A MEMBRANE ANALOGY FOR INVESTIGATING COMPRESSIBLE FLOW

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I

INTRODUCTION

The analogy established between shear stress in a twisted member and a slightly inflated membrane extended above the same boundary is well known.[24] Since only long and tedious approximate methods exist for calculating the shear stress in non-circular members, this analogy has been quite valuable.

Recently an analogy was established between subsonic compressible gas flow and a soap membrane which is not inflated.[25] The object of this paper is to extend this analogy and to check it experimentally.

In the derivation of the analogy, the exact equation of a membrane subjected to uniform tension and the basic equation of steady, potential flow of a compressible, non-viscous fluid are used. The derivations of these equations are given in another section of the thesis. Euler's equation, which provides a method for finding the equations of minimum surfaces, will be used in the derivation of the exact equation of an open soap bubble existing between given boundaries.[9,29] Any membrane which is subjected to uniform tension is a minimum surface. The basic equation of steady, potential flow of a compressible, non-viscous fluid is derived from fundamental principles and concepts of fluid mechanics.[2,25]

In the experimental verification of the analogy, a circular cylinder will be used. The potential functions around the boundary of the cylinder will be represented by vertical heights above some datum plane. In a similar manner, the potential functions in the free stream will be represented by heights above the same datum. If the cylinder is not to influence the value of the free stream potential function, the distance from the cylinder to the point where the free stream potential function is to be calculated must be large. This distance is assumed to be infinitely large.

The problem of finding a soap solution which will form this large bubble is tremendous. Large bubbles formed from these solutions last only for a short period of time.

An accurate determination of the slopes of the membrane at the boundary is required. This may be accomplished by a number of methods. The best method available is through the use of a microscope-telescope arrangement, although it requires a large amount of calculations.

The analogy may be applied to any steady, potential flow of a compressible, inviscid fluid in the subsonic velocity range. The circular cylinder problem is used only for experimental verification of the analogy. Aerodynamicists are interested primarily in fluid flow about airfoils. Airfoils which are of classical interest are the Joukowsky series of airfoils which are transformations from circular cylinders.

II

THE REVIEW OF LITERATURE

Lee, G. H., An Introduction to Experimental Stress Analysis, John Wiley & Sons, Inc., 1950, pp 225-231.

The author gives the existing membrane analogy between shear stress in torsional members and a slightly inflated soap bubble extended above the same boundary. He discusses how the boundary for the soap membrane should be formed. Also two different soap solutions which are quite stable are given. The first solution consists of 2 grams of sodium oleate per liter of water with 30 cc. of glycerin added. The other solution is 1 gram of sodium oleate per 2 liters of water with 6 cc. of glycerin added for each liter of solution.

To lengthen the life of any soap membrane, the rate of evaporation must be controlled. All known soap solutions evaporate and cause the membrane to burst. The rate of evaporation is decreased by sealing the soap membrane in its container.

Heaslet, M. A., "Compressible Potential Flow With Circulation About a Circular Cylinder," National Advisory Committee for Aeronautics, Report 780, 1944.

There are several known methods for the approximate solution of the basic equation of two-dimensional, irrotational potential flow of a compressible fluid. The more important ones are the method of small perturbations, the Rayleigh-Janzen method, and the hodograph method.

The author gives the results of the three methods; although he derives only the Rayleigh-Janzen method. The Rayleigh-Janzen method is essentially a perturbation method. The solution of the basic differential equation for two-dimensional, compressible flow is assumed in the form of an infinite power series in powers of the free stream Mach number, M. The free stream Mach number is defined as the free stream velocity divided by the velocity of sound in the free stream.

The solution is carried only to the term containing the fourth power of the Mach number. Graphs are given which show the results of the three methods of solution considered.

Liepmann, H. W., and Puckett, A. E., Introduction to Aerodynamics of a Compressible Fluid, John Wiley and Sons, Inc., 1947, pp 157-188.

The authors give various approximate solutions of the basic differential equation of steady, potential flow of a compressible, non-viscous fluid.

The solution for the potential function is derived by introducing a thickness parameter. The final solution is in powers of this thickness parameter.

Next the solution is found by the Rayleigh-Janzen method. The final solution in this case is in terms of powers of the free stream Mach number.

The hodograph method is also discussed and applied to the basic differential equation. Both the Chaplygin and the Karman-Tsien methods transform the basic differential equation to the hodograph plane. These two solutions are also given by the authors.

Dewar, J., "Soap Bubbles of Long Duration," *Journal of the Franklin Institute*, Vol. 188, No. 6, 1919, pp 713-749, and "Studies on Liquid Films," *Journal of the Franklin Institute*, Vol. 193, No. 2, 1922, pp 145-188.

These articles are discussions of techniques used to obtain soap bubbles lasting up to one year. The fact that surroundings which are free from dust particles and other foreign matter is of utmost importance is emphasized by the author.

The results of a number of experiments with various sized bubbles are given in tabular form. One example cites bubbles twenty cm. in diameter which lasted from 28 to 40 days.

Some of the solutions which were used contained five per cent potassium oleate, fifty per cent glycerine and forty-five per cent water; five per cent ammonium oleate, fifty per cent glycerine, and forty-five per cent water; three and one-half per cent ammonium oleate, thirty-three per cent glycerine, and sixty-three and one-half per cent water; and twenty-five per cent glycerine, four per cent alcohol; five per cent soap, and sixty-six per cent water.

III

THEORY

A- The Differential Equation of an Open Soap Membrane

For the analogy between shear stress in a twisted member and a soap bubble extended above the same boundary, the exact equation of the soap bubble is not used. The equation of the soap bubble, neglecting the squares of the slopes $\left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right)$, is

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} + \frac{p}{T} = 0 . \qquad (1)$$

In this equation p is the pressure to which the membrane is subjected, T the uniform tension in the membrane and z the infinitesimal displacement of the soap bubble after the slight pressure is applied. The analogous equation for the stress function ϕ is

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + 2 = 0 . \qquad (2)$$

Euler's equation will give the equation of the surface S, which has a minimum area for any fixed boundary. The soap membrane represents the minimum surface between the given boundary over which the membrane is stretched. The formula for the surface area (S) is

$$S = \iint_A \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy \qquad (3)$$

where x and y are the coordinates of a point and z is the height of the surface above the point (x,y). A is the projection of the surface S in the x-y plane over which the function is integrated.

Generally

$$I = \iint_A F\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) dx\, dy \tag{4}$$

where $F\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right)$ is a twice differentiable function with respect to its variables which is a necessary requirement for the derivation of Euler's equation. The dependent variable z is a function of x and y; $z = z\,(x,y)$. For simplicity of manipulation, introduce the following notation:

$$p = \frac{\partial z}{\partial x} \quad \text{and} \quad q = \frac{\partial z}{\partial y}. \tag{5}$$

The general equation is now of the form

$$I = \iint_A F(x, y, z, p, q)\; dx\; dy. \tag{4a}$$

For the derivation, first select any surface $\theta = \theta\,(x,y)$ where $\theta\,(x,y) = 0$ on curve B_1 as shown in Figure 1. Curve B_1 is the projection of the curve B in the x-y plane, where z(x,y) passes through curve B. Choose any set of surfaces (S_u) so that, for any value of u, the surfaces will pass through curve B. The equation of such a set of surfaces (S_u) is given by

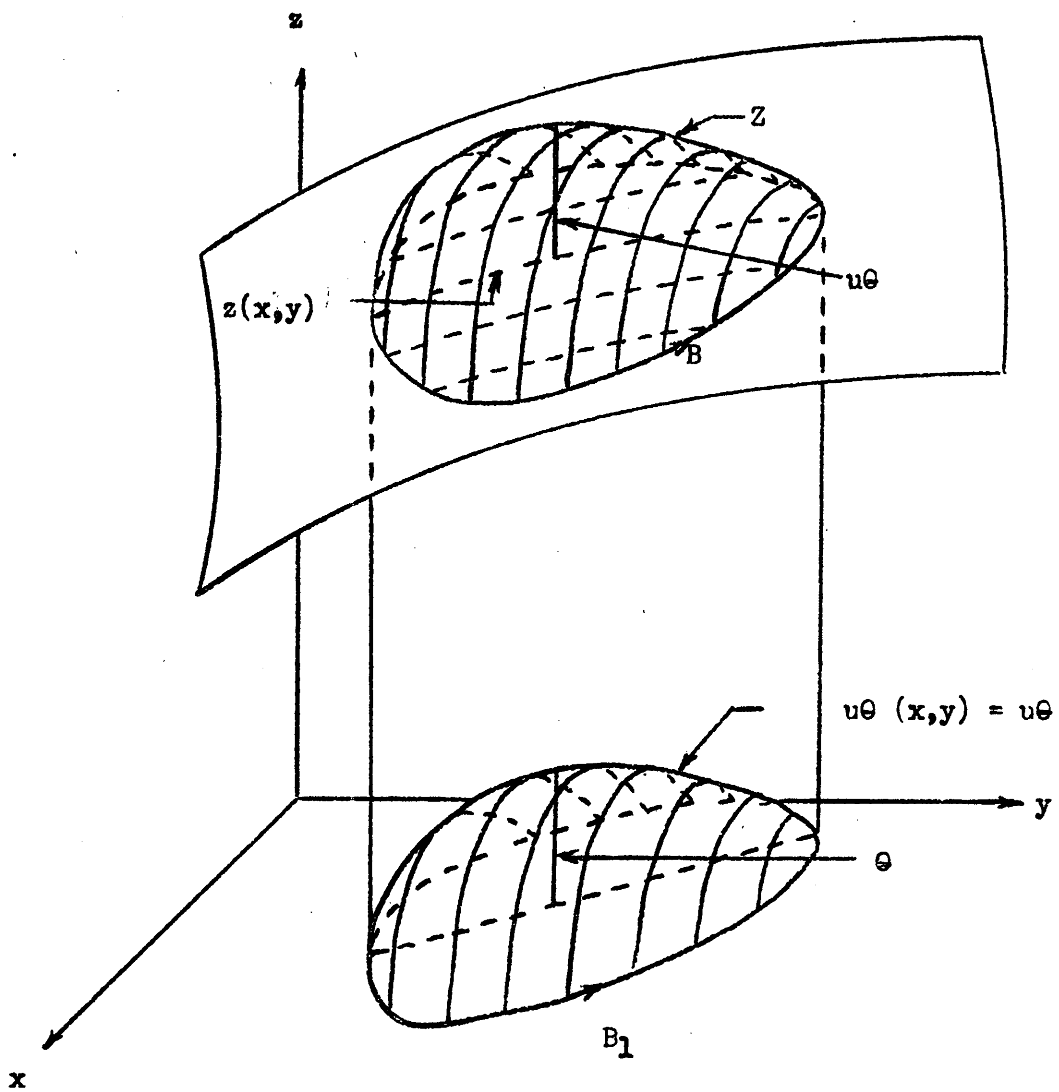


Figure 1. Surface diagram.

$$Z(x,y) = z(x,y) + u\,\theta(x,y). \tag{6}$$

For any one of these surfaces,

$$P = \frac{\partial Z}{\partial x} = p + u\,\frac{\partial \theta}{\partial x} \quad \text{and} \quad Q = \frac{\partial Z}{\partial y} = q + u\,\frac{\partial \theta}{\partial y}. \tag{7}$$

This gives the integral

$$I = \iint_A F(x, y, Z, P, Q)\,dx\,dy. \tag{8}$$

Differentiating this under the integral sign gives

$$\frac{dI}{du} = \iint_A \left(\frac{\partial F}{\partial Z}\,\theta + \frac{\partial F}{\partial P}\,\theta_x + \frac{\partial F}{\partial Q}\,\theta_y \right) dx\,dy. \tag{9}$$

Taking the last two terms under the integral sign and integrating by parts, produces

$$\int_{B_1} \frac{\partial F}{\partial P}\,\theta_x\,dx = \frac{\partial F}{\partial P}\,\theta - \int_{B_1} \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial P}\right)\theta\,dx \tag{9a}$$

and

$$\int_{B_1} \frac{\partial F}{\partial Q}\,\theta_y\,dy = \frac{\partial F}{\partial Q}\,\theta - \int_{B_1} \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial Q}\right)\theta\,dy. \tag{9b}$$

Substituting these back into $\frac{dI}{du}$, gives:

$$\frac{dI}{du} = \iint_A \left[\frac{\partial F}{\partial Z}\,\theta - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial P}\right)\theta - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial Q}\right)\theta\right] dx\,dy + \int_{B_1} \frac{\partial F}{\partial P}\,\theta\,dy + \int_{B_1} \frac{\partial F}{\partial Q}\,\theta\,dx. \tag{10}$$

The last two terms are zero since they are to be computed on the boundary curve B_1 in the x-y plane. In the x-y plane, $\theta(x,y) = 0$. Equation (10) now becomes:

$$\frac{dI}{du} = \iint_A \left[\frac{\partial F}{\partial z} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial P}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial Q}\right)\right] \theta \; dx \; dy. \qquad (11)$$

From equation (6), it is seen that when u = 0; z = z, P = p and Q = q. Equation (11) now becomes

$$\frac{dI}{du} = \iint_A \left[\frac{\partial F}{\partial z} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial p}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial q}\right)\right] \theta \; dx \; dy. \qquad (12)$$

If the integral I is to be an extremal, then $\frac{dI}{du}$ must be equal to zero. With this condition applied, equation (12) is now equal to zero.

$$\iint_A \left[\frac{\partial F}{\partial z} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial p}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial q}\right)\right] \theta \; dx \; dy = 0 \qquad (13)$$

Since θ was chosen arbitrarily, the value of θ is not zero. The bracketed term in equation (13) is continuous and must be equal to zero if equation (13) is equal to zero. Therefore, the <u>necessary</u> condition for a minimum is

$$\frac{\partial F}{\partial z} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial p}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial q}\right) = 0. \qquad (14)$$

Using the formula for computing the surface area (S):

$$S = \iint_A \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \; dx \; dy \qquad (15)$$

with Euler's equation (14), the exact equation of an open soap bubble or any membrane with uniform surface tension may be derived.

Making the following substitutions:

$$p = \frac{\partial z}{\partial x} \; ; \quad q = \frac{\partial z}{\partial y} \; ; \quad F = \sqrt{1 + p^2 + q^2} \qquad (16)$$

in equation (15), gives

$$S = \iint_A \sqrt{1 + p^2 + q^2} \quad dx\, dy. \qquad (17)$$

Using the above derived Euler equation:

$$\frac{\partial^2 z}{\partial x^2}\left[1 + \left(\frac{\partial z}{\partial y}\right)^2\right] + \frac{\partial^2 z}{\partial y^2}\left[1 + \left(\frac{\partial z}{\partial x}\right)^2\right] - 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\frac{\partial^2 z}{\partial x \partial y} = 0. \qquad (18)$$

The solution of equation (18) gives the minimum surface over any boundary. Equation (18) is the exact equation of an open soap bubble since a membrane stretched over a given boundary with uniform tension is the minimum surface above the boundary.

B. <u>Basic Differential Equation of Subsonic Compressible Flow</u>.

For an analogy to exist, the basic equation of steady, potential flow of a compressible, non-viscous fluid must be of the same form as equation (18) for an open soap bubble. This basic equation may be derived from fluid mechanical considerations.(2,25)

Consider the Euler equation for the steady, potential flow of a non-viscous gas, when outside forces are neglected,

$$\rho \; grad \; \frac{\bar{v}^2}{2} + grad \; p = 0 \qquad (19)$$

the continuity equation for steady flow

$$div \; (\rho \bar{v}) = 0 \qquad (20)$$

and the formula for the velocity of sound

$$c = \sqrt{\frac{dp}{d\rho}} \qquad (21)$$

which may be expressed in another form

$$c^2 \; grad \; \rho = grad \; p. \qquad (22)$$

In equations (19), (20) and (22), p is pressure, ρ is density and $\bar{v}$ the velocity vector. It is seen from the continuity equation (20) that

$$\bar{v} \cdot grad \; \rho = -\rho \; div \; \bar{v}. \qquad (23)$$

Equation (22) may be expressed as

$$\bar{v} \cdot grad \; \rho = \frac{\bar{v}}{c^2} \cdot grad \; p. \qquad (24)$$

From equations (23) and (24), the following relation exists

$$\bar{v} \cdot grad \; p = -c^2 \rho \; div \; \bar{v}. \qquad (25)$$

Multiplying equation (19) by the velocity vector gives

$$\bar{v} \cdot grad \; p = -\rho \; \bar{v} \cdot grad \; \frac{\bar{v}^2}{2} \qquad (26)$$

Combining equations (25) and (26), produces

$$\rho \bar{v} \cdot grad \frac{\bar{v}^2}{2} = c^2 \rho \; div \; \bar{v} \qquad (27)$$

or

$$\bar{v} \cdot grad \frac{\bar{v}^2}{2} - c^2 \; div \; \bar{v} = 0. \qquad (28)$$

Introduce the ϕ potential function into equation (28) by the relationship $\bar{v} = grad \, \phi$.

$$grad \, \phi \cdot \left\{ grad \left[\tfrac{1}{2} (grad \, \phi)^2 \right] \right\} - c^2 \; div \; grad \, \phi = 0 \qquad (29)$$

Transferring equation (29) into its scalar form, gives

$$\frac{\partial^2 \phi}{\partial x^2} \left[c^2 - \left(\frac{\partial \phi}{\partial x}\right)^2 \right] + \frac{\partial^2 \phi}{\partial y^2} \left[c^2 - \left(\frac{\partial \phi}{\partial y}\right)^2 \right] - 2 \frac{\partial^2 \phi}{\partial x \partial y} \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial y} = 0. \qquad (30)$$

In order to get equation (30) into the form of equation (18), use,

$$a^2 = c^2 - v^2 \qquad (31)$$

where all terms are velocity squared terms.

This may be written in the form

$$a^2 = c^2 - \left(\frac{\partial \phi}{\partial x}\right)^2 - \left(\frac{\partial \phi}{\partial y}\right)^2 \qquad (32)$$

or

$$c^2 = a^2 + \left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2. \qquad (33)$$

Substituting the value of c^2 from equation (33) into equation (30), gives the result

$$\frac{\partial^2 \phi}{\partial x^2} \left[a^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 \right] + \frac{\partial^2 \phi}{\partial y^2} \left[a^2 + \left(\frac{\partial \phi}{\partial x}\right)^2 \right] - 2 \frac{\partial^2 \phi}{\partial x \partial y} \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial y} = 0. \qquad (34)$$

This is the basic equation of steady, potential flow of a compressible, inviscid fluid.

C. Comparison of the Differential Equations of an Open Soap Membrane and Subsonic Compressible Flow.

If equation (18) and (24) are compared, it is seen that they are of the same form. Before an analogy can be established, both equations must be dimensionless. To get equation (18) into dimensionless form, introduce

$$x = XL \; ; \quad y = YL \; ; \quad z = ZL \tag{25}$$

where X, Y and Z are non-dimensional coordinates and L has the dimension of feet. After introduction of these values, equation (18) has the following form:

$$\frac{\partial^2 Z}{\partial X^2}\left[1+\left(\frac{\partial Z}{\partial Y}\right)^2\right]+\frac{\partial^2 Z}{\partial Y^2}\left[1+\left(\frac{\partial Z}{\partial X}\right)^2\right]-2\frac{\partial^2 Z}{\partial X \partial Y}\frac{\partial Z}{\partial X}\frac{\partial Z}{\partial Y}=0. \tag{26}$$

In order to get equation (24) into its non-dimensional form, introduce

$$x = XL \; ; \; y = YL \; ; \; \phi = \Psi L c_o \; ; \; c = C c_o \; ; \; a = A c_o \tag{27}$$

where X and Y are non-dimensional coordinates, Ψ is the non-dimensional potential and C and A are the non-dimensional velocities. The dimensions of the other terms are: L in feet and c_o in ft/sec.

Equation (24) now reads

$$\frac{\partial^2 \Psi}{\partial X^2}\left[A^2+\left(\frac{\partial \Psi}{\partial Y}\right)^2\right]+\frac{\partial^2 \Psi}{\partial Y^2}\left[A^2+\left(\frac{\partial \Psi}{\partial X}\right)^2\right]-2\frac{\partial^2 \Psi}{\partial X \partial Y}\frac{\partial \Psi}{\partial X}\frac{\partial \Psi}{\partial Y}=0. \tag{28}$$

There is a perfect analogy between equations (26) and (28) if $A^2 = 1$; or since $a = Ac_o$, if $a = c_o$.

By means of equation (33), **the above relation gives**

$$C^2 = C_o^2 + V^2 \tag{29}$$

which is the relation between sound velocity and velocity of flow. If equation (39) is satisfied, the above analogy is perfect.

The velocity of sound at rest ($c = c_\infty$), at the reference point ($c = c_o$) and at any other point in the flow ($c = c$) is connected with the velocity of the flow at rest ($v = o$), at the reference point ($v = v_o$) and at any point in the flow ($v = v$) by the equations,

$$C_\infty^2 = C_o^2 + \frac{\gamma-1}{2} V_o^2 = C^2 + \frac{\gamma-1}{2} V^2. \tag{40}$$

Equation (39) can be transformed to

$$C_\infty^2 - C_o^2 = \frac{\gamma+1}{2} V^2.$$

This result indicates that, since c_∞ and c_o are constants, either the flow velocity must be constant or $c_\infty = c_o$ and $\gamma = -1$. The first possibility has no physical meaning since the flow velocity does vary from point to point. The second case requires the introduction of a gas with an adiabatic gas constant of (-1).

The Mach number is

$$M = \frac{V}{C} = \sqrt{\frac{C_\infty^2 - C_o^2}{C_\infty^2 + \frac{\gamma-1}{2} C_o^2}}$$

From equation (40) it follows that

$$C_\infty > C_o$$

therefore $M < 1$ if $\gamma > -1$

The analogy according to the last result is valid in the subsonic range, with constant Mach number.

If the adiabatic gas constant, $\gamma = -1$, then equation (40) becomes

$$c_\infty^2 = c_o^2 - V_o^2 = c^2 - V^2 \tag{41}$$

By use of equation (39) for $\gamma = -1$,

$$c_\infty = c_o ,$$

therefore in (41), $V_o = 0$, i.e. the reference point turns out to be the point, where the flow is at rest.

For the Hooke gas ($\gamma = -1$) the general relation between density and velocity[27]

$$\rho = \rho_o \left[1 - \frac{\gamma - 1}{2} \frac{V^2}{c_o^2}\right]^{\frac{1}{\gamma - 1}} \tag{42}$$

becomes

$$\rho = \frac{\rho_o}{\sqrt{1 + V^2/c_o^2}} . \tag{43}$$

From (41) and the above relation it follows that

$$c^2 = c_o^2 + V^2 = c_o^2 (\rho_o/\rho)^2 .$$

Using the definition of c^2

$$c^2 = dp/d\rho$$

one obtains the differential equation $\frac{dp}{d\rho} = c_o^2 \frac{\rho_o^2}{\rho^2}$, from which

$$p = K - \frac{c_o^2 \rho_o^2}{\rho} . \tag{44}$$

Figure 2 shows a plot of this formula and the adiabatic relation, $\frac{p}{p_o} = \left(\frac{\rho}{\rho_o}\right)^\gamma$. It is clear that by the $\gamma = -1$ assumption the actual adiabatic curve is substituted by its tangent.[4]

Considering equations (42) and (43), eliminating c_o by $c_o^2 = c^2 - v^2$ and introducing $M = \frac{v}{c}$, one obtains

$$\frac{\rho}{\rho_o} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{-\frac{1}{\gamma - 1}} \tag{45}$$

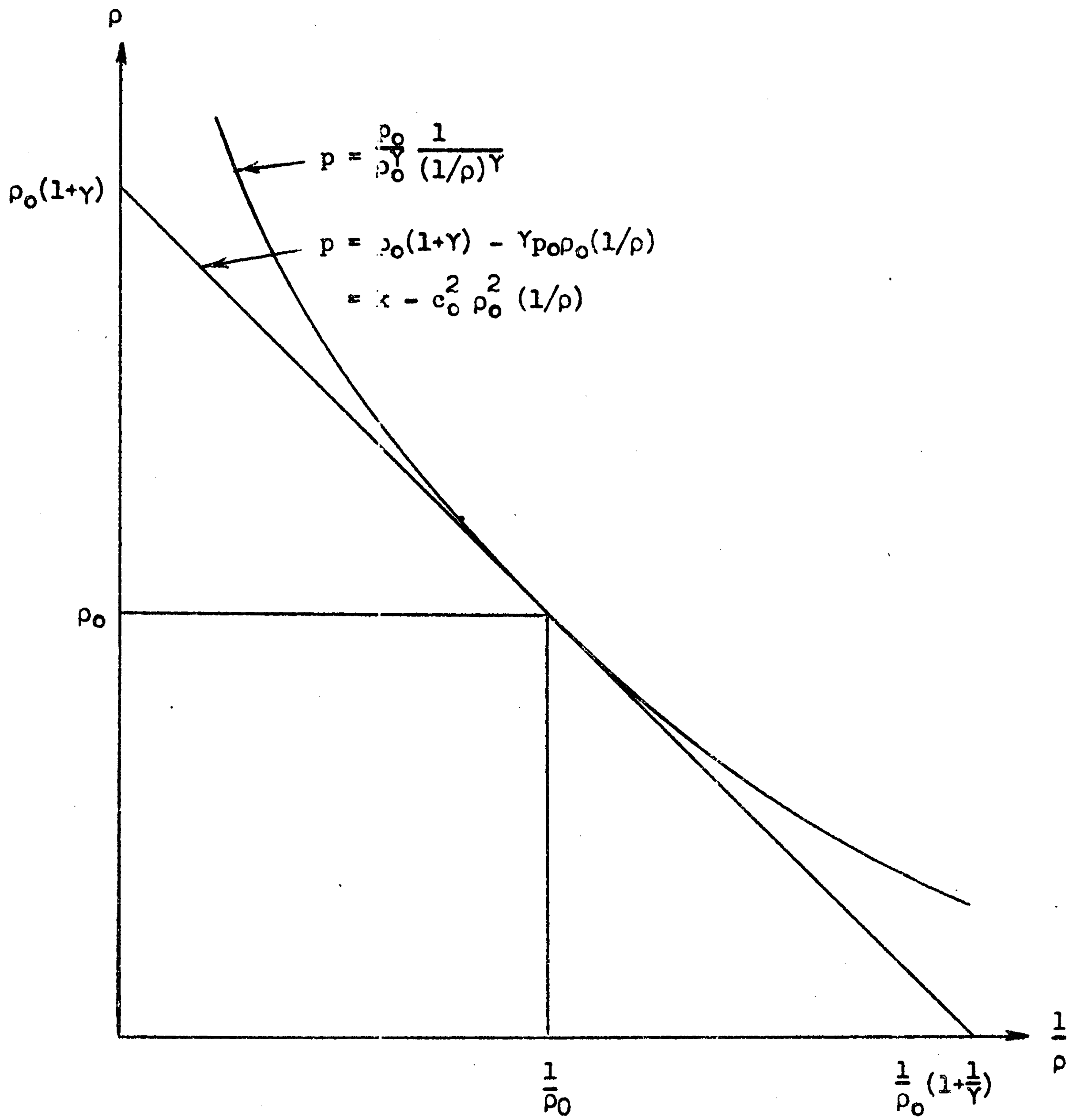


Figure 2. Plot of the adiabatic gas law

and

$$\frac{\rho}{\rho_0} = (1 - M^2)^{\frac{1}{2}} . \tag{46}$$

Equation (45) is the general relation between density and Mach number, while equation (46) connects density and Mach number for the Hooke gas. It is interesting to note that the power series expansions are equivalent if the fourth and higher powers of the Mach number are neglected.

$$\frac{\rho}{\rho_0} = (1 + \frac{\gamma - 1}{2} M^2)^{-\frac{1}{\gamma - 1}} = 1 - \frac{1}{2} M^2 + \frac{1}{8} \gamma M^4 \tag{47}$$

$$\frac{\rho}{\rho_0} = (1 - M^2)^{\frac{1}{2}} = 1 - \frac{1}{2} M^2 - \frac{1}{8} M^4 \tag{48}$$

Figure 3. shows the $\frac{\rho}{\rho_0} = f(M)$ relation for $\gamma = 1.4$ and $\gamma = -1$. If the deviation is defined by

$$\epsilon = 100 \; \frac{\left(\frac{\rho}{\rho_0}\right)_{\gamma = 1.4} - \left(\frac{\rho}{\rho_0}\right)_{\gamma = -1}}{\left(\frac{\rho}{\rho_0}\right)_{\gamma = 1.4}} , \tag{49}$$

then $\epsilon < 1.7\%$ if $M < 0.5$ and $\epsilon < 10\%$ if $M < 0.7$. This indicates that for $M < 0.5$ the error introduced by the $\gamma = -1$ assumption is expected to be less than the experimental errors.

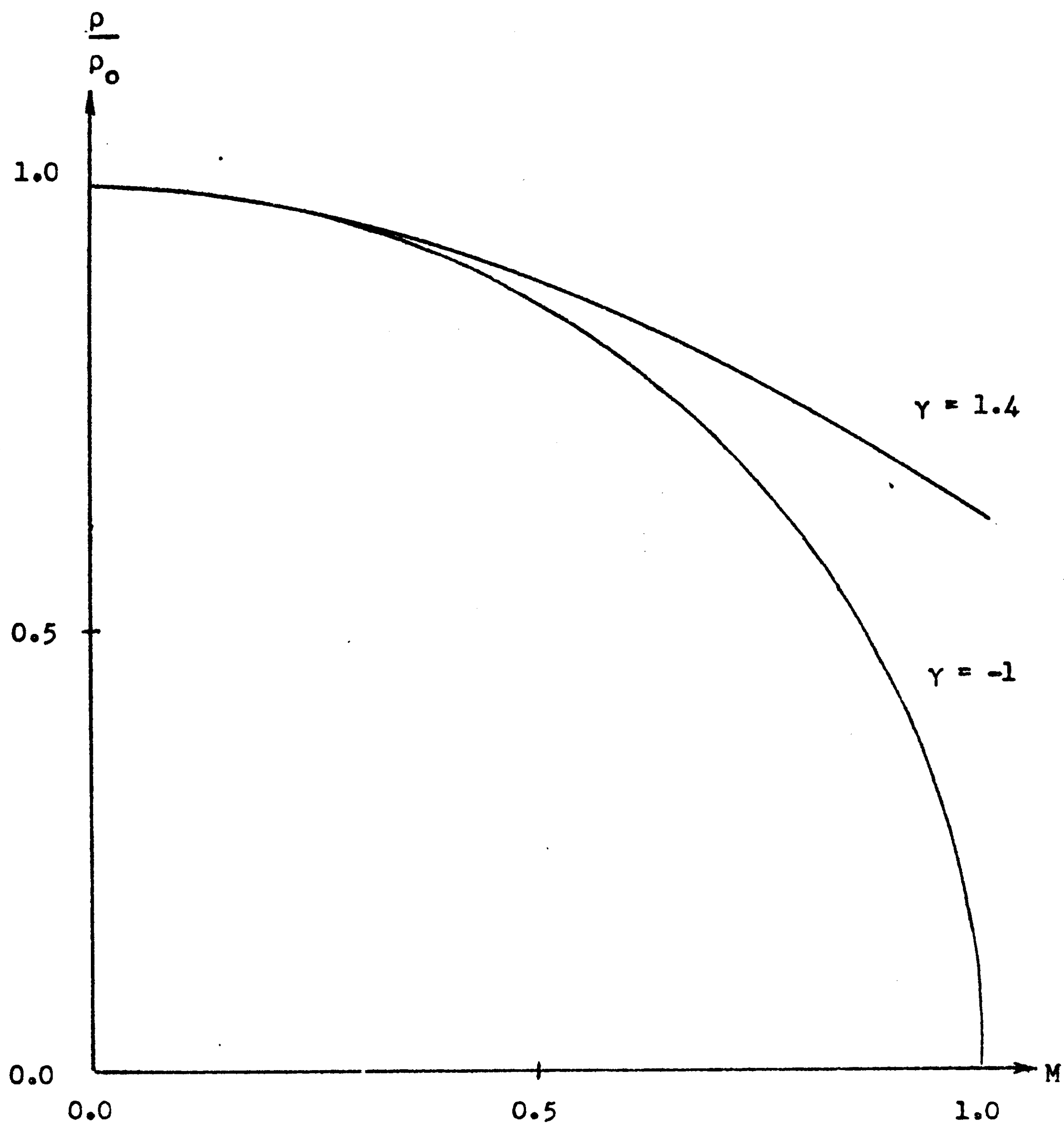


Figure 3. Plot of Mach number vs. density ratio.

IV

EXPERIMENTAL VERIFICATION OF THE ANALOGY

A. A Proposed Experiment

In this part of the thesis, an experiment is proposed which makes use of the analogy given above. A membrane (automatically satisfying the requirements of a minimal surface) is stretched between boundaries given by the fluid mechanical part of the analogy. First a circular cylinder is chosen for which theoretical as well as experimental results are available for comparison. Uniform flow is supposed at infinity, which makes the outside boundaries simple. The prescribed value of the potential function on the circular cylinder is computed from any theoretical solution of the basic differential equation of steady, potential flow of air around the cylinder, neglecting viscosity and considering compressibility.[12]

The slopes of the membrane are measured by means of a telescope supported by a glass plate at a constant height above the datum plane on which the boundaries are erected. The telescope is fixed to the glass plate at a constant angle of 45 degrees. The glass plate is not fixed but may be moved to any desired position. To prevent refraction, a hole is drilled in the glass plate where the line of sight of the telescope passes through the glass.

A long slot is also cut in the glass plate beyond the hole through which the line of sight passes. This slot is in the direction of the line of sight of the telescope. The slot is necessary for the insertion of a depth gage.

The depth gage is used to measure the height of any point for which the slope is desired. A lucite linear scale is fitted into this slot after the depth gage is removed. The scale is formed on the bottom surface to prevent refraction due to light passing through the scale. A light source is located above the scale, and the reflection of the scale from the bubble is read by means of the telescope. The slope of the soap membrane at any point is determined from these measurements.

Figure 4 shows the problem set-up. The operation procedure is as follows. First sight on any desired point by means of the telescope and the depth gage. The depth gage is used to locate and measure the elevation of the desired point. Once the telescope is focused on the desired point, the depth gage is removed, and the lucite scale is placed in the slot. Extreme care must be taken during this operation. If the glass plate is moved, the point on which the telescope is focused will be lost. The light source is then placed above the scale, and the reflection of the scale from the soap membrane is read by means of the telescope.

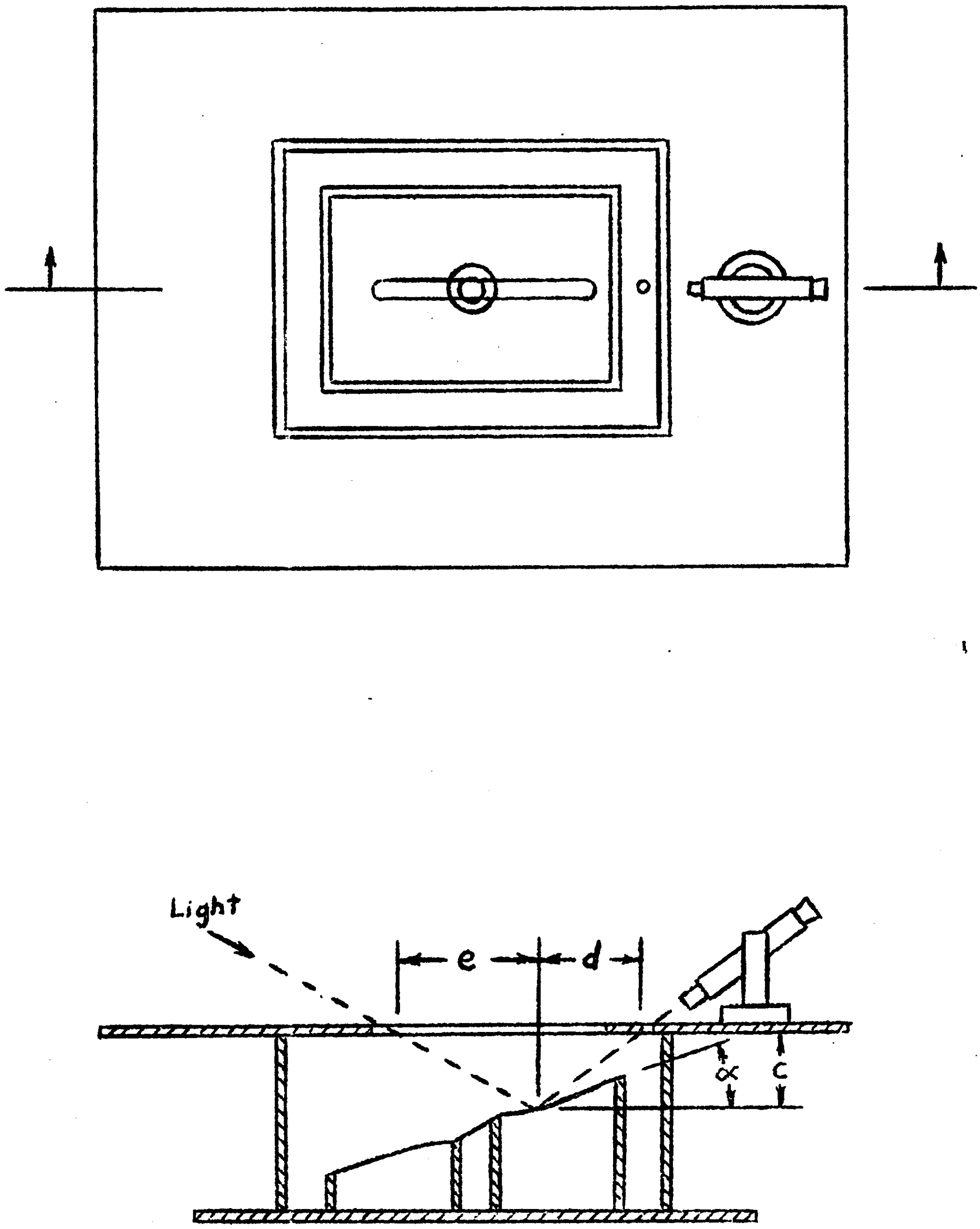


Figure 4. Diagram of Apparatus

The slope of the membrane is then computed by the following formula:

$$\alpha = \frac{1}{2} \arctan\left(\frac{c(e+d)}{c^2 - ed}\right).$$

Figure 4 defines the symbols used in the above formula.

The above procedure is repeated for all desired points. In this experiment the slopes of the membrane at points around the circular cylinder are desired. These slopes give the velocity components of the flow. The measured values of the velocity components may be compared with both theoretical and experimental values obtained by other methods.

B. <u>Compressible Potential Function Around A Circular Cylinder</u>.

The values of the potential function around the circular cylinder were calculated by an approximate method.[12] The method used was the Rayleigh-Janzen method.

This method like most other approximate methods consists of a series expansion. The potential function is expanded in terms of even powers of the free stream Mach number. The first term of the expansion (containing the zero power of the Mach number) is the solution for the incompressible flow problem. Each succeeding term is a corrective term for the effect of compressibility. Classical mathematics is used to determine each required term of the expansion. The terms are solutions of Poisson equations with boundary conditions for the circular cylinder.

A large amount of work is required in the determination of each term, and each succeeding term is considerably longer than the preceding one. After the algebraic expression for each term is derived, the work involved in calculating the numerical values for any given problem is tremendous.

Table 1 gives the results obtained for a cylinder with a radius of one inch. Only three terms of the expansion were considered. The third term affects the results very little since the Mach number to the fourth power is the coefficient for this term. A free stream Mach number of three-tenths was chosen for this calculation. The critical Mach number is reached on the cylinder if the free stream Mach number is approximately forty-two hundredths.

Figure 5 is a plot of the compressible potential function around the circular cylinder. Figure 6 gives the incompressible potential function around the cylinder, and figure 7 shows the difference between the compressible and incompressible potential functions.

The tangential velocity for compressible flow around the circular cylinder was calculated from velocity equations derived by the Rayleigh-Janzen Method. These results are given in table 2, and figure 8 is a plot of the results obtained.

TABLE 1

VALUES OF POTENTIAL AROUND CIRCULAR

CYLINDER WITHOUT CIRCULATION

$M = .3$; $U = 334.8$ ft/ sec; $c_0 = 1116$ ft/ sec; $R = 1$ in.

Φ_o = Incompressible potential function

Φ = Compressible potential function

θ = Angle around cylinder

θ	ϕ_0	ϕ_1	ϕ_2	Φ_0	Φ	$\Phi-\Phi_0$
0	2.0000	.5000	.7026	669.60	686.57	16.97
10	1.9696	.5122	.7085	659.42	676.76	17.34
20	1.8794	.5431	.7344	629.22	647.57	18.35
30	1.7320	.5774	.7938	579.87	599.43	19.56
40	1.5320	.5941	.8793	512.91	533.20	20.29
50	1.2856	.5730	.9472	430.42	450.24	19.82
60	1.0000	.5001	.9273	334.80	352.38	17.58
70	.6840	.3724	.7587	229.00	242.29	13.29
80	.3474	.1992	.4311	116.31	123.47	7.16
90	.0000	.0000	.0000	0.00	0.00	0.00
100	- .3474	-.1992	-.4311	-116.31	-123.47	- 7.16
110	- .6840	-.3724	-.7587	-229.00	-242.29	-13.29
120	-1.0000	-.5001	-.9273	-334.80	-352.38	-17.58
130	-1.2856	-.5730	-.9472	-430.42	-450.24	-19.82
140	-1.5320	-.5941	-.8793	-512.91	-533.20	-20.29
150	-1.7320	-.5774	-.7938	-579.87	-599.43	-19.56
160	-1.8794	-.5431	-.7344	-629.22	-647.57	-18.35

TABLE 1 (CONTINUED)

θ	ϕ_0	ϕ_1	ϕ_2	Φ_0	Φ	$\Phi-\Phi_0$
170	-1.9696	-.5122	-.7085	-659.42	-676.76	-17.34
180	-2.0000	-.5000	-.7026	-669.60	-686.57	-16.97
190	-1.9696	-.5122	-.7085	-659.42	-676.76	-17.34
200	-1.8794	-.5431	-.7344	-629.22	-647.57	-18.35
210	-1.7320	-.5774	-.7938	-579.87	-599.43	-19.56
220	-1.5320	-.5941	-.8793	-512.91	-533.20	-20.29
230	-1.2856	-.5730	-.9472	-430.42	-450.24	-19.82
240	-1.0000	-.5001	-.9273	-334.80	-352.38	-17.58
250	- .6840	-.3724	-.7587	-229.00	-242.29	-13.29
260	- .3474	-.1992	-.4311	-116.31	-123.47	- 7.16
270	.0000	.0000	.0000	0.00	0.00	0.00
280	.3474	.1992	.4311	116.31	123.47	7.16
290	.6840	.3724	.7587	229.00	242.29	13.29
300	1.0000	.5001	.9273	334.80	352.38	17.58
310	1.2856	.5730	.9472	430.42	450.24	19.82
320	1.5320	.5941	.8793	512.91	533.20	20.29
330	1.7320	.5774	.7938	579.87	599.43	19.56
340	1.8794	.5431	.7344	629.22	647.57	18.35
350	1.9696	.5122	.7085	659.42	676.76	17.34
360	2.0000	.5000	.7026	669.60	686.57	16.97

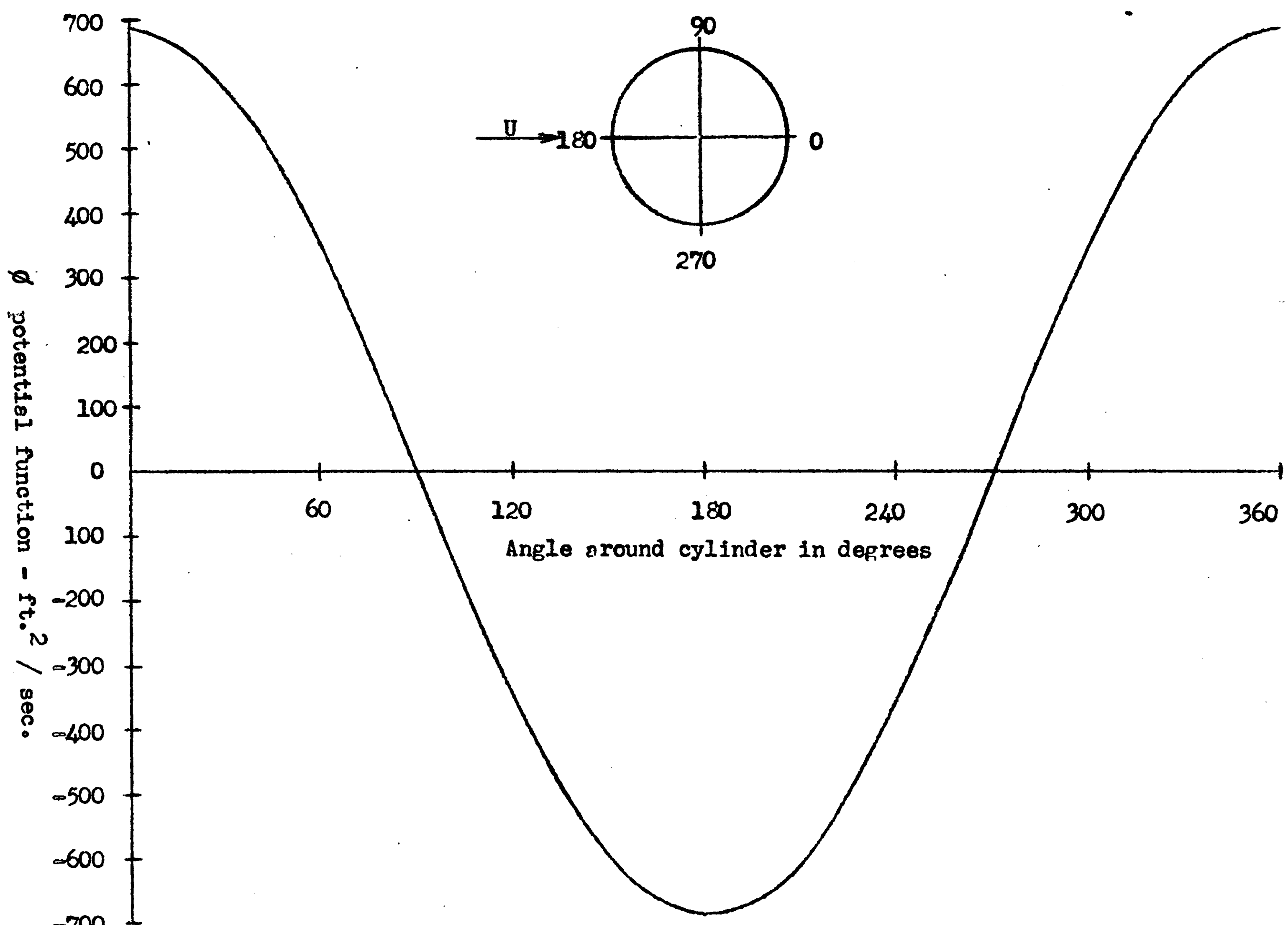


Figure 5. Plot of the compressible potential function around a circular cylinder without circulation.

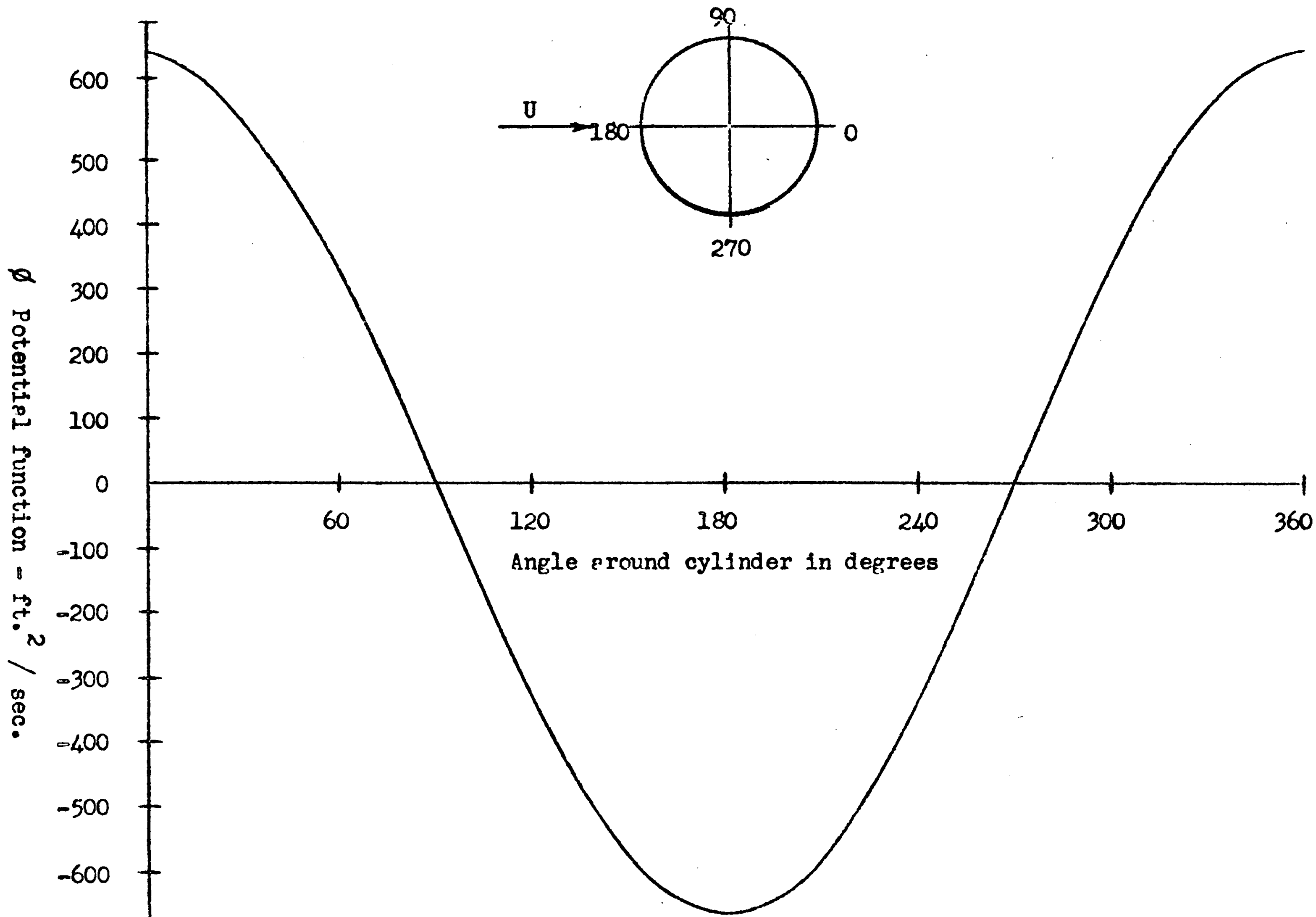


Figure 6. Plot of the incompressible potential function around a circular cylinder without circulation.

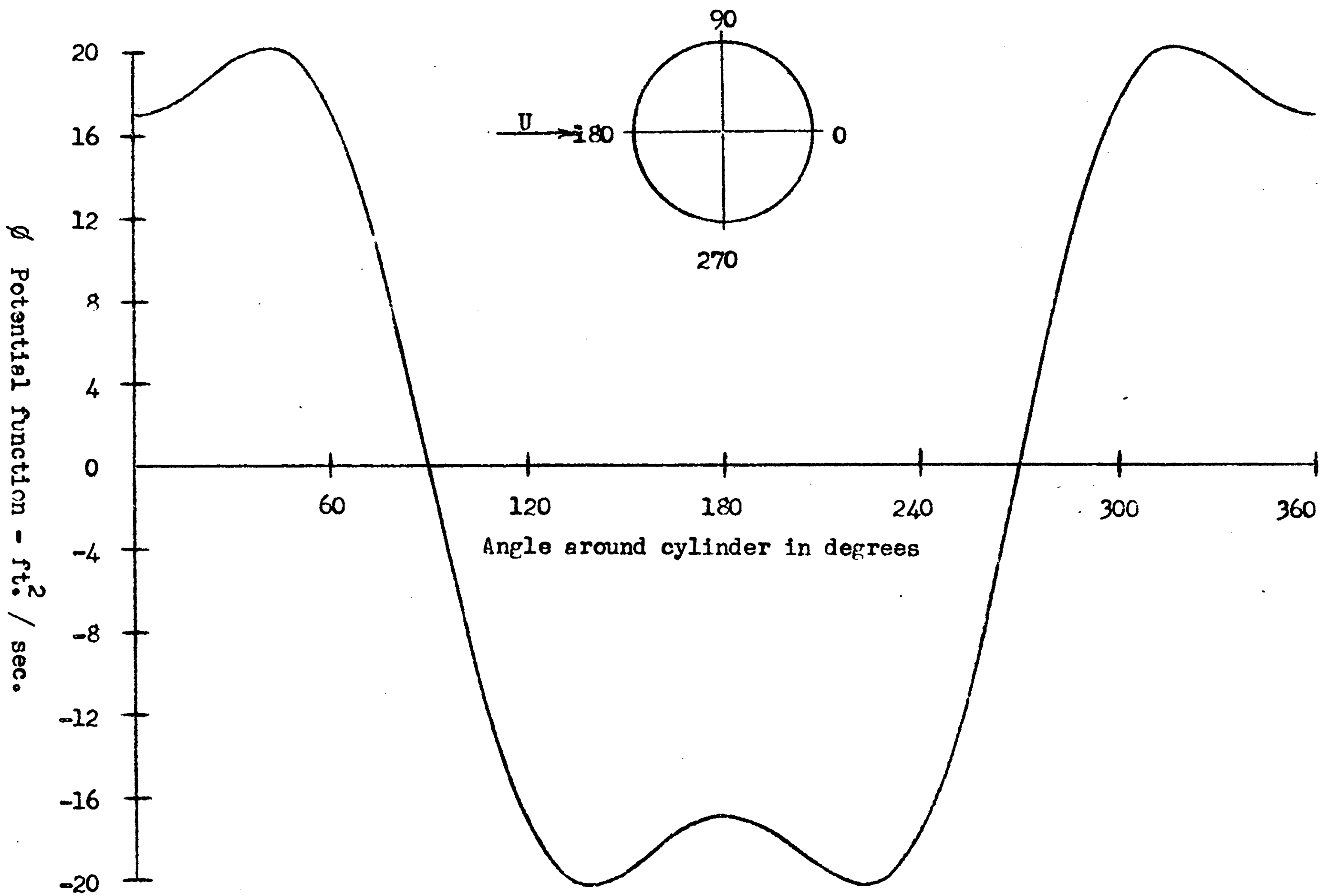


Figure 7. Plot of the difference between the incompressible and the compressible potential function around a circular cylinder without circulation.

TABLE 2

TANGENTIAL VELOCITY FOR COMPRESSIBLE FLOW

AROUND A CIRCULAR CYLINDER WITHOUT CIRCULATION

$M = 0.3$; Radius of cylinder $= 1$; $U = 334.8$ ft/sec;

$c_0 = 1116$ ft/sec

θ	Velocity	θ	Velocity
0	0.00	190	112.06
10	112.06	200	222.19
20	222.19	210	328.59
30	328.58	220	428.93
40	428.93	230	520.20
50	520.20	240	598.62
60	598.62	250	659.55
70	659.55	260	698.37
80	698.37	270	711.75
90	711.75	280	698.37
100	698.37	290	659.55
110	659.55	300	598.62
120	598.62	310	520.20
130	520.20	320	428.93
140	428.93	330	328.59
150	328.59	340	222.19
160	222.19	350	112.06
170	112.06	360	0.00
180	0.00		

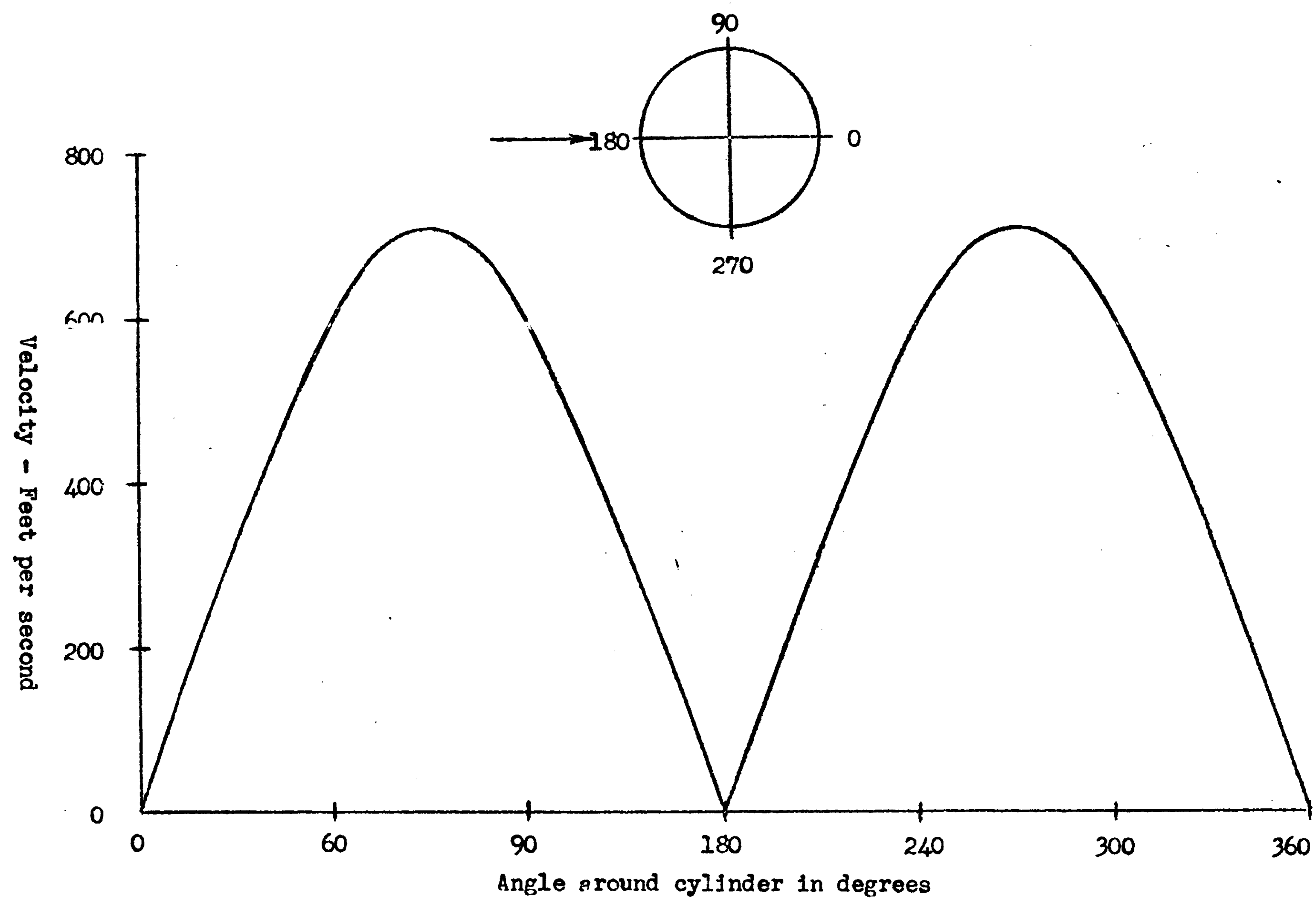


Figure 8. Plot of the tangential velocity for compressible flow around a circular cylinder without circulation.

C. Membrane Boundaries

The inside boundary of the soap membrane is formed from the values of the compressible potential function around the circular cylinder as determined by the Rayleigh-Jansen Method. A steel cylinder whose radius is one inch was used. The height of the cylinder was made everywhere equal to the potential function. The base of the cylinder was used as a datum whose value was minus 3000 feet squared per second.

A square outside boundary was used for ease of fabrication. The square is fourteen inches on a side. Undisturbed flow was assumed at this distance from the cylinder since the effect of the cylinder on this boundary is negligible. Figures 9 and 10 show the undisturbed boundary heights. The points plotted are values determined by the Rayleigh-Jansen Method. Table 3 gives the values for the potential function on the outside boundary as calculated. Figure 11 is a plot of the points on the boundary that correspond to the calculated values.

The outside of the circular cylinder boundary and the inside of the square boundary were undercut at an angle to prevent the soap bubble from moving down the boundaries.

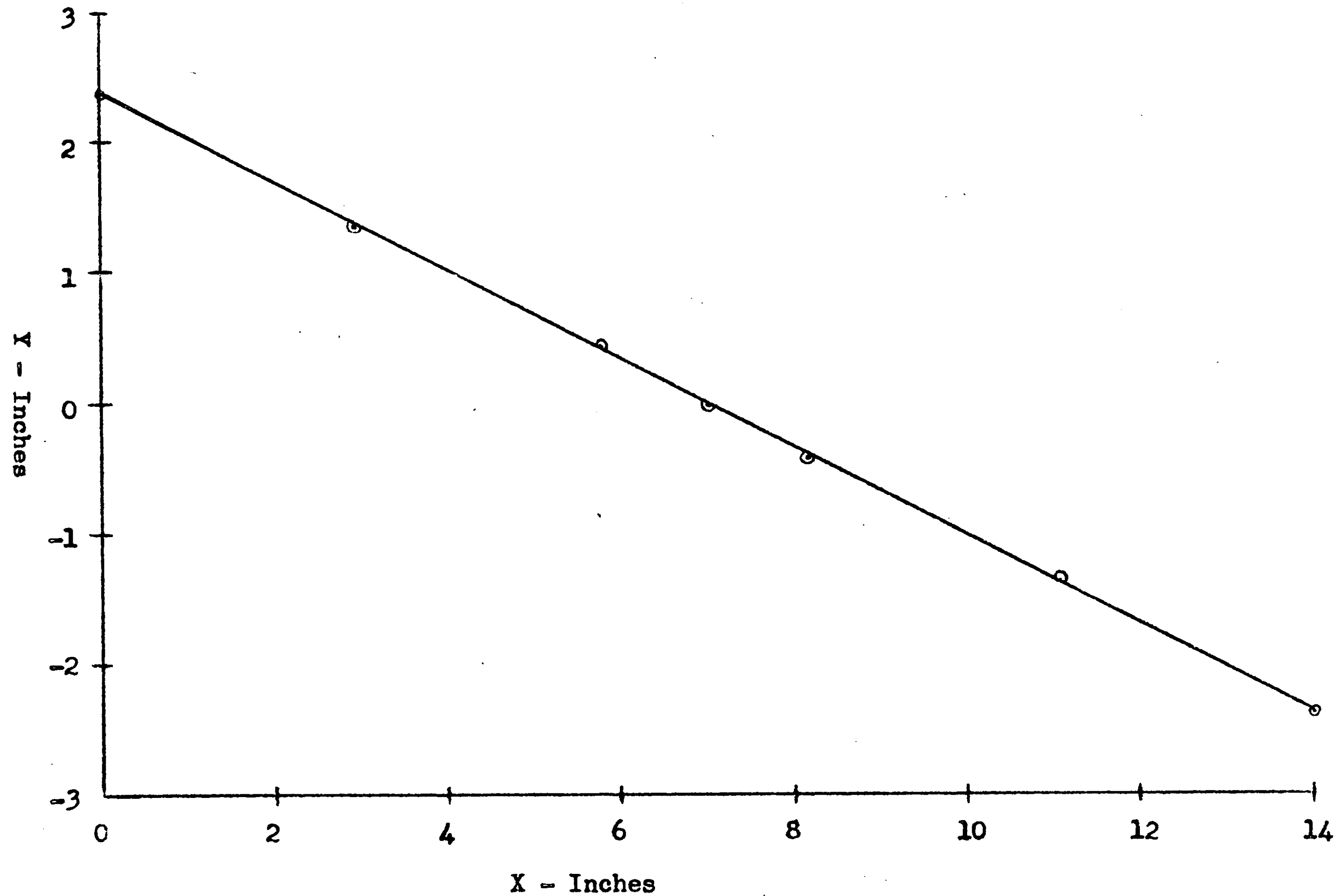


Figure 9. Shape of the outside boundry in the direction of flow.

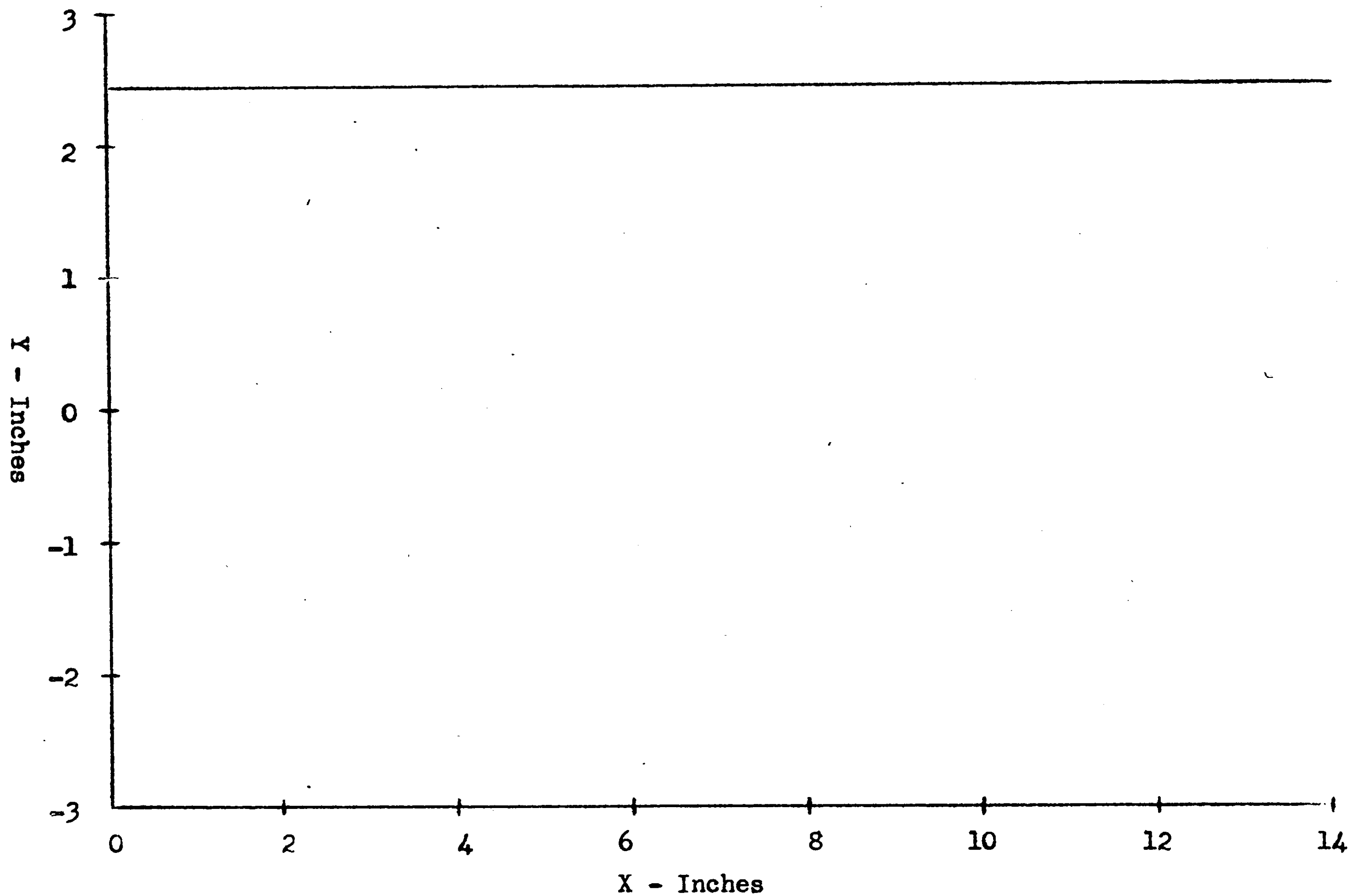


Figure 10. Shape of the outside boundary across the direction of flow.

TABLE III

OUTSIDE BOUNDARY VALUES OF THE POTENTIAL FUNCTION

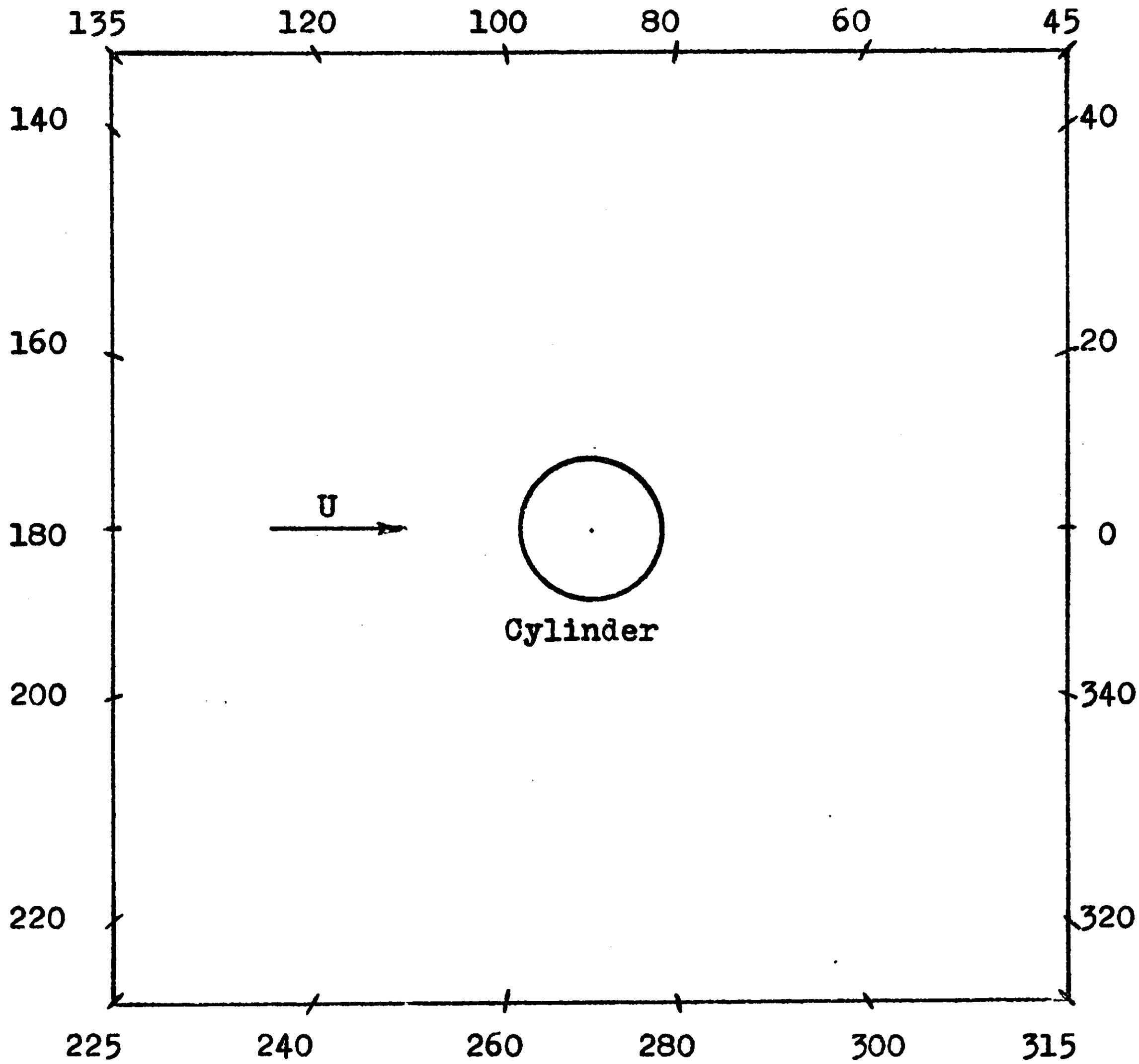
$M = 0.3$; $U = 334.8$ ft/sec; $c_o = 1116$ ft/sec; Radius = 1 in.

Φ_o = Incompressible potential function

Φ = Compressible potential function

θ = Angle around cylinder

θ	θ	ϕ_c	ϕ	Φ_o	Φ
0	360	7.1429	.1193	2391.44	2395.04
20	340	7.1259	.1188	2385.75	2389.33
40	320	7.0835	.1040	2371.56	2374.69
45	315	7.0717	.0948	2367.61	2370.46
60	300	4.1034	.0973	1373.82	1376.75
80	280	1.2591	.0437	421.55	422.86
90	270	0.0000	.0000	0.00	0.00
100	260	-1.2591	-.0437	- 421.55	- 422.86
120	240	-4.1034	-.0973	-1373.82	-1376.75
135	225	-7.0717	-.0948	-2367.61	-2370.46
140	220	-7.0835	-.1040	-2371.56	-2374.69
160	200	-7.1259	-.1188	-2385.75	-2389.33
180	180	-7.1429	-.1193	-2391.44	-2395.04



Scale: One-half

Figure 11. Location of outside boundary points

D. Soap Solutions

Many soap solutions were used in the attempt to find a solution which was stable. None of the solutions available were very stable. In fact, none of the solutions used gave bubbles which were stable enough to allow any measurements to be made.

The first solution given in Table 4 proved to be the best, although this solution gave membranes which lasted for only ten seconds under ordinary conditions. In an attempt to increase the stability of the soap membrane, the apparatus was moved into the one hundred percent humidity room of the Applied Mechanics Department of the Virginia Polytechnic Institute. The dust content of the air and the rate of evaporation of the soap solution are important factors in the stability of a soap membrane.[6,7] The increased humidity decreased both of these undesirable factors, but the membranes still lasted only thirty seconds which is not a long enough period of time to permit any measurements.

Table 4 gives the various solutions which have been used for soap bubbles. In addition a number of commercial soap bubble solutions were tested. None of these solutions produced membranes with the required stability.

Table 4. Soap Solutions

Solution No.

1.

a. 400 parts Orvus WA Paste
60 parts Triethanolamine

b. 99.5 parts water
.5 parts CMC High Viscosity

Bubble Solution: 20 parts a.
80 parts b.

2. 25 parts nonionic (Triton)
75 parts water
35 parts glycerine

3. 5 parts coconut oil potash soap
2 parts glycerine
1 part Methyl Cellulose
92 parts water

4. 1 part pure castile or palm oil soap
8 parts distilled water
4 parts pure glycerine

5. 2 parts castile soap
30 parts glycerine
40 parts water

6. 25 parts hard soap
15 parts glycerine
1000 parts water

Dissolve soap in the water, add glycerine, and mix thoroughly. On standing the liquid becomes clear at the bottom. The clear liquid is drawn off and keeps indefinitely. It is this portion that is used for making tough, long-lasting soap bubbles.

7. 10 gm. Sodium Oleate
400 cc. water
100 cc. glycerine

Dissolve oleate in water with occasional shaking. Do not heat. Then add the glycerine and allow to stand for two to three days. Remove the clearer portion and add one drop of stronger ammonium hydroxide. Do not filter.

8. 10 cc. of pure oleic acid
76 cc. water
54 cc. glycerine

Shake well and add:

3.23 grms. of TEA in
30 cc. of water

9. 15 parts coconut potash soap (anhyd)
2 parts Gum arabic
6 parts glycerine
0.3 parts basic dye
76.7 parts water

10. 6 parts gelatin
50 parts water
12 parts glycerine
13 parts diethylene glycol
7 parts denatured alcohol
12 parts Nac. N.R.S.F.
Orvus WA Paste

Soak the gelatin in the water until swelled and then heat until dissolved. Mix the glycerine, glycol and alcohol and add this solution to the gelatin portion. Finally dissolve in the Nacconol.

11. 5% sodium alkyl sulfate or sodium alkyl aryl sulfonate
4% crude CMC (Carbory methyl cellulose)
91% water

12. 3.5% Ammonium oleate
33 % glycerine
63.5% water

13. 5% ammonium oleate
50% glycerine
45% water

14. 1% ammonium oleate
10% glycerine
89% water

15. 50% glycerine
2.5% soap
47.5% water

16. 25% glycerine
4% alcohol
5% soap
66% water

17. 5% potassium oleate
50% glycerine
45% water

18. 1 cc. triethanolamine oleate
5 cc. water
5 cc. glycerine

19. Various proportions of Atlantic Ultra Wash and ethylene glycol

20. Various proportions of Atlantic Ultra Wash and glycerine

21. 2 gm. sodium oleate
1 liter water
30 cc. glycerine

22. 1 gm. sodium oleate
2 liters water
6 cc. glycerine per liter of solution

E. Rubber Membranes

After none of the soap solutions proved to be sufficiently stable, a rubber membrane was tested. One-fifth inch squares were ruled on an unstretched thin rubber sheet. The rubber sheet was then placed on the boundaries and stretched a nominal amount.

To insure uniform tension, the membrane was stretched at all points until each square was a perfect square and every square was the same size. Also all lines had to remain straight. This required innumerable adjustments at the boundaries. Since the rubber membrane is subjected to uniform tension, it satisfies the equation of the soap membrane and may be substituted for the soap membrane.

Aluminum angle was fastened to the outside square boundary, and the membrane was clamped to the angle by small, strong paper clamps. An aluminum plate was cut to the exact shape of the inside circular boundary. The rubber membrane was clamped between the boundary and the aluminum plate. The plate was secured to the boundary by means of a screw in the center.

Time did not permit the actual measurement of the slopes around the circular cylinder; therefore, no results are available for comparison with the theoretical results obtained.

F. Slope Measurements

The slope of the membrane gives the velocity components of the flow at all points. The only points of interest are the slopes at the boundary of the circular cylinder.

As suggested in a previous section of the thesis, the slopes may be measured with a telescope. If a rubber membrane is used, it must be coated with some substance which will reflect light. Another method of slope determination for this problem is contour mapping. Contour mapping may prove very useful if a rubber membrane is used rather than a soap bubble.

V.

FUTURE RECOMMENDATIONS

A number of suggestions which may prove helpful in future experimental work are given below.

Soap membranes which are stable present the most difficult problem. There are a number of possible solutions to this problem. First, a solution which is sufficiently stable may be found through continued research. Second, the size of the soap membrane may be decreased. If the size is decreased to the point where the present solutions are sufficiently stable, the experimental errors will increase considerably. The most promising solution is that the entire apparatus be enclosed in glass, lucite, or some other transparent material. The air in the enclosure could then be purified, getting rid of all dust particles and other foreign matter. Perfectly clean surrounding air is necessary for soap membranes of long duration. (6,7)

VI

SUMMARY

The results can best be summarized in a table showing the corresponding quantities in the analogy presented in this thesis.

Table 5

	Compressible flow	Soap Membrane
Basic equations	$\Psi_{XX}(1+\Psi_Y^2)+\Psi_{YY}(1+\Psi_X^2)$ $-2\Psi_{XY}\Psi_X\Psi_Y=0$	$Z_{XX}(1+Z_Y^2)+Z_{YY}(1+Z_X^2)$ $-2Z_{XY}Z_XZ_Y=0$
Symbols	Ψ : modified potential function X,Y: coordinates in the reference plane	Z: Height of soap membrane X,Y,:coordinates in the reference plane
Assumptions	adiabatic gas constant, $\gamma = -1$	soap membrane is unloaded
Derivatives	velocity components: $\frac{U_X}{C_0}$ and $\frac{U_Y}{C_0}$	Slopes: t_1 and t_2
Symbols	U_X = velocity component in the x direction U_y = velocity component in the y direction C_0 = velocity of sound at rest	$t_1 = \frac{\partial z}{\partial x}$ $t_2 = \frac{\partial z}{\partial y}$
Ratios	$\sqrt{\frac{U_X^2 + U_Y^2}{C^2}} = M$	$\sqrt{\frac{t_1^2+t_2^2}{1+t_1^2+t_2^2}} = \sin\alpha$
Symbols	M: Mach number c: local sound velocity	α : angle between surface normal and z axis.

VII

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VIII

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