Some Formation Problems for Linear Elastic Materials

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ABSTRACT

Some equations of linear elasticity are developed, including those specific to certain actuator structures considered in formation theory. The invariance of the strain-energy under the transformation from rectangular to spherical coordinates is then established for use in two specific formation problems. The first problem, involving an elastic structure with a cylindrical equilibrium configuration, is formulated in two dimensions using polar coordinates. It is shown that L^2 controls suffice to obtain boundary displacements in $H^{1/2}$. The second problem has a spherical equilibrium configuration and utilizes the elastic equations in spherical coordinates. Results similar to those obtained in the two dimensional case are indicated for the three dimensional problem.

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Chapter 1

Introduction

In this paper we look at two problems in linear formation theory. Formation theory is an area of control theory concerned with, among other things, the shape control of linear elastic materials. We begin in Chapter 2 with a brief introduction to linear elasticity in rectangular coordinates followed in Chapter 3 by an introduction to those aspects of formation theory relevant to the current discussion. Since the formation problems we consider in Chapters 5 and 7 have cylindrical and spherical equilibrium configurations, we examine in Chapter 4 the coordinate transformation of the elastic equations obtained in Chapters 2 and 3. Chapter 6 contains some graphical results based on the work in Chapter 5 and Chapter 8 contains some suggestions for future study in this area.

Chapter 2

Introduction to Linear Elasticity

2.1 General Comments and Notation

Many scientists have made significant contributions to the development of the theory of elasticity. What follows is a brief look at some events that provided the way for future research and that are relevant to understanding some fundamental ideas in the theory of elasticity. For a more comprehensive view the interested reader is referred to Love [7], Todhunter and Pearson [14], and Timoshenko [13], from which most of what follows is obtained. We will then discuss terminology and notation.

2.1.1 Some Background

Elasticity is a mathematical theory with empirical roots going back to Galileo Galilei and Robert Hooke in the 1600's. Galileo investigated the resistance of horizontal and vertical suspended beams to failure under their own weight and applied loads in his *Discorsi e Dimostrazioni matematiche* of 1638 [7]. Through experimentation in the mid to late 17th century, Robert Hooke in England and E. Maroitte in France independently discovered the proportionality of what are now referred to as stress and strain (more on stress and strain later). Then, in the 18th and 19th centuries, the mathematical theory of elasticity became the subject of extensive study by many world renown scientists.

Claude Navier is credited with first discovering the general equations of elasticity in 1821. Following Isaac Newton, Navier postulated a "structure" theory of elastic bodies in which "molecules" or more simply "material points" are presumed to interact in specified ways, being thus responsible for observable elastic phenomena. Navier's assumption of material points as centers of force, and similar assumptions of Siméon Poisson and of Augustin-Louis Cauchy in his later work on crystalline bodies, leads to elastic equations involving just one constant in the case of *isotropic* bodies. *Isotropic* elastic materials are presumed to exhibit the same stress-strain relationship regardless of direction within the body.

The equations found by Navier can be included as a special case of the equations of elasticity developed by Cauchy in 1822 and later derived by George Green and independently by George Gabriel Stokes using the principle of conservation of energy. Here no assumptions are made concerning interactions of material points and the elastic equations so derived contain two constants in the isotropic case. Stokes attributed the distinct constants to the tendency of an elastic body to resist compression (changes in volume *only*) and shear (changes in shape *only*). These constants are referred to as the *bulk modulus* (or the *modulus of compression*), denoted K, and the *shear modulus* (or the *rigidity*), denoted G, respectively. Thomas Young, in 1807, first commented on this property of elastic bodies to differentially resist expansion (or contraction) and shear and was the first to define a material's *modulus of elasticity* [7]. This modulus, now known as *Young's modulus*, denoted E, is the ratio of one-dimensional extensional stress to extensional strain.

2.1.2 Terminology and Notation

When subject to forces or loads nearly all commonly encountered material bodies tend to deform or experience changes in size and shape. The term *elasticity* refers to the tendency of certain types of material bodies to return to their original equilibrium configuration upon release of applied forces. The *stresses* in the body are a function of the loading and the corresponding changes in size and shape are elucidated in terms of relative displacements or *strains* (more precise notions of stress and strain are provided later). In *linear* elasticity we assume that components of stress are linear functions of the components of strain. This assumption, known as *Hooke's Law*, governs the standard mass-spring system in which the force of the spring is proportional to displacement of the mass. To ensure that we remain within the realm of linear elasticity, we assume that the strains are "small" in the sense that two points which are "close" at equilibrium remain "close" under load. We also require that the body be *perfectly* elastic in the sense that all (nonequilibrium) strains disappear upon removal of applied loads. In addition, the deformation must occur adiabatically, "that is to say in such a way that no heat is gained or lost by any element of the body." [7]

The natural complexities of the subject are compounded by the use of different terms for the same phenomena, the use of similar or the same terms to refer to distinct phenomena, and the use of many different notations, in particular, for the components of stress and strain (see [7], pp. 614-616). We have endeavored to conform with current notation and conventions for mathematics and elasticity. In cases of conflicting usage our choices will be clearly stated. We use σ to denote stress, ε to denote strains, and subscripts are used as descriptors. For example, u_x will be used to denote the x-component of the displacement vector, while $\partial u/\partial x$ will denote the partial derivative of u with respect to x.

2.2 Strain and the Strain-Displacement Equations

In this section and the following we will examine what happens within an elastic body subject to an arbitrary applied load and develop the notions of stress and strain in rectangular coordinates. It is instructive and useful to carry out the following calculations with respect to arbitrary orthogonal curvilinear coordinates, however this level of generality, combined with the natural complexities associated with modeling elastic solids, will unnecessarily muddy the waters. The results of such a course, for example from [7], pp. 51-58, will be used when cylindrical and spherical geometries are considered in Chapters 5 and 7, respectively. We will now develop the notion of strain as a measure of *relative* displacements within an elastic body, the relative movement of material points being the source of elastic potential energy.

We consider a three dimensional elastic solid $\mathcal{S}$ which initially occupies the region $\mathcal{R} \subseteq \mathbb{R}^3$. At equilibrium each coordinate point $P(x, y, z) \in \mathcal{R}$ corresponds to a material point $\mathcal{M} \in \mathcal{S}$. Under an arbitrary load our solid $\mathcal{S}$ is deformed and now occupies a new region $\tilde{\mathcal{R}}$. We may view this deformation as a smooth vector field, $\mathbf{D} : \mathcal{R} \mapsto \tilde{\mathcal{R}}$. Under the action of $\mathbf{D}$ the material point $\mathcal{M}$ is relocated to coordinate point $\tilde{P}(\tilde{x}, \tilde{y}, \tilde{z}) \in \tilde{\mathcal{R}}$. Assuming that $\mathbf{D}$ has rectangular components D_x, D_y, D_z, i.e.

$$\begin{aligned}\tilde{x} &= D_x(x, y, z)\\ \tilde{y} &= D_y(x, y, z)\\ \tilde{z} &= D_z(x, y, z),\end{aligned}$$

then the movement of material point $\mathcal{M}$ can be described by the displacement vector field, $\mathbf{u}(x, y, z)$,

$$\begin{aligned}\mathbf{u}(x, y, z) &= \Big(D_x(x, y, z) - x\Big)\mathbf{i} + \Big(D_y(x, y, z) - y\Big)\mathbf{j} + \Big(D_z(x, y, z) - z\Big)\mathbf{k}\\ &\equiv u_x(x, y, z)\mathbf{i} + u_y(x, y, z)\mathbf{j} + u_z(x, y, z)\mathbf{k},\end{aligned} \tag{2.1}$$

here $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are the standard Cartesian unit vectors and u_x, u_y, u_z are the x, y, z components, respectively, of the displacement vector $\mathbf{u}$.

We now consider the movement of two particular material points with coordinates $\bar{P}(\bar{x}, \bar{y}, \bar{z})$, $\bar{Q}(\bar{x} + \delta\bar{x}, \bar{y} + \delta\bar{y}, \bar{z} + \delta\bar{z}) \in \mathcal{R}$, respectively. Under loading $\bar{P} \mapsto \tilde{P}(\tilde{x}, \tilde{y}, \tilde{z})$ and $\bar{Q} \mapsto \tilde{Q}(\tilde{x} + \delta\tilde{x}, \tilde{y} + \delta\tilde{y}, \tilde{z} + \delta\tilde{z})$, where $\tilde{x} = D_x(\bar{x}, \bar{y}, \bar{z})$, $\tilde{y} = D_y(\bar{x}, \bar{y}, \bar{z})$, $\tilde{z} = D_z(\bar{x}, \bar{y}, \bar{z})$ and

$$\begin{aligned}
\tilde{x} + \delta\tilde{x} &= D_x(\bar{x} + \delta\bar{x}, \bar{y} + \delta\bar{y}, \bar{z} + \delta\bar{z}) && (2.2)\\
\tilde{y} + \delta\tilde{y} &= D_y(\bar{x} + \delta\bar{x}, \bar{y} + \delta\bar{y}, \bar{z} + \delta\bar{z}) && (2.3)\\
\tilde{z} + \delta\tilde{z} &= D_z(\bar{x} + \delta\bar{x}, \bar{y} + \delta\bar{y}, \bar{z} + \delta\bar{z}). && (2.4)
\end{aligned}$$

Considering (2.2) we write

$$\begin{aligned}
\tilde{x} + \delta\tilde{x} &= D_x(\bar{x} + \delta\bar{x}, \bar{y} + \delta\bar{y}, \bar{z} + \delta\bar{z})\\
&= D_x(\bar{x}, \bar{y}, \bar{z}) + \frac{\partial D_x}{\partial x}\delta\bar{x} + \frac{\partial D_x}{\partial y}\delta\bar{y} + \frac{\partial D_x}{\partial z}\delta\bar{z}\\
&\quad + \frac{1}{2}\left(\frac{\partial^2 D_x}{\partial x^2}\delta\bar{x}^2 + \frac{\partial^2 D_x}{\partial y \partial x}\delta\bar{x}\delta\bar{y} + \frac{\partial^2 D_x}{\partial z \partial x}\delta\bar{x}\delta\bar{z} + \ \ldots\ \right)\\
&\quad + \ \ldots\\
&\approx D_x(\bar{x}, \bar{y}, \bar{z}) + \frac{\partial D_x}{\partial x}\delta\bar{x} + \frac{\partial D_x}{\partial y}\delta\bar{y} + \frac{\partial D_x}{\partial z}\delta\bar{z}.
\end{aligned}$$

Therefore,

$$\delta\tilde{x} \approx \frac{\partial D_x}{\partial x}\delta\bar{x} + \frac{\partial D_x}{\partial y}\delta\bar{y} + \frac{\partial D_x}{\partial z}\delta\bar{z}, \tag{2.5}$$

as long as the quantities $|\delta\bar{x}|^2$, $|\delta\bar{x}\delta\bar{y}|$, $|\delta\bar{x}\delta\bar{z}|$, . . . are small compared to $|\delta\bar{x}|$, $|\delta\bar{y}|$, $|\delta\bar{z}|$. The same calculation using (2.3) and (2.4) provides the corresponding approximations for $\delta\tilde{y}$ and $\delta\tilde{z}$,

$$\begin{aligned}
\delta\tilde{y} &\approx \frac{\partial D_y}{\partial x}\delta\bar{x} + \frac{\partial D_y}{\partial y}\delta\bar{y} + \frac{\partial D_y}{\partial z}\delta\bar{z} && (2.6)\\
\delta\tilde{z} &\approx \frac{\partial D_z}{\partial x}\delta\bar{x} + \frac{\partial D_z}{\partial y}\delta\bar{y} + \frac{\partial D_z}{\partial z}\delta\bar{z}. && (2.7)
\end{aligned}$$

We use unadorned double vertical bars, $||\cdot||$, to denote the Euclidean distance between two points or the Euclidean length of a vector. For example, if $A(x_a, y_a, z_a)$, $B(x_b, y_b, z_b) \in \mathcal{R}$, then $||AB||$ denotes the distance between A and B and $||\vec{AB}||$ the length of vector $\vec{AB} \equiv (x_b - x_a)\mathbf{i} + (y_b - y_a)\mathbf{j} + (z_b - z_a)\mathbf{k}$,

$$||AB|| = ||\vec{AB}|| \equiv \sqrt{(x_b - x_a)^2 + (y_b - y_a)^2 + (z_b - z_a)^2}.$$

We now use the approximations (2.5)-(2.7) to calculate

$$
\begin{aligned}
&||\tilde{P}\tilde{Q}||^2 - ||\bar{P}\bar{Q}||^2 \\
&= \delta\tilde{x}^2 + \delta\tilde{y}^2 + \delta\tilde{z}^2 - \delta\bar{x}^2 - \delta\bar{y}^2 - \delta\bar{z}^2 \\
&\approx \left(\frac{\partial D_x}{\partial x}\delta\bar{x} + \frac{\partial D_x}{\partial y}\delta\bar{y} + \frac{\partial D_x}{\partial z}\delta\bar{z}\right)^2 + \left(\frac{\partial D_y}{\partial x}\delta\bar{x} + \frac{\partial D_y}{\partial y}\delta\bar{y} + \frac{\partial D_y}{\partial z}\delta\bar{z}\right)^2 \\
&\quad + \left(\frac{\partial D_z}{\partial x}\delta\bar{x} + \frac{\partial D_z}{\partial y}\delta\bar{y} + \frac{\partial D_z}{\partial z}\delta\bar{z}\right)^2 - \delta\bar{x}^2 - \delta\bar{y}^2 - \delta\bar{z}^2 \\
&= \left[{\frac{\partial D_x}{\partial x}}^2 + {\frac{\partial D_y}{\partial x}}^2 + {\frac{\partial D_z}{\partial x}}^2 - 1\right]\delta\bar{x}^2 + \left[{\frac{\partial D_x}{\partial y}}^2 + {\frac{\partial D_y}{\partial y}}^2 + {\frac{\partial D_z}{\partial y}}^2 - 1\right]\delta\bar{y}^2 \\
&\quad + \left[{\frac{\partial D_x}{\partial z}}^2 + {\frac{\partial D_y}{\partial z}}^2 + {\frac{\partial D_z}{\partial z}}^2 - 1\right]\delta\bar{z}^2 + 2\left[\frac{\partial D_x}{\partial x}\frac{\partial D_x}{\partial y} + \frac{\partial D_y}{\partial x}\frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial x}\frac{\partial D_z}{\partial y}\right]\delta\bar{x}\delta\bar{y} \\
&\quad + 2\left[\frac{\partial D_x}{\partial y}\frac{\partial D_x}{\partial z} + \frac{\partial D_y}{\partial y}\frac{\partial D_y}{\partial z} + \frac{\partial D_z}{\partial y}\frac{\partial D_z}{\partial z}\right]\delta\bar{y}\delta\bar{z} \\
&\quad + 2\left[\frac{\partial D_x}{\partial x}\frac{\partial D_x}{\partial z} + \frac{\partial D_y}{\partial x}\frac{\partial D_y}{\partial z} + \frac{\partial D_z}{\partial x}\frac{\partial D_z}{\partial z}\right]\delta\bar{z}\delta\bar{x}
\end{aligned}
$$

From (2.1) we have $D_x = u_x + x$, $D_y = u_y + y$, $D_z = u_z + z$ and so

$$
\begin{aligned}
&||\tilde{P}\tilde{Q}||^2 - ||\bar{P}\bar{Q}||^2 \\
&\approx \left[\left(\frac{\partial u_x}{\partial x} + 1\right)^2 + {\frac{\partial u_y}{\partial x}}^2 + {\frac{\partial u_z}{\partial x}}^2 - 1\right]\delta\bar{x}^2 + \left[{\frac{\partial u_x}{\partial y}}^2 + \left(\frac{\partial u_y}{\partial y} + 1\right)^2 + {\frac{\partial u_z}{\partial y}}^2 - 1\right]\delta\bar{y}^2 \\
&\quad + \left[{\frac{\partial u_x}{\partial z}}^2 + {\frac{\partial u_y}{\partial z}}^2 + \left(\frac{\partial u_z}{\partial z} + 1\right)^2 - 1\right]\delta\bar{z}^2 \\
&\quad + 2\left[\left(\frac{\partial u_x}{\partial x} + 1\right)\frac{\partial u_x}{\partial y} + \left(\frac{\partial u_y}{\partial y} + 1\right)\frac{\partial u_y}{\partial x} + \frac{\partial u_z}{\partial x}\frac{\partial u_z}{\partial y}\right]\delta\bar{x}\delta\bar{y} \\
&\quad + 2\left[\frac{\partial u_x}{\partial y}\frac{\partial u_x}{\partial z} + \left(\frac{\partial u_y}{\partial y} + 1\right)\frac{\partial u_y}{\partial z} + \left(\frac{\partial u_z}{\partial z} + 1\right)\frac{\partial u_z}{\partial y}\right]\delta\bar{y}\delta\bar{z} \\
&\quad + 2\left[\left(\frac{\partial u_x}{\partial x} + 1\right)\frac{\partial u_x}{\partial z} + \frac{\partial u_y}{\partial x}\frac{\partial u_y}{\partial z} + \left(\frac{\partial u_z}{\partial z} + 1\right)\frac{\partial D_z}{\partial x}\right]\delta\bar{z}\delta\bar{x}.
\end{aligned}
$$

Assuming that the first partial derivatives of u_x, u_y, and u_z are small enough that their squares and products may be disregarded we have

$$\begin{aligned} ||\tilde{P}\tilde{Q}||^2 - ||\bar{P}\bar{Q}||^2 \approx\ & 2\Bigg[\frac{\partial u_x}{\partial x}\delta\bar{x}^2 + \frac{\partial u_y}{\partial y}\delta\bar{y}^2 + \frac{\partial u_z}{\partial z}\delta\bar{z}^2 + \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)\delta\bar{x}\delta\bar{y} \\ & + \left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right)\delta\bar{y}\delta\bar{z} + \left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)\delta\bar{z}\delta\bar{x}\Bigg] \end{aligned} \tag{2.8}$$

From this expression we define the rectangular *components of strain*, ε_{xx}, ε_{xy}, ε_{xz}, ε_{yx}, ε_{yy}, ε_{yz}, ε_{zx}, ε_{zy}, ε_{zz}, as

$$\begin{gathered} \varepsilon_{xx} \equiv \frac{\partial u_x}{\partial x} \qquad \varepsilon_{yy} \equiv \frac{\partial u_y}{\partial y} \qquad \varepsilon_{zz} \equiv \frac{\partial u_z}{\partial z} \\ \varepsilon_{yx} = \varepsilon_{xy} \equiv \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \\ \varepsilon_{zy} = \varepsilon_{yz} \equiv \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \\ \varepsilon_{xz} = \varepsilon_{zx} \equiv \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right), \end{gathered} \tag{2.9}$$

so that

$$\begin{aligned} ||\tilde{P}\tilde{Q}||^2 - ||\bar{P}\bar{Q}||^2 \approx\ & 2\Big[\varepsilon_{xx}\delta\bar{x}^2 + \varepsilon_{yy}\delta\bar{y}^2 + \varepsilon_{zz}\delta\bar{z}^2 + \varepsilon_{xy}\delta\bar{x}\delta\bar{y} + \varepsilon_{yx}\delta\bar{y}\delta\bar{x} \\ & + \varepsilon_{yz}\delta\bar{y}\delta\bar{z} + \varepsilon_{zy}\delta\bar{z}\delta\bar{y} + \varepsilon_{zx}\delta\bar{z}\delta\bar{x} + \varepsilon_{xz}\delta\bar{x}\delta\bar{z}\Big]. \end{aligned} \tag{2.10}$$

A physical interpretation of Equations (2.9) helps distinguish the two types of strains: the *extensional strains*, ε_{xx}, ε_{yy}, ε_{zz}, and the *shear strains*, ε_{xy}, ε_{yx}, ε_{yz}, ε_{zy}, ε_{zx}, ε_{xz}. For example, the extensional strain, ε_{xx}, gives the relative *elongation* in the x-direction of "elastic fibers" initially parallel to **i**. The shear strain ε_{xy} can be associated with the *rotation* of fibers as follows: if f_x and f_y are fibers in the xy-plane, initially parallel to **i** and **j**, respectively, then ε_{xy} is one half the change in the angle between f_x and f_y [4]. Equations (2.9) are called the *strain-displacement equations.* (NOTE: It is not uncommon for the shear strains to be defined without the coefficient $\frac{1}{2}$ in the last three definitions of (2.9). This is done in [7], [13]. In this case the shear strain ε_{xy} would be the *total* change in the angle between fibers f_x and f_y.)

2.3 Stress and Hooke's Law

Stress in a three dimensional elastic body can be thought of as the mechanism by which force is transmitted from one portion of the body to another via (real or imagined) plane surfaces. We again consider a material point $\mathcal{M}$ in elastic solid $\mathcal{S}$, we let $\mathcal{A}_k$, $k = 1, 2, 3 \ldots$, be a sequence of plane area elements containing $\mathcal{M}$ with area A_k, and we denote by $\mathbf{F}_k$ the net force acting on $\mathcal{A}_k$. We assume that all area elements have the same normal vector $\mathbf{n}$ and that A_k, $||\mathbf{F}_k|| \to 0$ as $k \to \infty$. The *stress vector* or *traction*, $\mathbf{T_n}$, with respect to the plane with normal vector $\mathbf{n}$ and containing the point $\mathcal{M}$, is defined as [4],

$$\mathbf{T_n} = \lim_{k\to\infty} \frac{\mathbf{F}_k}{A_k}.$$

Note that, in general, $\mathbf{T_n}$ and $\mathbf{n}$ are *not* parallel.

The *state of stress* at a point is determined when the traction on every plane containing the point is known. Knowing the tractions, $\mathbf{T_i}$, $\mathbf{T_j}$, $\mathbf{T_k}$, on planes parallel to the coordinate planes (the yz, xz, xy-planes, respectively) is sufficient to determine the state of stress at a point [4]. When we resolve the tractions $\mathbf{T_i}$, $\mathbf{T_j}$, $\mathbf{T_k}$ into their $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ components we have

$$\begin{aligned} \mathbf{T_i} &= \sigma_{xx}\mathbf{i} + \sigma_{xy}\mathbf{j} + \sigma_{xz}\mathbf{k} \\ \mathbf{T_j} &= \sigma_{yx}\mathbf{i} + \sigma_{yy}\mathbf{j} + \sigma_{yz}\mathbf{k} \\ \mathbf{T_k} &= \sigma_{zx}\mathbf{i} + \sigma_{zy}\mathbf{j} + \sigma_{zz}\mathbf{k}, \end{aligned} \tag{2.11}$$

where σ_{xx}, σ_{xy}, σ_{xz}, σ_{yx}, σ_{yy}, σ_{yz}, σ_{zx}, σ_{zy}, σ_{zz}, are called the rectangular *components of stress.*

The First and Second Laws of Thermodynamics may be used to prove the existence of a function, W, called the *strain-energy function* or *potential energy density*, with the property that [7],

$$\begin{array}{lll} \sigma_{xx} = \dfrac{\partial W}{\partial \varepsilon_{xx}} & \sigma_{xy} = \dfrac{\partial W}{\partial \varepsilon_{xy}} & \sigma_{xz} = \dfrac{\partial W}{\partial \varepsilon_{xz}} \\[2ex] \sigma_{yx} = \dfrac{\partial W}{\partial \varepsilon_{yx}} & \sigma_{yy} = \dfrac{\partial W}{\partial \varepsilon_{yy}} & \sigma_{yz} = \dfrac{\partial W}{\partial \varepsilon_{yz}} \\[2ex] \sigma_{zx} = \dfrac{\partial W}{\partial \varepsilon_{zx}} & \sigma_{zy} = \dfrac{\partial W}{\partial \varepsilon_{zy}} & \sigma_{zz} = \dfrac{\partial W}{\partial \varepsilon_{zz}}. \end{array} \tag{2.12}$$

The strain-energy function has units energy per unit volume. Note that since $\varepsilon_{xy} = \varepsilon_{yx}$, $\varepsilon_{yz} = \varepsilon_{yz}$, $\varepsilon_{zx} = \varepsilon_{xz}$, Equations (2.12) imply that $\sigma_{xy} = \sigma_{yx}$, $\sigma_{yz} = \sigma_{yz}$, $\sigma_{zx} = \sigma_{xz}$.

Assuming Hooke's Law holds, the six distinct components of stress can be expressed as linear combinations of the six distinct components of strain via some 36 constants:

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} & c_{46} \\ c_{51} & c_{52} & c_{53} & c_{54} & c_{55} & c_{56} \\ c_{61} & c_{62} & c_{63} & c_{64} & c_{65} & c_{66} \end{pmatrix} \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{pmatrix}. \tag{2.13}$$

Assuming that W is continuously differentiable, calculations such as

$$c_{34} = \frac{\partial \sigma_{zz}}{\partial \varepsilon_{xy}} = \frac{\partial}{\partial \varepsilon_{xy}} \left[\frac{\partial W}{\partial \varepsilon_{zz}} \right] = \frac{\partial}{\partial \varepsilon_{zz}} \left[\frac{\partial W}{\partial \varepsilon_{xy}} \right] = \frac{\partial \sigma_{xy}}{\partial \varepsilon_{zz}} = c_{43},$$

show that the coefficient matrix (c_{ij}) is symmetric. Therefore, Hooke's law in its most general form, Equations (2.13) with $c_{ij} = c_{ji}$, requires 21 elastic constants.

If we assume the material is isotropic, then only 2 elastic constants are needed. It is common to use the *Lamé constants*, λ and μ, named after Gabriel Lamé. After publication of Lamé's *Leçons sur la théorie mathématique de l'élasticité des corps solides* in 1852, the 2 constant model of isotropic elastic solids was generally accepted [7]. Using the Lamé constants the stress-strain equations take the form

$$\begin{aligned} \sigma_{xx} &= (\lambda + 2\mu)\varepsilon_{xx} + \lambda\varepsilon_{yy} + \lambda\varepsilon_{zz} \\ \sigma_{yy} &= \lambda\varepsilon_{xx} + (\lambda + 2\mu)\varepsilon_{yy} + \lambda\varepsilon_{zz} \\ \sigma_{zz} &= \lambda\varepsilon_{xx} + \lambda\varepsilon_{yy} + (\lambda + 2\mu)\varepsilon_{zz} \\ \sigma_{xy} &= 2\mu\varepsilon_{xy} \\ \sigma_{yz} &= 2\mu\varepsilon_{yz} \\ \sigma_{zx} &= 2\mu\varepsilon_{zx}. \end{aligned} \tag{2.14}$$

We can express the shear modulus, G, bulk modulus, K, and Young's modulus, E in terms of the Lamé constants as follows,

$$\begin{aligned} K &= \lambda + \tfrac{2}{3}\mu \\ G &= \mu \\ E &= \frac{\mu(2\mu + 3\lambda)}{\lambda + \mu} \\ \nu &= \frac{\lambda}{2(\lambda + \mu)}. \end{aligned}$$

The last equation defines *Poisson's ratio*, ν, which is the ratio of lateral contraction to axial expansion due to uniaxial stress. For example, if $\sigma_{xx} \neq 0$ and all other stresses are zero, then equations (2.14) are equivalent to

$$\frac{\sigma_{xx}}{\varepsilon_{xx}} = \frac{\mu(2\mu + 3\lambda)}{\lambda + \mu} \tag{2.15}$$

$$-\frac{\varepsilon_{yy}}{\varepsilon_{xx}} = -\frac{\varepsilon_{zz}}{\varepsilon_{xx}} = \frac{\lambda}{2(\lambda + \mu)}. \tag{2.16}$$

The right-hand side of (2.15) is Young's modulus while the right-hand side of (2.16) is Poisson's ratio.

2.4 The Strain-Energy Function and Potential Energy

We may now derive an expression for the strain-energy function, W, as a quadratic function of the components of strain and find the potential energy, $\mathcal{V}_{\mathcal{S}}$, of any isotropic linear elastic solid $\mathcal{S}$.

The strain-energy function, W, is a function of the nine components of strain ε_{xx}, ε_{yy}, ε_{zz}, ε_{xy}, ε_{yz}, ε_{zx}, ε_{yx}, ε_{zy}, ε_{xz}, which, for our purposes in this section, we shall refer to as ε_1, ε_2, ε_3, ε_4, ε_5, ε_6, ε_7, ε_8, ε_9, respectively. This allows us to write the total differential of W as

$$dW = \sum_{i=1}^{9} \frac{\partial W}{\partial \varepsilon_i} d\varepsilon_i. \tag{2.17}$$

Note that W **cannot** be obtained by integrating both sides of (2.17):

$$W(\varepsilon_1, \ . \ . \ . \ , \varepsilon_9) \neq \sum_{i=1}^{9} \int_{\varepsilon_i} \frac{\partial W}{\partial \varepsilon_i} d\varepsilon_i.$$

Since the first partial derivatives of W are linear functions of $\varepsilon_1, \ . \ . \ . \ , \varepsilon_9$, i.e.

$$\frac{\partial W}{\partial \varepsilon_i} = \sum_{j=1}^{9} c_{ij}\varepsilon_j, \tag{2.18}$$

it can be shown that W is in fact given by

$$W(\varepsilon_1, \ . \ . \ . \ , \varepsilon_9) = \sum_{i=1}^{9} \int_{\varepsilon_i} \frac{\partial W}{\partial \varepsilon_i} d\varepsilon_i - \sum_{i=1}^{8} \sum_{j=i+1}^{9} c_{ij}\varepsilon_i\varepsilon_j.$$

This in turn can be shown equivalent to

$$W(\varepsilon_1, \ . \ . \ . \ , \varepsilon_9) = \tfrac{1}{2} \sum_{i=1}^{9} c_{ii}\varepsilon_i^2 + \sum_{i=1}^{8} \sum_{j=i+1}^{9} c_{ij}\varepsilon_i\varepsilon_j.$$

Comparing (2.18) to (2.14), we see that

$$\begin{gathered} c_{11} = c_{22} = c_{33} = \lambda + 2\mu \\ c_{12} = c_{13} = c_{23} = \lambda \\ c_{i4} = c_{i5} = \ . \ . \ . \ = c_{i9} = 0, \quad i = 1, 2, 3 \\ c_{44} = c_{55} = c_{66} = c_{77} = c_{88} = c_{99} = 2\mu \\ c_{ii+1} = c_{ii+2} = \ . \ . \ . \ = c_{i9} = 0, \quad i = 4, 5, 6, 7, 8, 9. \end{gathered}$$

We thus obtain the strain-energy function in the form

$$\begin{aligned} W \ &= \ \tfrac{1}{2}(\lambda + 2\mu)(\varepsilon_{xx}^2 + \varepsilon_{yy}^2 + \varepsilon_{zz}^2) + \lambda(\varepsilon_{xx}\varepsilon_{yy} + \varepsilon_{yy}\varepsilon_{zz} + \varepsilon_{zz}\varepsilon_{xx}) \\ &\quad + 2\mu(\varepsilon_{xy}^2 + \varepsilon_{yz}^2 + \varepsilon_{zx}^2). \end{aligned} \tag{2.19}$$

Various other forms of the strain-energy are possible. Some commonly encountered forms are shown below:

$$W = \tfrac{1}{2}(\lambda + 2\mu)(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz})^2 + 2\mu(\varepsilon_{yz}^2 + \varepsilon_{zx}^2 + \varepsilon_{xy}^2 - \varepsilon_{yy}\varepsilon_{zz} - \varepsilon_{zz}\varepsilon_{xx} - \varepsilon_{xx}\varepsilon_{yy}) \tag{2.20}$$

$$\begin{aligned} &= \tfrac{1}{6}(3\lambda + 2\mu)(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz})^2 + \tfrac{1}{3}\mu\left[(\varepsilon_{yy} - \varepsilon_{zz})^2 + (\varepsilon_{zz} - \varepsilon_{xx})^2 + (\varepsilon_{xx} - \varepsilon_{yy})^2\right] \\ &\quad + 2\mu(\varepsilon_{yz}^2 + \varepsilon_{zx}^2 + \varepsilon_{xy}^2) \end{aligned} \tag{2.21}$$

$$= \tfrac{1}{2}\lambda(\varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz})^2+\mu(\varepsilon_{xx}^2+\varepsilon_{yy}^2+\varepsilon_{zz}^2+2\varepsilon_{yz}^2+2\varepsilon_{zx}^2+2\varepsilon_{xy}^2) \tag{2.22}$$

$$\begin{aligned} &= \tfrac{1}{6}(3\lambda+2\mu)(\varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz})^2+\mu\Big[\tfrac{1}{2}(\varepsilon_{xx}-\varepsilon_{yy})^2+2\varepsilon_{yz}^2+2\varepsilon_{zx}^2+2\varepsilon_{xy}^2\Big] \\ &\quad + \tfrac{1}{6}\mu(\varepsilon_{xx}+\varepsilon_{yy}-2\varepsilon_{zz})^2. \end{aligned} \tag{2.23}$$

The potential energy of the elastic solid $\mathcal{S}$, $\mathcal{V}_{\mathcal{S}}$, is found by integrating the strain-energy function over the equilibrium configuration, $\mathcal{R}$, of the elastic solid: $\mathcal{V}_{\mathcal{S}} = \int_{\mathcal{R}} W dV$. Using (2.22) for the strain-energy function we obtain the potential energy in rectangular coordinates as a function of the strains,

$$\mathcal{V}_{\mathcal{S}} = \tfrac{1}{2}\int\limits_{\mathcal{R}}\Big\{\lambda(\varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz})^2+2\mu(\varepsilon_{xx}^2+\varepsilon_{yy}^2+\varepsilon_{zz}^2+2\varepsilon_{xy}^2+2\varepsilon_{yz}^2+2\varepsilon_{zx}^2)\Big\}dxdydz. \tag{2.24}$$

To obtain the potential energy as a function of the displacements we use the strain-displacement equations (2.9) in (2.24):

$$\begin{aligned} \mathcal{V}_{\mathcal{S}} &= \tfrac{1}{2}\int\limits_{\mathcal{R}}\Bigg\{\lambda\Big(\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}+\frac{\partial u_z}{\partial z}\Big)^2+\mu\Bigg[2\frac{\partial u_x}{\partial x}^2+2\frac{\partial u_y}{\partial y}^2+2\frac{\partial u_z}{\partial z}^2+\Big(\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\Big)^2 \\ &\qquad + \Big(\frac{\partial u_y}{\partial z}+\frac{\partial u_z}{\partial y}\Big)^2+\Big(\frac{\partial u_z}{\partial x}+\frac{\partial u_x}{\partial z}\Big)^2\Bigg]\Bigg\}dxdydz. \end{aligned} \tag{2.25}$$

Another form of the potential energy which will be useful in Chapter 4 is found by noting that the strain-energy function can be expressed as

$$W = \tfrac{1}{2}(\sigma_{xx}\varepsilon_{xx}+\sigma_{yy}\varepsilon_{yy}+\sigma_{zz}\varepsilon_{zz}+2\sigma_{xy}\varepsilon_{xy}+2\sigma_{yz}\varepsilon_{yz}+2\sigma_{zx}\varepsilon_{zx}) \tag{2.26}$$

Writing $\boldsymbol{\sigma}$ for the *stress matrix*,

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}, \tag{2.27}$$

and $\boldsymbol{\varepsilon}$ for the *strain matrix*,

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix}, \tag{2.28}$$

we have

$$W = \tfrac{1}{2} Tr(\boldsymbol{\sigma}\boldsymbol{\varepsilon}), \tag{2.29}$$

where $Tr(\mathbf{A})$, the *trace* of square matrix $\mathbf{A}$, is the sum of the diagonal elements of $\mathbf{A}$. We may therefore write

$$\mathcal{V}_{\mathcal{S}} = \frac{1}{2} \int\limits_{\mathcal{R}} Tr(\boldsymbol{\sigma}\boldsymbol{\varepsilon}) dx dy dz, \tag{2.30}$$

for the potential energy of the elastic solid $\mathcal{S}$ in rectangular coordinates.

Chapter 3

Introduction to Formation Theory

3.1 Introduction

Formation theory is an area of control theory concerned with the design, modeling, and control of elastic bodies with attached and/or embedded actuators. The term formation is used to emphasize that one of our goals is to seek classes of target configurations achievable with a prescribed, but somewhat general, actuator structure. Here we will focus on a specific type of actuator, called a "monotropic" actuator. Consideration of other modes of actuation may be found in [10]. Much of what follows is derived from [11].

The elastic bodies we seek to model are composed of two parts; the actuator material and the "matrix" material into which the actuators are embedded. The actuator structure will be divided into "families" which share certain geometric and elastic properties. The controls for each family will consist of actuator material densities and "signals" used to maintain actuators in a nonequilibrium state. We assume that the dimensions of the actuators are small enough, compared to the dimensions of the body, that the matrix material displaced due to their presence is negligible. We will therefore use (2.25) for the potential energy of the matrix material. Our goal in this chapter is to describe the actuator structure and derive an expression for the actuator potential energy, the total potential energy being the sum of the matrix and actuator potential energies.

3.2 The Actuators

In this section we will describe the actuators and some assumptions made when modeling the actuator structure. We will consider a specific type of actuator, henceforth referred to as a "monotropic" actuator. A monotropic actuator may be thought of as a filament or fiber or a very small rod with equilibrium length ℓ and uniform cross-section of area A. The dimensions of the cross-section (the lateral dimensions) are assumed to be much smaller than ℓ. The length of the actuator will increase or decrease an amount $\delta\ell$ in response to a control signal, u, and we assume that the *fractional change in length*, $\delta\ell/\ell$, is proportional to u. Therefore, the length of the actuator under the influence of the *control only* is $\ell + \delta\ell = \ell(1 + \kappa u)$, where $\kappa > 0$ is a constant. The length $\ell(1 + \kappa u)$ will be referred to as the *controlled equilibrium length.* We also assume that the lateral contraction or *Poisson effect* due to u is negligible, i.e. Poisson's ratio $\nu \approx 0$. This implies that the cross-sectional area, A, remains the same upon application of the control signal, u, and that the orientation of the long axis of the actuator determines the nature of the stress produced.

3.3 Actuator Families

We assume that the collection of actuators is divided into groups which we refer to as *families.* Many characteristics may distinguish one family from another, but here we will restrict our attention to the monotropic actuators described above, with each family of actuators having, in general, its own control signal, orientation, distribution density, and Young's modulus. In this section we quantify these characteristics.

Let $\mathcal{R}$ be the region in $\mathbb{R}^3$, that our elastic solid, $\mathcal{S}$, initially occupies and let $\mathcal{F}$ be a family of actuators. Let $\mathbf{x} = (x\ y\ z)^{\mathrm{T}}$ be an element of $\mathcal{R}$ and let $a_{\mathbf{x}}$ be an actuator in $\mathcal{F}$ near $\mathbf{x}$. We then define the *distribution density*, d, and the *orientation*, $\boldsymbol{\varphi}$, for family $\mathcal{F}$ as

$$d(\mathbf{x}) \equiv ||\mathbf{F}(\mathbf{x})|| \qquad \boldsymbol{\varphi}(\mathbf{x}) \equiv \frac{\mathbf{F}(\mathbf{x})}{||\mathbf{F}(\mathbf{x})||},$$

where $\mathbf{F}$ is a piecewise, continuously differentiable vector field on $\mathcal{R}$. The orientation, $\boldsymbol{\varphi}(\mathbf{x})$, is the unit vector in the direction of the long axis of actuator $a_{\mathbf{x}}$ and the distribution density, $d(\mathbf{x})$, is the density of actuator material at $\mathbf{x}$.

We let $u(\mathbf{x})$ be the control signal associated with family $\mathcal{F}$. This means that the actuator $a_{\mathbf{x}}$ in $\mathcal{F}$ with equilibrium length ℓ has controlled equilibrium length $\ell(1 + \kappa u(\mathbf{x}))$. This is the length of $a_{\mathbf{x}}$ in the absence of other deformations of the matrix material.

3.4 The Actuator Potential Energy

In this section we derive an expression for the potential energy, $\mathcal{V}_{\mathcal{F}}$, of the family of actuators $\mathcal{F}$. We assume that the actuators are securely embedded in the matrix structure so that the assemblage behaves as one under deformation and that the actuators are numerous enough that summation over a family may be approximated by integration over the region $\mathcal{R}$.

The potential energy per unit volume of an actuator as described in §3.2 can be expressed as one half the product of the stress, σ, and strain, ε. The potential energy of an actuator is therefore,

$$\mathcal{V} = \tfrac{1}{2}\sigma\varepsilon V,$$

where V is the volume of the actuator. In one dimension the strain is just the fractional change in length, $\varepsilon = \delta\ell/\ell$, and the stress, by Hooke's law, is $\sigma = E\varepsilon$. This gives us

$$\begin{aligned}\mathcal{V} &= \tfrac{1}{2}E\varepsilon^2 V \\ &= \tfrac{1}{2}E\left(\frac{\delta\ell}{\ell}\right)^2 A\ell \\ &= \tfrac{1}{2}EA\frac{\delta\ell^2}{\ell}.\end{aligned} \tag{3.1}$$

Now the change in length of the actuator, $\delta\ell$, depends on two influences. As an embedded element in the matrix material, the actuator is subject to the same deformation as the matrix. This is described in section §2.2 using the scalar components of vector fields $\mathbf{D}$ and $\mathbf{u}$. That calculation is repeated here using vector notation and the variables associated with the actuators. The actuator is also subject to changes due to the control, u, associated with the family $\mathcal{F}$. This effect is realized via the controlled equilibrium length, $\ell(1 + \kappa u(\mathbf{x}))$, and will enter the calculation later.

We assume that one end of the actuator is located at $\bar{\mathbf{x}}$ and the other at $\bar{\mathbf{x}} + \delta\bar{\mathbf{x}}$. Since the actuator has length ℓ and orientation $\boldsymbol{\varphi}(\bar{\mathbf{x}})$ we have $\delta\bar{\mathbf{x}} = \ell\boldsymbol{\varphi}(\bar{\mathbf{x}})$. The endpoints of the actuator are relocated due to deformation of the matrix to $\tilde{\mathbf{x}} = \mathbf{D}(\bar{\mathbf{x}})$ and $\tilde{\mathbf{x}} + \delta\tilde{\mathbf{x}} = \mathbf{D}(\bar{\mathbf{x}} + \delta\bar{\mathbf{x}})$. As before we write

$$\begin{aligned}\tilde{\mathbf{x}} + \delta\tilde{\mathbf{x}} &= \mathbf{D}(\bar{\mathbf{x}} + \delta\bar{\mathbf{x}}) \\ &= \mathbf{D}(\bar{\mathbf{x}}) + \frac{\partial\mathbf{D}}{\partial\mathbf{x}}(\bar{\mathbf{x}})\delta\bar{\mathbf{x}} + \ \ldots \\ &\approx \tilde{\mathbf{x}} + \frac{\partial\mathbf{D}}{\partial\mathbf{x}}(\bar{\mathbf{x}})\delta\bar{\mathbf{x}},\end{aligned}$$

where $\partial\mathbf{D}/\partial\mathbf{x}$ is the Jacobian of $\mathbf{D}$,

$$\frac{\partial \mathbf{D}}{\partial \mathbf{x}} \equiv \begin{pmatrix} \dfrac{\partial D_x}{\partial x} & \dfrac{\partial D_x}{\partial y} & \dfrac{\partial D_x}{\partial z} \\ \dfrac{\partial D_y}{\partial x} & \dfrac{\partial D_y}{\partial y} & \dfrac{\partial D_y}{\partial z} \\ \dfrac{\partial D_z}{\partial x} & \dfrac{\partial D_z}{\partial y} & \dfrac{\partial D_z}{\partial z} \end{pmatrix}.$$

Therefore,

$$\delta\tilde{\mathbf{x}} \approx \frac{\partial \mathbf{D}}{\partial \mathbf{x}}(\bar{\mathbf{x}})\delta\bar{\mathbf{x}}.$$

Since the displacement field, $\mathbf{u}$, is defined in (2.1) by $\mathbf{D}(\mathbf{x}) = \mathbf{u}(\mathbf{x}) + \mathbf{x}$ we have

$$\frac{\partial \mathbf{D}}{\partial \mathbf{x}} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}} + \mathbf{I},$$

where $\mathbf{I}$ is the identity matrix. We therefore write

$$\delta\tilde{\mathbf{x}} \approx \left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}(\bar{\mathbf{x}}) + \mathbf{I}\right)\delta\bar{\mathbf{x}},$$

and calculate the square of the length of the actuator under the influence of $\mathbf{D}$,

$$\begin{aligned} ||\delta\tilde{\mathbf{x}}||^2 &= \delta\tilde{\mathbf{x}}^{\mathrm{T}}\delta\tilde{\mathbf{x}} \\ &\approx \delta\bar{\mathbf{x}}^{\mathrm{T}}\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}^{\mathrm{T}}(\bar{\mathbf{x}}) + \mathbf{I}\right)\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}(\bar{\mathbf{x}}) + \mathbf{I}\right)\delta\bar{\mathbf{x}} \\ &= \delta\bar{\mathbf{x}}^{\mathrm{T}}\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}^{\mathrm{T}}(\bar{\mathbf{x}})\;\frac{\partial \mathbf{u}}{\partial \mathbf{x}}(\bar{\mathbf{x}}) + \frac{\partial \mathbf{u}}{\partial \mathbf{x}}(\bar{\mathbf{x}}) + \frac{\partial \mathbf{u}}{\partial \mathbf{x}}^{\mathrm{T}}(\bar{\mathbf{x}}) + \mathbf{I}\right)\delta\bar{\mathbf{x}} \qquad (3.2) \\ &\approx \delta\bar{\mathbf{x}}^{\mathrm{T}}\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}(\bar{\mathbf{x}}) + \frac{\partial \mathbf{u}}{\partial \mathbf{x}}^{\mathrm{T}}(\bar{\mathbf{x}}) + \mathbf{I}\right)\delta\bar{\mathbf{x}} \\ &= 2\delta\bar{\mathbf{x}}^{\mathrm{T}}\,\frac{1}{2}\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}(\bar{\mathbf{x}}) + \frac{\partial \mathbf{u}}{\partial \mathbf{x}}^{\mathrm{T}}(\bar{\mathbf{x}})\right)\delta\bar{\mathbf{x}} + \delta\bar{\mathbf{x}}^{\mathrm{T}}\delta\bar{\mathbf{x}}. \end{aligned}$$

We note that

$$\frac{1}{2}\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}+\frac{\partial \mathbf{u}}{\partial \mathbf{x}}^{\mathrm{T}}\right) = \begin{pmatrix} \frac{\partial u_x}{\partial x} & \frac{1}{2}\left(\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u_x}{\partial z}+\frac{\partial u_z}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u_y}{\partial x}+\frac{\partial u_x}{\partial y}\right) & \frac{\partial u_y}{\partial y} & \frac{1}{2}\left(\frac{\partial u_y}{\partial z}+\frac{\partial u_z}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial u_z}{\partial x}+\frac{\partial u_x}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial u_z}{\partial y}+\frac{\partial u_y}{\partial z}\right) & \frac{\partial u_z}{\partial z} \end{pmatrix}$$

$$= \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix} = \boldsymbol{\varepsilon}.$$

Therefore,

$$||\delta\tilde{\mathbf{x}}||^2 \approx 2\delta\bar{\mathbf{x}}^{\mathrm{T}}\boldsymbol{\varepsilon}(\bar{\mathbf{x}})\delta\bar{\mathbf{x}} + ||\delta\bar{\mathbf{x}}||^2. \tag{3.3}$$

Since the actuator has length ℓ and orientation $\boldsymbol{\varphi}(\bar{\mathbf{x}})$, with $||\boldsymbol{\varphi}(\bar{\mathbf{x}})|| = 1$, we use $\delta\bar{\mathbf{x}} = \ell\boldsymbol{\varphi}(\bar{\mathbf{x}})$ in (3.3) to obtain

$$\begin{aligned} ||\delta\tilde{\mathbf{x}}||^2 &\approx 2\ell^2\boldsymbol{\varphi}^{\mathrm{T}}(\bar{\mathbf{x}})\boldsymbol{\varepsilon}(\bar{\mathbf{x}})\boldsymbol{\varphi}(\bar{\mathbf{x}}) + \ell^2 \\ ||\delta\tilde{\mathbf{x}}|| &\approx \ell\sqrt{1+2\boldsymbol{\varphi}^{\mathrm{T}}(\bar{\mathbf{x}})\boldsymbol{\varepsilon}(\bar{\mathbf{x}})\boldsymbol{\varphi}(\bar{\mathbf{x}})} \\ &\approx \ell\left(1+\boldsymbol{\varphi}^{\mathrm{T}}(\bar{\mathbf{x}})\boldsymbol{\varepsilon}(\bar{\mathbf{x}})\boldsymbol{\varphi}(\bar{\mathbf{x}})\right). \end{aligned}$$

Since the change in length of the actuator, $\delta\ell$, is the length due to deformation of the matrix minus the controlled equilibrium length, we have

$$\begin{aligned} \delta\ell &= ||\delta\tilde{\mathbf{x}}|| - \ell(1+\kappa u(\mathbf{x})) \\ &\approx \ell\left(1+\boldsymbol{\varphi}^{\mathrm{T}}(\bar{\mathbf{x}})\boldsymbol{\varepsilon}(\bar{\mathbf{x}})\boldsymbol{\varphi}(\bar{\mathbf{x}})\right) - \ell(1+\kappa u(\mathbf{x})) \\ &= \ell\left(\boldsymbol{\varphi}^{\mathrm{T}}(\bar{\mathbf{x}})\boldsymbol{\varepsilon}(\bar{\mathbf{x}})\boldsymbol{\varphi}(\bar{\mathbf{x}}) - \kappa u(\mathbf{x})\right). \end{aligned} \tag{3.4}$$

Using (3.4) for $\delta\ell$ in (3.1), we obtain the potential energy of an actuator, accounting for both the deformation of the matrix material and the control associated with the corresponding actuator family,

$$\mathcal{V} = \tfrac{1}{2} E A \ell \left(\boldsymbol{\varphi}^{\mathrm{T}}(\bar{\mathbf{x}}) \boldsymbol{\varepsilon}(\bar{\mathbf{x}}) \boldsymbol{\varphi}(\bar{\mathbf{x}}) - \kappa u(\mathbf{x}) \right)^2 . \tag{3.5}$$

To obtain the potential energy for the family of actuators, $\mathcal{F}$, we sum expressions of the type (3.5) over the entire family. As mentioned earlier, we are assuming that the actuators composing each family are so numerous that this summation may be replaced by integration of (3.5) over the region $\mathcal{R}$. If the right hand side of (3.5) is multiplied and divided by a differential volume element $\Delta V(\mathbf{x})$, we see that the expression $A\ell/\Delta V(\mathbf{x})$ is just the volume density of actuator material. As $\Delta V(\mathbf{x}) \to 0$ this expression becomes $d(\mathbf{x})$, the density of actuator material at the point $\mathbf{x}$. We thus obtain the potential energy of the actuator family $\mathcal{F}$ in the form

$$\mathcal{V}_{\mathcal{F}} = \tfrac{1}{2} \int\limits_{\mathcal{R}} E d(\mathbf{x}) \left(\boldsymbol{\varphi}^{\mathrm{T}}(\mathbf{x}) \boldsymbol{\varepsilon}(\mathbf{x}) \boldsymbol{\varphi}(\mathbf{x}) - \kappa u(\mathbf{x}) \right)^2 dV. \tag{3.6}$$

3.5 The Total Potential Energy

The total potential energy of an elastic body as described above is the sum of the matrix potential energy, $\mathcal{V}_{\mathcal{S}}$, and the potential energies of each of the actuator families. We assume there are n families of actuators, $\mathcal{F}_i$, $i = 1, 2, \ldots, n$, each with Young's modulus E_i, density d_i, orientation $\boldsymbol{\varphi}_i$, and scaled control $\kappa_i u_i$. Using (2.30) for the matrix potential energy and (3.6) for the potential energy of each actuator family, we have the total potential energy,

$$\begin{aligned}
\mathcal{V} &= \mathcal{V}_{\mathcal{S}} + \sum_{i=1}^{n} \mathcal{V}_{\mathcal{F}_i} \\
&= \tfrac{1}{2} \int\limits_{\mathcal{R}} Tr(\boldsymbol{\sigma}(\mathbf{x}) \boldsymbol{\varepsilon}(\mathbf{x})) dV + \tfrac{1}{2} \sum_{i=1}^{n} \int\limits_{\mathcal{R}} E_i d_i(\mathbf{x}) \left(\boldsymbol{\varphi}_i^{\mathrm{T}}(\mathbf{x}) \boldsymbol{\varepsilon}(\mathbf{x}) \boldsymbol{\varphi}_i(\mathbf{x}) - \kappa_i u_i(\mathbf{x}) \right)^2 dV \\
&= \tfrac{1}{2} \int\limits_{\mathcal{R}} \left\{ Tr(\boldsymbol{\sigma}(\mathbf{x}) \boldsymbol{\varepsilon}(\mathbf{x})) + \sum_{i=1}^{n} E_i d_i(\mathbf{x}) \left(\boldsymbol{\varphi}_i^{\mathrm{T}}(\mathbf{x}) \boldsymbol{\varepsilon}(\mathbf{x}) \boldsymbol{\varphi}_i(\mathbf{x}) - \kappa_i u_i(\mathbf{x}) \right)^2 \right\} dV.
\end{aligned}$$

Chapter 4

Coordinate Transformations

4.1 Introduction

In subsequent chapters we will consider specific elastic solids with cylindrical and spherical equilibrium configurations. We will now examine how the potential energy changes due to the transformation from rectangular to spherical coordinates. In particular, we will show that the strain-energy, as a function of the strains, is invariant under this transformation. This result extends to all transformations from one orthogonal coordinate system to another, but this level of generality is not necessary here. We will then derive the potential energy expression for the cylindrical configuration from the corresponding expression in spherical coordinates when needed in Chapter 4.

4.2 Spherical Coordinates

We use (r, θ, ϕ) to represent the spherical coordinates of a point P in $\mathbb{R}^3$ with rectangular coordinates (x, y, z). Here $r \in [0, \infty)$ is the length of the line segment OP, from the origin O to P, $\theta \in [0, \pi]$ is the "latitude" angle measured from the positive z-axis to OP, and $\phi \in [0, 2\pi)$ is the "longitude" angle measured counterclockwise from the positive x-axis to the orthogonal projection of OP onto the xy-plane. We have the following relationships between the rectangular coordinates x, y, z and these spherical coordinates:

$$\begin{array}{lll} x = r\sin\theta\cos\phi & y = r\sin\theta\sin\phi & z = r\cos\theta \\ r^2 = x^2 + y^2 + z^2 & \tan\phi = \dfrac{y}{x} & \cos\theta = \dfrac{z}{r}. \end{array} \tag{4.1}$$

4.3 Coordinate Transformation as a Mapping

The transformation from spherical to rectangular coordinates, viewed as a mapping with components $x(r,\theta,\phi) = r\sin\theta\cos\phi$, $y(r,\theta,\phi) = r\sin\theta\sin\phi$, and $z(r,\theta,\phi) = r\cos\theta$, has Jacobian, $\mathbf{J}$, given by

$$\mathbf{J} = \begin{pmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} & \dfrac{\partial x}{\partial \phi} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} & \dfrac{\partial y}{\partial \phi} \\ \dfrac{\partial z}{\partial r} & \dfrac{\partial z}{\partial \theta} & \dfrac{\partial z}{\partial \phi} \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix}.$$

The (functional) inverse of this mapping, the transformation from rectangular to spherical coordinates, has Jacobian

$$\begin{pmatrix} \dfrac{\partial r}{\partial x} & \dfrac{\partial r}{\partial y} & \dfrac{\partial r}{\partial z} \\ \dfrac{\partial \theta}{\partial x} & \dfrac{\partial \theta}{\partial y} & \dfrac{\partial \theta}{\partial z} \\ \dfrac{\partial \phi}{\partial x} & \dfrac{\partial \phi}{\partial y} & \dfrac{\partial \phi}{\partial z} \end{pmatrix},$$

which may be shown equal to $\mathbf{J}^{-1}$, the matrix inverse of $\mathbf{J}$.

In what follows it will be useful to decompose $\mathbf{J}$ into the product of two square matrices, $\mathbf{R}$ and $\mathbf{B}$, as follows,

$$\mathbf{J} = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r\sin\theta \end{pmatrix} \equiv \mathbf{RB}. \tag{4.2}$$

Note that $|\mathbf{R}|$, the determinant of $\mathbf{R}$, is 1 and so

$$dV = dxdydz = |\mathbf{J}|drd\theta d\phi = |\mathbf{B}|drd\theta d\phi = r^2\sin\theta drd\theta d\phi.$$

The Jacobian, $\mathbf{J}$, also plays an important role relating partial differentiation with respect to rectangular coordinates and partial differentiation with respect to spherical coordinates. Using the chain rule for partial differentiation of a function of $r(x,y,z)$, $\theta(x,y,z)$, and $\phi(x,y,z)$ gives us

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi}$$

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial \phi}$$

$$\frac{\partial}{\partial z} = \frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial z}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial z}\frac{\partial}{\partial \phi},$$

which can be written in matrix form as

$$\begin{pmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{pmatrix} = (\mathbf{J}^{-1})^{\mathrm{T}} \begin{pmatrix} \dfrac{\partial}{\partial r} \\ \dfrac{\partial}{\partial \theta} \\ \dfrac{\partial}{\partial \phi} \end{pmatrix}.$$

Now we have $(\mathbf{J}^{-1})^{\mathrm{T}} = (\mathbf{B}^{-1}\mathbf{R}^{-1})^{\mathrm{T}} = (\mathbf{R}^{-1})^{\mathrm{T}}(\mathbf{B}^{-1})^{\mathrm{T}}$ and it can be shown that $\mathbf{R}^{-1} = \mathbf{R}^{\mathrm{T}}$, so that $(\mathbf{R}^{-1})^{\mathrm{T}} = \mathbf{R}$. Also, $(\mathbf{B}^{-1})^{\mathrm{T}} = \mathbf{B}^{-1}$ since $\mathbf{B}$, and thus $\mathbf{B}^{-1}$, is diagonal. Therefore, $(\mathbf{J}^{-1})^{\mathrm{T}} = \mathbf{R}\mathbf{B}^{-1}$, giving us

$$\begin{aligned} \begin{pmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{pmatrix} &= \mathbf{R}\mathbf{B}^{-1} \begin{pmatrix} \dfrac{\partial}{\partial r} \\ \dfrac{\partial}{\partial \theta} \\ \dfrac{\partial}{\partial \phi} \end{pmatrix} \\ &= \mathbf{R} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \dfrac{1}{r} & 0 \\ 0 & 0 & \dfrac{1}{r\sin\theta} \end{pmatrix} \begin{pmatrix} \dfrac{\partial}{\partial r} \\ \dfrac{\partial}{\partial \theta} \\ \dfrac{\partial}{\partial \phi} \end{pmatrix} \end{aligned}$$

$$= \mathbf{R} \begin{pmatrix} \dfrac{\partial}{\partial r} \\ \dfrac{1}{r}\dfrac{\partial}{\partial \theta} \\ \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial \phi} \end{pmatrix} . \tag{4.3}$$

We will use this result in §4.6.

4.4 Spherical Unit Vectors

We now look at the relationship between the rectangular unit vectors, $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, and the unit vectors corresponding to increasing values of r, θ, and ϕ, denoted $\hat{\mathbf{e}}_r$, $\hat{\mathbf{e}}_\theta$, and $\hat{\mathbf{e}}_\phi$, respectively. Based on the definitions of r, θ, and ϕ we see that the unit vectors $\hat{\mathbf{e}}_\theta$, $\hat{\mathbf{e}}_\phi$, $\hat{\mathbf{e}}_r$ may be obtained from $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, respectively, by a rotation ϕ in the xy-plane followed by a rotation θ in the zr-plane, where the "r-axis" is coincident with the line segment OP, cf. §4.2. Using the symbolic 3×1 "vectors" $(\hat{\mathbf{e}}_\theta \;\; \hat{\mathbf{e}}_\phi \;\; \hat{\mathbf{e}}_r)^{\mathrm{T}}$ and $(\mathbf{i} \;\; \mathbf{j} \;\; \mathbf{k})^{\mathrm{T}}$ we obtain

$$\begin{pmatrix} \hat{\mathbf{e}}_\theta \\ \hat{\mathbf{e}}_\phi \\ \hat{\mathbf{e}}_r \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{pmatrix} .$$

To get these in the "right" order we multiply on the left by the identity matrix with the rows shifted down one,

$$\begin{aligned} \begin{pmatrix} \hat{\mathbf{e}}_r \\ \hat{\mathbf{e}}_\theta \\ \hat{\mathbf{e}}_\phi \end{pmatrix} &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{pmatrix} \\ &= \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix} \begin{pmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{pmatrix} \end{aligned}$$

Therefore,

$$\begin{pmatrix} \hat{\mathbf{e}}_r \\ \hat{\mathbf{e}}_\theta \\ \hat{\mathbf{e}}_\phi \end{pmatrix} = \mathbf{R}^{-1} \begin{pmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{pmatrix} , \tag{4.4}$$

where $\mathbf{R}^{-1}$ is the inverse of the matrix defined in (4.2).

4.5 Spherical Components of Displacement

In §2.2 we introduced the rectangular displacement vector, $\mathbf{u}(x,y,z)$,

$$\mathbf{u}(x,y,z) = u_x(x,y,z)\mathbf{i} + u_y(x,y,z)\mathbf{j} + u_z(x,y,z)\mathbf{k}.$$

We use $\mathbf{u}_s$ to denote the spherical displacement vector and u_r, u_θ, u_ϕ to denote its r, θ, ϕ components, respectively,

$$\mathbf{u}_s(r,\theta,\phi) = u_r(r,\theta,\phi)\hat{\mathbf{e}}_r + u_\theta(r,\theta,\phi)\hat{\mathbf{e}}_\theta + u_\phi(r,\theta,\phi)\hat{\mathbf{e}}_\phi.$$

Using (4.4) we have

$$\begin{aligned}
\mathbf{u}_s(r,\theta,\phi) &= u_r(r,\theta,\phi)\hat{\mathbf{e}}_r + u_\theta(r,\theta,\phi)\hat{\mathbf{e}}_\theta + u_\phi(r,\theta,\phi)\hat{\mathbf{e}}_\phi \\
&= u_r(r,\theta,\phi)(\sin\theta\cos\phi\mathbf{i} + \sin\theta\sin\phi\mathbf{j} + \cos\theta\mathbf{k}) \\
&\quad + u_\theta(r,\theta,\phi)(\cos\theta\cos\phi\mathbf{i} + \cos\theta\sin\phi\mathbf{j} - \sin\theta\mathbf{k}) \\
&\quad + u_\phi(r,\theta,\phi)(-\sin\phi\mathbf{i} + \cos\phi\mathbf{j}), \\
&= \Big(u_r(r,\theta,\phi)\sin\theta\cos\phi + u_\theta(r,\theta,\phi)\cos\theta\cos\phi - u_\phi(r,\theta,\phi)\sin\phi\Big)\mathbf{i} \\
&\quad + \Big(u_r(r,\theta,\phi)\sin\theta\sin\phi + u_\theta(r,\theta,\phi)\cos\theta\sin\phi + u_\phi(r,\theta,\phi)\cos\phi\Big)\mathbf{j} \\
&\quad + \Big(u_r(r,\theta,\phi)\cos\theta - u_\theta(r,\theta,\phi)\sin\theta\Big)\mathbf{k}.
\end{aligned}$$

But $\mathbf{u}_s(r,\theta,\phi) = \mathbf{u}(r\sin\theta\cos\phi, r\sin\theta\sin\phi, rcos\theta)$, so we have

$$\begin{aligned}
&u_x(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta) \\
&\qquad = u_r(r,\theta,\phi)\sin\theta\cos\phi + u_\theta(r,\theta,\phi)\cos\theta\cos\phi - u_\phi(r,\theta,\phi)\sin\phi \\
&u_y(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta) \\
&\qquad = u_r(r,\theta,\phi)\sin\theta\sin\phi + u_\theta(r,\theta,\phi)\cos\theta\sin\phi + u_\phi(r,\theta,\phi)\cos\phi \\
&u_z(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta) \\
&\qquad = u_r(r,\theta,\phi)\cos\theta \ - u_\theta(r,\theta,\phi)\sin\theta.
\end{aligned}$$

We will use this result in matrix form,

$$\begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} = \mathbf{R} \begin{pmatrix} u_r \\ u_\theta \\ u_\phi \end{pmatrix}. \tag{4.5}$$

4.6 Transformation of the Strain Matrix

In this section we derive an expression for the rectangular strain matrix in spherical coordinates.

Some notation will make our task less cumbersome. We define two matrix differential operators ∇ and ∇_s as follows,

$$\nabla \equiv \begin{pmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{pmatrix} \qquad \nabla_s \equiv \begin{pmatrix} \dfrac{\partial}{\partial r} \\ \dfrac{1}{r}\dfrac{\partial}{\partial \theta} \\ \dfrac{1}{r \sin\theta}\dfrac{\partial}{\partial \phi} \end{pmatrix}.$$

We must exercise caution when using ∇ and ∇_s. Since both are 3×1 matrices, they may be "muliplied" on the left by $n \times 3$ matrices and on the right by $1 \times m$ matrices. We interpret multiplication on the left as creating a new differential operator. For example, if

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix},$$

then $\mathbf{A}\nabla$ is

$$\mathbf{A}\nabla = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \begin{pmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{pmatrix} = \begin{pmatrix} a_{11}\dfrac{\partial}{\partial x} + a_{12}\dfrac{\partial}{\partial y} + a_{13}\dfrac{\partial}{\partial z} \\ a_{21}\dfrac{\partial}{\partial x} + a_{22}\dfrac{\partial}{\partial y} + a_{23}\dfrac{\partial}{\partial z} \end{pmatrix},$$

a 2×1 matrix differential operator.

"Multiplication" on the right is performed according to the ordinary rules of matrix multiplication, but now juxtaposition of elements, with elements of ∇ or ∇_s on the left, is understood as differentiation. The expression $\nabla\mathbf{A}$ is undefined because ∇ has fewer rows than $\mathbf{A}$ has columns, but we could have an expression such as $\nabla^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$. We interpret this as a differentiation,

$$\begin{aligned}\nabla^{\mathrm{T}}\mathbf{A}^{\mathrm{T}} &= \begin{pmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \end{pmatrix} \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{pmatrix} \\ &= \begin{pmatrix} \dfrac{\partial a_{11}}{\partial x} + \dfrac{\partial a_{12}}{\partial y} + \dfrac{\partial a_{13}}{\partial z} & \dfrac{\partial a_{21}}{\partial x} + \dfrac{\partial a_{22}}{\partial y} + \dfrac{\partial a_{23}}{\partial z} \end{pmatrix}.\end{aligned}$$

Note that the above examples also show that the familiar rule, $(\mathbf{AB})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$, does not hold for matrix differential operators. Ambiguous expressions such as $\nabla\mathbf{AB}$ will not be used. We will either write $\nabla[\mathbf{A}]\mathbf{B}$ to indicate $\mathbf{A}$ is being differentiated, the result being multiplied by $\mathbf{B}$, or $\nabla[\mathbf{AB}]$ to indicate differentiation of the product $\mathbf{AB}$.

We now demonstrate the following,

Theorem 4.1 *The strain matrix in rectangular coordinates, $\boldsymbol{\varepsilon}$, is similar to the strain matrix in spherical coordinates,*

$$\boldsymbol{\varepsilon}_s = \begin{pmatrix} \varepsilon_{rr} & \varepsilon_{r\theta} & \varepsilon_{r\phi} \\ \varepsilon_{\theta r} & \varepsilon_{\theta\theta} & \varepsilon_{\theta\phi} \\ \varepsilon_{\phi r} & \varepsilon_{\phi\theta} & \varepsilon_{\phi\phi} \end{pmatrix},$$

via the matrix $\mathbf{R}$ defined in (4.2). In other words, $\boldsymbol{\varepsilon} = \mathbf{R}\,\boldsymbol{\varepsilon}_s\mathbf{R}^{-1}$.

Proof. Since the strain matrix, $\boldsymbol{\varepsilon}$, is given by

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left(\frac{\partial\mathbf{u}}{\partial\mathbf{x}} + \frac{\partial\mathbf{u}}{\partial\mathbf{x}}^{\mathrm{T}}\right),$$

we first consider how the transpose of $\partial\mathbf{u}/\partial\mathbf{x}$, the Jacobian of the displacement vector $\mathbf{u}$, transforms. We use the notation above along with (4.3) and (4.5) to calculate

$$\frac{\partial\mathbf{u}}{\partial\mathbf{x}}^{\mathrm{T}} = \nabla(u_x \;\; u_y \;\; u_z)$$

$$
\begin{aligned}
&= \mathbf{R}\nabla_s\Big[(u_r \quad u_\theta \quad u_\phi)\mathbf{R}^{-1}\Big] \\
&= \mathbf{R}\nabla_s\Big[(u_r \quad u_\theta \quad u_\phi)\Big]\mathbf{R}^{-1} + \mathbf{R}\left\{\begin{pmatrix} u_r \\ u_\theta \\ u_\phi \end{pmatrix}\nabla_s^{\mathrm{T}}\right\}^{\mathrm{T}}\mathbf{R}^{-1} \\
&= \mathbf{R}\begin{pmatrix} \dfrac{\partial u_r}{\partial r} & \dfrac{\partial u_\theta}{\partial r} & \dfrac{\partial u_\phi}{\partial r} \\ \dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta} & \dfrac{1}{r}\dfrac{\partial u_\theta}{\partial \theta} & \dfrac{1}{r}\dfrac{\partial u_\phi}{\partial \theta} \\ \dfrac{1}{r\sin\theta}\dfrac{\partial u_r}{\partial \phi} & \dfrac{1}{r\sin\theta}\dfrac{\partial u_\theta}{\partial \phi} & \dfrac{1}{r\sin\theta}\dfrac{\partial u_\phi}{\partial \phi} \end{pmatrix}\mathbf{R}^{-1} \\
&\quad + \mathbf{R}\begin{pmatrix} u_r\dfrac{\partial}{\partial r} & u_\theta\dfrac{\partial}{\partial r} & u_\phi\dfrac{\partial}{\partial r} \\ \dfrac{u_r}{r}\dfrac{\partial}{\partial \theta} & \dfrac{u_\theta}{r}\dfrac{\partial}{\partial \theta} & \dfrac{u_\phi}{r}\dfrac{\partial}{\partial \theta} \\ \dfrac{u_r}{r\sin\theta}\dfrac{\partial}{\partial \phi} & \dfrac{u_\theta}{r\sin\theta}\dfrac{\partial}{\partial \phi} & \dfrac{u_\phi}{r\sin\theta}\dfrac{\partial}{\partial \phi} \end{pmatrix}\mathbf{R}^{-1}
\end{aligned}
\tag{4.6}
$$

We now focus on the "product" $(\ \cdot\ )\mathbf{R}^{-1}$ in the second term of (4.6). To keep the following calculation manageable, we will write c in place of cos and s in place of sin. We have

$$
\begin{pmatrix} u_r\dfrac{\partial}{\partial r} & u_\theta\dfrac{\partial}{\partial r} & u_\phi\dfrac{\partial}{\partial r} \\ \dfrac{u_r}{r}\dfrac{\partial}{\partial \theta} & \dfrac{u_\theta}{r}\dfrac{\partial}{\partial \theta} & \dfrac{u_\phi}{r}\dfrac{\partial}{\partial \theta} \\ \dfrac{u_r}{r\mathrm{s}\theta}\dfrac{\partial}{\partial \phi} & \dfrac{u_\theta}{r\mathrm{s}\theta}\dfrac{\partial}{\partial \phi} & \dfrac{u_\phi}{r\mathrm{s}\theta}\dfrac{\partial}{\partial \phi} \end{pmatrix}\mathbf{R}^{-1}
$$

$$
= \begin{pmatrix} u_r\dfrac{\partial}{\partial r} & u_\theta\dfrac{\partial}{\partial r} & u_\phi\dfrac{\partial}{\partial r} \\ \dfrac{u_r}{r}\dfrac{\partial}{\partial \theta} & \dfrac{u_\theta}{r}\dfrac{\partial}{\partial \theta} & \dfrac{u_\phi}{r}\dfrac{\partial}{\partial \theta} \\ \dfrac{u_r}{r\mathrm{s}\theta}\dfrac{\partial}{\partial \phi} & \dfrac{u_\theta}{r\mathrm{s}\theta}\dfrac{\partial}{\partial \phi} & \dfrac{u_\phi}{r\mathrm{s}\theta}\dfrac{\partial}{\partial \phi} \end{pmatrix}\begin{pmatrix} \mathrm{s}\theta\mathrm{c}\phi & \mathrm{s}\theta\mathrm{s}\phi & \mathrm{c}\theta \\ \mathrm{c}\theta\mathrm{c}\phi & \mathrm{c}\theta\mathrm{s}\phi & -\mathrm{s}\theta \\ -\mathrm{s}\phi & \mathrm{c}\phi & 0 \end{pmatrix}
$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ \dfrac{\mathrm{c}\phi(u_r\mathrm{c}\theta - u_\theta\mathrm{s}\theta)}{r} & \dfrac{\mathrm{s}\phi(u_r\mathrm{c}\theta - u_\theta\mathrm{s}\theta)}{r} & \dfrac{-u_r\mathrm{s}\theta - u_\theta\mathrm{c}\theta}{r} \\ \dfrac{-\mathrm{s}\phi(u_r\mathrm{s}\theta + u_\theta\mathrm{c}\theta) - u_\phi\mathrm{c}\phi}{r\mathrm{s}\theta} & \dfrac{\mathrm{c}\phi(u_r\mathrm{s}\theta + u_\theta\mathrm{c}\theta) - u_\phi\mathrm{s}\phi}{r\mathrm{s}\theta} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ -\dfrac{u_\theta}{r} & \dfrac{u_r}{r} & 0 \\ -\dfrac{u_\phi}{r} & -\cot\theta\dfrac{u_\phi}{r} & \dfrac{u_r}{r} + \cot\theta\dfrac{u_\theta}{r} \end{pmatrix} \mathbf{R}^{-1}.$$

Using the expression above in (4.6) yields

$$\begin{aligned} \frac{\partial \mathbf{u}^{\mathrm{T}}}{\partial \mathbf{x}} &= \mathbf{R} \begin{pmatrix} \dfrac{\partial u_r}{\partial r} & \dfrac{\partial u_\theta}{\partial r} & \dfrac{\partial u_\phi}{\partial r} \\ \dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta} & \dfrac{1}{r}\dfrac{\partial u_\theta}{\partial \theta} & \dfrac{1}{r}\dfrac{\partial u_\phi}{\partial \theta} \\ \dfrac{1}{r\sin\theta}\dfrac{\partial u_r}{\partial \phi} & \dfrac{1}{r\sin\theta}\dfrac{\partial u_\theta}{\partial \phi} & \dfrac{1}{r\sin\theta}\dfrac{\partial u_\phi}{\partial \phi} \end{pmatrix} \mathbf{R}^{-1} \\ &\quad + \mathbf{R} \begin{pmatrix} 0 & 0 & 0 \\ -\dfrac{u_\theta}{r} & \dfrac{u_r}{r} & 0 \\ -\dfrac{u_\phi}{r} & -\cot\theta\dfrac{u_\phi}{r} & \dfrac{u_r}{r} + \cot\theta\dfrac{u_\theta}{r} \end{pmatrix} \mathbf{R}^{-1} \\ &= \mathbf{R} \begin{pmatrix} \dfrac{\partial u_r}{\partial r} & \dfrac{\partial u_\theta}{\partial r} & \dfrac{\partial u_\phi}{\partial r} \\ \dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta} - \dfrac{u_\theta}{r} & \dfrac{1}{r}\dfrac{\partial u_\theta}{\partial \theta} + \dfrac{u_r}{r} & \dfrac{1}{r}\dfrac{\partial u_\phi}{\partial \theta} \\ \dfrac{1}{r\sin\theta}\dfrac{\partial u_r}{\partial \phi} - \dfrac{u_\phi}{r} & \dfrac{1}{r\sin\theta}\dfrac{\partial u_\theta}{\partial \phi} - \cot\theta\dfrac{u_\phi}{r} & \dfrac{1}{r\sin\theta}\dfrac{\partial u_\phi}{\partial \phi} + \cot\theta\dfrac{u_\phi}{r} + \dfrac{u_r}{r} \end{pmatrix} \mathbf{R}^{-1} \\ &\equiv \mathbf{R}\, \boldsymbol{\mathcal{J}}_s^{\mathrm{T}}\, \mathbf{R}^{-1}. \end{aligned}$$

We may now compute the rectangular strain matrix,

$$\begin{aligned}\boldsymbol{\varepsilon} &= \frac{1}{2}\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}} + \frac{\partial \mathbf{u}}{\partial \mathbf{x}}^{\mathrm{T}}\right) \\ &= \tfrac{1}{2}\Big(\mathbf{R}\ \boldsymbol{\mathcal{J}}_s\ \mathbf{R}^{-1} + \mathbf{R}\ \boldsymbol{\mathcal{J}}_s^{\mathrm{T}}\ \mathbf{R}^{-1}\Big),\ \text{since } \mathbf{R}^{-1} = \mathbf{R}^{\mathrm{T}} \\ &= \mathbf{R}\ \tfrac{1}{2}\Big(\boldsymbol{\mathcal{J}}_s + \boldsymbol{\mathcal{J}}_s^{\mathrm{T}}\Big)\ \mathbf{R}^{-1}.\end{aligned}$$

It is shown in [7] that the components of the matrix $\tfrac{1}{2}\Big(\boldsymbol{\mathcal{J}}_s + \boldsymbol{\mathcal{J}}_s^{\mathrm{T}}\Big)$ are the components of strain referred to spherical coordinates (keeping in mind that Love defines the shear strains, $e_{\alpha\beta}$ (his notation), α and β distinct, to be *double* what we call the shear strains, cf. the discussion at the end of §2.2). The spherical strain matrix, $\boldsymbol{\varepsilon}_s$, is thus,

$$\boldsymbol{\varepsilon}_s = \tfrac{1}{2}\Big(\boldsymbol{\mathcal{J}}_s + \boldsymbol{\mathcal{J}}_s^{\mathrm{T}}\Big), \tag{4.7}$$

and so

$$\boldsymbol{\varepsilon} = \mathbf{R}\boldsymbol{\varepsilon}_s\mathbf{R}^{-1}. \tag{4.8}$$

This is the desired result. ■

From (4.7), we find the spherical components of strain,

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r} \qquad \varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \qquad \varepsilon_{\phi\phi} = \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \cot\theta\frac{u_\theta}{r} + \frac{u_r}{r} \tag{4.9}$$

$$\begin{gathered}\varepsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r}\right) \qquad \varepsilon_{\theta\phi} = \frac{1}{2}\left(\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi} - \cot\theta\frac{u_\phi}{r} + \frac{1}{r}\frac{\partial u_\phi}{\partial \theta}\right) \\ \varepsilon_{\phi r} = \frac{1}{2}\left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \phi} - \frac{u_\phi}{r} + \frac{\partial u_\phi}{\partial r}\right).\end{gathered} \tag{4.10}$$

The physical interpretations of the spherical extensional strains, (4.9), and the spherical shear strains, (4.10), are exactly analogous to those of the rectangular extensional and shear strains, respectively, described at the end of §2.2.

4.7 Invariance of the Strain-Energy Function

We are now in a position to prove the main result of this chapter.

Theorem 4.2 *The strain-energy function is invariant under the transformation from rectangular to spherical coordinates.*

What we mean by this is that as a function of the components of strain, the strain-energy is the same whether we use the rectangular strains or the spherical strains,

$$\begin{aligned} W &= W(\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \varepsilon_{xy}, \varepsilon_{yx}, \varepsilon_{yz}, \varepsilon_{zy}, \varepsilon_{zx}, \varepsilon_{xz}) \\ &= W(\varepsilon_{rr}, \varepsilon_{\theta\theta}, \varepsilon_{\phi\phi}, \varepsilon_{r\theta}, \varepsilon_{\theta r}, \varepsilon_{\theta\phi}, \varepsilon_{\phi\theta}, \varepsilon_{\phi r}, \varepsilon_{r\phi}). \end{aligned}$$

Proof. It was shown in §2.4 that the strain-energy function, W, can be written as

$$W = \tfrac{1}{2} Tr(\boldsymbol{\sigma}\boldsymbol{\varepsilon}), \tag{4.11}$$

where $\boldsymbol{\sigma}$ is the stress matrix and $\boldsymbol{\varepsilon}$ is the strain matrix. Using the stress-strain equations, (2.14), we can rewrite the stress matrix,

$$\begin{aligned} \boldsymbol{\sigma} &= \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \\ &= \begin{pmatrix} (\lambda+2\mu)\varepsilon_{xx} + \lambda\varepsilon_{yy} + \lambda\varepsilon_{zz} & 2\mu\varepsilon_{xy} & 2\mu\varepsilon_{xz} \\ 2\mu\varepsilon_{yx} & \lambda\varepsilon_{xx} + (\lambda+2\mu)\varepsilon_{yy} + \lambda\varepsilon_{zz} & 2\mu\varepsilon_{yz} \\ 2\mu\varepsilon_{zx} & 2\mu\varepsilon_{zy} & \lambda\varepsilon_{xx} + \lambda\varepsilon_{yy} + (\lambda+2\mu)\varepsilon_{zz} \end{pmatrix} \\ &= 2\mu\boldsymbol{\varepsilon} + \lambda e\mathbf{I}, \end{aligned}$$

where $e \equiv \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = Tr(\boldsymbol{\varepsilon})$ is called the *dilatation.*

Using the results of Theorem 4.1 we have

$$\begin{aligned} \boldsymbol{\sigma} &= 2\mu\boldsymbol{\varepsilon} + \lambda Tr(\boldsymbol{\varepsilon})\mathbf{I} \\ &= 2\mu\mathbf{R}\boldsymbol{\varepsilon}_s\mathbf{R}^{-1} + \lambda Tr(\mathbf{R}\boldsymbol{\varepsilon}_s\mathbf{R}^{-1})\mathbf{R}\mathbf{R}^{-1} \\ &= 2\mu\mathbf{R}\boldsymbol{\varepsilon}_s\mathbf{R}^{-1} + \lambda Tr(\boldsymbol{\varepsilon}_s)\mathbf{R}\mathbf{R}^{-1} \\ &= \mathbf{R}\Big(2\mu\boldsymbol{\varepsilon}_s + \lambda Tr(\boldsymbol{\varepsilon}_s)\mathbf{I}\Big)\mathbf{R}^{-1} \\ &\equiv \mathbf{R}\boldsymbol{\sigma}_s\mathbf{R}^{-1}, \end{aligned}$$

where $\boldsymbol{\sigma}_s = 2\mu\boldsymbol{\varepsilon}_s + \lambda Tr(\boldsymbol{\varepsilon}_s)\mathbf{I}$ is the spherical stress matrix,

$$\boldsymbol{\sigma}_s = \begin{pmatrix} \sigma_{rr} & \sigma_{r\theta} & \sigma_{r\phi} \\ \sigma_{\theta r} & \sigma_{\theta\theta} & \sigma_{\theta\phi} \\ \sigma_{\phi r} & \sigma_{\phi\theta} & \sigma_{\phi\phi} \end{pmatrix}.$$

We therefore have

$$\begin{aligned} W &= \tfrac{1}{2}Tr(\boldsymbol{\sigma}\boldsymbol{\varepsilon}) \\ &= \tfrac{1}{2}Tr(\mathbf{R}\boldsymbol{\sigma}_s\mathbf{R}^{-1}\mathbf{R}\boldsymbol{\varepsilon}_s\mathbf{R}^{-1}) \\ &= \tfrac{1}{2}Tr(\mathbf{R}\boldsymbol{\sigma}_s\boldsymbol{\varepsilon}_s\mathbf{R}^{-1}) \\ &= \tfrac{1}{2}Tr(\boldsymbol{\sigma}_s\boldsymbol{\varepsilon}_s), \end{aligned}$$

and the result is proven. ■

The strain-energy function in spherical coordinates is thus,

$$W = \tfrac{1}{2}\lambda(\varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{\phi\phi})^2 + \mu(\varepsilon_{rr}^2 + \varepsilon_{\theta\theta}^2 + \varepsilon_{\phi\phi}^2 + 2\varepsilon_{\theta\phi}^2 + 2\varepsilon_{\phi r}^2 + 2\varepsilon_{r\theta}^2), \qquad (4.12)$$

and the potential energy of an elastic solid $\mathcal{S}$ with equilibrium configuration $\mathcal{R}$ is

$$\begin{aligned} \mathcal{V}_{\mathcal{S}} &= \tfrac{1}{2}\int\limits_{\mathcal{R}} \Big\{\lambda(\varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{\phi\phi})^2 \\ &\qquad + 2\mu(\varepsilon_{rr}^2 + \varepsilon_{\theta\theta}^2 + \varepsilon_{\phi\phi}^2 + 2\varepsilon_{\theta\phi}^2 + 2\varepsilon_{\phi r}^2 + 2\varepsilon_{r\theta}^2)\Big\} r^2 \sin\theta dr d\theta d\phi, \qquad (4.13) \end{aligned}$$

where the spherical strains, as functions of the spherical components of displacement, u_r, u_θ, and u_ϕ, are given by (4.9)-(4.10).

4.8 The Actuator Potential Energy in Spherical Coordinates

Conversion of the actuator potential energy to spherical coordinates closely follows the conversion of the matrix potential energy.

The potential energy of the actuator family $\mathcal{F}$ is given by (3.6) and is repeated below,

$$\mathcal{V}_{\mathcal{F}} = \tfrac{1}{2} \int\limits_{\mathcal{R}} Ed(\mathbf{x}) \left(\boldsymbol{\varphi}^{\mathrm{T}}(\mathbf{x}) \boldsymbol{\varepsilon}(\mathbf{x}) \boldsymbol{\varphi}(\mathbf{x}) - \kappa u(\mathbf{x}) \right)^2 dV. \tag{4.14}$$

We saw in Theorem 4.1 that the rectangular strain matrix transforms according to (4.8),

$$\boldsymbol{\varepsilon} = \mathbf{R}\, \boldsymbol{\varepsilon}_s \mathbf{R}^{-1}. \tag{4.15}$$

We now concern ourselves with the quantities u, d, and $\boldsymbol{\varphi}$.

The spherical control will be denoted by $u_{\mathcal{F}}$ to emphasize its association with the actuator family $\mathcal{F}$. We have $u_{\mathcal{F}}(\boldsymbol{\varrho}) = u(\mathbf{x}(\boldsymbol{\varrho}))$, where $\boldsymbol{\varrho} \equiv (r \quad \theta \quad \phi)^{\mathrm{T}}$ and we write $\mathbf{x}(\boldsymbol{\varrho})$ to emphasize that x, y, and z are considered as functions of r, θ, and ϕ, cf. (4.1).

Now recall from §3.3 that the distribution density, d, and the orientation, $\boldsymbol{\varphi}$, are defined in terms of a vector field, $\mathbf{F}$, as

$$d(\mathbf{x}) = ||\mathbf{F}(\mathbf{x})|| \tag{4.16}$$

$$\boldsymbol{\varphi}(\mathbf{x}) = \frac{\mathbf{F}(\mathbf{x})}{||\mathbf{F}(\mathbf{x})||}. \tag{4.17}$$

We assume that $\mathbf{F}(\mathbf{x}) = f_x(\mathbf{x})\mathbf{i} + f_y(\mathbf{x})\mathbf{j} + f_z(\mathbf{x})\mathbf{k}$ and that its counterpart in spherical coordinates is $\mathbf{F}_s(\boldsymbol{\varrho}) = f_r(\boldsymbol{\varrho})\hat{\mathbf{e}}_r + f_\theta(\boldsymbol{\varrho})\hat{\mathbf{e}}_\theta + f_\phi(\boldsymbol{\varrho})\hat{\mathbf{e}}_\phi$. Since $\mathbf{F}$ and $\mathbf{F}_s$ are two different representations of the same vector, we have that $||\mathbf{F}|| = ||\mathbf{F}_s||$ and so

$$\begin{aligned} d(\mathbf{x}(\boldsymbol{\varrho})) &= \sqrt{f_x^2(\mathbf{x}(\boldsymbol{\varrho})) + f_y^2(\mathbf{x}(\boldsymbol{\varrho})) + f_z^2(\mathbf{x}(\boldsymbol{\varrho}))} \\ &= \sqrt{f_r^2(\boldsymbol{\varrho}) + f_\theta^2(\boldsymbol{\varrho}) + f_\phi^2(\boldsymbol{\varrho})} \\ &\equiv d_{\mathcal{F}}(\boldsymbol{\varrho}). \end{aligned}$$

Since $\boldsymbol{\varphi}(\mathbf{x})$ is a unit vector which describes the orientation of actuators near $\mathbf{x}$ it is convenient to view its components $\varphi_x(\mathbf{x})$, $\varphi_y(\mathbf{x})$, and $\varphi_z(\mathbf{x})$ as direction cosines. In other words, $\varphi_x(\mathbf{x})$, $\varphi_y(\mathbf{x})$, $\varphi_z(\mathbf{x})$ are the cosines of the angles between the unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, respectively, and the long axis of the actuators near $\mathbf{x}$. Since we will introduce specific actuator families in subsequent chapters with reference to cylindrical and spherical coordinates it is useful to view the orientation *angles* (not their cosines) as the dependent variables. We thus write the spherical orientation vector, denoted $\boldsymbol{\varphi}_{\mathcal{F}}$, in component form as

$$\boldsymbol{\varphi}_{\mathcal{F}}(\boldsymbol{\varrho}) = \cos\alpha(\boldsymbol{\varrho})\ \hat{\mathbf{e}}_r + \cos\beta(\boldsymbol{\varrho})\ \hat{\mathbf{e}}_\theta + \cos\gamma(\boldsymbol{\varrho})\ \hat{\mathbf{e}}_\phi, \tag{4.18}$$

where $\cos^2\alpha(\boldsymbol{\varrho}) + \cos^2\beta(\boldsymbol{\varrho}) + \cos^2\gamma(\boldsymbol{\varrho}) = 1$ and $\alpha(\boldsymbol{\varrho})$, $\beta(\boldsymbol{\varrho})$, $\gamma(\boldsymbol{\varrho})$ are the angles an actuator located at $\boldsymbol{\varrho}$ would make with $\hat{\mathbf{e}}_r$, $\hat{\mathbf{e}}_\theta$, $\hat{\mathbf{e}}_\phi$, respectively. Following the derivation in §4.5, we have

$$\begin{pmatrix} \varphi_x \\ \varphi_y \\ \varphi_z \end{pmatrix} = \mathbf{R} \begin{pmatrix} \cos\alpha \\ \cos\beta \\ \cos\gamma \end{pmatrix}. \tag{4.19}$$

Using (4.15) and (4.19) we calculate

$$\begin{aligned} \boldsymbol{\varphi}^{\mathrm{T}}\boldsymbol{\varepsilon}\boldsymbol{\varphi} &= (\cos\alpha \quad \cos\beta \quad \cos\gamma\ )\mathbf{R}^{-1}\mathbf{R}\ \boldsymbol{\varepsilon}_s\mathbf{R}^{-1}\mathbf{R}\begin{pmatrix} \cos\alpha \\ \cos\beta \\ \cos\gamma \end{pmatrix} \\ &= (\cos\alpha \quad \cos\beta \quad \cos\gamma\ )\boldsymbol{\varepsilon}_s\begin{pmatrix} \cos\alpha \\ \cos\beta \\ \cos\gamma \end{pmatrix} \\ &= (\cos\alpha \quad \cos\beta \quad \cos\gamma\ )\begin{pmatrix} \varepsilon_{rr} & \varepsilon_{r\theta} & \varepsilon_{r\phi} \\ \varepsilon_{\theta r} & \varepsilon_{\theta\theta} & \varepsilon_{\theta\phi} \\ \varepsilon_{\phi r} & \varepsilon_{\phi\theta} & \varepsilon_{\phi\phi} \end{pmatrix}\begin{pmatrix} \cos\alpha \\ \cos\beta \\ \cos\gamma \end{pmatrix} \\ &= \cos^2\alpha\ \varepsilon_{rr} + \cos^2\beta\ \varepsilon_{\theta\theta} + \cos^2\gamma\ \varepsilon_{\phi\phi} \\ &\quad + 2\cos\alpha\cos\beta\ \varepsilon_{r\theta} + 2\cos\beta\cos\gamma\ \varepsilon_{\theta\phi} + 2\cos\gamma\cos\alpha\ \varepsilon_{\phi r}. \end{aligned}$$

Therefore, the integrand of (4.14) can be written as

$$\begin{aligned} Ed\Big(\boldsymbol{\varphi}^{\mathrm{T}}\boldsymbol{\varepsilon}\boldsymbol{\varphi} - \kappa u\Big)^2 &= Ed_{\mathcal{F}}\Big(\cos^2\alpha\ \varepsilon_{rr} + \cos^2\beta\ \varepsilon_{\theta\theta} + \cos^2\gamma\ \varepsilon_{\phi\phi} + 2\cos\alpha\cos\beta\ \varepsilon_{r\theta} \\ &\quad + 2\cos\alpha\cos\beta\ \varepsilon_{r\theta} + 2\cos\beta\cos\gamma\ \varepsilon_{\theta\phi} + 2\cos\gamma\cos\alpha\ \varepsilon_{\phi r} - \kappa u_{\mathcal{F}}\Big)^2. \end{aligned}$$

The potential energy of the actuator family $\mathcal{F}$ in spherical coordinates is thus,

$$\begin{aligned}\mathcal{V}_{\mathcal{F}} &= \tfrac{1}{2}\int\limits_{\mathcal{R}} E_{\mathcal{F}} d_{\mathcal{F}}(\boldsymbol{\varrho})\Big(\cos^2\alpha\ \varepsilon_{rr} + \cos^2\beta\ \varepsilon_{\theta\theta} + \cos^2\gamma\ \varepsilon_{\phi\phi} + 2\cos\alpha\cos\beta\ \varepsilon_{r\theta} \\ &\qquad + 2\cos\beta\cos\gamma\ \varepsilon_{\theta\phi} + 2\cos\gamma\cos\alpha\ \varepsilon_{\phi r} - \kappa_{\mathcal{F}} u_{\mathcal{F}}(\boldsymbol{\varrho})\Big)^2 r^2 \sin\theta dr d\theta d\phi, \quad (4.20)\end{aligned}$$

where we use $E_{\mathcal{F}}$ and $\kappa_{\mathcal{F}}$ in place of E and κ to emphasize that these constants, along with $d_{\mathcal{F}}$, $u_{\mathcal{F}}$, α, β, and γ, may vary from one actuator family to another.

Chapter 5

A Two Dimensional Formation Problem

5.1 Introduction

In this chapter we examine an elastic structure consisting of an elastic solid or matrix, $\mathcal{S}$, with two families of actuators embedded in the solid and one family of actuators attached to the surface of the solid. The equilibrium configuration of the structure is a right circular cylinder of cross sectional radius 1 with a rigid coaxial right circular cylindrical core of cross sectional radius $r_i < 1$. We assume that the axis of the cylinder is coincident with the z-axis and that a state of *plane strain* exists in the cross-sections, i.e. the strains ε_{zz}, ε_{yz}, ε_{zx} are zero. In order to maintain this state of plane strain it is necessary to assume that the cylinder is bounded above and below by fixed planes perpendicular to the z-axis. The equilibrium configuration of the matrix material is therefore an annulus, $\mathcal{A}$, with inner radius r_i and outer radius 1, which is firmly attached at the inner boundary to a rigid core. In this way displacements at the inner boundary vanish and we avoid potential singularities at the origin O. The equilibrium configuration of the matrix, $\mathcal{A}$, the inner boundary, $\mathcal{I}$, and the outer boundary, $\mathcal{O}$, are thus

$$\begin{aligned}
\mathcal{A} &= \{(x,y)|\ r_i^2 < x^2 + y^2 < 1\} \\
\mathcal{I} &= \{(x,y)|\ x^2 + y^2 = r_i\} \\
\mathcal{O} &= \{(x,y)|\ x^2 + y^2 = 1\}.
\end{aligned}$$

Our objective is to use the various actuator families and their corresponding controls to map the outer boundary to a *target configuration*, $\mathcal{T}$. Ultimately we seek general classes of target configurations with respect to a given actuator structure and equilibrium configuration. In what follows we examine a particular class of target configurations suggested by the current cylindrical equilibrium configuration. We hope that this specific example will provide leads in the attempt to identify more general classes of achievable target configurations.

5.2 Polar Coordinates

We use (r, ϕ) to represent the polar coordinates of a point P in $\mathbb{R}^2$ with rectangular coordinates (x, y). Here r is the length of the line segment OP and $\phi \in [0, 2\pi)$ is the angle measured counterclockwise from the positive x-axis to OP. These polar coordinates are obtained from the spherical coordinates introduced in §4.2 by setting $\theta = \frac{\pi}{2}$ in (4.1). In this way we obtain the relationships between rectangular coordinates x, y and polar coordinates r, ϕ:

$$\begin{aligned} x &= r\cos\phi \qquad & y &= r\sin\phi \\ r^2 &= x^2 + y^2 \qquad & \tan\phi &= \frac{y}{x}. \end{aligned} \tag{5.1}$$

Using these polar coordinates the equilibrium configuration, the inner boundary, and the outer boundary are

$$\mathcal{A} = \{(r, \phi) |\ r_i < r < 1\}$$

$$\mathcal{I} = \{(r, \phi) |\ r = r_i\}$$

$$\mathcal{O} = \{(r, \phi) |\ r = 1\},$$

and we seek to achieve a target configuration of the type

$$\mathcal{T} = \{(r, \phi) |\ r = B(\phi)\},$$

where B, the *target function*, is 2π-periodic with $B(\phi) > r_i$ for all $\phi \in [0, 2\pi)$. We write $\mathcal{A}^c$ to denote the closure of $\mathcal{A}$, $\mathcal{A}^c = \{(r, \phi) |\ r_i \leq r \leq 1\}$.

The displacement vector in polar coordinates, $\mathbf{u}_p$, is

$$\mathbf{u}_p(r, \phi) = u_r(r, \phi)\hat{\mathbf{e}}_r + u_\phi(r, \phi)\hat{\mathbf{e}}_\phi, \tag{5.2}$$

where the polar unit vectors $\hat{\mathbf{e}}_r$, $\hat{\mathbf{e}}_\phi$ are obtained from rectangular unit vectors $\mathbf{i}$, $\mathbf{j}$, respectively, by a counterclockwise rotation of angle ϕ in the xy-plane.

We find the polar strain-displacement equations from the corresponding spherical strain-displacement equations, (4.9)-(4.10), by setting $\varepsilon_{r\theta} = \varepsilon_{\theta\theta} = \varepsilon_{\theta\phi} = 0$ and $\theta = \frac{\pi}{2}$,

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r} \qquad \varepsilon_{r\phi} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \phi} - \frac{u_\phi}{r} + \frac{\partial u_\phi}{\partial r}\right) \qquad \varepsilon_{\phi\phi} = \frac{1}{r}\frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r}. \tag{5.3}$$

The stress-strain equations are

$$\begin{aligned}
\sigma_{rr} &= (\lambda + 2\mu)\varepsilon_{rr} + \lambda\varepsilon_{\phi\phi} &(5.4)\\
\sigma_{\phi\phi} &= \lambda\varepsilon_{rr} + (\lambda + 2\mu)\varepsilon_{\phi\phi} &(5.5)\\
\sigma_{zz} &= \lambda(\varepsilon_{rr} + \varepsilon_{\phi\phi}) &(5.6)\\
\sigma_{r\phi} &= 2\mu\varepsilon_{r\phi}. &(5.7)
\end{aligned}$$

The stress σ_{zz}, normal to the cross sections, is a result of the rigid planes bounding the cylinder above and below [12].

Using (4.12) with $\varepsilon_{r\theta} = \varepsilon_{\theta\theta} = \varepsilon_{\theta\phi} = 0$, we find the strain-energy of the matrix in polar coordinates,

$$W = \tfrac{1}{2}\lambda(\varepsilon_{rr} + \varepsilon_{\phi\phi})^2 + \mu\left(\varepsilon_{rr}^2 + \varepsilon_{\phi\phi}^2 + 2\varepsilon_{r\phi}^2\right). \tag{5.8}$$

Elementary manipulations show that (5.8) may be written as,

$$W = \tfrac{1}{2}(\lambda + \mu)(\varepsilon_{rr} + \varepsilon_{\phi\phi})^2 + \mu\left[\tfrac{1}{2}(\varepsilon_{rr} - \varepsilon_{\phi\phi})^2 + 2\varepsilon_{r\phi}^2\right]. \tag{5.9}$$

Using (5.3) in (5.9) and integrating over $\mathcal{A}$ gives us the matrix potential energy,

$$\begin{aligned}
\mathcal{V}_{\mathcal{S}} = \tfrac{1}{2}\int_0^{2\pi}\int_{r_i}^{1}\Bigg\{&(\lambda+\mu)\Big(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\phi}{\partial \phi}\Big)^2 + \mu\Big(\frac{\partial u_r}{\partial r} - \frac{u_r}{r} - \frac{1}{r}\frac{\partial u_\phi}{\partial \phi}\Big)^2 \\
&+ \mu\Big(\frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \phi}\Big)^2\Bigg\} r\,dr\,d\phi.
\end{aligned} \tag{5.10}$$

The potential energy of the actuator family $\mathcal{F}$ in spherical coordinates is given by (4.20). We substitute $\varepsilon_{r\theta} = \varepsilon_{\theta\theta} = \varepsilon_{\theta\phi} = 0$ and $\theta = \frac{\pi}{2}$ to obtain an expression for the potential energy of $\mathcal{F}$ in polar coordinates,

$$\mathcal{V}_{\mathcal{F}} = \tfrac{1}{2}\int_{\mathcal{A}} E_{\mathcal{F}} d_{\mathcal{F}}\Big(\cos^2\alpha\ \varepsilon_{rr} + \cos^2\gamma\ \varepsilon_{\phi\phi} + 2\cos\alpha\cos\gamma\ \varepsilon_{r\phi} - \kappa_{\mathcal{F}} u_{\mathcal{F}}\Big)^2 r\,dr\,d\phi. \tag{5.11}$$

In general, $d_{\mathcal{F}}$, α, γ, and $u_{\mathcal{F}}$ are functions of both r and ϕ, but in the following we assume that $d_{\mathcal{F}} = d_{\mathcal{F}}(r)$.

We refer to the embedded actuator families by their orientation with respect to the equilibrium configuration of the solid. The *radial actuators*, $\mathcal{F}_r$, are parallel to the unit vector $\hat{\mathbf{e}}_r$ and the *circumferential actuators*, $\mathcal{F}_\phi$, are parallel to $\hat{\mathbf{e}}_\phi$. Radial actuators correspond to $\alpha = 0$, $\gamma = \frac{\pi}{2}$ and circumferential actuators correspond to $\alpha = \frac{\pi}{2}$, $\gamma = 0$. To simplify the notation we define

$$\begin{aligned} E_r(r) &\equiv E_{\mathcal{F}_r} d_{\mathcal{F}_r}(r) \qquad & c_r(r,\phi) &\equiv \kappa_{\mathcal{F}_r} u_{\mathcal{F}_r}(r,\phi) \\ E_\phi(r) &\equiv E_{\mathcal{F}_\phi} d_{\mathcal{F}_\phi}(r) \qquad & c_\phi(r,\phi) &\equiv \kappa_{\mathcal{F}_r} u_{\mathcal{F}_r}(r,\phi). \end{aligned}$$

We will refer to c_r as the *radial control* and c_ϕ as the *circumferential control.* We therefore have the potential energies of the radial actuator family, $\mathcal{V}_r$, and the circumferential actuator family, $\mathcal{V}_\phi$, in the form

$$\mathcal{V}_r = \tfrac{1}{2} \int_0^{2\pi} \int_{r_i}^{1} E_r(r) \left(\varepsilon_{rr} - c_r\right)^2 r dr d\phi \tag{5.12}$$

$$\mathcal{V}_\phi = \tfrac{1}{2} \int_0^{2\pi} \int_{r_i}^{1} E_\phi(r) \left(\varepsilon_{\phi\phi} - c_\phi\right)^2 r dr d\phi. \tag{5.13}$$

Substituting from (5.3) we obtain

$$\mathcal{V}_r = \tfrac{1}{2} \int_0^{2\pi} \int_{r_i}^{1} E_r(r) \left(\frac{\partial u_r}{\partial r} - c_r\right)^2 r dr d\phi \tag{5.14}$$

$$\mathcal{V}_\phi = \tfrac{1}{2} \int_0^{2\pi} \int_{r_i}^{1} E_\phi(r) \left(\frac{1}{r}\frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r} - c_\phi\right)^2 r dr d\phi. \tag{5.15}$$

A third actuator family, the *boundary actuators*, will consist of circumferential actuators attached to the outer boundary, $\mathcal{O}$, of the annulus. The potential energy of the boundary actuators, $\mathcal{V}_{\mathcal{O}}$, is (5.15) specialized to the curve $\mathcal{O}$,

$$\mathcal{V}_{\mathcal{O}} = \tfrac{1}{2} \int_0^{2\pi} E_{\mathcal{O}} \left(\frac{\partial u_\phi}{\partial \phi}(1,\phi) + u_r(1,\phi) - c_{\mathcal{O}}(\phi)\right)^2 d\phi. \tag{5.16}$$

Here the constant $E_{\mathcal{O}}$ is the product of the Young's modulus and the density of the boundary actuator material and $c_{\mathcal{O}}$ is the *boundary control.*

The potential energy of the composite structure is the sum of the matrix potential energy and the radial, circumferential, and boundary actuator potential energies,

$$\mathcal{V} = \mathcal{V}_{\mathcal{S}} + \mathcal{V}_r + \mathcal{V}_\phi + \mathcal{V}_{\mathcal{O}}. \tag{5.17}$$

Using (5.10), (5.14), (5.15), and (5.16) in (5.17), we obtain the total potential energy as a function of the polar displacements:

$$\begin{aligned} \mathcal{V} \;=\; & \tfrac{1}{2}\int_0^{2\pi}\int_{r_i}^{1}\Bigg\{(\lambda+\mu)\Big(\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\phi}{\partial \phi}\Big)^2+\mu\Big(\frac{\partial u_r}{\partial r}-\frac{u_r}{r}-\frac{1}{r}\frac{\partial u_\phi}{\partial \phi}\Big)^2 \\ & \qquad +\mu\Big(\frac{\partial u_\phi}{\partial r}-\frac{u_\phi}{r}+\frac{1}{r}\frac{\partial u_r}{\partial \phi}\Big)^2+E_r(r)\Big(\frac{\partial u_r}{\partial r}-c_r\Big)^2 \\ & \qquad +E_\phi(r)\Big(\frac{1}{r}\frac{\partial u_\phi}{\partial \phi}+\frac{u_r}{r}-c_\phi\Big)^2\Bigg\}r\,dr\,d\phi \\ & +\tfrac{1}{2}\int_0^{2\pi}E_{\mathcal{O}}\Big(\frac{\partial u_\phi}{\partial \phi}(1,\phi)+u_r(1,\phi)-c_{\mathcal{O}}\Big)^2 d\phi. \end{aligned} \tag{5.18}$$

5.3 Formation Problem

The problem we address is the following.

The Formation Problem. *Using the radial, circumferential, and boundary actuators and the corresponding controls, c_r, c_ϕ, $c_{\mathcal{O}}$, find displacements u_r and u_ϕ on $\mathcal{A}$ that minimize the potential energy, (5.18), vanish on the inner boundary, $\mathcal{I}$, and map the outer boundary, $\mathcal{O}$, to the target configuration $\mathcal{T}$.*

We obtain necessary conditions for the minimization of (5.18) using the calculus of variations, the result of which is two partial differential equations (PDEs) in u_r and u_ϕ and two natural boundary conditions (NBCs) at $r = 1$. The *radial PDE* and *radial NBC* are

$$\begin{aligned} & \frac{\partial}{\partial r}\Big[r\Big(\lambda+2\mu+E_r(r)\Big)\frac{\partial u_r}{\partial r}\Big]+\frac{\mu}{r}\frac{\partial^2 u_r}{\partial \phi^2}+(\lambda+\mu)\frac{\partial^2 u_\phi}{\partial \phi\partial r} \\ & -\frac{\lambda+3\mu+E_\phi(r)}{r}\left(\frac{\partial u_\phi}{\partial \phi}+u_r\right)+\frac{\mu}{r}u_r+E_\phi(r)c_\phi-\frac{\partial}{\partial r}\Big[rE_r(r)c_r\Big]=0 \end{aligned}$$

$$\Big(\lambda + 2\mu + E_r(1)\Big)\frac{\partial u_r}{\partial r}(1,\phi) + (E_{\mathcal{O}} + \lambda)\left(\frac{\partial u_\phi}{\partial \phi}(1,\phi) + u_r(1,\phi)\right)$$

$$- \, E_{\mathcal{O}} c_{\mathcal{O}}(\phi) - E_r(1) c_r(1,\phi) = 0,$$

and the *circumferential PDE* and *circumferential NBC* are

$$\frac{\lambda + 2\mu + E_\phi(r)}{r}\frac{\partial}{\partial \phi}\left[\frac{\partial u_\phi}{\partial \phi} + u_r\right] + \mu\frac{\partial}{\partial r}\left[r\frac{\partial u_\phi}{\partial r}\right] + (\lambda + \mu)\frac{\partial^2 u_r}{\partial \phi \partial r}$$

$$+ \, \frac{\mu}{r}\left(\frac{\partial u_r}{\partial \phi} - u_\phi\right) - E_\phi(r)\frac{\partial c_\phi}{\partial \phi} = 0$$

$$E_{\mathcal{O}}\frac{\partial^2 u_\phi}{\partial \phi^2}(1,\phi) + \mu\left(u_\phi(1,\phi) - \frac{\partial u_\phi}{\partial r}(1,\phi)\right) + (E_{\mathcal{O}} - \mu)\frac{\partial u_r}{\partial \phi}(1,\phi) - E_{\mathcal{O}}\frac{dc_{\mathcal{O}}}{d\phi} = 0.$$

Since the displacements are presumed to vanish on $\mathcal{I}$ we obtain the two inner boundary conditions (IBCs),

$$u_r(r_i,\phi) = 0$$

$$u_\phi(r_i,\phi) = 0.$$

The problem as stated makes obtaining the outer boundary conditions (OBCs) somewhat more difficult than the IBCs. Without additional restrictions, the requirement that $\mathcal{O} \mapsto \mathcal{T}$ leads to a complicated relation involving the outer boundary displacements, $u_r(1,\phi)$ and $u_\phi(1,\phi)$. We can show that if the point $(1,\phi)$ is displaced to the point $(B(\phi + \delta\phi), \phi + \delta\phi)$, then

$$u_r(1,\phi) + 1 = B(\phi + \delta\phi)\cos\delta\phi$$

$$u_\phi(1,\phi) = B(\phi + \delta\phi)\sin\delta\phi,$$

which are equivalent to

$$u_r(1,\phi) + 1 = \sqrt{B\left(\phi + \tan^{-1}\frac{u_\phi(1,\phi)}{u_r(1,\phi) + 1}\right)^2 - u_\phi(1,\phi)^2}. \tag{5.19}$$

As an alternative to (5.19), we obtain linear OBCs by making the *a priori* assumption that on $\mathcal{O}$ the displacement is purely radial, i.e. $u_\phi(1,\phi)$ vanishes identically on $\mathcal{O}$. We therefore have the OBCs,

$$\begin{gathered} u_r(1,\phi) = B(\phi) - 1 \\ u_\phi(1,\phi) = 0. \end{gathered}$$

We will refer to $B(\phi) - 1$ as the *target displacement.*

We thus seek displacements u_r and u_ϕ on $\mathcal{A}$ that satisfy the following boundary value problem (BVP) consisting of two second order PDEs, two NBCs, two OBCs, and two IBCs,

$$\begin{gathered} \frac{\partial}{\partial r}\left[r\Big(\lambda+2\mu+E_r(r)\Big)\frac{\partial u_r}{\partial r}\right]+\frac{\mu}{r}\frac{\partial^2 u_r}{\partial\phi^2}+(\lambda+\mu)\frac{\partial^2 u_\phi}{\partial\phi\partial r} \\ -\frac{\lambda+3\mu+E_\phi(r)}{r}\left(u_r+\frac{\partial u_\phi}{\partial\phi}\right)+\frac{\mu}{r}u_r+E_\phi(r)c_\phi-\frac{\partial}{\partial r}\Big[rE_r(r)c_r\Big]=0 \end{gathered} \tag{5.20}$$

$$\begin{gathered} \frac{\lambda+2\mu+E_\phi(r)}{r}\frac{\partial}{\partial\phi}\left[\frac{\partial u_\phi}{\partial\phi}+u_r\right]+\mu\frac{\partial}{\partial r}\left[r\frac{\partial u_\phi}{\partial r}\right]+(\lambda+\mu)\frac{\partial^2 u_r}{\partial\phi\partial r} \\ +\frac{\mu}{r}\left(\frac{\partial u_r}{\partial\phi}-u_\phi\right)-E_\phi(r)\frac{\partial c_\phi}{\partial\phi}=0 \end{gathered} \tag{5.21}$$

$$\begin{gathered} \Big(\lambda+2\mu+E_r(1)\Big)\frac{\partial u_r}{\partial r}(1,\phi)+(E_{\mathcal{O}}+\lambda)\left(\frac{\partial u_\phi}{\partial\phi}(1,\phi)+u_r(1,\phi)\right) \\ -E_{\mathcal{O}}c_{\mathcal{O}}(\phi)-E_r(1)c_r(1,\phi)=0 \end{gathered} \tag{5.22}$$

$$E_{\mathcal{O}}\frac{\partial^2 u_\phi}{\partial\phi^2}(1,\phi)+\mu\left(u_\phi(1,\phi)-\frac{\partial u_\phi}{\partial r}(1,\phi)\right)+(E_{\mathcal{O}}-\mu)\frac{\partial u_r}{\partial\phi}(1,\phi)-E_{\mathcal{O}}\frac{dc_{\mathcal{O}}}{d\phi}=0 \tag{5.23}$$

$$u_r(1,\phi) \;=\; B(\phi)-1 \tag{5.24}$$

$$
\begin{aligned}
u_\phi(1,\phi) &= 0 && (5.25)\\
u_r(r_i,\phi) &= 0 && (5.26)\\
u_\phi(r_i,\phi) &= 0. && (5.27)
\end{aligned}
$$

5.4 The Circumferential PDE and BCs

We now focus on the circumferential displacement, u_ϕ, choosing the circumferential control, c_ϕ, and the boundary control, $c_{\mathcal{O}}$, in such a way that the trivial solution, $u_\phi \equiv 0$, uniquely satisfies (5.21), (5.23), (5.25), and (5.27).

We use

$$c_\phi(r,\phi) = \frac{1}{E_\phi(r)}\left[\frac{\lambda+3\mu+E_\phi(r)}{r}u_r(r,\phi) + (\lambda+\mu)\frac{\partial u_r}{\partial r}(r,\phi)\right] \tag{5.28}$$

$$c_{\mathcal{O}}(\phi) = \frac{E_{\mathcal{O}}-\mu}{E_{\mathcal{O}}}u_r(1,\phi), \tag{5.29}$$

for the circumferential and boundary controls in (5.21) and (5.23) to obtain

$$\frac{\lambda+2\mu+E_\phi(r)}{r}\frac{\partial^2 u_\phi}{\partial\phi^2} + \mu\frac{\partial}{\partial r}\left[r\frac{\partial u_\phi}{\partial r}\right] - \frac{\mu}{r}u_\phi = 0 \tag{5.30}$$

$$E_{\mathcal{O}}\frac{\partial^2 u_\phi}{\partial\phi^2}(1,\phi) + \mu\left(u_\phi(1,\phi) - \frac{\partial u_\phi}{\partial r}(1,\phi)\right) = 0. \tag{5.31}$$

The BVP (5.30)-(5.31) along with the boundary conditions $u_\phi(r_i,\phi) = u_\phi(1,\phi) = 0$ clearly admits the trivial solution $u_\phi \equiv 0$. To show that the trivial solution is unique we assume that u_ϕ can be written in separated form as

$$u_\phi(r,\phi) = R(r)\Phi(\phi). \tag{5.32}$$

Substituting (5.32) into (5.30) we obtain the ordinary differential equations (ODEs)

$$\Phi'' + k^2\Phi = 0 \tag{5.33}$$

$$r^2R'' + rR' - f(r)R = 0, \tag{5.34}$$

where the separation constant, k^2, $k = 0, 1, 2, \ . \ . \ . \ \infty$, is chosen to ensure that solutions are 2π-periodic in ϕ and f is defined by

$$f(r) \equiv 1 + k^2 \frac{\lambda + 2\mu + E_\phi(r)}{\mu}.$$

The 2π-periodic solutions of (5.33) are $\cos k\phi$ and $\sin k\phi$, $k \neq 0$ and 1 for $k = 0$. The solutions of (5.34) are therefore the Fourier sine and cosine coefficients of the Fourier series expansion of u_ϕ.

Now if $\Phi \equiv 0$, then $u_\phi \equiv 0$ and we are done. Otherwise, assuming that $\Phi \not\equiv 0$, (5.25) implies that $R(1) = 0$ and (5.31) implies that $R'(1) = 0$. Therefore, with $E_\phi(r)$ continuous on $[r_i, 1]$, the unique solution of (5.34) satisfying $R(1) = R'(1) = 0$ is the trivial solution $R \equiv 0$. The Fourier coefficients of u_ϕ are all identically zero and so $u_\phi \equiv 0$.

5.5 The Radial Boundary Value Problem

We now use (5.28), (5.29), and $u_\phi \equiv 0$ in (5.20) and (5.22) obtaining

$$\frac{\partial}{\partial r}\left[r\Big(\lambda + 2\mu + E_r(r)\Big)\frac{\partial u_r}{\partial r}\right] + \frac{\mu}{r}\frac{\partial^2 u_r}{\partial \phi^2} + (\lambda + \mu)\frac{\partial u_r}{\partial r} + \frac{\mu}{r}u_r - \frac{\partial}{\partial r}\Big[rE_r(r)c_r\Big] = 0$$

$$\Big(\lambda + 2\mu + E_r(1)\Big)\frac{\partial u_r}{\partial r}(1,\phi) + (\lambda + \mu)u_r(1,\phi) - E_r(1)c_r(1,\phi) = 0.$$

We rename the radial control variable in order to write the radial pde in self-adjoint form. The control c_r is replaced by $c_r + \tilde{c}_r$, where $\tilde{c}_r$ is given by

$$\tilde{c}_r(r,\phi) = \frac{\lambda + \mu}{rE_r(r)}u_r(r,\phi).$$

The boundary-value problem we wish to solve is:

$$\frac{\partial}{\partial r}\left[r\Big(\lambda + 2\mu + E_r(r)\Big)\frac{\partial u_r}{\partial r}\right] + \frac{\mu}{r}\frac{\partial^2 u_r}{\partial \phi^2} + \frac{\mu}{r}u_r - \frac{\partial}{\partial r}\Big[rE_r(r)c_r\Big] = 0 \tag{5.35}$$

$$\Big(\lambda + 2\mu + E_r(1)\Big)\frac{\partial u_r}{\partial r}(1,\phi) - E_r(1)c_r(1,\phi) = 0 \tag{5.36}$$

$$u_r(1,\phi) = B(\phi) - 1 \tag{5.37}$$

$$u_r(r_i,\phi) = 0. \tag{5.38}$$

5.6 The Primal and Dual Systems

Separating variables in (5.35), as in §5.4 for (5.30), leads to the consideration of Fourier series expansions for the radial displacement, u_r, radial control, c_r, and target displacement, $B-1$. We therefore expand u_r, c_r, and $B-1$ in Fourier series and use the convergence properties of $B-1$ to establish the convergence of the series for c_r and u_r. We use

$$u_r(r,\phi) = a_0(r) + \sum_{k=1}^{\infty}(a_k(r)\cos k\phi + b_k(r)\sin k\phi)$$

$$c_r(r,\phi) = c_0(r) + \sum_{k=1}^{\infty}(c_k(r)\cos k\phi + d_k(r)\sin k\phi)$$

$$B(\phi) - 1 = \alpha_0 + \sum_{k=1}^{\infty}(\alpha_k\cos k\phi + \beta_k\sin k\phi), \tag{5.39}$$

in (5.35)-(5.38) to obtain, for $k = 0, 1, 2, \ \ldots, \infty$, ordinary differential equations and boundary conditions relating the Fourier cosine coefficients (FCC) of the radial displacement, radial control, and target displacement,

$$\frac{d}{dr}\Big[r\Big(\lambda + 2\mu + E_r(r)\Big)a_k{}'(r)\Big] - \frac{\mu}{r}\Big(k^2 - 1\Big)\,a_k(r) - \frac{d}{dr}\Big[rE_r(r)c_k(r)\Big] = 0 \tag{5.40}$$

$$\Big(\lambda + 2\mu + E_r(1)\Big)a_k{}'(1) - E_r(1)c_k(1) = 0 \tag{5.41}$$

$$a_k(1) = \alpha_k \tag{5.42}$$

$$a_k(r_i) = 0. \tag{5.43}$$

The Fourier sine coefficients satisfy the same equations with a_k replaced by b_k, c_k by d_k, and α_k by β_k. We refer to (5.40)-(5.43) as the *primal system.*

Now the homogeneous version of the primal differential equation, (5.40), is self-adjoint. We let σ_k denote the solution of the adjoint or *dual system* defined by:

$$\frac{d}{dr}\Big[r\Big(\lambda + 2\mu + E_r(r)\Big)\sigma_k{}'(r)\Big] - \frac{\mu}{r}\Big(k^2 - 1\Big)\,\sigma_k(r) = 0 \tag{5.44}$$

$$\sigma_k{}'(1) = 1 \tag{5.45}$$

$$\sigma_k(r_i) = 0. \tag{5.46}$$

Multiplying the primal differential equation by σ_k, integrating from r_i to 1, and using integration by parts as needed we find an expression for the FCC of the target displacement,

$$\alpha_k = \frac{1}{\lambda + 2\mu + E_r(1)} \int_{r_i}^{1} {\sigma_k}'(\rho) E_r(\rho) c_k(\rho) \rho d\rho. \tag{5.47}$$

We now set $c_k(r) = \xi_k {\sigma_k}'(r)$, where ξ_k is a constant, in (5.47) to obtain

$$\xi_k = \frac{\lambda + 2\mu + E_r(1)}{\int_{r_i}^{1} {\sigma_k}'(\rho)^2 E_r(\rho) \rho d\rho} \alpha_k.$$

The FCC of the radial control must therefore satisfy

$$c_k(r) = \frac{\lambda + 2\mu + E_r(1)}{\int_{r_i}^{1} {\sigma_k}'(\rho)^2 E_r(\rho) \rho d\rho} \, \alpha_k {\sigma_k}'(r). \tag{5.48}$$

This tells us how the FCC of the target displacement, α_k, are related to the FCC of the radial control, c_k, via the solution of the dual system, σ_k.

5.7 Convergence of the Fourier Series for c_r and u_r

We now establish the convergence of the Fourier series of the control c_r in $L^2(\mathcal{A}^c)$ via asymptotic properties of σ_k and α_k. As before we focus on the cosine coefficients.

Using Parseval's Identity we calculate the square of the L^2 norm of the radial control,

$$\begin{aligned} ||c_r||^2_{L^2(\mathcal{A}^c)} &\equiv \int_0^{2\pi} \int_{r_i}^{1} c_r(r,\phi)^2 r dr d\phi \\ &= 2\pi \int_{r_i}^{1} c_0(r)^2 r dr + \pi \sum_{k=1}^{\infty} \left\{ \int_{r_i}^{1} c_k(r)^2 r dr + \int_{r_i}^{1} d_k(r)^2 r dr \right\}. \end{aligned} \tag{5.49}$$

Using (5.48) we have

$$\sum_{k=0}^{\infty}\int_{r_i}^{1} c_k(r)^2 r dr = \sum_{k=0}^{\infty}\left(\frac{\big(\lambda+2\mu+E_r(1)\big)\alpha_k}{\int_{r_i}^{1}\sigma_k'(r)^2 E_r(r) r dr}\right)^2 \int_{r_i}^{1}\sigma_k'(r)^2 r dr. \tag{5.50}$$

We do not proceed with the general case at this time, but assume that $E_r(r) \equiv E_r$, a constant, in order to obtain an explicit asymptotic relationship between the norm of the radial control, c_r, and the Fourier coefficients of the target displacement. Using $E_r(r) \equiv E_r$ in (5.50) we obtain

$$\sum_{k=0}^{\infty}\int_{r_i}^{1} c_k(r)^2 r dr = \left(\frac{\lambda+2\mu+E_r}{E_r}\right)^2 \sum_{k=0}^{\infty}\frac{\alpha_k^2}{\int_{r_i}^{1}\sigma_k'(r)^2 r dr}. \tag{5.51}$$

Note that later we will use the fact that the above expression holds with d_k and β_k in place of c_k and α_k, respectively.

We therefore need to establish a lower bound on the square of the $L^2([r_i,1])$ norm of σ_k',

$$||\sigma_k'||^2_{L^2([r_i,1])} \equiv \int_{r_i}^{1}\sigma_k'(r)^2 r dr.$$

We obtain the needed bound by solving the dual system with $E_r(r) \equiv E_r$ for σ_k. The dual differential equation, (5.44), becomes an Euler equation,

$$r^2\sigma_k'' + r\sigma_k' - \epsilon_k^2\sigma_k = 0, \tag{5.52}$$

where

$$\epsilon_k \equiv \epsilon\sqrt{k^2-1} \quad \text{and} \quad \epsilon \equiv \sqrt{\frac{\mu}{\lambda+2\mu+E_r}}. \tag{5.53}$$

A fundamental set of solutions to (5.52) for $k = 1$ is $\{1, \ln r\}$ and for $k \neq 1$ we obtain $\{r^{\epsilon_k}, r^{-\epsilon_k}\}$. It is therefore convenient to use the fundamental set of solutions $\{\cosh(\epsilon_k \ln r), \sinh(\epsilon_k \ln r)/\epsilon_k\}$ which is valid (in the limiting sense when $k = 1$) for all $k = 0, 1, 2, \ldots, \infty$. Now the solution of (5.52) satisfying the boundary conditions (5.45), (5.46) can be written

$$\sigma_k(r) = \frac{\sinh\left(\epsilon_k \ln \frac{r}{r_i}\right)}{\epsilon_k \cosh(\epsilon_k \ln r_i)}, \tag{5.54}$$

from which we calculate σ_k' and $||\sigma_k'||^2_{L^2([r_i,1])}$,

$$\sigma_k{}'(r) = \frac{\cosh\left(\epsilon_k \ln \frac{r}{r_i}\right)}{r\cosh(\epsilon_k \ln r_i)} \tag{5.55}$$

$$||\sigma_k{}'||^2_{L^2([r_i,1])} = -\frac{\epsilon_k \ln r_i + \cosh(\epsilon_k \ln r_i)\sinh(\epsilon_k \ln r_i)}{2\epsilon_k \cosh^2(\epsilon_k \ln r_i)}. \tag{5.56}$$

Note that since $r_i < 1$ the right hand side of (5.56) is indeed positive and the expressions (5.54)-(5.56) hold (in the limiting sense when $k = 1$) for all integers $k \geq 0$.

We now establish the lower bound on $||\sigma_k{}'||^2_{L^2(\mathcal{A}^c)}$. Since

$$2||\sigma_k{}'||^2_{L^2([r_i,1])} = \frac{\ln \frac{1}{r_i}}{\cosh^2(\epsilon_k \ln r_i)} + \frac{1}{\epsilon_k}\tanh\left(\epsilon_k \ln \frac{1}{r_i}\right),$$

we calculate

$$\begin{aligned}\cosh^2(\epsilon_k \ln r_i) = \tfrac{1}{4}\left(r_i^{\epsilon_k} + r_i^{-\epsilon_k}\right)^2 \quad &< \quad \tfrac{1}{4}\left(2r_i^{-\epsilon_k}\right)^2 \quad \text{when } k > 1 \\ &= \quad r_i^{-2\epsilon_k}.\end{aligned}$$

Also for $k > 1$, we have

$$\tanh\left(\epsilon_k \ln \frac{1}{r_i}\right) = \frac{r_i^{-\epsilon_k} - r_i^{\epsilon_k}}{r_i^{-\epsilon_k} + r_i^{\epsilon_k}} = \frac{1 - r_i^{2\epsilon_k}}{1 + r_i^{2\epsilon_k}} = 1 - \frac{2r_i^{2\epsilon_k}}{1 + r_i^{2\epsilon_k}} > 1 - 2r_i^{2\epsilon_k}.$$

This gives us

$$\begin{aligned}2||\sigma_k{}'||^2_{L^2([r_i,1])} &= \frac{\ln \frac{1}{r_i}}{\cosh^2(\epsilon_k \ln r_i)} + \frac{1}{\epsilon_k}\tanh\left(\epsilon_k \ln \frac{1}{r_i}\right) \\ &> r_i^{2\epsilon_k} \ln \frac{1}{r_i} + \frac{1}{\epsilon_k}\left(1 - 2r_i^{2\epsilon_k}\right) \\ &= \frac{1}{\epsilon_k} + r_i^{2\epsilon_k}\left(\ln \frac{1}{r_i} - \frac{2}{\epsilon_k}\right).\end{aligned}$$

Now there exists a positive integer $K > 1$ such that $\ln \frac{1}{r_i} - \frac{2}{\epsilon_k} > 0$ for all $k > K$ and so

$$2||\sigma_k{}'||^2_{L^2([r_i,1])} > \frac{1}{\epsilon_k} = \frac{1}{\epsilon\sqrt{k^2 - 1}} > \frac{1}{\epsilon k}, \qquad \forall \;\; k > K. \tag{5.57}$$

Therefore, using (5.57) in (5.51) and the corresponding expression for d_k, we obtain from (5.49) an estimate for the norm of the radial control,

$$||c_r||^2_{L^2(\mathcal{A}^c)} = \int_0^{2\pi}\int_{r_i}^{1} c_r(r,\phi)^2 r dr d\phi < \tilde{M}\sum_{k=0}^{\infty} k\left(\alpha_k^2 + \beta_k^2\right), \tag{5.58}$$

where $\tilde{M} > 0$ is a constant.

It is known [1] that the Sobolev space $H^p([0,2\pi))$ consists of periodic functions,

$$f(\phi) = a_0 + \sum_{k=1}^{\infty}(a_k \cos k\phi + b_k \sin k\phi),$$

for which

$$\sum_{k=1}^{\infty} k^{2p}\left(|a_k|^2 + |b_k|^2\right) < \infty.$$

From (5.58) we see that $B \in H^{1/2}([0,2\pi))$ implies $c_r \in L^2(\mathcal{A}^c)$. Corresponding results for c_ϕ and $c_\mathcal{O}$ follow from (5.28) and (5.29). We have thus established the following theorem.

Theorem 5.1 *Suppose $E_r(r) \equiv E_r$, a constant, let $E_\phi \in C([r_i,1])$ and let target function $B \in H^{1/2}([0,2\pi))$. Then the Formation Problem, cf. §5.3, obtains with radial and circumferential controls $c_r, c_\phi \in L^2(\mathcal{A}^c)$ and boundary control $c_\mathcal{O} \in H^{1/2}([0,2\pi))$.*

Furthermore, assuming that $B \in C^\infty([0,2\pi))$ enables us to establish an exponential decay rate for the $L^2(\mathcal{A}^c)$ norm of c_r. We use integration by parts to calculate α_k, the FCC of the outer boundary displacement $u_r(1,\phi) = B(\phi) - 1$, for $k > 0$ as follows:

$$\begin{aligned}
\pi\alpha_k &= \int_0^{2\pi}(B(\phi)-1)\cos k\phi d\phi \\
&= \frac{1}{k}\int_0^{2\pi} B'(\phi)\sin k\phi d\phi \\
&= -\frac{1}{k^2}\int_0^{2\pi} B''(\phi)\cos k\phi d\phi
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{k^3}\int_0^{2\pi} B'''(\phi)\sin k\phi d\phi \\
&= \ \ldots \\
&= \frac{\cos(n+1)\frac{\pi}{2}+\sin(n+1)\frac{\pi}{2}}{k^n}\int_0^{2\pi} B^{(n)}(\phi)\begin{matrix}\cos k\phi \\ \sin k\phi\end{matrix} d\phi \qquad \begin{matrix} n \text{ even} \\ n \text{ odd}\end{matrix},
\end{aligned}
$$

where n is any positive integer. Therefore, for $k > 0$,

$$|\alpha_k| \le \frac{1}{\pi k^n}\int_0^{2\pi} \left|B^{(n)}(\phi)\right| d\phi,$$

and so there exists an $\varepsilon > 0$ such that

$$\alpha_k = O\left[(1+\varepsilon)^{-k}\right] \quad \text{as } k \to \infty.$$

The same estimate applies to the Fourier sine coefficients, β_k, of the outer boundary displacement and so from (5.58) we obtain

$$||c_r||^2_{L^2} < M\sum_{k=0}^{\infty}\zeta_k^2,$$

where $M > 0$ is a constant and $\zeta_k = O\left[k(1+\varepsilon)^{-k}\right]$ as $k \to \infty$.

Chapter 6

Some Graphical Results

In this chapter we look at some computations based on the results of the proceeding chapter. To that end we must specify several quantities, both variable and constant, and then we solve the equations satisfied by the Fourier coefficients of the radial displacement, u_r, and the radial control, c_r.

6.1 Equilibrium Configuration, Target Configuration, Elastic Constants, and Actuator Densities

First we specify quantities associated with the physical configuration of the elastic structure. In particular, we assume that the inner radius is $r_i = 0.1$ and the target configuration, $\mathcal{T}$, is an ellipse,

$$\begin{aligned} \mathcal{T} &= \Big\{(x, y)|\ x^2 + 4y^2 = 4\Big\} \\ &= \{(r, \phi)|\ r = B(\phi)\}, \end{aligned}$$

where

$$B(\phi) \equiv \frac{2}{\sqrt{\cos^2\phi + 4\sin^2\phi}}. \tag{6.1}$$

We assume that the matrix material is “rubber” [5] with Lamé constants $\lambda = 3250$ N/cm^2 and $\mu = 70$ N/cm^2 and that the actuator material is “quartz” [5], a piezoelectric with Young’s modulus 10^7 N/cm^2. Note that Poisson’s ratio for quartz, $\nu = 0.007$, also from [5], is compatible with our assumption in §3.2 that $\nu \approx 0$. We also assume all families of actuators are embedded with the same (constant) density, $d = 0.01$, so that $E_r = E_\phi = E_{\mathcal{O}} = 10^7 d = 10^5$ N/cm^2.

6.2 The Fourier Coefficients of the Outer Boundary Displacement $B(\phi) - 1$

Recall that the Fourier series expansion of the boundary displacement was expressed in §5.6 as

$$B(\phi) - 1 = \alpha_0 + \sum_{k=1}^{\infty} (\alpha_k \cos k\phi + \beta_k \sin k\phi),$$

where the Fourier coefficients are given by

$$\begin{aligned} \alpha_0 &= \frac{1}{2\pi} \int_0^{2\pi} (B(\phi) - 1) d\phi \\ \alpha_k &= \frac{1}{\pi} \int_0^{2\pi} (B(\phi) - 1) \cos k\phi \; d\phi \\ \beta_k &= \frac{1}{\pi} \int_0^{2\pi} (B(\phi) - 1) \sin k\phi \; d\phi. \end{aligned}$$

The calculation of the Fourier coefficients is greatly simplified by the particular form (6.1), which is graphed in Figure 6.1. We first note that B is an even function, so the Fourier sine coefficients, β_k, are all identically zero. We may also deduce that $\alpha_{2i+1} = 0$ for $i = 0, 1, 2, \ldots, \infty$, as follows.

- The function B and the functions $\cos(2i + 1)\phi$, $i = 0, 1, 2, \ldots, \infty$, are even with respect to (wrt) the line $\phi = \pi$ so that

$$\alpha_{2i+1} = \frac{1}{\pi} \int_0^{2\pi} (B(\phi) - 1) \cos(2i + 1)\phi \; d\phi = \frac{2}{\pi} \int_0^{\pi} (B(\phi) - 1) \cos(2i + 1)\phi \; d\phi.$$

- The function B is even wrt the line $\phi = \frac{\pi}{2}$, while the functions $\cos(2i + 1)\phi$, $i = 0, 1, 2, \ldots, \infty$, are odd, so that the integrands, $(B(\phi) - 1)\cos(2i + 1)\phi$, are odd wrt $\phi = \frac{\pi}{2}$ and thus

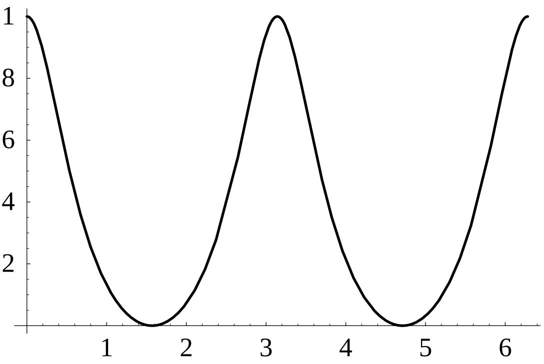

Figure 6.1: The outer boundary displacement $u_r(1,\phi) = B(\phi) - 1$ for $\phi \in [0, 2\pi]$.

$$\int_0^{\pi/2} (B(\phi) - 1)\cos(2i+1)\phi \; d\phi = -\int_{\pi/2}^{\pi} (B(\phi) - 1)\cos(2i+1)\phi \; d\phi.$$

- We therefore obtain

$$\begin{aligned} \alpha_{2i+1} &= \frac{2}{\pi}\int_0^{\pi} (B(\phi) - 1)\cos(2i+1)\phi \; d\phi \\ &= \frac{2}{\pi}\int_0^{\pi/2} (B(\phi) - 1)\cos(2i+1)\phi \; d\phi + \frac{2}{\pi}\int_{\pi/2}^{\pi} (B(\phi) - 1)\cos(2i+1)\phi \; d\phi \\ &= 0. \end{aligned}$$

Similar considerations show that the FCC for $k = 2i$, $i > 0$, can be written

$$\alpha_{2i} = \frac{4}{\pi}\int_0^{\pi/2} (B(\phi) - 1)\cos 2i\phi \; d\phi.$$

Now comparing the graph of the boundary displacement with the graphs of the $n^{\underline{\text{th}}}$ partial sums of its Fourier series,

$$S^n_{B-1}(\phi) \equiv \alpha_0 + \sum_{k=1}^{n} (\alpha_k \cos k\phi + \beta_k \sin k\phi) = \sum_{k=0}^{n/2} \alpha_{2k} \cos 2k\phi,$$

cf. Figure 6.2, we find that $S^{10}_{B-1}(\phi)$ provides a reasonable approximation to $B(\phi) - 1$.

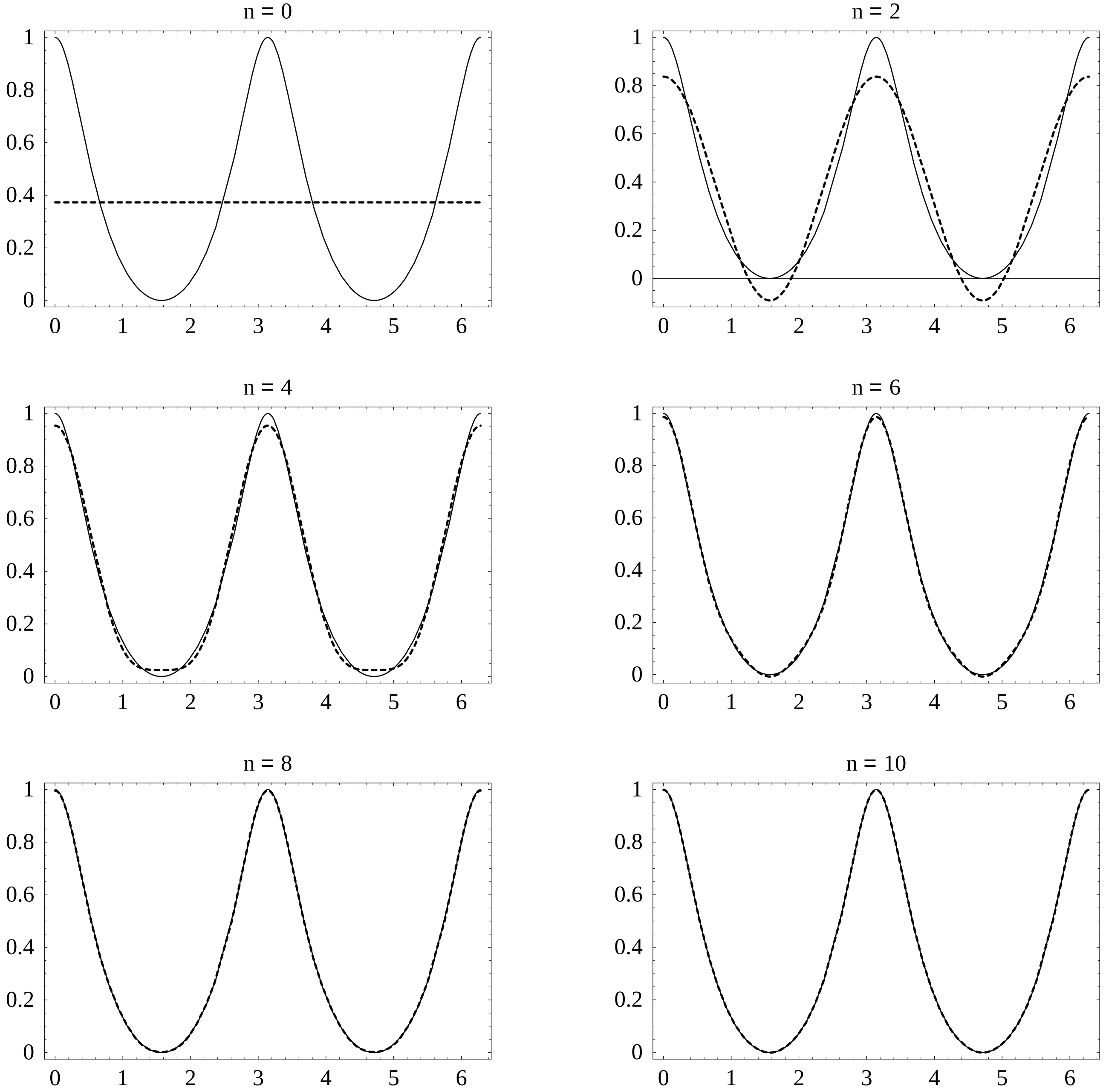

Figure 6.2: Comparison of the partial sums $S^n_{B-1}(\phi)$ and $B(\phi) - 1$ for $n = 0, 2, \ldots, 10$.

6.3 The Fourier Expansions of u_r and c_r

Recall that in §5.6 we expanded u_r and c_r in Fourier series as

$$u_r(r,\phi) = a_0(r) + \sum_{k=1}^{\infty}(a_k(r)\cos k\phi + b_k(r)\sin k\phi)$$

$$c_r(r,\phi) = c_0(r) + \sum_{k=1}^{\infty}(c_k(r)\cos k\phi + d_k(r)\sin k\phi),$$

and showed that the FCC a_k and c_k satisfy the ODE and BCs, (5.40)-(5.43), which for the case $E_r(r) \equiv E_r$ have the form

$$r^2 {a_k}'' + r{a_k}' - \epsilon_k^2 a_k = \frac{E_r}{\lambda + 2\mu + E_r}\frac{d}{dr}\Big[rc_k(r)\Big] \tag{6.2}$$

$${a_k}'(1) = \frac{E_r}{\lambda + 2\mu + E_r}c_k(1) \tag{6.3}$$

$$a_k(1) = \alpha_k \tag{6.4}$$

$$a_k(r_i) = 0, \tag{6.5}$$

where

$$\epsilon_k \equiv \sqrt{\frac{\mu(k^2-1)}{\lambda + 2\mu + E_r}}.$$

Equations (6.2)-(6.5) hold with the sine coefficients b_k, d_k, and β_k in place of a_k, c_k, and α_k, respectively.

We also showed that $c_k(r) \propto \alpha_k$, cf. (5.48), and a similar calculation reveals that $d_k(r) \propto \beta_k$. Now since $\beta_k = 0$ for all k, we have $d_k \equiv 0$ and so b_k satisfies the homogeneous versions of (6.2)-(6.4) and (6.5). This implies that $b_k \equiv 0$ for all k and we need only consider the FCC a_k and c_k.

Now the solution of (6.2)-(6.4) is

$$a_k(r) = \alpha_k \cosh(\epsilon_k \ln r) - \frac{E_r}{\lambda + 2\mu + E_r}\int_r^1 c_k(\rho)\cosh\left(\epsilon_k \ln\frac{\rho}{r}\right)d\rho, \tag{6.6}$$

and in order to satisfy (6.5) we require that

$$\alpha_k \cosh(\epsilon_k \ln r_i) = \frac{E_r}{\lambda + 2\mu + E_r} \int_{r_i}^{1} c_k(\rho) \cosh\left(\epsilon_k \ln \frac{\rho}{r_i}\right) d\rho. \tag{6.7}$$

Note that (6.7) is equivalent to (5.47). We thus find an explicit representation for c_k of the form

$$c_k(r) = \alpha_k \psi_k \frac{\lambda + 2\mu + E_r}{E_r} \, \frac{\cosh\left(\epsilon_k \ln \frac{r}{r_i}\right)}{r}, \tag{6.8}$$

where

$$\psi_k \equiv -\frac{\epsilon_k \cosh(\epsilon_k \ln r_i)}{\epsilon_k \ln r_i + \sinh(\epsilon_k \ln r_i) \cosh(\epsilon_k \ln r_i)}. \tag{6.9}$$

We now use (6.8) for c_k in (6.6) to obtain

$$\begin{aligned} a_k(r) &= \frac{\alpha_k}{2}\Big(2 + \psi_k \cosh(\epsilon_k \ln r_i) \ln r\Big) \cosh(\epsilon_k \ln r) \\ &+ \frac{\alpha_k \psi_k}{2\epsilon_k}\Big(\cosh(\epsilon_k \ln r_i) - \epsilon_k \sinh(\epsilon_k \ln r_i) \ln r\Big) \sinh(\epsilon_k \ln r). \end{aligned}$$

If we denote by $S^n_{(\cdot)}(r,\phi)$ the $n^{\underline{\text{th}}}$ partial sum of the Fourier series for the function $(\cdot)(r,\phi)$, then we have,

$$S^n_{u_r}(r,\phi) \equiv a_0(r) + \sum_{k=1}^{n} (a_k(r) \cos k\phi + b_k(r) \sin k\phi) = \sum_{k=0}^{n/2} a_{2k}(r) \cos 2k\phi$$

$$S^n_{c_r}(r,\phi) \equiv c_0(r) + \sum_{k=1}^{n} (c_k(r) \cos k\phi + d_k(r) \sin k\phi) = \sum_{k=0}^{n/2} c_{2k}(r) \cos 2k\phi.$$

Based on Figure 6.2 and the results from §5.7, choosing $n = 20$ should provide sufficient accuracy.

Now recall that in §5.5 we used the feedback-modified radial control, $c_r \mapsto c_r + \tilde{c}_r$, where $\tilde{c}_r = (\lambda + \mu) u_r / r E_r$. The radial control should therefore be approximated by

$$\tilde{S}^n_{c_r}(r,\phi) = S^n_{c_r}(r,\phi) + \frac{\lambda + \mu}{r E_r} S^n_{u_r}(r,\phi).$$

The graphs of $S^{20}_{u_r}(r,\phi)$ and $\tilde{S}^{20}_{c_r}(r,\phi)$ are shown in Figures 6.3 and 6.4.

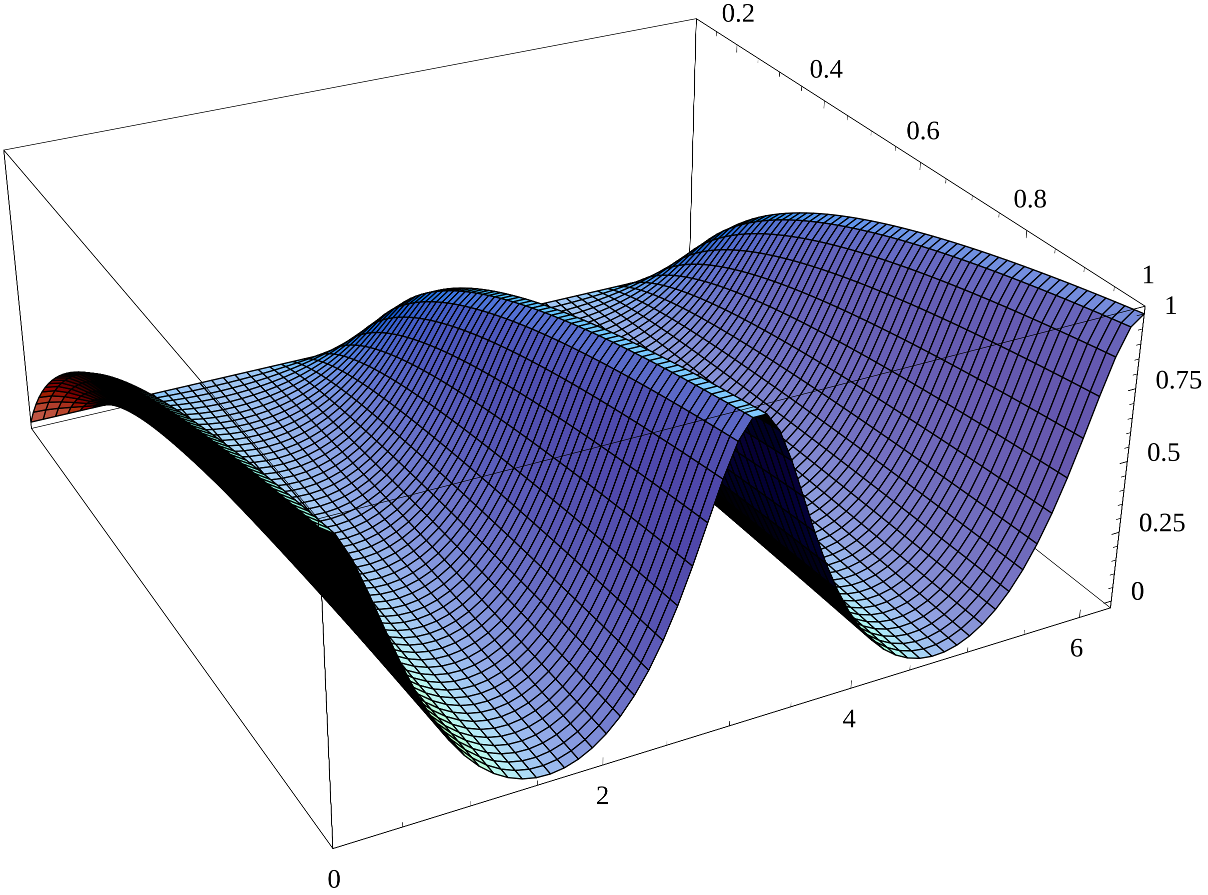

Figure 6.3: The graph of the approximation $S_{u_r}^{20}(r, \phi)$ to the radial displacement, $u_r(r, \phi)$, on $[r_i, 1] \times [0, 2\pi]$.

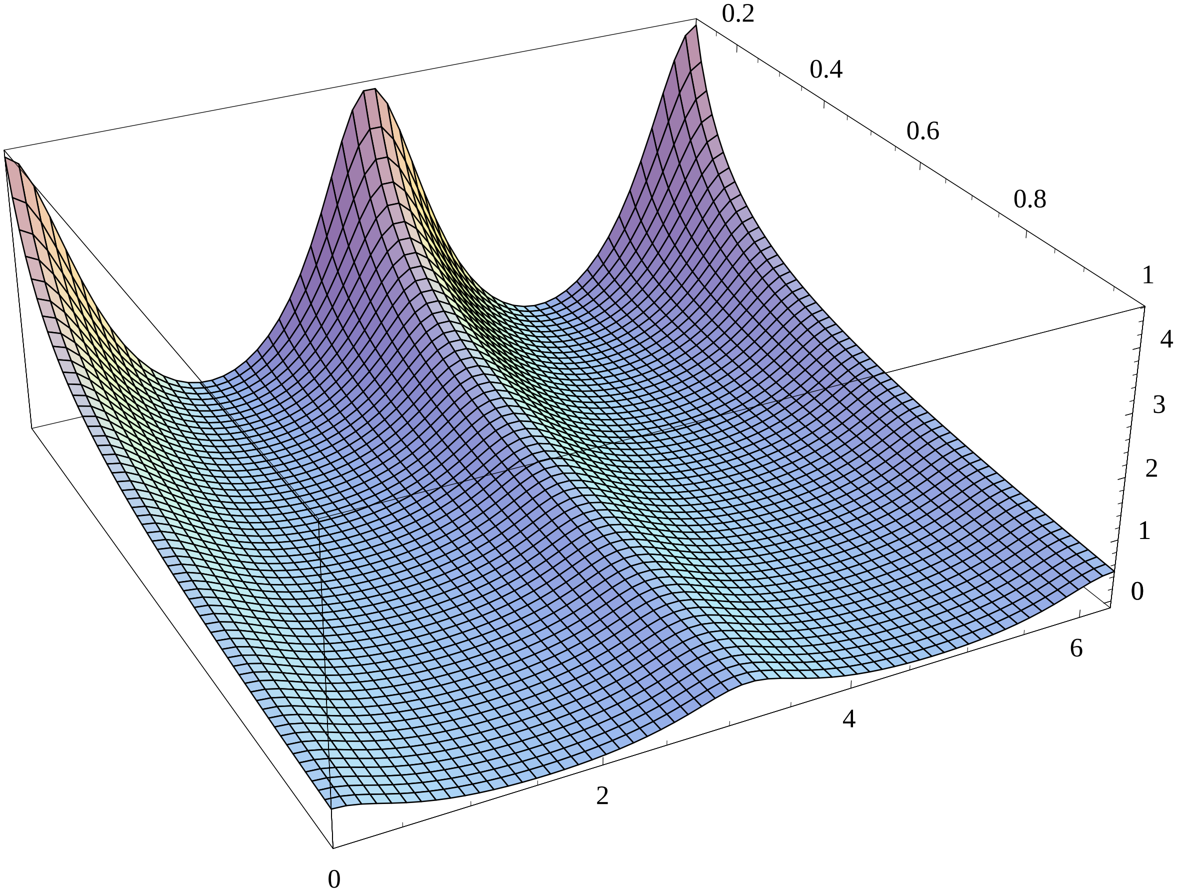

Figure 6.4: The graph of the approximation $\tilde{S}^{20}_{c_r}(r,\phi)$ to the radial control, $c_r(r,\phi)$, on $[r_i,1]\times[0,2\pi]$.

6.4 Other Graphs

We use $S^{20}_{u_r}$ to graph some of the other quantities that depend on u_r. In particular, we look at the circumferential control

$$c_\phi(r,\phi) = \frac{1}{E_\phi}\left[\frac{\lambda + 3\mu + E_\phi}{r} u_r(r,\phi) + (\lambda+\mu)\frac{\partial u_r}{\partial r}(r,\phi)\right],$$

in Figure 6.5, the strains

$$\begin{aligned}
\varepsilon_{rr} &= \frac{\partial u_r}{\partial r} \\
\varepsilon_{r\phi} &= \frac{1}{2r}\frac{\partial u_r}{\partial \phi} \\
\varepsilon_{\phi\phi} &= \frac{u_r}{r},
\end{aligned}$$

in Figures 6.6-6.8, and the stresses

$$\begin{aligned}
\sigma_{rr} &= (\lambda + 2\mu)\varepsilon_{rr} + \lambda\varepsilon_{\phi\phi} \\
\sigma_{r\phi} &= 2\mu\varepsilon_{r\phi} \\
\sigma_{\phi\phi} &= \lambda\varepsilon_{rr} + (\lambda + 2\mu)\varepsilon_{\phi\phi} \\
\sigma_{zz} &= \lambda(\varepsilon_{rr} + \varepsilon_{\phi\phi}),
\end{aligned}$$

in Figures 6.9-6.12.

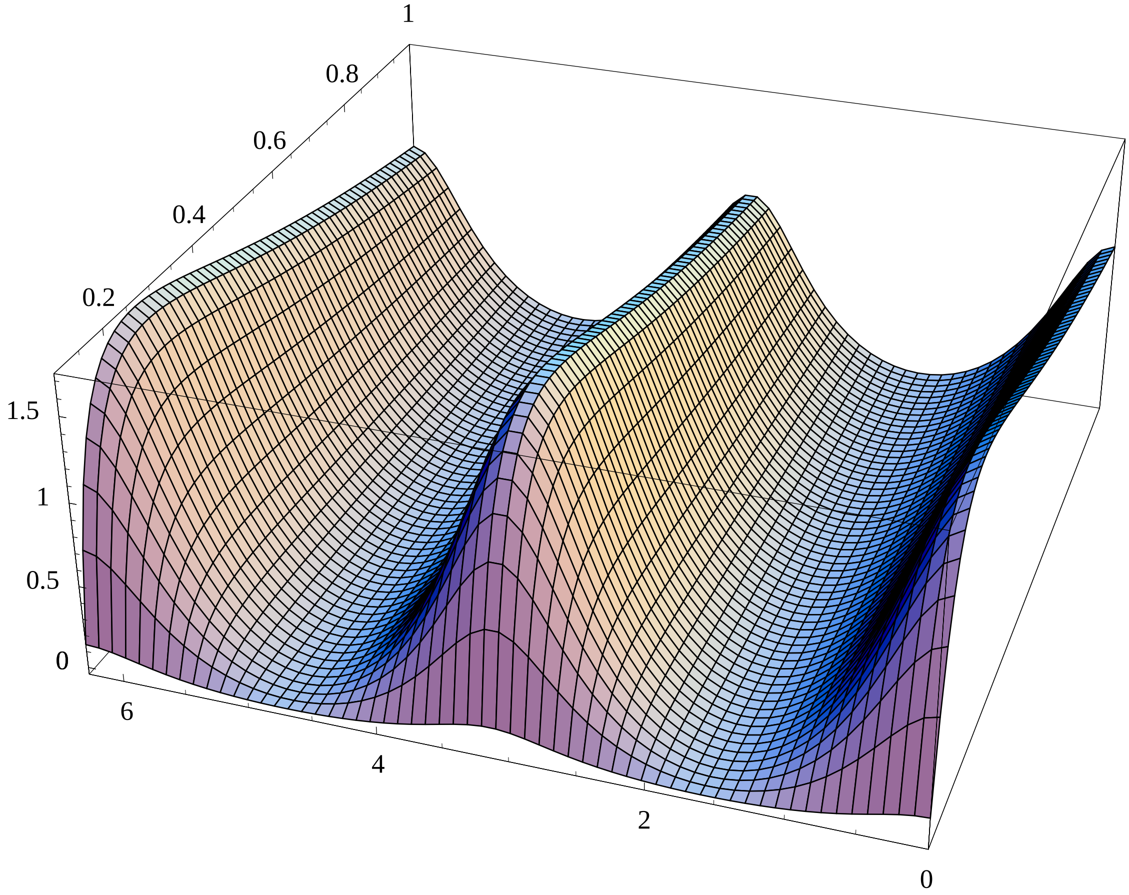

Figure 6.5: An approximate graph of the circumferential control $c_\phi(r, \phi)$ on $[r_i, 1] \times [0, 2\pi]$.

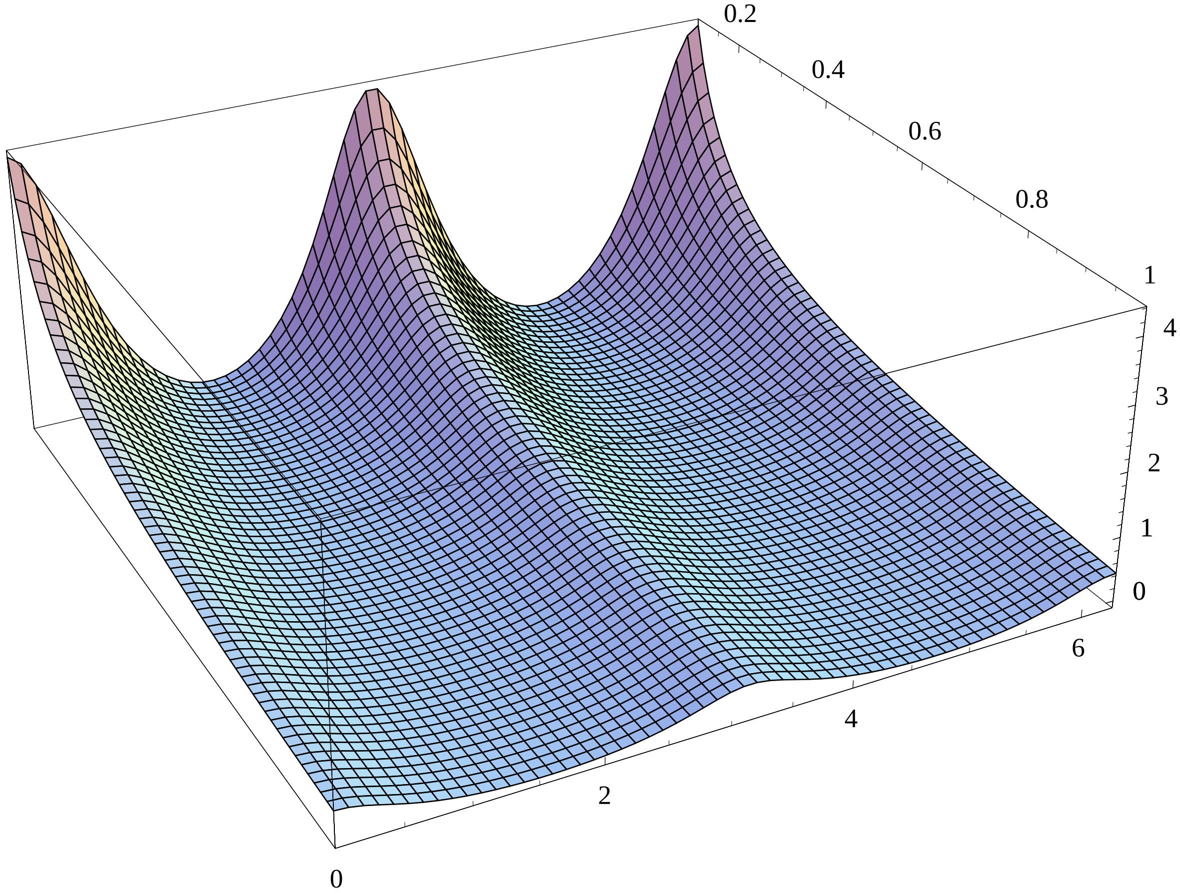

Figure 6.6: An approximate graph of the extensional strain $\varepsilon_{rr}(r, \phi)$ on $[r_i, 1] \times [0, 2\pi]$.

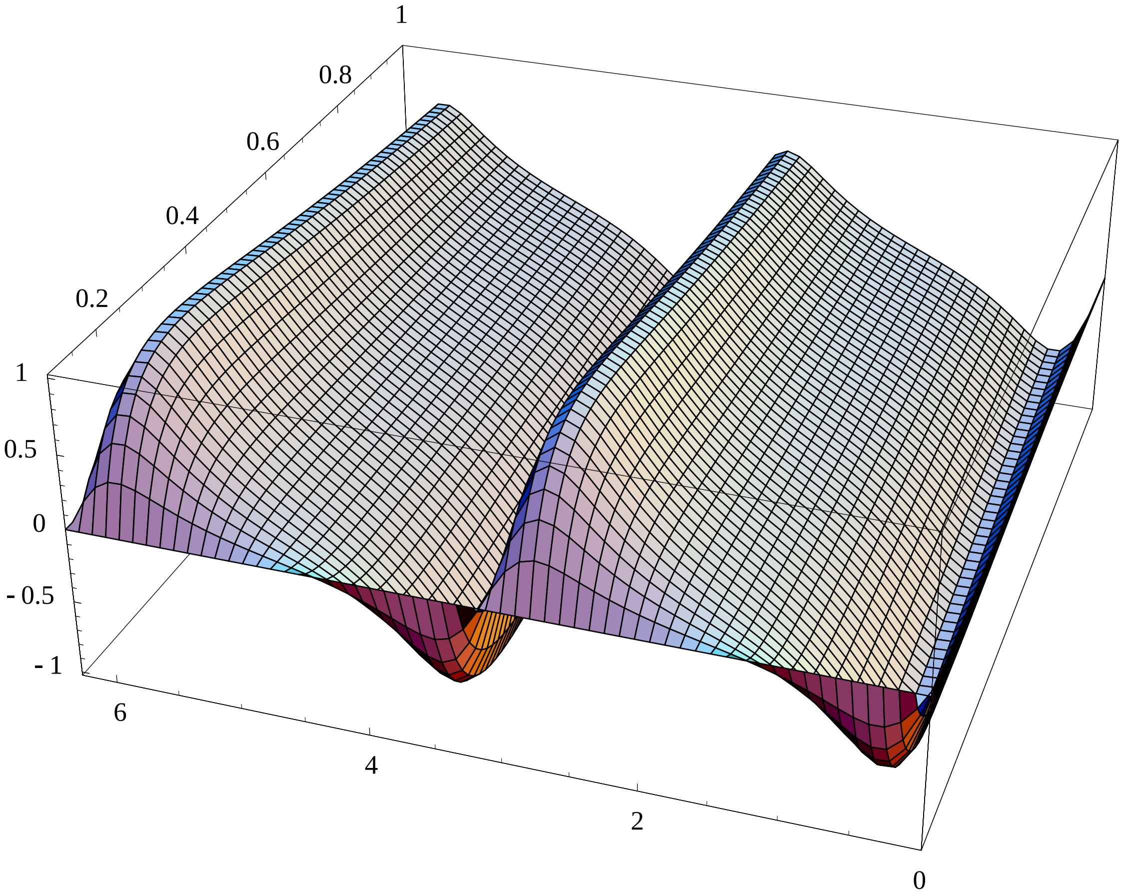

Figure 6.7: An approximate graph of the shear strain $\varepsilon_{r\phi}(r, \phi)$ on $[r_i, 1] \times [0, 2\pi]$.

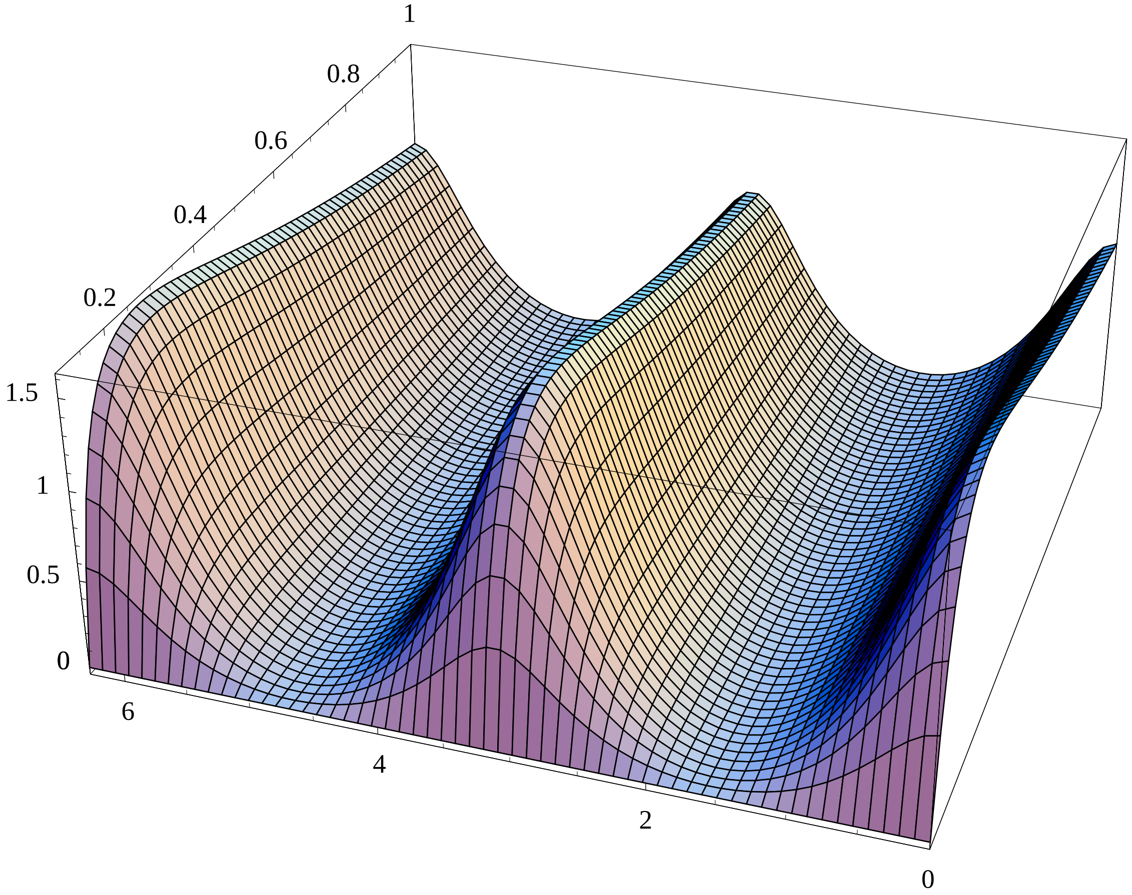

Figure 6.8: An approximate graph of the extensional strain $\varepsilon_{\phi\phi}(r, \phi)$ on $[r_i, 1] \times [0, 2\pi]$.

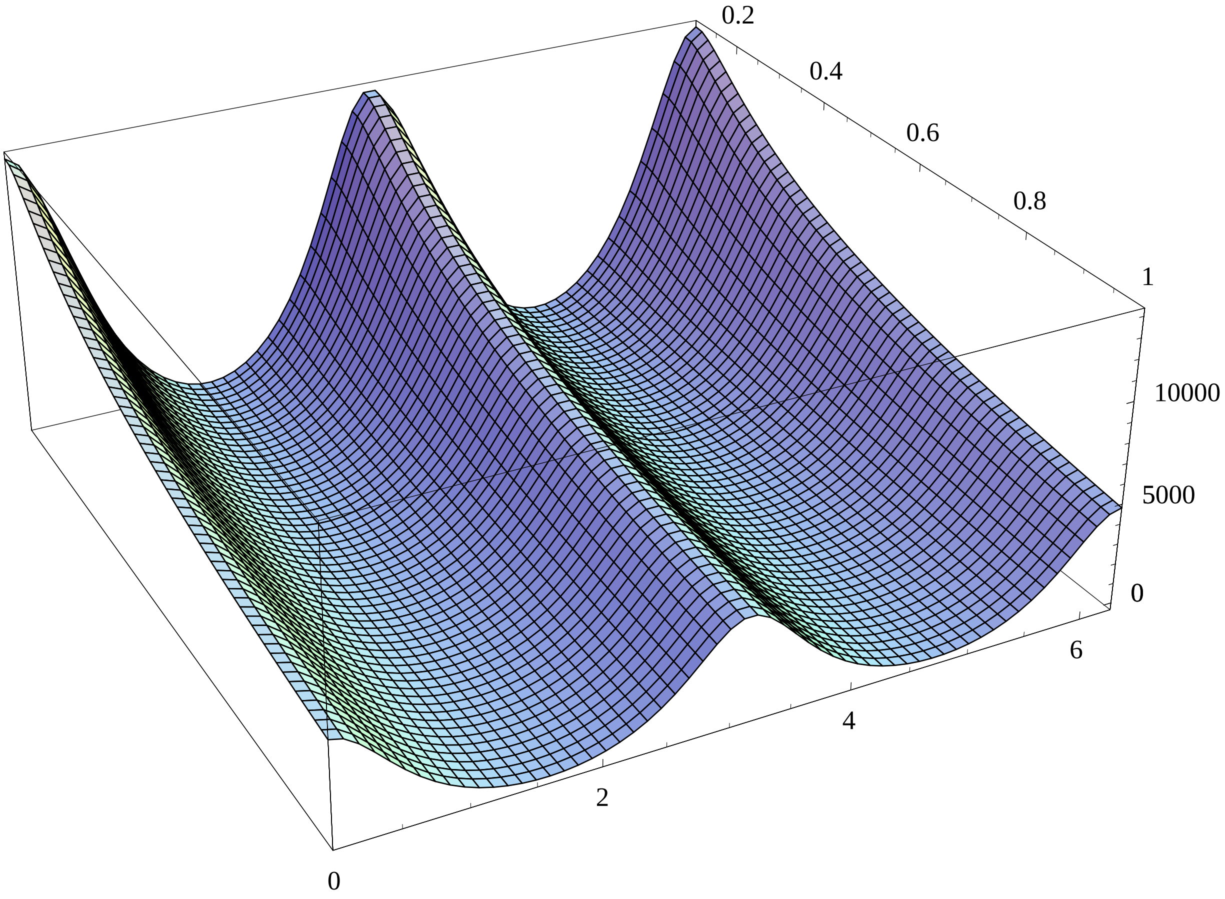

Figure 6.9: An approximate graph of the extensional stress $\sigma_{rr}(r, \phi)$ on $[r_i, 1] \times [0, 2\pi]$.

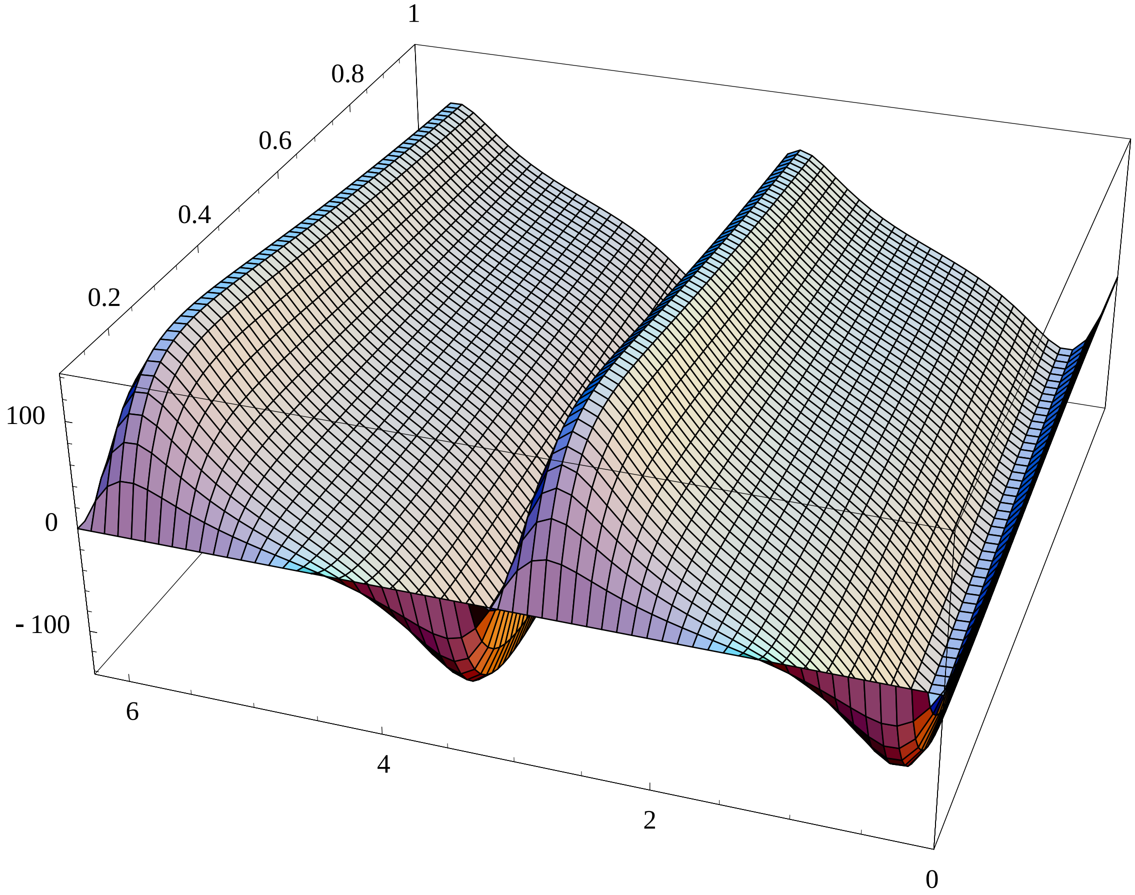

Figure 6.10: An approximate graph of the shear stress $\sigma_{r\phi}(r,\phi)$ on $[r_i, 1] \times [0, 2\pi]$.

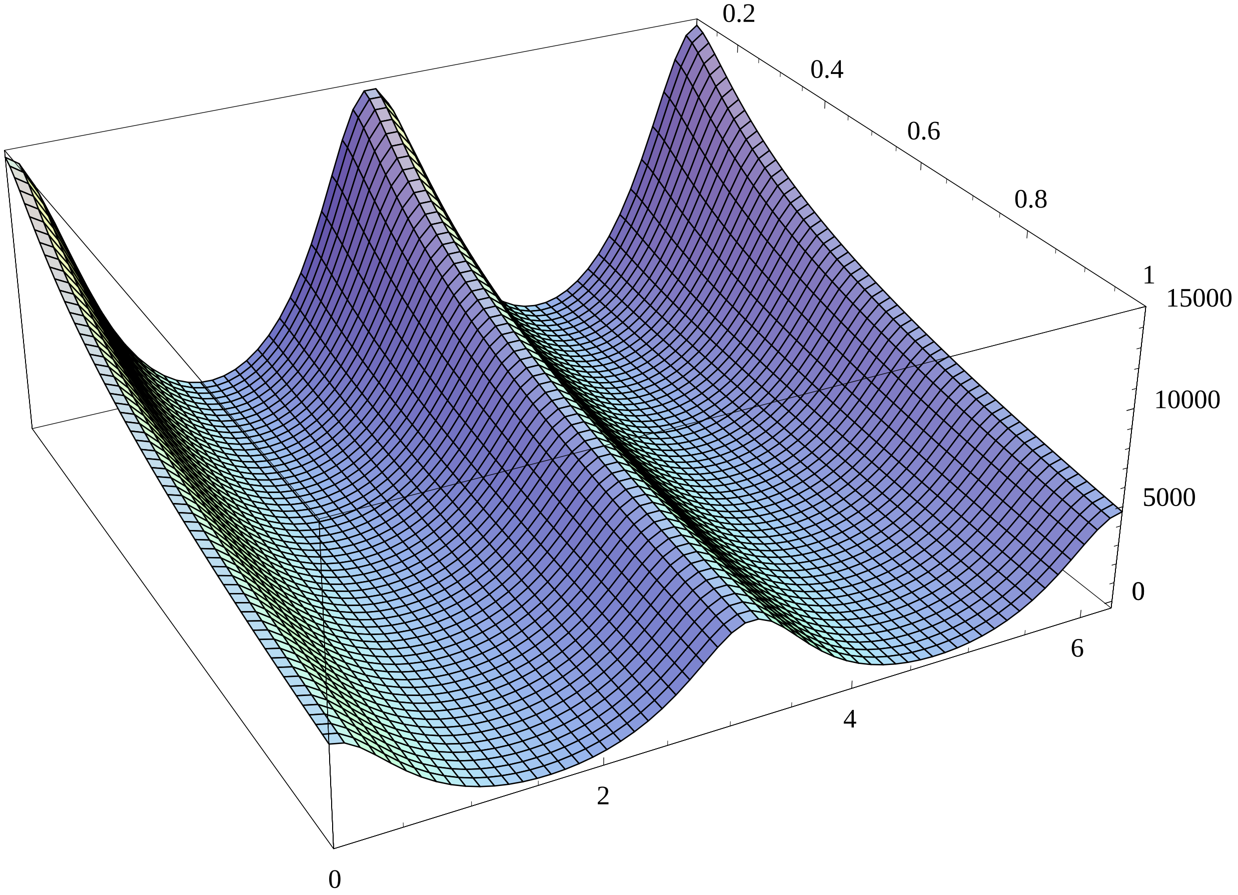

Figure 6.11: An approximate graph of the extensional stress $\sigma_{\phi\phi}(r, \phi)$ on $[r_i, 1] \times [0, 2\pi]$.

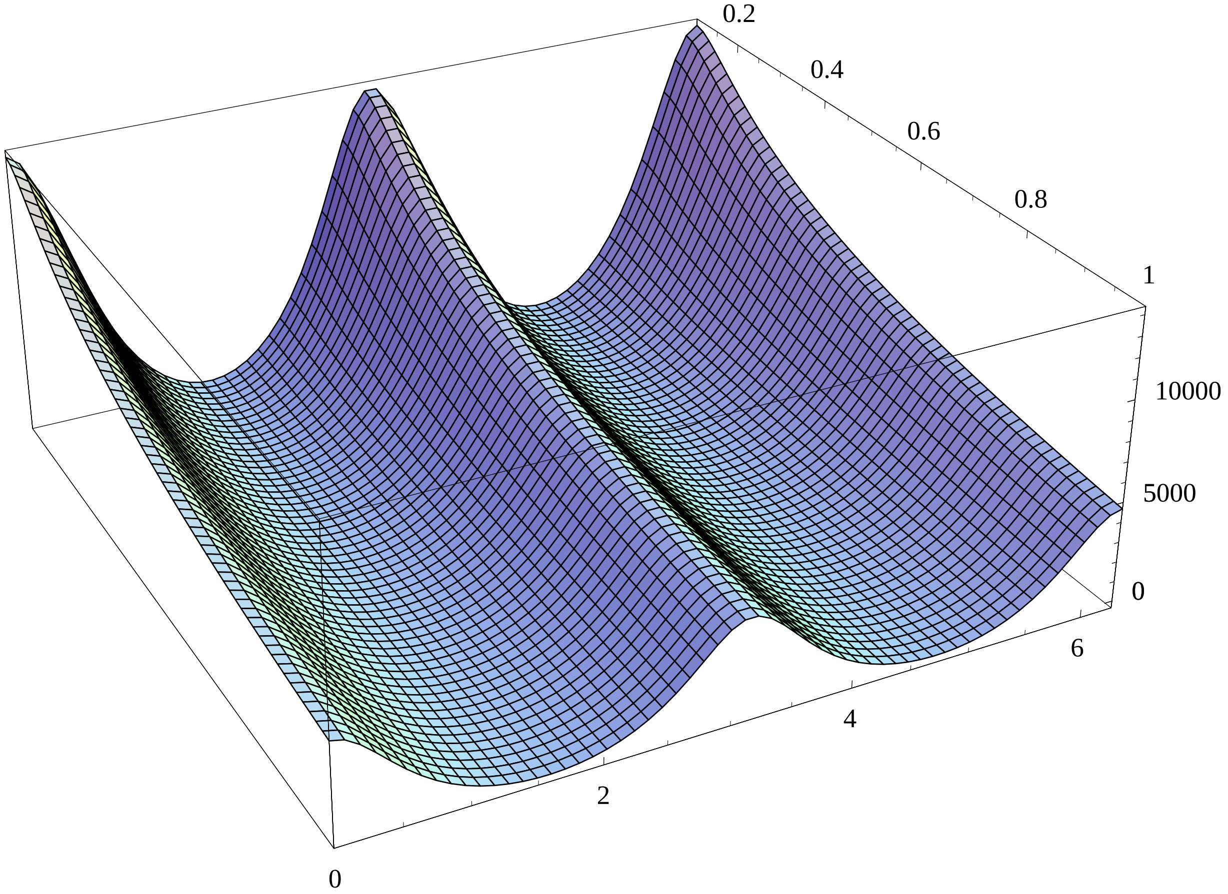

Figure 6.12: An approximate graph of the extensional stress $\sigma_{zz}(r, \phi)$ on $[r_i, 1] \times [0, 2\pi]$.

Chapter 7

The Three Dimensional Problem

7.1 Introduction

Here, as in Chapter 5, we examine an elastic structure consisting of a matrix with attached and embedded actuator families. The equilibrium configuration of the structure is a sphere of radius 1 with a rigid concentric spherical core of radius $r_i < 1$. At equilibrium, the matrix material occupies the region

$$\mathcal{R} = \{(x, y, z)|\ r_i^2 < x^2 + y^2 + z^2 < 1\},$$

and the inner and outer boundaries of the matrix are

$$\mathcal{I} = \{(x, y, z)|\ x^2 + y^2 + z^2 = r_i\}$$
$$\mathcal{O} = \{(x, y, z)|\ x^2 + y^2 + z^2 = 1\}.$$

Using the spherical coordinates defined in §4.2 we have

$$\mathcal{R} = \{(r, \theta, \phi)|\ r_i < r < 1\}$$
$$\mathcal{I} = \{(r, \theta, \phi)|\ r = r_i\}$$
$$\mathcal{O} = \{(r, \theta, \phi)|\ r = 1\},$$

and we use $\mathcal{R}^c$ to denote the closure of $\mathcal{R}$, $\mathcal{R}^c = \{(r, \theta, \phi)|\ r_i \leq r \leq 1\}$.

Our objective is to use the various actuator families and their corresponding controls to map the outer boundary to a target configuration, $\mathcal{T}$, of the form

$$\mathcal{T} = \{(r, \theta, \phi)|\ r = B(\theta, \phi)\},$$

where B is 2π-periodic in ϕ and $B(\theta, \phi) > r_i$ for all $(\theta, \phi) \in [0, \pi] \times [0, 2\pi)$.

7.2 The Matrix Potential Energy

Using (2.21) for the strain-energy with x, y, z replaced by r, θ, ϕ, respectively, and integrating over the equilibrium configuration of the solid we obtain the potential energy of the matrix, $\mathcal{V}_{\mathcal{S}}$, as a function of the strains,

$$\begin{aligned}\mathcal{V}_{\mathcal{S}} \;=\; & \tfrac{1}{2}\int_0^{2\pi}\int_0^{\pi}\int_{r_i}^{1}\Big\{\left(\lambda+\tfrac{2}{3}\mu\right)\left(\varepsilon_{rr}+\varepsilon_{\theta\theta}+\varepsilon_{\phi\phi}\right)^2+\tfrac{2}{3}\mu\left[\left(\varepsilon_{rr}-\varepsilon_{\theta\theta}\right)^2+\left(\varepsilon_{\theta\theta}-\varepsilon_{\phi\phi}\right)^2+\left(\varepsilon_{\phi\phi}-\varepsilon_{rr}\right)^2\right]\\ & \qquad + 4\mu\left(\varepsilon_{r\theta}^2+\varepsilon_{\theta\phi}^2+\varepsilon_{\phi r}^2\right)\Big\}r^2\sin\theta\, dr\, d\theta\, d\phi.\end{aligned}$$

We obtain the potential energy of the matrix as a function of the displacements using the spherical strain-displacement equations, (4.9)-(4.10),

$$\begin{aligned}\mathcal{V}_{\mathcal{S}} \;=\; & \tfrac{1}{2}\int_0^{2\pi}\int_0^{\pi}\int_{r_i}^{1}\Bigg\{\left(\lambda+\tfrac{2}{3}\mu\right)\left(\frac{\partial u_r}{\partial r}+2\frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\cot\theta\frac{u_\theta}{r}+\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi}\right)^2\\ & + \tfrac{2}{3}\mu\Bigg[\left(\frac{\partial u_r}{\partial r}-\frac{u_r}{r}-\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}\right)^2+\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}-\cot\theta\frac{u_\phi}{r}-\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi}\right)^2\\ & \qquad + \left(\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi}+\cot\theta\frac{u_\theta}{r}+\frac{u_r}{r}-\frac{\partial u_r}{\partial r}\right)^2\Bigg] \qquad\qquad (7.1)\\ & + \mu\Bigg[\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta}-\frac{u_\theta}{r}+\frac{\partial u_\theta}{\partial r}\right)^2+\left(\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi}-\cot\theta\frac{u_\phi}{r}+\frac{1}{r}\frac{\partial u_\phi}{\partial \theta}\right)^2\\ & \qquad + \left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \phi}-\frac{u_\phi}{r}+\frac{\partial u_\phi}{\partial r}\right)^2\Bigg]\Bigg\}r^2\sin\theta\, dr\, d\theta\, d\phi.\end{aligned}$$

7.3 Actuator Families

As in the two dimensional case, each embedded actuator family consists of actuators aligned with a particular unit vector and each attached actuator family corresponds to a unit vector tangent to the equilibrium configuration of the outer boundary $\mathcal{O}$. The three embedded actuator families are the *radial actuators*, $\mathcal{F}_r$, parallel to $\hat{\mathbf{e}}_r$, the *latitudinal actuators*, $\mathcal{F}_\theta$, parallel to $\hat{\mathbf{e}}_\theta$, and the *longitudinal actuators*, $\mathcal{F}_\phi$, parallel to $\hat{\mathbf{e}}_\phi$. The two attached actuator families are the *latitudinal boundary actuators*, $\mathcal{F}_{\mathcal{O}_\theta}$, and the *longitudinal boundary actuators*, $\mathcal{F}_{\mathcal{O}_\phi}$, parallel to $\hat{\mathbf{e}}_\theta$ and $\hat{\mathbf{e}}_\phi$, respectively, on $\mathcal{O}$.

In §4.8 we found the potential energy of the actuator family $\mathcal{F}$ in spherical coordinates,

$$\begin{aligned}\mathcal{V}_{\mathcal{F}} \;=\; & \tfrac{1}{2}\int\limits_{\mathcal{R}} E_{\mathcal{F}} d_{\mathcal{F}}\Big(\cos^2\alpha\ \varepsilon_{rr} + \cos^2\beta\ \varepsilon_{\theta\theta} + \cos^2\gamma\ \varepsilon_{\phi\phi} + 2\cos\alpha\cos\beta\ \varepsilon_{r\theta} \\ & + 2\cos\beta\cos\gamma\ \varepsilon_{\theta\phi} + 2\cos\gamma\cos\alpha\ \varepsilon_{\phi r} - \kappa_{\mathcal{F}} u_{\mathcal{F}}\Big)^2 r^2 \sin\theta dr d\theta d\phi, \qquad (7.2)\end{aligned}$$

where $E_{\mathcal{F}}$ is the Young's modulus of the actuator material, $d_{\mathcal{F}}$ is the density of actuator material, α, β, and γ are the orientation angles with respect to the spherical unit vectors $\hat{\mathbf{e}}_r$, $\hat{\mathbf{e}}_\theta$, $\hat{\mathbf{e}}_\phi$, and $\kappa_{\mathcal{F}} u_{\mathcal{F}}$ is the scaled control.

For the radial actuators we have $\alpha = 0$, $\beta = \gamma = \frac{\pi}{2}$, for latitudinal actuators $\beta = 0$, $\gamma = \alpha = \frac{\pi}{2}$, and latitudinal actuators correspond to $\gamma = 0$, $\alpha = \beta = \frac{\pi}{2}$. Using the notation

$$\begin{aligned} E_r(r) &\equiv E_{\mathcal{F}_r} d_{\mathcal{F}_r}(r) \qquad & c_r(r,\theta,\phi) &\equiv \kappa_{\mathcal{F}_r} u_{\mathcal{F}_r}(r,\theta,\phi) \\ E_\theta(r) &\equiv E_{\mathcal{F}_\theta} d_{\mathcal{F}_\theta}(r) \qquad & c_\theta(r,\theta,\phi) &\equiv \kappa_{\mathcal{F}_\theta} u_{\mathcal{F}_\theta}(r,\theta,\phi) \\ E_\phi(r) &\equiv E_{\mathcal{F}_\phi} d_{\mathcal{F}_\phi}(r) \qquad & c_\phi(r,\theta,\phi) &\equiv \kappa_{\mathcal{F}_\phi} u_{\mathcal{F}_\phi}(r,\theta,\phi), \end{aligned}$$

in (7.2) and using the spherical strain-displacement equations, we obtain the potential energies, $\mathcal{V}_r$, $\mathcal{V}_\theta$, and $\mathcal{V}_\phi$, of the radial, latitudinal, and longitudinal actuator families,

$$\mathcal{V}_r = \tfrac{1}{2}\int\limits_0^{2\pi}\int\limits_0^{\pi}\int\limits_{r_i}^{1} E_r(r)\left(\frac{\partial u_r}{\partial r} - c_r\right)^2 r^2 \sin\theta dr d\theta d\phi \qquad (7.3)$$

$$\mathcal{V}_\theta = \tfrac{1}{2}\int\limits_0^{2\pi}\int\limits_0^{\pi}\int\limits_{r_i}^{1} E_\theta(r)\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} - c_\theta\right)^2 r^2 \sin\theta dr d\theta d\phi \qquad (7.4)$$

$$\mathcal{V}_\phi = \tfrac{1}{2}\int\limits_0^{2\pi}\int\limits_0^{\pi}\int\limits_{r_i}^{1} E_\phi(r)\left(\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \cot\theta\frac{u_\theta}{r} + \frac{u_r}{r} - c_\phi\right)^2 r^2 \sin\theta dr d\theta d\phi. \qquad (7.5)$$

We call c_r the *radial control*, c_θ the *latitudinal control*, and c_ϕ the *longitudinal control.*

The potential energy expressions for the attached actuator families are found from the corresponding expressions for the embedded families specialized to the surface $\mathcal{O}$. Using (7.4) we find the potential energy of the latitudinal boundary actuator family, $\mathcal{V}_{\mathcal{O}_\theta}$,

$$\mathcal{V}_{\mathcal{O}_\theta} = \tfrac{1}{2}\int\limits_0^{2\pi}\int\limits_0^{\pi} E_{\mathcal{O}_\theta}\left(\frac{\partial u_\theta}{\partial \theta}(1,\theta,\phi) + u_r(1,\theta,\phi) - c_{\mathcal{O}_\theta}(\theta,\phi)\right)^2 \sin\theta d\theta d\phi, \qquad (7.6)$$

where the constant $E_{\mathcal{O}_\theta}$ is the product of the Young's modulus and density of the latitudinal boundary actuator material and $c_{\mathcal{O}_\theta}$ is the *latitudinal boundary control.* Using (7.5) we obtain the potential energy of the longitudinal boundary actuator family, $\mathcal{V}_{\mathcal{O}_\phi}$,

$$\mathcal{V}_{\mathcal{O}_\phi} = \tfrac{1}{2}\int_0^{2\pi}\int_0^{\pi} E_{\mathcal{O}_\phi}\left(\frac{1}{\sin\theta}\frac{\partial u_\phi}{\partial\phi}(1,\theta,\phi) + \cot\theta\ u_\theta(1,\theta,\phi) + u_r(1,\theta,\phi) - c_{\mathcal{O}_\phi}(\theta,\phi)\right)^2 \sin\theta d\theta d\phi, \tag{7.7}$$

where the constant $E_{\mathcal{O}_\phi}$ is the product of the Young's modulus and density of the longitudinal boundary actuator material and $c_{\mathcal{O}_\phi}$ is the *longitudinal boundary control.*

7.4 The Total Potential Energy

The total potential energy is the sum of the matrix potential energy and the actuator family potential energies:

$$\mathcal{V} = \mathcal{V}_{\mathcal{S}} + \mathcal{V}_r + \mathcal{V}_\theta + \mathcal{V}_\phi + \mathcal{V}_{\mathcal{O}_\theta} + \mathcal{V}_{\mathcal{O}_\phi}.$$

Using (7.1) and (7.3)-(7.7) we obtain the total potential energy in terms of the spherical strains,

$$
\begin{aligned}
\mathcal{V} \;=\; & \tfrac{1}{2}\int_0^{2\pi}\int_0^{\pi}\int_{r_i}^{1}\Bigg\{\Big(\lambda+\tfrac{2}{3}\mu\Big)\left(\frac{\partial u_r}{\partial r}+2\frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\cot\theta\frac{u_\theta}{r}+\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}\right)^2 \\
& \quad + \tfrac{2}{3}\mu\Bigg[\left(\frac{\partial u_r}{\partial r}-\frac{u_r}{r}-\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\right)^2+\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}-\cot\theta\frac{u_\phi}{r}-\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}\right)^2 \\
& \qquad + \left(\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}+\cot\theta\frac{u_\theta}{r}+\frac{u_r}{r}-\frac{\partial u_r}{\partial r}\right)^2\Bigg] \\
& \quad + \mu\Bigg[\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta}{r}+\frac{\partial u_\theta}{\partial r}\right)^2+\left(\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi}-\cot\theta\frac{u_\phi}{r}+\frac{1}{r}\frac{\partial u_\phi}{\partial\theta}\right)^2 \\
& \qquad + \left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\phi}-\frac{u_\phi}{r}+\frac{\partial u_\phi}{\partial r}\right)^2\Bigg] \\
& \quad + E_r(r)\left(\frac{\partial u_r}{\partial r}-c_r\right)^2+E_\theta(r)\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}-c_\theta\right)^2 \\
& \quad + E_\phi(r)\left(\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}+\cot\theta\frac{u_\theta}{r}+\frac{u_r}{r}-c_\phi\right)^2\Bigg\}r^2\sin\theta\,dr\,d\theta\,d\phi \\
& + \tfrac{1}{2}\int_0^{2\pi}\int_0^{\pi}\Bigg\{E_{\mathcal{O}_\theta}\left(\frac{\partial u_\theta}{\partial\theta}(1,\theta,\phi)+u_r(1,\theta,\phi)-c_{\mathcal{O}_\theta}\right)^2 \\
& \quad + E_{\mathcal{O}_\phi}\left(\frac{1}{\sin\theta}\frac{\partial u_\phi}{\partial\phi}(1,\theta,\phi)+\cot\theta\; u_\theta(1,\theta,\phi)+u_r(1,\theta,\phi)-c_{\mathcal{O}_\phi}\right)^2\Bigg\}\sin\theta\,d\theta\,d\phi.
\end{aligned}
\tag{7.8}
$$

7.5 Formation Problem

We address the following problem.

> **The Formation Problem**. *Using the radial, longitudinal, latitudinal, and boundary actuators and the corresponding controls,* c_r*,* c_θ*,* c_ϕ*,* $c_{\mathcal{O}_\theta}$*, and* $c_{\mathcal{O}_\phi}$*, find displacements* u_r*,* u_θ*, and* u_ϕ *on* $\mathcal{R}$ *that minimize the potential energy (7.8), vanish on the inner boundary,* $\mathcal{I}$*, and map the outer boundary,* $\mathcal{O}$*, to the target configuration* $\mathcal{T}$*.*

Necessary conditions for minimization of the potential energy (7.8) require that the displacements satisfy three PDEs and three NBCs on the surface $\mathcal{O}$. The *radial PDE* and *radial NBC* are

$$\begin{aligned}
&\frac{\partial}{\partial r}\left[r^2\Big(\lambda+2\mu+E_r(r)\Big)\frac{\partial u_r}{\partial r}\right]+\frac{\mu}{\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\left(\frac{\partial u_r}{\partial\theta}-u_\theta\right)\right]\\
&+\frac{\mu}{\sin^2\theta}\frac{\partial}{\partial\phi}\left[\frac{\partial u_r}{\partial\phi}-\sin\theta\ u_\phi\right]+\frac{r(\lambda+\mu)}{\sin\theta}\left(\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial u_\theta}{\partial r}\right]+\frac{\partial^2 u_\phi}{\partial\phi\partial r}\right)\\
&-\Big(\lambda+2\mu+E_\phi(r)\Big)\left(\frac{1}{\sin\theta}\frac{\partial u_\phi}{\partial\phi}+\cot\theta\ u_\theta+u_r\right)+rE_\phi(r)c_\phi\\
&-\Big(\lambda+2\mu+E_\theta(r)\Big)\left(\frac{\partial u_\theta}{\partial\theta}+u_r\right)+rE_\theta(r)c_\theta-\frac{\partial}{\partial r}\Big[r^2E_r(r)c_r\Big]=0
\end{aligned}\tag{7.9}$$

$$\begin{aligned}
&\Big(\lambda+2\mu+E_r(1)\Big)\frac{\partial u_r}{\partial r}(1,\theta,\phi)+(\lambda+E_{\mathcal{O}_\theta})\left(\frac{\partial u_\theta}{\partial\theta}(1,\theta,\phi)+u_r(1,\theta,\phi)\right)\\
&+(\lambda+E_{\mathcal{O}_\phi})\left(\frac{1}{\sin\theta}\frac{\partial u_\phi}{\partial\phi}(1,\theta,\phi)+\cot\theta\ u_\theta(1,\theta,\phi)+u_r(1,\theta,\phi)\right)\\
&-E_{\mathcal{O}_\theta}c_{\mathcal{O}_\theta}(\theta,\phi)-E_{\mathcal{O}_\phi}c_{\mathcal{O}_\phi}(\theta,\phi)-E_r(1)c_r(1,\theta,\phi)=0.
\end{aligned}\tag{7.10}$$

The *latitudinal PDE* and *latitudinal NBC* are

$$\begin{aligned}
&\frac{\lambda+2\mu+E_\theta(r)}{\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\left(\frac{\partial u_\theta}{\partial\theta}+u_r\right)\right]+\frac{\mu}{\sin^2\theta}\frac{\partial}{\partial\phi}\left[\frac{\partial u_\theta}{\partial\phi}-\cos\theta\ u_\phi\right]\\
&+\mu\frac{\partial}{\partial r}\left[r^2\frac{\partial u_\theta}{\partial r}\right]+(\lambda+\mu)\left(\frac{\partial}{\partial r}\left[r\frac{\partial u_r}{\partial\theta}\right]+\frac{1}{\sin\theta}\frac{\partial^2 u_\phi}{\partial\phi\partial\theta}-u_\theta\right)\\
&-\cot\theta\left\{\Big(\lambda+2\mu+E_\phi(r)\Big)\left(\frac{1}{\sin\theta}\frac{\partial u_\phi}{\partial\phi}+\cot\theta\ u_\theta+u_r\right)-rE_\phi(r)c_\phi\right\}\\
&+\mu\left(\frac{\partial u_r}{\partial\theta}-u_\theta\right)-\frac{rE_\theta(r)}{\sin\theta}\frac{\partial}{\partial\theta}\Big[\sin\theta\ c_\theta\Big]=0
\end{aligned}\tag{7.11}$$

$$\begin{aligned}
&\mu \sin\theta \left(\frac{\partial u_r}{\partial \theta}(1,\theta,\phi) - u_\theta(1,\theta,\phi) + \frac{\partial u_\theta}{\partial r}(1,\theta,\phi) \right) \\
&+ E_{\mathcal{O}_\phi} \cos\theta \left(\frac{1}{\sin\theta} \frac{\partial u_\phi}{\partial \phi}(1,\theta,\phi) + \cot\theta \; u_\theta(1,\theta,\phi) + u_r(1,\theta,\phi) - c_{\mathcal{O}_\theta}(\theta,\phi) \right) \\
&- E_{\mathcal{O}_\theta} \frac{\partial}{\partial \theta} \left[\sin\theta \left(\frac{\partial u_\theta}{\partial \theta}(1,\theta,\phi) + u_r(1,\theta,\phi) \right) \right] + E_{\mathcal{O}_\theta} \frac{\partial}{\partial \theta} \Big[\sin\theta \; c_{\mathcal{O}_\theta}(\theta,\phi) \Big] = 0.
\end{aligned} \tag{7.12}$$

The *longitudinal PDE* and *longitudinal NBC* are

$$\begin{aligned}
&\frac{\lambda + 2\mu + E_\phi(r)}{\sin\theta} \frac{\partial}{\partial \phi} \left[\frac{1}{\sin\theta} \frac{\partial u_\phi}{\partial \phi} + \cot\theta \; u_\theta + u_r \right] + \mu \frac{\partial}{\partial r} \left[r^2 \frac{\partial u_\phi}{\partial r} \right] \\
&+ \frac{\mu}{\sin\theta} \frac{\partial}{\partial \theta} \left[\sin\theta \frac{\partial u_\phi}{\partial \theta} \right] + \frac{\lambda + \mu}{\sin\theta} \left(\frac{\partial}{\partial r} \left[r \frac{\partial u_r}{\partial \phi} \right] + \frac{\partial^2 u_\theta}{\partial \theta \partial \phi} \right) \\
&+ \frac{\mu}{\sin\theta} \left(\frac{\partial u_r}{\partial \phi} + \cot\theta \frac{\partial u_\theta}{\partial \phi} - \frac{u_\phi}{\sin\theta} \right) - \frac{r E_\phi(r)}{\sin\theta} \frac{\partial c_\phi}{\partial \phi} = 0
\end{aligned} \tag{7.13}$$

$$\begin{aligned}
&\mu \sin\theta \left(\frac{1}{\sin\theta} \frac{\partial u_r}{\partial \phi}(1,\theta,\phi) - u_\phi(1,\theta,\phi) + \frac{\partial u_\phi}{\partial r}(1,\theta,\phi) \right) \\
&- E_{\mathcal{O}_\phi} \frac{\partial}{\partial \phi} \left[\frac{1}{\sin\theta} \frac{\partial u_\phi}{\partial \phi}(1,\theta,\phi) + \cot\theta \; u_\theta(1,\theta,\phi) + u_r(1,\theta,\phi) - c_{\mathcal{O}_\phi}(\theta,\phi) \right] = 0.
\end{aligned} \tag{7.14}$$

Since the displacements are assumed to vanish on $\mathcal{I}$, we obtain the the three IBCs

$$u_r(r_i,\theta,\phi) = 0 \tag{7.15}$$
$$u_\theta(r_i,\theta,\phi) = 0 \tag{7.16}$$
$$u_\phi(r_i,\theta,\phi) = 0, \tag{7.17}$$

and, as in Chapter 5, we assume that the displacement on the outer boundary, $\mathcal{O}$, is purely radial. This implies that $u_\theta(1,\theta,\phi) = u_\phi(1,\theta,\phi) \equiv 0$ and the OBCs are

$$u_r(1,\theta,\phi) = B(\theta,\phi) - 1 \tag{7.18}$$
$$u_\theta(1,\theta,\phi) = 0 \tag{7.19}$$
$$u_\phi(1,\theta,\phi) = 0. \tag{7.20}$$

We therefore seek displacements u_r, u_θ, and u_ϕ on $\mathcal{R}$ satisfying the BVP (7.9)-(7.20).

7.6 Solution Strategy

We will proceed along the same lines as the two dimensional case, attempting to control the nonradial displacements, u_θ and u_ϕ, to zero throughout $\mathcal{R}$. The more complex nature of Equations (7.9)-(7.14) compared to Equations (5.20)-(5.23) leads to several difficulties not present in the two dimensional case. To alleviate some difficulties we assume that E_r, E_θ, and E_ϕ are constant, but other problems are not so easily ameliorated. A complete resolution of all such difficulties is not currently at hand, but the nature and potential resolution of each is addressed as we proceed.

7.7 The Longitudinal PDE and BCs

We now choose the longitudinal controls, c_ϕ and $c_{\mathcal{O}_\phi}$, so that $u_\phi \equiv 0$ uniquely satisfies (7.13), (7.14), (7.17), and (7.20).

We use

$$c_\phi(r,\theta,\phi) = \frac{1}{rE_\phi}\left[\Big(\lambda + 3\mu + E_\phi\Big)(\cot\theta\ u_\theta + u_r) + (\lambda+\mu)\left(r\frac{\partial u_r}{\partial r} + \frac{\partial u_\theta}{\partial \theta} + u_r\right)\right] \tag{7.21}$$

$$c_{\mathcal{O}_\phi}(\theta,\phi) = \frac{E_{\mathcal{O}_\phi} - \mu}{E_{\mathcal{O}_\phi}} u_r(1,\theta,\phi) + \cot\theta\ u_\theta(1,\theta,\phi), \tag{7.22}$$

in (7.13) and (7.14) to obtain

$$\frac{\lambda + 2\mu + E_\phi}{\sin^2\theta}\frac{\partial^2 u_\phi}{\partial \phi^2} + \mu\frac{\partial}{\partial r}\left[r^2\frac{\partial u_\phi}{\partial r}\right] + \frac{\mu}{\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial u_\phi}{\partial\theta}\right] - \frac{\mu}{\sin^2\theta}u_\phi = 0 \tag{7.23}$$

$$\frac{E_{\mathcal{O}_\phi}}{\sin^2\theta}\frac{\partial^2 u_\phi}{\partial\phi^2}(1,\theta,\phi) + \mu\left(u_\phi(1,\theta,\phi) - \frac{\partial u_\phi}{\partial r}(1,\theta,\phi)\right) = 0. \tag{7.24}$$

Equations (7.23), (7.24), along with $u_\phi(r_i,\theta,\phi) = u_\phi(1,\theta,\phi) = 0$, clearly admit the trivial solution, $u_\phi \equiv 0$. To show that the trivial solution is unique we assume that u_ϕ can be written in separated form as

$$u_\phi(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi). \tag{7.25}$$

Substituting (7.25) into (7.23) we obtain the ODEs

$$\Phi''(\phi) + \kappa_1 \Phi(\phi) = 0 \tag{7.26}$$

$$r^2 R''(r) + 2rR'(r) - \kappa_2 R(r) = 0 \tag{7.27}$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\Big[\sin\theta\ \Theta'(\theta)\Big] + \left(\kappa_2 - \frac{1+\kappa_1/\mu_\phi}{\sin^2\theta}\right)\Theta(\theta) = 0, \tag{7.28}$$

where κ_1 and κ_2 are separation constants and

$$\mu_\phi \equiv \frac{\mu}{\lambda + 2\mu + E_\phi}.$$

Ideally, κ_1 and κ_2 are chosen so that the product, $\Theta\Phi$, of solutions Θ of (7.28) and solutions Φ of (7.26) is a *surface spherical harmonic*. Surface spherical harmonics are the two dimensional analogue of the circular harmonics sine and cosine. However, the term $1 + \kappa_1/\mu_\phi$ prevents this from happening for general values of the elastic constants.

Alternatively, we may guarantee the existence of bounded solutions of (7.28) by setting $\kappa_2 = k(k+1)$ and $1 + \kappa_1/\mu_\phi = m^2$, where n and m are nonnegative integers [8]. We will see this equation in §7.10 and detail its solutions there. Now, with $\kappa_1 = \mu_\phi(m^2 - 1)$, Equation (7.26) certainly admits bounded solutions and so we focus on solving (7.27).

The OBC (7.20) implies that $R(1) = 0$ and this along with the NBC (7.24) implies that $R'(1) = 0$. Therefore, the unique solution of (7.27) is $R \equiv 0$ and thus $u_\phi \equiv 0$ uniquely satisfies (7.13), (7.14), (7.17), and (7.20) with c_ϕ and $c_{\mathcal{O}_\phi}$ given by (7.21) and (7.22), respectively.

7.8 The Latitudinal PDE and BCs

We now use (7.21), (7.22), and $u_\phi \equiv 0$ in (7.11) and (7.12) to obtain

$$\begin{aligned}
&\frac{\lambda + 2\mu + E_\theta}{\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\left(\frac{\partial u_\theta}{\partial\theta} + u_r\right)\right] + \frac{\mu}{\sin^2\theta}\frac{\partial^2 u_\theta}{\partial\phi^2} + \mu\frac{\partial}{\partial r}\left[r^2\frac{\partial u_\theta}{\partial r}\right] \\
&+ (\lambda+\mu)\left(\frac{\partial}{\partial\theta}\left[\frac{\partial}{\partial r}\Big[ru_r\Big]\right] + \cot\theta\frac{\partial}{\partial r}\Big[ru_r\Big] + \cot\theta\frac{\partial u_\theta}{\partial\theta} - u_\theta\right) + \mu\left(\frac{\partial u_r}{\partial\theta} - u_\theta\right) \\
&+ \mu\cot\theta(\cot\theta\ u_\theta + u_r) - \frac{rE_\theta}{\sin\theta}\frac{\partial}{\partial\theta}\Big[\sin\theta\ c_\theta\Big] = 0
\end{aligned} \tag{7.29}$$

$$\begin{aligned} &\mu \sin\theta \left(\frac{\partial u_r}{\partial \theta}(1,\theta,\phi) - u_\theta(1,\theta,\phi) + \frac{\partial u_\theta}{\partial r}(1,\theta,\phi) \right) + \mu \cos\theta \; u_r \\ &- E_{\mathcal{O}_\theta} \frac{\partial}{\partial \theta} \left[\sin\theta \left(\frac{\partial u_\theta}{\partial \theta}(1,\theta,\phi) + u_r(1,\theta,\phi) - c_{\mathcal{O}_\theta}(\theta,\phi) \right) \right] = 0. \end{aligned} \tag{7.30}$$

We choose the latitudinal controls, c_θ and $c_{\mathcal{O}_\theta}$, so that $u_\theta \equiv 0$ uniquely satisfies (7.29), (7.30), (7.16), and (7.19).

We use

$$c_\theta(r,\theta,\phi) = \frac{1}{rE_\theta} \left[\Big(\lambda + 3\mu + E_\theta\Big) u_r + (\lambda + \mu) \left(r \frac{\partial u_r}{\partial r} + u_r + \cot\theta \; u_\theta \right) \right] \tag{7.31}$$

$$c_{\mathcal{O}_\theta}(\theta,\phi) = \frac{E_{\mathcal{O}_\theta} - \mu}{E_{\mathcal{O}_\theta}} u_r(1,\theta,\phi), \tag{7.32}$$

in (7.29) and (7.30) to obtain

$$\frac{\lambda + 2\mu + E_\theta}{\sin\theta} \frac{\partial}{\partial \theta} \left[\sin\theta \frac{\partial u_\theta}{\partial \theta} \right] + \frac{\mu}{\sin^2\theta} \frac{\partial^2 u_\theta}{\partial \phi^2} + \mu \frac{\partial}{\partial r} \left[r^2 \frac{\partial u_\theta}{\partial r} \right] + \mu \left(\cot^2\theta - 1 \right) u_\theta = 0 \tag{7.33}$$

$$\frac{E_{\mathcal{O}_\theta}}{\sin\theta} \frac{\partial}{\partial \theta} \left[\sin\theta \frac{\partial u_\theta}{\partial \theta}(1,\theta,\phi) \right] + \mu \left(u_\theta(1,\theta,\phi) - \frac{\partial u_\theta}{\partial r}(1,\theta,\phi) \right) = 0. \tag{7.34}$$

The BVP (7.33)-(7.34), along with the boundary conditions $u_\theta(r_i,\theta,\phi) = u_\theta(1,\theta,\phi) = 0$, clearly admits the trivial solution, $u_\theta \equiv 0$. To show that the trivial solution is unique we assume that u_θ can be written in separated form as

$$u_\theta(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi). \tag{7.35}$$

Substituting (7.35) into (7.33) we obtain the ODEs

$$\Phi''(\phi) + \kappa_1 \Phi(\phi) = 0 \tag{7.36}$$

$$r^2 R''(r) + 2rR'(r) - \kappa_2 R(r) = 0 \tag{7.37}$$

$$\frac{1}{\sin\theta} \frac{d}{d\theta} \Big[\sin\theta \; \Theta'(\theta) \Big] + \mu_\theta \left(\kappa_2 - 1 + \frac{\cos^2\theta - \kappa_1}{\sin^2\theta} \right) \Theta(\theta) = 0, \tag{7.38}$$

where κ_1 and κ_2 are separation constants and

$$\mu_\theta \equiv \frac{\mu}{\lambda + 2\mu + E_\theta}.$$

Again, we would like to choose κ_1 and κ_2 so that the product $\Theta\Phi$ is a surface spherical harmonic, but this is not possible.

A full accounting of solutions of (7.38) is not complete, but assuming that bounded solutions exist, we may proceed exactly as in §7.7. Using (7.35) in the BCs (7.19) and (7.30) shows that $R(1) = R'(1) = 0$ and so the unique solution of (7.37) is $R \equiv 0$. Therefore, $u_\theta \equiv 0$ uniquely satisfies (7.11), (7.12), (7.16), and (7.19) with c_θ and $c_{\mathcal{O}_\theta}$ given by (7.31) and (7.32), respectively.

7.9 The Radial Boundary Value Problem

We now use $u_\theta \equiv 0$, $u_\phi \equiv 0$, and the expressions (7.31), (7.21) for the latitudinal and longitudinal controls, respectively, in the radial PDE, (7.9), to obtain

$$\begin{aligned}(\lambda + 2\mu + E_r)\frac{\partial}{\partial r}\left[r^2\frac{\partial u_r}{\partial r}\right] + \frac{\mu}{\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial u_r}{\partial\theta}\right] + \frac{\mu}{\sin^2\theta}\frac{\partial^2 u_r}{\partial\phi^2} \\ + 2(\lambda+\mu)\frac{\partial}{\partial r}[ru_r] + 2\mu u_r - \frac{\partial}{\partial r}\left[r^2 E_r(r) c_r\right] = 0.\end{aligned} \tag{7.39}$$

To write this equation in self-adjoint form we add a feedback term to the radial control, $c_r \mapsto c_r + \tilde{c}_r$, with

$$\tilde{c}_r(r,\theta,\phi) = 2\frac{\lambda+\mu}{rE_r(r)}u_r(r,\theta,\phi).$$

Using $c_r + \tilde{c}_r$, $u_\theta \equiv 0$, $u_\phi \equiv 0$, and the expressions (7.32), (7.22) for the latitudinal and longitudinal boundary controls in the radial NBC, (7.10), we obtain the following BVP for the radial displacement u_r:

$$\frac{\partial}{\partial r}\left[r^2\frac{\partial u_r}{\partial r}\right] + \frac{\epsilon^2}{\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial u_r}{\partial\theta}\right] + \frac{\epsilon^2}{\sin^2\theta}\frac{\partial^2 u_r}{\partial\phi^2} + 2\epsilon^2 u_r - E_1\frac{\partial}{\partial r}\left[r^2 c_r\right] = 0 \tag{7.40}$$

$$\frac{\partial u_r}{\partial r}(1,\theta,\phi) = E_1 c_r(1,\theta,\phi) \tag{7.41}$$

$$u_r(1,\theta,\phi) = B(\theta,\phi) - 1 \tag{7.42}$$

$$u_r(r_i, \theta, \phi) = 0, \tag{7.43}$$

where,

$$\epsilon^2 \equiv \frac{\mu}{\lambda + 2\mu + E_r} \qquad \text{and} \qquad E_1 \equiv \frac{E_r}{\lambda + 2\mu + E_r}. \tag{7.44}$$

7.10 Separation of the Radial Equation

We assume that the radial displacement, u_r, and the radial control, c_r, can be written in separated form as

$$\begin{aligned} u_r(r, \theta, \phi) &= U(r)\Theta(\theta)\Phi(\phi) \\ c_r(r, \theta, \phi) &= C(r)\Theta(\theta)\Phi(\phi), \end{aligned}$$

and substitute these expressions into (7.40) to obtain

$$\Phi''(\phi) + \kappa_1 \Phi(\phi) = 0 \tag{7.45}$$

$$r^2 U''(r) + 2rU'(r) + \epsilon^2(2 - \kappa_2)U(r) = E_1 \frac{d}{dr}\left[r^2 C(r)\right] \tag{7.46}$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\Big[\sin\theta\ \Theta'(\theta)\Big] + \left(\kappa_2 - \frac{\kappa_1}{\sin^2\theta}\right)\Theta(\theta) = 0, \tag{7.47}$$

where κ_1 and κ_2 are separation constants.

Since we require solutions 2π-periodic in ϕ we need $\kappa_1 = m^2$, where $m = 0, 1, 2, \ldots$. This gives us periodic solutions of (7.45) of the form $\Phi(\phi) = c_1 \cos m\phi + c_2 \sin m\phi$. If we make the substitution $\xi = \cos\theta$ in (7.47) we obtain *Legendre's associated differential equation*:

$$\left(1 - \xi^2\right)\Xi''(\xi) - 2\xi\Xi'(\xi) + \left(\kappa_2 - \frac{\kappa_1}{1 - \xi^2}\right)\Xi(\xi) = 0. \tag{7.48}$$

Here $\Xi(\xi) \equiv \Theta(\theta)$ and $\Xi' \equiv d\Xi/d\xi$.

We are only interested in (7.48) when $\kappa_2 = k(k+1)$, $k = 0, 1, 2, \ldots$, since otherwise solutions become unbounded as $\xi \to -1$ or $\xi \to 1$, [8]. Independent solutions of

$$\left(1-\xi^2\right)\Xi''(\xi)-2\xi\Xi'(\xi)+\left(k(k+1)-\frac{m^2}{1-\xi^2}\right)\Xi(\xi)=0 \tag{7.49}$$

are P_k^m and Q_k^m, the *associated Legendre functions, of degree k and order m, of the first and second kinds*, respectively. These functions are given by

$$P_k^m(\xi)=\left(\xi^2-1\right)^{\frac{m}{2}}\frac{d^m P_k(\xi)}{d\xi^m} \tag{7.50}$$

$$Q_k^m(\xi)=(-1)^m\left(\xi^2-1\right)^{\frac{m}{3}}\frac{d^m Q_k(\xi)}{d\xi^m},$$

where $k,m\in\mathbb{N}$, $m\leq k$, and P_k, Q_k are the *Legendre functions, of degree k, of the first and second kinds*, respectively. It is known [8] that the Legendre functions of the second kind, Q_k and Q_k^m, converge for $|\xi|>1$, so we do not consider them further. The functions P_k, valid on $[-1,1]$, may be expressed by *Rodrigues' formula*,

$$P_k(\xi)=\frac{1}{2^k k!}\frac{d^k}{d\xi^k}\left(\xi^2-1\right)^k \qquad \forall\ k\in\mathbb{N}, \tag{7.51}$$

and for $k=0$, we have $P_0=1$.

Now $P_k^m(\xi)$ as defined by (7.50) is complex-valued for $-1<\xi<1$, so we prefer to use what is known as *Ferrers' associated Legendre function of the first kind*, T_k^m, given by

$$T_k^m(\xi)=(-1)^m\left(1-\xi^2\right)^{\frac{m}{2}}\frac{d^m P_k(\xi)}{d\xi^m} \tag{7.52}$$

for positive integers k and m, with $m\leq k$. For convenience, we define $T_k^0\equiv P_k$ for $k=0,1,2,\ \ldots$. The functions T_k^m are orthogonal on $[-1,1]$, i.e.

$$\int_{-1}^{1} T_i^m(\xi)T_j^m(\xi)d\xi=0$$

for nonnegative integers i,j, $i\neq j$, and $m=0,1,2,\ \ldots$, but they are not normalized. In [8] we find that their L^2 norm squared is

$$\int_{-1}^{1}\left\{T_k^m(\xi)\right\}^2 d\xi=\frac{(k+m)!}{(k-m)!}\frac{2}{2k+1}. \tag{7.53}$$

Now, with $\kappa_2=k(k+1)$, we have Equation (7.46) in the form

$$r^2U''(r) + 2rU'(r) + \epsilon^2\Big(2 - k(k+1)\Big)U(r) = E_1\frac{d}{dr}\Big[r^2C(r)\Big], \tag{7.54}$$

and we see another interesting result not present in the two dimensional case. The functions $T_k^m(\cos\theta)\cos m\phi$ and $T_k^m(\cos\theta)\sin m\phi$ are surface spherical harmonics and so we obtain u_r in separated form as a spherical harmonic series,

$$u_r(r,\theta,\phi) = \sum_{k=0}^{\infty}\sum_{m=0}^{k}\Big(a_k^m(r)\cos m\phi + b_k^m(r)\sin m\phi\Big)T_k^m(\cos\theta). \tag{7.55}$$

But the solutions of (7.54) are indexed by k *only*, so that we actually have coefficient functions $a_k(r) \equiv a_k^m(r)$ and $b_k(r) \equiv b_k^m(r)$ which do not depend on m. This appears to restrict the class of possible radial displacements and target functions $B(\theta,\phi)$, since if we write

$$B\theta,\phi) - 1 = \sum_{k=0}^{\infty}\sum_{m=0}^{k}\Big(\alpha_k^m\cos m\phi + \beta_k^m\sin m\phi\Big)T_k^m(\cos\theta), \tag{7.56}$$

then the boundary condition (7.42) implies

$$\begin{aligned}\alpha_k^0 = \alpha_k^1 = \alpha_k^2 = \ \ldots\ = \alpha_k^k = a_k(1)\\ \beta_k^0 = \beta_k^1 = \beta_k^2 = \ \ldots\ = \beta_k^k = b_k(1).\end{aligned}$$

The consequences, if any, of considering a restricted class of harmonic series, in which the coefficients depend on k and not m, are unclear at the present time.

We proceed with this case assuming that the radial displacement, u_r, the radial control, c_r, and the boundary displacement, $B-1$, are expanded in spherical harmonic series of the form

$$u_r(r,\theta,\phi) = \sum_{k=0}^{\infty}\sum_{m=0}^{k}\Big(a_k(r)\cos m\phi + b_k(r)\sin m\phi\Big)T_k^m(\cos\theta) \tag{7.57}$$

$$c_r(r,\theta,\phi) = \sum_{k=0}^{\infty}\sum_{m=0}^{k}\Big(c_k(r)\cos m\phi + d_k(r)\sin m\phi\Big)T_k^m(\cos\theta) \tag{7.58}$$

$$B\theta,\phi) - 1 = \sum_{k=0}^{\infty}\sum_{m=0}^{k}\Big(\alpha_k\cos m\phi + \beta_k\sin m\phi\Big)T_k^m(\cos\theta), \tag{7.59}$$

where it is understood that $b_0(r) = d_0(r) = \beta_0 \equiv 0$. As before, we focus on the relations involving the a_k, c_k, and α_k, realizing that the same relations hold for the b_k, d_k, and β_k. We thus consider the BVP,

$$r^2a_k''(r) + 2ra_k'(r) + \epsilon^2\Big(2 - k(k+1)\Big)a_k(r) = E_1\frac{d}{dr}\Big[r^2c_k(r)\Big] \tag{7.60}$$

$${a_k}'(1) = E_1c_k(1) \tag{7.61}$$

$$a_k(1) = \alpha_1 \tag{7.62}$$

$$a_k(r_i) = 0, \tag{7.63}$$

which we refer to as the *primal system.*

7.11 The Dual System

We may obtain the same type of relationship between the boundary displacement coefficients, α_k, and the control coefficients, c_k, via solutions of the adjoint equation as in the two dimensional case. We let σ_k denote the solution of the adjoint or *dual system* defined by

$$r^2\sigma_k''(r) + 2r\sigma_k'(r) + \epsilon^2\Big(2 - k(k+1)\Big)\sigma_k(r) = 0 \tag{7.64}$$

$${\sigma_k}'(1) = 1 \tag{7.65}$$

$$\sigma_k(r_i) = 0. \tag{7.66}$$

We multiply the primal differential equation, (7.60), by σ_k and integrate from r_i to 1 to obtain

$$\alpha_k = E_1\int\limits_{r_i}^{1} {\sigma_k}'(\rho)c_k(\rho)\rho^2 d\rho. \tag{7.67}$$

We now let c_k be a constant multiple of ${\sigma_k}'$ to obtain

$$c_k(r) = \frac{\alpha_k{\sigma_k}'(r)}{E_1\int\limits_{r_i}^{1}{\sigma_k}'(\rho)^2\rho^2 d\rho}. \tag{7.68}$$

7.12 Estimating the L^2 Norm of the Radial Control

We now investigate the convergence of the series expansion of the radial control, c_r, in $L^2(\mathcal{R}^c)$ via asymptotic properties of σ_k and α_k.

Using Parseval's Identity and (7.53) we calculate the square of the $L^2(\mathcal{R}^c)$ norm of the radial control,

$$\begin{aligned}
||c_r||^2_{L^2(\mathcal{R}^c)} &\equiv \int_0^{2\pi}\int_0^{\pi}\int_{r_i}^{1} c_r(r,\phi)^2 r^2 \sin\theta dr d\theta d\phi \\
&= \sum_{n=0}^{\infty}\int_{r_i}^{1} c_n(r)^2 r^2 dr \sum_{m=0}^{n}\int_0^{\pi} T_n^m(\cos\theta)^2 \sin\theta d\theta \int_0^{2\pi}\cos^2 m\phi d\phi \\
&\quad + \sum_{n=0}^{\infty}\int_{r_i}^{1} d_n(r)^2 r^2 dr \sum_{m=0}^{n}\int_0^{\pi} T_n^m(\cos\theta)^2 \sin\theta d\theta \int_0^{2\pi}\sin^2 m\phi d\phi \\
&= \sum_{n=0}^{\infty}\int_{r_i}^{1} c_n(r)^2 r^2 dr \frac{2\pi}{2n+1}\left(2+\sum_{m=1}^{n}\frac{(n+m)!}{(n-m)!}\right) \\
&\quad + \sum_{n=0}^{\infty}\int_{r_i}^{1} d_n(r)^2 r^2 dr \frac{2\pi}{2n+1}\sum_{m=1}^{n}\frac{(n+m)!}{(n-m)!} \\
&= \sum_{n=0}^{\infty}\frac{2\pi}{2n+1}\left(2+\sum_{m=1}^{n}\frac{(n+m)!}{(n-m)!}\right)\left\{\int_{r_i}^{1} c_n(r)^2 r^2 dr + \int_{r_i}^{1} d_n(r)^2 r^2 dr\right\} \\
&\equiv \sum_{n=0}^{\infty}\delta_n\left\{\int_{r_i}^{1} c_n(r)^2 r^2 dr + \int_{r_i}^{1} d_n(r)^2 r^2 dr\right\}.
\end{aligned}$$

Using (7.68) and the analogous result for d_k we have

$$||c_r||^2_{L^2(\mathcal{R}^c)} = \sum_{n=0}^{\infty}\frac{\delta_n\left(\alpha_n^2+\beta_n^2\right)}{E_1^2\int_{r_i}^{1}\sigma_n{}'(r)^2 r^2 dr}. \tag{7.69}$$

We thus seek a lower bound on $\int_{r_i}^{1}\sigma_k{}'(r)^2 r^2 dr$. The dual differential equation, (7.64), is an Euler equation and as such admits solutions of the form r^p, where p is a solution of the characteristic equation

$$p^2 + p + \epsilon^2(2 - k(k+1)) = 0. \tag{7.70}$$

For $k = 0$ the solutions of (7.64) depend on the elastic constants via ϵ^2, cf. (7.44), and for $k = 1$ we have the fundamental set of solutions $\{1, r^{-1}\}$, but for large values of k the solutions are qualitatively the same as the solutions of the dual system ODE, (5.44), in the two dimensional case. We therefore expect that

$$\int_{r_i}^{1} {\sigma_k}'(r)^2 r^2 dr = O\left[k^{-1}\right] \quad \text{as } k \to \infty,$$

so that the estimate

$$||c_r||^2_{L^2(\mathcal{R}^c)} < M \sum_{k=0}^{\infty} k\left(\alpha_k^2 + \beta_k^2\right), \tag{7.71}$$

where $M > 0$ is a constant, may follow.

It is clear that we have merely indicated the feasibility of this result based on the results of Chapter 5 and the calculations performed here. A complete solution of the Formation Problem with spherical equilibrium configuration must wait for the future.

Chapter 8

Future Work

It is clear from the work in Chapter 7 that results similar to those obtained for the two dimensional formation problem in Chapter 5 are likely. The establishment of these results for the three dimensional formation problem is the initial focus of future work. An extension of the results in Chapter 5 to include the case of radially dependent density (using $E_r(r)$ instead of the constant E_r) and more general equilibrium configurations is also desirable. We may also make use of other elastic models to include temperature dependence, anisotropic matrix materials, and nonlinear elastic effects and address some of the other goals of formation theory including optimization of the actuator structure, optimal control of the boundary displacement, and dynamic (time-dependent) boundary control.

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