TOPOLOGY OPTIMIZATION WITH SIMULTANEOUS ANALYSIS AND DESIGN

by

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(ABSTRACT)

Strategies for topology optimization of trusses and plane stress domains for minimum weight subject to stress and displacement constraints by Simultaneous Analysis and Design (SAND) are considered. The ground structure approach is used. For the truss topology optimization, a penalty function formulation of SAND is compared with an augmented Lagrangian formulation. The efficiency of SAND in handling combinations of general constraints for truss topology optimization is tested. A strategy for obtaining an optimal topology by minimizing the compliance of the truss is compared with a direct weight minimization solution to satisfy stress and displacement constraints. It is shown that for some problems, starting from the ground structure and using SAND is better than starting from a minimum compliance topology design and optimizing only the cross sections for minimum weight under stress and displacement constraints. One case where the SAND approach could not predict a singular topology obtained by compliance minimization is discussed in detail. A member elimination strategy to save CPU time is developed .

For the plane stress topology optimization problem, the ground structure is obtained by using 3 noded constant stress triangular elements. A chess board pattern is observed in the optimal topologies which may be attributed to the triangular elements. Some suggestions for future research are made.

ACKNOWLEDGMENTS

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I take pride in dedicating this dissertation to my father Subramanian and my late mother Meenakshi, who dreamed this dream for me and supported me all along in my academic career.

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1. INTRODUCTION

1.1 Structural Optimization - Classifications

Structural optimization is the science of designing structures to meet certain requirements or objectives while satisfying certain limitations or constraints that are imposed. With the advent of digital computers and the finite element method, research interest in structural optimization received a boost. The efforts focused mainly in developing numerical methods appropriate for use on the computers. Structural optimization can be classified broadly under three categories

1. Sizing optimization,
2. Shape optimization, and
3. Topology optimization.

Sizing optimization involves optimizing structural sizes (e.g., areas of cross sections of truss members, thicknesses of plates, etc.,) without changing the spatial lay-out of the structure and hence the finite element mesh. Variables such as the areas and thicknesses are called sizing design variables. Since the finite element mesh is fixed, sizing optimization is simple to implement. However, by keeping the geometry fixed only limited benefits can be reaped in the design. Allowing the geometry of the design domain to change during the course of optimization is called *Shape optimization.* Inherently, this is more difficult than sizing optimization. If the structure is modeled using finite elements, geometry changes require that the finite element mesh be regenerated repeatedly. Variables that describe the structural shape (shape variables) are also used as design variables in addition to the structural sizes. The sensitivity of the objective function and the constraints to the shape variables is expensive to compute. Despite the computational expense, the benefits of changing the geometry are quite significant. Reference [1] reviews some of the research on shape optimization.

When optimization of structures involves decisions on connectivity of a domain (number of holes) or on if and how individual members are connected with one another, we enter the 3rd classification of structural optimization, namely *Topology optimization*. Seeking member connectivity in addition to structural sizes makes the topology optimization problem difficult and challenging. However it provides additional gains in efficiency. This is illustrated by an example in Ref. [21]. The familiar 10 bar truss was optimized for minimum weight subjected to stress and a single displacement constraints. The optimization for sizing produced a design heavier than the one produced by topology optimization as shown in figure 1.2. It is clear that topology optimization removed the redundant members and reduced the weight.

1.2 Historical Background on Structural Topology Optimization

The first work done on designing an optimal topology for any structure was by Michell [2], early in this century. In this classical work Michell laid out theorems for designing optimal truss lay-outs for single loading conditions subject to stress constraints. These optimal trusses were statically determinate, consisting of a large number of members and sometimes even unstable. Hence, the Michell trusses were often impractical. However, in addition to pioneering research on topology optimization, Michell provided sufficient conditions for optimality of structural layouts that have been also used in recent years (e.g., see Lev [13]).

Research on topology optimization stalled for more than half a century after that, mainly because the mathematical theory behind the Michell trusses is very complex and also because of the fact that these trusses were impractical.

In the sixties Michell structures were re investigated by Hemp [3], Prager and Shield [4] for the design of minimum weight structures. Rozvany and Adidam [5], Rozvany [6],and Rozvany and Prager [9] used Michell's theory for the design of grillages and continued this work through the seventies. The reader is referred to the books by Rozvany [7], [8] for a detailed description of the foundations of the analytical optimal layout theory for trusses, beams, grillages and plates. The theory due to Prager expressed the problem in terms of *generalized stresses* (local stress, shear force, bending moment, etc.) and *generalized strains* (local strains, shearing strains, bending strains etc.). A cost function was then constructed by taking into account the structural weight, fabrication and handling costs, etc., A set of optimality criteria were developed for the minimality of this cost function satisfying the static and kinematic requirements. The optimal structural topologies developed were like Michell structures. They were characterized by a large number of elements, and were statically determinate. For more practical structures, topology optimization has always been based on some approximations. These approximation include but are not limited to :

a) approximate analysis models (plastic analysis, rigid),

b) considering only a few simple constraints,

c) simple objective functions (e.g., weight, compliance),

d) simple structural systems (trusses),

e) a limited number of loading cases.

Most of the research in topology optimization has been concentrated on the design of trusses, and for good reason. Trusses are simple yet nontrivial structures that can be analyzed and optimized easily. For a given set of nodal points there can be several ways of connecting them by truss members a natural problem for topology optimization. Starting from Michell [2], several researchers have proposed various solutions (exact as well as approximate) for truss topology optimization which are classified in the review paper by Kirsch [12].

In recent years a lot of effort has been put into designing light weight structures. This has spurred a renewed interest in topology optimization. After Michell, the first serious work on topology design was by Dorn, Gomory and Greenberg [14]. They introduced the concept of *ground structure* for automatic topology design of truss structures. Topology optimization by definition involves removal or retention of elements. This means that the optimization problem should have some way to keep track of the presence or absence of a particular member. Integer variables which function like switches can be used to keep track of the presence or absence [14]. However since efficient methods were not available to tackle the mixing on integer variables with other continuous variables like the member sizes, this was not a practical approach. The ground structure approach of [14] offered a solution to this problem. It defines an admissible set of nodal points in the structural domain where each of these nodes is connected with every other node by uniform truss members creating a highly connected structure that can be used as the initial design for topology optimization (see figure 1.4). The optimal truss topology will be obtained as a subset of this ground structure. The ground structure approach transforms the topology optimization problem into a large sizing optimization problem where integer variables are not needed and most of the cross sectional areas reduce to zero.

The objective of the study of [14], was to design a minimum weight truss subject to stress constraints. Member forces were considered as design variables which rendered the problem linear in the design variables. By using the linear programming (LP) method, Dorn, et al., obtained optimum topologies for a single loading case. If a design variable (member force) was non-basic, the corresponding member area was set to zero and was removed from the structure. Other cross sectional areas were obtained by dividing the absolute values of the member forces by the allowable stresses. These steps rendered the optimal design statically determinate and fully stressed.

Dobbs and Felton [15], extended the work of Dorn et al., to design statically indeterminate trusses subject to multiple loading cases. By considering the member cross sectional areas as design variables, they ended up with a linear objective function and nonlinear constraints. By using nonlinear programming (NLP) and heuristics they solved the problem. Members whose cross sectional areas were approaching zero were identified and removed but no proof that these members will not come back later was given.

Sheu and Schmit [16], used a branch and bound technique for the topology optimization of trusses. An LP problem was initially solved to get a lower bound on the minimum form a set of candidate topologies and then the most promising configurations from these are refined using NLP. This scheme is general since both stress and displacement constraints can be considered and multiple loading cases were considered too. However they found that in problems where the displacement constraints dominated, more than 50% of all the candidate topologies needed to be refined by using NLP. This proved computationally expensive. Hence, to make the cost non prohibitive, the initial set of candidate topologies had to have limited number of configurations only.

Majid and Elliot [17], used a "steepest descent alternate mode algorithm" to optimize ground structures subject to stress, displacement and buckling constraints for multiple loading cases. Initially the ground structure is analyzed. Then a series of influence coefficients is created by applying unit loads at the end of each member in the ground structure. Using these coefficients and some theorems on structural variation, the structure can be reanalyzed efficiently. The theorems and coefficients are used to eliminate members that are not needed and to detect any possible instability that may arise by removing a member. This method is however limited by the size of the problem. This method was later extended by Majid and Saka [18], to topology optimization of rigidly jointed frames.

Saka [19] considered in his work joint displacements as design variables in addition to member cross sectional areas and joint coordinates thus doing away with the structural analysis. The sensitivities were obtained by simple algebraic expressions. This concept is a simultaneous analysis and design approach that has also been used in this dissertation. Saka, by linearizing the problem and using the simplex method in a sequential manner was able to find optimal topologies for various truss configurations for multiple loading cases. To reduce the computational cost, he used for a ground structure certain practical configurations selected from experience and previous knowledge of the problem instead of the one described in [14].

When truss topologies are designed for a single loading case with the allowable stresses in tension and compression at the same level, the optimum topology is statically determinate. This was proved by Sved [10] and Barta [11]. This means that the plastic design optimum is also the elastic design optimum.

Kirsch [20], used plastic design to simplify the problem of truss topology optimization for a single loading case. In plastic design of trusses one does not need to satisfy elastic compatibility equations and the problem turns out to be linear in the cross sectional areas. Linear programming (LP) algorithms which are readily available can be used. This approach considers as design variables the cross sectional areas of the truss members and the member forces, the weight of the truss as the objective function and stress constraints and upper and lower bounds on the areas as simple side constraints. The objective function and constraints are linear in the design variables.

The solution to this simple LP problem is not usually the true optimum for multiple loading cases and problems with displacement constraints. Modifications are required as a second stage to take care of the elastic compatibility by solving the actual nonlinear problem by NLP methods. However this LP solution represents a lower bound on the optimum. Despite its approximate nature, this approach has the advantage that the use of LP process being very simple, large structures (for single loading case) can be optimized easily.

The optimum truss topology retains only a fraction of the members from the ground structure. The cross sectional areas of the other members are reduced to zero by the optimization algorithm. This vanishing of members may cause the stiffness matrix of the structure to become singular. It cannot be inverted or factored and the optimization problem becomes non differentiable.

In the plastic design for truss topology, since the compatibility conditions are neglected, the structure stiffness matrix is not needed and so the singularities do not pose a problem. This is another big advantage of using the LP based approach of Kirsch.

Ringertz [21], used a similar strategy of neglecting the compatibility and solved a LP problem to obtain a topology, then improved the cross-sections by solving the complete non-linear problem. He also used the LP solution as a starting point [22], for a branch and bound algorithm to get optimal truss topologies for multiple loading cases.

Fully stressed design procedures have been tried for topology optimization of trusses by Barnes, Topping and Wakefield [23], [24]. Starting with a ground structure and by using a stress ratio method they resized the truss members after each elastic analysis. With this technique they were able to drive most of the member cross sectional areas to zero. For a single loading case and with the allowable stresses in tension and compression on the members at the same level, they were able to derive statically determinate layouts that compared well with the layouts obtained by using LP based techniques. For multiple loading cases or for different stress allowables the resulting structures were statically indeterminate. The comparison with LP based techniques was not good. In this method there is no objective function being considered for the derivation of the optimality criteria and the trusses are optimized for member stresses.

The above mentioned stress ratio method is an example of a method that uses *Optimality criteria* for the design of structures. Optimality criteria are conditions that are satisfied by the optimal design. The stress ratio method is an example of a method based on an intuitive optimality criterion. The criterion is the requirement of a fully stressed design.

When structures are optimized for minimum weight subject to a single loading case and if the allowable stresses in tension and compression are at the same level, the optimum design will have minimized compliance also [30]. The compliance of a structure is the work done by the applied loads on the displacements so that small compliance means large stiffness. Minimizing the compliance of a structure for a given weight or equivalently minimizing the weight for a given compliance is computationally very cheap for two reasons. First, the calculation of derivatives of the compliance does not require the solution of the equilibrium equations. Secondly, optimization for a single constraint can be performed very efficiently by optimality criteria methods. For these reasons minimum compliance optimization has been quite popular. For example, Ben-Tal and Bendsøe [29] used this approach to design topologies for trusses subject to a weight budget. They based their approach on the work done by Taylor and Rossow [30] for the topology design of trusses by an optimality criteria based method. Taylor and Rossow also used an algorithm to identify active bars in the truss. Bendsøe and Ben-Tal used a displacement based formulation and the algorithm to identify active bars. The topology optimization problem is to find a set of active bars that minimize the mean compliance of the truss. The problem is converted into an equivalent problem using an optimality criterion based formulation. A steepest descent method is used for the minimization. The equivalent problem does not require the global stiffness matrix and so its singularities do not affect the optimization. The compliance minimization method does not consider any individual stress and displacement constraints. For general problems with both displacement and stress constraints, topologies are first obtained for minimum compliance without consideration of the stress constraints and then the cross sectional areas are resized for handling the constraints. This two stage strategy does not always produce the optimal design as will be seen from the examples in chapter 3 of this dissertation.

Rozvany and Zhou [26], have developed continuum-based optimality criteria methods (COC) for designing topologies and generalized shape optimization [27]. By applying the Euler-Lagrange equations in infinite dimensional design spaces (continua) and by using techniques of calculus of variations, control-theory and energy theorems they derived optimality criteria for shape optimization problems [28] . The optimality criteria were differential equations in the generalized stress resultants (e.g. bending moments). The design variables were the member sizes and the generalized forces. The methods are computationally efficient and are capable of predicting topologies that agree well with analytical solutions [26]. While large structures can be optimized efficiently without much computational cost, the COC methods suffered from a drawback that the for each type of structure and for different design conditions, the optimality criteria had to be analytically derived by a lengthy process. For discrete structures a discretized version of the COC method (DCOC) was used. The derivation of the optimality criteria was simpler in this case. The DCOC algorithm is capable of handling large size problems with stress and displacement constraints. The singularity problem of the stiffness matrix was overcome in this method by using extremely small minimum gage (10^{-12} inches) on the sizing variables.

Another class of methods for topology optimization has been becoming very popular in recent years for the optimization of 2-D and 3-D structures. This is based on the

Homogenization method. Starting from a 2-D continuum as the ground structure and by altering the material distribution at the micro structural level (e.g., see Bendsøe [31] shape design and topology design problems have been solved for two and three dimensional structures. The shape and topology optimization problem were nested into one and were solved as a material distribution problem. The initial design domain was a 2-D or 3-D continuum that was completely filled with a homogeneous isotropic material. It was then discretized using some special finite elements that are partially filled and partially void. The design variables for optimization were the dimensions of the void and its orientation in space. The objective function was the mean compliance of the total structure. The optimization problem was to get the best topology that minimized the mean compliance subject to a constraint on the percentage of material volume that can fill the domain. By altering the size and orientation of the voids internal boundaries (new holes) can be created in the isotropic continuum. The optimization problem can be viewed as the determination of the distribution in space of an anisotropic material that can carry the given loads and meet other requirements. The homogenization idea is to create and fill the design domain with this porous anisotropic material by periodically distributing an infinite number of infinitesimally small voids.

The advantages of this method are :

1. The main advantage of this method with compared to traditional shape optimization is that the shape and topology (e.g., number of holes) need not be known a priori. Using mathematical programming methods for shape optimization, it is not possible to predict voids. That means that the method will not introduce holes in the domain. The homogenization method is capable of doing this.
2. The optimization process can start with any arbitrary 2-D or 3-D domain. For shape optimization problems the FE mesh need not be updated.

The output from the homogenization method is the design domain consisting of elements with variable densities. Elements with small densities are interpreted as voids and elements with high densities as solid structure. The final designs that are output by this approach are only a non smooth estimate of the final shape of the structure. This method hence can be the first stage or the preprocessor of a 2-stage approach. The second stage will be a traditional shape optimization method where the information from the first stage is used to refine the shape.

Bendsøe and Kikuchi [32], developed a strategy using the homogenization method, where truss like topologies are obtained starting from a plate like domain by requiring the percentage of material volume to be low. They used this approach for the design of fillets.

Suzuki and Kikuchi [33] modified this approach to consider multiple loading cases. Later they extended the work for three dimensional shells and applied the method for designing automobile bodies [34].

Diaz and Bendsøe [35] have used the homogenization approach to design truss topologies for multiple loading cases by using a composite objective function which is the weighted sum of individual load cases. Several other research works on using the homogenization method for topology optimization are reported in the literature (see for example [36]– [41]).

This dissertation introduces an approach similar to that taken by Saka [19] for topology design. It is called the **Simultaneous Analysis and Design** approach. The approach eliminates the need for repeated analysis by considering the equilibrium equations of structural analysis as equality constraints and the displacements as design variables in addition to the sizing design variables. Saka linearized the optimization problem and solved it using LP techniques. The SAND approach here uses nonlinear programming solution methods.

The SAND method presented in this dissertation is also a natural way to avoid the singularity problem due to vanishing members. The method does not require the assembly or factoring of the global stiffness matrix. Hence the singularities are not a problem. The other advantage of the SAND approach is that unlike the compliance minimization method, SAND is a general method capable of handling multiple constraints. So, general problems with stress and displacement constraints can be solved in a single stage.

1.3 Historical background on simultaneous analysis and design

Structural optimization problems were originally solved based on the calculus of variations. A problem would be solved by first obtaining the Euler-Lagrange optimality differential equations and then analytically solving them simultaneously with the differential equations of the analysis. This method is still employed for optimizing individual structural elements [43]. With the advent of high speed electronic computers and the finite element method (FEM) for structural analysis, a transformation took place in structural optimization. Numerical methods of optimization were being developed for structural design using computers, and an approach called here the "Nested Approach" became the standard for optimization. The nested approach is represented in Fig 1.1. An optimization iteration begins by computing the structural responses for a given design variable set. The gradients of these responses with respect to the design variables are then calculated and are used to direct the optimization process and update the variables. This means that structural equations are solved repeatedly once for each optimization iteration. This nested approach is based on the use of efficient elimination methods like the Gaussian elimination method for solving the structural equations. Since techniques for optimization were not competitive with these elimination techniques, the nested approach that keeps the analysis and optimization at separate levels was very popular.

When iterative methods are used for analysis there is a way of integrating analysis and design. This was originally investigated by Rizk [44] for aerodynamic design problems. The two levels of analysis and optimization are still kept separate. The flow chart for this approach is shown in figure 1.3. The method takes advantage of the iterative analysis scheme. The analysis iterations, instead of running to full convergence are stopped after a small number of iterations, with the number carefully chosen so that meaningful information about the sensitivities can be obtained. This approach was applied by Barthelemy et al., [45] for hole shape optimization in a thick plate subjected to in-plane loads modeled by three dimensional finite elements. An element-by-element preconditioned conjugate gradient (EBE-PCG) method was used for the analysis. It was shown that this method was substantially cheaper computationally when a large number of elements through the thickness was used.

When the two levels are completely merged into one so that we solve one big optimization problem and perform no analysis, we call it the **Simultaneous Analysis and Design** (SAND) approach. This approach was initiated by Fox and Schmit [46],[47] in the mid sixties. They converted the optimization problem into an unconstrained problem by using a penalty function approach and then employed the conjugate gradient (CG) method for minimization. However, The discretized equilibrium equations are inherently ill-conditioned and the CG technique performed poorly because of this. Interest in SAND waned for a while.

Recently, preconditioning techniques have been developed and the resulting preconditioned conjugate gradient (PCG) methods are highly competitive with methods like Gaussian elimination [48] for poorly banded problems like the ones that arise in the discretization of three dimensional structures. Hence, interest in integrating analysis and design has been rekindled.

Haftka [49] revisited SAND with a penalty function formulation using an Element by element preconditioned conjugate gradient (EBE-PCG) algorithm due to Hughes et al. [50]. He used it to solve linear elastic truss and non-linear panel collapse problems. He showed that the SAND approach with EBE-PCG is more efficient computationally than the standard nested approach employing Gaussian elimination. Although the inherent ill-conditioning associated with the equilibrium equations that slowed down the regular CG method was alleviated using the EBE-PCG scheme, the SAND method was found to be very sensitive to certain parameters used to tune the algorithm.

Haftka and Kamat [51], used SAND to design structures where the equations of equilibrium were nonlinear. For the design of a 72 bar truss and an antenna truss they showed that the SAND approach using a penalty function solution or a projected Lagrangian solution was computationally superior to a nested approach using a projected Lagrangian solution or a generalized reduced gradient method.

Shin, Haftka and Plaut, [52] showed that it was feasible to use this approach for eigenvalue maximization problems. They designed optimum columns for a given foundation and optimum foundations for a given column.

The computational cost associated with SAND varies with the number of design variables in a nonlinear fashion like other mathematical programming methods. The addition of displacement design variables increases the dimensionality of the problem and hence the computation time. For designs under multiple loading cases SAND may not be the best method since the number of displacement design variables will very high. Chibani [53], tried to alleviate this problem by using a two-level SAND approach and geometric programming for the design of space trusses for multiple loading cases.

Ringertz [54] used the SAND approach for the design of space trusses with geometric nonlinearities. In one approach all the equilibrium equations were considered as equality constraints for the optimization. In a variation of this approach, only some of the equilibrium equations were considered for the optimization. Buckling constraints were also considered.

Smaoui and Schmit [55] used the SAND approach for the design of three dimensional trusses with geometric nonlinearities and geometric imperfections. Constraints on static displacements, stresses and local buckling were considered. The SAND problem was solved using a reduced gradient method.

Orozco and Ghattas [56] showed that when the projected Lagrangian algorithm is used and if the sparsity of the matrices in the problem (at least the sparsity of the Jacobian) is exploited, the SAND approach can be computationally superior to a nested approach that uses a SQP algorithm for the solution.

1.4 Present Work-objectives

This dissertation primarily focuses on using SAND for obtaining optimal topologies. The objectives of this dissertation are:

1. To obtain topologies by weight minimization subject to stress and displacement constraints starting from a ground structure and to compare these topologies with those obtained by the two stage process of optimizing topology by compliance minimization followed by sizing optimization of the minimum compliance topology.
2. To reduce the computational time associated with the SAND approach by using an augmented Lagrangian approach and developing a member elimination strategy to identify and eliminate periodically unwanted members from the ground structure.

The organization of the dissertation is as below:

Solution strategies for SAND are reviewed in Chapter 2. The relative advantages and disadvantages of each strategy is discussed.

In Chapter 3 the SAND approach for obtaining optimal truss topologies using the ground structure approach will be explained. The solutions are compared in terms of the geometry and layout to those in Ref. [29] to validate the method. The solution strategies are compared in terms of computational expense. A member elimination strategy to help reduce the computer time is developed and tested. The two stage approach of compliance minimization followed by sizing optimization is compared to a direct sizing optimization of the ground structure.

Chapter 4 applies the SAND approach for deriving optimal topologies for plane stress problems.

Chapter 5 offers concluding remarks and some suggestions for future research.

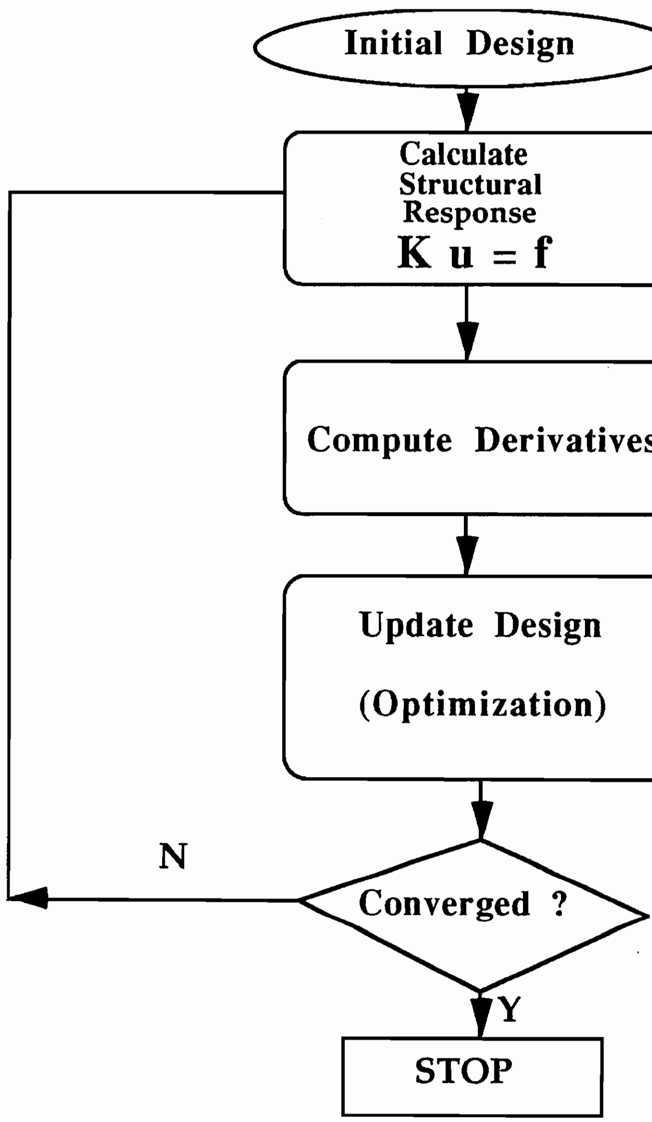


Figure 1.1 The nested approach

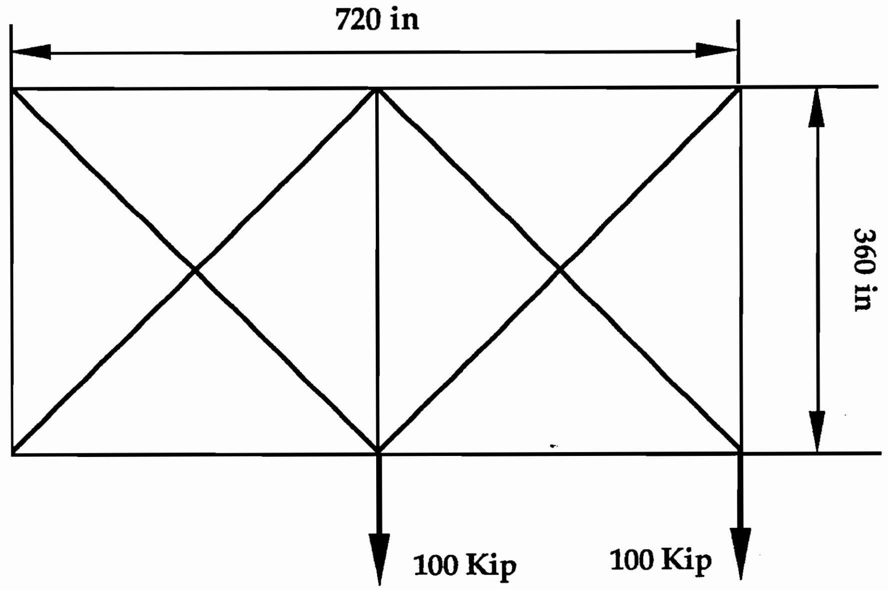


Figure 1.2 a) Sizing design. Weight = 5061 lb

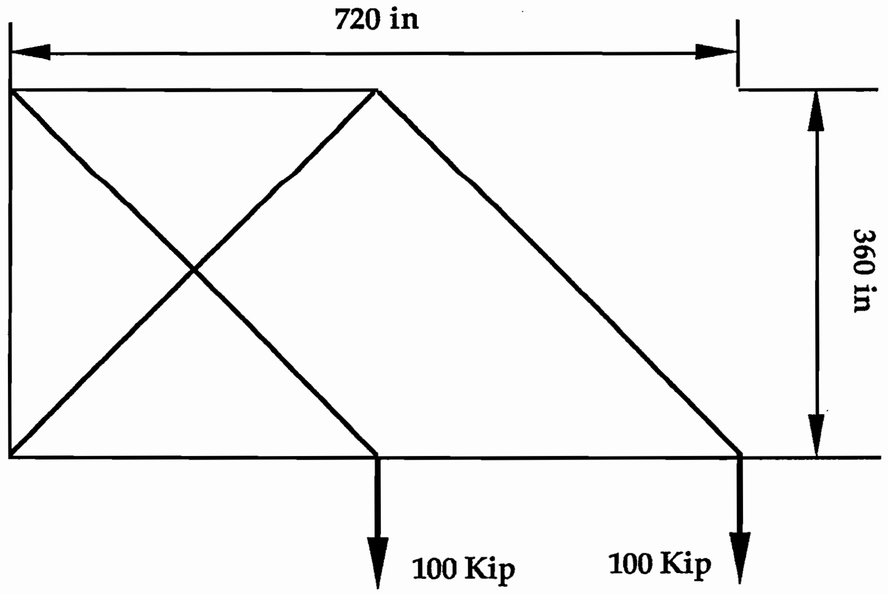


Figure 1.2 b) Topology design. Weight = 4900 lb

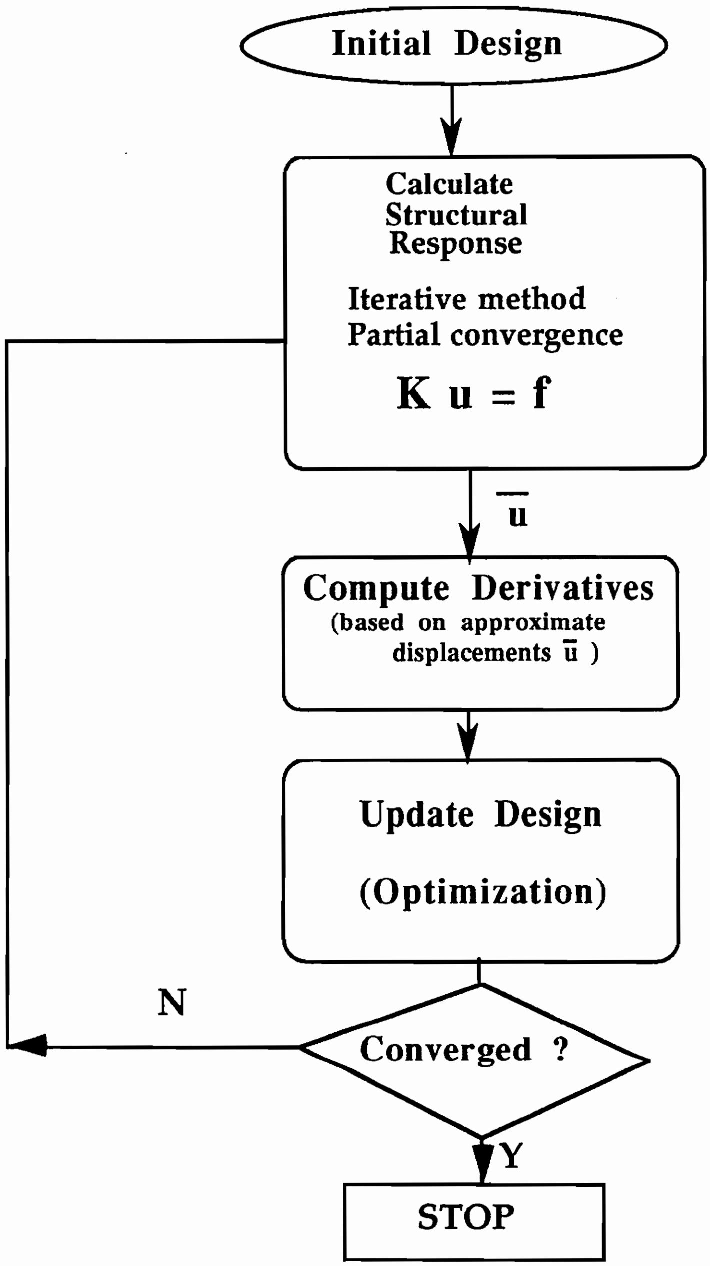


Figure 1.3 The integrated analysis and design approach

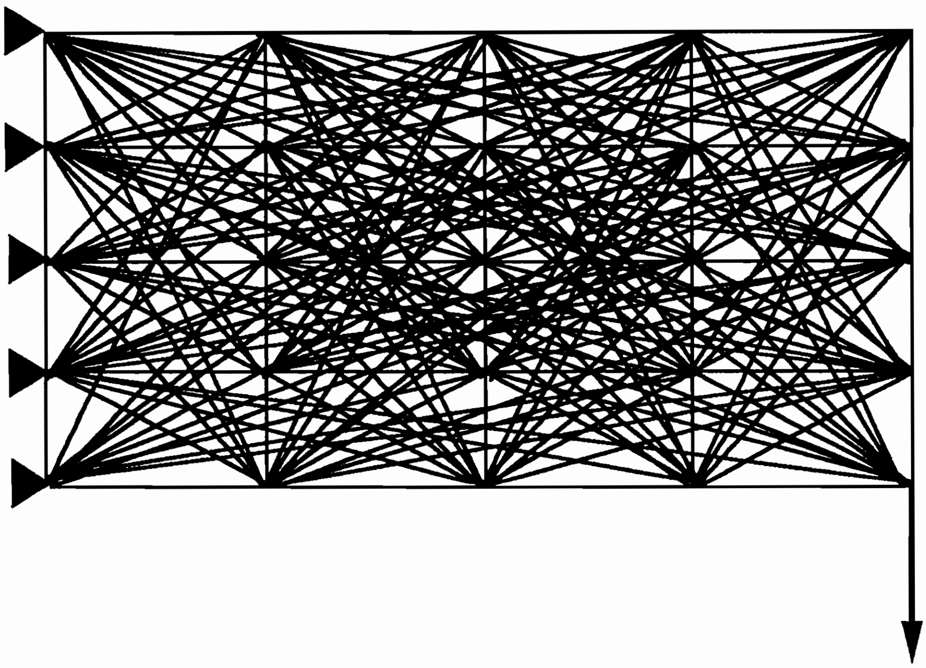


Figure 1.4 An example of a ground structure

2. SOLUTION STRATEGIES FOR SIMULTANEOUS ANALYSIS AND DESIGN

2.1 The Simultaneous Analysis and Design Problem

A structural optimization problem can be mathematically represented as

Find **x** to

$$\begin{aligned} &\text{minimize} \quad V(\mathbf{x}) \\ &\text{subject to} \quad g_j(\mathbf{x},\mathbf{u}) \geq 0 \qquad j = 1,\ldots,n_i \quad \text{and} \\ &\qquad\qquad h_k(\mathbf{x},\mathbf{u}) = 0 \qquad k = 1,\ldots,n_{eq} \end{aligned} \tag{2.1}$$

where g_j are some stress, displacement, buckling or frequency constraints, V(**x**) is the objective function to be optimized (minimum weight, minimum compliance), **u** is the displacement vector and **x** is the vector of design variables. The vector **u** is normally found as a solution to a set of algebraic equations (equations of equilibrium) or as a solution to a minimization (of the total potential energy) problem. The equations of equilibrium can be represented by

$$\mathbf{R}\ (\mathbf{x},\ \mathbf{u},\ \mathbf{P}) = 0. \tag{2.2}$$

The system of equations represented by **R** can be in general linear or nonlinear equations. **P** is the applied load vector. The problem is usually solved in a nested approach shown in Fig. 1.1. That is, the problem is solved by repeatedly calculating **u** using the above equation (2.2) and its derivatives with respect to the components of **x**, x_j by either differentiating (2.2) or by finite differences. Based on the information about **u** and its derivatives some optimization procedure can be employed to improve the design **x**. This approach involves solving (2.2) at least once every optimization iteration.

The Simultaneous analysis and design (SAND) formulation considered in the present work eliminates the need for repeatedly solving (2.2) by considering both $\mathbf{x}$ and $\mathbf{u}$ as design variables and treating the equilibrium equations of equation (2.2) as equality constraints for the optimization problem. This can be mathematically represented as

$$
\begin{aligned}
&\text{Find } \mathbf{x} \text{ and } \mathbf{u} \text{ to} \\
&\text{minimize} \quad \mathrm{V}(\mathbf{x}) \\
&\text{subject to} \quad g_j(\mathbf{x},\mathbf{u}) \geq 0 \qquad j = 1, \ldots, n_i \quad , \\
&\qquad h_k(\mathbf{x},\mathbf{u}) = 0 \qquad k = 1, \ldots, n_{eq} \quad \text{and} \\
&\qquad \mathbf{R}(\mathbf{x},\mathbf{u},\mathbf{P}) = 0
\end{aligned}
\tag{2.3}
$$

The addition of the displacements as design variables increases the dimensionality of the problem. For the problems considered in this dissertation, there are no equality constraints h_k other than the equilibrium constraints. Hence in the following discussion we omit them.

2.2 Solution strategies

The constrained problem expressed in (2.3) is can be solved directly by methods for constrained minimization or by using methods for unconstrained minimization by first converting (2.3) into an equivalent unconstrained problem. Saka [19], solved the truss topology optimization problem by linearizing it and using sequential linear programming (SLP) approach. Haftka and Kamat [51], used the projected Lagrangian method (sequential quadratic programming, SQP) and a penalty function formulation. This method uses a quadratic approximation of the Lagrangian function and uses a quadratic programming (QP) algorithm to find the minimum. Orozco and Ghattas [56], used a commercially available software (MINOS) to solve the problem by exploiting the sparsity of the matrices present in the problem. Haftka [49], converted equation (2.3) by using the **penalty**

function approach and solved it using an element-by-element preconditioned conjugate gradient method. In this dissertation, we follow initially the same approach as [49]. By the standard application of the penalty function technique ([57], pp. 190–195) the SAND formulation can be transformed into,

$$\text{minimize} \qquad \phi = \mathrm{V}(\mathbf{x}) + r\sum_{j=1}^{m} p[g_j(\mathbf{x},\mathbf{u})] + \frac{c}{\sqrt{r}}\mathbf{R}^T\mathbf{R}, \tag{2.4}$$

for $r = r_1, r_2, \ldots$, where $r_i \to 0$ and $\mathbf{R}$ is a vector of residuals when the equilibrium equations are not satisfied exactly. The coefficient r is the penalty parameter. The penalty function approach forms a composite objective function ϕ by combining the original objective function and penalties associated with the violation of constraints. The penalty associated with the inequality constraints is $r\sum_{j=1}^{m} p[g_j(\mathbf{x},\mathbf{u})]$ and the penalty associated with the vector of equilibrium constraints is $\frac{c}{\sqrt{r}}\mathbf{R}^T\mathbf{R}$.

The different types of penalty function are:

1. Exterior penalty function
2. Interior penalty function
3. Extended interior penalty function

The exterior penalty function assigns a penalty to a constraint only when it is violated (i.e. exterior to the feasible domain). The interior penalty function assigns penalties in the interior of the feasible domain. It keeps the design in the feasible region. The disadvantage of this method is that it always requires a feasible starting design. Often, it may not be possible to find one or during the course of the optimization we may encounter an infeasible design. It then becomes difficult to get back into the feasible domain. So, a combination of the two called the extended interior penalty function can be used to advantage. An example is the penalty function described in Eqn (2.4) due to Haftka and Starnes [58]. It is defined as

$$p[g_j] = \begin{cases} 1/g_0[(g_j/g_0)^2 - 3(g_j/g_0) + 3] & g_j < g_0 \\ 1/g_j & g_j \geq g_0 \end{cases} . \quad (2.5)$$

where g_0 is a transition parameter that defines the boundary between the interior (feasible) and exterior (infeasible) parts of the penalty function. As the value of r is decreased, the minimum of ϕ approaches the minimum of the original constrained optimization problem. However, for small values of r, (near the minimum) the curvature of the penalty function ϕ also increases due to the $\frac{c}{\sqrt{r}}\mathbf{R}^T\mathbf{R}$ term. This leads to numerical difficulties in solving equation (2.4). By using a sequence of values of r that progressively decreases i.e., for $r = r_1, r_2, \ldots$, where $r_i \to 0$, we can repeatedly solve equation (2.4) by using the minimum obtained for bigger values of r as the starting point for smaller values of r. The ill-conditioning associated with decreasing values of r is counterbalanced by the availability of a good starting point. For the minimization of ϕ any method suitable for unconstrained minimization can be used. The performance of the algorithm to minimize ϕ depends a great deal on the choice of the constants r_1 and c which are chosen initially. For smooth minimization it is important that the contributions of the objective function and the penalties due to the equality and inequality constraints are well balanced. To make the choice independent of the initial design choice, the coefficients are computed by effecting the balance in the final value of the penalty function. From earlier experiences in solving similar problems we can get an estimate of the final value of the objective function and then compute r_1 and c to balance it. To compute the penalties associated with the inequality constraints we use a rule of thumb that for a problem with n design variables at least $n/4$ of the constraints will be critical ($g_j = 0$) at the optimum. Balancing the penalty due to these constraints with the original objective function, we get

$$\mathrm{V}(\mathbf{x}) = r_1 \left(\frac{n}{4}\right)\left(\frac{3}{g_0}\right) \quad (2.6)$$

or

$$r_0 = \mathrm{V}(\mathbf{x}) \left(\frac{4}{n}\right) \left(\frac{g_0}{3}\right). \tag{2.7}$$

Thus we can get an initial estimate of the penalty parameter r_1. Equating the $\frac{c}{\sqrt{r}}\mathbf{R}^T\mathbf{R}$ term with the objective function and knowing the value of r_1, we can compute the initial value of c.

The algorithm used to minimize ϕ is the conjugate gradient (CG) method [60] developed originally by Hestenes and Stiefel, as a method for solving systems of linear equations. This was the method Fox [46] and his co-researchers used in their attempt to integrate analysis and design. In using the CG method, the problem arises from the last term. For linear elasticity problems, the equilibrium equations are of the form,

$$\mathbf{R} = \mathbf{Ku} - \mathbf{f} = 0. \tag{2.8}$$

Expanding the last term in ϕ we get,

$$\mathbf{R}^T\mathbf{R} = (\mathbf{Ku} - \mathbf{f})^T(\mathbf{Ku} - \mathbf{f}). \tag{2.9}$$

The second derivative matrix of $\mathbf{R}^T\mathbf{R}$ with respect to $\mathbf{u}$ is $\mathbf{K}^T\mathbf{K}$. The condition number of a matrix is the ratio of its largest eigenvalue to its smallest. If the condition number of a matrix is high, it is an ill-conditioned matrix. The conjugate gradient method converges slowly for ill-conditioned matrices. The stiffness matrix $\mathbf{K}$ that appears in equation (2.9) is ill-conditioned. The condition number of the matrix $\mathbf{K}^T\mathbf{K}$ is the square of the condition number of $\mathbf{K}$ which makes it extremely ill-conditioned. Hence, the minimization of ϕ will proceed very slowly.

In recent years, methods for alleviating this ill-conditioning and its effect on iterative processes have been developed. The process is called *preconditioning*. To lessen the ill-conditioning in ϕ due to the last term, the term $\mathbf{R}^T\mathbf{R}$ can be replaced by a term $\mathbf{R}^T\mathbf{B}^{-1}\mathbf{R}$, where the matrix **B** is called the preconditioner. The penalty function ϕ now take the form

$$\text{minimize} \qquad \phi = \mathrm{V}(\mathbf{x}) + r\sum_{j=1}^{m} p[g_j(\mathbf{x},\mathbf{u})] + \frac{c}{\sqrt{r}}\mathbf{R}^T\mathbf{B}^{-1}\mathbf{R}. \tag{2.10}$$

The preconditioner **B** should have certain properties to be able to lessen the effect of ill-conditioning. It should be an easily invertible or factorizable matrix and it should be also possible to calculate $\mathbf{B}^{-1}\mathbf{R}$ inexpensively. Otherwise we don't gain from the transformation. The ideal choice for **B** is therefore a cheap-to-invert approximation to **K**. We will discuss various choices for the preconditioner in the next section. Another way to overcome the ill-conditioning is to use methods other than the penalty function formulation. An alternative [51] is to use a projected Lagrangian method.

In the penalty function we have defined by equation (2.10) there is another type of ill-conditioning. As we decrease the penalty parameter r in successive cycles, the $\frac{c}{\sqrt{r}}$ keeps increasing. Theoretically, the minimum of ϕ is reached at $r = 0$ when $\frac{c}{\sqrt{r}} = \infty$. For small values of r, as we saw earlier, the curvature of the penalty function is high which makes the problem ill-conditioned and the optimization process slows down.

This type of ill-conditioning can be remedied by using an algorithm called the **augmented Lagrangian (AL) algorithm** ([60], [57] pp. 198–201) , [61] , [62] . The algorithm adds to equation (2.10) a term $\boldsymbol{\lambda}^T\mathbf{R}$, where $\boldsymbol{\lambda}$ is a vector of Lagrange multipliers for the equilibrium constraints. The composite objective function now becomes

$$\phi = \mathrm{V}(\mathbf{x}) + r\sum_{j=1}^{m} p[g_j(\mathbf{x},\mathbf{u})] + \frac{c}{\sqrt{r}}\mathbf{R}^T\mathbf{B}^{-1}\mathbf{R} - \boldsymbol{\lambda}^T\mathbf{R}\ . \tag{2.11}$$

Initially, the minimization is started with $\boldsymbol{\lambda} = 0$ and the multiplier vector is updated after each cycle (every time r is reduced) based on the first order necessary condition for the stationarity of ϕ. Differentiating equation (2.11) with respect to the design variable x_i we get,

$$\frac{\partial \phi}{\partial x_i} = \frac{\partial \mathrm{V}}{\partial x_i} - r\sum_{j=1}^{m} \frac{\partial p[g_j]}{\partial x_i} + \frac{2c}{\sqrt{r}}\frac{\partial \mathbf{R}^T}{\partial x_i}\mathbf{B}^{-1}\mathbf{R} - \boldsymbol{\lambda}^T\frac{\partial \mathbf{R}}{\partial x_i}\ . \tag{2.12}$$

The Lagrange multipliers at the optimum satisfy the exact condition [60]

$$\frac{\partial \mathrm{V}}{\partial x_i} + \frac{2c}{\sqrt{r}}\frac{\partial \mathbf{R}^T}{\partial x_i}\mathbf{B}^{-1}\mathbf{R} - \boldsymbol{\lambda}^{*T}\frac{\partial \mathbf{R}}{\partial x_i} = 0. \tag{2.13}$$

Comparing Eq. (2.12) and Eq. (2.13), we expect that

$$\boldsymbol{\lambda} - 2(\frac{c}{\sqrt{r}})\mathbf{B}^{-1}\mathbf{R} \rightarrow \boldsymbol{\lambda}^*. \tag{2.14}$$

as we approach the minimum. Based on this relation, Hestenes [60] suggested that Eq. (2.14) be used as a recurrence relation to update the Lagrange multipliers as

$$\boldsymbol{\lambda}^{(k+1)} = \boldsymbol{\lambda}^{(k)} - 2(\frac{c_k}{\sqrt{r_k}})\mathbf{B}^{-1}\mathbf{R}\ . \tag{2.15}$$

Once the Lagrange multipliers converge close to their optimal value, they are capable of reducing $\mathbf{R}$ without further reductions of the penalty parameter r. Hence $\frac{c}{\sqrt{r}}$ will not grow and the associated ill-conditioning will not increase. Reference [62] cites various numerical experiments where the AL algorithm has fared very well compared to the penalty function approach. In the next chapter we will show substantial savings in CPU time by using the AL algorithm compared to the penalty function approach. Reference [62] also discusses the convergence rate of the AL algorithm. It has been shown that the rate at which the design variables converge is at best the rate at which the Lagrange multipliers in $\boldsymbol{\lambda}$ converge to their actual value. Since we use a first order necessary condition to update $\boldsymbol{\lambda}$ we can at most expect a linear rate of convergence. To help $\boldsymbol{\lambda}$ converge faster, second order conditions have been suggested [63] for updating.

2.3 Types of preconditioners

When the PF or AL formulation is used for the SAND approach, preconditioning becomes important in order to alleviate the effects of ill-conditioning. There are several choices for the preconditioner $\mathbf{B}$. They are described below:

2.3.1 Diagonal preconditioner

This simple preconditioner is also known as Jacobi acceleration. This preconditioner is defined by

$$\mathbf{B} = diag(\mathbf{K}), \tag{2.16}$$

where $\mathbf{K}$ is the global stiffness matrix of the structure. This preconditioner basically multiplies the vector $\mathbf{R}$ by the inverse of the diagonal of $\mathbf{K}$.

2.3.2 Symmetric Gauss-Seidel preconditioner

The global stiffness matrix is decomposed as

$$\mathbf{K} = \mathbf{L} + \mathbf{W} + \mathbf{L}^T, \tag{2.17}$$

where $\mathbf{L}$ is a lower triangular matrix with zero diagonal entries and $\mathbf{W}$ is the diagonal of $\mathbf{K}$. The preconditioner is obtained as

$$\mathbf{B} = \frac{1}{2}\left(2\mathbf{W} + \mathbf{L}\right)\mathbf{D}^{-1}\left(2\mathbf{W} + \mathbf{L}^T\right) \tag{2.18}$$

The above two preconditioners are obtained by simple methods. However, the Gauss-Siedel precondiotioner requires the global stiffness matrix to be assembled. In methods like topology optimization where structural elements vanish during the course of the optimization, the global stiffness matrix may become singular along the way. Hence, these preconditioners have to be computed based on the initial design. Also structures with many elements storing the global stiffness matrix itself may pose a problem.

2.3.3 Element-By-Element (EBE) preconditioners.

About 10 years ago Hughes and his co-researchers [50] developed a new class of preconditioners that do not require the assembly or factorization of the global **K**. These are called *Element by Element (EBE) preconditioners.* The preconditioners are generated based on the approximate factorization of element matrices.

The global stiffness matrix **K** is the summation of all the element stiffness matrices

$$\mathbf{K} = \sum_{e=1}^{nele} \mathbf{K}^e. \tag{2.19}$$

We now represent **K** as,

$$\mathbf{K} = \mathbf{W}^{-1/2}\mathbf{C}\mathbf{W}^{-1/2}, \tag{2.20}$$

where **W** is a diagonal matrix formed out of the diagonal elements of **K**

$$\mathbf{W} = \sum_{e=1}^{nele} \mathbf{D}^e, \tag{2.21}$$

where $\mathbf{D}^e$ are the diagonals of the element stiffness matrices and so

$$\mathbf{C} = \mathbf{W}^{-1/2}\mathbf{K}\mathbf{W}^{-1/2}. \tag{2.22}$$

From equation (2.19),

$$\mathbf{C} = \mathbf{W}^{-1/2}\sum_{e=1}^{nele} \mathbf{K}^e\ \mathbf{W}^{-1/2}. \tag{2.23}$$

Adding and subtracting the unit matrix **I** we get

$$\mathbf{C} = \mathbf{W}^{-1/2}\sum_{e=1}^{nele} \mathbf{K}^e\ \mathbf{W}^{-1/2} - \mathbf{I} + \mathbf{I}. \tag{2.24}$$

By the definition of **W** and $\mathbf{D}^e$ we have,

$$\mathbf{C} = \mathbf{W}^{-1/2}\sum_{e=1}^{nele} \mathbf{K}^e\ \mathbf{W}^{-1/2} - \mathbf{W}^{-1/2}\sum_{e=1}^{nele} \mathbf{D}^e\ \mathbf{W}^{-1/2} + \mathbf{I}, \tag{2.25}$$

or

$$\mathbf{C} = \mathbf{W}^{-1/2}\sum_{e=1}^{nele}(\mathbf{K}^e - \mathbf{D}^e)\mathbf{W}^{-1/2} + \mathbf{I}. \tag{2.26}$$

Defining

$$\bar{\boldsymbol{K}}^e = \mathbf{W}^{-1/2}\left(\mathbf{K}^e - \mathbf{D}^e\right)\mathbf{W}^{-1/2}, \tag{2.27}$$

we get

$$\mathbf{C} = (\mathbf{I} + \sum_{e=1}^{nele}(\bar{\boldsymbol{K}}^e) = \mathbf{I} + \bar{\boldsymbol{K}}. \tag{2.28}$$

Now we approximate $\mathbf{K}$ as a product of the element matrices using the following approximation which is valid for a series of numbers h_i, where $h_i << 1$

$$\prod_{i=1}^{n}(1 + h_i) = (1 + h_1)(1 + h_2)\ldots(1 + h_n) \cong 1 + \sum_{i=1}^{n} h_i. \tag{2.29}$$

This first order approximation is used to express the sum I+ $\sum_{e=1}^{nele}(\bar{\boldsymbol{K}}^e)$ as a product and so

$$\mathbf{C} = \mathbf{I} + \sum_{e=1}^{nele}(\bar{\boldsymbol{K}}^e) \cong \prod_{i=1}^{nele}(\mathbf{I} + \bar{\boldsymbol{K}}^e). \tag{2.30}$$

The sum I+ $\bar{\boldsymbol{K}}^e$ for each element is factorized as

$$\mathbf{I} + \bar{\boldsymbol{K}}^e = \mathbf{L}^e\mathbf{D}^e(\mathbf{L})^{eT}, \tag{2.31}$$

and since $\mathbf{L}^e$and $\mathbf{D}^e$are close to the unit matrix we can also change the order of multiplication and get

$$\mathbf{C} \cong \bar{\boldsymbol{C}} = \prod_{i=1}^{nele}\mathbf{L}^e\prod_{i=1}^{nele}\mathbf{D}^e\prod_{i=nele}^{1}(\mathbf{L}^e)^T. \tag{2.32}$$

Finally we obtain **B** as

$$\mathbf{B} \cong \mathbf{K} = \mathbf{W}^{1/2}\bar{\boldsymbol{C}}\mathbf{W}^{1/2} \tag{2.33}$$

and $\mathbf{B}$ is indeed cheap to invert because it is a product of diagonal matrices and small element matrices.

In this work the equilibrium term of the composite objective function ϕ has been preconditioned with this EBE preconditioner given by equation (2.33).

2.3.4 Cholesky Element-By-Element (EBE) preconditioner.

We define as before

$$\bar{\boldsymbol{K}}^e = \mathbf{W}^{-1/2}\,(\,\mathbf{K}^e - \mathbf{D}^e)\,\mathbf{W}^{-1/2} \tag{2.34}$$

and factor $\bar{\boldsymbol{K}}^e + \mathbf{I}$. Let $\mathbf{L}_p$ be its lower triangular factor. We then obtain $\mathbf{B}$ as

$$\mathbf{B} = \mathbf{W}^{1/2} \prod_{i=1}^{nele} \mathbf{L}_p \prod_{i=nele}^{1} (\mathbf{L})_p{}^T \mathbf{W}^{1/2}. \tag{2.35}$$

2.3.4 Crout Element-By-Element (EBE) preconditioner.

If the matrix $\mathbf{K}$ is not symmetric, then we use this preconditioner. We generate $\bar{\boldsymbol{K}}^e$ matrix as before and factor it. Let $\mathbf{L}_p$, $\mathbf{D}_p$ and $\mathbf{U}_p$ be the lower, diagonal and upper factors of $\bar{\boldsymbol{K}}^e$. We then get $\mathbf{B}$ as

$$\mathbf{B} = \mathbf{W}^{1/2} \prod_{i=1}^{nele} \mathbf{L}_p \prod_{i=1}^{nele} \mathbf{D}_p \prod_{i=nele}^{1} (\mathbf{U})_p \mathbf{W}^{1/2}. \tag{2.36}$$

The Crout EBE preconditioner gives a better approximation of $\mathbf{K}$ than the Cholesky preconditioner, and has been shown to improve convergence of the algorithm. Hence, we in this work use a modified version of Crout EBE preconditioner even though the stiffness matrix $\mathbf{K}$ is symmetric.

$$\mathbf{B} = \mathbf{W}^{1/2} \prod_{i=1}^{nele} \mathbf{L}_p \prod_{i=1}^{nele} \mathbf{D}_p \prod_{i=nele}^{1} (\mathbf{L})_p{}^T \mathbf{W}^{1/2}. \tag{2.37}$$

This is the same preconditioner as expressed by equation (2.33).

2.4 Beale's restarted CG algorithm

The augmented Lagrangian function ϕ which has been preconditioned by the Crout's EBE preconditioner, is minimized by the Beale's restarted CG algorithm [64] . The steps of the algorithm to minimize a function f($\mathbf{x}_0$) are as follows:

1. Given the initial design variable vector $\mathbf{x}_0$, define the initial direction $\mathbf{s}_0$ as the steepest descent direction given by,

$$\mathbf{s}_0 = -\nabla f(\mathbf{x}_0) = \mathbf{g}_0.$$

Let k be the iteration index and t be the restart index. Initially, set $k = 0$ and $t = 0$ and begin iterations by incrementing k.

2. For $k \geq 1$ the direction vector $\mathbf{s}_k$ is given by the Beale's formula [64]

$$\mathbf{s}_k = -\mathbf{g}_k + \beta_k \mathbf{s}_{k-1} + \gamma_k \mathbf{s}_t, \tag{2.38}$$

where

$$\mathbf{g}_k = -\nabla f(\mathbf{x}_k), \tag{2.39}$$

and

$$\beta_k = \frac{\mathbf{g}_k{}^T[\mathbf{g}_k - \mathbf{g}_{k-1}]}{\mathbf{s}_{k-1}^t[\mathbf{g}_k - \mathbf{g}_{k-1}]}, \tag{2.40}$$

$$\gamma_k = \frac{\mathbf{g}_k{}^T[\mathbf{g}_{t+1} - \mathbf{g}_t]}{\mathbf{s}_t{}^T[\mathbf{g}_{t+1} - \mathbf{g}_t]}, \qquad if\ k > t+1 \tag{2.41}$$

$$\text{and} \quad \gamma_k = 0, \qquad if\ k = t+1 \quad . \tag{2.42}$$

The CG algorithm requires that the vectors $\mathbf{g}_k$ and $\mathbf{g}_{k-1}$ be orthogonal. Due to numerical rounding off, the vectors lose orthogonality after certain number of iterations. The algorithm is then restarted from the previous best solution. It is done by testing,

3. For $k \geq 1$ the inequality

$$|\mathbf{g}^T{}_{k-1}\mathbf{g}_k| \geq 0.2||\mathbf{g}_k||^2. \tag{2.43}$$

If this inequality holds then enough orthogonality has been lost between $\mathbf{g}_{k-1}$ and $\mathbf{g}_k$ and a restart is needed. This is achieved by setting $t = k - 1$ and proceeding.

4. For $k \geq t + 1$ the direction $\mathbf{s}_k$ is checked to guarantee a sufficiently large derivative by testing the inequalities

$$-1.2||\mathbf{g}_k||^2 \leq \mathbf{s}^T{}_k\mathbf{g}_k| \geq -0.8||\mathbf{g}_k||^2. \tag{2.44}$$

If these inequalities do not hold then the algorithm is restarted by setting $t = k - 1$. The CG algorithm in exact precision finds the minimum for a function of n variables in n or less iterations. However, using a digital computer with finite precision, this may not be possible. So,

5. if $k - t \geq n$, then the algorithm is restarted by setting $t = k - 1$.

6. The algorithm is successfully terminated when $||\mathbf{g}_{k-1}||$ is sufficiently small.

3. TRUSS TOPOLOGY OPTIMIZATION WITH SIMULTANEOUS ANALYSIS AND DESIGN

In this chapter the simultaneous analysis and design approach is applied to the topology design of trusses. We use the ground structure approach of Ref. [14]. The SAND approach is applied to truss topology design

1.) for minimum compliance and
2.) for minimum weight subject to constraints on the member stresses and nodal displacements.

For the compliance minimization problem the SAND approach is formulated using a penalty function approach and the weight minimization problem by using both penalty function and augmented Lagrangian formulations.

3.1 Ground structure approach

In the ground structure approach the design domain is divided into a finite number admissible nodal points. Each of these points represents a possible joint for truss members. For a given layout of grid points the ground structure is formed by connecting each node to every other node by truss members. The ground structure will be used as the initial design and the final optimum topology is obtained as a subset of this ground structure. In this dissertation a variation of the ground structure is used. The ground structure as proposed by reference [[14]] can have members overlapping. By avoiding the duplication while connecting the nodes, a ground structure with less members can be formed. For example, for the 5 X 5 rectangular grid, the ground structure of [[14]] will have $25(24)/2 = 300$ members, whereas without overlapping members it has only 196 members. The ground structure for the case where the rectangular design domain has been divided into 25 grid points is shown in Figure 3.1.

3.2 SAND approach for compliance minimization

The minimization problem [42] involves the minimization of compliance $\mathbf{f}^T \mathbf{u}$ (maximization of stiffness) for a given volume V of the truss, where, $\mathbf{f}$ and $\mathbf{u}$ are the force and displacement vectors, respectively. Denoting the elemental volumes as x_i, the problem is formulated as

$$
\begin{aligned}
&\underset{\mathbf{x},\mathbf{u}}{\text{minimize}} \quad \mathbf{f}^T\mathbf{u} \\
&\text{subject to} \quad g_j(\mathbf{x}) = x_j/x_0 \geq 0, \quad j = 1,\ldots,m, \\
&\qquad g_{m+1}(\mathbf{x}) = 1 - \sum_{j=1}^{m} x_i/V \geq 0, \qquad \text{and} \\
&\sum_{i=1}^{m} \mathrm{x}_i \mathrm{K}_i \mathbf{u} - \mathbf{f} = \mathbf{0}\,,
\end{aligned}
\tag{3.1}
$$

where x_0 is a reference element volume and K_i is the stiffness matrix per unit volume of the i th truss element.

The above constrained minimization problem is then converted into an unconstrained minimization problem by using a penalty function technique.

The advantages of the compliance minimization formulation are :

- The calculation of derivatives of the compliance does not require the solution of the equilibrium equations. Hence it is easily computed.
- Secondly, optimization for a single constraint (volume constraint here) can be performed very efficiently.

The main disadvantage of the compliance minimization formulation is that it does not address the general problem with stress and displacement constraints. When designs under stress and displacement constraints are required, a two stage approach is used. The topology is obtained first by compliance minimization and then the truss members are resized for the actual objective function and constraints.

The simultaneous analysis and design (SAND) approach on the other hand, is a more general method that can solve actual problems with stress and displacement constraints. This method is discussed in the next sections.

3.3 The SAND approach with weight minimization

The truss topology optimization problem by weight minimization is to minimize the volume (hence the weight) of the truss subject to stress and displacement constraints. For a single loading case with the allowable stress levels in tension and compression being at the same level, this is equivalent to the compliance minimization problem subject to a weight (volume) budget.

The weight minimization problem is formulated as

$$\begin{aligned} &\underset{\mathbf{x},\mathbf{u}}{\text{minimize}} \quad W(\mathbf{x}) \\ &\text{subject to} \quad g_j(\mathbf{x},\mathbf{u}) \geq 0, \quad j = 1, \ldots, m, \quad \text{and} \\ &\mathbf{R} = \mathbf{K}\mathbf{u} - \mathbf{f} = \mathbf{0}\,, \end{aligned} \tag{3.2}$$

where W is the weight, g_j are stress or displacement constraints and $\mathbf{R}$ is the residual vector when the equilibrium equations are not satisfied.

Due to the addition of the displacements as design variables, The SAND method generally increases the number of design variables substantially. However for truss topology problems this is less of a problem since the ground structure approach leads to a very large number of cross sectional area design variables and comparatively fewer displacement design variables. For example, for the ground structure shown in figure 3.1, the 196 cross sectional areas are augmented by only 40 displacement variables.

The weight minimization problem is then converted into an unconstrained optimization problem by using the penalty function (PF) formulation and the augmented Lagrangian (AL) (see Chapter 2) formulation.

3.4 Example problems

The truss topology optimization problem is to find the optimal truss to transmit an applied load to the supports, as a subset of the initial ground structure. In this work the problem of finding the optimal truss to transmit a vertical load, applied at the lower right hand corner, to the simply supported nodes on the left was used to demonstrate the use of SAND.

As a first example a ground structure similar to the one shown in figure 3.1 with a horizontal length of 720 inches and a height of 360 inches (aspect ratio of 2:1) was used. All the truss elements had an elastic modulus of 10^4 ksi and a density of 0.1 lb/in^3. The truss was loaded with a vertical point load of 100 kips. The objective function was the weight of the truss and only stress constraints were considered with allowable stresses of 25 ksi in tension and compression. The member cross sectional areas in the ground structure were uniform and chosen to satisfy the stress constraints.

Three approaches based on the SAND approach were used to find the optimal topology :

a) Compliance minimization for a given weight using a PF solution scheme

b) Weight minimization subject to stress constraints using PF solution

c) Weight minimization subject to stress constraints using AL solution

It was mentioned in chapter 2 that weight minimization subject to uniform stress allowables in tension and compression produces a design with maximized stiffness or minimized compliance. Hence the approach (a) is equivalent to both (b) and (c).

The optimal truss designs obtained in each of these cases were compared with those in Ref. [29] in terms of geometry, layout, and the nondimensional compliance η defined as

$$\eta = (\mathbf{f}^T\mathbf{u})VE/(\| \mathbf{f} \|^2 L^2), \tag{3.3}$$

Table 3.1 Comparison of SAND based algorithms for aspect ratio 2:1 problem

Problem Size	Non-Dimensional compliance			IBM 3090 CPU time (sec)		
	Compliance minimization	Weight minimization		Compliance minimization	Weight minimization	
	PF	PF	AL	PF	PF	AL
4 X 3	16.447	16.447	16.448	31	13	10
5 X 5	14.342	14.344	14.344	621	298	116
7 X 5	14.123	14.122	14.130	1196	740	561

where V is the volume, E the elastic modulus and L the horizontal length of the truss.

The non-dimensional compliances and the computation times for the 2:1 aspect ratio problem are shown in Table 3.1.

Column 1 describes the problem size. In columns 2, 3 and 4, the non-dimensional compliances obtained with the three methods are compared. As we increase the number of nodes in the ground structure, the non dimensional compliance decreases but seems to be converging. This is reasonable since by increasing the number of grid points and the number of members in the ground structure we get closer to the absolute optimum. The agreement between weight and compliance minimization is good, confirming the theoretical result that weight minimization with uniform stress constraints is equivalent to compliance minimization. Columns 5, 6 and 7 compare the CPU time used by the three methods on an IBM-3090 computer. As expected, the AL approach is more efficient that the PF approach. The large advantage of the weight minimization formulation over compliance minimization formulation (though they are theoretically equivalent and the fact that compliance minimization formulation is computationally efficient) may be due to programming idiosyncrasies as the results were obtained with different computer programs. Comparison of columns 6 and 7 shows that the AL formulation is computationally more efficient than the PF formulation.

The optimal trusses obtained with these three formulations by using three different grid sizes are shown in figures 3.2, 3.3 and 3.4 respectively. The thicknesses of the lines in the figures are proportional to the cross sectional areas of the corresponding truss members they represent. It is seen that all three methods predict the same optimal topology for the 4×3 and the 7×5 grid, whereas the weight minimization optimum topology for the 5×5 is different from the compliance minimum optimum though the non dimensional compliance value is the same in both cases. It is possible that for this case the optimum design is not unique, or that one of the results is a near optimum.

3.5 Truss topology optimization for stress and displacement constraints.

Since compliance minimization can be performed very efficiently by specialized methods [29], [42], it makes sense to try to use it also for more general problems. This means that we can first find an optimum topology by compliance minimization, and then resize members to take care of the actual objective function and constraints. The SAND approach, on the other hand, is applicable directly to general stress and displacement constraints starting from the ground structure. The two approaches are compared for the previous example with a displacement constraint added to the stress constraints .

The test case constraint is a constraint on the horizontal displacement at the corner (See Fig.3.1) where only the vertical load is applied in addition to the stress constraints. The horizontal displacement at this node was constrained to be 0.002 L, which is 80% of the displacement at that node in the minimum compliance optimum.

Table 3.2. Comparison of optimal weights for x displacement at loading node $\leq 2 \times 10^{-3}$ L in (80% of optimum compliance design)

Problem Size	Direct Optimization of Ground Structure lb	Sizing Optimization of Compliance Topology lb
4 X 3	1264	1264
5 X 5	1140	1140
7 X 5	1130	1136

The final designs obtained with both approaches are compared in Table 3.2.

The first column describes the problem. The second column shows the optimal weight obtained by direct optimization of the ground structure. The third column gives the weight obtained by sizing optimization of the minimum compliance topology. For this displacement constraint case it is seen that sizing optimization of the minimum compliance optimum topology is as good as direct optimization starting from the ground structure. This changed when the displacement was required to be less than or equal to 50 % of the unconstrained displacement. At first the optimal designs turned out to be mechanisms since the displacement constraint is trivially satisfied then. To avoid this, a small horizontal load of 0.1 lb was additionally introduced at the lower right hand corner and the trusses were reoptimized.

Table 3.3 shows that sizing optimization of the optimum compliance topology leads to a heavier design than the one obtained from direct optimization of the ground structure

Table 3.3 Comparison of optimal weights for x displacement at loading node $\leq$ 1.25×10^{-3} L in (50% of optimum compliance design)

Problem Size	Direct Optimization of Ground Structure	Sizing Optimization of Compliance Topology
4 X 3	1433	1546
5 X 5	1274	1209
7 X 5	1259	1304

for the 4×3 grid and the 7×5 grid. However, for the 5×5 grid direct optimization yielded a heavier design compared to sizing optimization. It was thought that this could be a local minimum. To see if this was indeed a local minimum a test was performed. Let $\mathbf{x}_{CM}$ represent the cross sectional areas in the optimal compliance topology and $\mathbf{x}_{WM}$ the cross sectional areas from the weight minimization topology. The designs on the line connecting $\mathbf{x}_{CM}$ and $\mathbf{x}_{WM}$ are given by all convex combinations of the two vectors. This combination is given by

$$\mathbf{x} = \alpha \mathbf{x}_{CM} + (1 - \alpha)\mathbf{x}_{WM}. \qquad 0 \leq \alpha \leq 1 \tag{3.4}$$

The resulting truss with cross sectional areas given by $\mathbf{x}$ was analyzed to get the displacements and stresses. The amounts by which these stresses violate the constraints are computed The percentage violations of the constraints were plotted as a function of α and is shown in figure. 3.5. The weight of the trusses for different values of α is also plotted. It can be seen that near $\alpha = 0$ which is the compliance minimization topology, the constraint violation is very high. As we move away from this point and approach the weight minimization topology by increasing α, the constraint violation decreases.

This means that the compliance minimization topology may be a **singular truss** [[20]] in the design space. A singular truss is an optimum solution that has no stress constraint violation. However, if we add a very slender member to this solution the stress will be very high in that member and the stress constraint will be violated heavily as is the case for small values of α. The constraint violations can be eliminated only by large increases in the cross sectional areas of the truss members. This means that the singular topology cannot be reached by gradually reducing the areas of the additional members. Reference [29] arrived at this topology by using the optimality criteria method. When the convex combination for $\alpha = 0.6$ was used as the initial design for weight minimization with SAND, we were able to converge to the compliance minimum singular topology optimum. For other values of α the optimizer could converge only to the heavier weight minimization topology.

The optimum designs are shown in Fig. 3.6 for the 4×3 grid, in Fig. 3.7 for the 5×5 and in Fig. 3.8 for the 7×5 grid. It is clear that direct optimization changed the topology to remove the horizontal member at the loading point since this makes the displacement constraint easy to satisfy. In Figure 3.8(a) we see that the optimum topology is still a mechanism.

The allowable limit on the displacement was next set to be less than 25 % of the unconstrained displacement. The trend was similar to the previous case but the optimal trusses obtained by sizing optimization were much heavier than the SAND optima for the 4×3 and 7×5 cases. The results are shown in Table 3.4 and in figures 3.9-3.11. It can be seen that the weight minimization of the ground structure predicts topologies that are in general more complicated but lighter than those predicted by the two stage approach. This is exemplified in figure 3.12 for the 7×5 grid.

For this case of displacement constraint, it is seen that the weight minimization of the 4×3 ground structure using the SAND approach and considering all the constraints

Table 3.4 Comparison of optimal weights for x displacement at loading node $\leq$ 0.625×10^{-3} L in (25% of optimum compliance design)

Problem Size	Direct Optimization of Ground Structure	Sizing Optimization of Compliance Topology
4 X 3	1648	2316
5 X 5	1448	1300
7 X 5	1441	1760

from the beginning predicted an optimal truss which was 40% lighter than the resized optimum from the compliance minimized topology.

From the discussion above we infer two main points.

(i) Using the compliance minimization approach in a two-stage process to obtain optimal topologies for general problems can lead to substantially inferior designs.

(ii) Though weight minimization using the SAND approach produced lighter optimal trusses for the 4×3 and 7×5 ground structures, it failed to do so for the 5×5 ground structure. SAND, being a mathematical programming method, is capable of predicting only local minima, and is not able to predict the singular topology for the 5×5 grid obtained by compliance minimization.

3.6 Member elimination strategy

The ground structure shown in figure 3.1 has 196 truss members. The optimal truss that predicted of the SAND approach starting from this ground structure has 27 members. The cross sectional areas of most of the members present in the initial structure have been reduced to zero by the optimizer. A significant amount of computation time is spent by the optimizer in reducing the cross sectional areas of those members that do not appear in the final design from their initial value to zero. In the latter part of the process the optimization procedure has to spend a lot of time in making very small cross sectional areas even smaller.

The cost of reducing the unwanted areas to zero could be alleviated if it was possible to identify and eliminate those unimportant members early on in the optimization process. This was the motivation for developing a member elimination strategy.

The first strategy attempted was based on the cross sectional areas of the members. After a predetermined number of cycles (reductions of the penalty parameter r), the cross sectional areas of the truss members were compared and the maximum was identified. All the members with areas less than 1% of this maximum were then eliminated from the design. The finite element mesh was then resized and the optimization process was resumed with this reduced structure. This strategy worked well for the 4×3 grid, but for the 5×5 and 7×5 truss this strategy eliminated some of the members that were required to be present in the final design. These members had very small cross sections but were fully stressed to 25 ksi. Elimination of these members produced nonoptimal topologies.

With the realization that member cross sectional area alone cannot be used as criterion for eliminating members, a second strategy was developed and tested. After every five optimization cycles (i.e. after the penalty parameter r has been reduced five times) the element cross sections and the elemental stresses were compared and the maxima identified. Elements were then eliminated based on the following criterion. An element was removed if its cross sectional area was less than 1% of the maximum area in the current design, and if simultaneously the elemental stress was less than 75% of the maximum stress. This strategy combining cross sectional areas and the stresses identified and effectively weeded out most of the unimportant members. The optimization process was resumed after redefining the finite element model for the truss.

Table 3.5. Member elimination strategy for 2 : 1 geometry. (Weight minimization with AL). (Compare with Table 3.1 to see effect on CPU time)

Problem Size	Number of Members			IBM 3090 CPU time (Seconds)
	In ground structure	After 5 cycles	In final design	
4 X 3	47	6	6	10
5 X 5	196	27	27	86
7 X 5	384	102	23	222

The problems described in Table 3.1 were reoptimized with the member elimination strategy applied after every five optimization cycles. The final topologies are the same but there was considerable savings in CPU time as shown in Table 3.5.

The first column of Table 3.4 describes the problem. The second column shows the number of elements in the ground structure. The third column shows the number of members left after the elimination strategy was used once after five optimization cycles. The fourth column shows the number of members in the final optimal design. The fifth column shows the CPU time used in an IBM-3090 computer. The savings in CPU time by employing the elimination strategy are compared with those in the last column of Table 3.1 and can be seen to range up to 60%. It is also seen that the savings in CPU time increase with problem size.

The aspect ratio of the ground structures for all the problems considered thus far was 1:2. Ref. [42] describes the problem of transmitting a vertical force to a parallel line of supports, with a 8:5 aspect ratio grid. The same problems were solved here to check the effect of grid refinement on the nondimensional compliance and CPU time.

The problems were solved by weight minimization using the AL formulation and the member elimination strategy. The optimal topologies are shown in Figure 3.11. These are different from the 3×3 and the 5×5 topologies given in [42]. but the non dimensional compliances are same as those in [42]. Table 3.6 shows the values obtained.

Table 3.6. Comparison of SAND based algorithm with member elimination strategy for the 8 : 5 aspect ratio grid (Weight minimization with AL)

Problem	Number of Members			Nondimensional	IBM 3090 CPU time
Size	In ground structure	After 5 cycles	In final design	Compliance	(Seconds)
2 X 2	5	5	2	14.631	0.24
3 X 3	26	18	13	13.323	5.15
5 X 5	196	30	17	11.179	61
7 X 7	754	325	33	11.071	995
9 X 9	2104	205	60	10.960	2942

The first column of Table 3.6 describes the problem. The second column shows the number of members in the ground structure. Column 3 shows the number of members after 5 optimization cycles and column 4 shows the number of members in the final design. Column 5 gives the nondimensional compliances for each case. Column 6 shows the CPU time in seconds in the IBM 3090. It is seen from the table that as the problem size increases the CPU time also increases and the non dimensional compliance decreases. The CPU time taken by the SAND approach for weight minimization seems to be vary as the number of design variables to the power of 1.7. The compliance minimization approach of Ref. [29] is computationally superior to the SAND approach hence it can handle problems with large number of design variables without the prohibitive cost experienced by SAND.

3.7 Conclusions

In this chapter a SAND formulation was applied to the problem of truss topology design for minimum weight subject to stress and displacement constraints. The generality of the SAND formulation is an advantage over specialized methods available for compliance minimization that can be computationally less expensive than SAND. It was demonstrated that the topology which is optimal for compliance minimization may not be optimal for a combination of stress and displacement constraints. Using an optimal compliance topology and optimizing cross sectional areas to minimize weight was shown to result in a weight penalty of up to 40% for one case of displacement and stress constraints.

Two strategies for alleviating the computational cost of SAND approach were implemented. They are the use of an augmented Lagrangian algorithm and progressive elimination of members with small cross sectional areas. Together these strategies reduced computational cost by up to 60%, where larger savings were obtained for larger problem sizes.

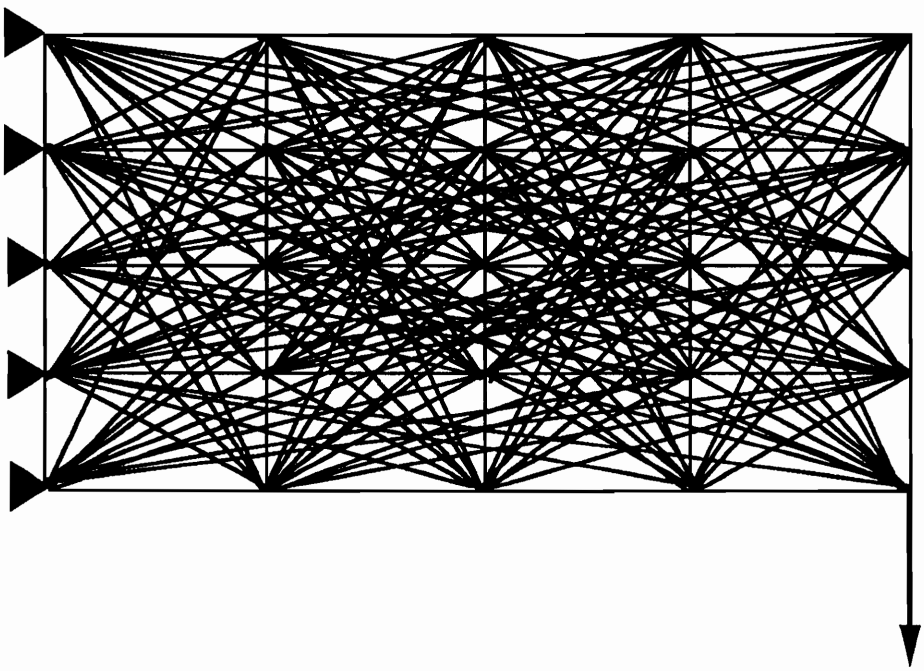


Figure 3.1 Ground structure for the 5 X 5 grid (196 members)

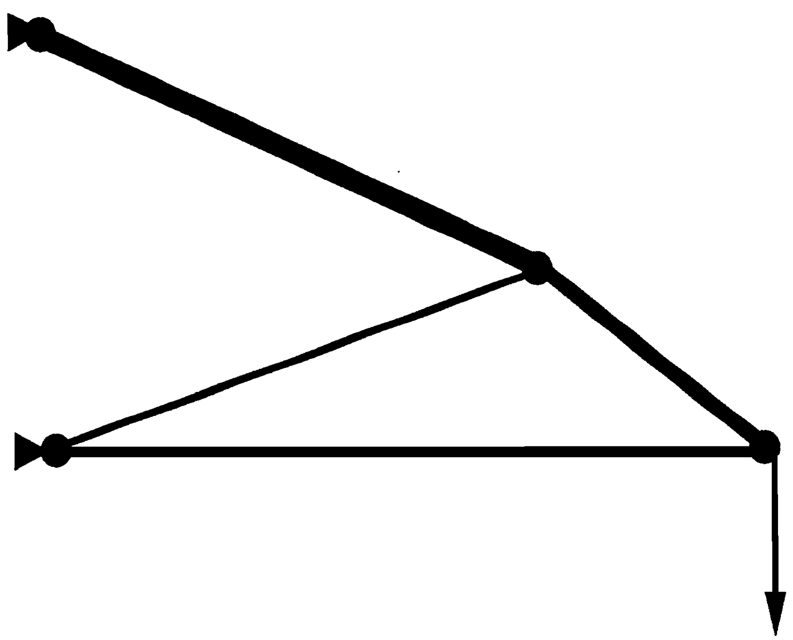


Figure 3.2 Optimum truss topology obtained for the 4 X 3 grid by all three methods

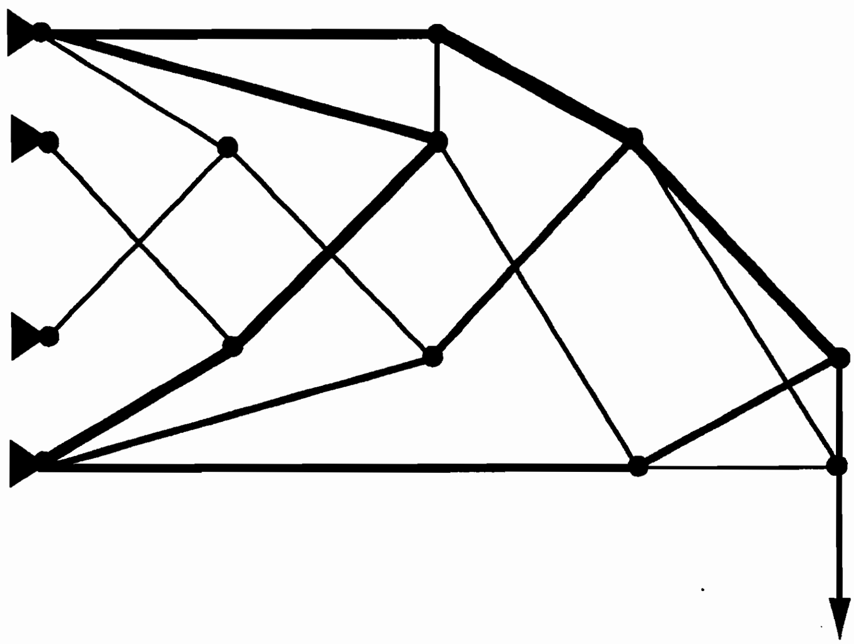


Figure 3.3 (a) Optimum truss topology obtained for the 5 X 5 grid by compliance minimization

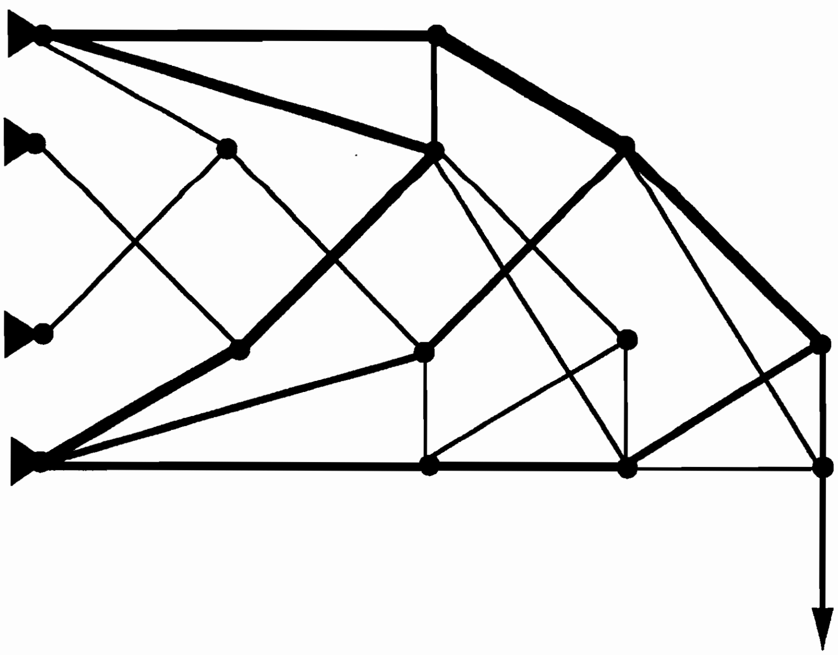


Figure 3.3 (b) Optimum truss topology obtained for the 5 X 5 grid by weight minimization

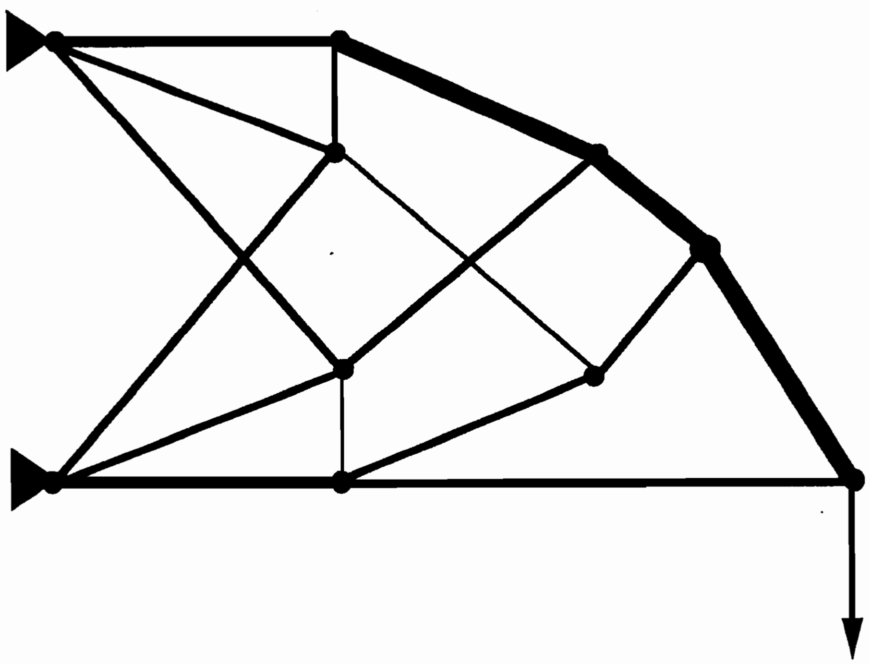


Figure 3.4 Optimum truss topology obtained for the 7 X 5 grid by all three methods

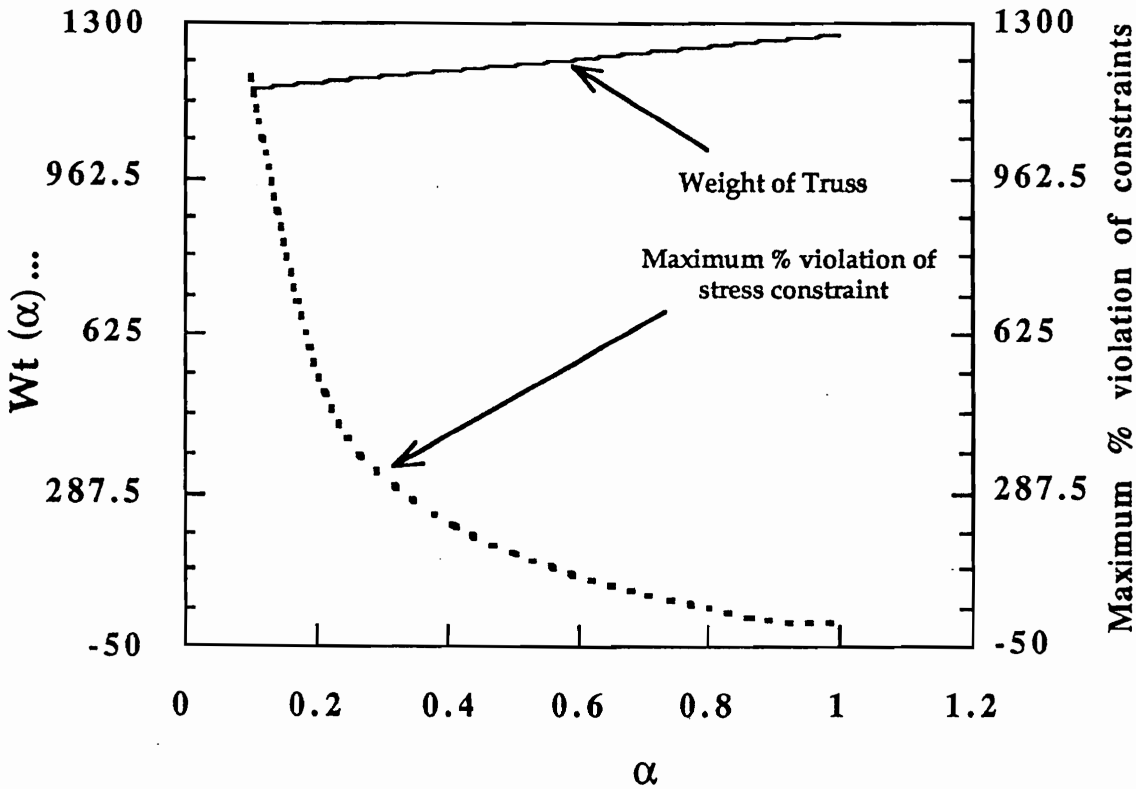
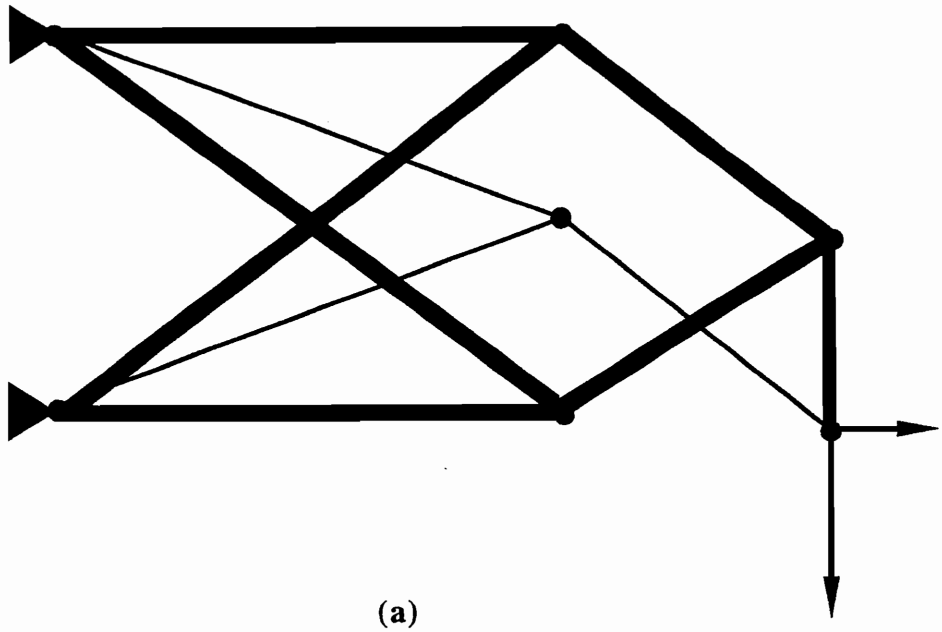
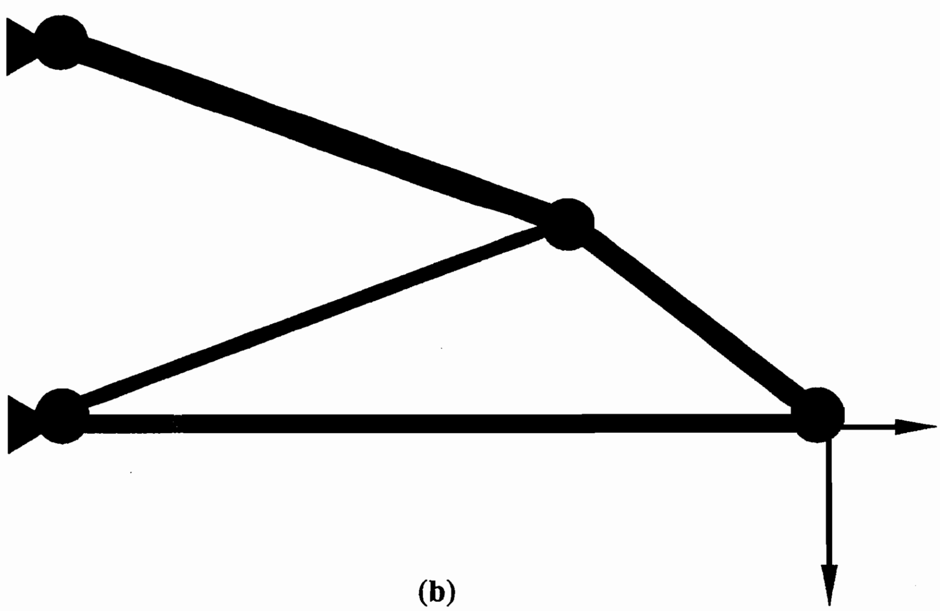


Figure 3.5. Test for local minimum

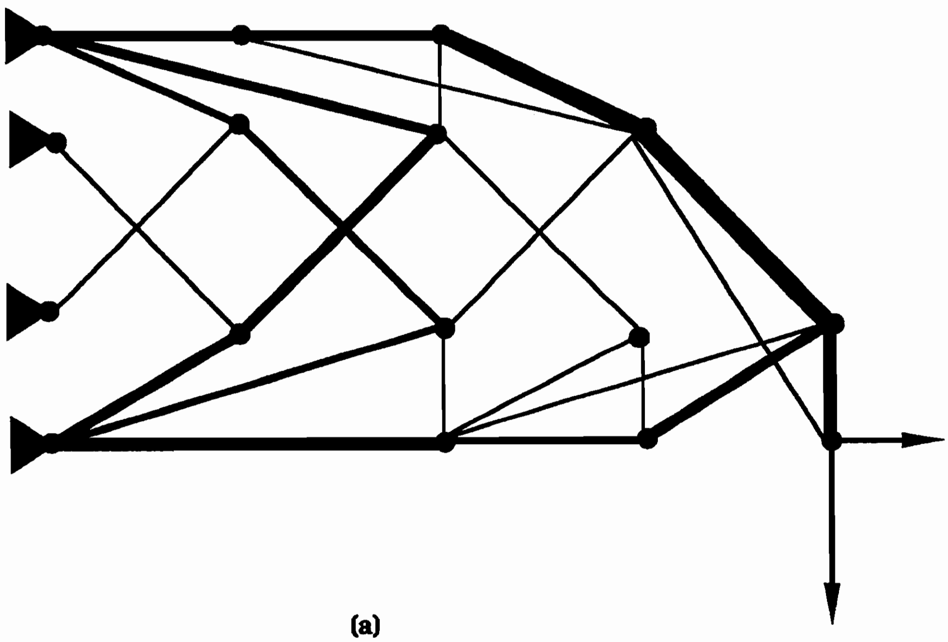


(a)

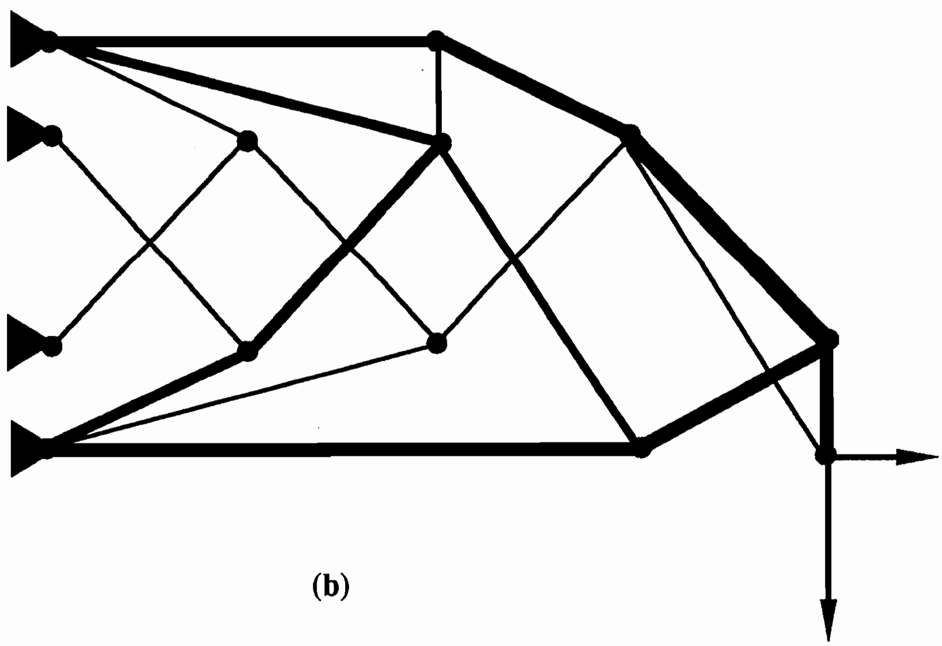


(b)

Fig 3.6. Optimal trusses obtained for the 4 X 3 grid with the 50% horizontal displacement constraint at the load with **(a)** SAND **(b)** Sizing optimization of optimum minimum compliance topology

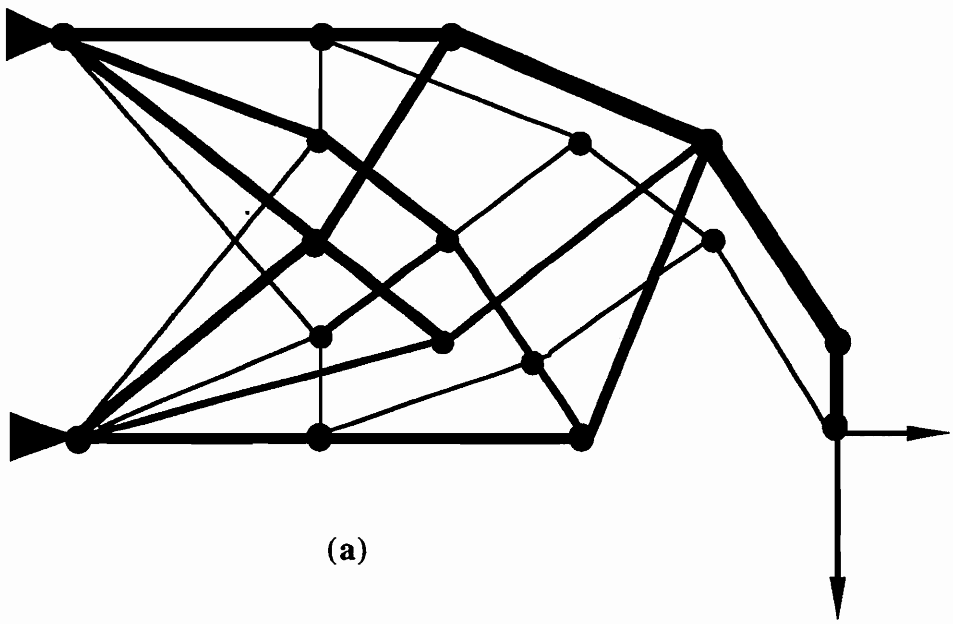


(a)

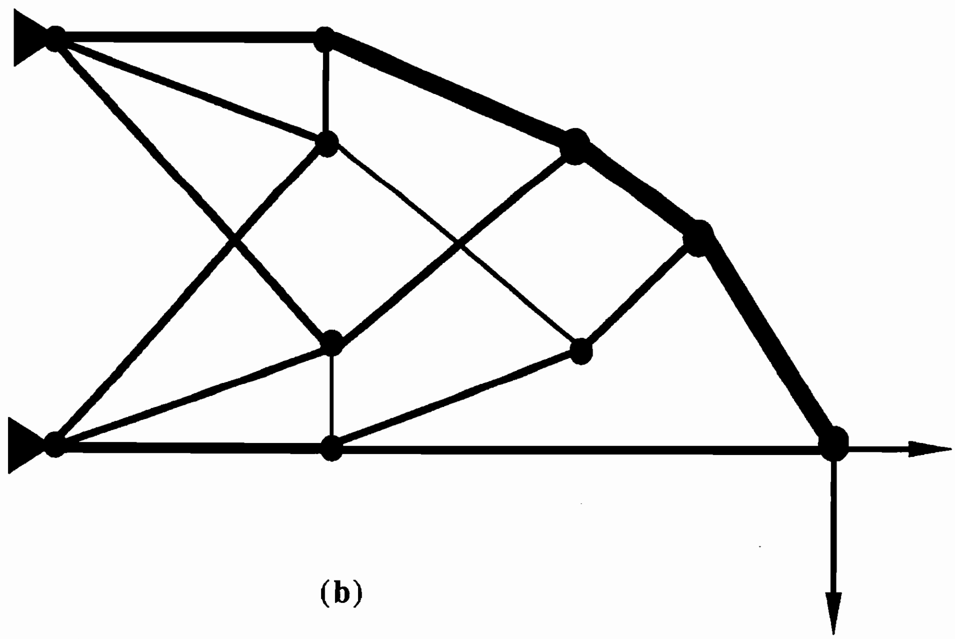


(b)

Fig 3.7. Optimal trusses obtained for the 5 X 5 grid with the 50% horizontal displacement constraint at the load with **(a)** SAND **(b)** Sizing optimization of optimum minimum compliance topology



(a)



(b)

Fig 3.8. Optimal trusses obtained for the 7 X 5 grid with the 50% horizontal displacement constraint at the load with **(a)** SAND **(b)** Sizing optimization of optimum minimum compliance topology

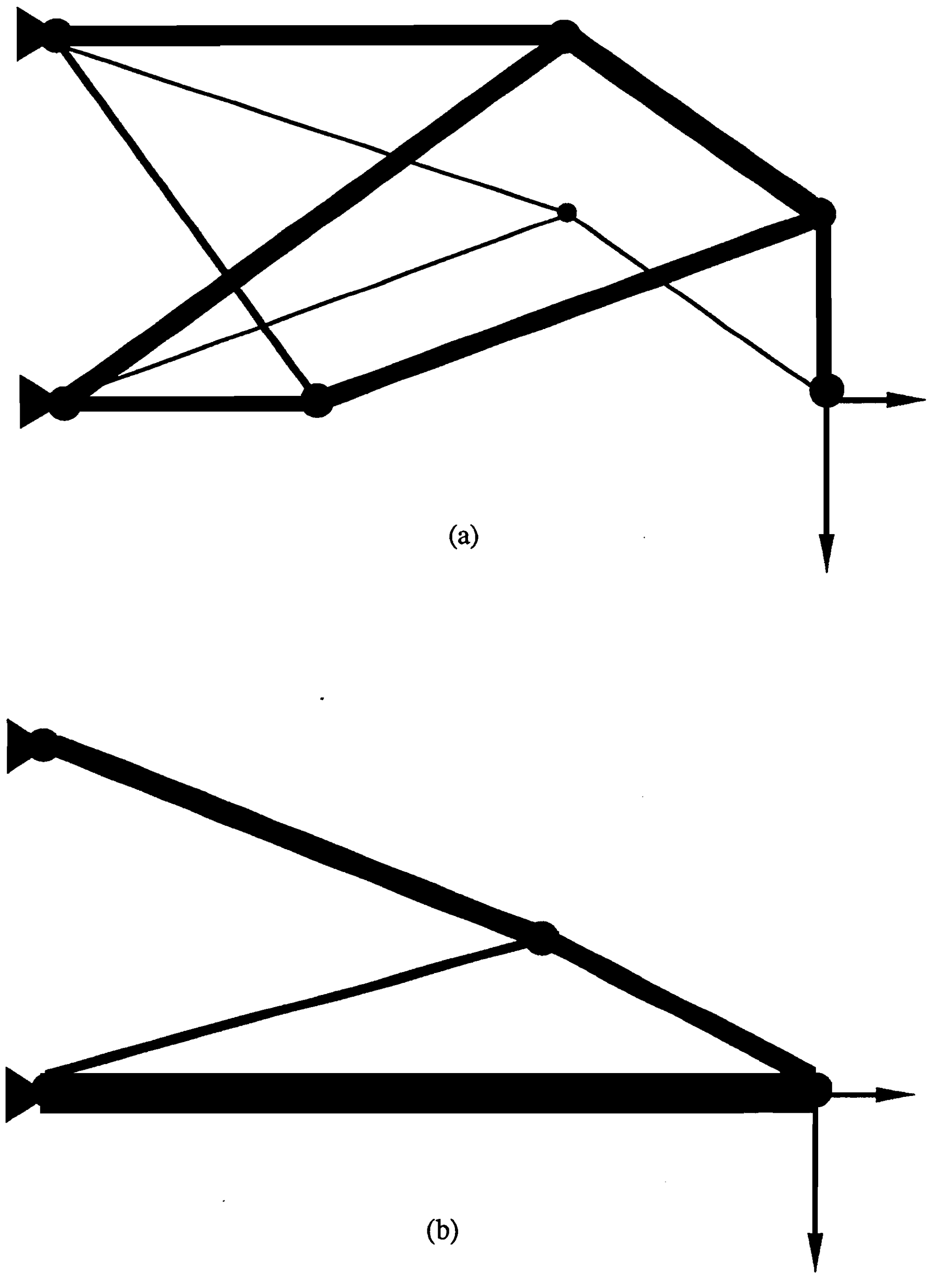
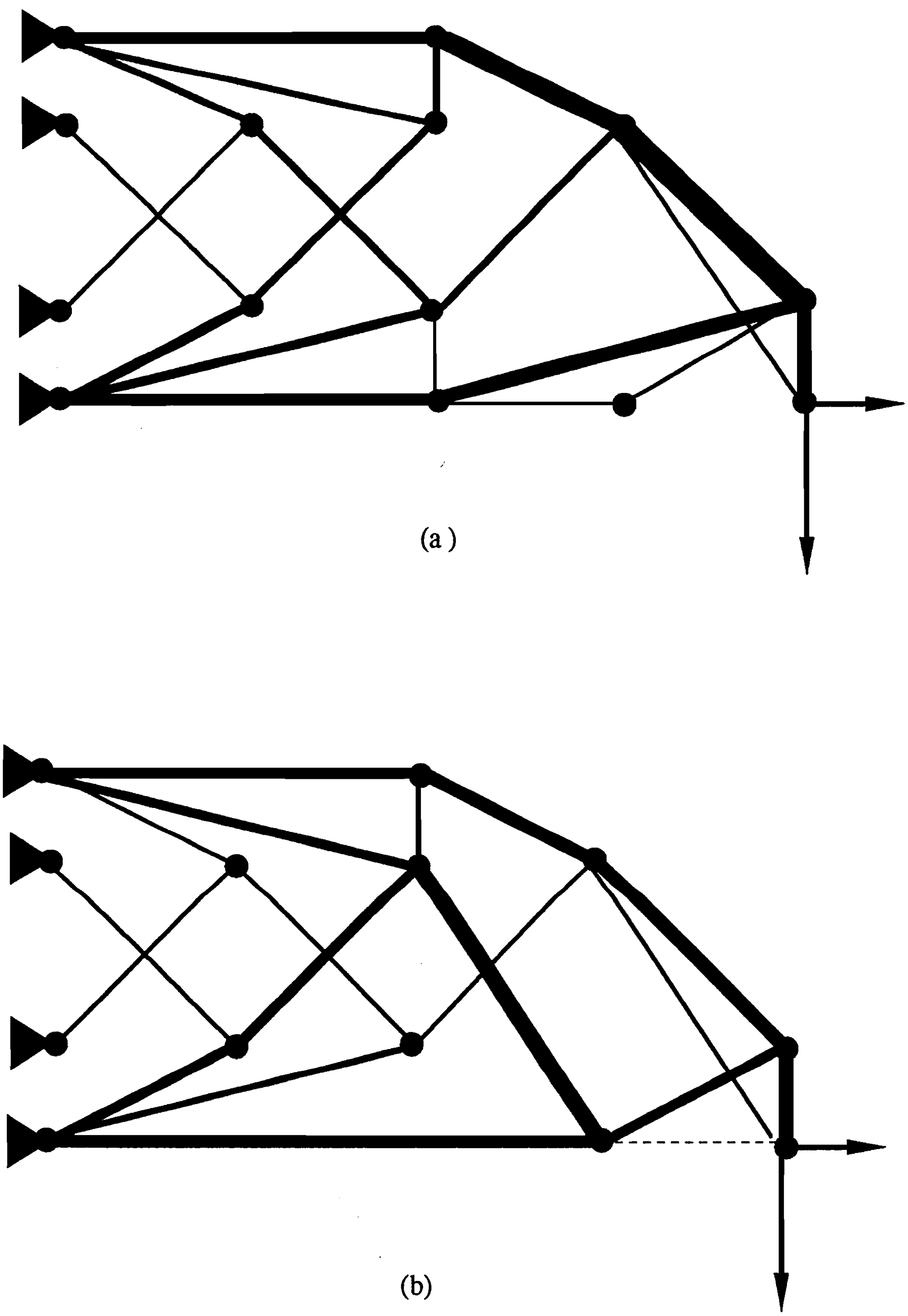


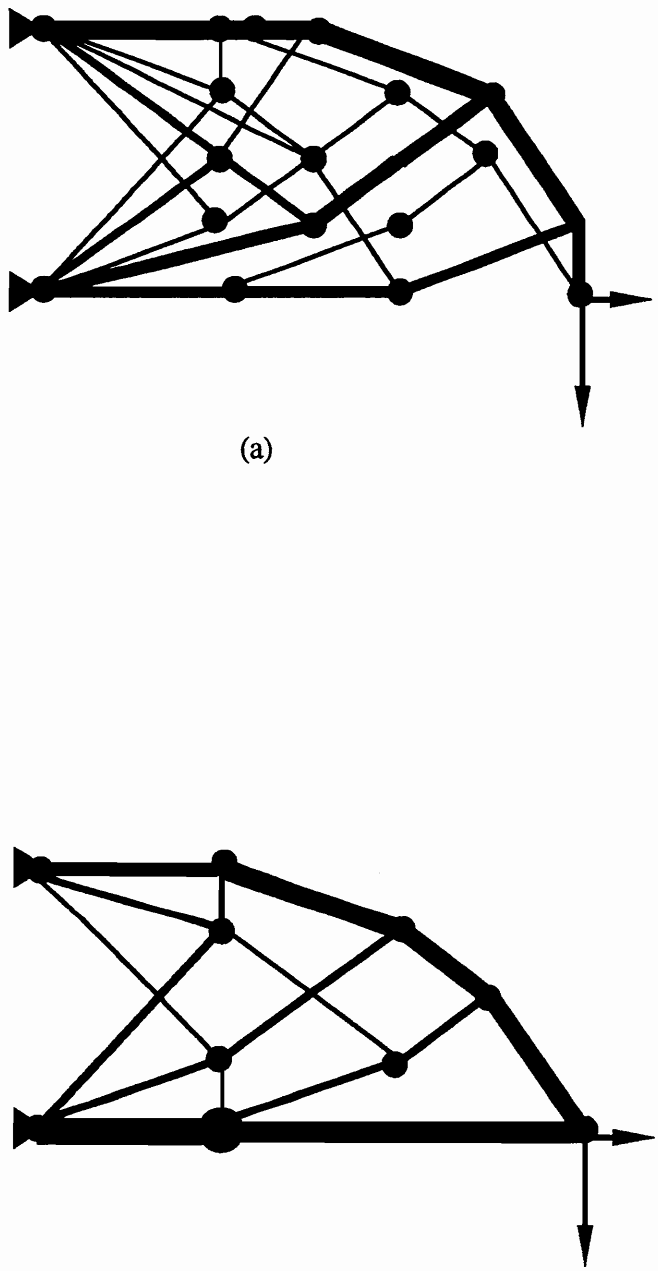
Fig 3.9. Optimal trusses obtained for the 4 X 3 grid with the 25 % horizontal displacement constraint at the load with **(a)** SAND **(b)** Sizing optimization of optimum minimum compliance topology



(a)

(b)

Fig 3.10. Optimal trusses obtained for the 5 X 5 grid with the 25% horizontal displacement constraint at the load with **(a)** SAND **(b)** Sizing optimization of optimum minimum compliance topology



(a)

(b)

Fig 3.11. Optimal trusses obtained for the 7 X 5 grid with the 25% horizontal displacement constraint at the load with **(a)** SAND **(b)** Sizing optimization of optimum minimum compliance topology

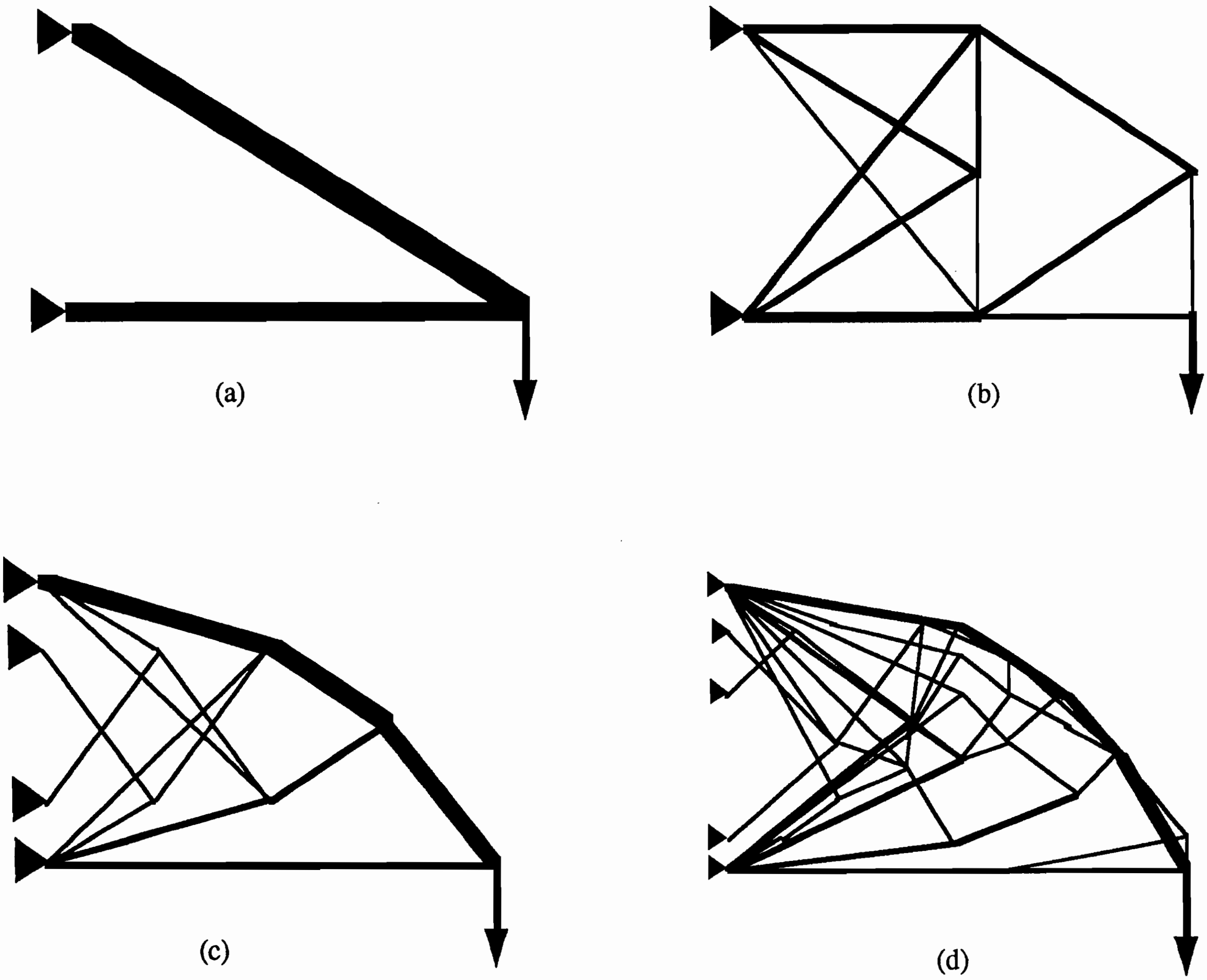


Fig . 3.12 Optimal topologies obtained with all methods for **(a)** 2 X 2 grid **(b)** 3 X 3 grid **(c)** 5 X 5 grid and **(d)** 9 X 9 grid with weight minimization only

4. 2-D TOPOLOGY OPTIMIZATION WITH SIMULTANEOUS ANALYSIS AND DESIGN

4.1 Application of SAND to plane stress problems

The SAND approach was shown to be an efficient method for the topology design of trusses in the last chapter. In chapter 1 the homogenization method [32] was introduced as a method of developing structural topologies from two dimensional domains. This approach is used for "generalized shape optimization", which encompasses both traditional shape optimization and topology optimization. The homogenization approach considers the density of the elements as design variables and the final topology is a variable density design. The topology optimization problem is solved as a material distribution problem.

Generalized shape optimization has also been accomplished by the approach of Rozvany and Zhou [27]. By applying the continuum based optimality criteria (COC) approach, shape and topology optimization has been performed starting from 2-D rectangular domains to obtain topologies that are interpreted to resemble trusses. Problems involving large number of elements have been efficiently solved by this method. The method uses the elemental thicknesses as design variables.

One of the objectives of this dissertation is to test the ability of the SAND approach to solve topology optimization of plane stress problems. The computational efficiency of the SAND algorithm to design topologies for plane stress problems is the key question in terms of the feasibility of using it. The CPU time using the SAND approach varies nonlinearly with the number of design variables. In the plane stress formulation a reasonable mesh refinement involves a large number of design variables. This is because of the fact that SAND uses the nodal displacements as design variables and refining the finite element model increases the number of displacement design variables. Hence, the issue of computational efficiency in using the SAND approach to design topologies for plane stress problems is very important.

The ground structure approach is used. We start from a plane stress domain and use the elemental thicknesses as design variables. The topology optimization problem is solved as a sizing (thickness)optimization problem by using the augmented Lagrangian formulation. The optimizer will reduce some elemental thicknesses to zero, vary the rest and create a variable thickness plate as the final topology. The variable thickness plate that is obtained as the final design can be easily translated into a practical physically realizable structure. The variable *density plate* that comes out of the homogenization method requires special skills of interpretation to be interpreted as a physical structure.

4.2 Results and discussion

For the examples considered for plane stress topology optimization, the design domain used is the same as for truss topology optimization. This means that the design domain consists of a rectangular plate of dimensions 720 inches by 360 inches discretized using constant stress 3 noded triangular elements. The material of the plate has a Poisson's ratio of 0.3 and Young's modulus of 10^4 ksi and a density of 0.1 lb/in^3. The ground structure for a 20 $\times$ 10 mesh (20 nodes along the x direction and 10 nodes along the y direction) is shown in figure 4.1. A vertical load of 100 kips was applied at the lower right hand corner (see fig 4.1) of the plate. The left edge of the plate is clamped.

The plane stress topology optimization problem that is considered in this dissertation is to design a plate of minimum weight to transmit the applied load to the clamped support at the left subject to a constraint on the vertical displacement at the load application point. This displacement is constrained to be less than or equal to 6.527 $\times$ 10^{-4} inches which is 50 % of the displacement at that node resulting from an analysis of the initial design domain with a constant thickness of 1.0 inch. This displacement constraint is equivalent to a compliance constraint, since it is applied to the displacement component collinear with the applied vertical load. However, the SAND approach can deal with more general displacement constraints.

The first case was a 3×3 mesh consisting of 18 elements. The plate thickness was chosen to be 1.0 inches in order not to violate the displacement constraint. The initial design which was a plate of constant thickness weighed $720 \times 360 \times 1 \times 0.1 = 25920$ lbs. The optimal topology for this case is shown in Fig. 4.2. It can be seen that most of the elements in the initial design are retained in the final topology also. The final design weighed about 2500 lbs. Similar to the truss topology problems, the nondimensional compliance η is defined by,

$$\eta = (\mathbf{f}^T\mathbf{u})VE/(\|\mathbf{f}\|^2L^2), \tag{4.1}$$

where V is the volume, E the elastic modulus and L the horizontal length of the plate.

The discretization of the plate was refined further to a 5×5, 7×7, 10×5, 10×10, and 20×20 grids successively. The results are summarized in Table 4.1.

Column 1 of table 4.1 shows the problem size. Column 2 shows the number of elements in the initial design. Column 3 shows the weight of the optimum topology. Column 4 shows the nondimensional compliance values for these meshes. The weight and nondimensional compliance increase with the number of elements but they seem to be converge. This trend is due to the fact that when a small number of elements are used in the finite element model, it is too stiff. That is the compliance predicted by the finite element model is too low. When the mesh is refined this problem is resolved and we converge towards the optimum compliance for the design domain considered. The convergence of the nondimensional compliance are to be compared to the trend

Table 4.1. Effect of mesh refinement on the optimal topology and CPU time

Problem size	Number of elements in ground structure	Weight of optimum topology (lbs)	Nondimensional compliance η	CPU time * (seconds)
3 X 3	18	2499	6.78	14
5 X 5	32	2801	7.60	69
7 X 7	72	3192	8.66	1017
10 X 5	72	3232	8.77	560
10 X 10	162	3675	9.97	2270
20 X 10	342	3852	10.45	8300

* IBM 3090

shown in table 3.1. In Table 3.1 the non dimensional compliance value decreases as the number of elements in the initial design increases. Comparing Table 3.1 with 4.1 it appears that the nondimensional compliances converge to the same value in both cases. Column 4 shows the CPU time in an IBM 3090 computer taken by the SAND approach using an augmented Lagrangian solution process. The CPU time varies approximately as the number of elements in the initial design to the power of 2.16. For truss topology optimization problems table 3.1 shows that the CPU time varies approximately as number of truss elements in the ground structure to the power of 1.72.

The optimal topologies are shown in figure 4.2 for the 3 $\times$ 3 mesh, in figure 4.3 for the 5 $\times$ 5 mesh, in figure 4.4 for the 7 $\times$ 7 mesh, in figure 4.5 for the 10 $\times$ 5 mesh, in figure 4.6 for the 10 $\times$ 10 mesh, and in figure 4.7 for the 20 $\times$ 10 mesh. As the mesh is refined the number of elements with thicknesses smaller than 10^{-2} inches in the optimal topology increases. However it is seen that all the optimum topologies display alternating thick and thin elements (characterized by dark and light shades in the figures) in a so called "chess board " pattern. This is more evident in the cases where the domain discretization is not very fine. This may be attributed to the fact that the design domain has been discretized by using the most basic triangular elements. By using higher order elements this problem may be overcome. As the mesh is refined we also see that more elements vanish close to the support on the left.

In chapter 1 and chapter 2 it was noted that the SAND approach was particularly sensitive to the tuning parameters (the coefficients multiplying the penalty terms). For the topology optimization of plane stress problems the experience was that the SAND approach was particularly sensitive to the coefficient c (see chapter 3) multiplying the penalty due to the equilibrium equations. If this coefficient was chosen too small then the equilibrium equations were violated in the earlier optimization iterations in order to satisfy the displacement constraint easily. In later iterations when $\frac{c}{\sqrt{r}}$ became large the optimizer tried to satisfy the equilibrium equations but found it hard to do. A lot of computer time is spent to find the best value for the displacements $\mathbf{u}$ to satisfy the equilibrium equations. To get a good estimate of the initial value for the coefficient c, the 3×3, 5×5 and 7×7 problems were run by a different optimization package and the actual number of active constraints at the optimum were found. The coefficient c was computed from this estimate for these problems. From this information, the value of c was extrapolated for larger problems that are reported in the table 4.1. This choice of c prevented the violation of the equilibrium equations and helped reduce the computation time.

The SAND algorithm is also sensitive to the rate of reduction of the penalty parameter r (see chapter 3). Decreasing r very fast will result in ill-conditioning due to increasing $\frac{c}{\sqrt{r}}$. Reducing r slowly will result in slow convergence. For the plane stress topology problems it was found that reducing r by a factor of 10 for successive optimization iterations resulted in the best convergence.

Reference [62] mentions that for augmented Lagrangian methods it may be necessary to choose the initial value of the Lagrange multipliers λ, in a close neighborhood of their optimal value. Since the optimal values were not known, certain non zero values were tried as the initial values of λ but in general the performance of the SAND algorithm worsened with these non zero values.

From column 5 of table 4.1 we see that while both the 7×7 and 10×5 grids have 72 elements in the initial designs, the former took almost 50 % more time to arrive at the same optimal compliance. This may be due to the fact that the elements in the 10×5 grid have a better aspect ratio of 1 while the elements in the 7×7 mesh have an aspect ratio of $\frac{1}{2}$ which is poor. Hence for finer discretizations it may be better to use elements with aspect ratio of 1.

The performance of the SAND algorithm to predict optimal topologies for plane stress problems is not very satisfactory at present. Certain modifications may improve the performance. They include:

a. Methods that utilize the sparsity of the matrices involved [56] can be used instead of the augmented Lagrangian formulation to reduce the CPU time.

b. Higher order elements can be used instead of the 3 noded triangular elements which may eliminate the "chess board"pattern seen in the optimal topologies here.

c. Instead of using a completely displacement based formulation like here, a mixed formulation (in terms of forces and stresses) may also help reduce the computational difficulties since we will not be using the stiffness matrix and the ill conditioning associated with it will no longer pose a problem.

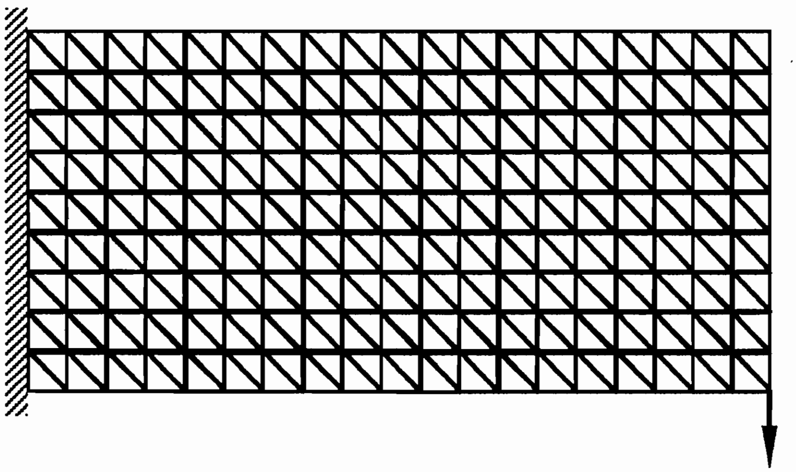


Figure 4.1 The ground structure for the 20 X 10 grid (342 elements)

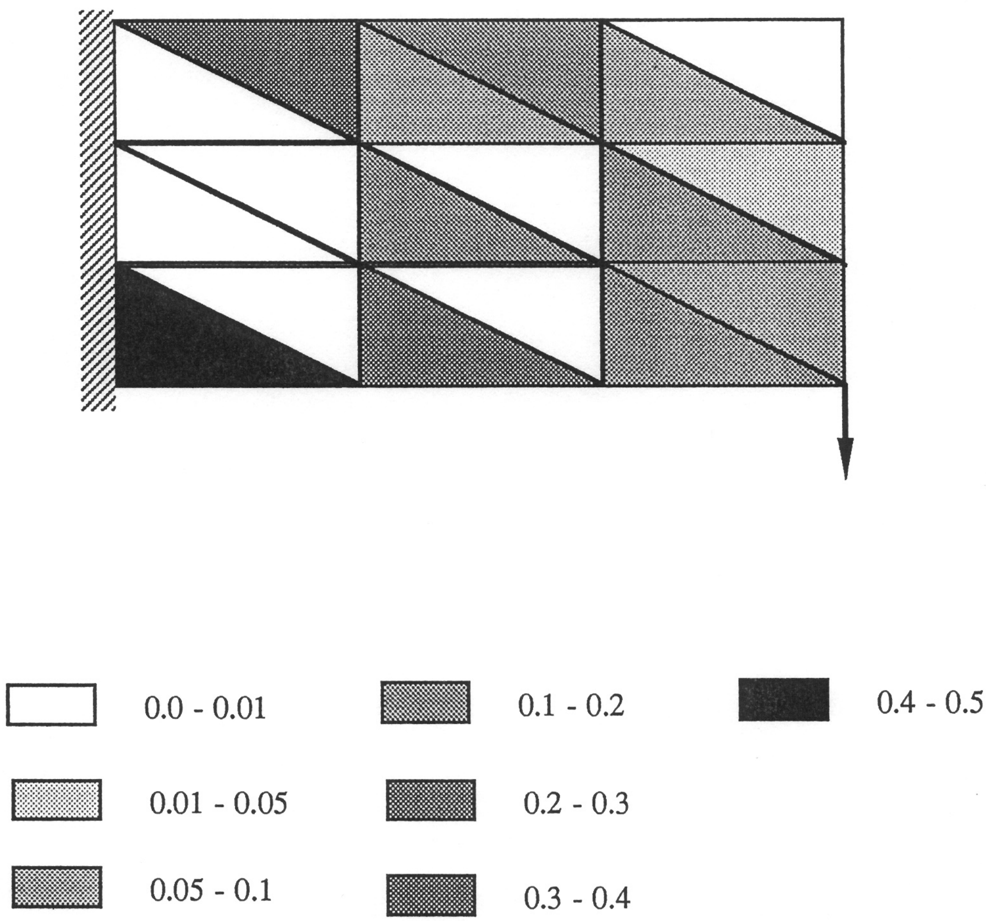


Figure 4.2 Thickness distribution in the 3 X 3 optimum topology

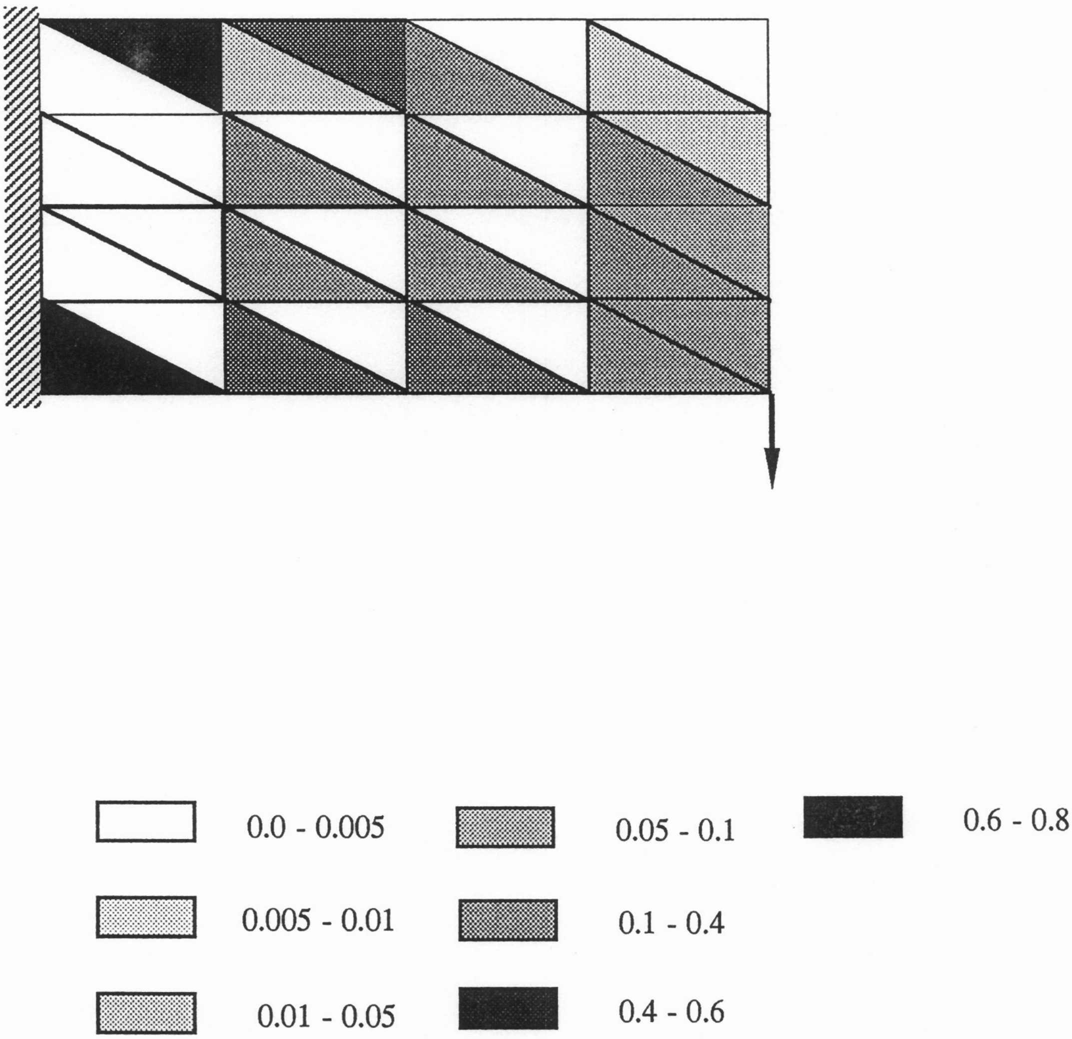


Figure 4.3 Thickness distribution in the 5 X 5 optimal topology

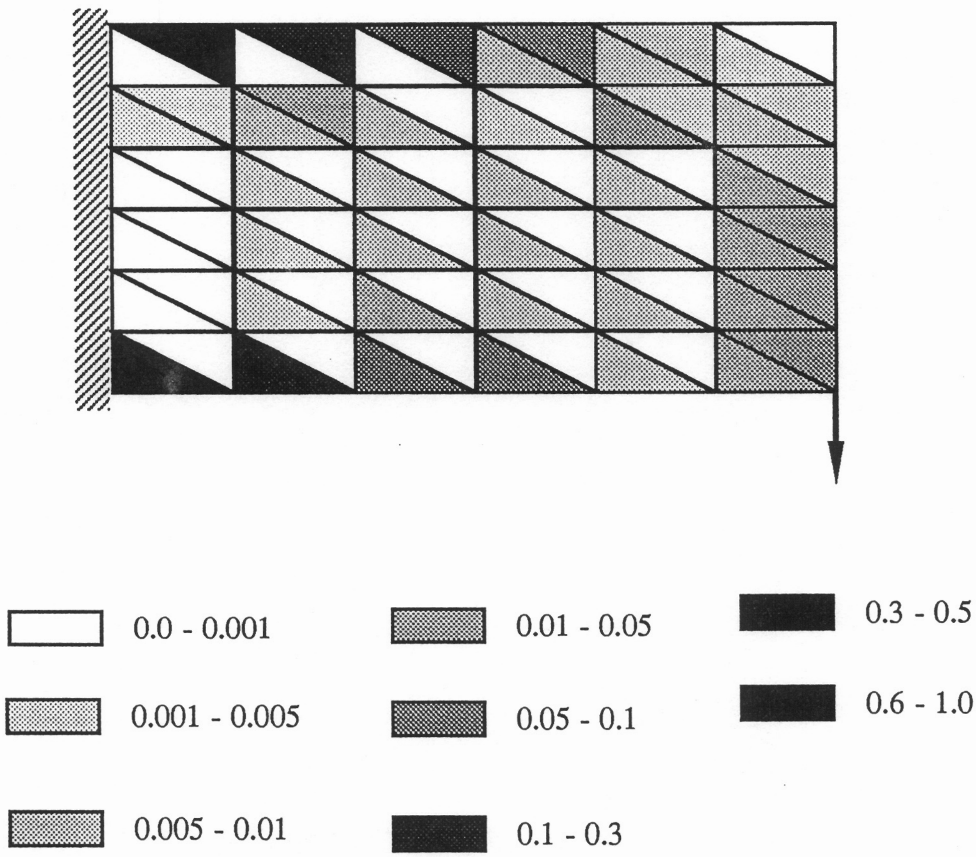


Figure 4.4 Thickness distribution in the 7 X 7 optimal topology

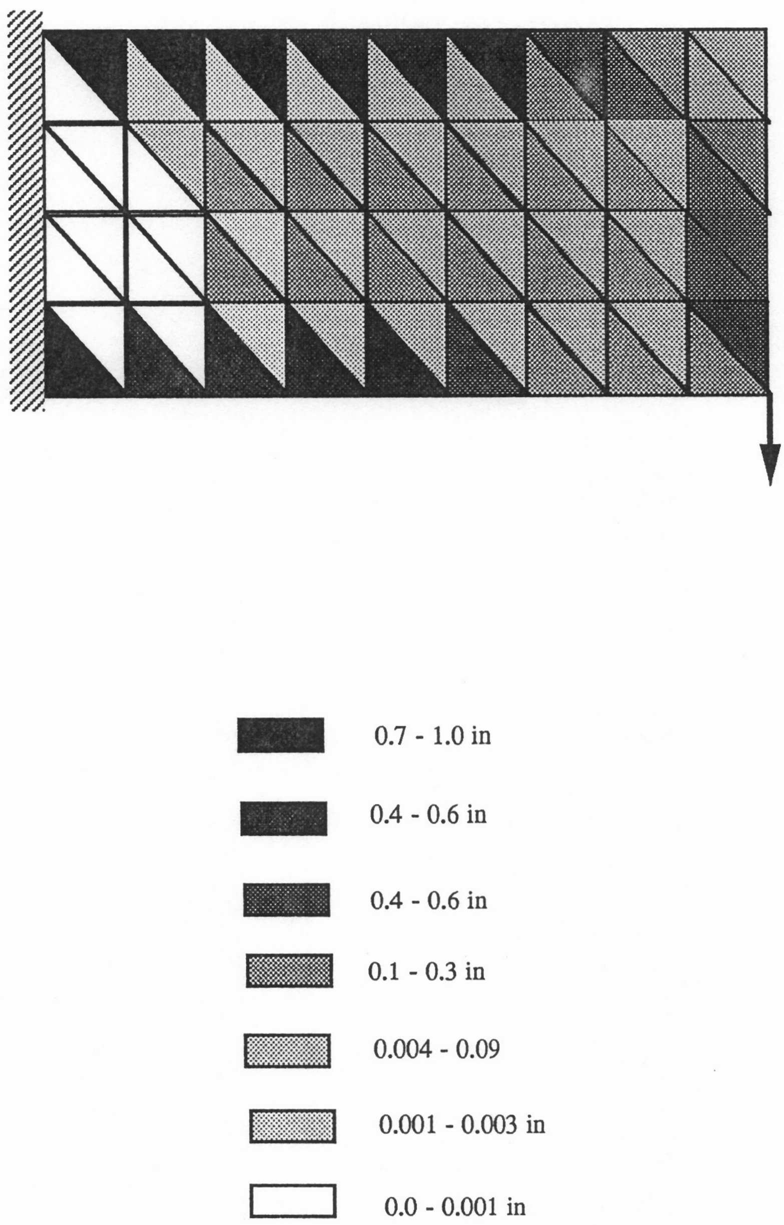


Figure 4.5 Thickness distribution in the 10 X 5 optimum topology

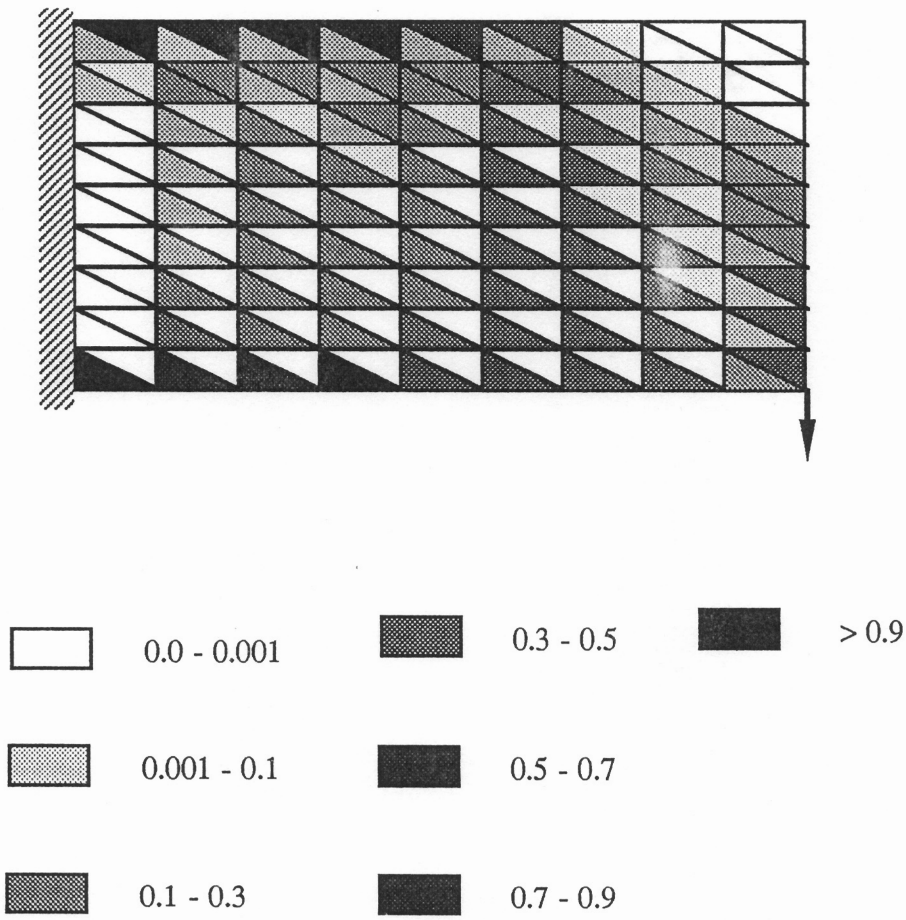


Figure 4.6 Thickness distriburion in the 10 X 10 optimum topology

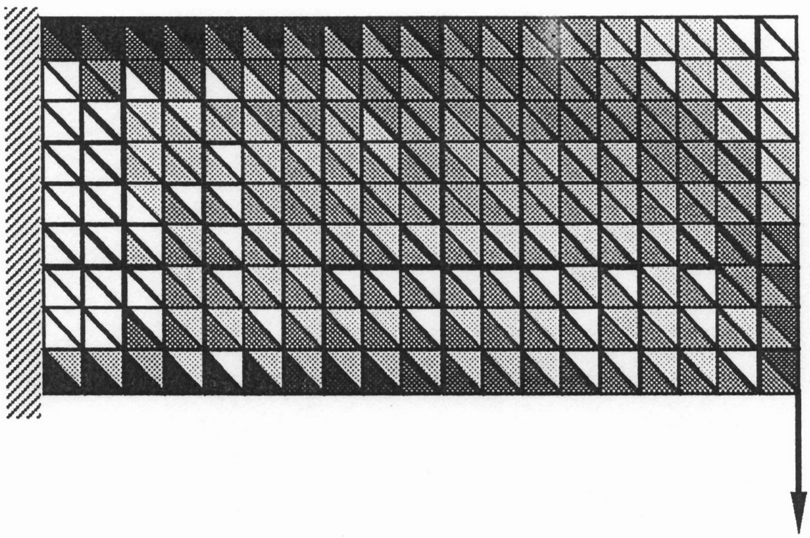


Figure 4.7 Thickness distribution in the 20 X 10 optimal topology

5. CONCLUSIONS

In this dissertation a mathematical programming approach called the simultaneous analysis and design was presented as a general method for topology optimization. The generality of this method was contrasted with the efficiency of the compliance minimization method. The compliance minimization method is computationally efficient in handling stress and displacement constraints in a two stage approach of obtaining an optimal compliance topology and then optimizing the cross sectional areas to minimize weight. However, It was shown to result in a weight penalty of up to 40% for one case of displacement and stress constraints.

The computational cost varies almost quadratically with the number of members in the ground structure for truss topology optimization problems. Two strategies for alleviating the computational cost of SAND approach were implemented. They are the use of an augmented Lagrangian algorithm and progressive elimination of members with small cross sectional areas. Together these strategies reduced computational cost by up to 60%, where bigger savings were obtained for larger problem sizes.

The SAND approach was also implemented for the topology optimization of plane stress problems. Since only low order elements were used in the finite element model, the optimal results displayed a so called chess board pattern. By using higher order elements this problem may be eliminated. The CPU time varied more than quadratically with the number of elements in the finite element model.

The SAND approach is a viable approach for topology optimization. Its generality and the capability to handle stress and displacement constraints is an advantage over methods like compliance minimization method and homogenization methods that are currently employed for topology optimization. Future research can concentrate on different formulations and different finite elements for the plane stress problem to make the method more efficient.

REFERENCES

[1] Haftka, R.T., and Grandhi, R. V.," Structural shape optimization-A survey", *Computer methods in applied mechanics and engineering*, **57**, 1986, pp. 91-106.

[2] Michell, A. G. M., " The limits of economy in frame structures ", *Philo. Mag. Sect 6*, **8**, 1904, pp. 589-597.

[3] Hemp, W. S., Studies in the theory of Michell structures, *Proc. 11th int. congress appl. mech*, Munich, 1964, pp. 621-628.

[4] Prager, W. and Shield, R. T., " Optimal design of multipurpose structures", *Int. Jou. of solids and structures* , **4**, 1968, pp. 469-475.

[5] Rozvany, G. I. N., and Adidam, S. R., "Rectangular grillages of least weight", *Journal of the engg. mech. division, ASCE*, **Vol.98**, No. EM6, 1972, pp. 1337-1352.

[6] Rozvany, G. I. N., " Optimal topography of force transmission by flexure", *Journal of applied mech.*, **40**, 4, 1973, pp. 983-987.

[7] Rozvany, G. I. N., " Structural design via optimality criteria", *Kluwer academic publishers*, Dordrecht, 1989.

[8] Rozvany, G. I. N., " Optimal design of flexural systems", *Pergamon press*, Oxford, 1976.

[9] Prager, W., and Rozvany, G. I. N., " Optimal layout of grillages", *Journal of structural mechanics*, **Vol. 5**, No. 1, 1977, pp. 1-18.

[10] Sved, G., " The minimum weight of certain redundant structures", *Austr. journal of appl. sci.*, **5**, 1954, pp. 1-8

[11] Barta, J., " The minimum weight of certain redundant structures", *Acta tech. acad. sci. Hung.*, **18**, 1/2, 1957, pp. 67-76

[12] Kirsch. U., "Optimal topologies of structures", *applied mechanics reviews*, **Vol. 42**, No. 8, 1989, pp. 223-239.

[13] Lev,. O. E., " Topology and optimality of certain trusses", *Journal of the structural division, ASCE*, **Vol. 107**, No. ST2, 1981, pp. 383-393

[14] Dorn, W., Gomory, R. and Greenberg, H., " Automatic Design of Optimal Structures", *J. de Mecanique*, **Vol. 3**, No. 1, 1964, pp. 25-52.

[15] Dobbs, M. and Felton, I., " Optimization of truss geometry ", *Journal of the structural division, ASCE*, **Vol. 95**, No. ST10, 1969, pp. 2105-2118.

[16] Sheu, C. and Schmit, L., " Minimum weight design of elastic redundant trusses under multiple static loading cases ", *AIAA journal*, **Vol. 10**, No. 2, 1972, pp. 155-162.

[17] Majid, K. I., and Elliot, D. W. C, " Topological design of pin jointed structures by nonlinear programming ", *Proceedings of the Instt. of civil eng.*, **55**, Part 2, 1973, pp. 129-149.

[18] Majid, K. I., and Saka, M. P., " Optimum design of rigidly jointed frames", *Proceedings of the symposium on applications of computer methods in engineering*, UCLA, 1977.

[19] Saka, M. P., " Shape optimization of trusses ", *Journal of the structural division, ASCE*, **Vol. 106**, No. ST5, 1981, pp. 2105-2118.

[20] Kirsch, U., " Optimal Topologies of Truss Structures", *Comp. Meth. Appl. Mech. Engrg.*, **Vol. 72**, No. 1, 1989, pp. 15-28.

[21] Ringertz, U. T., " On Topology Optimization of Trusses", *Engrg. Optimization.*, **Vol. 9**, No. 3, 1985, pp. 209-217.

[22] Ringertz, U. T., "A branch and bound algorithm for Topology Optimization of Truss structures", *Engrg. Optimization.*, **Vol. 10**, No. 3, 1986, pp. 111-124.

[23] Barnes, M. R., Topping, B. H. V. and Wakefield, D. S., " Aspects of form-finding by dynamic relaxation", *International conference on the behavior of slender structures*, London, 1977.

[24] Topping, B. H. V., " Shape optimization of skeletal structures : A review", *Journal of the structural division, ASCE*, **Vol. 109**, No. 8, 1983, pp. 1933-1951.

[25] Khot, N.S., Berke, L., " Structural optimization using optimality criteria methods. In: *new directions in structural design*, Atrek, E., Gallagher, R. H., Ragsdell, K. M., and Zienkiewicz, O. C., (editors), Chichester, Wiley, 1984, pp. 47-74.

[26] Rozvany, G. I. N and Zhou, M. ," Applications of the COC Algorithm in Layout Optimization", *Proc. Int. Conf. Engrg. Optimization in Design Processes*., Karlsruhe, 1990, Lecture Notes in Engrg. Springer Vetes.

[27] Rozvany, G. I. N and Zhou, M. ,"Layout and generalized shape optimization by iterative COC methods", *NATO/ DFG ASI Lecture Notes on Optimization of Large Structural Systems*, Berchtesgaden, G. I. N. Rozvany (editor), **Vol. 3**, 1991, pp. 81-95.

[28] Zhou, M. and Rozvany, G. I. N .,"The COC algorithm, part II; Topological, geometrical and generalized shape optimization", *Computer methods in applied mechanics and engineering*, **Vol. 89**, 1991, No. 3, pp. 309-336.

[29] Ben-Tal, A. and Bendsøe, M. P., " Truss Topology Optimization by a Displacements Based Optimality Criterion Approach", *NATO/ DFG ASI Lecture Notes on Optimization of Large Structural Systems*, Berchtesgaden, G. I. N. Rozvany (editor), **Vol. 1**, 1991, pp. 67-81.

[30] Taylor, J.E., and Rossow, M. P., " Optimal truss design based on an algorithm using optimality criteria", *Int. Jour. solids and str.*, **13**, 1977, pp. 913-923.

[31] Bendsøe, M. P. " Optimal shape design as a material distribution problem", *Structural optimization.*, 1, 1989, pp. 193-202.

[32] Bendsøe, M. P. and Kikuchi, N. " Generating Optimal Topologies in Structural Design using a Homogenization Method", *Comp. Meth. Appl. Mech. Engrg.*, **Vol. 71**, No. 2, 1988, pp. 197-224.

[33] Suzuki, K., and Kikuchi, N., " A homogenization method for shape and topology optimization", *Computer methods in applied mechanics and engineering*, **93**, 1991, pp. 291-318.

[34] Suzuki, K., and Kikuchi, N., "Layout optimization using the homogenization method: generalized layout design of three-dimensional shells for car bodies", *NATO/ DFG ASI Lecture Notes on Optimization of Large Structural Systems*, Berchtesgaden, G. I. N. Rozvany (editor), **Vol. 3**, 1991, pp. 110-126.

[35] Diaz, A. and Bendsøe, M. P., " Shape optimization of structures for multiple loading conditions using a homogenization method", *Structural optimization*, **Vol. 4**, No.1, 1992, pp. 17-22.

[36] Diaz, A. and Belding, B., " On optimum truss lay-out by a homogenization method", *ASME transactions of mechanical design* (to appear).

[37] Diaz, A. and Kikuchi, N., " Solutions to shape and topology eigenvalue optimization problems using a homogenization method", *Int. J. Num. Meth. Engg.* (to appear).

[38] Francfort, G., and Murat, F., " Homogenization and optimal bounds in linear elasticity", *Arch. Rat. mech. Anal*, **94**, 1986, pp. 307-334.

[39] Guedes, J. M., and Kikuchi, N., " Pre and post processing for materials based on the homogenization method with adaptive finite element methods", *Comput. Meth. Appl. Mech. Engrg*, **57**, 1986, pp. 143-198.

[40] Kikuchi, N., and Suzuki, K., " Structural optimization of a linearly elastic structure using the homogenization method", in: *Composite media and homogenization theory*, G. Dal Maso and C. F. Dell Antonio, (editors), Birkhauser, Boston, 1991, pp. 183-204.

[41] Jog, C. S., Haber, R. B., and Bendsøe, M. P.," Topology design using a material with self optimizing microstructure", *Preprint*, Dept. of theoretical and applied mech. University of Illinois at Urbana-Champaign , 1992.

[42] Bendsøe, M. P., Ben-Tal, A., and Haftka, R. T., " New Displacement-Based Methods For Optimal Truss Topology Design", *Proceedings of the AIAA/ ASME/ ASCE/ AHS/ ASC 32nd Structures, Structural Dynamics, and Materials Conference*, Baltimore, **Part 1**, 1991, pp. 684-696.

[43] Plaut, R.H., Johnson, L. W., and Olhoff, N., " Bimodal optimization of compressed columns of elastic foundations", *Journal of applied mechanics*, **Vol. 53**, 1986, pp. 130-134.

[44] Rizk, M. H., " The single cycle scheme - A new approach to numerical optimization", *AIAA journal*, **Vol. 21**, December 1983, pp.1640-1647.

[45] Barthelemy, B., Haftka, R. T., Madapur, U., and Sankaranarayanan, S., " Integrated analysis and design using 3-D finite elements", *AIAA journal*, **29**, No. 5, 1991, pp. 791-797.

[46] Schmit, L. A., and Fox, R. L.,"An integrated approach to structural synthesis and analysis", *AIAA Journal*, **Vol. 3**, June 1965, pp. 1104-1112.

[47] Fox, R. L. and Schmit, L. A., 'Advances in the integrated approach to structural design", *Journal of spacecraft and rockets*, 3, June 1966, pp. 858-866.

[48] Muller, A., " Element-by-element iterative procedures in structural analysis", *Ph. D dissertation*, Stanford university, 1985, pp. 39-48.

[49] Haftka, R. T. " Simultaneous Analysis and Design ", *AIAA Journal*, **Vol. 23**, No. 7, 1985, pp. 1099-1103.

[50] Hughes, T. J. R., Winget, J., Levit, I., and Tezduyar, T. E., "New alternating direction procedures in finite element analysis based on EBE approximate factorization", *Computer methods for nonlinear solids and structural mechanics*, edited by S. Atluri and N. Perrone, AMD, **Vol. 54**, 1983, pp. 75-109.

[51] Haftka, R. T., and Kamat, M. P., " Simultaneous nonlinear structural analysis and design", *computational mechanics*, **4**, pp. 409-416.

[52] Shin, Y. S., Haftka, R. T., and Plaut, R. H., " Simultaneous analysis and design for eigenvalue maximization", *AIAA journal*, **Vol 26**, No. 6, 1987, pp. 738-744.

[53] Chibani, L., " Optimal design of structures", *Springer Velag*, Berlin, Heidelberg, 1989.

[54] Ringertz, U. T., " Optimization of structures with nonlinear response", *Engg. optim*, **Vol. 14**, 1989, pp. 179-188.

[55] Smaoui, H., and Schmit, L. A., " An integrated approach to the synthesis of non-linear structures.", *Int. Jour. for numer. meth. in engg.*, **Vol 26**, 1988, pp. 555-570.

[56] Orozco, C. E., and Ghattas, O., " Sparse approach to simultaneous analysis and design of geometrically nonlinear structures", *AIAA journal*, **Vol 30**, No. 7, 1992, pp. 1877-1885.

[57] Haftka, R.T., Gürdal, Z., "Elements of structural optimization", Third revised edition, *Kluwer academic publishers* , Dordrecht, 1991.

[58] Haftka, R. T. and Starnes, J. H. Jr., " Applications of a quadratic extended interior penalty function for structural optimization", *AIAA Journal*, **Vol. 14**, 1976, pp. 718-724.

[59] Haftka, R. T., "Damage tolerant design using collapse techniques", *AIAA Journal*, **Vol. 21**, No. 10, 1983, pp. 1462-1466.

[60] Hestenes, M. R., " Multiplier and gradient methods ", *Journal of optimization theory and applications*, **4**, 1969, pp. 303-320.

[61] Gill, P. E., Murray, W and Wright, M. H.," Practical Optimization", *Academic Press*, New York, 1981, pp. 225-232

[62] Bertsekas, D. P., ,"Multiplier methods : A survey", *Automatica*, **Vol. 12**, 1976, pp. 133-145.

[63] Tapia, R. A., "Diagonalized multiplier methods and quasi-Newton methods for constrained minimization", *Optim. theory and appl.*, **22**, 1977, pp. 135-194.

[64] Gill, P. E. and Murray, W., " Conjugate-Gradient Methods for Large Scale Non-linear Optimization", *Technical Report 79-15*, Systems Optimization Lab., Dept. of Operations Res., Stanford Univ., 1979, pp. 10-12.

[65] Powell, M. J. D., " Restart procedures for the conjugate gradient methods", *Mathematical Prog.*, **Vol. 11**, 1976, pp.42-49.

[66] Pederson, P., " On the minimum mass layout of trusses", *AGARD conf. proc. No. 36 on symposium on structural optimization*, 1970.

[67] Pederson, P., " On the optimal layout of multi-purpose trusses", *computers and structures*, **Vol. 2**, 1972, pp. 695-712.

[68] Rozvany, G. I. N., and Gollub, W., " Michell layouts for various combinations of line supports, Part I ", *Int. Journal of mech. sci*, **(to appear)**

[69] Rozvany, G. I. N., Gollub, W., and Zhou, M., " Michell layouts for various combinations of line supports, Part II ", *Int. Journal of mech. sci*, **(to appear)**

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