition 2.23 below is an analogue of this observation referring to the left zero structure of W. We shall use in this section the following immediate corollary of Proposition 2.23.

Corollary 2.19 *Let W_1 and W_2 be $m \times n_1$ and $m \times n_2$ rational matrix functions, let $\lambda \in \mathbf{C}_\infty$, and let $k_1, k_2, ..., k_{n_1}$ and $l_1, l_2, ..., l_{n_2}$ be partial multiplicities of the zero at λ of W_1 and W_2. If the column spaces of W_1 and W_2 are orthogonal at λ, then $k_1, k_2, ..., k_{n_1}, l_1, l_2, ..., l_{n_2}$ are the partial multiplicities of the zero at λ of the rational matrix function $[\, W_1 \quad W_2 \,]$.*

In particular, if all columns of W are orthogonal on $\sigma \subset \mathbf{C}_\infty$, Propositions 2.17 and 2.19 specialize as follows (cf. Result 2 in [VK] and Ex. 6.5-20 in [K]).

Proposition 2.20 *Let $\sigma \subset \mathbf{C}_\infty$ and let W be a left invertible rational matrix function whose columns are orthogonal on σ. Choose $\lambda \in \sigma$ and let $r_1, r_2, ..., r_n \in \mathcal{R}^n$ be such that*

$$W(z) = \tilde{W}(z) \begin{bmatrix} r_1(z) & & & \\ & r_2(z) & & \\ & & \ddots & \\ & & & r_n(z) \end{bmatrix} \qquad (6)$$
$$\equiv \tilde{W}(z)\tilde{D}(z)$$

and $\tilde{W}$ has columns that are analytic and nonzero at λ. Suppose $r_{i_1}, r_{i_2}, ..., r_{i_k}$ vanish at λ and $r_{j_1}, r_{j_2}, ..., r_{j_l}$ have a pole at λ. Then

(i) the partial multiplicities of the zero of W at λ coincide with the orders of zeros at λ of $r_{i_1}, r_{i_2}, ..., r_{i_k}$,

(ii) the partial multiplicities of the pole of W at λ coincide with the orders of poles at λ of $r_{j_1}, r_{j_2}, ..., r_{j_l}$,

(iii) the columns $j_1, j_2, ..., j_l$ of $\tilde{W}$ form a canonical set of right pole functions for W at λ.

2.3 Discrete left zero structure

Let W be an $m \times n$ rational matrix function and let $\lambda \in \mathbf{C}_\infty$. We say that a function $\phi \in \mathcal{R}^{1\times m}$ is a <u>left null function</u> for W at λ of <u>order</u> k, k a positive integer, if

(i) ϕ is analytic and nonzero at λ,

(ii) ϕW has a zero at λ of order k,

(iii) ϕ is orthogonal to W^{ol} at λ.

In view of Proposition 2.6 conditions (i)-(iii) above are equivalent to (i), (ii) and

(iii') $\phi(\lambda) \notin W^{ol}(\lambda)$.

A set of left null functions $\{\phi_1, \phi_2, ..., \phi_\eta\}$ for an $m \times n$ rational matrix function W at $\lambda \in \mathbf{C}_\infty$ of orders $k_1, k_2, ..., k_\eta$, respectively, is called a <u>canonical set</u> of <u>left null functions</u> for W at λ if

(i) $\phi_1(\lambda), \phi_2(\lambda), ..., \phi_\eta(\lambda)$ are linearly independent,

(ii) $\phi_1, \phi_2, ..., \phi_\eta$ are contained in an orthogonal complement of W^{ol} in $(\mathcal{R}^{1\times m}, \{\lambda\})$,

(iii) $\sum_{i=1}^{\eta} k_i$ is maximal subject to conditions (i) and (ii).

We note that if $\tau = (D_\kappa, A_\kappa, B_\kappa)$ is a left kernel triple for W, then in view of Propositions 2.6 and 2.7 the condition (ii) above is equivalent to the linear independence of rows of the matrix

$$\begin{bmatrix} \pi(\lambda) \\ \phi_1(\lambda) \\ \phi_2(\lambda) \\ \vdots \\ \phi_\eta(\lambda) \end{bmatrix}$$

where $\pi(\lambda)$ is a basis for $W^{ol}(\lambda)$ obtained from τ.

We show first that the canonical sets of left null functions for $W \in \mathcal{R}^{m\times n}$ at $\lambda \in \mathbf{C}_\infty$ can be projected onto orthogonal complements of W^{ol} in $(\mathcal{R}^{1\times m}, \{\lambda\})$.

Proposition 2.20 *Let $\{\phi_1, \phi_2, ..., \phi_\eta\}$ be a canonical set of left null functions for an $m \times n$ rational matrix function W at $\lambda \in \mathbf{C}_\infty$ of orders $k_1, k_2, ..., k_\eta$ respectively and let Λ be an orthogonal complement of W^{ol} in $(\mathcal{R}^{1\times m}, \{\lambda\})$. Let $\tilde{\phi}_1, \tilde{\phi}_2, ..., \tilde{\phi}_\eta$ be the projections of $\phi_1, \phi_2, ..., \phi_\eta$ onto Λ along W^{ol}. Then $\{\tilde{\phi}_1, \tilde{\phi}_2, ..., \tilde{\phi}_\eta\}$ is a canonical* *set of left null functions for W at λ of orders $k_1, k_2, ..., k_\eta$.*

Proof Choose orthonormal bases $\{v_1, v_2, ..., v_k\}$ and $\{w_1, w_2, ..., w_l\}$ for $(\Lambda, \{\lambda\})$ and $(W^{ol}, \{\lambda\})$. Then the rational matrix function

$$Q = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \\ w_1 \\ w_2 \\ \vdots \\ w_l \end{bmatrix}$$

is regular and analytic at λ. By Proposition 2.6, Q does not have a zero at λ and hence Q^{-1} is analytic at λ. Consequently, the functions $\psi_1 = \phi_1 Q^{-1}, \psi_2 = \phi_2 Q^{-1}, ..., \psi_\eta = \phi_\eta Q^{-1}$ are analytic at λ. If $\psi_i = [\, \psi_{i1} \quad \psi_{i2} \quad \ldots \quad \psi_{im} \,]$ $(\psi_{ij} \in \mathcal{R})$, let $\tilde{\phi}_i = \sum_{j=1}^{k} \psi_{ij} v_j$ $(i = 1, 2, ..., \eta)$. Then $\tilde{\phi}_1, \tilde{\phi}_1, ..., \tilde{\phi}_\eta$ are the projections of $\phi_1, \phi_2, ..., \phi_\eta$ onto Λ along W^{ol} and $\tilde{\phi}_1, \tilde{\phi}_2, ..., \tilde{\phi}_\eta$ are analytic at λ. Since span$\{\tilde{\phi}_1(\lambda), \tilde{\phi}_2(\lambda), ..., \tilde{\phi}_\eta(\lambda)\}$ = span$\{\phi_1(\lambda), \phi_2(\lambda), ..., \phi_\eta(\lambda)\}$ modulo $W^{ol}(\lambda)$, the vectors $\tilde{\phi}_1(\lambda), \tilde{\phi}_2(\lambda), ..., \tilde{\phi}_\eta(\lambda)$ are linearly independent. Since $\tilde{\phi}_i W = \phi_i W$ $(i = 1, 2, ..., \eta)$, the orders of $\tilde{\phi}_i$ and ϕ_i as left null functions for W at λ are equal. Finally, suppose that there exists a left null function $\tilde{\phi}_{\eta+1} \in \Lambda$ for W at λ such that $\tilde{\phi}_{\eta+1} \notin$ span $\{\tilde{\phi}_1(\lambda), \tilde{\phi}_2(\lambda), ..., \tilde{\phi}_\eta(\lambda)\}$ or such that $\tilde{\phi}_{\eta+1}$ should replace some function ϕ_i in the canonical set of left null functions $\{\phi_1, \phi_2, ..., \phi_\eta\}$ because of order considerations. Then, reversing the argument above, we can find a left null function $\phi_{\eta+1}$ for W at λ such that $\phi_1, \phi_2, ..., \phi_{\eta+1}$ are contained in a subspace orthogonal to W^{ol} at λ and either $\phi_{\eta+1}(\lambda) \notin$ span $\{\phi_1(\lambda), \phi_2(\lambda), ..., \phi_\eta(\lambda)\}$ or $\phi_{\eta+1}$ should replace some functions in $\{\phi_1, \phi_2, ..., \phi_\eta\}$ because of order considerations. Since $\{\phi_1, \phi_2, ..., \phi_\eta\}$ is a canonical set of left null functions for W at λ, this is a contradiction. It follows that $\{\tilde{\phi}_1, \tilde{\phi}_2, ..., \tilde{\phi}_\eta\}$ is a canonical set of left null functions for W at λ and the proof is complete.

$\square$

Proposition 2.22 *Let $\lambda \in \mathbf{C}_\infty$ be a zero of an $m \times n$ rational matrix function W. Then the orders of the functions in any canonical set of left null functions for W at λ are equal to the partial multiplicities of the zero of W at λ.*

Proof We assume without loss of generality that $\lambda \in \mathbf{C}$. Let EDF be a Smith-McMillan factorization of W and suppose that the first k diagonal entries of D are nonzero and all the other entries of D are zero. Then W^{ol} is spanned by the last $m-k$ rows of E^{-1}. By Proposition 2.6 the space Λ spanned by the first k rows of E^{-1} is orthogonal to W^{ol} at λ. Clearly, the rows of E^{-1} corresponding to the nonzero entries of D which vanish at λ are the left null functions for W at λ of orders equal to the partial zero multipicities of W at λ. Hence, by Proposition 2.20 the orders of functions in any canonical set of left null functions for W at λ are equal to the partial multiplicities of the zero of W at λ.

$\square$

Proposition 2.20 implies also the following analogue of Proposition 2.17.

Proposition 2.23 *Let W_1 and W_2 be $m \times n_1$ and $m \times n_2$ rational matrix functions, let $\lambda \in \mathbf{C}_\infty$, and suppose that the column space of W_1 is orthogonal at λ to the column space of W_2. Then there exist canonical sets Φ_1 and Φ_2 of left null functions at λ for W_1 and W_2, respectively, such that every left null function $\phi \in \Phi_1$ for W_1 (resp. $\phi \in \Phi_2$ for W_2) at λ of order k is a left null function at λ of order k for the rational matrix function $W = [W_1 \quad W_2]$, and $\Phi = \Phi_1 \cup \Phi_2$ is a canonical set of left null functions at λ for W.*

Proof Let Λ_i be the column space of W_i $(i = 1, 2)$ and let Λ_3 be an orthogonal complement of $\Lambda_1 + \Lambda_2$ in $\mathcal{R}^m$. Let

$$\Omega_1 = (\Lambda_2 \oplus \Lambda_3)^\circ,$$

$$\Omega_2 = (\Lambda_1 \oplus \Lambda_3)^\circ,$$

$$\Omega_3 = (\Lambda_1 \oplus \Lambda_2)^\circ.$$

Then, by Theorem 2.11, Ω_i is orthogonal to Ω_j at λ whenever $i \neq j$ and $\Omega_1 \oplus \Omega_2 \oplus \Omega_3 = \mathcal{R}^{1\times m}$. Also,

$$W_1^{ol} = \Omega_2 \oplus \Omega_3,$$

$$W_2^{ol} = \Omega_1 \oplus \Omega_3,$$

$$W^{ol} = \Omega_3.$$

Choose a canonical set $\tilde{\Phi}_i$ of left null functions at λ for W_i and project functions in $\tilde{\Phi}_i$ onto Ω_i along W_i^{ol} to get Φ_i $(i = 1,2)$. Let $\Phi = \Phi_1 \cup \Phi_2$. If $\phi \in \Phi_1$ is a left null function for W_1 at λ of order k, then ϕW_1 has a zero at λ of order k and ϕW_2 vanishes identically. Since Ω_1 is orthogonal to W^{ol} at λ, ϕ is a left null function for W at λ of order k. Similarly, every left null function ϕ for W_2 at λ of order k is a left null function for W at λ of order k. Let ϕ be a left null function at λ for W of order k. We may assume $\phi \in \Omega_1 \oplus \Omega_2$. Let $\phi = \phi_{\Omega_1} + \phi_{\Omega_2}$ with $\phi_{\Omega_1} \in \Omega_1$. Then

$$(\phi_{\Omega_1} + \phi_{\Omega_2})[\,W_1 \quad W_2\,] = [\,(\phi_{\Omega_1} + \phi_{\Omega_2})W_1 \quad (\phi_{\Omega_1} + \phi_{\Omega_2})W_2\,]$$

$$= [\,\phi_{\Omega_1} W_1 \quad \phi_{\Omega_2} W_2\,]$$

vanishes to order k at λ. So each of $\phi_{\Omega_1} W_1$ and $\phi_{\Omega_2} W_2$ vanish to order at least k at λ and $\phi(\lambda)$ is contained in the span of values at λ of null functions in Φ of order at least k. Since, by Proposition 2.6, the values at λ of functions in Φ are linearly independent, Φ is a canonical set of left null functions for W at λ.

□

A canonical set of left null functions for an $m \times n$ rational matrix function W at λ can be found similarly as in the regular case (see [BGR1]) with the additional constraint that the functions have to be contained in a subspace Λ orthogonal to W^{ol}

at λ. That is, choose an orthogonal complement Λ of W^{ol} in $(\mathcal{R}^{1\times m}, \{\lambda\})$. Find a left null function $\phi_l \in \Lambda$ for W of maximal possible order. Inductively, given left null functions $\phi_1, \phi_2, ..., \phi_\gamma \in \Lambda$ for W at λ, find a left null function $\phi_{\gamma+1}$ for W at λ of maximal possible order such that $\phi_{\gamma+1} \in \Lambda$ and $\phi_{\gamma+1}(\lambda) \notin$ span $\{\phi_1(\lambda), \phi_2(\lambda), ..., \phi_\gamma(\lambda)\}$. Continue until the span of values at λ of left null functions for W at λ which are contained in Λ has been exhausted.

Let $\lambda \in \mathbf{C}_\infty$ be a zero of an $m \times n$ rational matrix function W and let $\{\phi_1, \phi_2, ..., \phi_\eta\}$ be a canonical set of left null functions for W at λ of orders $k_1, k_2, ..., k_\eta$, respectively. Let $\phi_{i,j} \in \mathbf{C}^{1\times n}$ be such that

$$\phi(z) = \begin{cases} \displaystyle\sum_{j=0}^{\infty} (z-\lambda)^j \phi_{i,j}, & \text{if } \lambda \in \mathbf{C} \\ \displaystyle\sum_{j=0}^{\infty} z^{-j} \phi_{i,j}, & \text{if } \lambda = \infty. \end{cases}$$

Any ordered pair of matrices (A, B), where

$$A = \begin{cases} S \begin{bmatrix} J_{k_1}(\lambda) & & & \\ & J_{k_2}(\lambda) & & \\ & & \ddots & \\ & & & J_{k_\eta}(\lambda) \end{bmatrix} S^{-1}, & \text{if } \lambda \in \mathbf{C} \\ S \begin{bmatrix} J_{k_1}(0) & & & \\ & J_{k_2}(0) & & \\ & & \ddots & \\ & & & J_{k_\eta}(0) \end{bmatrix} S^{-1}, & \text{if } \lambda = \infty \end{cases} \tag{7}$$

and

$$B = S \begin{bmatrix} \phi_{1,k_1-1} \\ \phi_{1,k_1-2} \\ \vdots \\ \phi_{1,0} \\ \phi_{2,k_2-1} \\ \phi_{2,k_2-2} \\ \vdots \\ \phi_{2,0} \\ \vdots \\ \phi_{\eta,k_\eta-1} \\ \phi_{\eta,k_\eta-2} \\ \vdots \\ \phi_{\eta,0} \end{bmatrix}$$

for some invertible matrix S of an appropriate size, is called a <u>left null pair</u> for W at λ.

Let $\sigma \subset \mathbf{C}$ and let $\lambda_1, \lambda_2, ..., \lambda_r$ be the zeros of a rational matrix function W in σ. Let (A_i, B_i) be a left null pair for W at λ_i $(i = 1, 2, ..., r)$. Any ordered pair of matrices (A, B), where

$$A = S \begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_p \end{bmatrix} S^{-1}, \qquad B = S \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_p \end{bmatrix}$$

for some invertible matrix S of an appropriate size, is called a <u>left null pair</u> for W over σ (or a σ-left null pair for W).

We note that since the values at $\lambda \in \mathbf{C}$ of functions in any canonical set of left null functions for $W \in \mathcal{R}^{m \times n}$ at λ are linearly independent, a left null pair (A, B) for W over a $\sigma \subset \mathbf{C}$, or at infinity, is controllable (see Theorem 3.4 in [BGR1]), i.e. the matrix $[\,B \quad AB \quad \ldots \quad A_l B\,]$ has full column rank for sufficiently large integers l.

We shall need later the following property of left null pairs.

Lemma 2.24 *Let $\sigma \subset \mathbf{C}$, let (A, B) be a left null pair for a rational matrix function W over σ, and let $\{\lambda_1, \lambda_2, ..., \lambda_s\}$ be a finite set of points in $\mathbf{C}_\infty$. Suppose that the* *largest geometric multiplicity of a zero of W in σ equals κ. Then there exists a subspace Λ orthogonal to W^{ol} on $\{\lambda_1, \lambda_2, ..., \lambda_s\}$ and functions $\phi_1, \phi_2, ..., \phi_\kappa \in \Lambda$ such that the pair (A, B) is left-similar to the pair constructed from the Taylor coefficients of $\phi_1, \phi_2, ..., \phi_\kappa$ at the eigenvalues of A.*

Proof After augmenting the set $\{\lambda_1, \lambda_2, ..., \lambda_s\}$ if necessary, we may assume that $s > r$, the eigenvalues of A are $\{\lambda_1, \lambda_2, ..., \lambda_r\}$, and $\lambda_s = \infty$. Let $\{\psi_{i,1}, \psi_{i,2}, ..., \psi_{i,\eta_i}\}$ $(i = 1, 2, ..., r)$ be a canonical set of left null functions for W at λ_i such that the pair (A, B) is left-similar to the pair constructed from the coefficients in the Taylor expansions of $\psi_{i,j}$ at λ_i. We assume $\kappa = \eta_1 \geq \eta_2 \geq ... \geq \eta_r$. Let $k_{i,j}$ be the order of $\psi_{i,j}$ as a left null function for W at λ_i $(i = 1, 2, ..., r,\ j = 1, 2, ..., \eta_i)$, and let

$$\begin{aligned}\psi_j(z) = {} & (z - \lambda_2)^{k_{2,j}}(z - \lambda_3)^{k_{3,j}} ... (z - \lambda_{\gamma_j})^{k_{\gamma_j,j}}\psi_{1,j} + \\ & + (z - \lambda_1)^{k_{1,j}}(z - \lambda_3)^{k_{3,j}} ... (z - \lambda_{\gamma_j})^{k_{\gamma_j,j}}\psi_{2,j} + ... + \\ & + (z - \lambda_1)^{k_{1,j}}(z - \lambda_2)^{k_{2,j}} ... (z - \lambda_{\gamma_j - 1})^{k_{\gamma_j-1,j}}\psi_{\gamma_j,j}\end{aligned} \tag{8}$$

where γ_j is the largest integer such that $\eta_{\gamma_j} \geq j$ $(j = 1, 2, ..., \kappa)$. Then ψ_j is a left null function for W at the points $\lambda_1, \lambda_2, ..., \lambda_{\gamma_j}$ of orders $k_{1,j}, k_{2,j}, ..., k_{\gamma_j,j}$ respectively.

We claim that the pair (A, B) is left-similar to the pair constructed from the canonical sets $\{\psi_1, \psi_2, ..., \psi_{\eta_1}\}$, $\{\psi_1, \psi_2, ..., \psi_{\eta_2}\}$, ..., $\{\psi_1, \psi_2, ..., \psi_{\eta_r}\}$ of left null functions for W at $\lambda_1, \lambda_2, ..., \lambda_r$. To prove the claim it suffices to show that if $\psi_{i,j}^\gamma$ and ψ_j^γ are the γ'th coefficients in the Taylor expansions of $\psi_{i,j}$ and ψ_j at λ_i $(1 \leq i \leq r$, $1 \leq j \leq \eta_i)$, then the pairs of matrices

$$\left(J_{k_{i,j}}(\lambda_i), \quad \begin{bmatrix} \psi_{i,j}^{k_{i,j}-1} \\ \psi_{i,j}^{k_{i,j}-2} \\ \vdots \\ \psi_{i,j}^{0} \end{bmatrix} \right) \quad \text{and} \quad \left(J_{k_{i,j}}(\lambda_i), \quad \begin{bmatrix} \psi_{j}^{k_{i,j}-1} \\ \psi_{j}^{k_{i,j}-2} \\ \vdots \\ \psi_{j}^{0} \end{bmatrix} \right)$$

are left-similar. Now it follows from (8) that

$$\psi_j(z) = p_{i,j}(z)\psi_{i,j}(z) + (z - \lambda_j)^{k_{i,j}} q_{i,j}(z)$$

where $p_{i,j}(z)$ is a scalar polynomial with $p_{i,j}(\lambda_i) \neq 0$ and where $q_{i,j}(z)$ is a vector function analytic at λ_i. Hence for $\gamma \in \{0,1,...,k_{i,j}-1\}$ we have $\psi_j^\gamma = \sum_{\mu+\nu=\gamma} a_\mu \psi_{i,j}^\nu$ where $a_0, a_1, ..., a_{k_{i,j}-1}$ are numbers such that $p_{i,j}(z) = \sum_{\mu=0}^{L} a_\mu(\lambda-\lambda_i)^\mu$ and $a_0 \neq 0$. Hence

$$\begin{bmatrix} \psi_j^{k_{i,j}-1} \\ \psi_j^{k_{i,j}-2} \\ \vdots \\ \psi_j^0 \end{bmatrix} = \begin{bmatrix} a_0 & a_1 & \dots & a_{k_{i,j}-1} \\ & a_0 & \dots & a_{k_{i,j}-2} \\ & & \ddots & \vdots \\ & & & a_0 \end{bmatrix} \begin{bmatrix} \psi_{i,j}^{k_{i,j}-1} \\ \psi_{i,j}^{k_{i,j}-2} \\ \vdots \\ \psi_{i,j}^0 \end{bmatrix}$$

$$\equiv S \begin{bmatrix} \psi_{i,j}^{k_{i,j}-1} \\ \psi_{i,j}^{k_{i,j}-2} \\ \vdots \\ \psi_{i,j}^0 \end{bmatrix}.$$

Since the inverse of S is the upper triangular matrix

$$\begin{bmatrix} b_0 & b_1 & \dots & b_{k_{i,j}-1} \\ & b_0 & \dots & a_{k_{i,j}-2} \\ & & \ddots & \vdots \\ & & & b_0 \end{bmatrix}$$

such that

$$\sum_{\mu+\nu=\gamma,\ \mu,\nu\geq 0} a_\mu b_\nu = \begin{cases} 1, & \text{if } \gamma = 0 \\ 0, & \text{if } \gamma = 1,2,...,k_{i,j}-1, \end{cases}$$

we have

$$SJ_{k_{i,j}}(\lambda_i)S^{-1} = S\left(\lambda_i + \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ & 0 & 1 & \dots & 0 \\ & & & \ddots & \vdots \\ & & & & 1 \\ & & & & 0 \end{bmatrix}\right)S^{-1}$$

$$= \lambda_i + \begin{bmatrix} a_0 & a_1 & a_2 & \dots & a_{k_{i,j}-1} \\ & a_0 & a_1 & \dots & a_{k_{i,j}-2} \\ & & & \ddots & \vdots \\ & & & & a_1 \\ & & & & a_0 \end{bmatrix} \begin{bmatrix} 0 & b_0 & b_1 & \dots & b_{k_{i,j}-2} \\ & 0 & b_0 & \dots & \vdots \\ & & & \ddots & \\ & & & & b_0 \\ & & & & 0 \end{bmatrix}$$

$$= \lambda_i + \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ & 0 & 1 & \dots & 0 \\ & & & \ddots & \vdots \\ & & & & 1 \\ & & & & 0 \end{bmatrix} = J_{k_{i,j}}(\lambda)$$

and the claim is established.

We now describe the *orthogonalization process* which transforms $\psi_1, \psi_2, ..., \psi_\kappa$ into functions $\phi_1, \phi_2, ..., \phi_\kappa$ with the required properties (cf. Proposition 2.6). We begin with ψ_1. Choose constant vectors $c_{1,r+1}, c_{1,r+2}, c_{1,s}$ such that $c_{1,j} \notin W^{ol}(\lambda_j)$ $(j = r+1, r+2, ..., s)$ and suppose that the integers $\mu_{1,j}$ are such that the rational vector function

$$\begin{cases} (z-\lambda_j)^{-\mu_{1,j}}\psi_1(z), & \text{if } j = r+1, r+2, ..., s-1 \\ z^{\mu_{1,j}}\psi_1(z), & \text{if } j = s \end{cases}$$

is analytic and nonzero at λ_j $(j = r+1, r+2, ..., s)$. We put

$$\begin{aligned} \phi_1(z) =& \psi_1(z) + \prod_{i=1}^{r}(z-\lambda_i)^{k_{i,1}+1} \cdot \\ & \cdot \Bigg(\sum_{j=r+1}^{s-1} (z-\lambda_j)^{\mu_{1,j}} \Big(c_{i,j} - [\psi_1(z)]_{\lambda_j} \Big/ \prod_{i=1}^{r} (\lambda_j - \lambda_i)^{k_{i,1}+1} \Big) + \\ & + z^{-\mu_{1,s} - \sum_{i=1}^{r}(k_{i,1}+1)} \Big(c_{1,s} - [\psi_1(z)]_\infty \Big) \Bigg). \end{aligned}$$

Then ϕ_1 is a left null function for W at λ_i of order $k_{i,1}$ $(i = 1, 2, ..., r)$, the first $k_{i,1}$ coefficients in Taylor expansions of ϕ_1 and ψ_1 at λ_i coincide $(i = 1, 2, ..., r)$, and the leading coefficients in the Laurent expansions of ϕ_1 at $\lambda_{r+1}, \lambda_{r+2}, ..., \lambda_s$ are $c_{1,r+1}, c_{1,r+2}, ..., c_{1,s}$. Inductively, suppose $\phi_1, \phi_2, ..., \phi_{j-1}$ with $j \leq \kappa$ are given and let $\gamma_j \leq r$ be the largest integer such that $\eta_{\gamma_j} \geq j$. Choose nonzero constant vectors $c_{j,\gamma_j+1}, c_{j,\gamma_j+2}, ..., c_{j,s}$ such that the span of $c_{j,\nu}$ and the leading coefficients in the Laurent expansions of $\phi_1, \phi_2, ..., \phi_{j-1}$ at λ_ν intersects trivially with $W^{ol}(\lambda_\nu)$ $(\nu = \gamma_{j+1}, \gamma_{j+2}, ..., \gamma_s)$, and suppose that the integers $\mu_{j,\nu}$ are such that the rational vector function

$$\begin{cases} (z-\lambda_\nu)^{-\mu_{j,\nu}}\psi_j(z), & \text{if } \nu = j+1, j+2, ..., s-1 \\ z^{\mu_{j,\nu}}\psi_j(z), & \text{if } \nu = s \end{cases}$$

is analytic and nonzero at λ_ν $(\nu = j+1, j+2, \ldots, s)$. Let

$$\phi_j(z) = \psi_j(z) + \prod_{i=1}^{\gamma_j}(z-\lambda_i)^{k_{i,j}+1}\cdot$$
$$\cdot\Bigg(\sum_{\nu=\gamma_j+1}^{s-1}(z-\lambda_\nu)^{\mu_{j,\nu}}\Big(c_{j,\nu} - [\psi_j(z)]_{\lambda_j}\Big/\prod_{i=1}^{\gamma_j}(\lambda_\nu-\lambda_i)^{k_{i,j}+1}\Big)+$$
$$+ z^{-\mu_{j,s}-\sum_{i=1}^{\gamma_j}(k_{i,j}+1)}\Big(c_{j,s} - [\psi_j(z)]_\infty\Big)\Bigg).$$

Finally, we put $\Lambda = \text{span }\{\phi_1, \phi_2, \ldots, \phi_\kappa\}$. The functions $\phi_1, \phi_2, \ldots, \phi_\kappa$ and Λ have the required properties.

□

We note that by choosing constant vectors $c_{i,j}$ in the orthogonalization process in the proof of Lemma 2.24, we have actually defined $\Lambda(\lambda_i)$ $(i = r+1, r+2, \ldots, s)$. Also, in general $\Lambda(\lambda_i)$ for $i = 2, 3, \ldots, r$ was not completely determined and depended only on the values of $\psi_{i,1}, \psi_{i,2}, \ldots, \psi_{i,\eta_i}$ at λ_i. For future reference, we state this in the following corollary.

Corollary 2.25 *Let $\sigma \subset \mathbf{C}$, let (A, B) be a left null pair for an $m \times n$ rational matrix function W over σ, and let $\{\lambda_1, \lambda_2, \ldots, \lambda_s\}$ be a finite set of points in $\mathbf{C}_\infty$. Suppose that the pair (A, B) is left-similar to the pair constructed from a canonical set of left null functions*

$$\{\{\phi_{\lambda j}\}:\ \lambda \text{ is an eigenvalue of } A,\ 1 \le j \le \eta_\lambda\}$$

and let

$$\Omega_\lambda = \text{span }\{\phi_{\lambda j}(\lambda):\ 1 \le j \le \eta_\lambda\}$$

for each eigenvalue λ of A. Let k be an integer such that $k \le m$ and k is greater than or equal to the largest geometric multiplicity of a zero of W in σ. Suppose that to every integer $i = 1, 2, \ldots, s$ there corresponds a k-dimensional subspace Λ_i of $\mathbf{C}^{1\times m}$ such that Λ_i contains Ω_{λ_i} whenever λ_i is an eigenvalue of A. Then we can find a *subspace Ξ of $\mathcal{R}^{1\times m}$ such that $\Xi(\lambda_i) = \Lambda_i$ and the pair (A,B) is left-similar to the left null pair constructed from a canonical set of left null functions for W in Ξ.*

It follows from Lemma 2.24 that any left null pair (A,B) for an $m \times n$ rational matrix function W over $\sigma \subset \mathbf{C}$ is left-similar to the pair constructed from functions which are contained in a subspace orthogonal to W^{ol} on $\sigma(A)$. It turns out that the left null pairs for W over σ constructed from functions in the same orthogonal complement of W^{ol} in $(\mathcal{R}^{1\times m}, \{\lambda_1, \lambda_2, ..., \lambda_r\})$, where $\lambda_1, \lambda_2, ..., \lambda_r$ are the zeros of W in σ, are necessarily left-similar.

Proposition 2.26 *Let (A_1, B_1) and (A_2, B_2) be two left null pairs for an $m \times n$ rational matrix function W over $\sigma \subset \mathbf{C}$ constructed from functions contained in the same orthogonal complement Λ of W^{ol} in $(\mathcal{R}^{1\times m}, \sigma(A_1))$. Then (A_1, B_1) and (A_2, B_2) are left-similar.*

Proof Let $\{v_1, v_2, ..., v_k\}$ be an orthonormal basis for $(\Lambda^{\circ}, \sigma(A_1))$. By Theorem 2.11, the rational matrix function

$$Q = [W \quad v_1 \quad v_2 \quad \dots \quad v_k]$$

is regular. Since the pairs (A_1, B_1) and (A_2, B_2) are left null pairs for Q over $\sigma(A_1)$, they are left-similar by Theorem 3.3 in [BGR1].

□

We note that left null pairs for a rational matrix function W over $\sigma \subset \mathbf{C}$ constructed from functions which are not contained in a subspace orthogonal to W^{ol} at the zeros of W in σ need not be left-similar. E.g. if

$$W(z) = \begin{bmatrix} z-1 \\ z-1 \end{bmatrix},$$

$(A_1, B_1) = ([1],\ [1\ \ 0])$ and $(A_2, B_2) = ([1],\ [0\ \ 1])$ are left null pairs for W over $\mathbf{C}$ which are not left-similar. A characterization of when two controllable pairs (A_1, B_1) and (A_2, B_2) are left null pairs for the same matrix polynomial W is given in [BCRo].

2.4 Right zero structure

Let W be an $m \times n$ rational matrix function. We will denote the right annihilator of W by W^{or}. The set of values at $\lambda \subset \mathbf{C}_\infty$ of functions in W^{or} which are analytic at λ will be denoted by $W^{or}(\lambda)$. A matrix polynomial whose columns form a minimal polynomial basis for W^{or} is called a right kernel polynomial for W. We will assume that the columns of a right kernel polynomial P are ordered according to decreasing degrees.

Proposition 2.27 *Let $P \in \mathcal{R}^{n\times k}$ be a matrix polynomial with columns orthogonal at infinity and suppose that the i'th column of P equals $\sum_{j=0}^{\eta_i} z^{\eta_i - j} P_{i,j}$ ($P_{i,j} \in \mathbf{C}^{n\times 1}$, $P_{i,0} \neq 0$ if $P_i \neq 0$, $i = 1, 2, ..., k$). Suppose that the first l columns of P have a pole at infinity and the last $k - l$ columns are constant. Then a minimal realization of P is given by the formula*

$$P(z) = P(0) +$$

$$\begin{bmatrix} P_{1,0} & P_{1,1} & \cdots & P_{1,\eta_1-1} & P_{2,0} & P_{2,1} & \cdots & P_{2,\eta_2-1} & \cdots & P_{l,0} & P_{l,1} & \cdots & P_{l,\eta_l-1} \end{bmatrix}$$

$$\cdot \left(z^{-1} - \begin{bmatrix} J_{\eta_1}(0) & & & \\ & J_{\eta_2}(0) & & \\ & & \ddots & \\ & & & J_{\eta_l}(0) \end{bmatrix} \right)^{-1} \begin{bmatrix} E_1 \\ E_2 \\ \vdots \\ E_l \end{bmatrix} \tag{9}$$

where E_i is the $\eta_i \times k$ matrix with 1 at the position $(i, 1)$ and zeros elsewhere.

We note that the matrix $\begin{bmatrix} E_1^T & E_2^T & \ldots & E_l^T \end{bmatrix}^T$ in the realization (9) of P can be easily reproduced from other matrices in (9). If P is a right kernel polynomial for a rational matrix function W, the ordered triple of matrices $(C_\kappa, A_\kappa, D_\kappa)$ where

$$C_\kappa = \begin{bmatrix} P_{1,0} & P_{1,1} & \dots & P_{1,\eta_1-1} & P_{2,0} & P_{2,1} & \dots & P_{l,0} & P_{l,1} & \dots & P_{l,\eta_l-1} \end{bmatrix},$$

$$A_\kappa = \begin{bmatrix} J_{\eta_1}(0) & & & \\ & J_{\eta_2}(0) & & \\ & & \ddots & \\ & & & J_{\eta_l}(0) \end{bmatrix} \quad \text{and} \quad D_\kappa = P(0)$$

will be called a <u>right kernel triple</u> for W.

We note that if W is a rational matrix function and $\lambda \in \mathbf{C}$, a basis for $W^{or}(\lambda)$ can be easily computed from a right kernel triple $(C_\kappa, A_\kappa, D_\kappa)$ for W; a basis for $W^{or}(\infty)$ can be read off directly from $(C_\kappa, A_\kappa, D_\kappa)$.

Let W be an $m \times n$ rational matrix function and let $\lambda \in \mathbf{C}_\infty$. A function $\phi \in \mathcal{R}^{n\times 1}$ is a <u>right null function</u> for W at λ of <u>order</u> k, k a positive integer, if

(i) ϕ is analytic and nonzero at λ,

(ii) $W\phi$ has a zero at λ of order k,

(iii) ϕ is orthogonal to W^{or} at λ.

Condition (iii) above can be replaced by

(iii') $\phi(\lambda) \notin W^{or}(\lambda)$.

A set of right null functions $\{\phi_1, \phi_2, ..., \phi_\eta\}$ for an $m \times n$ rational matrix function W at $\lambda \in \mathbf{C}_\infty$ of orders $k_1, k_2, ..., k_\eta$, respectively, is called a <u>canonical set of right null functions</u> for W at λ if

(i) $\phi_1(\lambda), \phi_2(\lambda), ..., \phi_\eta(\lambda)$ are linearly independent,

(ii) $\phi_1, \phi_2, ..., \phi_\eta$ are contained in an orthogonal complement of W^{or} in $(\mathcal{R}^{n\times 1}, \{\lambda\})$,

(iii) $\sum_{i=1}^{\eta} k_i$ is maximal subject to conditions (i) and (ii).

Condition (ii) above can be replaced with

(ii') span $\{\phi_1(\lambda), \phi_2(\lambda), ..., \phi_\eta(\lambda) \cap W^{or}(\lambda) = (0)$.

We note that the number of functions in a canonical set of right null functions for W at λ is equal to the geometric multiplicity of the zero of W at λ. The orders of functions in a canonical set of right null functions for W at λ coincide with the partial multiplicities of the zero of W at λ.

Let $\lambda \in \mathbf{C}_\infty$ be a zero of an $m \times n$ rational matrix function W and let $\{\phi_1, \phi_2, ..., \phi_\eta\}$ be a canonical set of right null functions for W at λ, of orders $k_1, k_2, ..., k_\eta$, respectively. Let $\phi_{i,j} \in \mathbf{C}^{n\times 1}$ be the j'th coefficient in the Taylor expansion of ϕ_i at λ. Any ordered pair of matrices (C, A), where C equals

$$\begin{bmatrix} \phi_{1,0} & \phi_{1,1} & \cdots & \phi_{1,k_1-1} & \phi_{2,0} & \phi_{2,1} & \cdots & \phi_{2,k_2-1} & \cdots & \phi_{\eta,0} & \phi_{\eta,1} & \cdots & \phi_{\eta,k_\eta-1} \end{bmatrix} S$$

and

$$A = \begin{cases} S^{-1}\begin{bmatrix} J_{k_1}(\lambda) & & & \\ & J_{k_2}(\lambda) & & \\ & & \ddots & \\ & & & J_{k_\eta}(\lambda) \end{bmatrix} S, & \text{if } \lambda \in \mathbf{C} \\ S^{-1}\begin{bmatrix} J_{k_1}(0) & & & \\ & J_{k_2}(0) & & \\ & & \ddots & \\ & & & J_{k_\eta}(0) \end{bmatrix} S, & \text{if } \lambda = \infty \end{cases}$$

for some invertible matrix S of an appropriate size, is called a <u>right null pair</u> for W at λ.

Let $\sigma \subset \mathbf{C}$ and let $\lambda_1, \lambda_2, ..., \lambda_r$ be the zeros of a rational matrix function W in σ. Let (C_i, A_i) be a right null pair for W at λ_i ($i = 1, 2, ..., r$). Any ordered pair of matrices (C, A), where

$$C = [\, C_1 \quad C_2 \quad \ldots \quad C_r \,]\, S \quad \text{and} \quad A = S^{-1}\begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_p \end{bmatrix} S$$

for some invertible matrix S of an appropriate size, is called a <u>right null pair</u> for W over σ (or a σ-right null pair for W).

Similarly as in the regular case, right null pairs are controllable. Right null pairs also have the following property.

Proposition 2.28 *Let $\sigma \subset \mathbf{C}$, let (C, A) be a right null pair for a rational matrix function W over σ, and let $\{\lambda_1, \lambda_2, ..., \lambda_s\}$ be a finite set of points in $\mathbf{C}_\infty$. Suppose that the largest geometric multiplicity of a zero of W in σ equals κ. Then there exists a subspace Λ orthogonal to W^{or} on $\{\lambda_1, \lambda_2, ..., \lambda_s\}$ and functions $\phi_1, \phi_2, ..., \phi_\kappa \in \Lambda$ such that the pair (C, A) is right-similar to the pair constructed from the Taylor coefficients of $\phi_1, \phi_2, ..., \phi_\kappa$ at the eigenvalues of A.*

Proposition 2.28 allows us, in particular, to associate with each right null pair (C, A) for a rational matrix function W over $\sigma \subset \mathbf{C}$ a subspace Λ of $\mathcal{R}^{m\times 1}$ which is orthogonal to W^{or} on $\sigma(A)$ such that (C, A) is right-similar to a pair constructed from right null functions for W which are contained in Λ. Right null pairs constructed from functions in a fixed orthogonal complement of W^{or} in $(\mathcal{R}^{n\times 1}, \{\lambda_1, \lambda_2, ..., \lambda_r\})$, where $\lambda_1, \lambda_2, ..., \lambda_r$ are all the zeros of W in σ, are right-similar.

Chapter III

Local Right Equivalence

Recall that two $m \times n$ rational matrix functions W and H are said to be right equivalent on $\sigma \subset \mathbf{C}_\infty$ if $W = HQ$ for some regular rational matrix function Q such that Q and Q^{-1} are analytic on σ. We note that if $\sigma = \mathbf{C}_\infty$, Liouville's theorem implies that a rational matrix function Q which has no poles in σ is constant. Thus, two rational matrix functions which are right equivalent on $\mathbf{C}_\infty$ differ by a constant right factor.

If a subset σ of the extended complex plane is given, the relation of right equivalence on σ divides all rational matrix functions of the same size into equivalence classes. The members of the same class have the same right pole and left zero structure: if W and H are right equivalent on σ, (C_π, A_π) is a right pole pair for W over σ if and only if (C_π, A_π) is a right pole pair for H and (A_ζ, B_ζ) is a left null pair for W over σ if and only if (A_ζ, B_ζ) is a left null pair for H over σ. As can be seen from the regular case (see Theorem 5.1 in [GK2]), the converse of the preceding statement does not hold: two $m \times n$ rational matrix functions with the same right-pole and left-zero structure on σ need not be right equivalent on σ.

Local right equivalence of rational matrix functions, that is right equivalence over a proper subset of $\mathbf{C}_\infty$, is a generalization of a well understood concept of equivalence of matrices over a principal ideal domain: A and B, matrices over a principal ideal domain $\mathcal{D}$, are said to be equivalent if $A = PBQ$ for some unimodular matrices over $\mathcal{D}$, P and Q (see [McD, J]). Morse studied in [Mor] a relation between matrices over a principal ideal domain which he called dynamic equivalence. Two matrices A and B over a principal ideal domain $\mathcal{D}$ are said to be dynamically equivalent if $A = BQ$

for some unimodular matrix over $\mathcal{D}$, Q. Plainly, dynamically equivalent matrices are equivalent. Our definition of right equivalence of rational matrix functions A and B extends the definition of dynamic equivalence to the case where A and B may have entries in a field properly including the principal ideal domain $\mathcal{D}$ (see Proposition 3.1 below).

A concept related with local right equivalence is that of the left null-pole subspace of a rational matrix function (see Section 3.1 below). Null-pole subspaces are also called singular subspaces in the literature. They were introduced in [GR3, GR4] for analytic matrix and operator functions. Null-pole subspaces of regular rational matrix functions were investigated in [BRan1], [BCR] and [BGR2].

In the case where σ contains both poles and zeros of a rational matrix function W, the description of the null-pole subspace requires besides a right pole pair and a left null pair for W over σ an extra invariant, called the null-pole coupling matrix. For the regular case, this first appears in a global form in [GKLR] in connection with minimal divisibility questions. In connection with null-pole subspace, it was first introduced for the regular case in [BRan1, BRan2]. For the nonregular case, it appears here for the first time.

Chapter III is organized as follows. In Section 3.1 we describe in more detail the concept of right equivalence of rational matrix functions on a subset σ of $\mathbf{C}_\infty$. Included here is an additional piece of structure, namely the null-pole coupling matrix, which, together with the right pole and left zero structure on $\sigma \subset \mathbf{C}$ already introduced, completely characterizes right equivalence on σ. In Section 3.2 we find a sufficient condition for minimality of the McMillan degree of a rational matrix function H which is right equivalent to a given function W on σ.

3.1 Left null-pole subspaces

If $\sigma \subset \mathbf{C}_\infty$, we will denote by $\mathcal{R}(\sigma)$ the integral domain of all functions in $\mathcal{R}$ which are analytic on σ. It is shown in [Mor] that $\mathcal{R}(\sigma)$ is in fact a principal ideal domain. We will denote by $\mathcal{R}^{m\times n}(\sigma)$ the set of matrices over $\mathcal{R}(\sigma)$. Using this notation we can characterize right equivalence of rational matrix functions as follows.

Proposition 3.1 *If $W, H \in \mathcal{R}^{m\times n}$ and $\sigma \subset \mathbf{C}_\infty$, the following are equivalent:*

(i) W, H are right equivalent on σ,

(ii) $W = HQ_1$ and $H = WQ_2$ for some $Q_1, Q_2 \in \mathcal{R}^{n\times n}(\sigma)$,

(iii) $W\mathcal{R}^{n\times 1}(\sigma) = H\mathcal{R}^{n\times 1}(\sigma)$.

Proof The implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ are immediate. Suppose (iii) holds. If $\sigma = \mathbf{C}_\infty$, $\mathcal{R}^{n\times 1}(\sigma) = \mathbf{C}^{n\times 1}$. Then $W\mathcal{R}^{n\times 1}(\sigma)$ is a finite diminsional C-vector space. Let $k = \dim W\mathbf{C}^{n\times 1}$. We have $W = \begin{bmatrix} \tilde{W} & 0 \end{bmatrix} P_W$ for some invertible constant matrix P_W and some $m \times (n-k)$ rational matrix function $\tilde{W}$ with columns linearly independent over C. Since $\dim (W\mathbf{C}^{n\times 1}) = \dim (H\mathbf{C}^{n\times 1})$, $H = \begin{bmatrix} \tilde{H} & 0 \end{bmatrix} P_H$ for some invertible constant matrix P_H and some $m \times (n-k)$ rational matrix function $\tilde{H}$ with columns linearly independent over C. Since columns of $\tilde{W}$ and $\tilde{H}$ form bases for the same vector space over C, $\tilde{W} = \tilde{H}P$ for some $(n-k)\times(n-k)$ invertible matrix P. It follows that

$$\begin{aligned} W &= \begin{bmatrix} \tilde{W} & 0 \end{bmatrix} P_W = \begin{bmatrix} \tilde{H} & 0 \end{bmatrix} \begin{bmatrix} P & \\ & I \end{bmatrix} P_W \\ &= HP_H^{-1} \begin{bmatrix} P & \\ & I \end{bmatrix} P_W \\ &\equiv HQ \end{aligned}$$

and W and H are right equivalent on $\mathbf{C}_\infty$.

Suppose σ is a proper subset of $\mathbf{C}_\infty$. After applying a suitable Möbius transformation, we may assume $\sigma \subset \mathbf{C}$. Let $E_W D_W F_W$ and $E_H D_H F_H$ be Smith-McMillan factorizations of W and H. Then

$$E_W D_W \mathcal{R}^{n\times 1}(\sigma) = E_H D_H \mathcal{R}^{n\times 1}(\sigma). \tag{1}$$

Let k be the number of nonzero diagonal entries in D_H. Comparing the dimensions of the free $\mathcal{R}(\sigma)$-modules on both sides of equality (1), we see that the number of nonzero diagonal entries in D_W is also equal to k. If $\tilde{D}_W$ and $\tilde{D}_H$ are $m \times k$ rational matrix functions such that $D_W = \begin{bmatrix} \tilde{D}_W & 0 \end{bmatrix}$ and $D_H = \begin{bmatrix} \tilde{D}_H & 0 \end{bmatrix}$, it follows from (1) that

$$\tilde{D}_W \mathcal{R}^{k\times 1}(\sigma) = E_W^{-1} E_H \tilde{D}_H \mathcal{R}^{k\times 1}(\sigma).$$

Let e_i be a constant $k \times 1$ vector with 1 at the i'th position and zeros elsewhere, and choose $f_1, f_2, ..., f_k \in \mathcal{R}^{k\times 1}(\sigma)$ such that $\tilde{D}_W e_i = E_W^{-1} E_H \tilde{D}_H f_i$ $(i = 1, 2, ..., k)$. Let $\tilde{Q} = [f_1 \quad f_2 \quad \cdots \quad f_k]$. $\tilde{Q}$ is a square matrix over $\mathcal{R}(\sigma)$. Since $\tilde{Q}$ relates two bases of $\mathcal{R}^{k\times 1}(\sigma)$, $\tilde{Q}$ is a unit in the ring of $k \times k$ matrices over $\mathcal{R}(\sigma)$. It follows that $\tilde{Q}$ is a k×k rational matrix function such that $\tilde{Q}$ and $\tilde{Q}^{-1}$ are analytic on σ and

$$E_W D_W = E_H \tilde{D}_H \tilde{Q}.$$

Hence

$$E_W D_W = E_H D_H \begin{bmatrix} \tilde{Q} & \\ & I \end{bmatrix}.$$

So

$$\begin{aligned} W &= E_W D_W F_W \\ &= E_H D_H F_H \left(F_H^{-1} \begin{bmatrix} \tilde{Q} & \\ & I \end{bmatrix} F_W \right) \\ &\equiv HQ. \end{aligned}$$

Thus, W and H are right equivalent on σ.

□

It follows from Proposition 3.1 that all $m \times n$ right equivalent on σ rational matrix functions viewed as maps $\mathcal{R}^{n\times 1}(\sigma) \to \mathcal{R}^{m\times 1}$ have the same image. Let $W \in \mathcal{R}^{m\times n}$

and let $\sigma \subset \mathbf{C}_\infty$. Since multiplication of matrices commutes with multiplication by scalars, $W\mathcal{R}^{n\times 1}(\sigma)$ is an $\mathcal{R}(\sigma)$-module. In particular, $W\mathcal{R}^{n\times 1}(\sigma)$ is a $\mathbf{C}$-vector space. After [BGR2] we shall call $W\mathcal{R}^{n\times 1}(\sigma)$ the left null-pole subspace for W over σ.

If $W \in \mathcal{R}^{m\times m}$ is a regular rational matrix function and $\sigma \subset \mathbf{C}$, the left null-pole subspace for W over σ is determined by a right pole pair (C_π, A_π) for W over σ, a left null pair (A_ζ, B_ζ) for W over σ and a matrix Γ which *couples* the pairs (C_π, A_π) and (A_ζ, B_ζ). The explicit representation of $W\mathcal{R}^{m\times 1}(\sigma)$ in terms of $(C_\pi, A_\pi), (A_\zeta, B_\zeta)$ and Γ is given by the formula (see Theorem 3.4.1 in [BGR2])

$$W\mathcal{R}^{m\times 1}(\sigma) = \{C_\pi(z - A_\pi)^{-1}x + h(z) : \; x \in \mathbf{C}^{n_\pi\times 1}, h \in \mathcal{R}^{m\times 1}(\sigma) \\ \text{and} \; \sum_{z_0\in\sigma} \mathrm{Res}_{z=z_0}(z - A_\zeta)^{-1}B_\zeta h(z) = \Gamma x\}$$

where n_π is the size of the matrix A_π.

The matrix Γ above is called a null-pole coupling matrix for (C_π, A_π) and (A_ζ, B_ζ) as the right-pole and left-null pairs for W over σ (see [BGR2]), or a coupling operator (see e.g. [BRan1]). If $W(z)$ is equal to $I + C(z-A)^{-1}B$ with the realization (A, B, C, I) minimal and a right pole pair (C_π, A_π) and a left null pair (A_ζ, B_ζ) for W over σ are given, the corresponding null-pole coupling matrix can be computed as follows (see [BRan1, GK2, BGR2]). Choose simple positively oriented contours γ_π and γ_ζ around $\sigma(A_\pi)$ and $\sigma(A_\zeta)$ such that $\sigma(A)\backslash\sigma(A_\pi)$ is outside γ_π and $\sigma(A - BC)\backslash\sigma(A_\zeta)$ is outside γ_ζ and let

$$\begin{aligned} \Theta &= (1/2\pi i)\int_{\gamma_\pi}(z - A)^{-1}dz \\ \Theta^\times &= (1/2\pi i)\int_{\gamma_\zeta}(z - A + BC)^{-1}dz. \end{aligned} \tag{2}$$

The null-pole coupling matrix for (C_π, A_π) and (A_ζ, B_ζ) as the right pole and left null pairs for W over σ is given by the formula $S_\zeta^{-1}\Theta^\times S_\pi^{-1}$ where S_ζ and S_π are the unique matrices such that

$$(C|\mathrm{Im}\Theta, A|\mathrm{Im}\Theta) = (C_\pi S_\pi, S_\pi^{-1}A_\pi S_\pi)$$

$$(A - BC|\mathrm{Im}\Theta^\times, \Theta^\times B) = (S_\zeta A_\zeta S_\zeta^{-1}, S_\zeta B_\zeta).$$

We note that if (C_π, A_π) and (A_ζ, B_ζ) are right pole and left null pairs over $\sigma \subset \mathbf{C}$ for some regular rational matrix function W and Γ is the corresponding null-pole coupling matrix, the equality

$$\Gamma A_\pi - A_\zeta \Gamma = B_\zeta C_\pi \tag{4}$$

holds (see [GK2,BGR2]).

We now extend the definition of a null-pole coupling matrix to the non-regular case. Let W be an $m \times n$ rational matrix function and let $\sigma \subset \mathbf{C}$. Let (C_π, A_π) and (A_ζ, B_ζ) be right pole and left null pairs for W over σ. By Lemma 2.24, there exist an orthogonal complement Λ of W^{ol} in $(\mathcal{R}^{1\times m}, \sigma(A_\pi)\cup\sigma(A_\zeta))$ such that the pair (A_ζ, B_ζ) is left-similar to a pair constructed from functions in Λ. Choose an orthonormal basis $\{v_1, v_2, ..., v_k\}$ for $(\Lambda^\circ, \sigma(A_\pi) \cup \sigma(A_\zeta))$. Fix a Smith-McMillan factorization EDF of W and let $\tilde{D}$ be the rational matrix function obtained from D after deleting the zero columns. Then (C_π, A_π) and (A_ζ, B_ζ) are right pole and left zero pairs for the regular rational matrix function $W_E = \begin{bmatrix} E\tilde{D} & v_1 & v_2 & \dots & v_k \end{bmatrix}$. We define the null-pole coupling matrix for (C_π, A_π) and (A_ζ, B_ζ), viewed as the right pole pair and left null pair for W_E over $\sigma(A_\pi) \cup \sigma(A_\zeta)$, to be the <u>null-pole coupling matrix</u> Γ for the <u>right pole pair</u> (C_π, A_π) and <u>left null pair</u> (A_ζ, B_ζ) for W over σ.

We need to show that Γ is well-defined. We will show that if we choose $\tilde{\Lambda}$ instead of Λ and $\tilde{v}_1, \tilde{v}_2, ..., \tilde{v}_k$ instead of $v_1, v_2, ..., v_k$ then $\tilde{W}_E = \begin{bmatrix} E\tilde{D} & \tilde{v}_1 & \tilde{v}_2 & \dots & \tilde{v}_k \end{bmatrix}$ is right equivalent to W_E on $\sigma(A_\pi) \cup \sigma(A_\zeta)$. Since W_E and $\tilde{W}_E$ are regular rational matrix functions of the same size,

$$\begin{bmatrix} E\tilde{D} & \tilde{v}_1 & \tilde{v}_2 & \dots & \tilde{v}_k \end{bmatrix} = \begin{bmatrix} E\tilde{D} & v_1 & v_2 & \dots & v_k \end{bmatrix} Q \tag{5}$$

for some regular rational matrix function Q. Let

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}$$

with Q_{22} of the size $k \times k$. Since the columns of W_E and $\tilde{W}_E$ form bases for $\mathcal{R}^{n\times 1}$, $Q_{11} = I$ and $Q_{21} = 0$. Suppose that some entry of Q_{22} has a pole at $\lambda \in \sigma(A_\pi) \cup \sigma(A_\zeta)$. Let Q_j be the corresponding column of Q. Then $W_E Q_j$ is the sum of a function in Λ° which has a pole at λ and a function in the column space of W. So $W_E Q_j$ has a pole at λ. Since, by (5), $W_E Q_j \in \{\tilde{v}_1, \tilde{v}_2, ..., \tilde{v}_k\}$, this is a contradiction. Thus, Q_{22} is analytic on $\sigma(A_\pi) \cup \sigma(A_\zeta)$. Suppose now that some entry in Q_{12} has a pole at $\lambda \in \sigma(A_\pi) \cup \sigma(A_\zeta)$. Let q_{ij} be the corresponding entry of Q. If the i'th diagonal entry of $\tilde{D}$ does not have a zero at λ, the orthogonality of the columns of W_E at λ implies that the j'th column of $W_E Q$ has a pole at λ. This again contradicts (5). Suppose that the i'th diagonal entry of $\tilde{D}$ has a zero at λ of order κ and let ϕ be the i'th row of E^{-1}. Then ϕ is a left null function for $\tilde{W}_E$ at λ of order κ. Since W_E and $\tilde{W}_E$ share a common left null pair, W_E and $\tilde{W}_E$ have the same left zero structure. Hence ϕ is a left null function at λ of order κ for W_E. Since Q_{22} is analytic on σ and $\phi\,[\,v_1 \quad v_2 \quad \ldots \quad v_k\,]$ has a zero at λ of order at least κ, $\phi W_E Q$ vanishes at λ to the order strictly less than κ or $\phi W_E Q$ does not vanish at λ at all, a contradiction. It follows that Q is analytic on $\sigma(A_\pi) \cup \sigma(A_\zeta)$. We show similarly that Q^{-1} is analytic on $\sigma(A_\pi) \cup \sigma(A_\zeta)$. Thus, W_E and $\tilde{W}_E$ are right equivalent on $\sigma(A_\pi) \cup \sigma(A_\zeta)$ and the null-pole coupling matrix Γ is well-defined.

We note that if W is a rational matrix function, $\sigma \subset \mathbf{C}$, and a right pole pair (C_π, A_π) and a left null pair (A_ζ, B_ζ) for W over σ are given, we can actually compute the corresponding null-pole coupling matrix as follows.

1. Find W_E as in the definition of null-pole coupling matrix.
2. Find $\tilde{W}_E$ which is right equivalent to W_E on $\sigma(A_\pi) \cup \sigma(A_\zeta)$ and has value I at infinity.
3. Find a minimal realization (A, B, C, I) for $\tilde{W}_E$.
4. Compute Γ using formulas (2) and (3).

The involved computations may be, however, extensive.

Let W be a rational matrix function and let $\sigma \subset \mathbf{C}$. If (C_π, A_π) is a σ-right pole pair for W, (A_ζ, B_ζ) is a σ-left null pair for W, and Γ is the corresponding null-pole coupling matrix, we will call the ordered triple

$$\tau = \{(C_\pi, A_\pi), (A_\zeta, B_\zeta), \Gamma\}$$

a left spectral triple for W over σ or a left σ-spectral triple for W.

We note that if $\tau = \{(C_\pi, A_\pi), (A_\zeta, B_\zeta), \Gamma\}$ is a left spectral triple for a rational matrix function W over $\sigma \subset \mathbf{C}$, then it follows from (4) and the definition of a null-pole coupling matrix that Γ satisfies the Sylvester equation

$$SA_\pi - A_\zeta S = B_\zeta C_\pi.$$

Also, by the properties of spectral triples for regular rational matrix functions, if T_π and T_ζ are invertible matrices of appropriate sizes then

$$\tilde{\tau} = \{(C_\pi T_\pi, T_\pi^{-1} A_\pi T_\pi), (T_\zeta A_\zeta T_\zeta^{-1}, T_\zeta B_\zeta), T_\zeta \Gamma T_\pi\} \tag{6}$$

is another left spectral triple for W over σ. If $\tilde{\tau}$ is any left spectral triple for W over σ and there exist matrices T_π and T_ζ such that (6) holds, $\tilde{\tau}$ and τ are said to be similar. If W is regular, all left σ-spectral triples for W are similar. Since rational matrix functions with nontrivial left annihilators may have left null pairs over σ which are not left-similar, a nonregular rational matrix function may have left spectral triples which are not left-similar.

We can now characterize local right equivalence of rational matrix functions in terms of spectral data.

Theorem 3.2 *Let $\sigma \subset \mathbf{C}$ and let W be an $m \times n$ rational matrix function with a left kernel triple τ_κ and a left spectral triple over σ $\tau_s = \{(C_\pi, A_\pi), (A_\zeta, B_\zeta), \Gamma\}$. Then an*

$m \times n$ rational matrix function H is right equivalent to W on σ if and only if τ_κ is a left kernel triple for H and τ_s is a left spectral triple for H over σ.

Proof Suppose first that W and H are right equivalent on σ. Then τ_κ is a left kernel triple for H, (C_π, A_π) is a right pole pair for W over σ and (A_ζ, B_ζ) is a left null pair for W over σ. Furthermore we can find regular rational matrix functions W_E and H_E such that W_E and H_E are right equivalent on $\sigma(A_\pi) \cup \sigma(A_\zeta)$, (C_π, A_π) and (A_ζ, B_ζ) are right pole and left null pairs for W and H over $\sigma(A_\pi) \cup \sigma(A_\zeta)$, and Γ is a null-pole coupling matrix for (C_π, A_π) and (A_ζ, B_ζ) viewed as right pole and left null pairs for W_E over σ. Hence Γ is a null-pole coupling matrix for (C_π, A_π) and (A_ζ, B_ζ) viewed as right pole and left null pairs for H over σ.

Suppose now that τ_κ is a left kernel triple for H and τ_s is a spectral triple for H over σ. Let $E_W D_W F_W$ and $E_H D_H F_H$ be Smith-McMillan factorizations of W and H and let $\tilde{D}_W$ and $\tilde{D}_H$ be rational matrix functions formed by nonzero columns of D_W and D_H. Choose an orthogonal complement Λ of the column space of W in $(\mathcal{R}^{m\times 1}, \sigma(A_\pi) \cup \sigma(A_\zeta))$ and let $\{v_1, v_2, ..., v_k\}$ be an orthonormal basis for $(\Lambda, \sigma(A_\pi) \cup \sigma(A_\zeta))$. By the definition of the null-pole coupling matrix, there exists a rational matrix function Q_E such that Q_E and Q_E^{-1} have no poles nor zeros on $\sigma(A_\pi) \cup \sigma(A_\zeta)$ and $W_E = H_E Q_E$ where

$$W_E = \begin{bmatrix} E_W \tilde{D}_W & v_1 & v_2 & \dots & v_k \end{bmatrix},$$
$$H_E = \begin{bmatrix} E_H \tilde{D}_H & v_1 & v_2 & \dots & v_k \end{bmatrix}.$$

Since the columns of W_E and H_E form bases for $\mathcal{R}^m$,

$$Q_E = \begin{bmatrix} Q_{11} & 0 \\ Q_{21} & I \end{bmatrix}.$$

Since the column spaces of $E_W \tilde{D}_W$ and $E_H \tilde{D}_H$ are the same, $Q_{21} = 0$. Since Q_E and Q_E^{-1} are analytic on $\sigma(A_\pi) \cup \sigma(A_\zeta)$, Q_{11} and Q_{11}^{-1} are analytic on $\sigma(A_\pi) \cup \sigma(A_\zeta)$. Since

$\tilde{D}_W$ and $\tilde{D}_H$ have no poles nor zeros in $\sigma\backslash(\sigma(A_\pi)\cup\sigma(A_\zeta))$, Q_{11} and Q_{11}^{-1} are analytic on the whole σ. So

$$\begin{aligned}
W &= E_W\begin{bmatrix}\tilde{D}_W & 0\end{bmatrix}F_W \\
&= E_H\begin{bmatrix}\tilde{D}_H & 0\end{bmatrix}\begin{bmatrix}Q_{11} & \\ & I\end{bmatrix}F_H \\
&= H\left(F_H^{-1}\begin{bmatrix}Q_{11} & \\ & I\end{bmatrix}F_H\right) \\
&\equiv HQ
\end{aligned}$$

with Q an $n\times n$ rational matrix function such that Q and Q^{-1} are analytic on σ. Thus, W and H are right equivalent on σ.

□

It follows from Theorem 3.2 that the left null-pole subspace for a rational matrix function W over $\sigma\subset\mathbf{C}$ is determined completely by a left kernel triple for W and a left spectral triple for W over σ. In fact, we can characterize the left null-pole subspace for W over σ as follows.

Proposition 3.3 *Let W be an $m\times n$ rational matrix function, let $\sigma\subset\mathbf{C}$ and let $\{(C_\pi,A_\pi),\ (A_\zeta,B_\zeta),\Gamma\}$ be a left spectral triple for W over σ. Let τ be a left kernel triple for W and let P be the matrix polynomial corresponding to τ. Then the left null-pole subspace for W over σ is given by the formula*

$$\begin{aligned}
W\mathcal{R}^{m\times 1}(\sigma) = \{C_\pi(z-A_\pi)^{-1}x+h(z):\ \ & x\in\mathbf{C}^{n_\pi\times 1}, h\in\mathcal{R}^{m\times 1}(\sigma)\quad \textit{and} \\
&\sum_{z_0\in\sigma} Res_{z=z_0}(z-A_\zeta)^{-1}B_\zeta h(z)=\Gamma x\}\cap\{f\in\mathcal{R}^{m\times 1}:\ Pf=0\}
\end{aligned} \tag{7}$$

where n_π is the size of the matrix A_π.

Proof By the definition of a null-pole coupling matrix Γ and the regular case,

$$\begin{aligned} W\mathcal{R}^{m\times 1}\Big(\sigma(A_\pi)\cup\sigma(A_\zeta)\Big) =&\{C_\pi(z-A_\pi)^{-1}x+h(z):\ x\in \mathbf{C}^{n_\pi\times 1},\\ & h\in\mathcal{R}^{m\times 1}\Big(\sigma(A_\pi)\cup\sigma(A_\zeta)\Big) \quad \text{and}\\ & \sum_{z_0\in\sigma(A_\pi)\cup\sigma(A_\zeta)} \operatorname{Res}_{z=z_0}(z-A_\zeta)^{-1}B_\zeta h(z)=\Gamma x\}\\ & \cap\{f\in\mathcal{R}^{m\times 1}:\ Pf=0\}\end{aligned}$$

Since W is right equivalent on $\sigma\backslash(\sigma(A_\pi)\cup\sigma(A_\zeta))$ to a rational matrix function whose nonzero columns form an orthonormal basis for $\Big(\{f\in\mathcal{R}^{m\times 1};\ Pf=0\},\sigma\backslash(\sigma(A_\pi)\cup\sigma(A_\zeta))\Big)$,

$$W\mathcal{R}^{m\times 1}\Big(\sigma\backslash(\sigma(A_\pi)\cup\sigma(A_\zeta))\Big)=\{f\in\mathcal{R}^{m\times 1}\Big(\sigma\backslash(\sigma(A_\pi)\cup\sigma(A_\zeta))\Big):\ Pf=0\}.$$

So

$$\begin{aligned} W\mathcal{R}^{m\times 1}(\sigma) &= W\mathcal{R}^{m\times 1}\Big(\sigma(A_\pi)\cup\sigma(A_\zeta)\Big)\cap W\mathcal{R}^{m\times 1}\Big(\sigma\backslash(\sigma(A_\pi)\cup\sigma(A_\zeta))\Big)\\ &=\{f\in\mathcal{R}^{m\times 1}:\ Pf=0\}\cap\Big(\{C_\pi(z-A_\pi)^{-1}x+h(z):\\ &\quad x\in\mathbf{C}^{n_\pi\times 1},\ h\in\mathcal{R}^{m\times 1}(\sigma(A_\pi)\cup\sigma(A_\zeta)) \text{ and}\\ &\quad \sum_{z_0\in(\sigma(A_\pi)\cup\sigma(A_\zeta))}\operatorname{Res}_{z=z_0}(z-A_\zeta)^{-1}B_\zeta h(z)=\Gamma x\}\\ &\quad \cap\mathcal{R}^{m\times 1}\Big(\sigma\backslash(\sigma(A_\pi)\cup\sigma(A_\zeta))\Big)\Big).\end{aligned}$$

Now $(z-A_\pi)^{-1}$ and $(z-A_\zeta)^{-1}$ are analytic on $\sigma\backslash(\sigma(A_\pi)\cup\sigma(A_\zeta))$, so if $h\in\mathcal{R}^{m\times 1}(\sigma(A_\pi)\cup\sigma(A_\zeta))$ is not analytic on $\sigma\backslash(\sigma(A_\pi)\cup\sigma(A_\zeta))$, $C_\pi(z-A_\pi)^{-1}x+h(z)$ is not analytic on $\sigma\backslash(\sigma(A_\pi)\cup\sigma(A_\zeta))$ for any $x\in\mathbf{C}^{n_\pi\times 1}$. Conversely, if $h\in\mathcal{R}^{m\times 1}(\sigma)$ then $C_\pi(z-A_\pi)^{-1}x+h(z)$ is analytic on $\sigma\backslash(\sigma(A_\pi)\cup\sigma(A_\zeta))$ for any $x\in\mathbf{C}^{n_\pi\times 1}$. It follows that

$$\mathcal{R}^{m\times 1}\Big(\sigma\backslash(\sigma(A_\pi)\cup\sigma(A_\zeta))\Big)\cap\{C_\pi(z-A_\pi)^{-1}x+h(z):$$

$$x\in\mathbf{C}^{n_\pi\times 1},\ h\in\mathcal{R}^{m\times 1}\Big(\sigma(A_\pi)\cup\sigma(A_\zeta)\Big)\text{ and}$$

$$\sum_{z_0\in\sigma(A_\pi)\cup\sigma(A_\zeta)}\mathrm{Res}_{z=z_0}(z-A_\zeta)^{-1}B_\zeta h(z)=\Gamma x\}$$

$$=\{C_\pi(z-A_\pi)^{-1}x+h(z):\ x\in\mathbf{C}^{n_\pi\times 1},\ h\in\mathcal{R}^{m\times 1}(\sigma)\text{ and}$$

$$\sum_{z_0\in\sigma}\mathrm{Res}_{z=z_0}(z-A_\zeta)^{-1}B_\zeta h(z)=\Gamma x\}.$$

So equality (7) holds.

□

3.2 Right equivalence and McMillan degree

In this chapter we continue to call the leading coefficient in the Laurent expansion at λ of a rational vector function h the leading coefficient of h at λ. We will denote the leading coefficient of a rational vector function h at λ by $[h]_\lambda$. Thus, e.g. $[0]_\infty=0$ and if in a neighborhood of infinity $h=\sum_{i=-\infty}^{k} z^i h_i$ with $h_k\neq 0$ then $[h]_\infty=h_k$. We will denote the multiplicity of a pole of a rational matrix function W at λ by $\delta(W,\lambda)$. The McMillan degree of W will be denoted by $\delta(W)$. Thus, $\delta(W)=\sum_{\lambda\in\mathbf{C}_\infty}\delta(W,\lambda)$.

We begin with the following lemma.

Lemma 3.4 *Let $H=[\,h_1\quad h_2\quad\ldots\quad h_n\,]$ be an $m\times n$ rational matrix function with the columns ordered according to decreasing pole multiplicity at $\lambda\in\mathbf{C}_\infty$. If $h_1,h_2,\ldots,h_l$ are orthogonal at λ and h_l has a pole at λ, then we can extend the set*

$$X=\{z^{-\delta(h_1,\lambda)}h_1,\ z^{-\delta(h_2,\lambda)}h_2,\ \ldots,\ z^{-\delta(h_l,\lambda)}h_l\}$$

to a canonical set of right pole functions for H at λ.

Proof Clearly $z^{-\delta(h_1,\lambda)}h_1,\ z^{-\delta(h_2,\lambda)}h_2,\ \ldots,\ z^{-\delta(h_l,\lambda)}h_l$ are right pole functions for H at λ of order $\delta(h_1,\lambda),\ \delta(h_2,\lambda),\ldots,\delta(h_l,\lambda)$ respectively. Since $z^{-\delta(h_1,\lambda)}H$ is analytic at λ, H has no right pole functions at λ of order greater than $\delta(h_1,\lambda)$. Suppose we are in the process of finding a canonical set of right pole functions for H at λ and we have chosen $z^{-\delta(h_1,\lambda)}h_1,\ z^{-\delta(h_2,\lambda)}h_2,\ \ldots,\ z^{-\delta(h_i,\lambda)}h_i$ $(1 \leq i < l)$. The rational vector function $z^{-\delta(h_{i+1},\lambda)}h_{i+1}$ is a right pole function for H at λ of order $\delta(h_{i+1},\lambda)$ with value at λ not contained in span $\{[h_1]_\lambda, [h_2]_\lambda, \ldots [h_i]_\lambda\}$. Since by Proposition 2.13 $[h_1]_\lambda, [h_2]_\lambda, \ldots, [h_i]_\lambda$ are linearly independent, the value at λ of any right pole function for H at λ of order greater than $\delta(h_{i+1},\lambda)$ is contained in span $\{[h_1]_\lambda, [h_2]_\lambda, \ldots, [h_i]_\lambda\}$. Thus, we can append $z^{-\delta(h_{i+1},\lambda)}h_{i+1}$ to the set $\{z^{-\delta(h_1,\lambda)}h_1,\ z^{-\delta(h_2,\lambda)}h_2,\ \ldots,\ z^{-\delta(h_i,\lambda)}h_i\}$. The lemma follows by induction.

□

In particular, if all the columns of H which have a pole at λ are orthogonal at λ, Lemma 3.4 specializes as follows.

Lemma 3.5 *Let H be an $m \times n$ rational matrix function function and suppose that $h_{i_1}, \ldots, h_{i_\kappa}$ are the columns of H which have a pole at λ. If $[h_{i_1}]_\lambda, [h_2]_\lambda, \ldots, [h_{i_\kappa}]_\lambda$ are linearly independent, then*

$$X = \{z^{-\delta(h_{i_1},\lambda)}h_{i_1}, z^{-\delta(h_{i_2},\lambda)}h_{i_2}, \ldots, z^{-\delta(h_{i_\kappa},\lambda)}h_{i_\kappa}\}$$

is a canonical set of right pole functions for H at λ.

Proof By Lemma 3.4 there is an extension of X to a canonical set of right pole functions for H at λ, $\tilde{X}$. Let k be a linear combination over polynomials in z^{-1} of the columns of H such that k has a pole at λ. Since the leading coefficients of the columns of h that have a pole at λ are linearly independent, $[k]_\lambda \in$ span $\{[h]_\lambda : h \in X\}$. It follows that $\tilde{X} = X$.

□

Lemma 3.5 has the following immediate consequence.

Corollary 3.6 *Let H be an $m \times n$ rational matrix function such that the leading coefficients of the columns of H that have a pole at λ are linearly independent. Then $\delta(H, \lambda)$ equals the sum of multiplicities of the poles at λ of the columns of H.*

We shall also need the following lemma.

Lemma 3.7 *Let H, K be $m \times n$ rational matrix functions which are right equivalent on $\mathbf{C}_\infty \backslash \{\lambda\}$ and let the columns of each of H, K be orthogonal at λ. Then $\delta(H) = \delta(K)$.*

Proof After applying a suitable Möbius transformation we, may assume $\lambda = \infty$. Then H and K are right equivalent on $\mathbf{C}$ and $H = KQ$ for some matrix polynomial Q such that $\det Q$ equals a nonzero constant. Let $i_1, i_2, ..., i_n$ be distinct integers such that the (i_j, j)'th entry of Q is a nonzero polynomial ($j = 1, 2, ..., n$). Then $\delta(h_j, \infty) \geq \delta(k_{i_j}, \infty)$. Using Corollary 3.6 we see that

$$\delta(H, \infty) = \sum_{j=1}^{n} \delta(h_j, \infty) \geq \sum_{j=1}^{n} \delta(k_{i_j}, \infty) = \sum_{i=1}^{n} \delta(k_i, \infty) = \delta(K, \infty).$$

We show similarly the inequality $\delta(K, \infty) \geq \delta(H, \infty)$. Thus, $\delta(H, \infty) = \delta(K, \infty)$. Since $\delta(H, \lambda) = \delta(K, \lambda)$ for every $\lambda \in \mathbf{C}$, $\delta(H) = \delta(K)$ as asserted.

□

We will show now that the columns of a rational matrix function K can be made orthogonal at any point λ of the extended complex plane without increasing the McMillan degree of K. The involved column operations, used by Forney in [F], do not affect the orthogonality of the columns of K on $\mathbf{C}_\infty \backslash \{\lambda\}$. Unlike the orthogonalization process utilized in the proof of Lemma 2.24, the operations in the proof of Theorem 3.8 do not change the span of the columns of K.

Theorem 3.8 *Let $K \in \mathcal{R}^{m\times n}$ and let $\lambda \in \mathbf{C}_\infty$. Then there exists an $H \in R^{m\times n}$ such that*

(i) H and K are right equivalent on $\mathbf{C}_\infty \backslash \{\lambda\}$,

(ii) $\delta(H) \leq \delta(K)$,

(iii) the columns of H are orthogonal at λ.

Proof After applying a suitable Möbius transformation, we may assume $\lambda = \infty$. Let $K = [\, k_1 \quad k_2 \quad \ldots \quad k_n \,]$ be an $m \times n$ rational matrix function with linearly dependent leading coefficients at infinity of nonzero columns. We assume without loss of generality that the columns of K are ordered according to decreasing degrees, where by the degree of a rational $m \times 1$ vector function k we understand $-\infty$ if $k = 0$, or the number η such that $z^{-\eta}k$ is analytic and nonzero at infinity if $k \neq 0$. It suffices to show that the McMillan degree of K does not increase due to a single operation which we will now describe. Let Φ be the collection of all submatrices $\left[\, k_{i_1} \quad k_{i_2} \quad \ldots \quad k_{i_{j_i}} \,\right]$ containing columns $k_{i_1}, k_{i_2}, \ldots, k_{i_{j_i}}$ of K whose leading coefficients at infinity $[k_{i_1}], \ldots, [k_{i_{j_i}}]$ form a linearly dependent subset of $\mathbf{C}^m$ that becomes linearly independent after removing any one element. Let p be the smallest integer such that the p'th column k_p of K is the last column of some matrix $\left[\, k_{i_1} \quad k_{i_2} \quad \ldots \quad k_p \,\right]$ in Φ. Clearly there is exactly one matrix in Φ with the last column k_p. The operation to be considered replaces k_{i_1} by

$$\tilde{k}_{i_1} = k_{i_1} - \alpha_{i_2} z^{\deg(k_{i_1})-\deg(k_{i_2})} k_{i_2} - \ldots - \alpha_p z^{\deg(k_{i_1})-\deg(k_p)} k_p \tag{8}$$

where $\alpha_{i_2}, \alpha_{i_3}, \ldots, \alpha_p$ are such that $[k_{i_1}]_\infty - \alpha_{i_2}[k_{i_2}]_\infty - \ldots - \alpha_p[k_p]_\infty = 0$. We note that such an operation can be carried out whenever the leading coefficients at infinity of nonzero columns of K are linearly dependent. Also, a finite number of such operations leads to a matrix function whose columns are orthogonal at infinity, that is a matrix function with linearly independent leading coefficients at infinity of nonzero columns (see Proposition 2.13). Indeed, let $q(z)$ be a scalar monic polynomial such that $K = (1/q)L$ for some matrix polynomial $L = [l_1, l_2, ..., l_n]$. Then $\tilde{k}_{i_1} = (1/q)\tilde{l}_{i_1}$ where

$$\tilde{l}_{i_1} = l_{i_1} - \alpha_{i_2} z^{\deg(l_{i_1})-\deg(l_{i_2})} l_{i_2} - \dots - \alpha_p z^{\deg(l_{i_1})-\deg(l_p)} l_p \tag{8'}$$

has lower degree than l_{i_1}. Plainly a finite number of operations like (8′) transforms L into a matrix polynomial with linearly independent leading coefficients of nonzero columns. Hence a finite number of operations like (8) transforms K into a rational matrix function with linearly independent leading coefficients at infinity of nonzero columns. Finally, the replacing of k_{i_1} by $\tilde{k}_{i_1}$ corresponds to multiplication on the right by a unimodular matrix polynomial, and so the resulting rational matrix function is right equivalent to K on $\mathbf{C}$. We denote the rational matrix function obtained by replacing the column k_{i_1} of K with $\tilde{k}_{i_1}$ by $\tilde{K}$.

Since K and $\tilde{K}$ are right equivalent on $\mathbf{C}$, we have $\delta(K, z) = \delta(\tilde{K}, z)$ for each $z \in \mathbf{C}$. We need to show $\delta(K, \infty) \geq \delta(\tilde{K}, \infty)$. We will show this by comparing canonical sets of right pole functions at infinity for K and $\tilde{K}$. We shall consider the case when k_p has a pole at infinity. The proof in the case when k_p is analytic at infinity is simpler.

By Lemma 3.4, we can choose a canonical set of right pole functions for K at infinity

$$X = \left\{ z^{-\delta(x_1,\infty)} x_1, \dots, z^{-\delta(x_{p-1},\infty)} x_{p-1}, z^{-\delta(x_{p+1},\infty)} x_{p+1}, \dots, z^{-\delta(x_s,\infty)} x_s \right\}$$

where $x_i = k_i$ if $i = 1, 2, ..., p-1$ and $z^{-\delta(x_i,\infty)} x_i$ is a right pole function for K at infinity of order $\delta(x_i, \infty)$ if $i \geq p+1$. We assume that $\delta(x_i, \infty) \geq \delta(x_j, \infty)$ whenever $i < j$. We may also assume that for each $i = p+1, ..., s$ x_i is a linear combination over polynomials in z^{-1} of the columns of K that have a pole at infinity. For notational convenience we put $x_p = k_p$.

Lemma 3.8.1 *For $1 \leq i \leq s$, $i \neq i_1$, $z^{-\delta(x_i,\infty)}x_i$ is a right pole function for $\tilde{K}$ at infinity.*

Proof If $h(z)$ is a pole function for K at infinity of order α, then $z^{\alpha}h(z)$ is a linear combination over polynomials in z^{-1} of columns of K which have a pole at infinity. If this linear combination does not contain k_{i_1}, then trivially $h(z)$ is also a pole function for $\tilde{K}(z)$ at infinity. Otherwise, the coefficient $q(z^{-1})$ of k_{i_1} has the form $z^{-\Delta}\tilde{q}(z^{-1})$ where $\Delta = \delta(x_{i_1}, \infty) - \delta(x_p, \infty)$ and $\tilde{q}$ is a polynomial. Then replace $q(z^{-1})k_{i_1}$ with

$$q(z^{-1})\tilde{k}_{i_1} + \alpha_{i_2}q(z^{-1})z^{\deg(k_{i_1})-\deg(k_{i_2})}k_{i_2} + \ldots + \alpha_p q(z^{-1})z^{\deg(k_{i_1})-\deg(k_p)}k_p. \tag{9}$$

Since the coefficients in (9) are polynomials in z^{-1}, we may conclude that $z^{\alpha}h(z)$is a linear combination (with coefficients equal to polynomials in z^{-1}) of columns of $\tilde{K}(z)$, and hence $h(z)$ is a pole function for $\tilde{K}(z)$ also in this case.

□

Continuation of proof of Theorem 3.8

While by Lemma 3.8.1 we know that $X\backslash\{z^{-\delta(x_{i_1},\infty)}x_{i_1}\}$ consists of pole functions for $\tilde{K}$ at infinity, it may happen that linear combinations (over polynomials in z^{-1}) of columns of $\tilde{K}$ (including the new column $\tilde{k}_{i_1}$) produce rational vector functions with a higher order pole at infinity. In this situation $X\backslash\{z^{-\delta(x_{i_1},\infty)}x_{i_1}\}$ is not a part of a canonical set of right pole functions for $\tilde{K}$ at $X\backslash\{z^{-\delta(x_{i_1},\infty)}x_{i_1}\}$. To overcome this difficulty, we will define a finite set $\{c_1, \ldots, c_\kappa\}$ of rational vector functions such that the set

$$X \cup \{z^{-\delta(x_p,\infty)}x_p\} \cup \{z^{-\delta(c_1,\infty)}c_1, \ldots, z^{-\delta(c_\kappa,\infty)}c_\kappa\}$$

contains a canonical set of right pole functions for $\tilde{K}$ at infinity. To define the $c_1, ..., c_\kappa$ put $c_1 = \tilde{k}_{i_1}$. Inductively, suppose that γ is a nonnegative integer and we are given a rational vector function c_γ. If c_γ is analytic at infinity or $[c_\gamma]_\infty \notin$ span $\{[x]_\infty : x \in X\}$, stop. Otherwise find the smallest integer j_γ such that $[c_\gamma]_\infty \in$ sp$\{[x_\eta]_\infty : 1 \le \eta \le j_\gamma, \eta \ne i_1\}$, choose numbers $\alpha_\eta (1 \le \eta \le j_\gamma, \eta \ne i_1)$ such that

$$[c_\gamma]_\infty = \alpha_1[x_1]_\infty + ... + \alpha_{i_1-1}[x_{i_1-1}]_\infty + \alpha_{i_1+1}[x_{i_1+1}]_\infty + ... + \alpha_{j_\gamma}[x_{j_\gamma}]_\infty$$

and put

$$c_{\gamma+1} = z^{\zeta_\gamma - \delta(c_\gamma,\infty)}c_\gamma - \alpha_1 z^{\zeta_\gamma-\delta(x_1,\infty)}x_1 - ... - \alpha_{j_\gamma} z^{\zeta_\gamma - \delta(x_{j_\gamma},\infty)}x_{j_\gamma} \tag{10}$$

where $\zeta_\gamma = \min\{\delta(c_\gamma,\infty), \delta(x_{j_\gamma},\infty)\}$. For the sake of definiteness we assume that $\delta(c_\kappa, \infty)$ is positive.

Let $\nu_1, ..., \nu_r$ with $1 \le \nu_1 < ... < \nu_r < \kappa$ be all the integers such that $\delta(c_{\nu_i}, \infty) > \delta(x_{j_{\nu_i}}, \infty)$ $(i = 1, ..., r)$.

Lemma 3.8.2 *Let $c_1, ..., c_\kappa$ be defined as above. Then the set of rational vector functions*

$$\tilde{X} = ((X \cup \{z^{-\delta(x_p,\infty)}x_p\})\backslash\{z^{-\delta(x_{i_1},\infty)}x_{i_1}, z^{-\delta(x_{j_{\nu_1}},\infty)}x_{j_{\nu_1}}, \ldots, z^{-\delta(x_{j_{\nu_r}},\infty)}x_{j_{\nu_r}}\})$$
$$\cup \{z^{-\delta(c_{\nu_1},\infty)}c_{\nu_1}, \ldots, z^{-\delta(c_{\nu_r},\infty)}c_{\nu_r}, z^{-\delta(c_\kappa,\infty)}c_\kappa\}$$

is a canonical set of right pole functions for $\tilde{K}$ at ∞.

Proof We argue by contradiction. Let l be the largest integer such that the l'th column of $\tilde{K}$ has a pole at infinity and suppose there exist scalar polynomials $q_1, ..., q_l$ such that

$$\phi(z) = q_1(z^{-1})k_1 + ... + q_{i_1}(z^{-1})\tilde{k}_{i_1} + q_{i_1+1}(z^{-1})k_{i_1+1} + ... + q_l(z^{-1})k_l \tag{11}$$

has a pole at infinity and either

$$[\phi]_\infty \notin \text{sp}\{[x]_\infty : x \in \tilde{X}\} \tag{12}$$

or

$$\begin{aligned}&\text{there is a right pole function } y \in \tilde{X} \text{ for } \tilde{K} \text{ at } \infty\\&\text{of order less than } \delta(\phi,\infty) \text{ such that}\\&[y]_\infty \in \mathrm{sp}(\{[\phi]_\infty\} \cup \{[x]_\infty : x \in \tilde{X}, x \neq y\}).\end{aligned} \tag{13}$$

We will show that there exists an integer $\mu \leq \kappa$ and scalar polynomials $q_1^{(\mu)}, \ldots, q_{s-p}^{(\mu)}$ such that

$$\begin{aligned}\phi(z) =& q_1^{(\mu)}(z^{-1})k_1(z) + \ldots + q_{i_1}^{(\mu)}(z^{-1})c_\mu(z) + q_{i_1+1}^{(\mu)}(z^{-1})k_{i_1+1}(z) + \ldots\\&+ q_l^{(\mu)}(z^{-1})k_l(z) + q_{l+1}^{(\mu)}(z^{-1})x_{p+1}(z) + \ldots + q_{s-p}^{(\mu)}(z^{-1})x_s(z)\end{aligned} \tag{14}$$

and

$$\delta(q_{i_1}^{(\mu)}(z^{-1})c_\mu(z), \infty) \leq \delta(\phi(z), \infty). \tag{15}$$

Note that by definition $c_1 = \tilde{k}_{i_1}$; hence, if $\delta(q_{i_1}(z^{-1})\tilde{k}_{i_1}(z), \infty) \leq \delta(\phi(z), \infty)$, then by (11) we may take $\mu = 1$ and have (14) and (15) satisfied. If $\delta(q_{i_1}(z^{-1})\tilde{k}_{i_1}(z), \infty) > \delta(\phi(z), \infty)$, we proceed by induction. Let $\gamma \leq \kappa$ be a positive integer and suppose that

$$\begin{aligned}\phi(z) =& q_1^{(\gamma)}(z^{-1})k_1(z) + \ldots + q_{i_1}^{(\gamma)}(z^{-1})c_\gamma(z) + q_{i_1+1}^{(\gamma)}(z^{-1})k_{i_1+1}(z) + \ldots\\&+ q_l^{(\gamma)}(z^{-1})k_l(z) + q_{l+1}^{(\gamma)}(z^{-1})x_{p+1}(z) + \ldots + q_{s-p}^{(\gamma)}(z^{-1})x_s(z)\end{aligned} \tag{16}$$

for some scalar polynomials $q_j^{(\gamma)}$. We note that (16) can be obtained directly from (11) when $\gamma = 1$. Suppose $\delta(q_{i_1}^{(\gamma)}(z^{-1})c_\gamma(z), \infty) > \delta(\phi, \infty)$. Then the leading terms in the Laurent expansions at infinity of $q_{i_1}^{(\gamma)}(z^{-1})c_\gamma(z)$ and $(-\phi(z) + q_{i_1}^{(\gamma)}(z^{-1})c_\gamma(z))$ are the same. In particular, $[c_\gamma]_\infty \in \mathrm{sp}\{[x]_\infty : x \in X\}$. Since we are assuming that $\delta(c_\kappa, \infty) > 0$, by the construction necessarily $[c_\kappa]_\infty \notin \mathrm{sp}\{[x]_\infty : x \in X\}$; hence $\gamma < \kappa$. Let $q_{i_1}^{(\gamma)}(z) = a_t z^t + a_{t+1}z^{t+1} + \ldots + a_\vartheta z^\vartheta$, $a_t \neq 0$. By construction, $\delta(c_\gamma, \infty) - \zeta_\gamma$ is the smallest integer η such that the leading term in the Laurent series at infinity for $z^{-\eta}c^\gamma(z)$ coincides with the leading term of the Laurent series at infinity for some linear combination (over polynomials in z^{-1}) of $x_1, \ldots, x_{i_1-1}, x_{i_1+1}, \ldots, x_s$; hence necessarily $t \geq \eta = \delta(c_\gamma, \infty) - \zeta_\gamma$. Hence by (10) we see that $q_{i_1}^{(\gamma)}(z^{-1})c_\gamma(z)$ is a linear combination over polynomials in z^{-1} of $x_1, ..., x_{i_1-1}, x_{i_1+1}, ...x_s, c_{\gamma+1}$. Consequently, we obtain a formula of the form (14) with $\gamma+1$ in place of γ. It follows by induction that there exists an integer $\mu \leq \kappa$ such that (14) and (15) hold.

If the inequality (15) is strict, let $\psi(z) = \phi(z) - q_{i_1}^{(\mu)}(z^{-1})c_\mu(z)$. Then $z^{-\delta(\psi,\infty)}\psi(z)$ is a right pole function for K at infinity. Since X is a canonical set of right pole functions for K at infinity, neither (12) nor (13) can happen, a contradiction. Next suppose we have equality in (15). Since $q_{i_1}^{(\mu)}$ is a polynomial, certainly $\delta(c_\mu, \infty) \geq \delta(q_{i_1}^{(\mu)}(z^{-1})c_\mu(z), \infty) = \delta(\phi(z), \infty)$. Hence if $[\phi]_\infty = [c_\mu]_\infty$, (12) is not possible since $[c_\mu]_\infty$ is in $\mathrm{sp}\{[x]_\infty : x \in \tilde{X}\}$ by the choice of $\tilde{X}$. Since either $z^{-\delta(c_\mu,\infty)}c_\mu$ is in $\tilde{X}$ or $[c_\mu]_\infty$ is in the span of leading coefficients at infinity of pole functions in $\tilde{X}$ of order at least $\delta(c_\mu, \infty)$, (13) is also not possible. Suppose $[\phi]_\infty \neq [c_\mu]_\infty$. If we let $\psi(z) = \phi(z) - q_{i_1}^{(\mu)}(z^{-1})c_\mu(z)$, then $\delta(\psi(z), \infty) = \delta(\phi(z), \infty)$ and $z^{-\delta(\psi,\infty)}\psi(z)$ is a right pole function for both $\tilde{K}$ and K at infinity. Since X is a canonical set of right pole functions for K at infinity, we have $[\psi]_\infty \in \mathrm{sp}\{[x_1]_\infty, ..., [x_{i_1-1}]_\infty, [x_{i_1+1}]_\infty, ..., [x_p]_\infty, ..., [x_j]_\infty\}$ for some j such that $\delta(x_j, \infty) \geq \delta(\psi, \infty)$. Note that $\mathrm{sp}\{[x]_\infty : x \in X\} \subset \mathrm{sp}\{[x]_\infty : x \in \tilde{X}\}$ by the choice of $\tilde{X}$. Hence (12) in this case is not possible. Since

$$[\psi]_\infty \in \mathrm{sp}\{[x_1]_\infty, ..., [x_{i_1-1}]_\infty, [x_{i_1+1}]_\infty, ..., [x_j]_\infty, [c_\mu]_\infty\}$$

where $\delta(p, \infty) \geq \delta(\phi, \infty)$ for $p = x_1, ..., x_{i_1-1}, x_{i_1+1}, ..., x_j, c_\mu$, (13) is not possible either.

□

Continuation of proof of Theorem 3.8

Since partial multiplicities of a pole at infinity of any rational matrix function W are equal to the orders of the functions in a canonical set of right pole functions for

W at infinity, comparing X and $\tilde{X}$ we see that

$$\begin{aligned}
&\delta(K,\infty)-\delta(\tilde{K},\infty)\geq \\
&\qquad \geq \delta(k_{i_1},\infty)+\sum_{i=1}^{r}\delta(x_{j_{\nu_i}},\infty)-\delta(x_p,\infty)-\sum_{i=1}^{r}\delta(c_{\nu_i},\infty)-\delta(c_\kappa,\infty) \\
&\qquad = [\delta(k_{i_1},\infty)-\delta(x_p,\infty)]-\sum_{i=1}^{r}[\delta(c_{\nu_i},\infty)-\delta(x_{j_{\nu_i}},\infty)]-\delta(c_\kappa,\infty) \\
&\qquad \equiv \Delta-\sum_{i=1}^{r}\Delta_i-\Delta_{r+1}.
\end{aligned}$$

Suppose that $\Delta-\sum_{i=1}^{r}\Delta_i<0$ and let μ be the smallest integer such that $\sum_{i=1}^{\mu}\Delta_i>\Delta$. Then

$$\begin{aligned}
\delta(z^{\Delta_1+\dots+\Delta_{\mu-1}-\Delta}c_{\nu_\mu},\infty)-\delta(x_{j_\mu},\infty) &= \sum_{i=1}^{\mu-1}\Delta_i-\Delta+\delta(c_{\nu_\mu},\infty)-\delta(x_{j_\mu},\infty) \\
&= \sum_{i=1}^{\mu}\Delta_i-\Delta \\
&> 0.
\end{aligned}$$

By the choice of j_μ,

$$\mathrm{sp}(\{[x_i]_\infty:\ i<j_{\nu_\mu}\}\cup\{[c_{\nu_\mu}]_\infty\})=\mathrm{sp}\{[x_i]_\infty:\ i\leq j_{\nu_\mu}\}.$$

Also, since $z^{-\Delta}c_1(z)=z^{-\Delta}\tilde{k}_{i_1}(z)$ is a linear combination over polynomials in z^{-1} of the columns of K, we see from (10) that $z^{\Delta_1+\dots+\Delta_{\mu-1}-\Delta}c_{\nu_\mu}$ is such a combination. It follows that X is not a canonical set of right pole functions for K at infinity, a contradiction. Thus, $\Delta-\sum_{i=1}^{r}\Delta_i\geq 0$. Similarly, $\Delta-\sum_{i=1}^{r+1}\Delta_i<0$ implies that $z^{\Delta_1+\dots+\Delta_r-\Delta}c_\kappa$ has a pole at infinity. Since $z^{\Delta_1+\dots+\Delta_r-\Delta}c_\kappa$ is a linear combination over polynomials in z^{-1} of the columns of K, this is a contradiction. It follows that $\delta(K,\infty)\geq\delta(\tilde{K},\infty)$ and the proof is complete.

□

In the proof of the next theorem we will need the following lemma.

Lemma 3.9 *Let $K\in\mathcal{R}^{m\times n}$ and let $\lambda_1,\lambda_2\in\mathbf{C}_\infty$. Then there exists a rational matrix function H such that* *(i) H and K are right equivalent on $\mathbf{C}_\infty \backslash \{\lambda_1, \lambda_2\}$,*

(ii) $\delta(H) \leq \delta(K)$,

(iii) H has neither a pole nor a zero at λ_1.

Proof It follows from Theorem 3.8 that there exists a rational matrix function W such that $\delta(W) \leq \delta(K)$, W and K are right equivalent on $\mathbf{C}_\infty \backslash \{\lambda_1, \lambda_2\}$ and the nonzero columns of W are orthogonal on $\{\lambda_1, \lambda_2\}$. For notational convenience we assume $\lambda_1, \lambda_2 \neq \infty$. Choose $\alpha_1, \alpha_2, ..., \alpha_n$ and $\beta_1, \beta_2, ..., \beta_n$ such that $W(z)$ is equal to

$$\tilde{W}(z) \begin{bmatrix} (z-\lambda_1)^{\alpha_1}(z-\lambda_2)^{\beta_1} & & & \\ & (z-\lambda_1)^{\alpha_2}(z-\lambda_2)^{\beta_2} & & \\ & & \ddots & \\ & & & (z-\lambda_1)^{\alpha_n}(z-\lambda_2)^{\beta_n} \end{bmatrix}$$

and all columns of $\tilde{W}$ are analytic and nonzero on $\{\lambda_1, \lambda_2\}$. Define a rational matrix function H by

$$H(z) = \tilde{W}(z) \begin{bmatrix} (z-\lambda_2)^{\alpha_1+\beta_1} & & & \\ & (z-\lambda_2)^{\alpha_2+\beta_2} & & \\ & & \ddots & \\ & & & (z-\lambda_2)^{\alpha_n+\beta_n} \end{bmatrix}.$$

Then H and K are right equivalent on $\mathbf{C}_\infty \backslash \{\lambda_1, \lambda_2\}$. By Corollary 2.20 H has neither a zero nor a pole at λ_1 and

$$\begin{aligned} \delta(H, \lambda_1) + \delta(H, \lambda_2) &= \delta(H, \lambda_2) \\ &= \sum_{\substack{1 \leq i \leq n \\ \alpha_i + \beta i > 0}} (\alpha_i + \beta i) \\ &\leq \sum_{\substack{1 \leq i \leq n \\ \alpha_i > 0}} \alpha_i + \sum_{\substack{1 \leq i \leq n \\ \beta_i > 0}} \beta_i \\ &= \delta(W, \lambda_1) + \delta(W, \lambda_2). \end{aligned}$$

So $\delta(H) \leq \delta(K)$.

$\square$

We can now prove the following theorem.

Theorem 3.10 *Let $W, H \in \mathcal{R}^{m\times n}$ be right equivalent on $\sigma \subset \mathbf{C}_\infty$ and let $\lambda \in \mathbf{C}_\infty\backslash\sigma$. If*

(i) H has no zeros nor poles in $\mathbf{C}_\infty\backslash(\sigma \cup \{\lambda\})$ and

(ii) the columns of H are orthogonal at λ

then H has the minimal McMillan degree among all rational matrix functions which are right equivalent to W on σ.

Proof Let H have the properties (i) and (ii) and let K be a rational matrix function which is right equivalent to W on σ. We show that $\delta(H) \le \delta(K)$. Applying Lemma 3.9 to K a finite number of times, we find a rational matrix function K_1 such that K_1 is right equivalent to W on σ, $\delta(K_1) \le \delta(K)$ and K_1 has no poles nor zeros in $\mathbf{C}_\infty\backslash(\sigma \cup \{\lambda\})$. Applying Theorem 3.8 to K_1, we find a rational matrix function K_2 such that $\delta(K_2) \le \delta(K_1)$, K_1 and K_2 are right equivalent on $\mathbf{C}_\infty\backslash(\sigma \cup \{\lambda\})$ and the columns of K_2 are orthogonal at λ. Then K_2 and H are right equivalent on $\mathbf{C}_\infty\backslash\{\lambda\}$ and it follows from Lemma 3.7 that $\delta(H) = \delta(K_2)$. So $\delta(H) \le \delta(K)$ as asserted.

$\square$

Considering the special cases when $\sigma = \emptyset$ or $\sigma = \mathbf{C}$, we obtain the following two corollaries.

Corollary 3.11 *A matrix polynomial whose columns form a minimal polynomial basis for a subspace V of $\mathcal{R}^m$ has the minimal McMillan degree among all rational matrix functions with the column space V.*

Corollary 3.12 *Let W be an $m \times n$ matrix polynomial and let H be a matrix polynomial that is right equivalent to W on $\mathbf{C}$ and whose columns are orthogonal at infinity. Then H has the minimal McMillan degree among all matrix polynomials with the same left zero structure on $\mathbf{C}$ as W.*

Theorem 3.10 gives rise to the algorithm for finding a minimal McMillan degree rational matrix function which is right equivalent to a given matrix function on $\sigma \subset \mathbf{C}_\infty$. Such an algorithm is described, together with an example, in [BR].

We conclude this chapter with two propositions which form an analogue of Theorem 3.8. In view of Corollary 2.20, the proof of Proposition 3.13 allows us to find the partial multiplicities of the pole and zero of a rational matrix function K at $\lambda \in \mathbf{C}$ without finding a Smith-McMillan factorization of K and without finding canonical sets of right pole and left null functions for K at λ (cf. [VK]).

Proposition 3.13 *Let $K \in \mathcal{R}^{m \times n}$ and let $\lambda \in \mathbf{C}$. Then there exists a rational matrix function H such that*

(i) H and K are right equivalent on $\mathbf{C}$,

(ii) the columns of H are orthogonal at λ.

Proof If nonzero columns $k_{i_1}, k_{i_2}, ..., k_{i_s}$ of K are not orthogonal at λ, find numbers $\alpha_1, \alpha_2, ..., \alpha_s$ such that $\alpha_1[k_{i_1}]_\lambda + \alpha_2[k_{i_2}]_\lambda + ... + \alpha_s[k_{i_s}]_\lambda = 0$. Then choose k_{i_q} such that $\|k_{i_q}\|_{z=\lambda} = \min \{\|k_{i_j}\|_{z=\lambda} : 1 \leq j \leq s\}$ and replace k_{i_q} by

$$\tilde{k}_{i_q} = \sum_{j=1}^{s} \alpha_j (z - \lambda)^{\alpha_j} k_{i_j}$$

where α_j $(1 \leq j \leq s)$ is such that $\|(z-\lambda)^{\alpha_j} k_{i_j}\|_{z=\lambda} = \|k_{i_j}\|_{z=\lambda}$ (cf. formula (8) in the proof of Theorem 3.8). Since $\alpha_j > 0$ $(1 \leq j \leq s)$, this operation corresponds to multiplication on the right by a unimodular matrix polynomial. By an argument as in the proof of Theorem 3.8, a finite number of such operations yields a rational matrix function H with columns orthogonal at λ.

□

Proposition 3.14 *Let $K \in \mathcal{R}^{m \times n}$ and let $\lambda \in \mathbf{C}$. Then there exists a rational matrix function H such that* *(i) H and K are right equivalent on* $\mathbf{C}$,

(ii) $\delta(H) \leq \delta(K)$,

(iii) the columns of H are orthogonal at λ.

Proof In view of Theorem 3.10, it suffices to multiply the rational matrix function H obtained as in Proposition 3.13 by a unimodular matrix polynomial such that the columns of H become, after multiplication, orthogonal at infinity.

□

Chapter IV

Local inverse spectral problem

In this chapter we solve the local inverse spectral problem for not necessarily regular rational matrix functions. The problem is as follows. Suppose we are given a triple of matrices $(A_\kappa, B_\kappa, D_\kappa)$ and two pairs of matrices (C_π, A_π) and (A_ζ, B_ζ). Under what conditions does there exist a rational matrix function W such that $(A_\kappa, B_\kappa, D_\kappa)$ is a left kernel triple for W, (A_ζ, B_ζ) is a left null pair for W over a subset σ of $\mathbf{C}$ and (C_π, A_π) is a right pole pair for W over σ? In other words, does there exist a rational matrix function W with a given left zero and right pole structure?

It has been shown in [Ro] that there always exists an analytic matrix function W with a given left zero structure. In fact, W can be taken to be a rational matrix function (see [BCRo]). The solution of the local inverse spectral problem has been known also in the case when the triple $(A_\kappa, B_\kappa, D_\kappa)$ is vacuous: if the pairs (A_ζ, B_ζ) and (C_π, A_π) are given and $\sigma \subset \mathbf{C}$ contains $\sigma(A_\zeta) \cup \sigma(A_\pi)$, then there exists a right invertible rational matrix function W such that (A_ζ, B_ζ) and (C_π, A_π) are left zero and right pole pairs for W over σ if and only if the pair (A_ζ, B_ζ) is controllable, the pair (C_π, A_π) is observable, and the Sylvester equation

$$SA_\pi - A_\zeta S = B_\zeta C_\pi \tag{1}$$

has a solution (see [GK2]). Moreover, for any solution Γ of equation (1) there exists a regular rational matrix function W with a left σ-spectral triple $\{(C_\pi, A_\pi),(A_\zeta, B_\zeta), \Gamma\}$. In this chapter we generalize these results to the case when the left kernel triple is nonvacuous.

Chapter IV contains two sections. In Section 4.1 we solve the basic local inverse spectral problem, that is we prove a necessary and sufficient condition for existence of a rational matrix function with a given right pole and left zero structure over $\sigma \subset \mathbf{C}$. In Section 4.2 we show that if (C_π, A_π), (A_ζ, B_ζ) and $(A_\kappa, B_\kappa, D_\kappa)$ satisfy certain normalization conditions and if Γ is any solution of equation (1), then there exists a rational matrix function W with a left kernel triple–$(A_\kappa, B_\kappa, D_\kappa)$ and a left spectral triple over σ–$\{(C_\pi, A_\pi), (A_\zeta, B_\zeta), \Gamma\}$.

4.1 Basic local inverse problem

Right pole and left null pairs and a left kernel triple for a rational matrix function have to satisfy certain obvious conditions. We summarize these conditions in the following proposition.

Proposition 4.1 *Suppose (C_π, A_π) and (A_ζ, B_ζ) are right pole and left null pairs over $\sigma \subset \mathbf{C}$ for a rational matrix function W and $(A_\kappa, B_\kappa, D_\kappa)$ is a left kernel triple for W. Then*

(i) the pair (C_π, A_π) is observable and $\sigma(A_\pi) \subset \sigma$,

(ii) the pair (A_ζ, B_ζ) is controllable and $\sigma(A_\zeta) \subset \sigma$,

(iii) A_κ is in Jordan form, $\sigma(A_\kappa) \subset \{0\}$, and the matrix polynomial P corresponding to $(A_\kappa, B_\kappa, D_\kappa)$ is such that P has full row rank at every $\lambda \in \mathbf{C}$ and the columns of P are orthogonal at infinity,

(iv) the rational matrix function $P(z)C_\pi(z - A_\pi)^{-1}$ is analytic on $\mathbf{C}$,

(v) if λ is an eigenvalue of A_ζ, $SA_\zeta S^{-1}$ is a Jordan form of A_ζ and $b_1, b_2, \ldots, b_r$ are the rows of SB_ζ corresponding to the last rows in Jordan blocks of $SA_\zeta S^{-1}$ with λ on the diagonal, then span $\{b_1, b_2, \ldots, b_r\}$ intersects trivially with $W^{ol}(\lambda)$, where W^{ol} is the row space of P.

Proof Assertions (i)-(iii) follow from the definitions of a right pole pair for W over σ, left null pair for W over σ, and a left kernel triple for W. To show (iv) note that after multiplying $P(z)C_\pi(z-A_\pi)^{-1}$ on the right by a constant invertible matrix, we may assume that A_π is in Jordan form and the columns of C_π contain coefficients of the Taylor expansions of right pole functions for W at the corresponding eigenvalues of A_π. We may also assume without loss of generality that A_π consists of a single Jordan cell of size $k \times k$ with λ on the diagonal and $P(z) = \sum_{i=0}^{l}(z-\lambda)^i P_i$ is a polynomial row vector function. Let the columns of C_π be $C_0, C_1, \ldots C_{k-1}$. Then there exist vectors $C_k, C_{k+1}, \ldots$ such that $\psi(z) = \sum_{j=1}^{\infty}(z-\lambda)^j C_j$ is a right pole function for W at λ. Since P annihilates ψ,

$$\sum_{\substack{i+j=\nu \\ 0\leq i\leq l \\ j\geq 0}} P_i C_{\nu-j} = 0 \qquad (\nu = 0, 1, \ldots).$$

Hence

$$P(z)C_\pi(z-A_\pi)^{-1} = \left(\sum_{i=1}^{l}(z-\lambda)^i P_i\right)\left[(z-\lambda)^{-1}C_0 \quad (z-\lambda)^{-2}C_0 + (z-\lambda)^{-1}C_1 \right.$$
$$\left. \ldots \quad (z-\lambda)^{-k}C_0 + (z-\lambda)^{-k+1}C_1 + \ldots + (z-\lambda)^{-1}C_{k-1}\right]$$

is analytic at λ and (iv) is established. Finally, assertion (v) follows from condition (iii) in the definition of a canonical set of left null functions for W at λ.

□

We can now state the solution of the local inverse spectral problem. The proof will be completed with the proof of Theorem 4.8 below.

Theorem 4.2 *Let $\sigma \subset \mathbf{C}$ and let (C_π, A_π), (A_ζ, B_ζ) and $(A_\kappa, B_\kappa, D_\kappa)$ satisfy conditions (i)-(v) in Proposition 4.1. Then there exists a rational matrix function W with the right pole and left null pairs over σ (C_π, A_π) and (A_ζ, B_ζ) and with the left kernel triple $(A_\kappa, B_\kappa, D_\kappa)$ if and only if the matrix equation in Γ*

$$\Gamma A_\pi - A_\zeta \Gamma = B_\zeta C_\pi \tag{2}$$

has a solution.

Proof If (C_π, A_π) is a right pole pair for a rational matrix function W over σ and (A_ζ, B_ζ) is a left null pair for W over σ, then clearly equation (2) has a solution. In fact, a null-pole coupling matrix for (C_π, A_π) and (A_ζ, B_ζ) as right pole and left null pairs for W over σ is one such solution.

Suppose equation (2) has a solution. Then, since (C_π, A_π) and (A_ζ, B_ζ) satisfy conditions (i) and (ii) in Proposition 4.1, by Theorem 2.3 in [GK2], there exists a regular rational matrix function H with the right pole and left null pairs over σ (C_π, A_π) and (A_ζ, B_ζ). Since (C_π, A_π), (A_ζ, B_ζ) and $(A_\kappa, B_\kappa, D_\kappa)$ satisfy conditions (iii)-(v) in Proposition 4.1, by Theorem 4.8 below, there exists a rational matrix function W with a left kernel triple–$(A_\kappa, B_\kappa, D_\kappa)$ and right pole and left null pairs over σ–(C_π, A_π) and (A_ζ, B_ζ).

□

We shall need below the following properties of orthogonal projections in $\mathcal{R}^n$.

Proposition 4.3 *Let σ be a subset of $\mathbf{C}_\infty$ and let Λ, Ω be orthogonal complements in $(\mathcal{R}^n, \sigma)$. If $h \in \mathcal{R}^n$ is analytic on σ then the projection of h onto Λ along Ω is analytic on σ.*

Proof Let $h_\Lambda \in \Lambda$ and $h_\Omega \in \Omega$ be such that $h = h_\Lambda + h_\Omega$ and suppose that h_Λ has a pole at $\lambda \in \sigma$. Then, since $[h_\Lambda]_\lambda$ and $[h_\Omega]_\lambda$ are linearly independent, h has a pole at λ, a contradiction.

□

Proposition 4.4 *Let $\lambda \in \mathbf{C}_\infty$ and let Λ, Ω be orthogonal complements in $(\mathcal{R}^n, \{\lambda\})$.*

Let $h \in \mathcal{R}^n$ have the Laurent expansion at λ

$$\begin{cases} \sum_{j=k}^{\infty}(z-\lambda)^j h_j, & \text{if } \lambda \in \mathbf{C} \\ \sum_{j=-\infty}^{k} z^j h_j, & \text{if } \lambda = \infty \end{cases}$$

with $h_k \neq 0$. If h_Λ is the projection of h onto Λ along Ω, then

$$h_\Lambda(z) = \begin{cases} \sum_{j=k}^{\infty}(z-\lambda)^j h_{\Lambda j}, & \text{if } \lambda \in \mathbf{C} \\ \sum_{j=-\infty}^{k} z^j h_{\Lambda j}, & \text{if } \lambda = \infty \end{cases} \tag{3}$$

where $h_{\Lambda k}$ is the projection of h_k along $\Omega(\lambda)$ onto $\Lambda(\lambda)$.

Proof In view of Proposition 4.3, h_Λ can be represented as in (3). Similarly, the projection h_Ω of h along Λ onto Ω has the Laurent expansion at λ

$$h_\Omega(z) = \begin{cases} \sum_{j=k}^{\infty}(z-\lambda)^j h_{\Omega j}, & \text{if } \lambda \in \mathbf{C}\ , \\ \sum_{j=-\infty}^{k} z^j h_{\Omega j}, & \text{if } \lambda = \infty\ . \end{cases}$$

Since $h_\Lambda \in \Lambda$ and $h_\Omega \in \Omega$, $h_{\Lambda k} \in \Lambda(\lambda)$ and $h_{\Omega k} \in \Omega(\lambda)$. (We note that $h_{\Lambda k}$ or $h_{\Omega k}$ may be equal to 0.) Since $h_k = h_{\Lambda k} + h_{\Omega k}$, the assertion follows.

□

Corollary 4.5 *Let $\lambda \in \mathbf{C}_\infty$ and let Λ, Ω be orthogonal complements in $(\mathcal{R}^n, \{\lambda\})$. Let $h \in \mathcal{R}^n$ and let h_λ be the projection of h along Ω onto Λ. If $[h]_{z=\lambda} \notin \Omega(\lambda)$, then*

$$\|h\|_{z=\lambda} = \|h_\Lambda\|_{z=\lambda}.$$

Let $\lambda \in \mathbf{C}_\infty$, let Λ and Ω be orthogonal complements in $(\mathcal{R}^{1\times m}, \{\lambda\})$, and let $h \in \mathcal{R}^{m\times 1}$. In view of Theorem 2.11, if h has a pole of multiplicity 1 at λ and $[h]_\lambda \in \Lambda^\circ(\lambda)$, then Proposition 4.4 implies that the projection of h along Ω° onto Λ° does not affect the singular part at λ of h. This observation can be generalized as follows.

Proposition 4.6 *Let $\lambda \in \mathbf{C}_\infty$ and suppose that the rows of a rational matrix function P form an orthonormal basis for a subspace Λ of $(\mathcal{R}^{1\times m}, \{\lambda\})$. Let $h \in \mathcal{R}^{m\times 1}$ and let h_{Λ° be a projection of h onto Λ° along some orthogonal complement Ω° of Λ° in $(\mathcal{R}^{m\times 1}, \{\lambda\})$. Then $h - h_{\Lambda^\circ}$ is analytic at λ whenever Ph is analytic at λ.*

Proof We have $h = h_{\Lambda^\circ} + h_{\Omega^\circ}$ with $h_{\Omega^\circ} \in \Omega^\circ$. Suppose h_{Ω° has a pole at λ. Then $[h_{\Omega^\circ}]_\lambda \in \Omega^\circ(\lambda)$, so $P(\lambda)[h_{\Omega^\circ}]_\lambda \neq 0$. Hence

$$P(h_{\Lambda^\circ} + h_{\Omega^\circ}) = Ph_{\Omega^\circ}$$

has a pole at λ. Since Ph is analytic at λ, this is a contradiction.

□

We shall also need below the following characterization of a canonical set of left null functions at $\lambda \in \mathbf{C}_\infty$ of a rational matrix function W.

Proposition 4.7 *Let W be an $m \times n$ rational matrix function, let V be the linear span over $\mathcal{R}$ of the columns of W and let $\lambda \in \mathbf{C}_\infty$. Suppose $\Xi_1, \Xi_2, ..., \Xi_l \in \mathcal{R}^{1\times m}$ have no poles at λ and for $i = 1, 2, ..., l$ $\Xi_i W$ has a zero at λ of order k_i, $k_i > 0$. If*

(i) $\Xi_1(\lambda), \Xi_2(\lambda), ..., \Xi_l(\lambda)$ are linearly independent,

(ii) total multiplicity of the zero of W at λ equals $\sum_{i=1}^{l} k_i$,

(iii) dim $(\Xi_1(\lambda)V(\lambda) + \Xi_2(\lambda)V(\lambda) + ... + \Xi_l(\lambda)V(\lambda)) = l$,

then Ξ_i is a left null function for W at λ of order k_i $(i = 1, 2, ..., l)$ and $\{\Xi_1, \Xi_2, ..., \Xi_l\}$ is a canonical set of left null functions for W at λ.

Proof Conditions (i) and (iii) imply that the subspace of $\mathcal{R}^{1\times m}$ spanned by $\Xi_1, \Xi_2, ..., \Xi_l$ is orthogonal to W^{ol} at λ. Hence each Ξ_i is a left null function for W at λ of order k_i and, by (ii), $\{\Xi_1, \Xi_2, ..., \Xi_l\}$ is a canonical set of left null functions for W at λ.

□

We can show now how to construct from a regular rational matrix function H a left invertible rational matrix function W with the same left null and right pole pairs over $\sigma \subset \mathbf{C}$ as H and a given left kernel triple $(A_\kappa, B_\kappa, D_\kappa)$.

Theorem 4.8 *Let H be a regular $m \times m$ rational matrix function with right pole and left null pairs over $\sigma \subset \mathbf{C}$ (C_π, A_π) and (A_ζ, B_ζ), respectively. If $(A_\kappa, B_\kappa, D_\kappa)$ is a triple of matrices which satisfy conditions (iii)-(v) in Proposition 4.1, then a left invertible rational matrix function W with right pole and left null pairs over σ (C_π, A_π) and (A_ζ, B_ζ) and a left kernel triple $(A_\kappa, B_\kappa, D_\kappa)$ can be constructed as follows.*

Step 1 *Find a Smith-McMillan factorization EDF of H and let $W_1 = ED$.*

Step 2 *Let d_i denote the i'th diagonal entry of D $(i = 1, 2, ..., m)$ and let ν and μ be the largest geometric multiplicity of a pole and of a zero of W_1 in σ respectively. Let η be the largest integer such that the sum of the geometric multiplicity of a pole and the geometric multiplicity of a zero of W_2 at some point $\lambda \in \sigma$ equals η. Define two $m \times \eta$ rational matrix functions V_L and V_R by*

$$V_L = [\, d_1 E_1 \quad d_2 E_2 \quad \ldots \quad d_\nu E_\nu \quad 0 \quad \ldots \quad 0\,],$$

$$V_R = \left[\, 0 \quad \ldots \quad 0 \quad d_{m-\mu+1} E_{m-\mu+1} \quad d_{m-\mu+2} E_{m-\mu+2} \quad \ldots \quad d_m E_m \,\right]$$

and let p_i $(i = \eta - \mu + 1, \eta - \mu + 2, ..., \nu)$ be the minimal degree monic polynomials such that p_i times the i'th column of V_L has the same zeros with the same multiplicities as the i'th column of V_R. We put

$$W_2 = V_L \begin{bmatrix} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & p_{\eta-\mu+1} & & & & & & \\ & & & p_{\eta-\mu+2} & & & & & \\ & & & & \ddots & & & & \\ & & & & & p_\nu & & & \\ & & & & & & 1 & & \\ & & & & & & & \ddots & \\ & & & & & & & & 1 \end{bmatrix} + V_R. \qquad (4)$$

Step 3 *Let $\lambda_1, \lambda_2, ..., \lambda_r$ be the points of $\sigma(A_\pi) \cup \sigma(A_\zeta)$. Call the columns of W_2 $w_1, w_2, ..., w_\eta$ and let the geometric multiplicities of the pole and zero of W_2 at λ_i be l and $\tilde{l}$. For each $i \in \{1, 2, ..., r\}$ find a subspace Λ_i of $\mathbf{C}^{1\times m}$ as follows. Let*

$$\Omega_1 = span\ \{[w_1]_{\lambda_i}, [w_2]_{\lambda_i}, ..., [w_l]_{\lambda_i}\}$$

$$\Omega_2 = span\ \{[w_{l+1}]_{\lambda_i}, [w_{l+2}]_{\lambda_i}, ..., [w_{\eta-\tilde{l}}]_{\lambda_i}\}$$

and let Ω be the subspace of $\mathbf{C}^{m\times 1}$ annihilated by the bottom $\tilde{l}$ rows of $E^{-1}(\lambda_i)$. Let W^{ol} be the row span of the matrix polynomial corresponding to $(A_\kappa, B_\kappa, D_\kappa)$ and let $\left(W^{ol}(\lambda_i)\right)^\circ$ be the subspace of $\mathbf{C}^{m\times 1}$ annihilated by $W^{ol}(\lambda_i)$ (cf. Proposition 2.4). Find a complement Ω_{pr} of $\left(W^{ol}(\lambda_i)\right)^\circ \cap \Omega$ in Ω such that the projection of Ω_2 along Ω_{pr} onto $\left(W^{ol}(\lambda_i)\right)^\circ \cap \Omega$ has dimension $\eta - \tilde{l} - l$ and intersects trivially with Ω_1. Let Λ_i be the subspace of $\mathbf{C}^{1\times m}$ which annihilates Ω_{pr}.

Using Corollary 2.25 find a subspace Ξ of $\mathcal{R}^{1\times m}$ such that $\Xi(\lambda_i) = \Lambda_i$ ($i = 1, 2, ..., r$) and the pair (A_ζ, B_ζ) is left-similar to the pair constructed from functions $\phi_{m-\mu+1}, \phi_{m-\mu+2}, ..., \phi_m$ which are contained in Ξ. Construct W_3 by projecting each column of W_2 along Ξ° onto $(W^{ol})^\circ$.

Step 4 *If*

$$\begin{aligned} t &\equiv dim\ (W^{ol})^\circ - dim\ (\text{column span of } W_3) \\ &> 0, \end{aligned}$$

proceed as follows. Let P be the matrix polynomial corresponding to $(A_\kappa, B_\kappa, D_\kappa)$ and let $\psi_1, \psi_2, ..., \psi_\gamma \in \Xi$ be such that the rational matrix function

$$R = \begin{bmatrix} P \\ \psi_1 \\ \psi_2 \\ \vdots \\ \psi_\gamma \\ \phi_{m-\mu+1} \\ \phi_{m-\mu+2} \\ \vdots \\ \phi_m \end{bmatrix}$$

has no zeros nor poles in $\sigma(A_\pi)\cup\sigma(A_\zeta)$. *Let* $U_1, U_2, ..., U_{\mu+\gamma}$ *be the last* $\mu+\gamma$ *columns of* R^{-1} *and suppose that the largest the geometric multiplicity of a zero of* W_3 *in* σ *is* t. *For each* $\lambda_i \in \{\lambda_1, \lambda_2, ..., \lambda_r\} = \sigma(A_\pi)\cup\sigma(A_\zeta)$ *choose* $v_{i1}, v_{i2}, ..., v_{it} \in \mathcal{R}^{m\times 1}$ *such that* $v_{i1}, v_{i2}, ..., v_{it}$ *are analytic on* σ *and*

$$\left(W^{al}(\lambda_i)\right)^{\circ} = span\ \{v_{i1}(\lambda_i), v_{i2}(\lambda_i), ..., v_{it}(\lambda_i)\} \oplus V(\lambda_i)$$

where V *is the column span of* W_3. *Using Lagrange interpolation find rational vector functions* $v_1, v_2, ..., v_t$ *such that* $v_j(\lambda_i) = v_{ij}(\lambda_i)$ *and* $\phi_i v_j$ *vanishes at* $\lambda \in \sigma$ *whenever* ϕ_i *is a left null function for* W_3 *at* λ *of order* k. *We put*

$$W_4 = [\,W_3 \quad v_1 \quad v_2 \quad \ldots \quad v_t\,].$$

Step 5 *Apply to* W_4 *Lemma 3.9 a finite number of times to get a rational matrix function* W *such that* W *is right equivalent to* W_4 *on* $\sigma(A_\pi)\cup\sigma(A_\zeta)$ *and* W *has no poles nor zeros in* $\sigma\backslash\left(\sigma(A_\pi)\cup\sigma(A_\zeta)\right)$.

Proof We note that since (C_π, A_π) and (A_ζ, B_ζ) are right pole and left null pairs for H over σ, (C_π, A_π) and (A_ζ, B_ζ) satisfy conditions (i) and (ii) in Proposition 4.1. Thus, (C_π, A_π), (A_ζ, B_ζ) and $(A_\kappa, B_\kappa, D_\kappa)$ satisfy conditions (i)-(v) in Proposition 4.1. For convenience, we shall refer to these conditions as simply conditions (i)-(v). For the sake of definiteness we assume that the number of rows in the matrix D_κ is k. We assume $m > k > 0$. We also put $n = m - k$. Thus, the size of the constructed rational matrix function W is $m \times n$.

We need to show that all steps in the algorithm are feasible, and that the resulting rational matrix function has a right pole pair over σ (C_π, A_π), left null pair over σ (A_ζ, B_ζ) and a left kernel triple $(A_\kappa, B_\kappa, D_\kappa)$. We shall discuss the algorithm step by step.

Step 1 Since W_1 and H are right equivalent on $\mathbf{C}$, W_1 has the right pole pair over σ (C_π, A_π) and the left null pair over σ (A_ζ, B_ζ). After applying appropriate similarity transformations to the pairs (C_π, A_π) and (A_ζ, B_ζ), we assume that

- A_π and A_ζ are in Jordan forms,
- (C_π, A_π) has been read off from the first ν columns of E, $E_1, E_2, ..., E_\nu$,
- (A_ζ, B_ζ) has been read off from the last μ rows of E^{-1}, $\Xi_{m-\mu+1}, \Xi_{m-\mu+2}, ..., \Xi_m$.

Step 2 Since $\eta \geq \nu$ and $\eta \geq \mu$, the $m \times \eta$ rational matrix functions V_L and V_R exist. We have to show that we can find the polynomials p_i occuring in the definition of W_2. We show first that $\eta \leq n$. Let $\lambda \in \sigma$ be such that η diagonal entries of D have either a pole or a zero at λ. The i columns of $E(\lambda)$ which correspond to the diagonal entries of D with a pole at λ are annihilated by $W^{ol}(\lambda)$ (see condition (iv)) and by the j rows of $E^{-1}(\lambda)$ which correspond to the diagonal entries of D with a zero at λ. Since, by condition (v), the span of the rows of $E^{-1}(\lambda)$ which correspond to the diagonal entries of D with a zero at λ intersects trivially with $W^{ol}(\lambda)$, $i + j + k \leq m$. So $\eta = i + j \leq m - k = n$.

Suppose now that we take the first η diagonal entries $d_1, d_2, ..., d_\eta$ of D, the last η diagonal entries $d_{m-\eta+1}, d_{m-\eta+2}, ..., d_m$ of D, and map the i'th component of the η-tuple $(d_1, d_2, ..., d_\eta)$ to the i'th component of the η-tuple $(d_{m-\eta+1}.d_{m-\eta+2}, ..., d_m)$:

$$\begin{array}{cccc} d_1 & d_2 & \cdots & d_\eta \\ \downarrow & \downarrow & & \downarrow \\ d_{m-\eta+1} & d_{m-\eta+2} & \cdots & d_m \end{array}.$$

Since $\eta \leq n < m$, $i < j$ whenever d_i is mapped to d_j. The preceding implication persists if we replace the last $\eta - \nu$ components of $(d_1, d_2, ..., d_\eta)$ and the first $\eta - \mu$ components of $(d_{m-\eta+1}, d_{m-\eta+2}, ..., d_m)$ by 0's. Thus, if the i'th columns of V_L and V_R are d_iE_i and d_jE_j then $i < j$ and, by the properties of the Smith-McMillan form of a rational matrix function, we can find a monic polynomial p_i of the least degree such that p_id_i has the same zeros in σ with the same multiplicities as d_j (or, equivalently, such that $p_i d_i E_i$ has the same zeros in σ with the same multiplicities as $d_j E_j$). So W_2 can be constructed in the way described in Step 2.

We list now the properties of W_2.

Lemma 4.8.1 *If W_2 is constructed as in Step 2, then the following hold.*

(i) The number of columns of W_2 is less than, or equal to, $m-k$.

(ii) W_2 has no zero columns.

(iii) The columns of W_2 are orthogonal on σ.

(iv) The i'th column of W_2 has the same poles in σ with the same multiplicities as the i'th column of W_1 $(1 \le i \le \nu)$ and the $(\eta - j)$'th column of W_2 has the same zeros in σ with the same multiplicities as the $(m-j)$'th column of W_1 $(1 \le j \le \mu)$.

(v) (C_π, A_π) is a right pole pair for W_2 over σ.

(vi) (A_ζ, B_ζ) is a left null pair for W_2 over σ.

Proof Inequality $\eta \le n$ has been indicated above and property (ii) follows from the choice of η.

Suppose that the first l columns of V_L have a pole at $\lambda \in \sigma$. Then by the choice of η at most $\eta - l$ columns of V_R have a zero at λ and, by construction, these columns are the last columns of V_R. Thus, it cannot happen that the i'th column of V_L has a pole at λ while the i'th column of V_R has a zero at λ. Consequently, the i'th column of W_2 has the same poles in σ with the same multiplicities as the i'th column of V_L and hence as the i'th column of W_1. By construction, the $(\eta - j)$'th column of W_2 has the same zeros in σ with the same multiplicities as the $(\eta - j)$'th column of V_R and hence as the $(m-j)$'th column of W_1. This establishes (iv).

We show now (iii). For each column w_i of W_2 we can find a scalar rational function $\tilde{d}_i$ such that $w_i = \tilde{d}_i \tilde{w}_i$ and $\tilde{w}_i$ has no poles nor zeros in σ. Let $\tilde{W}_2 =$

$[\,\tilde{w}_1 \quad \tilde{w}_2 \quad \ldots \quad \tilde{w}_1\,]$ so that

$$W_2 = \tilde{W}_2 \begin{bmatrix} \tilde{d}_1 & & & \\ & \tilde{d}_2 & & \\ & & \ddots & \\ & & & \tilde{d}_\eta \end{bmatrix}.$$

Since for each $\lambda \in \sigma$ the set $\{E_1(\lambda), E_2(\lambda), ..., E_m(\lambda)\}$ is linearly independent, the set

$$\{E_1(\lambda), E_2(\lambda), ..., E_{\eta-\mu}(\lambda), E_{\eta-\mu+1}(\lambda) + \alpha_1 E_{m-\mu+1}(\lambda), \ldots,$$
$$E_\nu(\lambda) + \alpha_{\mu+\nu-\eta} E_{m-\nu+\eta}(\lambda), E_{m-\eta+\nu+1}(\lambda), \ldots, E_m(\alpha)\}$$

is linearly independent for all constants $\alpha_1, \alpha_2, ..., \alpha_{\mu+\nu-\eta}$. Therefore it follows from (4) that $\tilde{W}_2(\lambda)$ has full column rank for all $\lambda \in \sigma$. By Proposition 2.13 the columns of W_2 are orthogonal on σ as asserted.

In view of (iv), in order to prove (v), it suffices to show that if $\tilde{d}_i$ has a pole at $\lambda \in \sigma$ of order l then we can find a right pole function ψ_i for W_2 at λ of order l such that the first l coefficients in the Taylor expansions at λ of ψ_i and E_i coincide. The latter assertion is obvious when the i'th column of V_R is 0 or the i'th column of V_R is analytic at λ. Suppose $d_j E_j$, the i'th column of V_R, has a pole at λ. Then the i'th column of W_2 is $p_i d_i E_i + d_j E_j$ and the j'th column of W_2 is $p_j d_j E_j + d_\kappa E_\kappa$ for some κ and scalar polynomials p_i, p_j with $p_i(\lambda) p_j(\lambda) \neq 0$. So

$$p_j(p_i d_i E_i + d_j E_j) - (p_j d_j E_j + d_\kappa E_\kappa) = p_i p_j d_i E_i - d_\kappa E_\kappa.$$

Inducting, if necessary, on κ we can find an $m \times 1$ rational vector function $\tilde{\psi} = p d_i E_i - d_\kappa E_\kappa$ with $d_\kappa(\lambda) \neq 0$ such that $\tilde{\psi} = W_2 \phi$ with ϕ a vector polynomial which does not vanish at λ. So $\psi = (pd_i)^{-1} \tilde{\psi}$ is a right pole function for W_2 at λ of order l and the first l coefficients in the Taylor expansions at λ of ψ and E_i coincide. Thus, (v) is established.

Suppose, finally, that $\{\Xi_{m-\tilde{l}+1}, \Xi_{m-\tilde{l}+2}, ..., \Xi_m\}$ is a canonical set of left null functions for W_1 at $\lambda \in \sigma$ of orders $l_1, l_2, ..., l_{\tilde{l}}$. Then the last $\tilde{l}$ diagonal entries of D vanish at λ to the orders $l_1, l_2, ..., l_{\tilde{l}}$. By (iv), the last $\tilde{l}$ diagonal entries of diag $(\tilde{d}_1, \tilde{d}_2, ..., \tilde{d}_\eta)$ vanish at λ to the orders $l_1, l_2, ..., l_{\tilde{l}}$. By (iii) and Corollary 2.20, the multiplicity of the zero at λ of W_2 is equal to $\sum_{i=1}^{\tilde{l}} l_i$. Now if $\Xi_i W_1$ vanishes at λ to order l_j then it follows from the construction that $\Xi_i W_2$ vanishes at λ to order l_j. By (iii), (iv) and Corollary 2.20 $l_1, l_2, ..., l_{\tilde{l}}$ are partial multiplicities of the zero of W_2 at λ. Plainly, $\Xi_{m-\tilde{l}+1}(\lambda), \Xi_{m-\tilde{l}+2}(\lambda), ..., \Xi_m(\lambda)$ are linearly independent. Since $\Xi_{m-\tilde{l}+i}(\lambda)$ annihilates span $\{\tilde{w}_1(\lambda), \tilde{w}_2(\lambda), ..., \tilde{w}_{\eta-\tilde{l}+i-1}(\lambda)\}$ and $\Xi_{m-\tilde{l}+i}(\lambda)\tilde{w}_{\eta-\tilde{l}+i}(\lambda) \neq 0$ for $i = 1, 2, ..., \tilde{l}$, it follows from Proposition 4.7 that $\{\Xi_{m-\tilde{l}+1}, \Xi_{m-\tilde{l}+2}, ..., \Xi_m\}$ is a canonical set of left null functions for W_2 at λ of orders $l_1, l_2, ..., l_{\tilde{l}}$. Hence (A_ζ, B_ζ) is a left null pair for W_2 over σ.

□

Continuation of proof of Theorem 4.8

Step 3 Choose $\lambda_i \in \sigma(A_\pi) \cup \sigma(A_\zeta)$ and let $l, \tilde{l}, \Omega_1, \Omega_2$ and Ω be as in Step 3 of the algorithm. By property (iii) of W_2, $\Omega_1 \cup \Omega_2 = (0)$. By condition (v), dim $\left((W^{ol}(\lambda_i))^\circ \cap \Omega\right) = m - k - \tilde{l}$ and hence

$$\begin{aligned} \dim\,(\Omega_1 + \Omega_2) &= \eta - \tilde{l} \\ &\leq n - \tilde{l} \\ &= m - k - \tilde{l} \\ &= \dim\,\left((W^{ol}(\lambda_i))^\circ \cap \Omega\right). \end{aligned}$$

Therefore we can find Ω_{pr} with the required property.

Let Λ_i be the annihilator of Ω_{pr} in $\mathbf{C}^{1\times m}$.

Lemma 4.8.2 *Λ_i has the following properties:*

(i) $\mathbf{C}^{1\times m} = \Lambda_i \oplus W^{ol}(\lambda_i)$,

(ii) the bottom $\tilde{l}$ rows of $E^{-1}(\lambda_i)$ are contained in Λ_i,

(iii) the projection of span $\{[w_1]_{\lambda_i}, [w_2]_{\lambda_i}, \ldots, [w_\eta]_{\lambda_i}\}$ *onto* $\left(W^{ol}(\lambda_i)\right)^\circ$ *along* Ω_{pr}*, the right annihilator of* Ω_i *in* $\mathrm{C}^{m\times 1}$*, has dimension* η.

Proof Property (i) is equivalent to the direct sum decomposition $\mathrm{C}^{m\times 1} = \Omega_{pr} \oplus \left(W^{ol}(\lambda_i)\right)^\circ$. To see that the latter decomposition holds, choose a basis $\{c_1, c_2, \ldots, c_k\}$ for $W^{ol}(\lambda_i)$ and vectors $c_{k+1}, c_{k+2}, \ldots, c_{m-\tilde{l}}$ such that the matrix

$$M = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{m-\tilde{l}} \\ \Xi_{m-\tilde{l}+1}(\lambda_i) \\ \Xi_{m-\tilde{l}+2}(\lambda_i) \\ \vdots \\ \Xi_m \end{bmatrix}$$

is invertible. Then the first $m-\tilde{l}$ columns of M^{-1} form a basis for Ω and the last $\tilde{l}$ columns of M^{-1} form a basis for a subspace $\tilde{\Omega} \subset \left(W^{ol}(\lambda_i)\right)^\circ$. So

$$\begin{aligned} \mathrm{C}^{m\times 1} &= \Omega \oplus \tilde{\Omega} \\ &= \Omega_{pr} \oplus ((W\mathrm{o}l(\lambda_i))^\circ \cap \Omega) \oplus \tilde{\Omega} \\ &= \Omega_{pr} \oplus (W\mathrm{o}l(\lambda_i))^\circ \end{aligned}$$

as asserted.

Property (ii) follows from the definition of Ω. Property (iii) follows from the fact that the projection of the first $\eta - \tilde{l}$ columns of $\tilde{W}_2$ onto $(W\mathrm{o}l(\lambda_i))^\circ$ along Ω_{pr} has full column rank and $\Xi_j(\lambda_i)w_k(\lambda_i) = \delta_{jk}$ for $m-\tilde{l}+1 \le j \le m$ and $1 \le k \le \eta$.

□

Continuation of proof of Theorem 4.8

It follows from Corollary 2.25 and parts (i) and (ii) of Lemma 4.8.2 that we can indeed construct Ξ. By Proposition 2.6 and Theorem 2.11, $(W^{ol})^\circ$ and Ξ° are orthogonal complements in $(\mathcal{R}^{m\times 1}, \sigma(A_\pi) \cup \sigma(A_\zeta))$. In particular, $(W^{ol})^\circ$ and Ξ° are algebraic complements in $\mathcal{R}^m$ and the construction of W_3 is possible.

We note that by condition (iv) and Proposition 4.6, (C_π, A_π) is a right pole pair for W_3 over $\sigma(A_\pi) \cup \sigma(A_\zeta)$. Also, by Proposition 4.7, (A_ζ, B_ζ) is a left null pair for W_3 over $\sigma(A_\pi) \cup \sigma(A_\zeta)$.

Step 4 The feasibility of construction steps 4 and 5 is clear.

Since $v_1, v_2, \ldots, v_t$ are analytic on σ, (C_π, A_π) is a right pole pair for W_4 over σ. Since the columns of W_4 are orthogonal on $\sigma(A_\pi) \cup \sigma(A_\zeta)$, by Proposition 2.23 (A_ζ, B_ζ) is a left null pair for W_4 over σ. By construction, $(A_\kappa, B_\kappa, D_\kappa)$ is a left kernel triple for W_4.

Step 5 Since W and W_4 differ by a right factor, $(A_\kappa, B_\kappa, D_\kappa)$ is a left kernel triple for W. Since W and W_4 are right equivalent on $\sigma(A_\pi) \cup \sigma(A_\zeta)$, (C_π, A_π) and (A_ζ, B_ζ) are right pole and left null pairs for W over $\sigma(A_\pi) \cup \sigma(A_\zeta)$. Since W has no zeros nor poles in $\sigma \setminus (\sigma(A_\pi) \cup \sigma(A_\zeta))$, (C_π, A_π) and (A_ζ, B_ζ) are right pole and left null pairs for W over σ.

□

We note that the rational matrix functions obtained in all steps of the algorithm in Theorem 4.8 have the same right pole and left null pairs over $\sigma(A_\pi) \cup \sigma(A_\zeta)$– (C_π, A_π) and (A_ζ, B_ζ). Steps 2, 3 and 4 affected the left kernel polynomial of the respective functions.

4.2 Functions with a given left null-pole subspace

Let $\sigma \subset \mathbf{C}$ and suppose (C_π, A_π), (A_ζ, B_ζ) and $(A_\kappa, B_\kappa, D_\kappa)$ satisfy conditions (i)-(v) in Proposition 4.1. By Theorem 4.2, there exists a rational matrix function with the right pole structure described by (C_π, A_π) and the left zero structure described by (A_ζ, B_ζ) and $(A_\kappa, B_\kappa, D_\kappa)$ whenever equation (1) has a solution. It turns out that a stronger assertion is true.

Theorem 4.9 *Let $\sigma \subset \mathbf{C}$, suppose that (C_π, A_π), (A_ζ, B_ζ) and $(A_\kappa, B_\kappa, D_\kappa)$ satisfy conditions (i)-(v) in Proposition 4.1, and let Γ be a solution of the Sylvester equation*

$$SA_\pi - A_\zeta S = B_\zeta C_\pi. \tag{5}$$

Then there exists a left-invertible rational matrix function W with a left kernel triple $(A_\kappa, B_\kappa, D_\kappa)$ and a left spectral triple over σ $\{(C_\pi, A_\pi), (A_\zeta, B_\zeta), \Gamma\}$.

Proof We may assume $\sigma(A_\pi) \cup \sigma(A_\zeta) \neq \emptyset$. Let P be a matrix polynomial corresponding to the triple $(A_\kappa, B_\kappa, D_\kappa)$. For the sake of definiteness, we assume that the size of P is $k \times m$. Let

$$\begin{aligned} S_{W_\sigma} = \{C_\pi(z - A_\pi)^{-1}x + h(z) :\ & h \in \mathcal{R}^{m\times 1}(\sigma) \\ & \text{and} \sum_{z_0 \in \sigma} \mathrm{Res}_{z=z_0}(z - A_\zeta)^{-1} B_\zeta h(z) = \Gamma x\} \\ \cap \{f \in \mathcal{R}^{m\times 1} :\ & Pf = 0\}. \end{aligned} \tag{6}$$

Since Γ satisfies equation (5), by Theorem 12.2.1 in [BGR3] S_{W_σ} is an $\mathcal{R}(\sigma)$-module. Clearly S_{W_σ} is a submodule of the $\mathcal{R}(\sigma)$-module

$$\begin{aligned} S_\sigma = \{C_\pi(z - A_\pi)^{-1}x + h(z) :\ & h \in \mathcal{R}^{m\times 1}(\sigma) \\ & \text{and} \sum_{z_0 \in \sigma} \mathrm{Res}_{z=z_0}(z - A_\zeta)^{-1} B_\zeta h(z) = \Gamma x\}. \end{aligned}$$

Since by Theorems 3.3.2 and 3.3.3 in [BGR2] the module S_σ is finitely generated and $\mathcal{R}(\sigma)$ is a principal ideal domain, S_{W_σ} is finitely generated. Since the torsion submodule of S_{W_σ} is trivial, by the fundamental structure theorem for finitely generated modules over a principal ideal domain (see e.g. [J]) S_{W_σ} is a free module. Choose a basis $\{w_1, w_2, ..., w_n\}$ for S_{W_σ} and define the rational matrix function W by

$$W = [\,w_1 \quad w_2 \quad \dots \quad w_n\,].$$

We show first that the rational matrix function W is left invertible, and the triple $(A_\kappa, B_\kappa, D_\kappa)$ is a left kernel triple for W. Since the columns of W are contained in a subspace of $\mathcal{R}^{m\times 1}$ which is annihilated by P, it suffices to show the equality $n = m-k$. Choose an algebraic basis $\{h_1, h_2, ..., h_{m-k}\}$ for an $\mathcal{R}$-vector space $\{f \in \mathcal{R}^{m\times 1} : Pf = 0\}$ and let q be a scalar polynomial such that the rational matrix functions

$$q(z)(z - A_\zeta)^{-1}B_\zeta \begin{bmatrix} h_1(z) & h_2(z) & \dots & h_{m-k}(z) \end{bmatrix}$$

and

$$q \begin{bmatrix} h_1 & h_2 & \dots & h_{m-k} \end{bmatrix}$$

are analytic on $\sigma(A_\pi) \cup \sigma(A_\zeta)$. Then $qh_1, qh_2, ..., qh_{m-k}$ are contained in S_{W_σ} and hence $n \geq m-k$. Suppose the last inequality is strict. Then the set $\{w_1, w_2, ..., w_n\}$ is linearly dependent over $\mathcal{R}$ and

$$r_1w_1 + r_2w_2 + \dots r_nw_n = 0 \tag{9}$$

for some scalar rational functions $r_1, r_2, ..., r_n$ not all equal to zero. After multiplying both sides of equality (9) by a scalar polynomial we may assume $r_1, r_2, ..., r_n \in \mathcal{R}\Big(\sigma(A_\pi) \cup \sigma(A_\zeta)\Big)$. But then equality (9) contradicts the direct sum decomposition

$$S_{W_\sigma} = \mathcal{R}(\sigma)w_1 \oplus \mathcal{R}(\sigma)w_2 \oplus \dots \oplus \mathcal{R}(\sigma)w_n.$$

Thus, $n = m-k$. So P is a left kernel polynomial for W and W is left invertible.

It remains to show that $\{(C_\pi, A_\pi), (A_\zeta, B_\zeta), \Gamma\}$ is a left spectral triple for W over σ. Let

$$\sigma_1 = \sigma(A_\pi) \cup \sigma(A_\zeta) \cup \{\lambda \in \sigma : \ W \text{ has a pole or zero at } \lambda\}.$$

Since (A_ζ, B_ζ) and $(A_\kappa, B_\kappa, D_\kappa)$ satisfy condition (v) in Proposition 4.1, by Corollary 2.25 we can find an orthogonal complement Ξ of the row span of P in $(\mathcal{R}^{1\times m}, \sigma_1)$ such that the pair (A_ζ, B_ζ) is left-similar to a pair constructed from functions in Ξ.

In view of Proposition 2.21 we can choose a left spectral triple $\{(\tilde{C}_\pi, \tilde{A}_\pi),(\tilde{A}_\zeta, \tilde{B}_\zeta), \tilde{\Gamma}\}$ for W over σ such that the pair $(\tilde{A}_\zeta, \tilde{B}_\zeta)$ is left-similar to a pair constructed from functions in Ξ.

By Proposition 3.3,

$$\begin{aligned} S_{W\sigma} = \{\tilde{C}_\pi(z-\tilde{A}_\pi)^{-1}x + h(z) : \ & h \in \mathcal{R}^{m\times 1}(\sigma) \\ & \text{and} \sum_{z_0\in\sigma} \mathrm{Res}_{z=z_0}(z-\tilde{A}_\zeta)^{-1}\tilde{B}_\zeta h(z) = \tilde{\Gamma}x\} \\ \cap \{f \in \mathcal{R}^{m\times 1} : \ & Pf = 0\}. \end{aligned} \tag{7}$$

Now (cf. Proposition 3.1.3 and Theorem 3.3.3 in [BGR2])

$$\begin{aligned} S_\sigma = \ & \{C_\pi(z-A_\pi)^{-1}x + h(z) : \ h \in \mathcal{R}^{m\times 1}(\sigma_1) \\ & \quad \text{and} \sum_{z_0\in\sigma_1} \mathrm{Res}_{z=z_0}(z-A_\zeta)^{-1}B_\zeta h(z) = \Gamma x\} \\ & \cap \{C_\pi(z-A_\pi)^{-1}x + h(z) : \ h \in \mathcal{R}^{m\times 1}(\sigma\backslash\sigma_1) \\ & \quad \text{and} \sum_{z_0\in\sigma\backslash\sigma_1} \mathrm{Res}_{z=z_0}(z-A_\zeta)^{-1}B_\zeta h(z) = \Gamma x\} \\ = \ & \{C_\pi(z-A_\pi)^{-1}x + h(z) : \ h \in \mathcal{R}^{m\times 1}(\sigma_1) \\ & \quad \text{and} \sum_{z_0\in\sigma_1} \mathrm{Res}_{z=z_0}(z-A_\zeta)^{-1}B_\zeta h(z) = \Gamma x\} \\ & \cap \mathcal{R}^{n\times 1}(\sigma\backslash\sigma_1) \\ \equiv \ & S_{\sigma_1} \cap \mathcal{R}^{n\times 1}(\sigma\backslash\sigma_1). \end{aligned}$$

Similarly,

$$\begin{aligned} \tilde{S}_\sigma \equiv \ & \{\tilde{C}_\pi(z-\tilde{A}_\pi)^{-1}x + h(z) : \ h \in \mathcal{R}^{m\times 1}(\sigma) \\ & \quad \text{and} \sum_{z_0\in\sigma} \mathrm{Res}_{z=z_0}(z-\tilde{A}_\zeta)^{-1}\tilde{B}_\zeta h(z) = \tilde{\Gamma}x\} \\ = \ & \tilde{S}_{\sigma_1} \cap \mathcal{R}^{n\times 1}(\sigma\backslash\sigma_1) \end{aligned}$$

where

$$\begin{aligned} \tilde{S}_{\sigma_1} = \{\tilde{C}_\pi(z-\tilde{A}_\pi)^{-1}x + h(z) : \ & h \in \mathcal{R}^{m\times 1}(\sigma_1) \\ & \text{and} \sum_{z_0\in\sigma_1} \mathrm{Res}_{z=z_0}(z-\tilde{A}_\zeta)^{-1}\tilde{B}_\zeta h(z) = \tilde{\Gamma}x\}. \end{aligned}$$

By Theorem 12.2.1 in [BGR3] S_{σ_1} and $\tilde{S}_{\sigma_1}$ are $\mathcal{R}(\sigma_1)$-modules. We claim that $S_{\sigma_1} = \tilde{S}_{\sigma_1}$. Indeed, suppose there exists $f \in S_{\sigma_1}\backslash\tilde{S}_{\sigma_1}$ and let p be a scalar polynomial with all zeros in $\sigma\backslash\sigma_1$ such that $pf \in \mathcal{R}^{m\times 1}(\sigma\backslash\sigma_1)$. Then $p \in \mathcal{R}(\sigma_1)$ and $p^{-1} \in \mathcal{R}(\sigma_1)$, so $pf \in S_{\sigma_1}$ and $pf \notin \tilde{S}_{\sigma_1}$. Hence $pf \in S_\sigma$ and $pf \notin \tilde{S}_\sigma$. Let f_W be the projection of pf along Ξ° onto $\{f \in \mathcal{R}^{m\times 1} : \ Pf = 0\}$. Then

$$pf = f_K + f_W \tag{8}$$

for some $f_K \in \Xi^\circ$. Since (C_π, A_π) and $(A_\kappa, B_\kappa, D_\kappa)$ satisfy condition (iv) in Proposition 4.1, $P(pf)$ is analytic on σ_1. Hence, by Proposition 4.6, $f_K \in \mathcal{R}^{m\times 1}(\sigma_1)$. Multiplying, if necessary, both sides of equality (8) by a scalar polynomial with all zeros in $\sigma\backslash\sigma_1$ we may assume $f_K \in \mathcal{R}^{m\times 1}(\sigma)$. Since the pairs (A_ζ, B_ζ) and $(\tilde{A}_\zeta, \tilde{B}_\zeta)$ are left similar to the pairs which are constructed from functions which annihilate f_K,

$$\sum_{z_0\in\sigma_1} \operatorname{Res}_{z=z_0}(z - A_\zeta)^{-1}B_\zeta f_K(z) = 0$$

and

$$\sum_{z_0\in\sigma_1} \operatorname{Res}_{z=z_0}(z - \tilde{A}_\zeta)^{-1}\tilde{B}_\zeta f_K(z) = 0.$$

Consequently,

$$\begin{aligned} f_W \in \{C_\pi(z - A_\pi)^{-1}x + h(z) : \ & h \in \mathcal{R}^{m\times 1}(\sigma) \\ & \text{and} \sum_{z_0\in\sigma} \operatorname{Res}_{z=z_0}(z - A_\zeta)^{-1}B_\zeta h(z) = \Gamma x\} \\ \cap \{f \in \mathcal{R}^{m\times 1} : \ & Pf = 0\} \end{aligned}$$

and

$$\begin{aligned} f_W \notin \{\tilde{C}_\pi(z - \tilde{A}_\pi)^{-1}x + h(z) : \ & h \in \mathcal{R}^{m\times 1}(\sigma) \\ & \text{and} \sum_{z_0\in\sigma} \operatorname{Res}_{z=z_0}(z - \tilde{A}_\zeta)^{-1}\tilde{B}_\zeta h(z) = \tilde{\Gamma} x\} \\ \cap \{f \in \mathcal{R}^{m\times 1} : \ & Pf = 0\}. \end{aligned}$$

In view of (6) and (7), this is a contradiction. Thus, $S_{\sigma_1} \subset \tilde{S}_{\sigma_1}$. Similarly, $\tilde{S}_{\sigma_1} \subset S_{\sigma_1}$. So $S_{\sigma_1} = \tilde{S}_{\sigma_1}$ as asserted and it follows from Theorem 12.2.4 in [BGR3] that the triples $\{(C_\pi, A_\pi),\ (A_\zeta, B_\zeta),\ \Gamma\}$ and $\{(\tilde{C}_\pi, \tilde{A}_\pi),\ (\tilde{A}_\zeta, \tilde{B}_\zeta),\ \tilde{\Gamma}\}$ are similar. Hence $\{(C_\pi, A_\pi),$

(A_ζ, B_ζ), $\Gamma\}$ is a left spectral triple for W over σ.

□

Theorem 4.2 can be proved as a corollary of Theorem 4.9. However, the proof of Theorem 4.9 relies on existence of a basis for a free module S_{W_σ} whereas the proof of Theorem 4.2 is based on the construction described in Theorem 4.8. This construction, apart from the Smith-McMillan factorization in Step 1, is long and tedious but involves mainly standard linear algebra procedures. Conceivably there should also be a constructive proof of the stronger Theorem 4.9. The ultimate goal is a realization formula for the solution of the interpolation problem in Theorems 4.2 and 4.9 in terms of spectral data. Such formulas exist for the regular case (see [BGR3]). Applications of such realization formulas to interpolation and factorization problems analogous to those developed for the regular case are anticipated.

Let $\sigma \subset \mathbf{C}$ and suppose that (C_π, A_π), (A_ζ, B_ζ) and $(A_\kappa, B_\kappa, D_\kappa)$ satisfy conditions (i)-(v) in Proposition 4.1. In view of Proposition 3.1 and Theorem 3.2 it follows from Theorem 4.9 that there is a one-to-one correspondence between solutions of equation (5) on the one hand and equivalence classes of right equivalent on σ left invertible rational matrix functions with right pole and left null pairs over σ equal to (C_π, A_π) and (A_ζ, B_ζ), respectively, and a left kernel triple equal to $(A_\kappa, B_\kappa, D_\kappa)$ on the other. In general, if the size of the matrix D_κ is $k \times m$, there is a one-to-one correspondence between solutions of equation (5) and equivalence classes of $m \times n$ rational matrix functions with a right pole and left null pairs over σ equal to (C_π, A_π) and (A_ζ, B_ζ) and with a left kernel triple $(A_\kappa, B_\kappa, D_\kappa)$ for any $n \geq m - k$.

Corollary 4.10 *Let $\sigma \subset \mathbf{C}$ and suppose that (C_π, A_π), (A_ζ, B_ζ) and $(A_\kappa, B_\kappa, D_\kappa)$ satisfy conditions (i)-(v) in Proposition 4.1. Then for every integer $n \geq m - k$ and* *each solution* Γ *of the Sylvester equation*

$$SA_\pi - A_\zeta S = B_\zeta C_\pi$$

there exists an $m \times n$ *rational matrix function* R *with a left kernel triple* $(A_\kappa, B_\kappa, D_\kappa)$ *and a left spectral triple over* σ $\{(C_\pi, A_\pi), (A_\zeta, B_\zeta), \Gamma\}$.

Proof Let W be the function given by Theorem 4.9. The rational matrix function

$$R = [W \quad 0]$$

where the size of the block matrix 0 is $m \times (n-k)$ has the required properties.

$\square$

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List of symbols

$\mathbf{C}_\infty$ – Extended complex plane, $\mathbf{C} \cup \infty$

$\mathbf{C}^n$ – Product of n copies of $\mathbf{C}$

$\mathbf{C}^{i\times j}$ – $i \times j$ matrices with entries in $\mathbf{C}$

$\mathcal{R}$ – Field of scalar rational functions

$\mathcal{R}^n$ – Product of n copies of $\mathcal{R}$ (viewed as a vector space over $\mathcal{R}$)

$\mathcal{R}^{i\times j}$ – $i \times j$ rational matrix functions

$\mathcal{R}(\sigma)$ – Subset of $\mathcal{R}$ formed by functions which are are analytic on σ, σ a subset of $\mathbf{C}_\infty$

$\mathcal{R}^{i\times j}(\sigma)$ – Functions in $\mathcal{R}^{i\times j}$ which are are analytic on σ, σ a subset of $\mathbf{C}_\infty$

$\mathcal{R}^{n*}$ – Algebraic dual of $\mathcal{R}^n$ (see Section 2.2.2)

$\delta(W, \lambda)$ – Multiplicity of a pole of a rational matrix function W at $\lambda \in \mathbf{C}_\infty$

$\delta(W)$ – McMillan degree of a rational matrix function W

(X, Y) – Right or left pole or null pair, X, Y matrices (see Sections 1.1, 1.2, 2.3 and 2.4)

$(A_\kappa, B_\kappa, D_\kappa)$ – Left kernel triple (see Section 2.1)

$J_k(\lambda)$ – $k \times k$ Jordan cell with $\lambda \in \mathbf{C}$ on the diagonal

$\mathrm{diag}_{i=1}^n A_i$ – (Block) diagonal matrix

$$\begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_n \end{bmatrix}$$

$|\cdot|_{z=\lambda}$ – Valuation of $\mathcal{R}$ at λ, $\lambda \in \mathbf{C}_\infty$ (see Section 2.2.1)

$\|\cdot\|_{z=\lambda}$ – Max norm on $\mathcal{R}^n$ induced by $|\cdot|_{z=\lambda}$, $\lambda \in \mathbf{C}_\infty$ (see Section 2.2.1)

(Λ, σ) – Subspace Λ of $\mathcal{R}^n$ with the topology generated by $\{\|\cdot\|_{z=\lambda} : \lambda \in \sigma\}$

$\Lambda(\lambda)$ – Subspace of $\mathbf{C}^n$ formed by the values at λ of all functions in Λ which are analytic at λ, Λ a subspace of $\mathcal{R}^n$

Λ° – Algebraic annihilator of Λ, Λ a subspace of $\mathcal{R}^n$ or $\mathbf{C}^n$

W^{ol} – Left annihilator of a rational matrix function W

W^{or} – Right annihilator of a rational matrix function W

$[h]_\lambda$ – Leading coefficient in the Laurent expansion at $\lambda \in \mathbf{C}_\infty$ of h, h a rational vector function

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