APPLICATION OF NON-UNIFORM SAMPLING TECHNIQUES TO DIGITAL FILTER SYNTHESIS

by

Joseph Siuming Tsui

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APPROVED:

Dr. J. A. Cadzow (Chairman)

Dr. L. L. Grigsby

Dr. R. L. Moose

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CHAPTER I

INTRODUCTION

It is well known that any bandlimited continuous time signsls f(t) can be represented by a discrete time sequence {f(kT)}. In order to obtain the sequence {f(kT)} the signal f(t) is 'sampled' at regular displacements (T) in time. That is, each number in the sequence {f(kT)} represents a sampled data of f(t) at the observation time kT (where k=0,1,2,.....). This process is known as the uniform sampling of the signal f(t) because each sample is taken at a uniform rate and is governed by the Uniform Sampling Theorem, which states that:

> A bandlimited signal f(t) with highest spectral component f_M is uniquely determined by the periodical samples taken every T seconds, and, the signal f(t) can be reconstructed from these samples with no distortion providing that the sampling interval T is such that, $T < 1/2f_M$.

However, in many applications a non-uniform sampling scheme is used to generate the data sequence. In this case, the sampling time 'T' is not a constant. One example is the Moving Target Indicator in radar signal processing, which employs the simple stagger sampling scheme in order to minimize the 'blind velocity' effect(1). The advantage of using a non-uniform sampling scheme over the uniform sampling scheme is that we have an extra degree of freedom in designing the filter. This freedom will enable us to synthesize a digital filter with extended periods. That is, with the uniform sampling scheme we will generate a digital filter with a periodic frequency response having fundamental period equal to $2\pi/T$; but, as it was found[1,2] the frequency response of the filter using non-uniform sampling scheme has a period of $2\pi P/T$, where P is the integer relating the non-uniform sampling spacing. We shall see (in an example) that the periodicity in the frequency response of a high pass filter based on a uniform sampling scheme will cause loss of information which has frequency components at the periodic notches of the response. The non-uniform sampling scheme, on the other hand, will tend to reduce this type of loss and hence can better process the signal.

In this analysis we shall mainly study the synthesis of digital filters using a special non-uniform sampling scheme -- the simple stagger sampling scheme. The objective is to minimize the periodical notches in the frequency response of the digital filter as one would obtain in using a uniform sampling scheme. To measure the goodness of approximating to a desired filter response, an error criterion is established. The properties of the error criterion function will be analysed, and, based on this criterion an algorithm is developed which will determine the optimal parameters (that is to minimize the error criterion) for the digital filter. Several numerical examples will be carried out in order to illustrate the concepts developed in the analysis, and with the aforementioned algorithm a high pass filter is simulated and its optimal parameters obtained.

CHAPTER II

THE TRANSFER FUNCTION FOR PROCESSING NON-UNIFORMLY SAMPLED DATA

2.1 The General Transfer Function

For a linear time varying digital operator with infinite impulse response $\{h_k(t_n)\}$, as shown in Figure 1, the relationship between the non-uniformly sampled input $\{x(t_n)\}$ and output $\{y\ (t_n)\}$ signals at a given observation time t, is given by the convolution sum;

$$y(t_n) = \sum_{k=-\infty}^{\infty} h_k(t_n)*x(t_{n-k}) \tag{2.1}$$

where:

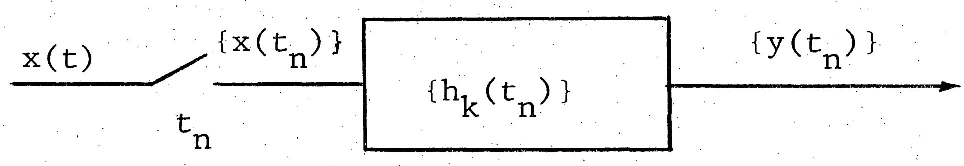
$\{t_n\}$ represents the time instants at which the continuous time signal is sampled,

h is the impulse response coefficient of the linear operator.

If the digital operator in equation (2.1) is assumed to be time invariant and causal (i.e., $h_k(t_n) = 0$, for $k < 0$, then the input and output relationship can be modified as:

$$y(t_n) = \sum_{k=0}^{\infty} h_k*x(t_{n-k}) \tag{2.2}$$

In order to obtain the sinusoidal response characteristics of this operator, the input signal x(t) is replaced



$$k = -\infty, \ldots, 0, \ldots, \infty$$

Figure 1: A Digital Linear Operator

by a standard complex exponential input signal:

$$x(t) = Ae^{j(\omega t+\theta)} \tag{2.3}$$

where, A, ω and θ are constants representing the amplitude, radian frequency and phase angle of the signal respectively. With this input signal, the response of the linear operator at the observation time t_n

$$y(t_n) = \sum_{k=0}^{\infty} h_k A e^{j(\omega t_{n-k}+\theta)}$$

By a simple algebraic manipulation, this expression can be modified to give the standard transfer function relationship, i.e.,

$$y(t_n) = \{ \sum_{k=0}^{\infty} h_k e^{-j\omega(t_n-t_{n-k})} \} Ae^{j(\omega t_n+\theta)} \tag{2.4}$$

Hence, the transfer function can be identified as:

$$H_{t_n}(\omega) = \sum_{k=0}^{\infty} h_k e^{-j\omega(t_n-t_{n-k})} \tag{2.5}$$

Note that in the uniform sampling case, where t_n=nT (n=0,1,2,...), this expression reduces to:

$$H(\omega) = \sum_{k=0}^{\infty} h_k e^{-j\omega kT} \tag{2.6}$$

which is the standard transfer function formula for a uniform sampling system. Also notice that the transfer function for the non-uniform sampling scheme (2.5) is dependent on the radian frequency of the signal and on the observation time t_n. The dependence on frequency, ω, enables the implementation of the various attenuation functions of a digital filter; such as, low pass, high pass and band pass filters. The second dependence, on time t_n, however, gives an extra degree of freedom in the design of the digital filter; and, it is this characteristic that enables the application of non-uniform sampling scheme to digital filter synthesis. Moreover, the variation of the transfer function as a function of observation time will depend on the particular non-uniform sampling scheme used.

2.2 Transfer Function for Simple Stagger Sampling Scheme

In this analysis, the simple stagger sampling scheme is used, whose sampling time instants are defined as:

$$t = \begin{cases} kT & \text{for } k=\text{even} \\ kT + \varepsilon & \text{for } k=\text{odd} \end{cases} \tag{2.7}$$

where, T is the "nominal" sampling time, and, ε is the offset parameter, such that $|\varepsilon| < T$.

To facilitate the analysis of the transfer function (2.5) for the simple stagger sampling scheme, the behavior of $t_n - t_{n-k}$ for different combinations of even and odd n,k's are determined[1,2] and is summarized in Table 1.

With the information from Table 1, the transfer function for the simple stagger sampling, equation (2.5), can be rewritten and decomposed into the summation of the partial sums over the odd and even indices. Hence, the filter transfer function (2.5) becomes:

$$H_{t_n}(\omega) = \sum_{k\varepsilon\Lambda_e} h_k e^{-j\omega(t_n - t_{n-k})} + \sum_{k\varepsilon\Lambda_o} h_k e^{-j\omega(t_n - t_{n-k})} \tag{2.8}$$

where Λ_e and Λ_o represent the even and odd set of non-negative integers respectively.

Applying the results of Table 1 into equation (2.8), the expression can be written as:

$$H_{t_n}(\omega) = \sum_{k\varepsilon\Lambda_e} h_k e^{-j\omega kT} + e^{j(-1)^n \omega\varepsilon} \sum_{k\varepsilon\Lambda_o} h_k e^{-j\omega kT} \tag{2.9}$$

or

$$H_{t_n}(\omega) = H_e(\omega) + e^{j(-1)^n \omega\varepsilon} \cdot H_o(\omega) \tag{2.10}$$

where

$$H_e(\omega) = \sum_{k\varepsilon\Lambda_e} h_k e^{-j\omega kT} \tag{2.11a}$$

and

TABLE 1

Values of $t_n - t_{n-k}$ (1).

n	k	$t_n - t_{n-k}$
even	even	kT
even	odd	$kT-\varepsilon$
odd	even	kT
odd	even	$kT+\varepsilon$

$$H_o(\omega) = \sum_{k\varepsilon\Lambda_o} h_k e^{-j\omega kT} \tag{2.11b}$$

From the results of expressions (2.9) and (2.10) it was found[1] that the transfer function is also a function of the offset parameter ε, and is independent of the observation time t_n. Hence, the transfer function for the simple stagger sampling scheme can be written as:

$$H_n(\varepsilon,\omega) = H_e(\omega) + e^{j(-1)^n \omega\varepsilon} H_o(\omega) \tag{2.12}$$

It has also been proven[2] that "the transfer function $H_n(\varepsilon,\omega)$ as given by equation (2.12) is a periodic function of ω if and only if the parameters ε is equal to qT/P, where P and q are both integers with q less than P. Moreover, the fundamental period is given by $2\pi P/T$ where T is the nominal sampling time. This periodicity characterization holds for both n odd and n even". (This is Theorem 1 as stated in [2]). Recall that the period of the transfer function in the case of uniform sampling is $2\pi/T$, this implies that the period of the transfer function in the non-uniform sampling case is "extended", for $P > 1$. This fact can be best illustrated by examining the frequency responses of a high pass filter based on uniform and the simple stagger sampling schemes.

Figure 2 shows the frequency response of an ideal digital high pass filter based on uniform sampling. In

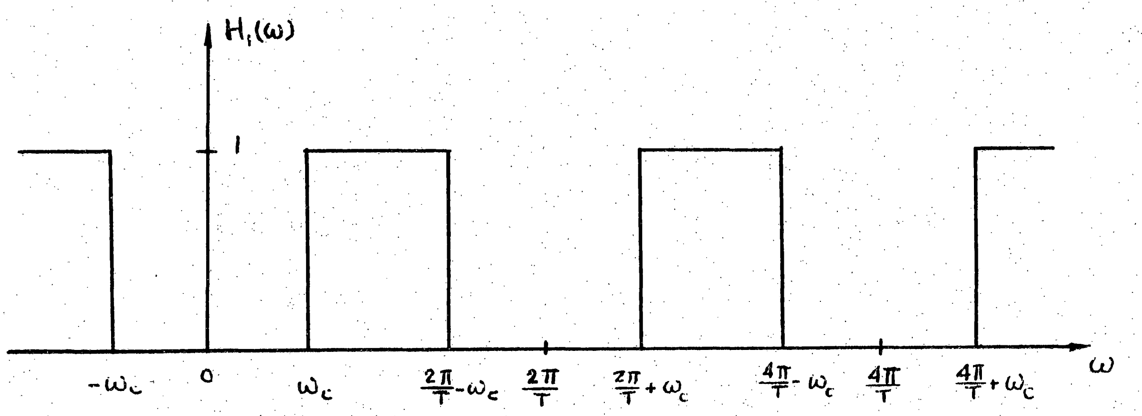


Figure 2: Ideal High Pass Filter (uniform sampling)

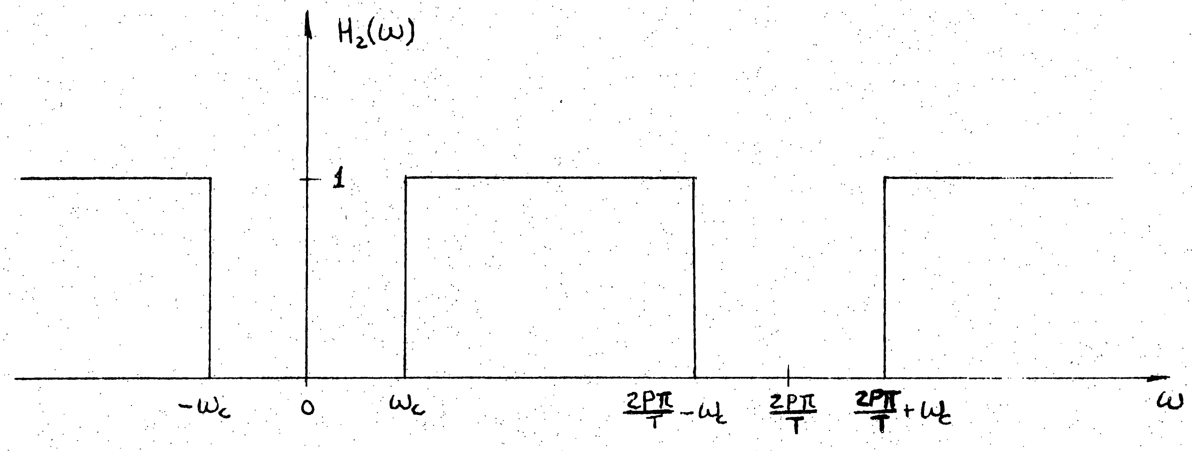


Figure 3: Ideal High Pass Filter (non-uniform sampling)

Figure 3, the frequency response of the same ideal digital filter based on simple stagger sampling is shown. It can be seen that the fundamental frequency of the filter has been "extended" to $2\pi P/T$ in the non-uniform sampling case. The fact that the fundamental period is extended is very important in the design of digital filters.

It can be seen that because of the periodical notches in digital filters, unnecessary loss of information (in the case of high pass filter) and aliasing (in the case of low pass filter) results. For example, if uniform sampling is used in a high pass filter design, then information between the frequencies $2\pi/T - \omega_c$ and $2\pi/T + \omega_c$ as well as their harmonics are rejected. However, if the non-uniform sampling technique is used, one can shift the notches to higher frequency ranges, so as not to affect the low frequency components of the signal, by selecting a proper value for P. This technique has been employed by radar engineers to implement the Moving Target Indicator Processor[1,2].

CHAPTER III

THE ERROR CRITERION FUNCTION

3.1 Development of the Minimum Error Criterion

In order to determine how good the weighting sequence $\{h_k\}$ can be selected so as to best approximate a desired transfer function behavior, some kind of criterion has to be established. We shall now consider the analysis for a general non-uniform sampling scheme, and then apply the results to the simple stagger sampling scheme. In this analysis, the criterion chosen for evaluating the goodness of approximation is the integral squared error (mean squared error) criterion, which is given by the following expression:

$$f(h) = \int_{\Omega} W(\omega)\,|H_d(\omega) - H_n(\omega)|^2 d\omega \tag{3.1}$$

where, $H_d(\omega)$ is the frequency response of the desired filter, $H_n(\omega)$ is the frequency response of the synthesized filter, Ω is the set of ω values over which the comparison between $H_d(\omega)$ and $H_n(\omega)$ is to be made, and $W(\omega)$ is a real non-negative weighting function. Note that expression (3.1) can also be written as:

$$f(h) = \int_{\Omega} W(\omega)\{H_d(\omega) - H_n(\omega)\}\{H_d(\omega) - H_n(\omega)\}^* d\omega \tag{3.2a}$$

$$= \int_{\Omega} W(\omega)\ \{|H_d(\omega)|^2 - H_d(\omega)\ H_n^*(\omega) -$$

$$H_n(\omega)\ H_d^*(\omega) + |H_n(\omega)|^2\}d\omega \qquad (3.2b)$$

where $(\ )^*$ denotes the complex conjugate of the argument.

Recall that for the most general case of non-uniform sampling, we have:

$$H_n(\omega) = \sum_{k=0}^{\infty} h_k e^{-j\omega(t_n - t_{n-k})} \qquad (2.5)$$

If this expression is substituted into equation (3.2) then we get:

$$f(h) = \int_{\Omega} W(\omega)\{H_d^2(\omega) - \sum_{k=0}^{\infty} h_k e^{-j\omega(t_n - t_{n-k})} H_d^*(\omega) -$$

$$H_d(\omega) \sum_{k=0}^{\infty} h_m^* e^{j\omega(t_n - t_{n-m})} + \sum_{k=0}^{\infty} h_k e^{-j\omega(t_n - t_{n-k})}$$

$$\sum_{m=0}^{\infty} h_m^* e^{j\omega(t_n - t_{n-m})}\}d\omega \qquad (3.3)$$

In general, $\{h_k\}$ is a complex valued sequence having coefficients $h_k = \alpha_k + j\beta_k$ for all k. (α_k and β_k are the real and imaginary parts of h_k respectively.) In order to minimize the squared error criterion, a necessary condition is that the partial derivatives of f(h) with respect to the α_k's and β_k's are all zero. That is:

$$\frac{\partial f(h)}{\alpha_k} = \frac{\partial f(h)}{\alpha\beta_k} = 0 \qquad \text{for } k=0,1,2... \tag{3.4}$$

Using equation (3.3) we can evaluate expression (3.4) as:

$$\frac{\partial f(h)}{\alpha\beta_k} = \int_\Omega W(\omega)\ \{-e^{-j\omega(t_n-t_{n-k})}H_d^*(\omega) - H_d(\omega)e^{j\omega(t_n-t_{n-k})}$$

$$+ e^{-j\omega(t_n-t_{n-k})} \sum_{m=0}^{\infty} h_m^* e^{j\omega(t_n-t_{n-m})}$$

$$+ \sum_{k=0}^{\infty} h_k e^{-j\omega(t_n-t_{n-k})} e^{j\omega(t_n-t_{n-m})} \}d\omega \tag{3.5}$$

If we equate equation (3.5) to zero, then we can rewrite this expression as:

$$\int_\Omega W(\omega)\,(\sum_{m=0}^{\infty} h_m^* e^{j\omega(t_n-t_{n-m})}\ e^{-j\omega(t_n-t_{n-k})}$$

$$+ \sum_{k=0}^{\infty} h_k e^{-j\omega(t_n-t_{n-k})} e^{j\omega(t_n-t_{n-m})})d\omega$$

$$= \int_\Omega W(\omega)\ \ (H_d^*(\omega)e^{-j\omega(t_n-t_{n-k})} + H_d(\omega)e^{j\omega(t_n-t_{n-k})})d\omega \tag{3.6}$$

Similarly, the second condition in equation (3.4) $\frac{\partial f(h)}{\partial\beta_k} = 0$ can also be expressed as:

$$\int_{\Omega} W(\omega) \ (\sum_{m=0}^{\infty} h_m^* e^{j\omega(t_n - t_{n-m})} e^{-j\omega(t_n - t_{n-k})}$$

$$- \sum_{k=0}^{\infty} h_k e^{-j\omega(t_n - t_{n-k})} e^{-j\omega(t_n - t_{n-m})}) d\omega$$

$$= \int_{\Omega} W(\omega) (H_d^*(\omega) e^{-j\omega(t_n - t_{n-k})} - H_d(\omega) e^{j\omega(t_n - t_{n-k})}) d\omega \tag{3.7}$$

By performing addition and subtraction on equations (3.6) and (3.7), one gets:

$$\int_{\Omega} W(\omega) \sum_{k=0}^{\infty} h_k^* e^{j\omega(t_n - t_{n-k})} e^{-j\omega(t_n - t_{n-m})} d\omega$$

$$= \int_{\Omega} W(\omega) \ H_d^*(\omega) e^{-j\omega(t_n - t_{n-m})} d\omega$$

and

$$\int_{\Omega} W(\omega) \sum_{k=0}^{\infty} h_k e^{-j\omega(t_n - t_{n-k})} e^{j\omega(t_n - t_{n-m})} d\omega$$

$$= \int_{\Omega} W(\omega) \ H_d(\omega) e^{j\omega(t_n - t_{n-m})} d\omega$$

Since $W(\omega)$ is chosen to be real, it is seen that the above two expressions are complex conjugates of each other. As a result, the condition for a minimum approximation error requires that:

$$\int_{\Omega} W(\omega) \sum_{k=0}^{\infty} h_k e^{-j\omega(t_n - t_{n-k})} e^{j\omega(t_n - t_{n-m})} d\omega$$

$$= \int_{\Omega} W(\omega)\ H_d(\omega)\ e^{j\omega(t_n - t_{n-m})} d\omega \qquad (3.8)$$

If we interchange the order of integration and summation we get:

$$\sum_{k=0}^{\infty} h_k \int_{\Omega} W(\omega) e^{j\omega(t_{n-k} - t_{n-m})} d\omega = \int_{\Omega} W(\omega)\ H_d(\omega) e^{j\omega(t_n - t_{n-m})} d\omega \qquad (3.9)$$

Now, let's define:

$$c_{mk} = \int_{\Omega} W(\omega) e^{j\omega(t_{n-k} - t_{n-m})} d\omega \qquad (3.10)$$

and

$$b_k = \int_{\Omega} W(\omega) H_d(\omega) e^{j\omega(t_n - t_{n-k})} d\omega \qquad (3.11)$$

then equation (3.9) becomes:

$$\sum_{k=0}^{\infty} h_k c_{mk} = b_m \qquad m,k=0,1,2,\ldots \qquad (3.12)$$

Note that equation (3.12) involves an infinite sum for implementing the causal infinite impulse response (IIR) filter. In practice, only a finite impulse response (FIR) filter is implemented. Hence, for a causal FIR filter equation (3.12) has to be modified as:

$$\sum_{k=0}^{N-1} h_k \; c_{mk} = b_m \quad m=0,1,2,...,N-1 \tag{3.13}$$

Expressed in vector form, this expression becomes:

$$C \; h = b \tag{3.14}$$

where N is the order of the causal FIR filter, C is the NxN matrix with c_{mk} as elements, h is the Nx1 vector formed by the weighting coefficient sequence $\{h_k\}$, and b is the Nx1 vector with b_k as elements.

The problem of selecting the optimal weighting coefficient vector of the transfer function h is now reduced to solving the system of N simultaneous linear equations (3.14) for h. That is we have:

$$h^o = C^{-1} \; b$$

where h^o is the optimal set of h_k's for a minimum value of error, and C^{-1} is the inverse of the matrix C.

3.2 Simple Stagger Sampling Application

We shall now restrict our interest to simple stagger sampling. If we explicitly include the functional dependency of the stagger sampling scheme parameter, as identified by the symbol ε, into the process of minimizing the approximation error, then the error criterion function f(h) becomes:

$$f(\varepsilon, h)$$

and $\quad C = C(\varepsilon)$

$$b = b(\varepsilon)$$

also $\quad h^o = h(\varepsilon) = C^{-1}(\varepsilon)\ b(\varepsilon) \qquad (3.15)$

A general algorithm for finding the optimal value of the stagger parameter ε is now developed:

Algorithm:

FINDING THE OPTIMAL VALUE OF THE STAGGER PARAMETER ε.

1. An arbitrary value of ε (between $\pm T$) is selected, then,
2. based on this value of ε; the matrix C and the vector b, equations (3.10) and (3.11), are evalued.
3. Equation (3.15) is now solved for the optimal set of coefficients $\{h_k\}$.
4. With this set of coefficients, the value of ε is varied in order to find an improvement (i.e., reducing the error criterion function). This new value of ε is obtained by perturbing the old value of ε in the direction of the negative gradient,

   i.e., $\quad \varepsilon_{new} = \varepsilon_{old} - \alpha \frac{df}{d\varepsilon}\Big|_{\varepsilon_{old}} \qquad (3.16)$

   where α is the step size.

5. After this new value is determined, the whole procedure is repeated until a minimum value of $f(h(\varepsilon),\varepsilon)$ is found. The minimum value of $f(h(\varepsilon),\varepsilon)$ is detected when no improvement in the error criterion is achieved by varying $h(\varepsilon)$ and ε.

3.3 Properties of $C(\varepsilon)$ and $b(\varepsilon)$

In this section we shall study the properties of the matrix $C(\varepsilon)$ and vector $b(\varepsilon)$. For illustrative purpose the aforementioned scheme is applied in synthesizing a high pass filter which has the ideal desired frequency response as shown in Figure 4.

In most radar applications, the stop band of the filter ($\omega < \omega_c$) is very small compared to the pass band (about 1% of the frequency spectrum out to $2\pi P/T$). Since the stop band is a very important feature of the high pass filter and in this case, it constitutes only a small fraction of the entire spectrum[2], a special weighting function $W(\omega)$ has to be used in order to emphasize this desired stop band characteristic. The weighting function used will have to weigh very heavy errors in the region of the stop band and the transition to the pass band. The weighting function that is used in this analysis is shown in Figure 5.

If we choose $\{\Omega: \ \omega \leq \omega_h\}$ as the set of frequencies for comparison, where $\omega_c < \omega_1 < \omega_h$, then the elements c_{mk} and b_k can be evaluated. Inserting the above information into

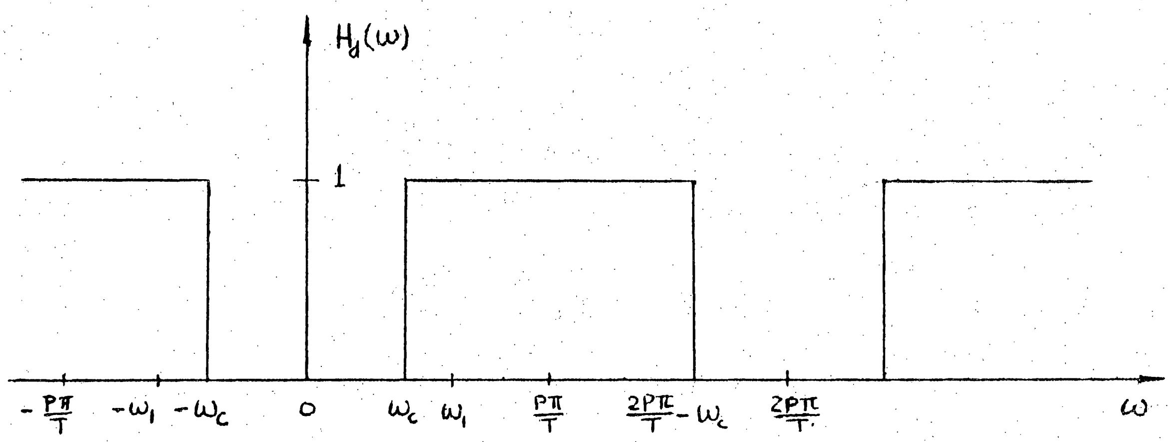


Figure 4: Ideal desired frequency response of HPF.

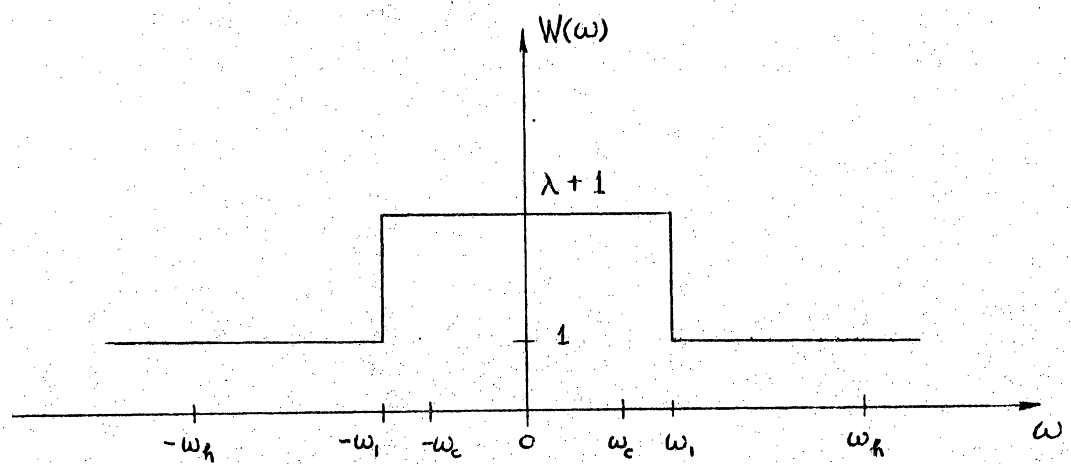


Figure 5: The weighting function

equation (3.10) and (3.11) one gets:

$$C_{mk} = \int_{-\omega_h}^{\omega_h} e^{j\omega(t_{n-k}-t_{n-m})} d\omega + \lambda \int_{-\omega_1}^{\omega_1} e^{j\omega(t_{n-k}-t_{n-m})} d\omega$$

$$= \begin{cases} \dfrac{e^{j\omega(t_{n-k}-t_{n-m})} \Big|_{-\omega_h}^{\omega_h} - e^{j\omega(t_{n-k}-t_{n-m})} \Big|_{-\omega_1}^{\omega_1}}{j(t_{n-k}-t_{n-m})} & m \neq k \\ \displaystyle\int_{-\omega_h}^{\omega_h} d\omega \quad + \quad \lambda \int_{-\omega_h}^{\omega_1} d\omega & m = k \end{cases}$$

$$= \begin{cases} \dfrac{2}{t_{n-k}t_{n-m}} \{\sin[\omega_h(t_{n-k}-t_{n-m})] \\ \qquad\qquad + \lambda \sin[\omega_1(t_{n-k}-t_{n-m})]\} & m \neq k \\ 2(\omega_h + \lambda\omega_1) & m = k \end{cases}$$

(3.17)

Solving equation (3.11) gives,

$$b_m = \int_{-\omega_h}^{\omega_c} e^{j\omega(t_n - t_{n-m})} d\omega + \lambda \int_{-\omega_1}^{\omega_c} e^{j\omega(t_n - t_{n-m})} d\omega$$

$$+ \lambda \int_{\omega_c}^{\omega_1} e^{j\omega(t_n - t_{n-m})} d\omega + \int_{+\omega_c}^{\omega_h} e^{j\omega(t_n - t_{n-m})} d\omega$$

$$= \begin{cases} \lambda \left. \dfrac{e^{j\omega(t_n - t_{n-m})}}{j(t_n - t_{n-m})} \right|_{\omega_c}^{\omega_1} + \left. \dfrac{e^{j\omega(t_n - t_{n-m})}}{j(t_n - t_{n-m})} \right|_{-\omega_1}^{\omega_c} & \\ \qquad\qquad & m \neq 0 \\ + \left. \dfrac{e^{j\omega(t_n - t_{n-m})}}{j(t_n - t_{n-m})} \right|_{\omega_c}^{\omega_h} + \left. \dfrac{e^{j\omega(t_n - t_{n-m})}}{j(t_n - t_{n-m})} \right|_{-\omega_h}^{\omega_c} & \\ \displaystyle\int_{-\omega_h}^{\omega_c} d\omega + \lambda \int_{-\omega_1}^{-\omega_c} d\omega + \lambda \int_{\omega_c}^{\omega_1} d\omega + \int_{\omega_c}^{\omega_h} d\omega & m = 0 \end{cases}$$

$$= \begin{cases} \dfrac{2}{t_n - t_{n-m}} \{\lambda \sin[\omega_1(t_n - t_{n-m})] - \lambda \sin[\omega_c(t_n - t_{n-m})] & \\ \qquad + \sin[\omega_h(t_n - t_{n-m})] - \sin[\omega_c(t_n - t_{n-m})]\} & m \neq 0 \\ 2\lambda(\omega_1 - \omega_c) + 2(\omega_h - \omega_c) & m = 0 \end{cases} \qquad (3.18)$$

If we choose N to be the order of the FIR filter and t_N to be the observation time, then the expressions $t_{n-k}-t_{n-m}$ and t_n-t_{n-m} in equations (3.17) and (3.18) becomes $t_{N-k}-t_{N-m}$ and t_N-t_{N-m} respectively. With reference to Table 1 with N odd, the different values of the expressions are summarized in Table 2.

Substituting the results from Table 2 into equation (3.17) yields:

$$c_{mk}=\begin{cases} \dfrac{2\sin[\omega_h((m-k)T+\varepsilon)]+2\lambda\sin[\omega_1((m-k)T+\varepsilon)]}{(m-k)T+\varepsilon} & \begin{array}{l}m=\text{odd}\\ k=\text{even}\end{array} \\[2ex] \dfrac{2\sin[\omega_h((m-k)T-\varepsilon)]+2\lambda\sin[\omega_1((m-k)T-\varepsilon)]}{(m-k)T+\varepsilon} & \begin{array}{l}m=\text{even}\\ k=\text{odd}\end{array} \\[2ex] \dfrac{2\sin[\omega_h(m-k)T]+2\lambda\sin[\omega_1(m-k)T]}{(m-k)T+\varepsilon} & m,k=\text{even} \\[2ex] 2(\omega_h+\lambda\omega_1) & m=k \end{cases} \tag{3.19}$$

Similarly, equation (3.18) becomes,

$$\begin{cases} \dfrac{2}{mT+\varepsilon}\{\lambda\sin[\omega_1(mT+\varepsilon)]-\lambda\sin[\omega_c(mT+\varepsilon)]+ \\[2ex] \qquad \sin[\omega_h(mT+\varepsilon)]-\sin[\omega_c(mT+\varepsilon)]\} & m=\text{odd} \end{cases}$$

$$b_m = \begin{cases} \frac{2}{mT}\{\lambda\sin(\omega_1 mT) - \lambda\sin(\omega_c mT) + \\ \qquad \sin(\omega_h mT) - \sin(\omega_c mT)\} & m=\text{even} \\ \\ 2[\lambda(\omega_1 - \omega_c) + (\omega_h + \omega_c)] & m=0 \end{cases} \tag{3.20}$$

A close observation of equation (3.19) shows that there are some special relationships between the elements of the matrix $C(\varepsilon)$. This can be demonstrated by the following examples:

Let m=1, k=2, then by Table 2 and equation (3.19), this gives,

$$c_{12} = \frac{2\sin[(-T + \varepsilon)\omega_h] + 2\lambda\sin[(-T + \varepsilon)\omega_1]}{(-T + \varepsilon)}$$

$$= \frac{2\sin\ [-(T - \varepsilon)\omega_h] + 2\lambda\sin\ [-(T - \varepsilon)\omega_1]}{-(T - \varepsilon)}$$

If we now, let m=2, k=1 then,

$$c_{21} = \frac{2\sin[(T - \varepsilon)\omega_h] + 2\lambda\sin[(T - \varepsilon)\omega_1]}{(T - \varepsilon)}$$

Since $\sin(x) = -\sin(-x)$, therefore, $c_{12} = c_{21}$. Similarly, c_{mk} can be shown to equal c_{km} for all m and k. This implies that matrix C has identical terms on both sides of the diagonal. Moreover, the diagonal elements are given by $2(\lambda\omega_1 + \omega_h)$ for m=k, equation (3.19), is constant for all m and k, i.e., the diagonal elements are the same and are constant. Hence, the matrix $C(\varepsilon)$ can be said to be conjugate symmetric, and n therefore Hermitian (i.e., $C^{\dagger}(\varepsilon)=C(\varepsilon)$), this also results in a similar characterization of $C^{\dagger}(\varepsilon)$: $(C^{-1}(\varepsilon))^{\dagger}=C^{\dagger}(\varepsilon)$); $((\ )^{\dagger}$ denotes the complex conjugate transpose of the argument). This symmetry property of $C(\varepsilon)$ will be used when we examine the properties of the error criterion function.

3.4 Properties of the Error Criterion Function

Since the error function provides a measure of goodness of the digital filter synthesized, a study of its properties is very beneficial in terms of minimizing the approximation error. By expanding equation (3.1) one obtains:

$$f(h) = \int_{\Omega} W(\omega)(H_d(\omega) - H_n(\omega))(H_d^*(\omega) - H_n^*(\omega))d\omega$$

$$= \int_{\Omega} W(\omega)H_d(\omega)H_d^*(\omega)d\omega + \int_{\Omega} W(\omega)H_d(\omega)H_n^*(\omega)d\omega -$$

$$\int_{\Omega} W(\omega)H_n(\omega)H_d^*(\omega)d\omega + \int_{\Omega} W(\omega)H_n(\omega)H_n^*(\omega)d\omega$$

TABLE 2

Values of $t_{N-k}-t_{N-m}$ and t_N-t_{N-m} for N=odd

m,k		$t_{N-k}-t_{N-m}$	t_N-t_{N-m}
m=odd,	k=even	$(m-k)T + \varepsilon$	$mT + \varepsilon$
m-even,	k=odd	$(m-k)T - \varepsilon$	mT
m=odd,	k=odd	$(m-k)T$	$mT + \varepsilon$
m=even,	k=even	$(m-k)T$	mT

$$= \int_{\Omega} W(\omega)|H_d(\omega)|^2 d\omega - \int_{\Omega} W(\omega)H_d(\omega)H_n^*(\omega)d\omega -$$

$$\int_{\Omega} W(\omega)H_n(\omega)H_d^*(\omega)\, d\omega + \int W(\omega)|H_n(\omega)|^2 d\omega \qquad (3.21)$$

Substituting equation (2.5) into the above expression, the error function can be written as:

$$f(h) = \int_{\Omega} W(\omega)|H_d(\omega)|^2 d\omega - \sum_{m=0}^{N-1} h_m^* \int_{\Omega} W(\omega)H_d(\omega)e^{j\omega(t_n - t_{n-m})}d\omega$$

$$- \sum_{k=0}^{N-1} h_k \int_{\Omega} W(\omega)H_d^*(\omega)e^{-j\omega(t_n - t_{n-k})}d\omega$$

$$+ \sum_{k=0}^{N-1} h_k \sum_{m=0}^{N-1} h_m^* \int_{\Omega} W(\omega)e^{j\omega(t_{n-k} - t_{n-m})}d\omega \qquad (3.22)$$

If we define:

$$a = \int_{\Omega} W(\omega)|H_d(\omega)|^2 d\omega \qquad (3.23)$$

and substitute the b_k's and c_{mk}'s as defined in equations (3.10) and (3.11) into equation (3.22), then this expression can be simplified to:

$$f(h) = a - \sum_{k=0}^{N-1} h_k^* b_k - \sum_{k=0}^{N-1} b_k^* h_k + \sum_{k=0}^{N-1} \sum_{m=0}^{N-1} h_k^* c_{mk} h_m \qquad (3.24)$$

Expressed in vector form this becomes:

$$f(h) = a - h^{\dagger}b - b^{\dagger}h + h^{\dagger}Ch \tag{3.25}$$

and the Nxl vector h will hereby refered to as the filter coefficient vector.

Note that the last term in equation (3.24) $(\sum_{k=0}^{N-1} \sum_{m=0}^{N-1} h_k^* C_{mk} h_m)$ is always positive, because it represents the squared magnitude term, $\int_{\Omega} W(\omega)|H_n(\omega)|^2 d\omega$ in equation (3.21). It then follows that the product $h^{\dagger}ch$ is positive. Since C is a NxN Hermitian symmetric matrix (section III.2), therefore, it is evident that C is a positive definite matrix in order that the above statement be true.

As was done in the previous section, we shall now introduce the explicit dependence of the stagger sampling parameter ε into the error criterion function, that is,

$$f(\varepsilon,h) = a - h^{\dagger}b(\varepsilon) - b^{\dagger}(\varepsilon)h + h^{\dagger}C(\varepsilon)h \tag{3.26}$$

The task now is to select the value of ε and h which will minimize the criterion $f(\varepsilon,h)$. Recall that the optimal transfer coefficient vector h is:

$$h^{o} = h(\varepsilon) = C^{-1}(\varepsilon)b(\varepsilon) \tag{3.15}$$

then it is clear that for a particular value of ε:

$$f(\varepsilon,h(\varepsilon)) \leq f(\varepsilon,h) \tag{3.27}$$

The justification for expression (3.27) follows from the fact that $C(\varepsilon)$ is a positive definite NxN matrix, hence, Rank $\{C(\varepsilon)\}$ is equal to N and thus the inverse of $C(\varepsilon)$ exists. If the inverse of $C(\varepsilon)$ exists, then the solution of $h(\varepsilon)$, equation (3.15), is unique. Therefore, there is only one minimum value of $f(\varepsilon,h)$ for a particular value of ε.

Hence, for the optimal filter coefficient vector $h(\varepsilon)$.

$$\min_h f(\varepsilon,h) = f(\varepsilon,h(\varepsilon)) \leq f(\varepsilon,h) \tag{3.28}$$

In order to study the effects of varying the parameters ε and h on the error criterion function $f(\varepsilon,h)$, one has to examine the derivatives of the function with respect to the variables ε and h's. In previous sections, it was shown that for a given value of ε, the gradient of the functional with respect to h will yield an optimal solution for h as:

$$h^o = h(\varepsilon) = C^{-1}(\varepsilon)b(\varepsilon) \tag{3.15}$$

Now, if we take the derivative of the functional equation (3.26), with respect to ε, we can go under two possible conditions. First, we can take the derivative of equation (3.26) with respect to ε while holding h constant, and then evaluate the derivative with the optimal set of h coefficients, i.e., at $h = h(\varepsilon)$. Or, we can take the derivative with respect to ε for the error criterion function with the optimal set of h coefficients as a variable of the function (i.e., $f(\varepsilon,h) = f(\varepsilon,h(\varepsilon))$).

For the first procedure we get:

$$\frac{d}{d\varepsilon}f(\varepsilon,h) = \frac{d}{d\varepsilon}\left[a - b^{\dagger}(\varepsilon)\, h - h^{\dagger}b(\varepsilon) + h^{\dagger}C(\varepsilon)\, h\right]$$

$$= -\frac{db^{\dagger}(\varepsilon)}{d\varepsilon}\, h - h^{\dagger}\frac{db(\varepsilon)}{d\varepsilon} + h^{\dagger}\frac{dC(\varepsilon)}{d\varepsilon}\, h \tag{3.29}$$

$$\therefore \frac{d}{d\varepsilon} f(\varepsilon,h)\Big|_{h=h(\varepsilon)} = -\frac{db^{\dagger}(\varepsilon)}{d\varepsilon}\left[C^{-1}(\varepsilon)b(\varepsilon)\right]-\left[C^{-1}(\varepsilon)b(\varepsilon)\right]^{\dagger}$$

$$\frac{db(\varepsilon)}{d\varepsilon} + \left[C^{-1}(\varepsilon)b(\varepsilon)\right]^{\dagger}\frac{dC(\varepsilon)}{d\varepsilon}\left[C^{-1}(\varepsilon)b(\varepsilon)\right]$$

Since $C(\varepsilon)$ is symmetric (as shown in section II.2) so will $C(\varepsilon)$, thus the expression can be modified and becomes:

$$\frac{d}{d\varepsilon} f(\varepsilon,h)\Big|_{h=h(\varepsilon)} = -b^{\dagger}(\varepsilon)C^{-1}(\varepsilon)\,\frac{db(\varepsilon)}{d\varepsilon} - \frac{db^{\dagger}(\varepsilon)}{d}\,C^{-1}(\varepsilon)b(\varepsilon) +$$

$$b^{\dagger}(\varepsilon)C^{-1}(\varepsilon)\,\frac{dC(\varepsilon)}{d\varepsilon}\,C^{-1}(\varepsilon)b(\varepsilon) \tag{3.30}$$

Note that:

$$I = C(\varepsilon)C^{-1}(\varepsilon)$$

where I is the identity matrix. If we take the derivative with respect to ε on both sides of this identity, we get:

$$0 = \frac{dC(\varepsilon)}{d\varepsilon} C^{-1}(\varepsilon) + C(\varepsilon) \frac{dC^{-1}(\varepsilon)}{d\varepsilon}$$

which can be written as:

$$\frac{dC(\varepsilon)}{d\varepsilon} C^{-1}(\varepsilon) = - C(\varepsilon) \frac{dC^{-1}(\varepsilon)}{d\varepsilon} \tag{3.31}$$

Substituting this into equation (3.30) yields:

$$\begin{aligned}
\left.\frac{df(\varepsilon,h)}{d\varepsilon}\right|_{h=h(\varepsilon)} &= - b^{\dagger}(\varepsilon) C^{-1}(\varepsilon)\frac{db(\varepsilon)}{d\varepsilon} - \frac{db^{\dagger}(\varepsilon)}{d\varepsilon} C^{-1}(\varepsilon) b(\varepsilon) \\
&\quad - b^{\dagger}(\varepsilon) C^{-1}(\varepsilon)C(\varepsilon) \frac{dC^{-1}(\varepsilon)}{d\varepsilon} b(\varepsilon) \\
&= - b^{\dagger}(\varepsilon)C^{-1}(\varepsilon)\frac{db(\varepsilon)}{d\varepsilon} - \frac{db^{\dagger}(\varepsilon)}{d\varepsilon} C^{-1}(\varepsilon) b(\varepsilon) \\
&\quad - b^{\dagger}(\varepsilon) \frac{dC^{-1}(\varepsilon)}{d\varepsilon} b(\varepsilon) \\
&= - b^{\dagger}(\varepsilon)\left[C^{-1}(\varepsilon)\frac{db(\varepsilon)}{d\varepsilon} + \frac{dC^{-1}(\varepsilon)}{d\varepsilon} b(\varepsilon)\right] \\
&\quad - \frac{db^{\dagger}(\varepsilon)}{d\varepsilon} C^{-1}(\varepsilon) b(\varepsilon) \\
&= - b^{\dagger}(\varepsilon)\frac{d}{d\varepsilon} \left[C^{-1}(\varepsilon) b(\varepsilon)\right] - \frac{db^{\dagger}(\varepsilon)}{d\varepsilon}C^{-1}(\varepsilon)b(\varepsilon)
\end{aligned}$$

$$= - b^{\dagger}(\varepsilon) \frac{dh(\varepsilon)}{d\varepsilon} - \frac{db^{\dagger}(\varepsilon)}{d\varepsilon} h(\varepsilon)$$

$$= - \frac{d}{d\varepsilon} [b^{\dagger}(\varepsilon)h(\varepsilon)] \qquad (3.32)$$

For a minimum value of $f(\varepsilon,h)$ one would anticipate that $\frac{d}{d\varepsilon}f(\varepsilon,h) \Big|_{h=h(\varepsilon)} = 0$, and then solve for the value of ε which will satisfy this condition.

For the second procedure we shall take the derivative of the error criterion function with respect to ε, which has $h(\varepsilon)$ as its argument, that is:

$$\frac{d}{d\varepsilon}f(\varepsilon,h(\varepsilon)) = \frac{d}{d\varepsilon} [a - h^{\dagger}(\varepsilon)\, b(\varepsilon) - b^{\dagger}(\varepsilon)\, h(\varepsilon)$$

$$+ h^{\dagger}(\varepsilon)\, C(\varepsilon)\, h(\varepsilon)]$$

With $h(\varepsilon) = C^{-1}(\varepsilon)\, b(\varepsilon)$, this can be written as:

$$f(\varepsilon,h(\varepsilon)) = \frac{d}{d\varepsilon} \{a - [C^{-1}(\varepsilon)b(\varepsilon)]^{\dagger}b(\varepsilon)$$

$$- b^{\dagger}(\varepsilon)\, [C^{-1}(\varepsilon)b(\varepsilon)]$$

$$+ [C^{-1}(\varepsilon)b(\varepsilon)]^{\dagger}C(\varepsilon)[C^{-1}(\varepsilon)\, b(\varepsilon)]\}$$

$$= \frac{d}{d\varepsilon} \{a - b^{\dagger}(\varepsilon)C^{-1}(\varepsilon)\, b(\varepsilon)$$

$$- b^{\dagger}(\varepsilon)C^{-1}(\varepsilon)\, b(\varepsilon)$$

$$+ b(\varepsilon)C^{-1}(\varepsilon)\ C(\varepsilon)C^{-1}(\varepsilon)\ b(\varepsilon)\}$$

$$= \frac{d}{d\varepsilon}\{a - b^{\dagger}(\varepsilon)C^{-1}(\varepsilon)\ b(\varepsilon) \tag{3.33}$$

$$= -\frac{d}{d\varepsilon}[b^{\dagger}(\varepsilon)C^{-1}(\varepsilon)\ b(\varepsilon)]$$

$$= -\frac{d}{d\varepsilon}[b^{\dagger}(\varepsilon)\ h(\varepsilon)] \tag{3.34}$$

Surprisingly, this is the same result as obtained in the first procedure, equation (3.32). However, a second thought would tell us that they have to be equal, because they represent the same point on the error criterion function for the same value of ε, i.e.,

$$f(\varepsilon,h)\Big|_{h=h(\varepsilon)} = f(\varepsilon,h(\varepsilon))$$

and hence they should have the same slope with respect to ε. Therefore, we can conclude that:

$$\frac{d}{d\varepsilon}f(\varepsilon,h)\Big|_{h=h(\varepsilon)} = \frac{d}{d\varepsilon}f(\varepsilon,h(\varepsilon)) = -\frac{d}{d\varepsilon}[b^{\dagger}(\varepsilon)h(\varepsilon)] \tag{3.35}$$

Furthermore, we shall refer the total derivative of the error function with respect to ε as $f_{\varepsilon}(\varepsilon,h(\varepsilon))$ and thus:

$$f_{\varepsilon}(\varepsilon,h(\varepsilon)) = -\frac{d}{d\varepsilon}[b^{\dagger}(\varepsilon)h(\varepsilon)]$$

$$= -\frac{db^{\dagger}(\varepsilon)}{d\varepsilon}h(\varepsilon) - b^{\dagger}(\varepsilon)\frac{dh(\varepsilon)}{d\varepsilon} \tag{3.36}$$

Recall that: $C(\varepsilon)h(\varepsilon) = b(\varepsilon)$ hence, if we take the derivative with respect to ε on both sides of this identity, we get:

$$\frac{dC(\varepsilon)}{d\varepsilon}h(\varepsilon) + C(\varepsilon)\frac{dh(\varepsilon)}{d\varepsilon} = \frac{db(\varepsilon)}{d\varepsilon}$$

$$\therefore \frac{dh(\varepsilon)}{d\varepsilon} = C^{-1}(\varepsilon)\left[\frac{db(\varepsilon)}{d\varepsilon} - \frac{dC(\varepsilon)}{d\varepsilon}h(\varepsilon)\right] \tag{3.37}$$

Substituting this into equation (3.34), yields:

$$f_\varepsilon(\varepsilon,h(\varepsilon)) = -\frac{db^\dagger(\varepsilon)}{d\varepsilon}h(\varepsilon) - b^\dagger(\varepsilon)\,C^{-1}(\varepsilon)\left[\frac{db(\varepsilon)}{d\varepsilon} - \frac{dC(\varepsilon)}{d\varepsilon}h(\varepsilon)\right] \tag{3.38}$$

This will be the relationship used to generate the derivative of the error criterion function with respect to ε in the simulations.

Before leaving this section we shall recall that the error criterion function is a function of the two variables ε (the simple stagger sampling parameter) and h (the transfer function coefficient vector). Also, it was found that the h_k's of h, would also be a function of ε if h is chosen to be optimal. Hence, in order to achieve the best approximation to a desired filter, one has to select the value of ε that would minimize the error criterion function. The technique used to determine this optimal value of ε is as outlined in the Algorithm in section III.2. This algorithm will form the sole basis of the numerical examples presented in the next section.

CHAPTER IV

NUMERICAL EXAMPLES AND DISCUSSIONS

In order to demonstrate the concepts and observations developed in the previous sections, we shall consider the design of a high pass filter. This filter will have a functional relationship between the input and output signals as given by equation (2.1), i.e.,

$$y(t_n) = \sum_{k=0}^{N-1} h_k x(t_{n-k}) \tag{2.1}$$

where $y(t_n)$ is the output of the linear filter at time t_n; $x(t_n)$ is the input at time t_n; and h_k are the transfer function coefficients.

The design objective, now, is to select the value of the stagger sampling parameter ε of the simple stagger sampling and the filter transfer function coefficient vector h, so that the minimum error criterion described in section III can be satisfied. The desired frequency response of the digital high pass filter will be as shown in Figure 4. The interval set for comparison is: $\Omega = \{\omega : |\omega| \leq P\pi/T\}$. In a practical situation, this kind of filter -- '<u>the clutter rejection filter</u>' -- will have a very small notch, i.e., the relative width of the stop band; given by: $2\omega_c/(2\pi P/T) = \omega_c T/P\pi$ is small. In order to ensure the presence of the notch at low frequency in the implemented filter, one would choose a weighting function which weighs heavily around the regions of transition between the stop band and the pass band as well as the stop band itself. A typical weighting function of this type is shown in Figure 5, i.e.,

$$W(\omega) = \begin{cases} \lambda + 1 & |\omega| \leq \omega_1 \\ 1 & \text{otherwise} \end{cases} \tag{4.1}$$

in which $\lambda >> 1$ and $\omega_1 > \omega_c$. The value of λ has to be large in order to give a good match to the stop band at low frequencies in the frequency response of the ideal high pass filter. Note that a small value of λ will give a poor match and hence will not give a desired stop band at low frequency, i.e., it might result in an all pass filter, which is to be avoided.

In order to contrast the two different sampling schemes: uniform and simple stagger, the following set of parameters is used in synthesizing a high pass filter (Table 3). Where all ω's are in radians; with ε defined as $\varepsilon = qT/P$, a value of $q = 0$ would result in a filter implemented by uniform sampling, and for $q \neq 0$ the sampling scheme is the simple stagger sampling.

We shall now proceed by following the steps as outlined in the Algorithm of Chapter III. The b_k's and c_{mk}'s (equations (3.17) and (3.18), respectively) are now

TABLE 3

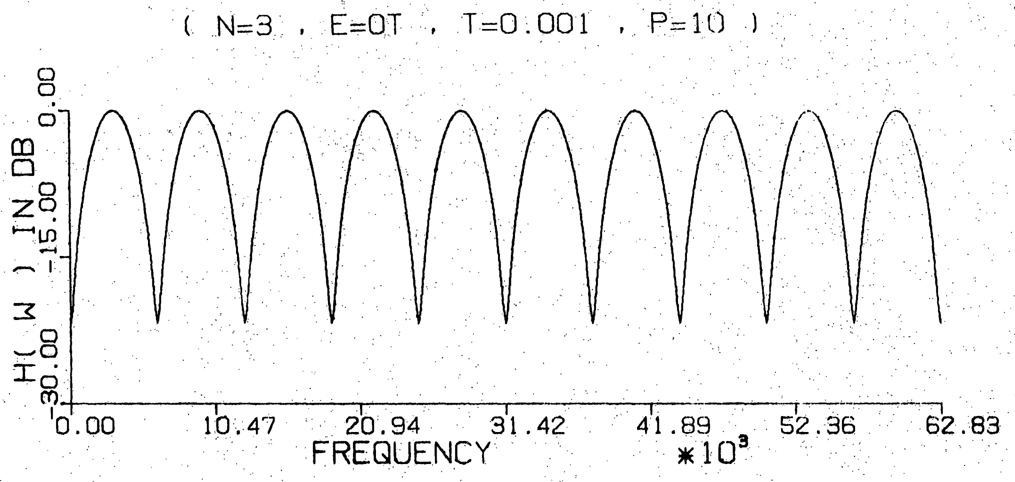
Parameters for a digital high pass filter

$\lambda = 10^6$	$n = 3, 9$
$\omega_1 = 3\omega_c$	$P = 10$
$\omega_h = P\pi/T$	$T = 0.001$ sec
$\omega_c = \pi/10T$	$q = 0, 1$
$\varepsilon = qT/P$	

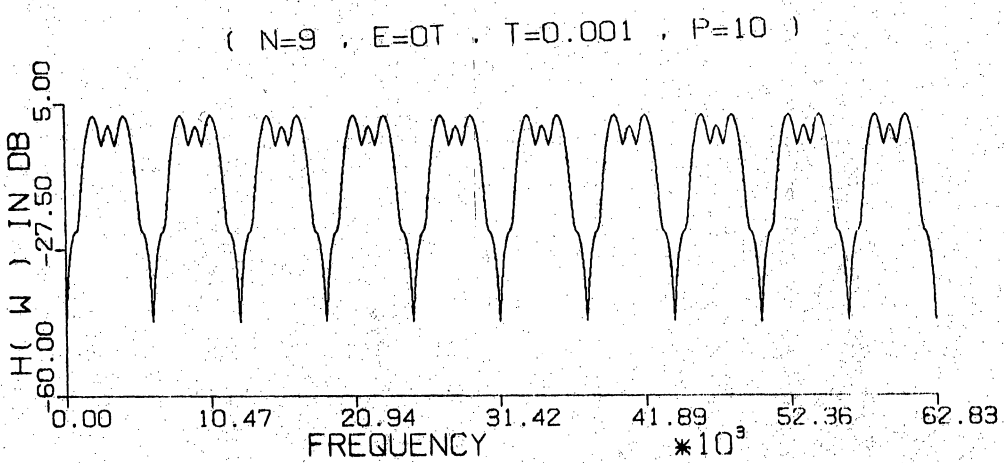
evaluated. Then solving the system of N equations in N unknowns (equation (3.11)), the optimal values of the filter transfer function coefficients for a given ε are obtained. Hence, the frequency response can be obtained by evaluating the Fourier Transform relationship:

$$H(\varepsilon) = \sum_{k=0}^{N-1} h_k e^{-j\omega(t_N - t_{N-k})} \tag{2.5}$$

Figure 6 shows the frequency response obtained with $N = 3$, 9 and $\varepsilon = 0$ (i.e., uniform sampling). Note that when N is increased a better roll off for the transition bands can be achieved which is common as in all (analog or digital) filter synthesis. If we now set $\varepsilon = 0.1T$ (simple stagger sampling, T is the uniform sampling period) then the frequency response in Figure 7 shows that the periodic notches are removed, although some 'troughs' are still observable in the pass band spectrum. Again, with a high order N, the frequency response of the filter is comparatively flatter, i.e., the troughs are reduced and hence we have a better approximation to the desired response. From this example one can see that the selection of the parameters N and ε will have definite influences on the coefficients h_k and hence on the frequency response and error criterion function of the filter. Later on, we shall study the influence of another important parameter

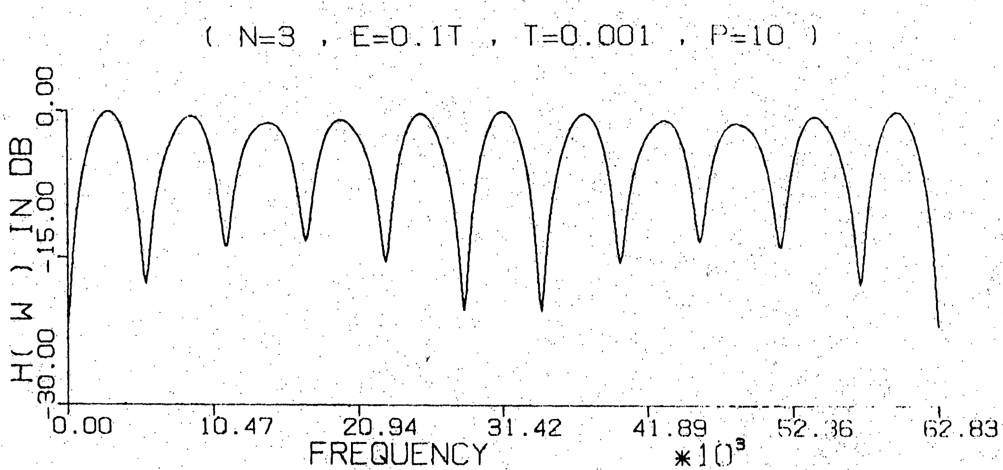


(a)

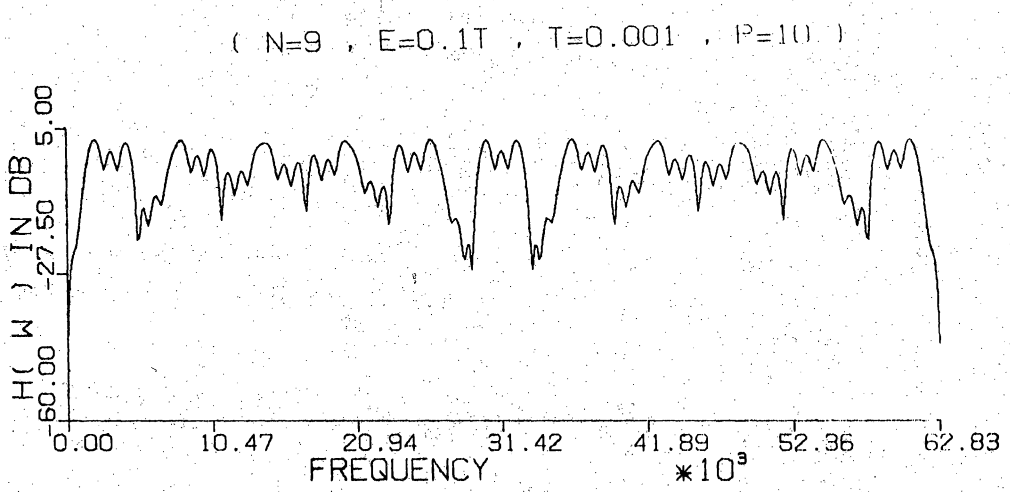


(b)

Figure 6 : Frequency reponses of a digital filter (uniform sampling)



(a)



(b)

Figure 7: Frequency response of a digital filter (non-uniform sampling)

λ, the constant in the weighting function, on the frequency response and approximation error of the filter.

In order to determine the optimal value of ε which will minimize the error criterion function, one now, has to perform step 4 of the Algorithm in Chapter III. That is, to find the direction of negative gradient of the error criterion function and perturb the value of ε accordingly. However, in the process of simulation it was suspected that the minimum value of $f(\varepsilon,h(\varepsilon))$ did not change drastically as the value of ε was varied between $\pm T$. This behavior is observed from a plot of $f(\varepsilon,h)$ for several different values of h. With the parameters as given in Table 4, $f(\varepsilon,h)$ is plotted and is shown in Figure 8.

It can be seen from Figure 8 that the minimum values of $f(\varepsilon,h)$ for all different values of h have more or less the same values (i.e., the difference between these minimum values is not large, and that these minimum values occur at the selected values of ($\varepsilon = \pm 0.7T$, $\pm 0.5T$, $\pm 0.1T$, 0). This confirms equation (3.28), which says:

$$\min_h f(\varepsilon,h) = f(\varepsilon,h(\varepsilon)) \tag{3.28}$$

If we recall equation (3.33) we see that:

$$f(\varepsilon,h(\varepsilon)) = a - b^{\dagger}(\varepsilon)h(\varepsilon) \tag{4.2}$$

TABLE 4

Parameters for the graph of $f(\varepsilon,h)$ vs ε.

$\lambda = 10^6$	$N = 9$
$\omega_1 = 3\ \omega_c$	$P = 10$
$\omega_h = P\pi/T$	$T = 1$ sec
$\omega_c = 3\pi/10T$	$q = \pm 7,\ \pm 5,\ \pm 1,\ 0$
$\varepsilon = \pm 0.7T,\ \pm 0.5T,\ \pm 0.1T,\ 0$	

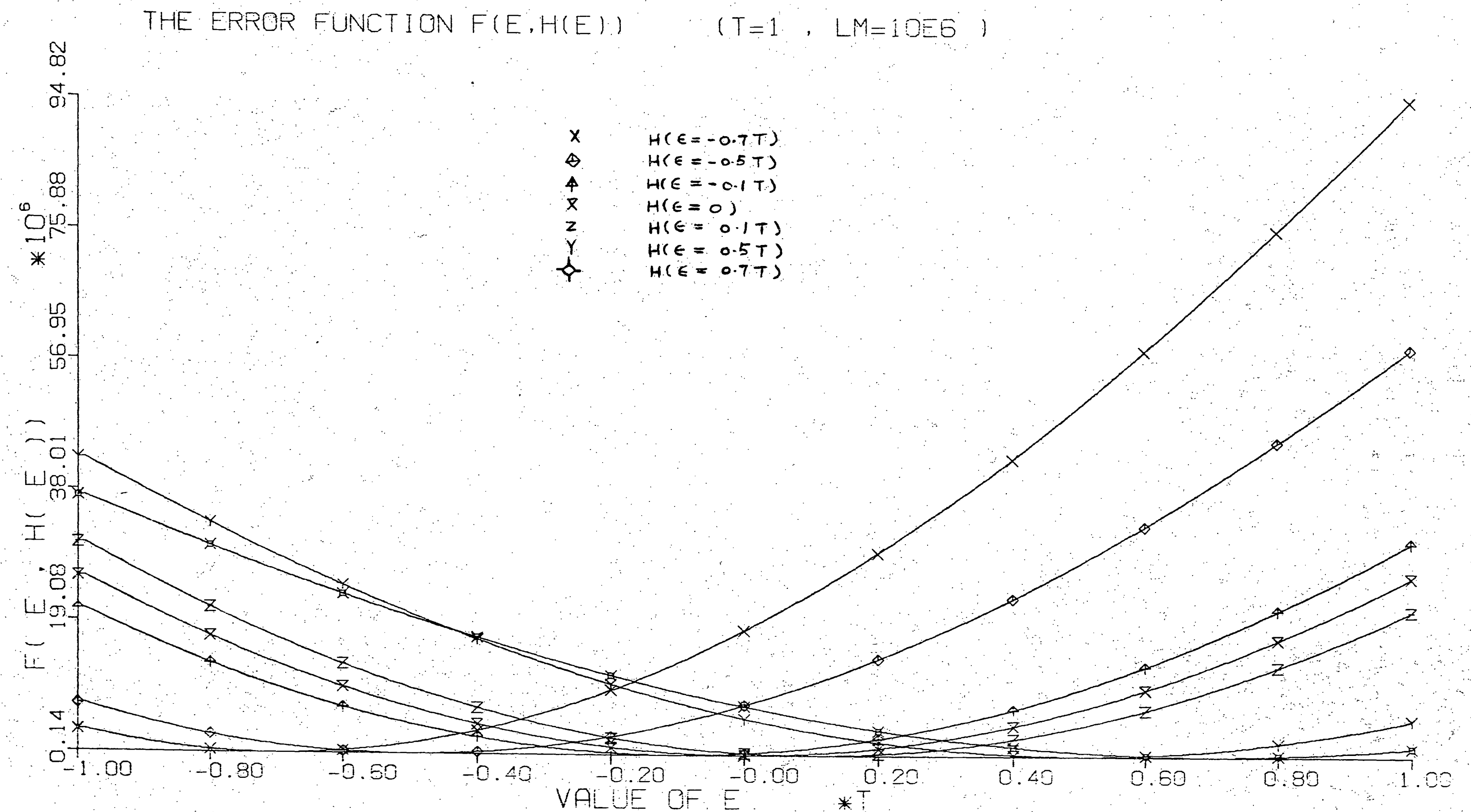


Figure 8: The Error Criterion Function ($\cdot = 10^6$)

Since a is independent of ε, and hence, in order that $f(\varepsilon,h(\varepsilon))$ to have approximately the same minimum value for different ε's, i.e., for minimum $f(\varepsilon,h(\varepsilon))$ tends to be constant over the range of ε interested, the product $b^{\dagger}(\varepsilon)h(\varepsilon)$ will have to be approximately constant. To verify this fact, a series of tables with different parameters is generated and is presented in Table 5(a-e). These tables show the values of the product $b^{\dagger}(\varepsilon)h(\varepsilon)$, the value of $f(\varepsilon,h(\varepsilon))$ and of $f(\varepsilon,h(\varepsilon))$ with $h(\varepsilon) = h(0)$, corresponding to a set of values of $\varepsilon = 0, \pm 0.1T, \pm 0.5T, \pm 0.7T$.

It can be seen that, although the product of $b^{\dagger}(\varepsilon)h(\varepsilon)$ 'tends' to have very little variation, the value of $f(\varepsilon,h(\varepsilon))$ does not necessarily have the same small order of variation. In fact, the variation in the value of $f(\varepsilon,h(\varepsilon))$ is larger than in the product of $b^{\dagger}(\varepsilon)h(\varepsilon)$. This is due to the fact that $f(\varepsilon,h(\varepsilon))$ has a smaller order than the product of $b^{\dagger}(\varepsilon)h(\varepsilon)$. Hence, a small change in $b^{\dagger}(\varepsilon)h(\varepsilon)$ may reflect in a large change in $f(\varepsilon,h(\varepsilon))$.

Based on these results, we can now hypothesis the following conjecture:

CONJECTURE:

For a particular filter transfer function coefficient vector, h, obtained optimally for a given value of $\bar{\varepsilon}$, the plot of $f(\varepsilon,h(\bar{\varepsilon}))$ appears to have a minimum at $\varepsilon = \bar{\varepsilon}$. It was then hypothesized that the condition

$$f(\bar{\varepsilon},h(\bar{\varepsilon})) \leq f(\varepsilon,h(\bar{\varepsilon}))$$

for all ε in the range $-T<\varepsilon<T$.

TABLE 5

values of $f(\epsilon, h(\epsilon))$ for different ϵ's

(a)

WC = 0.62832D 00   W1 = 0.18850D 01   WH = 0.47124D 02

ORDER = 9   LAMDA = 0.10D 08

P = 0.15D 02   T = 0.10D 01

E	B(E)*H(E)	F(E,H(E))	F(E,H(0))
-0.7T	0.22566D 08	0.25668D 07	0.10248D 09
-0.5T	0.22458D 08	0.26753D 07	0.54001D 08
-0.1T	0.22223D 08	0.29099D 07	0.49758D 07
0.0T	0.22165D 08	0.29675D 07	0.29675D 07
0.1T	0.22109D 08	0.30242D 07	0.50890D 07
0.5T	0.21889D 08	0.32442D 07	0.54431D 08
0.7T	0.21776D 08	0.33572D 07	0.10289D 09

(b)

WC = 0.31416D 03   W1 = 0.10996D 04   WH = 0.15708D 05

ORDER = 9   LAMDA = 0.10D 05

P = 0.50D 01   T = 0.10D-02

E	B(E)*H(E)	F(E,H(E))	F(E,H(0))
-0.7T	0.13520D 08	0.22177D 07	0.74094D 07
-0.5T	0.13659D 08	0.20785D 07	0.48133D 07
-0.1T	0.13667D 08	0.20706D 07	0.21827D 07
0.0T	0.13655D 08	0.20823D 07	0.20823D 07
0.1T	0.13642D 08	0.20949D 07	0.22069D 07
0.5T	0.13553D 08	0.21846D 07	0.49078D 07
0.7T	0.13453D 08	0.22838D 07	0.75037D 07

(c)

WC = 0.94248D 03   W1 = 0.32987D 04   WH = 0.15708D 05

ORDER = 9   LAMDA = 0.10D 06

P = 0.50D 01   T = 0.10D-02

E	B(E)*H(E)	F(E,H(E))	F(E,H(0))
-0.7T	0.37699D 09	0.94273D 08	0.15055D 09
-0.5T	0.38061D 09	0.90452D 08	0.12329D 09
-0.1T	0.39721D 09	0.74050D 08	0.75324D 08
0.0T	0.40099D 09	0.70276D 08	0.70276D 08
0.1T	0.40325D 09	0.68010D 08	0.69171D 08
0.5T	0.39623D 09	0.75037D 08	0.10184D 09
0.7T	0.38798D 09	0.83285D 08	0.13175D 09

(d)

WC = 0.31416D 00   W1 = 0.94248D 00   WH = 0.31416D 02

ORDER = 9   LAMDA = 0.10D 04

P = 0.10D 02   T = 0.10D 01

E	B(E)*H(E)	F(E,H(E))	F(E,H(0))
-0.7T	0.10827D 04	0.23490D 03	0.25402D 03
-0.5T	0.10814D 04	0.23616D 03	0.24538D 03
-0.1T	0.10819D 04	0.23565D 03	0.23599D 03
0.0T	0.10823D 04	0.23533D 03	0.23533D 03
0.1T	0.10826D 04	0.23503D 03	0.23535D 03
0.5T	0.10833D 04	0.23426D 03	0.24197D 03
0.7T	0.10834D 04	0.23415D 03	0.24894D 03

(e)

```
*****************************************************************
WC = 0.31416D 00      W1 = 0.94248D 00      WH = 0.31416D 02
          ORDER = 9               LAMDA = 0.10D 07
          P = 0.10D 02               T = 0.10D 01
*****************************************************************
   E         B(E)*H(E)        F(E,H(E))        F(E,H(0))
*****************************************************************

 -0.7T      0.11071D 07      0.14963D 06      0.15556D 08
 -0.5T      0.11108D 07      0.14585D 06      0.81051D 07
 -0.1T      0.11092D 07      0.14748D 06      0.46979D 06
  0.0T      0.11078D 07      0.14887D 06      0.14887D 06
  0.1T      0.11061D 07      0.15059D 06      0.47288D 06
  0.5T      0.10952D 07      0.16146D 06      0.81172D 07
  0.7T      0.10863D 07      0.17041D 06      0.15568D 08
```

However, as an example shown in Figure 9 for a small value of λ (i.e., 10^3), it illustrates that this condition is not true in general. The reason that Figure 8 indicated this condition is thought to be caused by the large value of λ (i.e., 10^6) used in the design.

Nevertheless, if we can very crudely regard these variations in the functional $f(\varepsilon,h(\varepsilon))$ with respect to different ε to be insignificant, then, we will be able to manipulate the value of ε in order to redistribute the error over the entire spectrum. That is, because only a small improvement in the error criterion function is obtained by varying the parameter ε, one can, instead of aiming to minimize the error criterion function, select the value of ε that will produce a frequency response that is best for the processing of a particular input signal spectrum.

To verify this, the filter with parameters as shown in Table 4 is synthesized. The value of q, again, has been varied for the different values in the set $\{q: \pm 7, 2, \pm 1\}$. The frequency response of the synthesized filters corresponding to each different value of ε is presented in Figure 10(a-e). Notice that the locations and the depths of the troughs are not the same in each case. This means, although we cannot totally eliminate the troughs, we can manipulate the value of the parameter ε, such that, an

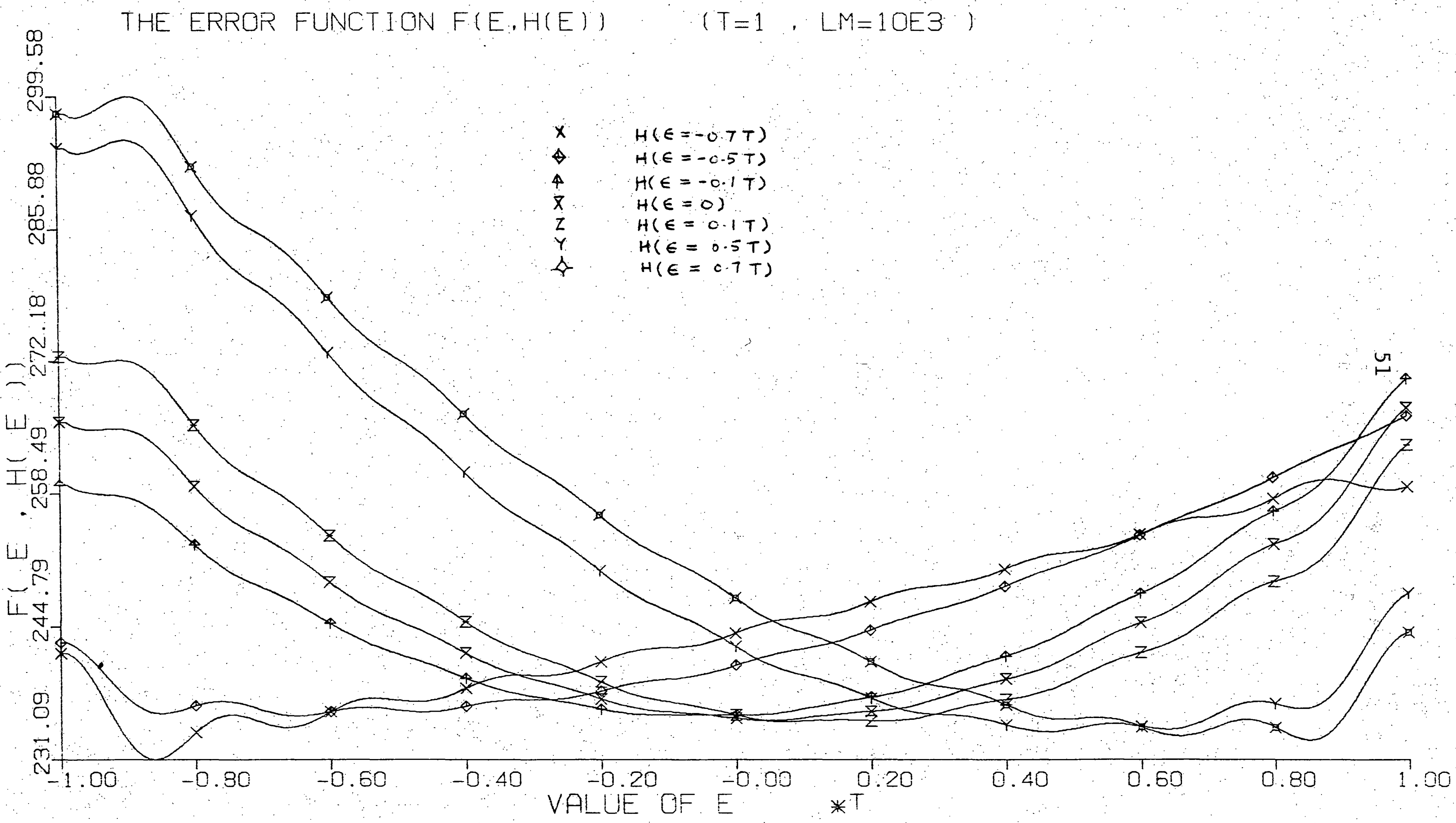


Figure 9: The Error Criterion Function ($\lambda = 10^3$)

Figure 10: Frequency Response of digital filter with different values of ε.

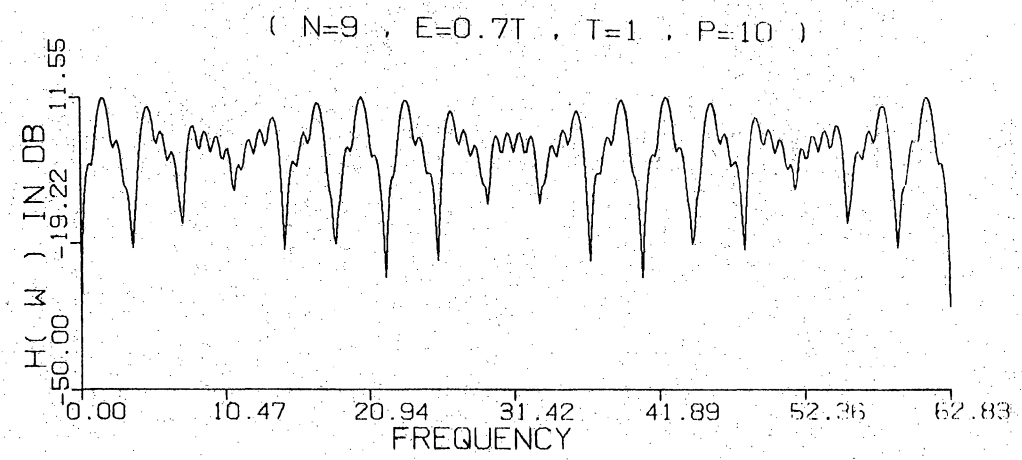


Figure 10(a)

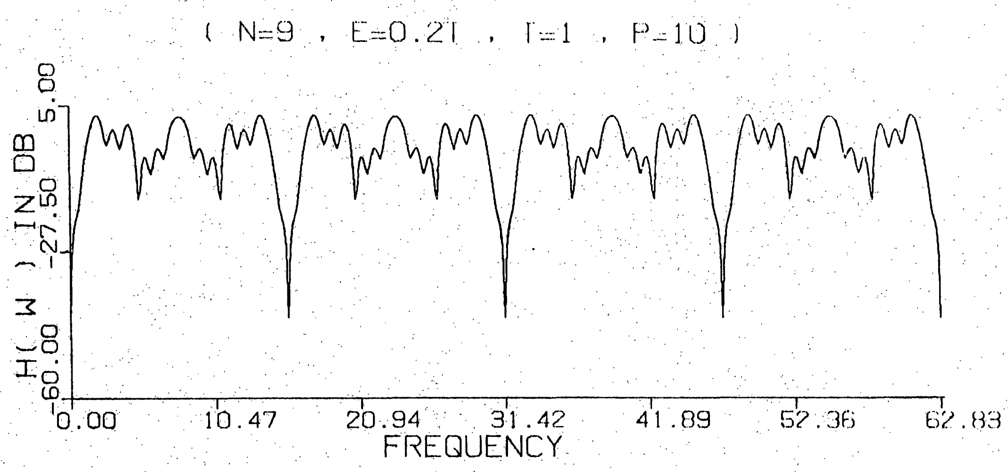


Figure 10(b)

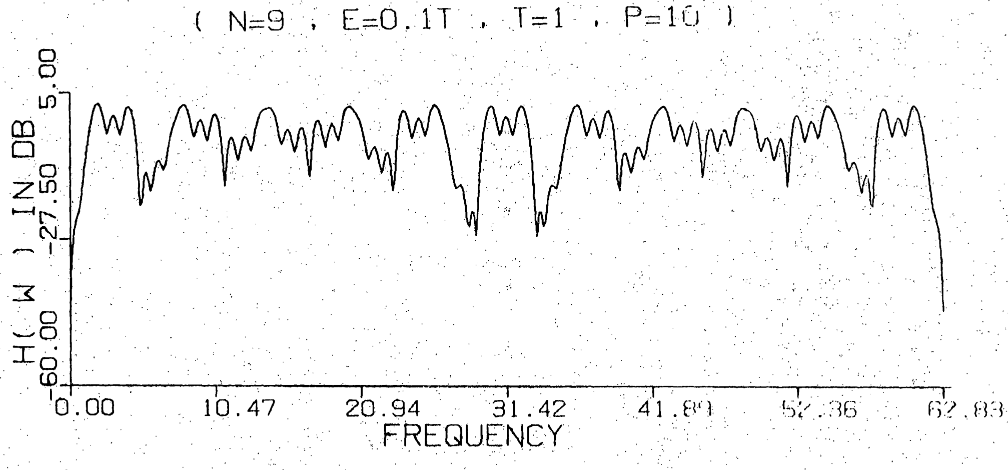


Figure 10(c)

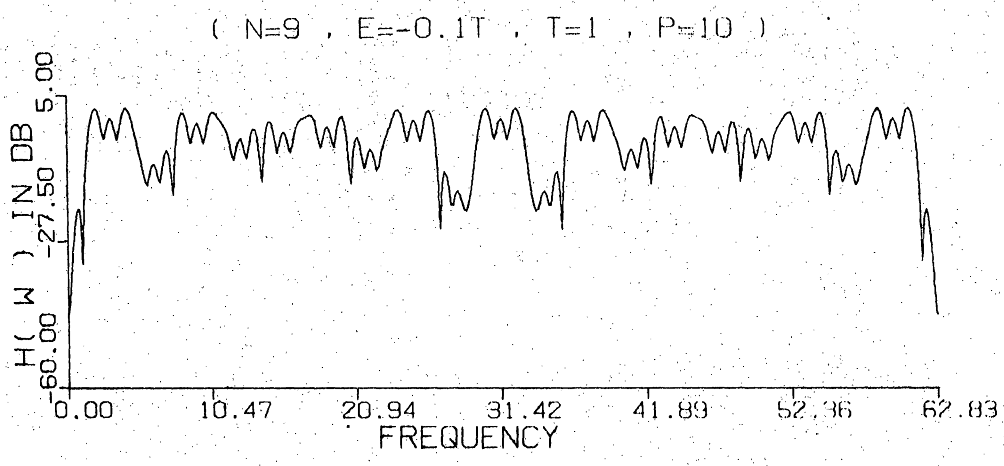


Figure 10(d)

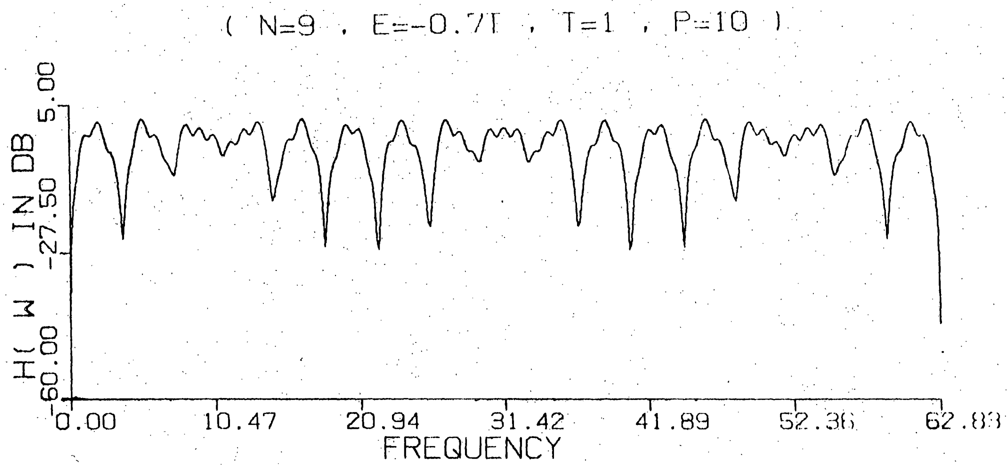

(N=9 , E=-0.7T , T=1 , P=10)
H(W) IN DB
5.00
-27.50
-60.00
0.00
10.47
20.94
31.42
41.89
52.36
62.83
FREQUENCY

Figure 10(e)

input spectrum can be processed with minimum distortion. This is seen in the following example.

In processing radar signals, some particular frequency components may be insignificant in the frequency spectrum concerned, (i.e., assume that signals may occur only at certain frequencies). Hence, one may tend to orient the troughs of the implemented filter response to those frequencies. Thus, the signal can be processed without much loss of information due to the presence of the troughs.

However, it can be seen that there is no systematic approach to select the parameter ε. One has to study all the response spectrum of the filter corresponding to all possible values of ε before the best value can be selected.

Nevertheless, if we proceed with the algorithm presented in Chapter III; we can always find an optimal value of ε which minimizes the error criterion function.

The filter presented in Figure 7b is now being solved for the optimal value of ε, and was found to be $\varepsilon^{o}=-0.38T$. With this optimal value of ε the frequency response of the filter is generated and is shown in Figure 11. If we compare Figure 6b (uniform sampling), Figure 7b (non-uniform stagger sampling with arbitrary ε) and Figure 11 (non-uniform stagger sampling with optimal ε) we can see that the periodic notches present in the frequency response based on

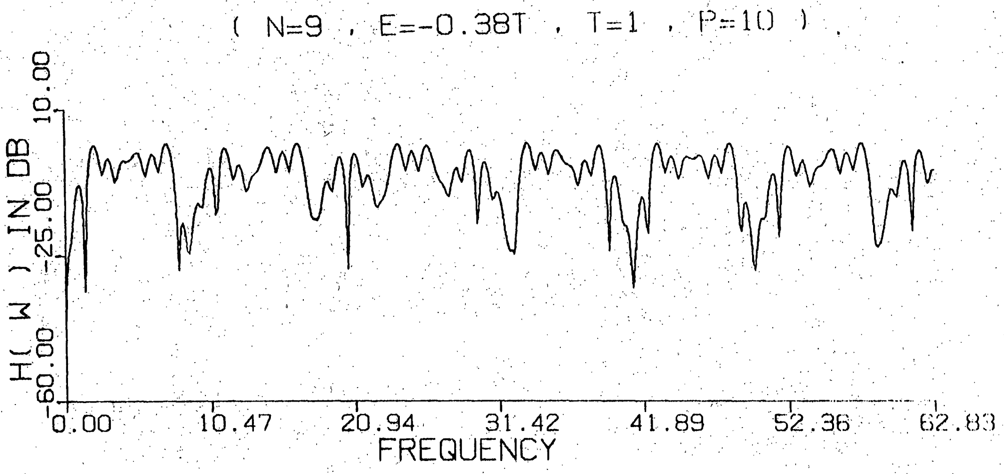


Figure 11: Frequency Response of digital filter with optimal ε.

uniform sampling is reduced in the non-uniform sampling cases. It can also be observed that in the non-uniform sampling case using the optimal value of ε (Figure 11) the locations of troughs that are originally present in the response with an arbitrary value of ε is redistributed over the entire spectrum. That is, with the optimal value of ε, the frequency response of the filter will have a better approximation to the desired response than the one using an arbitrary value of ε, as expected.

Note that, in Figure 11 the symmetry about the frequency $P\pi/T$ is lost. Since ε^o is equal to -0.38T in this case, which means the relationship $\varepsilon = qT/P$ gives $q/P = -0.38$ or $q = -38$, $P = 100$. However, we have originally used $P = 10$ (Table 3); therefore, this means that we have a further increase in the period. That is, with $\varepsilon = -0.38T$ the filter will have a frequency response that has a period of $100\pi/T$ rather than $10\pi/T$ as we have expected, and, that is the reason for lack of symmetry in the response shown in Figure 11 over the frequency range presented.

It was mentioned, that the constant λ in the weighting function $W(\omega)$, equation (4.1), has to be large in order to insure the notch in the stop band. This fact is illustrated by Figure 12 and 13. In Figure 12 we have used a value of λ equal to 10^3 and in Figure 13 a value of 10^6. It can be observed from Figure 12 that the response has a very small attenuation (-7 db) at low frequencies; that is,

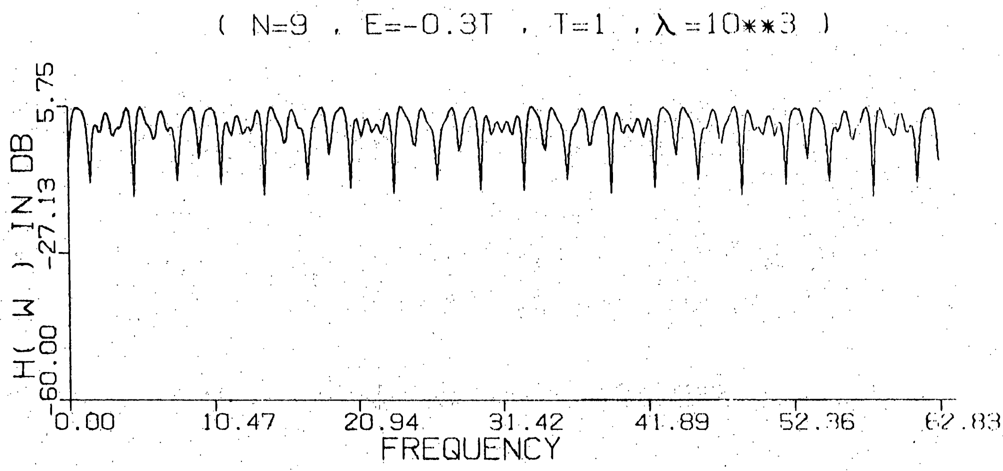


Figure 12 : Frequency response of filter with small λ.

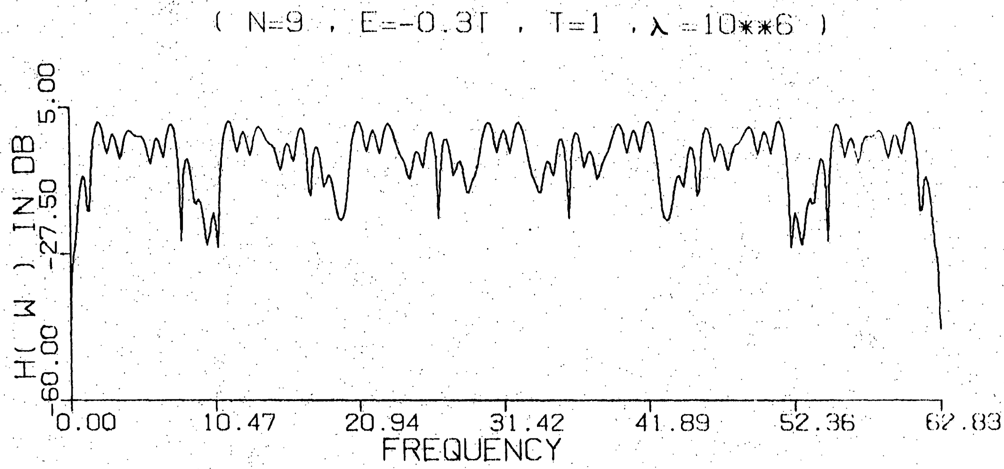


Figure 13: Frequency response of filter with large λ.

the stop band of the high pass filter is not achieved. Whereas with a large λ, the attenuation at low frequencies is high (-60 db) which, in effect, acquires the function of the stop band.

CHAPTER V

SUMMARY AND CONCLUSION

The loss of information and aliasing problems due to the periodicity of a digital filter based on uniform sampling scheme is found to be very undesirable in some applications in signal processing. In order to overcome this effect, one would tend to consider the digital filter based on a non-uniform sampling scheme, because the frequency response of the digital filter in this case will have an extended period. This effect is important in that we can design a filter with the proper extended period, so that the loss of information and aliasing problem can be minimized.

It was shown that, with the simple stagger sampling, an optimal value of the stagger parameter can always be found which minimizes an error criterion. That is, the optimal value of the error criterion function $f(\varepsilon,h)$ is given by:

$$\min_h f(\varepsilon,h) = f(\varepsilon,h(\varepsilon))$$

where h is the transfer function coefficients vector.

However, it was found that the optimal value of the error criterion function $f(\varepsilon,h(\varepsilon))$ does not have a large variation by varying the parameter ε for $|\varepsilon|\leq T$. This effect allows an extra degree of freedom in the filter design. Because by selecting a proper value of ε one can have a filter response that would best process a particular signal; however, this value of ε may not be the optimal value that would minimize the error criterion function.

A conjecture was put forth that the condition $f(\bar{\varepsilon},h(\bar{\varepsilon}))\leq f(\varepsilon,h(\bar{\varepsilon}))$ is true; that is, the minimum of $f(\varepsilon,h(\bar{\varepsilon}))$ occurs at $\bar{\varepsilon}$. However, this condition was shown to be false when the value of λ, the constant in the weighting function, was chosen to be small.

Only the simple stagger sampling scheme has been studied in this analysis. The properties and characteristics of this particular sampling scheme may not be common to other types of non-uniform sampling schemes. However, the advantages of using a non-uniform sampling scheme in synthesizing a digital filter are inevitable. Since this analysis does not give an exhaustive study on all possible non-uniform sampling schemes, further investigation should be carried out to examine the properties, characteristics and influences on the frequency response of a synthesized digital filter.

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APPLICATION OF NON-UNIFORM SAMPLING TECHNIQUES TO DIGITAL FILTER SYNTHESIS

by

Joseph Siuming Tsui

(ABSTRACT)

An investigation of the non-uniform sampling technique as applied to digital filter designs will be made. The objective of the design is to reduce the interference problems as one would encounter in using uniform sampling technique in the synthesis. An analysis of the error function which measures the goodness in approximating a desired frequency response will also be undertaken. An algorithm which determines the optimal parameters for a high pass filter will be developed and used to synthesize the particular high pass filter. The results of this design, the frequency response and its approximation error will be studied and evaluated.