

ACTIVE CONTROL OF DISTRIBUTED STRUCTURES

by

Lawrence M. Silverberg

Dissertation submitted to the Faculty of the
Virginia Polytechnic Institute and State University
in partial fulfillment of the requirements for the degree of
DOCTOR OF PHILOSOPHY
in
Engineering Mechanics

APPROVED:

L. Meirovitch, Chairman

D. T. Mook

M. P. Singh

R. A. Heller

W. L. Hallauer

June, 1983

Blacksburg, Virginia

ACKNOWLEDGMENTS

I am deeply indebted to Prof. Leonard Meirovitch for the many opportunities and guidance he has provided me in the course of my graduate education. Indeed, the contributions he has made to my education are substantial. All of his labor in my behalf is most sincerely appreciated.

I am grateful to Drs. R. A. Heller, M. P. Singh, D. T. Mook, and W. L. Hallauer for serving as members of my examining committee and for their review of this dissertation.

TABLE OF CONTENTS

	<u>Page</u>
ACKNOWLEDGEMENTS	ii
LIST OF TABLES	v
LIST OF FIGURES	vi
 CHAPTER	
1. INTRODUCTION	1
1.1 Preliminary Remarks	1
1.2 Outline	2
2. TWO BRACKETING THEOREMS CHARACTERIZING THE EIGENSOLUTION FOR THE H-VERSION OF THE FINITE ELEMENT METHOD	5
2.1 Introduction	5
2.2 The Classical Rayleigh-Ritz Method and the Inclusion Principle	7
2.3 The Inclusion Principle and the Hierarchical Finite Element Method	10
2.4 Second-Order (In Space) Problems	12
2.5 Fourth-Order (In Space) Problems	15
2.6 Two Bracketing Theorems for the Finite Element Method for Fourth-Order Problems	20
2.7 Numerical Example	23
2.8 Conclusions	24
3. GLOBAL OPTIMAL CONTROL OF SELF-ADJOINT DISTRIBUTED STRUCTURES	33
3.1 Introduction	33
3.2 Equations of Motion of a Self-Adjoint Distributed- Parameter System	35
3.3 Modal Equations of Motion	36
3.4 Controllability and Observability	38
3.5 Coupled Control	40
3.6 Natural Control	42
3.7 Control Sensitivity to System Parameters	47
3.8 Control Implementation	49
3.9 Natural Control of a Simply-Supported Beam	53
a. Distributed Control Force	54
b. Discrete Point Control Forces	55
c. Piecewise Constant Control Forces	55

	<u>Page</u>
d. Discrete Measurements	56
3.10 Natural Control of a Simply-Supported Rectangular Membrane	57
a. Distributed Control Force	58
b. Discrete Point Control Forces	59
c. Piecewise Constant Control Forces	59
d. Discrete Measurements	60
4. CONTROL OF NON-SELF-ADJOINT DISTRIBUTED STRUCTURES	80
4.1 Introduction	80
4.2 Equations of Motion of Non-Self-Adjoint Distributed Structures	82
4.3 Characterization of Non-Self-Adjoint Distributed Structures	84
4.4 Independent Modal-Space Control	86
a. Steady-State Solution	89
b. Transient Solution	89
4.5 Control Implementation	90
4.6 Numerical Example	93
4.7 Conclusions	94
5. BLOCK-INDEPENDENT CONTROL OF DISTRIBUTED STRUCTURES	100
5.1 Introduction	100
5.2 Equations of Motion for Self-Adjoint Distributed- Parameter Systems	102
5.3 Modal Equation for the DPS	103
5.4 Control Implementation	104
5.5 Block-Independent Control of the DPS	105
5.6 Linear Optimal Control	108
5.7 Numerical Example	109
5.8 Conclusions	111
REFERENCES	115
VITA	118

LIST OF TABLES

<u>Table</u>		<u>Page</u>
2.1	Computed eigenvalues for first- and second-order approximations	26
3.1	Open- and closed-loop eigenvalues for beam controlled with a distributed actuator	62
3.2	Closed-loop eigenvalues for beam controlled with discrete point actuators	63
3.3	Closed-loop eigenvalues for beam controlled with piecewise constant actuators	64
3.4	Closed-loop eigenvalues for beam controlled using discrete measurements	65
3.5	Open- and closed-loop eigenvalues for membrane controlled with a distributed actuator	66
3.6	Closed-loop eigenvalues for membrane controlled with discrete point actuators	67
3.7	Closed-loop eigenvalues for membrane controlled with piecewise constant actuators	68
3.8	Closed-loop eigenvalues for membrane controlled using discrete measurements	69
4.1	System parameters and control parameters for a whirling shaft	95
4.2	Open-loop eigenvalues for a whirling shaft	96
4.3	Closed-loop eigenvalues for a whirling shaft	97
5.1	Closed-loop poles	112

LIST OF FIGURES

<u>Figure</u>		<u>Page</u>
2.1	Typical finite element for second-order problems	27
2.2	Subdivision of a typical finite element for h- and p-versions for second-order problems	28
2.3	Typical finite element for fourth-order problems	29
2.4	Subdivision of a typical finite element for h- and p-versions for fourth-order problems	30
2.5	First and second bracketing theorems	31
2.6	Numerical example	32
3.1	Natural control of a simply supported beam	70
3.2	Free response of simply supported beam	71
3.3	Response of simply supported beam using a distributed actuator	72
3.4	Response of simply supported beam using 6 discrete actuators	73
3.5	Response of simply supported beam using 6 piecewise constant actuators	74
3.6	Natural control of a rectangular membrane	75
3.7	Free response of simply-supported rectangular membrane .	76
3.8	Response of simply-supported rectangular membrane using a distributed actuator	77
3.9	Response of simply supported rectangular membrane using 16 discrete actuators	78
3.10	Response of simply-supported rectangular membrane using 16 piecewise-constant actuators	79
4.1	Uniform shaft whirling at constant angular velocity Ω .	98
4.2	Controlled responses of whirling shaft	99

<u>Figure</u>		<u>Page</u>
5.1	Block-diagram for block-independent control	113
5.2	Response of beam and control effort	114

Chapter I

INTRODUCTION

1.1 PRELIMINARY REMARKS

The problem of control of distributed structures has received a great deal of attention in recent years. This can be attributed, at least in part, to the prospect of placing large flexible structures in space.

The motion of a distributed structure is characterized by its frequencies and mode shapes. Based on these quantities, it is of interest to design external forces on the distributed structure which will control its motion. Moreover, the control system is designed to be optimal in some manner. By optimal, it is implied that some given cost function associated with the control is minimized. For example, it is often desirable to control the motion of a structure with minimum control effort.

External forces at discrete points on the structure can be generated by means of actuators where the number of actuators varies. As more actuators are taken, the minimum of the cost function associated with the optimal control problem decreases. In the limit, the infinite set of discrete actuators represents a distributed actuator and minimization of the cost function leads to a truly optimal solution of the control problem. Indeed, it is not surprising that globally optimal control forces are distributed. Note that this solution is desirable even when the use of distributed forces is undesirable. Indeed, the

optimal solution can serve as a measure for purposes of comparison and leads to insights into the suboptimal problem of controlling a distributed system using discrete actuators.

The frequencies and mode shapes, also referred to as eigenquantities, are the solution to a differential eigenvalue problem associated with the distributed structure. In general, the differential eigenvalue problem admits no closed-form solution. Hence discretization must be used which reduces the differential eigenvalue problem to an algebraic eigenvalue problem. A commonly used discretization method is the finite element method. Convergence of the algebraic eigenvalue problem derived by the finite element method to the differential eigenvalue problem is important if one wishes to compute accurately the eigenquantities needed for control.

As a consequence of discretization, errors are introduced. Perhaps more significant, the system parameters for these structures are not known exactly, so that parameter errors are introduced, as well. Indeed, the performance of the control in the presence of these errors raises the question of control robustness.

The analysis presented in this dissertation has been divided into four chapters, each outlined in the next section.

1.2 OUTLINE

As mentioned earlier, convergence of the algebraic eigenvalue problem to the differential eigenvalue problem is important if one wishes to compute accurately the eigenquantities needed for control. In the second chapter, it is demonstrated that the classical inclusion

principle is in general not valid for the h-version of the finite element method. Whereas the inclusion principle is shown to be valid for second-order systems discretized by the h-version of the finite element method, provided linear interpolation functions are used, the principle is not valid for fourth-order systems. To characterize the convergence of the computed eigenvalues for fourth-order systems discretized by the h-version of the finite element method, two bracketing theorems are formulated. A numerical example, demonstrating these ideas, is presented.

A common approach to the problem of control of distributed structures amounts to treating the distributed structure as though it were discrete in space. As a consequence, the vast resource of past work in the control of discrete systems can be utilized. This approach, however, has several undesirable effects. In the first place, the control can lack many of those characteristics which are exclusively associated with a distributed system. For example, the use of distributed controls and distributed sensors must be ruled out, as they no longer can be handled by a discrete in space formulation. In the third chapter, an alternate approach, known as natural control, is developed. The optimal control forces are ideally distributed. However, it is often desirable to use discrete actuators. It is shown in this chapter how to implement natural control using discrete control forces and discrete measurements. Two numerical examples, illustrating the method, are given.

The structures considered in chapters two and three can be characterized by a single operator possessing a self-adjointness

property. However, gyroscopic forces, circulatory forces as well as damping each tend to destroy this property. In chapter four, the problem of control of distributed systems is extended to non-self-adjoint distributed systems. A numerical example, demonstrating the method, is presented.

The motion of a distributed structure is generally governed by a set of simultaneous partial differential equations which can be transformed into an infinite set of modal equations of motion. These equations are independent in the absence of feedback controls. However, it is common practice to design feedback controls which recouple the modal equations of motion. Control of this form is known as coupled control. An alternate approach is known as independent modal-space control. For independent modal-space control the modal equations remain decoupled even in the presence of control feedback. However, the method requires as many discrete actuators as controlled modes. As a compromise between coupled control and independent modal-space control, a block-independent control method is proposed. It is intended to reduce the computational difficulties encountered in coupled control and to relax the requirement on the number of actuators encountered in independent modal-space control. A numerical example is presented.

Chapter II

TWO BRACKETING THEOREMS CHARACTERIZING THE EIGENSOLUTION FOR THE H-VERSION OF THE FINITE ELEMENT METHOD

2.1 INTRODUCTION

The eigenvalue problem associated with distributed structures in general does not admit a closed form solution, so that the only alternative is an approximate solution. For self-adjoint systems, frequently used discretization procedures are the classical Rayleigh-Ritz method (Refs. 1 and 2) and the finite element method (Refs. 1-4), where the latter can be regarded as a special case of the former (although current use of the finite element method goes well beyond that of the classical Rayleigh-Ritz method). Because the discretized model is only an approximation of the distributed system, the question arises as to how the computed eigenvalues obtained by solving the algebraic eigenvalue problem for the discretized model relate to the actual eigenvalues that would have been obtained if the solution of the differential eigenvalue problem were possible.

In the classical Rayleigh-Ritz method, the eigensolution is assumed in the form of a finite series of global admissible functions multiplied by undetermined coefficients. The usual procedure results in an algebraic eigenvalue problem yielding approximations for the lowest eigenfunctions. The approximations can be improved by simply adding terms to the series. If just one term is added to the series, then the old and new eigenvalues satisfy a classical inclusion principle, according to which the new eigenvalues bracket the old

eigenvalues (Refs. 2, 5 and 6).

It is widely assumed that the classical inclusion principle governing the approximate eigenvalues produced through discretization via the classical Rayleigh-Ritz method is also valid for approximate eigenvalues obtained through application of the finite element method. In fact, the validity of the inclusion principle for the finite element method is often taken for granted. Yet no general proof has ever been produced. It turns out that, whereas the inclusion principle is valid for some special cases of the finite element method, it is not valid in general.

We must distinguish between two versions of the finite element method. The first one is the h-version, in which improved approximations are sought by refining the finite element mesh (Refs. 1-4). The second and less common version is the p-version of the finite element method, in which improved approximations are sought by increasing the number of admissible functions over given finite elements (Refs. 7-9). Of particular interest in the p-version are the hierarchical finite elements (Refs. 10-12), which have the property that they preserve the imbedding property of the coefficient matrix, which is characteristic of the classical Rayleigh-Ritz method. Quite recently, the classical inclusion principle was proved to be valid for the hierarchical finite element method (Ref. 13).

This chapter is concerned with the existence of an inclusion principle for the h-version of the finite element method. It is demonstrated that the classical inclusion principle is valid for second-order systems discretized by the h-version of the finite

element method, provided linear interpolation functions are used. More significantly, however, it demonstrates that the classical inclusion principle is not valid in general for fourth-order systems, which are important systems in structural dynamics. For fourth-order systems discretized by the h-version of the finite element method, the classical inclusion principle is replaced with two bracketing theorems characterizing convergence of the computed eigenvalues.

2.2 THE CLASSICAL RAYLEIGH-RITZ METHOD AND THE INCLUSION PRINCIPLE

The eigenvalue problem associated with a self-adjoint distributed-parameter system can be written in the form (Ref. 2)

$$L\phi(P) = \lambda_m(P) \phi(P) , \quad P \in D \quad (2.1)$$

where L is a self-adjoint differential operator of order $2p$, m is the mass density, ϕ is a function of the spatial position P and D is the domain of extension of the system. Moreover, $\phi(P)$ is subject to the boundary conditions

$$B_i \phi = 0 , \quad i = 1, 2, \dots, p , \quad P \in S \quad (2.2)$$

where B_i are differential operators of maximum order $2p-1$ and S is the boundary of D . The solution of Eqs. (2.1) and (2.2) consists of a denumerably infinite set of real eigenvalues λ_r and real eigenfunctions $\phi_r(P)$ ($r = 1, 2, \dots$). Assuming that L is positive definite, all the eigenvalues are positive. For convenience they can be arranged in ascending order, i.e., $\lambda_1 \leq \lambda_2 \leq \dots$.

The eigenvalue problem given by Eqs. (2.1) and (2.2) can be replaced by an equivalent problem consisting of rendering stationary

Rayleigh's quotient

$$R = \frac{\langle \phi, L\phi \rangle}{\langle \phi, m\phi \rangle} \quad (2.3)$$

where $\langle \phi, L\phi \rangle$ is an energy inner product related to the potential energy and $\langle \phi, m\phi \rangle$ is a weighted inner product related to the kinetic energy (Ref. 2). In future discussions, it is assumed that solutions of the eigenvalue problem are being sought by working with Eq. (2.3) instead of Eqs. (2.1) and (2.2).

We are concerned with the case in which the eigenvalue problem does not admit a closed-form solution and seek an approximate solution by the classical Rayleigh-Ritz method in the form

$$\phi^{(n)}(P) = \sum_{j=1}^n a_j \psi_j(P) \quad (2.4)$$

where a_j are undetermined coefficients and $\psi_j(P)$ are admissible functions taken from a complete set. Exact eigenfunctions $\phi_r(P)$ ($r = 1, 2, \dots$) can be obtained by letting $n \rightarrow \infty$. Practical considerations dictate that we limit n to a finite number, which is equivalent to imposing the constraints

$$a_{n+1} = a_{n+2} = \dots = 0 \quad (2.5)$$

on the distributed system.

Introducing Eq. (2.4) into (2.3) is tantamount to replacing a distributed system by a discrete system with n degrees of freedom, a process known as discretization. The usual Rayleigh-Ritz approach yields an algebraic eigenvalue problem of order n having the form (Ref. 2)

$$K_{a_r}^{(n)} = \lambda_r^{(n)} M_{a_r}^{(n)}, \quad r = 1, 2, \dots, n \quad (2.6)$$

where $K^{(n)}$ and $M^{(n)}$ are $n \times n$ positive definite real symmetric matrices with the entries

$$K_{ij}^{(n)} = \langle \psi_i, L\psi_j \rangle, \quad M_{ij}^{(n)} = \langle \psi_i, m\psi_j \rangle \quad i, j = 1, 2, \dots, n \quad (2.7)$$

The solution of the algebraic eigenvalue problem consists of the eigenvalues $\lambda_1^{(n)} \leq \lambda_2^{(n)} \leq \dots \leq \lambda_n^{(n)}$ and the eigenvectors $a_1^{(n)}, a_2^{(n)}, \dots, a_n^{(n)}$. The computed eigenvalues $\lambda_r^{(n)}$ ($r = 1, 2, \dots, n$) provide upper bounds for the actual eigenvalues λ_r ($r = 1, 2, \dots$). Moreover, the n components of a typical eigenvector $a_r^{(n)}$ can be introduced into Eq. (2.4) to obtain the computed eigenfunctions

$$\phi_r^{(n)}(P) = \sum_{j=1}^n a_{rj}^{(n)} \psi_j(P), \quad r = 1, 2, \dots, n \quad (2.8)$$

The estimates of the eigenvalues can be improved by increasing the number of terms in the approximation to $n+1$, so that the approximate solution has the form

$$\phi^{(n+1)}(P) = \sum_{j=1}^{n+1} a_j \psi_j(P) \quad (2.9)$$

which is equivalent to relaxing the constraint $a_{n+1} = 0$. The new eigenvalue problem is given by

$$K_{a_r}^{(n+1)} = \lambda_r^{(n+1)} M_{a_r}^{(n+1)} \quad (2.10)$$

where the matrices $K^{(n+1)}$ and $M^{(n+1)}$ possess the embedding property, i.e., they have the forms

$$K^{(n+1)} = \begin{bmatrix} K^{(n)} & x \\ & x \\ x & x & x \end{bmatrix}, \quad M^{(n+1)} = \begin{bmatrix} M^{(n)} & x \\ & x \\ x & x & x \end{bmatrix} \quad (2.11)$$

where the x 's in Eqs. (2.11) denote entries due to the additional admissible function $\psi_{n+1}(P)$. Hence, the embedding property implies that matrices $K^{(n+1)}$ and $M^{(n+1)}$ are obtained by adding one extra row and one extra column to the matrices $K^{(n)}$ and $M^{(n)}$, respectively.

The embedding property exhibited by the algebraic eigenvalue problems derived by the classical Rayleigh-Ritz method have significant implications. Indeed, it is shown in Ref. 2 that the two sets of computed eigenvalues $\lambda_1^{(n)}, \lambda_2^{(n)}, \dots, \lambda_n^{(n)}$ and $\lambda_1^{(n+1)}, \lambda_2^{(n+1)}, \dots, \lambda_{n+1}^{(n+1)}$ satisfy the chain of inequalities.

$$\lambda_1^{(n+1)} \leq \lambda_1^{(n)} \leq \lambda_2^{(n+1)} \leq \lambda_2^{(n)} \leq \dots \leq \lambda_n^{(n+1)} \leq \lambda_n^{(n)} \leq \lambda_{n+1}^{(n+1)} \quad (2.12)$$

We shall refer to inequalities (2.12) as the classical inclusion principle, or the inclusion principle. The finite element method can be regarded as a Rayleigh-Ritz method, at least for the type of self-adjoint systems under consideration. Hence, the question arises whether the above inclusion principle is valid for the finite element method.

2.3 THE INCLUSION PRINCIPLE AND THE HIERARCHICAL FINITE ELEMENT METHOD

The finite element method is a discretization procedure that can be used to transform the differential eigenvalue problem given by

Eqs. (2.1) and (2.2), to an algebraic one. Many of the implementation details of the finite element method are the same as those of the classical Rayleigh-Ritz method and its convergence characteristics can be conveniently demonstrated by treating it as a Rayleigh-Ritz Method. On the other hand, there are significant differences between the finite element method and the classical Rayleigh-Ritz method. In the way of similarities, the transition from Eqs. (2.1) and (2.2) to Eqs. (2.7) and (2.8) is the same in both methods. The basic differences lie in the nature of the admissible functions $\psi_j(P)$. In the classical Rayleigh-Ritz method, $\psi_j(P)$ are defined over the entire domain D and they tend to be relatively complicated functions. On the other hand, in the finite element method, $\psi_j(P)$ are defined over sub-domains h of D , known as finite elements, and they tend to be low order polynomials (Refs. 1-4). As shown in the preceding section, improvement of the eigensolution can be brought about in the classical Rayleigh-Ritz method by adding terms to series, Eq. (2.4). To improve the accuracy of the eigensolutions derived by the finite element method, the common approach is to refine the finite element mesh, which amounts to dividing the domain of extension of the system into a larger number of elements. This approach is known as the h-version of the finite element method. Another approach is to keep the number of elements constant and to increase the number of polynomials in a given element (Refs. 7-9). This latter approach combines features from both the classical Rayleigh-Ritz method and the finite element method and is referred to as the p-version of the finite element method. In using the p-version of the finite element

method, one is free to choose polynomials so that the embedding characteristics of Eq. (2.11) are preserved. This approach is known as the hierarchical finite element method (Refs. 10-12).

The question arises whether the inclusion principle holds true also for the h-version of the finite element method. The answer to this depends on the order of the operator L in Eq. (2.1) and on the nature of the admissible functions.

2.4 SECOND-ORDER (IN SPACE) PROBLEMS

Let us assume that the operator L is of order two, so that we are concerned essentially with a Sturm-Liouville problem. In this case the admissible functions must be differentiable once, so that the lowest-degree polynomials that are admissible are linear functions. Note that admissible functions in the finite element terminology are often referred to as interpolation functions.

Next, let us examine the relationship between the h-version and the p-version of the finite element method for linear interpolation functions. Figure 2.1 shows a typical element, in which w_1 and w_2 are nodal displacements and $w(\xi)$ is the displacement of an arbitrary point at a distance from the right node, where ξ is recognized as a nondimensional local coordinate. Let L_1 and L_2 be the interpolation functions

$$L_1 = \xi, \quad L_2 = 1 - \xi \quad (2.13)$$

The displacement $w(\xi)$ inside the element can be expressed in terms of the nodal displacements and the interpolation functions as follows:

$$w(\xi) = L_1(\xi)w_1 + L_2(\xi)w_2 \quad (2.14)$$

In the h-version of the finite element method, we subdivide the element into two elements, as shown in Fig. 2.2a. In the hierarchical finite element method, we simply add an extra interpolation function in the form of a triangle with height \bar{w}_3 . In the case of Fig. 2.2a, the displacement can be expressed as

$$w(\xi) = \begin{cases} L_1\left(\frac{\xi-h_2}{h_1}\right)w_1 + L_2\left(\frac{\xi-h_2}{h_1}\right)w_3 & , \quad h_2 \leq \xi \leq 1 \\ L_2\left(\frac{\xi}{h_2}\right)w_2 + L_1\left(\frac{\xi}{h_2}\right)w_3 & , \quad 0 \leq \xi \leq h_2 \end{cases} \quad (2.15)$$

On the other hand, considering Fig. 2.2b, we have for the hierarchical finite element method

$$w(\xi) = L_1(\xi)w_1 + L_2(\xi)w_2 + L_3(\xi)\bar{w}_3 \quad (2.16)$$

Considering

$$w_3 = w(h_2) = L_1(h_2)w_1 + L_2(h_2)w_2 + \bar{w}_3, \quad h_1 + h_2 = 1 \quad (2.17a)$$

$$L_3(h_2) = 1 \quad (2.17b)$$

we conclude that the representation of Fig. 2.2a is identical to that of Fig. 2.2b if we choose

$$L_3(\xi) = \begin{cases} L_1(\xi/h_2) & , \quad 0 \leq \xi \leq h_2 \\ L_2[(\xi-h_2)/h_1] & , \quad h_2 \leq \xi \leq 1 \end{cases} \quad (2.18)$$

and we note that $L_3(0) = L_3(1) = 0$, so that the term $L_3(\xi)\bar{w}_3$ does not affect any elements other than the one under consideration.

The mass and stiffness matrices $M^{(n+1)}$ and $K^{(n+1)}$, respectively, obtained by adding one hierarchical function to one of the finite elements possess the embedding property described earlier. Although the mass and stiffness matrices obtained by the h-version of the finite element method do not display this property explicitly, they do possess it implicitly. To explain this statement we recognize that the nodal coordinates of Fig. 2.2a can be expressed in terms of the coordinates of Fig. 2.2b as follows:

$$\begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} = C \begin{Bmatrix} w_1 \\ w_2 \\ \bar{w}_3 \end{Bmatrix} \quad (2.19)$$

where, from Eqs. (2.13) and (2.17a), the transformation matrix is

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ h_2 & h_1 & 1 \end{bmatrix} \quad (2.20)$$

The mass and stiffness matrices corresponding to Fig. 2.2a can be transformed to forms exhibiting the embedding characteristics by multiplying them on the left by C^T and on the right by C . Of course, this transformation applies only to the element that has been subdivided, and the remaining elements are left unchanged.

The above proof of the equivalence of the h-version of the finite element method and the hierarchical finite element method permits us to state the following:

Theorem

The inclusion principle is valid for second-order systems for the h-version of the finite element method, provided linear interpolation functions are used as admissible functions.

2.5 FOURTH-ORDER (IN SPACE) PROBLEMS

When the operator L in Eq. (2.1) is of order four, the interpolation functions used must be at least of order three (Ref. 2). The most widely used polynomials are the Hermite cubics. Before examining the validity of the inclusion principle for fourth-order problems treated by the finite element method, let us investigate the relation between the h-version and the p-version of the finite element method.

The displacement of any point in the element can be written in terms of the interpolation functions and nodal displacements (Fig. 2.3) in the form

$$w(\xi) = L_1(\xi)w_1 + L_2(\xi)\theta_1 + L_3(\xi)w_2 + L_4(\xi)\theta_2 \quad (2.21)$$

where

$$\begin{aligned} L_1 &= 3\xi^2 - 2\xi^3, & L_2 &= \xi^2 - \xi^3 \\ L_3 &= 1 - 3\xi^2 + 2\xi^3, & L_4 &= -\xi + 2\xi^2 - \xi^3 \end{aligned} \quad (2.22)$$

are Hermite cubics (Ref. 1).

In the case of the h-version of the finite element method, we subdivide the element into two elements and introduce the displacement w_3 and the rotation θ_3 at the new node (Fig. 2.4a). Then, following the procedure used in the previous section, we can write

$$w(\xi) = L_1\left(\frac{\xi-h_2}{h_1}\right)w_1 + L_2\left(\frac{\xi-h_2}{h_1}\right)\theta_1 + L_3\left(\frac{\xi-h_2}{h_1}\right)w_3 + L_4\left(\frac{\xi-h_2}{h_1}\right)\theta_3, \quad h_2 \leq \xi \leq 1 \quad (2.23a)$$

$$w(\xi) = L_3\left(\frac{\xi}{h_2}\right)w_2 + L_4\left(\frac{\xi}{h_2}\right)\theta_2 + L_1\left(\frac{\xi}{h_2}\right)w_3 + L_2\left(\frac{\xi}{h_2}\right)\theta_3, \quad 0 \leq \xi \leq h_2 \quad (2.23b)$$

On the other hand, in the case of the hierarchical finite element method, the displacement can be written from Fig. 2.4b in the form

$$w(\xi) = L_1(\xi)w_1 + L_2(\xi)\theta_1 + L_3(\xi)w_2 + L_4(\xi)\theta_2 + L_5(\xi)\bar{w}_3 + L_6(\xi)\bar{\theta}_3 \quad (2.24)$$

Next, we wish to determine $L_5(\xi)$ and $L_6(\xi)$, which must be done for each of the two subdomains, $0 \leq \xi \leq h_2$ and $h_2 \leq \xi \leq 1$.

Let us consider first the subdomain $0 \leq \xi \leq h_2$. From Eqs. (2.23b) and (2.24), we can write

$$\begin{aligned} w(h_2) &= L_3(1)w_2 + L_4(1)\theta_2 + L_1(1)w_3 + L_2(1)\theta_3 = w_3 \\ &= L_1(h_2)w_1 + L_2(h_2)\theta_1 + L_3(h_2)w_2 + L_4(h_2)\theta_2 \\ &\quad + L_5(h_2)\bar{w}_3 + L_6(h_2)\bar{\theta}_3 \end{aligned} \quad (2.25a)$$

$$\begin{aligned} \left. \frac{dw}{d\xi} \right|_{\xi=h_2} &= \left. \frac{d}{d\xi} L_3\left(\frac{\xi}{h_2}\right) \right|_{\xi=h_2} w_2 + \left. \frac{d}{d\xi} L_4\left(\frac{\xi}{h_2}\right) \right|_{\xi=h_2} \theta_2 \\ &\quad + \left. \frac{d}{d\xi} L_1\left(\frac{\xi}{h_2}\right) \right|_{\xi=h_2} w_3 + \left. \frac{d}{d\xi} L_2\left(\frac{\xi}{h_2}\right) \right|_{\xi=h_2} \theta_3 \\ &= \frac{1}{h_2} [L_3'(1)w_2 + L_4'(1)\theta_2 + L_1'(1)w_3 + L_2'(1)\theta_3] = -\frac{\theta_3}{h_2} \end{aligned}$$

$$\begin{aligned}
&= L_1'(h_2)w_1 + L_2'(h_2)\theta_1 + L_3'(h_2)w_2 + L_4'(h_2)\theta_2 \\
&\quad + L_5'(h_2)\bar{w}_3 + L_6'(h_2)\bar{\theta}_3
\end{aligned} \tag{2.25b}$$

where primes denote differentiations with respect to the argument.

Note that in arriving at Eqs. (2.25), Eqs. (2.22) have been considered.

Next, let us assume that

$$L_5(h_2) = 1 \quad , \quad L_6(h_2) = 0 \quad , \quad L_5'(h_2) = 0 \quad , \quad L_6'(h_2) = -1 \tag{2.26}$$

so that Eqs. (2.25) can be solved for \bar{w}_3 and $\bar{\theta}_3$, with the result

$$\begin{aligned}
\bar{w}_3 &= w_3 - L_1(h_2)w_1 - L_2(h_2)\theta_1 - L_3(h_2)w_2 - L_4(h_2)\theta_2 \\
\bar{\theta}_3 &= \frac{\theta_3}{h_2} + L_1'(h_2)w_1 + L_2'(h_2)\theta_1 + L_3'(h_2)w_2 + L_4'(h_2)\theta_2
\end{aligned} \tag{2.27}$$

Introducing Eqs. (2.27) into (2.24), we obtain for $0 \leq \xi \leq h_2$

$$\begin{aligned}
w(\xi) &= L_1(\xi)w_1 + L_2(\xi)\theta_1 + L_3(\xi)w_2 + L_4(\xi)\theta_2 + L_5(\xi)[w_3 \\
&\quad - L_1(h_2)w_1 - L_2(h_2)\theta_1 - L_3(h_2)w_2 - L_4(h_2)\theta_2] \\
&\quad + L_6(\xi) \left[\frac{\theta_3}{h_2} + L_1'(h_2)w_1 + L_2'(h_2)\theta_1 + L_3'(h_2)w_2 + L_4'(h_2)\theta_2 \right]
\end{aligned} \tag{2.28}$$

For Eqs. (2.23b) and (2.28) to be identical, the coefficients of

$w_1, \theta_1, \dots, w_3, \theta_3$ must be the same, or

$$0 = L_1(\xi) - L_5(\xi)L_1(h_2) + L_6(\xi)L_1'(h_2) \tag{2.29a}$$

$$0 = L_2(\xi) - L_5(\xi)L_2(h_2) + L_6(\xi)L_2'(h_2) \tag{2.29b}$$

$$L_3\left(\frac{\xi}{h_2}\right) = L_3(\xi) - L_5(\xi)L_3(h_2) + L_6(\xi)L_3'(h_2) \tag{2.29c}$$

$$L_4\left(\frac{\xi}{h_2}\right) = L_4(\xi) - L_5(\xi)L_4(h_2) + L_6(\xi)L_4'(h_2) \quad (2.29d)$$

$$L_1\left(\frac{\xi}{h_2}\right) = L_5(\xi) \quad (2.29e)$$

$$L_2\left(\frac{\xi}{h_2}\right) = \frac{1}{h_2} L_6(\xi) \quad (2.29f)$$

Following the same procedure as above, we obtain for $\underline{h_2 \leq \xi \leq 1}$

$$\begin{aligned} w(\xi) = & L_1(\xi)w_1 + L_2(\xi)\theta_1 + L_3(\xi)w_2 + L_4(\xi)\theta_2 + L_5(\xi)[w_3 - L_1(h_2)w_1 \\ & - L_2(h_2)\theta_1 - L_3(h_2)w_2 - L_4(h_2)\theta_2] + L_6(\xi) \left[\frac{\theta_3}{h_1} + L_1'(h_2)w_1 \right. \\ & \left. + L_2'(h_2)\theta_1 + L_3'(h_2)w_2 + L_4'(h_2)\theta_2 \right] \end{aligned} \quad (2.30)$$

so that, for Eqs. (2.23a) and (2.30) to be identical, we must have

$$L_1\left(\frac{\xi-h_2}{h_1}\right) = L_1(\xi) - L_5(\xi)L_1(h_2) + L_6(\xi)L_1'(h_2) \quad (2.31a)$$

$$L_2\left(\frac{\xi-h_2}{h_1}\right) = L_2(\xi) - L_5(\xi)L_2(h_2) + L_6(\xi)L_2'(h_2) \quad (2.31b)$$

$$0 = L_3(\xi) - L_5(\xi)L_3(h_2) + L_6(\xi)L_3'(h_2) \quad (2.31c)$$

$$0 = L_4(\xi) - L_5(\xi)L_4(h_2) + L_6(\xi)L_4'(h_2) \quad (2.31d)$$

$$L_3\left(\frac{\xi-h_2}{h_1}\right) = L_5(\xi) \quad (2.31e)$$

$$L_4\left(\frac{\xi-h_2}{h_1}\right) = \frac{1}{h_1} L_6(\xi) \quad (2.31f)$$

It is easy to verify that Eqs. (2.29) on the one hand and Eqs. (2.31) on the other hand are satisfied identically provided

$$L_5(\xi) = \begin{cases} L_3[(\xi-h_2)/h_1] & , \quad h_2 \leq \xi \leq 1 \\ L_1(\xi/h_2) & , \quad 0 \leq \xi \leq h_2 \end{cases} \quad (2.32a)$$

$$L_6(\xi) = \begin{cases} h_1 L_4[(\xi-h_2)/h_1] & , \quad h_2 \leq \xi \leq 1 \\ h_2 L_2(\xi/h_2) & , \quad 0 \leq \xi \leq h_2 \end{cases} \quad (2.32b)$$

Hence, if $L_5(\xi)$ and $L_6(\xi)$ are chosen according to Eqs. (2.32), the h-version of the finite element representation of Fig. 2.4a is identical to the hierarchical finite element representation of Fig. 2.4b. Moreover, we note that $L_5(0) = L_1(0) = 0$, $L_5(1) = L_3(1) = 0$, $L_6(0) = L_2(0) = 0$ and $L_6(1) = L_4(1) = 0$, so that the terms $L_5(\xi)\bar{w}_3$ and $L_6(\xi)\bar{\theta}_3$ do not affect indeed any elements other than those in question.

The relation between the coordinates of Fig. 2.4a and those of Fig. 2.4b can be expressed as

$$\begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \\ w_3 \\ \theta_3 \end{Bmatrix} = C \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \\ \bar{w}_3 \\ \bar{\theta}_3 \end{Bmatrix} \quad (2.33)$$

where, from Eqs. (2.22) and (2.27), the transformation matrix is

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 3h_2^2 - 2h_2^3 & h_2^2 - h_2^3 & 1 - 3h_2^2 + 2h_2^3 & -h_2 + 2h_2^2 - h_2^3 & 1 & 0 \\ -6h_2^2 + 6h_2^3 & -2h_2^2 + 3h_2^3 & +6h_2^2 - 6h_2^3 & h_2 - 4h_2^2 + 3h_2^3 & 0 & h_2 \end{bmatrix} \quad (2.34)$$

2.6 TWO BRACKETING THEOREMS FOR THE FINITE ELEMENT METHOD FOR FOURTH-ORDER PROBLEMS

In the preceding section, we have shown that the h -version of the finite element method can be rendered identical to the hierarchical finite element method for fourth-order problems. The question remains whether the inclusion principle, as given by inequalities (2.12) and demonstrated to hold for second-order problems, holds for fourth-order problems as well. Intuition may tempt one to believe that it should, but the answer is actually negative.

Whereas the h -version of the finite element and the hierarchical finite element method can be rendered identical for fourth-order systems, and the inclusion principle was demonstrated to be valid for the hierarchical finite element method (Ref. 13), one must be careful in extending the same conclusion to the h -version of the finite element method. The reason is that in fourth-order problems the equivalence just demonstrated is based on adding two hierarchical admissible functions to the approximation and not one. As a result, the embedding property must be stated in the form

$$K^{(n+2)} = \begin{bmatrix} & & & \\ & K^{(n)} & x & x \\ & & x & x \\ x & x & x & x \\ x & x & x & x \end{bmatrix}, \quad M^{(n+2)} = \begin{bmatrix} & & & \\ & M^{(n)} & x & x \\ & & x & x \\ x & x & x & x \\ x & x & x & x \end{bmatrix} \quad (2.35)$$

In fact, by definition of the hierarchical admissible functions, we have

$$K^{(n+1)} = \begin{bmatrix} & & \\ & K^{(n)} & x \\ x & x & x \end{bmatrix}, \quad M^{(n+1)} = \begin{bmatrix} & & \\ & M^{(n)} & x \\ x & x & x \end{bmatrix} \quad (2.36a)$$

$$K^{(n+2)} = \begin{bmatrix} & & \\ & K^{(n+1)} & x \\ x & x & x \end{bmatrix}, \quad M^{(n+2)} = \begin{bmatrix} & & \\ & M^{(n+1)} & x \\ x & x & x \end{bmatrix} \quad (2.36b)$$

Then, denoting the computed eigenvalues associated with $K^{(n+i)}$, $M^{(n+i)}$ system by $\lambda_1^{(n+i)}, \lambda_2^{(n+i)}, \dots, \lambda_{n+i}^{(n+i)}$ ($i = 0, 1, 2$), there exists one inclusion principle for the pair 0,1 and another one for the pair 1,2, or

$$\lambda_1^{(n+1)} \leq \lambda_1^{(n)} \leq \lambda_2^{(n+1)} \leq \lambda_2^{(n)} \leq \dots \leq \lambda_n^{(n)} \leq \lambda_{n+1}^{(n+1)} \quad (3.37a)$$

$$\lambda_1^{(n+2)} \leq \lambda_1^{(n+1)} \leq \lambda_2^{(n+2)} \leq \lambda_2^{(n+1)} \leq \dots \leq \lambda_{n+1}^{(n+1)} \leq \lambda_{n+2}^{(n+2)} \quad (3.37b)$$

Eliminating the eigenvalues associated with $K^{(n+1)}$, $M^{(n+1)}$ from inequalities (37), we can state the following:

First Bracketing Theorem for Fourth-Order Problems: If the order of the approximation in the h-version of the finite element method is increased by subdividing one element into two, the two sets of

computed eigenvalues satisfy the chain of inequalities

$$\lambda_1^{(n+2)} \leq \lambda_1^{(n)} \leq \lambda_3^{(n+2)} \leq \lambda_3^{(n)} \leq \lambda_5^{(n+2)} \leq \dots \leq \lambda_{n-1}^{(n+2)} \leq \lambda_{n-1}^{(n)} \leq \lambda_{n+1}^{(n+2)} \quad (2.38a)$$

$$\lambda_2^{(n+2)} \leq \lambda_2^{(n)} \leq \lambda_4^{(n+2)} \leq \lambda_4^{(n)} \leq \lambda_6^{(n+2)} \leq \dots \leq \lambda_n^{(n+2)} \leq \lambda_n^{(n)} \leq \lambda_{n+2}^{(n+2)} \quad (2.38b)$$

The above bracketing theorem is different from the inclusion principle associated with the classical Rayleigh-Ritz method, which was shown to be valid in section 2.3 for second-order problems in conjunction with the h-version of the finite element method using linear interpolation functions. Indeed, from inequalities (2.38), we conclude that no direct comparison between odd- and even-numbered eigenvalues corresponding to the approximations of order n and n+2 are possible. The reason for this can be explained by the fact that, with the addition of one finite element, the order of the approximation increases by two more eigenvalues than the old one and not just one more eigenvalue, so that the classical inclusion principle cannot possibly hold.

In view of the experience with the classical Rayleigh-Ritz method, one may be tempted to believe that, in adding one element in the case of fourth-order systems, that the new eigenvalues will be higher than the highest of the lower-order approximations and the remaining eigenvalues will simply satisfy the classical inclusion principle. This belief has a great deal of intuitive appeal, and in many cases it turns out to be true. However, this is not always the case. Indeed, from inequalities (2.37) we can state:

Second Bracketing Theorem for Fourth-Order Problems: Any two adjacent eigenvalues of the lower-order approximation bracket none, one or two of the eigenvalues of the highest-order approximation.

This second theorem can be demonstrated by the diagram of Fig. 2.5. Of course, due to the fact that in any Rayleigh-Ritz approximation the most dramatic improvement takes place in the higher eigenvalues, the odds favor the intuitive conclusion stated in the beginning of the paragraph. It is a mistake to take this for granted, however, as demonstrated in the numerical example presented in the following section.

2.7 NUMERICAL EXAMPLE

Let us consider a cantilever beam in bending vibration, in which case the operator L in Eq. (2.1) has the form

$$L = \frac{d^2}{dx^2} \left[EI(x) \frac{d^2}{dx^2} \right] \quad , \quad D : 0 < x < L \quad (2.39)$$

where we have replaced P by x because the problem is one-dimensional and we note that L is the length of the beam. Moreover, we assume that the system parameters have the distributions

$$EI(x) = EI(1 - 0.9 x/L) \quad , \quad m(x) = m \quad (2.40)$$

where $EI(x)$ can be identified as the flexural stiffness and $m(x)$ is the mass density.

As a first approximation, we divide the beam into two elements, as shown in Fig. 2.6a, and compute the lowest four approximate eigenvalues $\lambda_r^{(4)}$ ($r = 1, 2, 3, 4$) by the finite element method using the Hermite cubics of Eqs. (2.22) as interpolation functions. Next, as a

second approximation, we divide the right element in Fig. 2.6a into two elements, as shown in Fig. 2.6b, and compute the lowest six eigenvalues $\lambda_r^{(6)}$ ($r = 1, 2, \dots, 6$) in a similar fashion. The results are displayed in Table 2.1. From Table 2.1, we observe that

$$\lambda_1^{(6)} < \lambda_1^{(4)} < \lambda_2^{(6)} < \lambda_2^{(4)} < \lambda_3^{(6)} < \lambda_3^{(4)} < \lambda_4^{(6)} < \lambda_5^{(6)} < \lambda_4^{(4)} < \lambda_6^{(6)} \quad (2.41)$$

so that $\lambda_5^{(6)}$ is lower than $\lambda_4^{(4)}$, which is contrary to intuition.

Clearly, the classical inclusion principle, which is valid for second-order problems, is not valid for fourth-order problems. On the other hand, it is clear that

$$\lambda_1^{(6)} < \lambda_1^{(4)} < \lambda_3^{(6)} < \lambda_3^{(4)} < \lambda_5^{(6)} \quad (2.42a)$$

$$\lambda_2^{(6)} < \lambda_2^{(4)} < \lambda_4^{(6)} < \lambda_4^{(4)} < \lambda_6^{(6)} \quad (2.42b)$$

so that the First Bracketing Theorem developed in this chapter is verified. Moreover, from inequalities (2.41), we conclude that the Second Bracketing Theorem is satisfied, as expected.

2.8 CONCLUSIONS

It is generally assumed that the classical inclusion principle is applicable to distributed structures discretized by the h-version of the finite-element method, although no general proof has ever been produced. The absence of a general proof is consistent with the fact that the classical inclusion principle is not valid in general for the h-version of the finite element method. In particular, whereas the

inclusion principle is valid for second-order problems, provided linear interpolation functions are used, it is not valid for fourth-order problems.

This chapter demonstrates first that the h-version of the finite element method is equivalent to a special form of the hierarchical finite element method. This permits proof of the validity of the classical inclusion principle for second-order problems in conjunction with the h-version of the finite element method using linear interpolation functions. It also permits proof of two bracketing theorems characterizing the computed eigenvalues for fourth-order problems discretized by the h-version of the finite element method.

Table 2.1 Computed eigenvalues for first- and second-order approximations

r	1	2	3	4	5	6
$\lambda_r^{(4)} mL^4/EI$	3.1339	18.077	50.048	191.81	---	---
$\lambda_r^{(6)} mL^4/EI$	3.1326	16.720	46.772	91.795	170.41	316.47

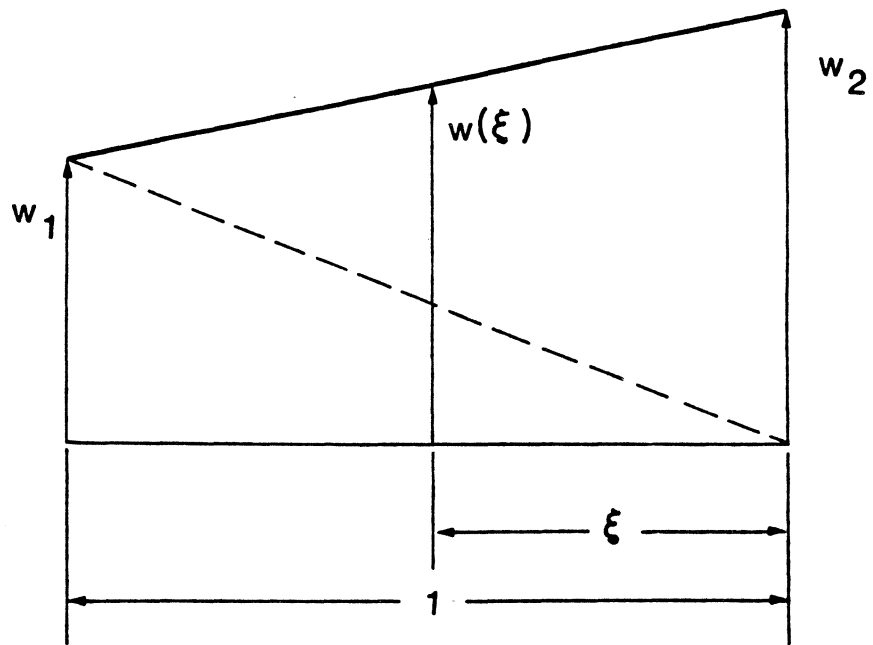
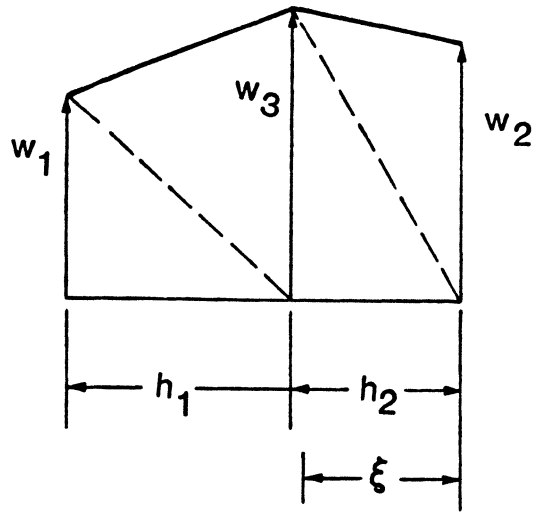
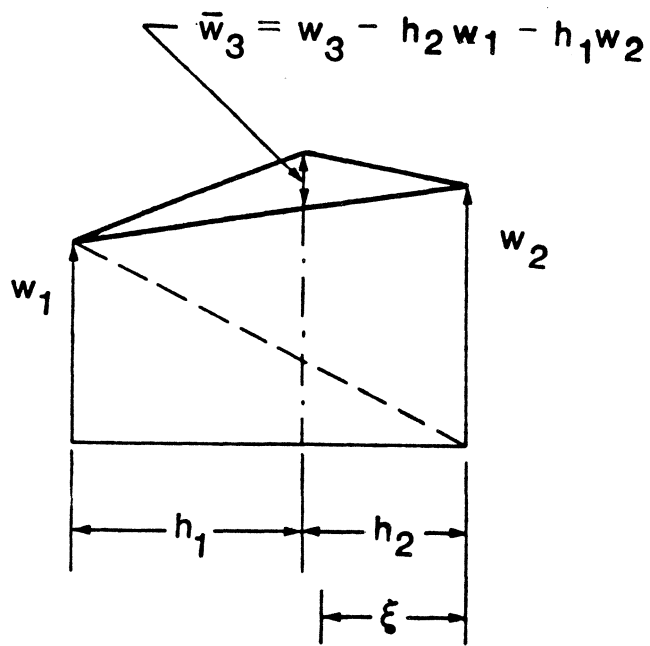


Figure 2.1 Typical finite element for second-order problems



(a)



(b)

Figure 2.2a,b Subdivision of a typical finite element for h- and p-versions for second-order problems

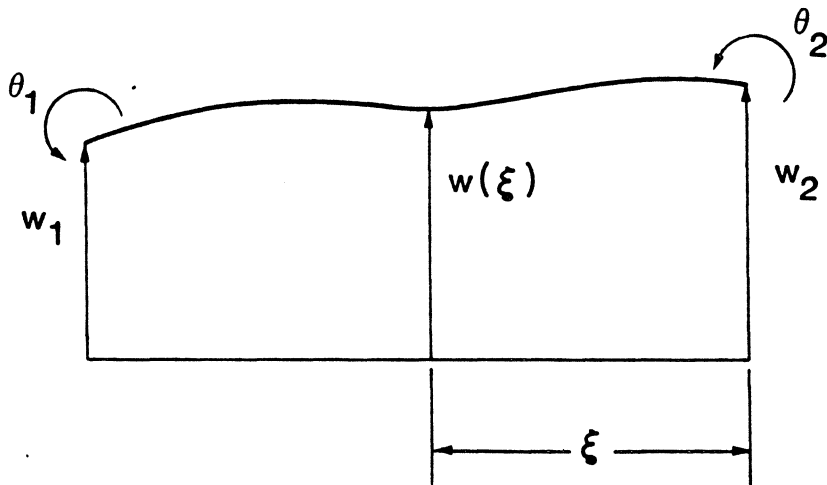
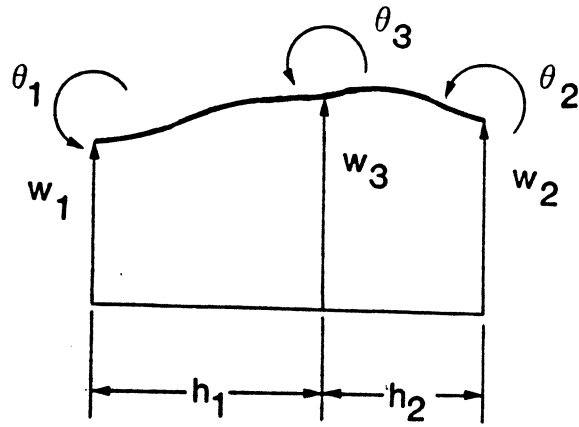
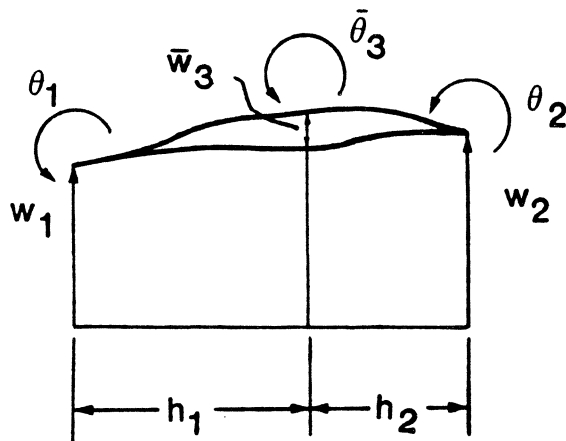


Figure 2.3 Typical finite element for fourth-order problems



(a)



(b)

Figure 2.4a,b Subdivision of a typical finite element for h- and p-versions for fourth-order problems

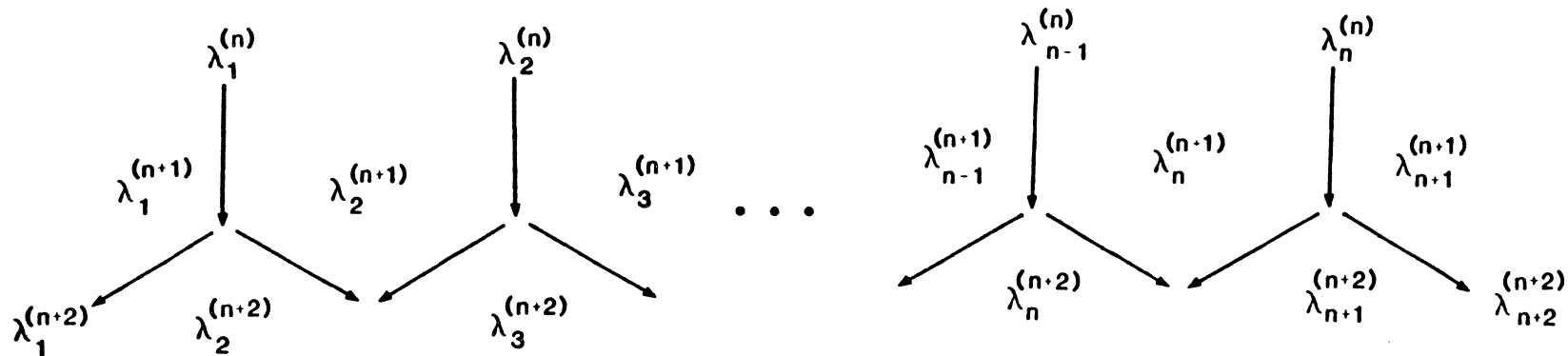
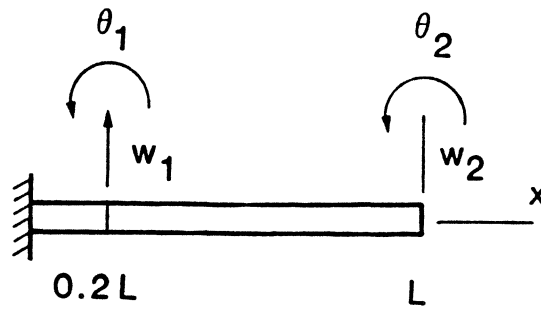
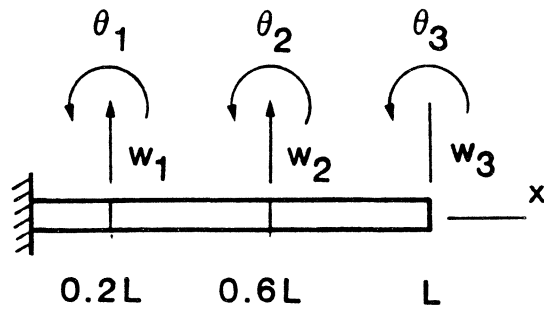


Figure 2.5 First and second bracketing theorems



(a)



(b)

Figure 2.6a,b Numerical example: first- and second-order approximations

Chapter III

GLOBAL OPTIMAL CONTROL OF SELF-ADJOINT DISTRIBUTED STRUCTURES

3.1 INTRODUCTION

The partial differential equations of motion of a distributed structure can be transformed into a set of independent second-order ordinary differential equations by means of a linear transformation. The resulting set of independent equations are called modal equations and the independent coordinates are called modal coordinates. In vibration analysis, the independent coordinates are also referred to as natural coordinates since the motion associated with a mode of vibration is a natural motion, i.e., a harmonic motion natural to the given distributed system. For the same reason, the frequencies of the natural motions are called natural frequencies.

The above behavior is characteristic of distributed structures upon which no control forces act. However, control feedback generally recouples the modal equations of motion so that the modal coordinates cease to be natural coordinates of the system. The question arises whether there exists a control in which the natural behavior of the distributed structure is not destroyed.

It is shown in this chapter that there exists such a natural control known as independent modal-space control. For independent modal-space control the natural coordinates of the open-loop (uncontrolled) system and the natural coordinates of the closed-loop (controlled) system are identical. Whereas the open-loop eigenvalues

are different from the closed-loop eigenvalues, the open- and closed-loop eigenfunctions are the same.

A widely accepted approach to control of distributed structures has been first to discretize the system in space leading to a discrete model of the distributed system. This practice arises from the inability to compute controls for distributed systems. However, the resulting discrete model contains erroneous information. Indeed, no discretized model can represent a distributed system since only a small subset of the eigenvalues and eigenfunctions are accurate while the remaining eigenvalues and eigenfunctions can be grossly in error.

In this chapter, it is demonstrated that natural control represents a method capable of controlling the distributed system in which the control gains admit closed-form expressions. Indeed, the full set of modes can be controlled, at least in theory. It is also shown that natural control provides a unique and globally optimal solution to the linear optimal control of the distributed system.

For natural control, the optimal control forces are ideally distributed. In practice, however, it is often desirable to use finite-dimensional controls as well as to use discrete measurements for feedback. To this end, it is shown how finite-dimensional controls can be constructed to imitate distributed control forces. Also, it is shown how to use a finite number of measurements for feedback.

Two numerical examples of natural control of a beam and a membrane are presented.

3.2 EQUATIONS OF MOTION OF A SELF-ADJOINT DISTRIBUTED-PARAMETER SYSTEM

Consider a distributed-parameter system and assume that its behavior is governed by the partial differential equation (Ref. 14)

$$\ddot{u}(P,t) + Au(P,t) = f(P,t) \quad , \quad P \in D \quad (3.1)$$

where $u(P,t)$ represents the displacement of the system at any point P in the spatial domain D and for any time t , $f(P,t)$ represents the distributed control force and A is a differential operator. Overdots represent derivatives with respect to t . The functions u and f are elements of some Hilbert space X with suitable norm $\| \cdot \|$ and inner product $\langle \cdot , \cdot \rangle$. In addition to Eq. (3.1), u must satisfy given boundary conditions at every point of the set S bounding D .

The interest lies in the case in which the operator A is self-adjoint. This implies that for any two admissible functions ψ_i and ψ_j in X satisfying all the system boundary conditions, as well as certain differentiability conditions, we have

$$\langle \psi_i , A\psi_j \rangle = \langle \psi_j , A\psi_i \rangle \quad (3.2)$$

Satisfaction of Eq. (3.2) can be verified through integration by parts in conjunction with full consideration of the boundary conditions. The operator $A \in L(X)$ is assumed to be positive definite and to admit a discrete spectrum, which implies that the system admits a countable set of eigenvalues λ_r and associated eigenfunctions $\phi_r(P)$ ($r = 1,2,\dots$). The eigenvalues λ_r are related to the system natural frequencies ω_r by $\lambda_r = \omega_r^2$ ($r = 1,2,\dots$). Moreover, the eigenfunctions are orthogonal and can be normalized so as to satisfy

$$\langle \phi_r, \phi_s \rangle = \delta_{rs} \quad , \quad \langle \phi_r, A\phi_s \rangle = \lambda_r \delta_{rs} \quad , \quad r,s = 1,2,\dots \quad (3.3a,b)$$

where δ_{rs} is the Kronecker delta.

3.3 MODAL EQUATIONS OF MOTION

As pointed out in Sec. 3.2, the set of eigenfunctions $\{\phi_r\}_{r=1}^{\infty}$ is orthogonal. Moreover, the set is complete on $X = \text{span}\{\phi_r\}$. Therefore, any element in X can be expressed as a linear combination of the system eigenfunctions. Expansions in terms of the system eigenfunctions can be carried out by means of the modal projection operators π_r defined as

$$\pi_r = \langle \cdot, \phi_r \rangle \phi_r \quad (3.4)$$

and we observe that

$$\begin{aligned} \pi_r \pi_s &= \langle \langle \cdot, \phi_s \rangle \phi_s, \phi_r \rangle \phi_r \\ &= \langle \cdot, \phi_s \rangle \langle \phi_s, \phi_r \rangle \phi_r = \pi_r \delta_{rs} \end{aligned} \quad (3.5)$$

Hence, the set of modal projection operators $\{\pi_r\}_{r=1}^{\infty}$ resolves the identity as well as the system operator A on X as follows:

$$1 = \sum_{r=1}^{\infty} \pi_r \quad , \quad A = \sum_{r=1}^{\infty} \lambda_r \pi_r \quad (3.6a,b)$$

where convergence is in the uniform topology of $L(X)$.

Next, define the sequence space Y with elements $y = (y_1, y_2, \dots)$ and refer to it as the modal space. Then, define the modal transformation $\theta \in L(X, Y)$ by

$$\theta x = y = (\langle x, \phi_1 \rangle, \langle x, \phi_2 \rangle, \dots) \quad (3.7a)$$

as well as the inverse transformation θ^{-1} by

$$\theta^{-1}y = x = \sum_{r=1}^{\infty} \phi_r y_r \quad (3.7b)$$

The function $u(P,t)$ in X can be expanded in terms of the system eigenfunctions as follows:

$$u(P,t) = \sum_{r=1}^{\infty} \phi_r(P) u_r(t), \quad u_r(t) = \langle u(P,t), \phi_r(P) \rangle, \\ r = 1, 2, \dots \quad (3.8a,b)$$

Likewise, the distributed force in X can be expanded in the form

$$f(P,t) = \sum_{r=1}^{\infty} \phi_r(P) f_r(t), \quad f_r(t) = \langle f(P,t), \phi_r(P) \rangle, \\ r = 1, 2, \dots \quad (3.9a,b)$$

Performing the modal transformation θ on Eq. (3.1) and considering Eqs. (3.8) and (3.9), we obtain

$$\ddot{u}_r(t) + \lambda_r u_r(t) = f_r(t), \quad r = 1, 2, \dots \quad (3.10)$$

where the time-dependent functions $u_r(t)$ ($r = 1, 2, \dots$) are known as modal displacements and $f_r(t)$ ($r = 1, 2, \dots$) are known as modal control forces. Equations (3.1) represent an infinite set of second-order ordinary differential equations called modal equations of motion. Use of Eqs. (3.10) to control the distributed system is often referred to as modal control. There are various types of modal control, depending on the manner in which the modal control forces $f_r(t)$ are generated.

3.4 CONTROLLABILITY AND OBSERVABILITY

The state of a distributed system at any point P and time t is defined by the vector $\underline{u}(P,t) = [\dot{u}(P,t) \ u(P,t)]^T$. Moreover, we can define an associated control vector $\underline{f}(P,t) = [f(P,t) \ 0]^T$. Then, the distributed system described by Eq. (3.1) is said to be completely state-controllable if, for any initial time t_0 , each initial state $\underline{u}(P,t_0)$ can be transferred to any final state $\underline{u}(P,t_f)$ in a finite time t_f , $t_f > t_0$, by means of an unconstrained control vector $\underline{f}(P,t)$. In modal control, however, we do not work with a distributed system described by the partial differential equation (3.1) but with one described by the infinite set of ordinary differential modal equations (3.10). Hence, the object is to define controllability in terms of modal quantities. To this end, let us introduce the modal state vectors $\underline{u}_r(t) = [\dot{u}_r(t) \ \omega_r u_r(t)]^T$ ($r = 1,2,\dots$), and recognize from Eq. (3.8a) that the state $\underline{u}(P,t)$ is related to the modal state vectors $\underline{u}_r(t)$ ($r = 1,2,\dots$) by means of a linear transformation. Moreover, the distributed control vector $f(P,t)$ is related to the modal control forces $f_r(t)$ ($r = 1,2,\dots$) by a similar linear transformation, as indicated by Eq. (3.9a). Then, the distributed system described by Eq. (3.1) is said to be completely modal state - controllable if, for any initial time t_0 , each and every initial modal state $\underline{u}_r(t_0)$ can be transferred to any final state $\underline{u}_r(t_f)$ in a finite time t_f , $t_f > t_0$, by means of unconstrained modal controls $f_r(t)$ ($r = 1,2,\dots$).

To derive a modal controllability criterion, we rewrite the modal equations (3.10) in the state form

$$\dot{\tilde{u}}_r(t) = A_r \tilde{u}_r(t) + B_r f_r(t), \quad r = 1, 2, \dots \quad (3.11)$$

where

$$A_r = \begin{bmatrix} 0 & -\omega_r \\ \omega_r & 0 \end{bmatrix}, \quad B_r = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad r = 1, 2, \dots \quad (3.12a,b)$$

and define the modal controllability matrices by

$$C_r = [B_r \ A_r B_r] = \begin{bmatrix} 1 & 0 \\ 0 & \omega_r \end{bmatrix}, \quad r = 1, 2, \dots \quad (3.13)$$

Then, Eq. (3.1) is modal-state controllable if and only if each and every controllability matrix C_r is of full rank 2. Clearly, the controllability matrices are of full rank, so that the distributed system is controllable. Next, consider the observability of distributed systems and assume that either the displacement $u(P,t)$ or the velocity $\dot{u}(P,t)$ is observable. From Eqs. (3.8b), we conclude that if the displacement $u(P,t)$ is observable, then the modal displacements $u_r(t)$ ($r = 1, 2, \dots$) are observable. Likewise, the fact that the velocity $\dot{u}(P,t)$ is observable implies that the modal velocities $\dot{u}_r(t)$ ($r = 1, 2, \dots$) are observable.

Assuming observation of the displacement $u(P,t)$, we can write the modal measurements

$$z_r(t) = C_r \tilde{u}_r(t), \quad C_r = [1 \ 0] \quad , \quad r = 1, 2, \dots \quad (3.14a,b)$$

The distributed system (3.1) is modal-observable provided the modal observability matrices

$$O_r = \begin{bmatrix} C_r \\ C_r A_r \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -\omega_r \end{bmatrix}, \quad r = 1, 2, \dots \quad (3.15)$$

are of full rank 2, which is indeed the case. Hence, the distributed system (3.1) with observable displacement is observable. If the velocity $\dot{u}(P,t)$ is observable, then the modal measurements equations remain in the form of Eq. (3.14a), but now Eq. (3.14b) is replaced by $C_r = [0, 1]$. The observability matrices are then

$$O_r = \begin{bmatrix} C_r \\ C_r A_r \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \omega_r & 0 \end{bmatrix}, \quad r = 1, 2, \dots \quad (3.16)$$

Hence, if either the displacement $u(P,t)$ or the velocity $\dot{u}(P,t)$ of the distributed system is observable, then the distributed system (3.1) is observable.

3.5 COUPLED CONTROL

Let us assume that the distributed control function can be written in the form of the linear feedback

$$f(P,t) = -G(P)u(P,t) - H(P)\dot{u}(P,t) \quad (3.17)$$

where $G \in L(X)$ and $H \in L(X)$ are self-adjoint control gain operators. The operators G and H admit discrete spectra and $\phi_r^*(P)$ and $\phi_r^{**}(P)$ represent the eigenfunctions associated with G and H , respectively. Then, introducing the projection operators

$$\pi_r^* = \langle \cdot, \phi_r^* \rangle \phi_r^*, \quad \pi_r^{**} = \langle \cdot, \phi_r^{**} \rangle \phi_r^{**} \quad (3.18a,b)$$

we can resolve G and H as follows:

$$G = \sum_{r=1}^{\infty} g_r \pi_r^* \quad , \quad H = \sum_{r=1}^{\infty} h_r \pi_r^{**} \quad (3.19a,b)$$

where g_r and h_r are eigenvalues associated with the control gains G and H , respectively. Hence, determination of the distributed linear control reduces to the determination of the operators π_r^* and π_r^{**} and of the control gain eigenvalues g_r and h_r ($r = 1, 2, \dots$).

Performing the modal transformation θ on Eq. (3.17) and considering Eqs. (3.8) and (3.9), we obtain the modal controls

$$f_r(t) = - \sum_{s=1}^{\infty} [g_{rs} u_s(t) + h_{rs} \dot{u}_s(t)] \quad , \quad r = 1, 2, \dots \quad (3.20)$$

where

$$g_{rs} = \langle G\phi_s, \phi_r \rangle \quad , \quad h_{rs} = \langle H\phi_s, \phi_r \rangle \quad (3.21a,b)$$

Inserting Eqs. (3.20) into (3.10), we obtain the closed-loop modal equations

$$\ddot{u}_r(t) + \lambda_r u_r(t) + \sum_{s=1}^{\infty} [g_{rs} u_s(t) + h_{rs} \dot{u}_s(t)] = 0 \quad , \quad r = 1, 2, \dots \quad (3.22)$$

In the case of an uncontrolled system, $f_r \equiv 0$ ($r = 1, 2, \dots$), or in the case of open-loop control, $f_r = f_r(t)$ ($r = 1, 2, \dots$), where the latter implies that the modal control forces depend explicitly on time and they do not depend on the modal displacements and modal velocities, Eqs. (3.10) are independent. In the case of feedback control of the type (3.17), resulting in modal control forces as given by Eqs. (3.20), the closed-loop modal equations are recoupled by the

control forces, as can be seen from Eqs. (3.22). Control of this type is referred to as coupled control.

3.6 NATURAL CONTROL

In vibration analysis, the partial differential equation describing the motion of a distributed-parameter system, or the set of ordinary differential equations governing the motion of a lumped-parameter system, can be transformed into a set of independent second-order ordinary differential equations by means a modal transformation of the type (3.8a). The resulting set of independent equations are called modal equations and the independent coordinates are known as modal coordinates. The independent coordinates are also called natural coordinates due to the fact that motion in any one of the modes of vibration is a natural motion, i.e., a harmonic motion characteristic of a given system and determined by the system parameters. For the same reason, the frequencies of the natural motions are called natural frequencies. The modal transformation leading to the independent set is unique to the system, and so are the natural coordinates.

The above behavior is typical of open-loop systems, in which the forces acting upon a system do not depend on the state of the system. Feedback control forces do depend on the state of the system, however, so that in closing the loop the question arises as to what happens to the natural coordinates. In the case of coupled control, the feedback controls recouple the modal equations, so that the modal coordinates cease to be natural coordinates. There exists a particular choice of feedback controls, namely, those prescribed by independent modal-space

control, for which the modal equations remain independent. Hence, in the case of independent modal-space control the natural coordinates of the open-loop system remain natural coordinates for the closed-loop system. For this reason, independent modal-space control will be referred to as natural control. We also observe that, whereas the closed-loop poles differ from the open-loop poles, for natural control the closed-loop eigenfunctions are the same as the open-loop eigenfunctions. In the following, we propose to discuss natural control in the context of the various terms and concepts introduced earlier in this dissertation.

Let us consider control gain operators having projections identical to the modal projections for the distributed system, so that

$$\pi_r^* = \pi_r^{**} = \pi_r \quad (3.23)$$

This choice of projections is tantamount to considering eigenfunctions associated with the control gain operators G and H identical to the system eigenfunctions, so that

$$\phi_r^* = \phi_r^{**} = \phi_r \quad (3.24)$$

Substituting Eqs. (3.24) and Eqs. (3.19) into Eq. (3.21), we conclude that the eigenvalues of G and H are related to g_{rs} and h_{rs} by

$$g_{rs} = g_r \delta_{rs} , \quad h_{rs} = h_r \delta_{rs} \quad (3.25)$$

Inserting Eqs. (3.25) into Eqs. (3.20), we obtain

$$f_r(t) = -g_r u_r(t) - h_r \dot{u}_r(t) , \quad r = 1, 2, \dots \quad (3.26)$$

Control of this form is known as independent modal-space or natural control. Introducing Eqs. (3.26) into Eqs. (3.10), the closed-loop

modals equations reduce to

$$\ddot{u}_r(t) + h_r \dot{u}_r(t) + (\lambda_r + g_r)u_r(t) = 0 \quad , \quad r = 1, 2, \dots \quad (3.27)$$

In contrast to the closed-loop equations for coupled control, Eqs. (3.22), the closed-loop equations (3.27) remain decoupled even in the presence of feedback control. Following are three theorems pertinent to natural control.

Theorem 1.--The eigenfunctions of the closed-loop system are the same as those of the open-loop system.

Because in the case of natural control the open-loop and closed-loop equations are independent, it follows that the modal transformation θ of A decouples both the open-loop and the closed-loop distributed system.

Theorem 2.--For natural control the pair of closed-loop eigenvalues corresponding to each mode can be selected independently of those for any other mode.

The pairs of closed-loop eigenvalues are solutions of the second-order polynomials

$$s_r^2 + h_r s_r + \lambda_r + g_r = 0 \quad , \quad r = 1, 2, \dots \quad (3.28)$$

Equations (3.28) follow from Eqs. (3.26). Furthermore, we can express the eigenvalues of G and H in terms of the closed-loop eigenvalues

$s_{1r} = \alpha_r + i\beta_r$ and $s_{2r} = \alpha_r - i\beta_r$, so that

$$g_r = \alpha_r^2 + \beta_r^2 - \lambda_r \quad , \quad h_r = -2\alpha_r \quad , \quad r = 1, 2, \dots \quad (3.29a, b)$$

Theorem 3.--Natural control, in which the global performance functional

$$J = \int_0^{\infty} [\langle \dot{u}, \dot{u} \rangle + \langle u, Au \rangle + \langle f, Rf \rangle] dt, \quad (3.30a)$$

$$R = \sum_{r=1}^{\infty} R_r \pi_r, \quad R_r > 0, \quad r = 1, 2, \dots \quad (3.30b)$$

is minimized, provides a unique and globally optimal solution to the linear optimal control of the distributed system.

First, note that the first two terms of the integrand in Eq. (3.30a) represent the total (kinetic + potential) energy of the distributed system and the last term represents the distributed control effort in which R denotes a weighting operator. Note that, from Eq. (3.30b), for equal modal weights, $R_1 = R_2 = \dots$, the weighting operator reduces to the identity multiplied by a scalar. Substituting Eqs. (3.18) and (3.9) into Eq. (3.30a), we obtain

$$J = \sum_{r=1}^{\infty} J_r, \quad J_r = \int_0^{\infty} (\dot{u}_r^2 + \lambda_r u_r^2 + Rf_r^2) dt, \quad r = 1, 2, \dots \quad (3.31)$$

It follows from Eq. (3.10) and (3.31) for unconstrained control forces $f(P, t)$ that

$$\min J = \min \sum_{r=1}^{\infty} J_r = \sum_{r=1}^{\infty} \min J_r \quad (3.32)$$

Hence, the minimization can be carried out independently for every modal equation. Performing the minimization and assuming linear control gain operators, Eq. (3.17), we obtain

$$G = \sum_{r=1}^{\infty} g_r \langle \cdot, \phi_r \rangle \phi_r, \quad H = \sum_{r=1}^{\infty} h_r \langle \cdot, \phi_r \rangle \phi_r \quad (3.33a,b)$$

where the control gain eigenvalues g_r and h_r have the closed-form expressions

$$g_r = -\omega_r^2 + \omega_r (\omega_r^2 + 1/R_r)^{1/2} \quad r = 1, 2, \dots \quad (3.34a,b)$$

$$h_r = [-2\omega_r^2 + 1/R_r + 2\omega_r (\omega_r^2 + 1/R_r)^{1/2}]^{1/2}$$

Equation (3.17), together with Eq. (3.33), represents a globally optimal solution to the linear optimal control problem in which the performance functional (3.30a) has been minimized. Moreover, it is well known that the solution to the linear optimal control problem is unique (Ref. 28). Therefore, natural control of the distributed system (3.1) in conjunction with the global performance index (3.30) provides a unique globally optimal closed-form solution.

For coupled control, closed-form optimal solutions are not available because the closed-loop eigenfunctions are not known in advance. Indeed, prior knowledge of the closed-loop eigenfunctions permits decoupling of the closed-loop modal equations, which in turn leads to a closed-form optimal solution. But, in the case of coupled control, the closed-loop eigenfunctions cannot be determined in advance but only after solving the optimal control problem. Hence, the solution to the coupled control problem cannot be obtained in closed-form.

By contrast natural control lends itself to a solution which not only exists but can also be computed. Indeed, natural control permits construction of a gain operator with known eigenfunctions. Advance

knowledge of the closed-loop system eigenfunctions permits in turn decoupling of the modal equations. Moreover, natural control represents a closed-form solution to the linear optimal control problem, in which the global performance measure (3.30) is minimized.

3.7 CONTROL SENSITIVITY TO SYSTEM PARAMETERS

In practice, the system parameters entering into the operator A are known only approximately. The uncertainty in the parameters can produce inaccuracies in the open- and closed-loop eigensolutions, in which case the modal equations (3.22) can be in error. We consider the case in which parameter errors lead to errors in the open-loop eigenvalues λ_r and not in the open-loop eigenfunctions.

The solution of the closed-loop modal equations has the exponential form

$$u_r(t) = u_r e^{st} \quad (3.35)$$

Introducing Eq. (3.35) into Eqs. (3.22), we obtain for coupled control the algebraic eigenvalue problem

$$(s_i^2 I + s_i H_0 + \Lambda + G_0) u_i = 0 \quad , \quad i = 1, 2, \dots \quad (3.36)$$

where I is an infinite-dimensional identity matrix, $\Lambda = \text{diag } \lambda_r$ ($r = 1, 2, \dots$) is an infinite-dimensional diagonal matrix of open-loop eigenvalues $H_0 = [h_{rs}]$ and $G_0 = [g_{rs}]$ are infinite-dimensional arbitrary matrices of control gains. Of course, s_i and u_i ($r = 1, 2, \dots$) are closed-loop eigenvalues and the associated infinite-dimensional eigenvectors, respectively.

Recalling that changes in the system parameters affect only the open-loop eigenvalues $\Lambda_r = \omega_r^2$ ($r = 1, 2, \dots$), we can check the sensitivity of the controller to changes in parameters by examining the partial derivatives $\partial s_i / \partial \omega_r$ ($i, r = 1, 2, \dots$). But, Eqs. (3.36) are coupled through the feedback matrices H_0 and G_0 . Because H_0 and G_0 are infinite matrices, no sensitivity assessment is possible for coupled control. One alternative is to truncate, which implies plant truncation. Truncation is likely to introduce errors in the closed-loop poles, making an assessment of the sensitivity very difficult, if not impossible.

For natural control, Eqs. (3.27) and (3.35) yield the infinite set of independent characteristic equations

$$s_r^2 + h_r s_r + \lambda_r + g_r = 0 \quad , \quad r = 1, 2, \dots \quad (3.37)$$

having the roots

$$s_r = -\frac{h_r}{2} \pm \sqrt{\left(\frac{h_r}{2}\right)^2 - \lambda_r - g_r} \quad , \quad r = 1, 2, \dots \quad (3.38)$$

The modal control gains g_r and h_r are such that the radical in Eq. (3.38) is imaginary. Moreover, the sensitivity is dictated by the real part of the closed-loop poles. Denoting the change in the natural frequency ω_r by $\delta\omega_r$, we can write

$$\begin{aligned} \operatorname{Re} s_r(\omega_r + \delta\omega_r) &= \operatorname{Re} s_r(\omega_r) + \frac{\operatorname{Re} s_r}{\partial\omega_r} \delta\omega_r + \frac{1}{2} \frac{\partial^2 \operatorname{Re} s_r}{\partial\omega_r^2} \delta\omega_r^2 \\ &+ \dots \quad \quad \quad r = 1, 2, \dots \quad (3.39) \end{aligned}$$

where $\operatorname{Re} s_r = -h_r/2$. For small changes $\delta\omega_r$, the nonlinear terms in $\delta\omega_r$ can be discarded, so that the sensitivity is controlled by

$$\frac{\partial \text{Re } s_r}{\partial \omega_r} = -\frac{1}{2} \frac{\partial h_r}{\partial \omega_r}, \quad r = 1, 2, \dots \quad (3.40)$$

For optimal control, h_r is given by Eq. (3.34b), so that Eq. (3.40) becomes

$$\frac{\partial \text{Re } s_r}{\partial \omega_r} = -\frac{1}{2} \frac{-4\omega_r + 2(\omega_r^2 + R_r^{-1})^{1/2} + 2\omega_r \frac{1}{2} (\omega_r^2 + R_r^{-1})^{-1/2} 2\omega_r}{2[-2\omega_r^2 + R_r^{-1} + 2\omega_r (\omega_r^2 + R_r^{-1})^{1/2}]^{1/2}} \quad r = 1, 2, \dots \quad (3.41)$$

In the case in which $\omega_r^2 R_r \gg 1$, we can write

$$(\omega_r^2 + R_r^{-1})^{\pm 1/2} \approx \omega_r \left(1 \pm \frac{1}{2} \frac{R_r^{-1}}{\omega_r^2}\right), \quad r = 1, 2, \dots \quad (3.42)$$

Hence, by designing the modal controls so that $\omega_r^2 R_r \gg 1$, Eqs. (3.41)

reduce to

$$\frac{\partial \text{Re } s_r}{\partial \omega_r} = 0, \quad r = 1, 2, \dots \quad (3.43)$$

so that the natural control is insensitive to changes in the system parameters.

3.8 CONTROL IMPLEMENTATION

It is common practice in coupled control of a distributed system to discretize the equations of motion, thereby reducing the distributed system to discrete. This practice arises from the inability to compute control gains for infinite-dimensional systems. However, the resulting discrete system contains erroneous information. Indeed, no discretized system can represent a distributed system exactly, so that only a

small subset of the closed-loop poles can be regarded as accurate and the remaining modes can be grossly in error. As a consequence, it is not possible to control the full set of modes of the discretized system.

In this chapter, it is demonstrated that natural control represents a method capable of controlling a distributed system in which the control gains admit closed-form expressions. Indeed, the full set of modes can be controlled, at least in theory.

Unlike other methods, the independent modal space control method permits computation of optimal control forces that are ideally distributed. Nevertheless, in practice it is often desirable to use finite-dimensional controls, i.e., controls generated by a finite number of actuators acting at discrete points or over given subdomains of the system. Likewise, it is often desirable to use discrete measurements. In spite of this, the open-loop system remains distributed.

In this section implementation of the control using finite-dimensional controls and discrete measurements is addressed. In section 3.6, it was demonstrated how the independent modal-space control method can generate distributed controls $f(P,t)$ for distributed-parameter systems. For feedback, observation of either the displacement $u(P,t)$ or the velocity $\dot{u}(P,t)$ is assumed. In practice, infinite-dimensional controls and observations may not be possible, in which case the interest lies in implementing a natural control with a finite number of actuators and sensors. The problem reduces to how to approximate best distributed controls and measurements by discrete controls and measurements.

Let us write the actual control force as a projection of the computed distributed control force

$$f(P,t) = Sf^*(P,t) \quad (3.44)$$

where S denotes the projection operator under consideration and $f^*(P,t)$ is the computed distributed control force. Similarly, the computed distributed displacement is a projection of the actual displacement, written

$$u^*(P,t) = Tu(P,t) \quad (3.45)$$

where T represents the projection operator at hand and $u^*(P,t)$ denotes the computed distributed displacement. Given the projections S and T , the closed-loop eigenvalues can be found.

Substituting Eq. (3.44) into (3.9b), we obtain

$$f_r = \langle Sf^*, \phi_r \rangle = \sum_{s=1}^{\infty} S_{rs} f_s^* \quad , \quad r = 1,2,\dots \quad (3.46)$$

where

$$S_{rs} = \langle S\phi_s, \phi_r \rangle \quad , \quad r,s = 1,2,\dots \quad (3.47)$$

and f_s^* ($s = 1,2,\dots$) are computed modal controls. Moreover, substituting Eq. (3.45) into (3.8b), we obtain

$$u_r^* = \langle Tu, \phi_r \rangle = \sum_{s=1}^{\infty} T_{rs} u_s \quad , \quad r = 1,2,\dots \quad (3.48)$$

where u_r^* ($r = 1,2,\dots$) are computed modal displacements and

$$T_{rs} = \langle T\phi_s, \phi_r \rangle \quad , \quad r,s = 1,2,\dots \quad (3.49)$$

The modal controls are generated in the form

$$f_r^*(t) = -g_r u_r^*(t) - h_r \dot{u}_r^*(t) \quad , \quad r = 1, 2, \dots \quad (3.50)$$

Introducing the notation

$$u_{1r} = \dot{u}_r \quad , \quad u_{24} = u_r \quad , \quad r = 1, 2, \dots$$

and considering Eqs. (3.10) in conjunction with Eqs. (3.46), (3.48)

and (3.50), we obtain the closed-loop modal equations

$$\dot{u}_{1r} = \sum_{t=1}^{\infty} (a_{11rt} u_{1t} + a_{12rt} u_{2t}) \quad , \quad r = 1, 2, \dots \quad (3.51a)$$

$$\dot{u}_{24} = \sum_{t=1}^{\infty} (a_{21rt} u_{1t} + a_{22rt} u_{2t}) \quad , \quad r = 1, 2, \dots \quad (3.51b)$$

where

$$\begin{aligned} a_{11rt} &= - \sum_{s=1}^{\infty} S_{rs} h_s^T s^T \quad , \quad a_{12rt} = -\lambda_r \delta_{rt} - \sum_{s=1}^{\infty} S_{rs} g_s^T s^T \\ a_{21rt} &= \delta_{rt} \quad , \quad a_{22rt} = 0 \end{aligned} \quad (3.52)$$

Thus, the closed-loop eigenvalues are the solution to the determinantal equation

$$\det(sI - A) = 0 \quad , \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad (3.53a,b)$$

where A_{ij} ($i, j = 1, 2$) have entries a_{ijrs} ($r, s = 1, 2, \dots$). In Eq. (3.44), we approximate a distributed control force by a finite-dimensional control force. In the process, the closed-loop eigenvalues of the natural control are altered, as indicated by Eqs. (3.53). In approximating a distributed control force by a finite-dimensional one, as indicated by Eq. (3.44), the closed-loop eigenvalues are no longer given by Eqs. (3.29) but by Eqs. (3.53). A distributed control function

can be expressed as an infinite linear combination of the system eigenfunctions. The finite-dimensional controls approximating distributed controls must be capable of reproducing the time histories of the modal control forces associated with the lower modes. The approximations of the higher modes need not be so accurate. Therefore the use of finite-dimensional controls leads naturally to a truncation of the set of modes to be controlled. This does not, however, imply a truncation of the control law. Indeed, if we control n modes, then we have simply

$$g_r = h_r = 0 \quad , \quad r = n+1, n+2, \dots \quad (3.54)$$

3.9 NATURAL CONTROL OF A SIMPLY-SUPPORTED BEAM

As a first example, we shall control a simply supported uniform beam of length $a = 10$ and with unit mass density and unit stiffness density. The system operator is

$$A = \partial^4 / \partial x^4 \quad , \quad 0 < x < a \quad (3.55)$$

and it admits eigenfunctions and frequencies in the closed-form (Ref. 2)

$$\phi_r(x) = \sqrt{\frac{2}{a}} \sin \frac{r\pi x}{a} \quad , \quad \omega_r = \left(\frac{r\pi}{a} \right)^2 \quad , \quad r = 1, 2, \dots \quad (3.56)$$

The displacement satisfies the geometric boundary condition $u(0,t) = u(a,t) = 0$ and the natural boundary conditions $\partial^2 u(0,t) / \partial x^2 = \partial^2 u(a,t) / \partial x^2 = 0$. We shall consider a frequency invariant control of the lowest n modes of vibration with uniform decay rate α , so that

$$\begin{aligned}
 g_r &= \alpha^2 \quad , \quad h_r = 2\alpha \quad , \quad r = 1, 2, \dots, n \\
 g_r &= h_r = 0 \quad , \quad r = n+1, n+2, \dots
 \end{aligned}
 \tag{3.57}$$

For the sake of this example, it will be assumed that the lowest 15 modes of vibration contribute significantly to the overall system response and the contribution to the motion of the remaining modes is neglected. The beam is given an initial unit impulse at $x_0 = 4$. The beam is controlled using distributed control forces, control forces acting at discrete points and piecewise constant control forces, as shown in Fig. 3.1. The distributed displacement and velocity are computed from discrete measurements of the displacement, slope, velocity and angular velocity.

a. Distributed Control Force

The free response of the beam has the closed-form solution

$$u(x, t) = \sum_{r=1}^{\infty} \frac{1}{\omega_r} \phi_r(x) \phi_r(x_0) \sin \omega_r t
 \tag{3.58}$$

and is shown in Fig. 3.2. The response of the beam using distributed controls admits the closed-form solution

$$\begin{aligned}
 u(x, t) &= \sum_{r=1}^n \frac{1}{\omega_r} \phi_r(x) \phi_r(x_0) e^{-\alpha t} \sin \omega_r t \\
 &+ \sum_{r=n+1}^{\infty} \frac{1}{\omega_r} \phi_r(x) \phi_r(x_0) \sin \omega_r t
 \end{aligned}
 \tag{3.59}$$

in which we control the lowest $n=6$ modes using frequency-invariant control, Eq. (3.57). The response is shown in Fig. 3.3. The open- and closed-loop eigenvalues are given in Table 3.1.

b. Discrete Point Control Forces

Let us subdivide the beam into n subdomains $D_i = [x_{i-1}, x_i]$, ($i = 1, 2, \dots, n$) in which $x_i = ia/n$. Discrete actuators are placed at the center of the subdomains at $\bar{x}_i = (i - \frac{1}{2})a/n$. The discrete control force is a projection of the distributed control forces. Hence Eq. (3.44) has the explicit form

$$f(x,t) = \sum_{r=1}^n f^*(\bar{x}_r, t) \frac{a}{n} \delta(x - \bar{x}_r) \quad (3.60)$$

Substituting Eq. (3.60) into Eq. (3.47), we obtain

$$S_{rs} = \frac{2}{n} \sum_{t=1}^n \sin \frac{r\pi\bar{x}_t}{a} \sin \frac{s\pi\bar{x}_t}{a} \quad (3.61)$$

The closed-loop eigenvalues corresponding to $n = 4, 6$ and 8 actuators are given in Table 3.2. The response of the beam controlled with $n = 6$ actuators is shown in Fig. 3.4.

c. Piecewise Constant Control Forces

Let us consider actuator forces constant over the subdomains D_i ($i = 1, 2, \dots, n$). Eq. (3.44) has the explicit form

$$f(x,t) = f^*(\bar{x}_r, t) \quad , \quad x \in D_r \quad , \quad r = 1, 2, \dots, n \quad (3.62)$$

Substituting Eq. (3.62) into (3.44), we obtain

$$S_{rs} = \frac{2}{r\pi} \sum_{t=1}^n \sin \frac{s\pi\bar{x}_t}{a} \cos \frac{r\pi x_{t-1}}{a} - \cos \frac{r\pi x_t}{a} \quad (3.63)$$

The closed-loop eigenvalues corresponding to $n = 4, 6$ and 8 piecewise constant actuators are given in Table 3.3. The response of the beam

controlled with $n = 6$ actuators is shown in Fig. 3.5. In the above cases, we have considered distributed measurements.

d. Discrete Measurements

Distributed measurements will be reconstructed from discrete measurements. To this end, we divide the beam into m subdomains $D_i = (x_{i-1}, x_i)$ ($i = 1, 2, \dots, m$) where $x_i = ia/m$. The computed distributed measurement in a given subdomain is expressed as a linear combination of interpolation functions multiplied by nodal measurements (Ref. 2). Equation (3.45) becomes

$$u^*(x_i, t) = L_1 u(x_{i-1}, t) + L_2 u'(x_{i-1}, t) + L_3 u(x_i, t) + L_4 u'(x_i, t), \quad x \in D_i, \quad i = 1, 2, \dots, m \quad (3.64)$$

where primes denote differentiations with respect to x and

$$\begin{aligned} L_1 &= 3\xi^2 - 2\xi^3, & L_2 &= (\xi^2 - \xi^3)a/m \\ L_3 &= 1 - 3\xi^2 + 2\xi^3, & L_4 &= (-\xi + 2\xi^2 - \xi^3)a/m \end{aligned} \quad (3.65)$$

in which $\xi = i - mx/a$ is a nondimensional variable. From Eqs. (3.64) and (3.49), we obtain

$$\begin{aligned} T_{ij} &= \frac{2}{m} \sum_{k=1}^m \{e_3 b(d_1 - d_3) + e_4 b[b(d_4 - d_2) - d_1] \\ &\quad + (3e_1 + e_2 - 3e_3 + 2e_4) [d_1 b + 2d_2 b^2 + 2(d_3 - d_1)b^3] \\ &\quad + (-2e_1 - e_2 + 2e_3 - e_4) [d_1 b - 3d_2 b^2 - 6d_1 b^3 \\ &\quad + 6(d_4 - d_2)b^4]\} \quad , \quad i, j = 1, 2, \dots \end{aligned} \quad (3.66)$$

where $b = m/i\pi$ and

$$\begin{aligned}
d_1 &= \cos \frac{i\pi x_{k-1}}{a}, \quad d_2 = \sin \frac{i\pi x_{k-1}}{a}, \quad d_3 = \cos \frac{i\pi x_k}{a}, \quad d_4 = \sin \frac{i\pi x_k}{a} \\
e_1 &= \sin \frac{j\pi x_{k-1}}{a}, \quad e_2 = \cos \frac{j\pi x_{k-1}}{a}, \quad e_3 = \sin \frac{j\pi x_k}{a}, \quad e_4 = \cos \frac{j\pi x_k}{a}
\end{aligned}
\tag{3.67}$$

Control of n modes using m discrete measurements is considered for the cases $(n = 4, m = 2)$, $(n = 6, m = 3)$ and $(n = 8, m = 4)$, in which the control force is distributed. The closed-loop eigenvalues are given in Table 3.4.

3.10 NATURAL CONTROL OF A SIMPLY-SUPPORTED RECTANGULAR MEMBRANE

As a second example, we shall control a simply-supported rectangular membrane. The membrane has the dimensions $a = 3$ in the x direction and $b = 2$ in the y direction, has unit mass density and unit stiffness. The system operator (Ref. 15)

$$A = -\partial^2 / \partial x^2 - \partial^2 / \partial y^2 \tag{3.68}$$

admits eigenfunctions and frequencies in the closed-form

$$\begin{aligned}
\phi_{rs}(x,y) &= \frac{2}{\sqrt{ab}} \sin \frac{r\pi x}{a} \sin \frac{s\pi y}{b}, \quad \omega_{rs} = \pi \left[\left(\frac{r}{a}\right)^2 + \left(\frac{s}{b}\right)^2 \right]^{1/2} \\
r,s &= 1,2,\dots
\end{aligned}
\tag{3.69}$$

The displacement satisfies the boundary conditions $u(x,0,t) = u(x,b,t) = 0$ and $u(0,y,t) = u(a,y,t) = 0$. The membrane is given an initial unit impulse at $x_0 = 1, y_0 = 1$. We consider linear optimal control, with the performance functional in the explicit form

$$J = \int_0^\infty \int_0^a \int_0^b [\dot{u}^2(x,t) - u(x,t) \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + \sum_{r=1}^\infty \sum_{s=1}^\infty R_{rs} f(x,y,t) \int_0^a \int_0^b f(x,y,t) \phi_{rs}(x,y) dx dy] dx dy dt \quad (3.70)$$

where $R_{rs} = 0.5$ ($r, s = 1, 2, \dots, n$) and $R_{rs} = \infty$ ($r, s = n+1, n+2, \dots$). It is assumed that the r, s modes ($r, s = 1, 2, \dots, 6$) contributed significantly to the overall system response and contribution to the motion of the remaining modes is neglected. The membrane is controlled with either distributed control forces, point control forces or piecewise constant control forces (see Fig. 3.6).

a. Distributed Control Force

The free response of the membrane has the closed-form solution

$$u(x,y,t) = \sum_{r=1}^\infty \sum_{s=1}^\infty \frac{1}{\omega_{rs}} \phi_{rs}(x,y) \phi_{rs}(x_0, y_0) \sin \omega_{rs} t \quad (3.71)$$

as shown in Fig. 3.7. The response of the membrane in the case of a distributed control force also admits a solution in the closed-form

$$u(x,y,t) = \sum_{r=1}^\infty \sum_{s=1}^\infty \phi_{rs}(x,y) u_{rs}(t) \quad r, s = 1, 2, \dots \quad (3.72)$$

$$u_{rs}(t) = \frac{1}{\gamma_{rs}} \phi_{rs}(x_0, y_0) e^{-\gamma_{rs} t} \sin \gamma_{rs} t$$

in which

$$\alpha_{rs} = \frac{1}{2} [-2\omega_{rs}^2 + 1/R_{rs} + 2\omega_{rs}(\omega_{rs}^2 + 1/R_{rs})^{1/2}]^{1/2} \quad r, s = 1, 2, \dots \quad (3.73)$$

$$\gamma_{rs} = \frac{1}{4} [2\omega_{rs}^2 - 1/R_{rs} + 2\omega_{rs}(\omega_{rs}^2 + 1/R_{rs})^{1/2}]^{1/2}$$

in which the modes $r, s = 1, 2, 3, 4$ are controlled, i.e., $R_{rs} = 0.5$ ($r, s = 1, 2, 3, 4$) and $R_{rs} = \infty$ ($r, s = 5, 6, \dots$). The response is shown in Fig. 3.8. The open- and closed-loop eigenvalues are given in Table 3.5.

b. Discrete Point Control Forces

Let us subdivide the rectangular membrane into the subdomains D_{ij} ($i, j = 1, 2, \dots, n$) where D_{ij} includes all points x, y such that $x_{i-1} < x < x_i$ and $y_{j-1} < y < y_j$, where $x_i = ia/n$, $y_j = jb/n$. We place discrete actuators at the centers of these subdomains at $\bar{x}_i = (i - \frac{1}{2})a/n$, $\bar{y}_j = (j - \frac{1}{2})b/n$. From Eq. (3.47), we find that

$$S_{rstu} = \frac{4}{n^2} \sum_{v=1}^n \sum_{w=1}^n \sin \frac{r\pi\bar{x}_v}{a} \sin \frac{s\pi\bar{y}_w}{b} \sin \frac{t\pi\bar{x}_v}{a} \sin \frac{u\pi\bar{y}_w}{b} \quad (3.74)$$

The closed-loop eigenvalues corresponding to 4, 9 and 16 discrete actuators are given in Table 3.6.

c. Piecewise Constant Control Forces

Let us consider control forces constant over each subdomain D_{ij} . Then, Eq. (3.44) has the explicit form

$$f(x, y, t) = f^*(\bar{x}_i, \bar{y}_j, t) \quad , \quad x, y \in D_{ij} \quad (i, j = 1, 2, \dots, n) \quad (3.75)$$

From Eq. (3.47), we have

$$S_{rstu} = \frac{4}{rs\pi^2} \sum_{v=1}^n \sum_{w=1}^n \sin \frac{t\pi\bar{x}_v}{a} \sin \frac{u\pi\bar{y}_w}{b} \left[\cos \frac{r\pi\bar{x}_v}{a} - \cos \frac{r\pi\bar{x}_{v-1}}{a} \right] \times \\ \left[\cos \frac{s\pi\bar{y}_w}{b} - \cos \frac{s\pi\bar{y}_{w-1}}{b} \right] \quad (3.76)$$

The closed-loop eigenvalues corresponding to 4, 9 and 16 piecewise constant actuators are given in Table 3.7. The response of the membrane

controlled with 16 actuators is shown in Fig. 3.10. In the above cases, measurements are assumed to be distributed.

d. Discrete Measurements

The distributed measurements are reconstructed for each sub-domain D_{ij} ($i, j = 1, 2, \dots, m$). As in the previous example, we express the computed distributed measurement as a linear combination of interpolation functions multiplied by nodal measurements (Ref. 2). Equation (3.45) is written as

$$\begin{aligned} u^*(x, y, t) = & L_1 u(x_{i-1}, y_{j-1}, t) + L_2 u(x_i, y_{j-1}, t) + L_3 u(x_i, y_j, t) \\ & + L_4 u(x_{i-1}, y_j, t) \quad , \quad x \in D_{ij} \end{aligned} \quad (3.77)$$

where

$$\begin{aligned} L_1 &= \frac{1}{4} (1 - \xi_1)(1 - \xi_2) \quad , \quad L_2 = \frac{1}{4} (1 + \xi_1)(1 - \xi_2) \\ L_3 &= \frac{1}{4} (1 + \xi_1)(1 + \xi_2) \quad , \quad L_4 = \frac{1}{4} (1 - \xi_1)(1 + \xi_2) \end{aligned} \quad (3.78)$$

in which $\xi_1 = 1 - 2i + 2xn/a$ and $\xi_2 = 1 - 2j + 2yn/b$ are nondimensional variables. Substituting Eq. (3.77) into (3.49), we obtain

$$T_{rstu} = \frac{1}{4n^2} \sum_{v=1}^m \sum_{w=1}^m [T_{x1} T_{y1} + T_{x2} T_{y2} + T_{x3} T_{y3} + T_{x4} T_{y4}] \quad (3.79)$$

where

$$T_{x1} = \sin \frac{t\pi x_{v-1}}{a} \left[\frac{4m}{s\pi} \cos \frac{r\pi x_{v-1}}{a} - \left(\frac{2m}{r\pi} \right)^2 \left(\sin \frac{r\pi x_v}{a} - \sin \frac{r\pi x_{v-1}}{a} \right) \right] \quad (3.80a)$$

$$T_{y1} = \sin \frac{u\pi y_{w-1}}{a} \left[\frac{4m}{s\pi} \cos \frac{s\pi y_{w-1}}{b} - \left(\frac{2m}{r\pi} \right)^2 \left(\sin \frac{r\pi y_w}{b} - \sin \frac{s\pi y_{w-1}}{b} \right) \right] \quad (3.80b)$$

$$T_{x2} = \sin \frac{t\pi x_v}{a} \left[-\frac{4m}{r\pi} \cos \frac{r\pi x_v}{a} + \left(\frac{2m}{r\pi}\right)^2 \left(\sin \frac{r\pi x_v}{a} - \sin \frac{r\pi x_{v-1}}{a} \right) \right] \quad (3.80c)$$

$$T_{y2} = \sin \frac{u\pi y_{w-1}}{a} \left[\frac{4m}{s\pi} \cos \frac{s\pi x_{w-1}}{a} - \left(\frac{2m}{s\pi}\right)^2 \left(\sin \frac{s\pi x_w}{b} - \sin \frac{s\pi x_{w-1}}{b} \right) \right] \quad (3.80d)$$

$$T_{x3} = \sin \frac{t\pi x_v}{a} \left[-\frac{4m}{r\pi} \cos \frac{r\pi x_v}{a} + \left(\frac{2m}{r\pi}\right)^2 \left(\sin \frac{r\pi x_v}{a} - \sin \frac{r\pi x_{v-1}}{a} \right) \right] \quad (3.80e)$$

$$T_{y3} = \sin \frac{u\pi y_w}{b} \left[-\frac{4m}{s\pi} \cos \frac{s\pi y_w}{b} + \left(\frac{2m}{s\pi}\right)^2 \left(\sin \frac{s\pi y_w}{b} - \sin \frac{s\pi y_{w-1}}{b} \right) \right] \quad (3.80f)$$

$$T_{x4} = \sin \frac{t\pi x_v}{a} \left[\frac{4m}{r\pi} \cos \frac{r\pi x_{v-1}}{a} - \left(\frac{2m}{r\pi}\right)^2 \left(\sin \frac{r\pi x_v}{a} - \sin \frac{r\pi x_{v-1}}{a} \right) \right] \quad (3.80g)$$

$$T_{y4} = \sin \frac{u\pi y_{w-1}}{b} \left[-\frac{4m}{s\pi} \cos \frac{s\pi y_w}{b} + \left(\frac{2m}{s\pi}\right)^2 \left(\sin \frac{r\pi x_v}{a} - \sin \frac{r\pi x_{v-1}}{a} \right) \right] \quad (3.80h)$$

The closed-loop eigenvalues are given in Table 3.8, in which the modes $(r, s = 1, 2, \dots, n)$ are controlled using distributed controls and measurements are taken over the elements D_{ij} ($i, j = 1, 2, \dots, m$) for the case $(n = 3, m = 7)$.

Table 3.1 Open- and closed-loop eigenvalues for beam controlled with a distributed actuator

	Open-loop	Closed-loop
1	$0 \pm .0986i$	$-1. \pm .0986i$
2	$0 \pm .3947i$	$-1. \pm .3947i$
3	$0 \pm .8882i$	$-1. \pm .8882i$
4	$0 \pm 1.579 i$	$-1. \pm 1.579 i$
5	$0 \pm 2.467 i$	$-1. \pm 2.467 i$
6	$0 \pm 3.553 i$	$-1. \pm 3.553 i$

Table 3.2 Closed-loop eigenvalues for beam controlled with discrete point actuators

	4	6	8
1	- .9999 ± .0987i	- .9999 ± .0987i	- .9999 ± .0987i
2	- .9999 ± .3947i	- .9999 ± .3947i	- .9999 ± .3947i
3	- .9999 ± .8882i	- .9999 ± .8882i	- .9999 ± .8882i
4	-1.999 ± .7025i	- .9999 ± 1.579 i	- .9999 ± 1.579 i
5	.0 ± 2.467 i	- .9999 ± 2.467 i	- .9999 ± 2.467 i
6	.0 ± 3.553 i	-1.999 ± 3.259 i	- .9999 ± 3.553 i
7	.0 ± 4.835 i	0. ± 4.835 i	- .9999 ± 4.835 i
8	.0 ± 6.316 i	0. ± 6.316 i	-1.9999 ± 6.155 i

Table 3.3 Closed-loop eigenvalues for beam controlled with piecewise constant actuators

	4	6	8
1	- .9744 ± .1860i	- .9885 ± .1448i	- .9935 ± .1269i
2	- .9002 ± .4955i	- .9548 ± .4459i	- .9744 ± .4251i
3	- .7842 ± .9788i	- .9002 ± .9374i	- .9431 ± .9179i
4	-1.273 ± 1.464 i	- .8269 ± 1.623 i	- .9002 ± 1.607 i
5	0. ± 2.467 i	- .7378 ± 2.506 i	- .8468 ± 2.493 i
6	0. ± 3.553 i	-1.273 ± 3.503 i	- .7841 ± 3.576 i
7	0. ± 4.835 i	0. ± 4.835 i	- .7135 ± 4.856 i
8	0. ± 6.316 i	0. ± 6.316 i	-1.273 ± 6.288 i

Table 3.4 Closed-loop eigenvalues for beam controlled using discrete measurements

	4	6	8
1	- .9951 ± .1258i	- .9984 ± .1060i	- .9993 ± .1011i
2	- .8105 ± .5562i	- .9795 ± .4222i	- .9923 ± .4044i
3	- .6351 ± 1.007 i	- .8104 ± .9707i	- .9653 ± .9090i
4	- .6079 ± 1.625 i	- .7383 ± 1.633 i	- .8104 ± 1.626 i
5	0. ± 2.467 i	- .5253 ± 2.516 i	- .7857 ± 2.495 i
6	0. ± 3.552 i	- .6078 ± 3.586 i	- .6378 ± 3.583 i
7	0. ± 4.835 i	0. ± 4.835 i	- .4680 ± 4.861 i
8	0. ± 6.316 i	0. ± 6.316 i	- .6078 ± 6.334 i

Table 3.5 Open- and closed-loop eigenvalues for membrane controlled with a distributed actuator

	Open-loop	Closed-loop
1,1	$0. \pm 1.887i$	$-0.971 \pm 1.873i$
2,1	$0. \pm 2.617i$	$-0.989 \pm 3.308i$
1,2	$0. \pm 3.331i$	$-0.994 \pm 4.826i$
3,1	$0. \pm 3.512i$	$-0.996 \pm 6.369i$
2,2	$0. \pm 3.775i$	$-0.983 \pm 2.611i$
3,3	$0. \pm 4.442i$	$-0.991 \pm 3.773i$
4,1	$0. \pm 4.473i$	$-0.995 \pm 5.155i$
1,3	$0. \pm 4.827i$	$-0.997 \pm 6.622i$
2,3	$0. \pm 5.156i$	$-0.990 \pm 3.509i$
4,2	$0. \pm 5.235i$	$-0.993 \pm 4.441i$
3,3	$0. \pm 5.663i$	$-0.996 \pm 5.662i$
4,3	$0. \pm 6.304i$	$-0.997 \pm 7.024i$
1,4	$0. \pm 6.369i$	$-0.994 \pm 4.472i$
2,4	$0. \pm 6.623i$	$-0.995 \pm 5.235i$
3,4	$0. \pm 7.024i$	$-0.996 \pm 6.304i$
4,4	$0. \pm 7.551i$	$-0.997 \pm 7.551i$

Table 3.6 Closed-loop eigenvalues for membrane controlled with discrete point actuators

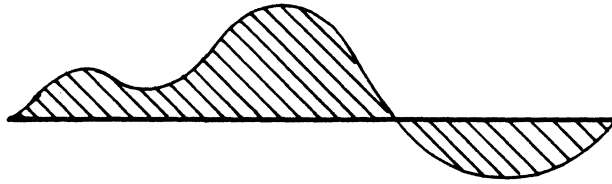
4	9	16
$-3.966 \pm 1.545i$	$-0.971 \pm 1.872i$	$-0.971 \pm 1.872i$
$-0.971 \pm 1.872i$	$-0.983 \pm 2.611i$	$-0.983 \pm 2.611i$
$-1.967 \pm 2.202i$	$-1.980 \pm 3.214i$	$-0.989 \pm 3.307i$
$-1.978 \pm 2.993i$	$-0.989 \pm 3.307i$	$-0.990 \pm 3.509i$
	$-0.991 \pm 3.773i$	$-0.991 \pm 3.773i$
	$-1.987 \pm 4.211i$	$-1.987 \pm 4.243i$
	$-3.984 \pm 4.486i$	$-0.993 \pm 4.441i$
	$-1.989 \pm 4.614i$	$-0.994 \pm 4.825i$
	$-1.990 \pm 4.958i$	$-1.990 \pm 5.040i$
		$-0.995 \pm 5.155i$
		$-0.996 \pm 5.662i$
		$-1.993 \pm 6.143i$
		$-1.993 \pm 6.210i$
		$-1.994 \pm 6.469i$
		$-3.990 \pm 6.711i$
		$-1.994 \pm 6.879i$

Table 3.7 Closed-loop eigenvalues for membrane controlled with piecewise constant actuators

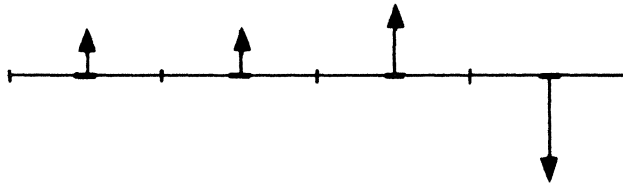
4	9	16
$-0.787 \pm 1.913i$	$-0.886 \pm 1.894i$	$-0.922 \pm 1.885i$
$-1.127 \pm 2.579i$	$-0.776 \pm 2.643i$	$-0.863 \pm 2.632i$
$-1.133 \pm 3.282i$	$-0.781 \pm 3.332i$	$-0.867 \pm 3.324i$
$-1.607 \pm 3.637i$	$-1.204 \pm 3.471i$	$-0.756 \pm 3.534i$
	$-0.678 \pm 3.801i$	$-0.803 \pm 3.793i$
	$-1.046 \pm 4.434i$	$-1.233 \pm 4.438i$
	$-1.209 \pm 4.798i$	$-0.701 \pm 4.464i$
	$-1.048 \pm 5.149i$	$-0.760 \pm 4.844i$
	$-1.614 \pm 5.572i$	$-0.702 \pm 5.175i$
		$-1.141 \pm 5.218i$
		$-0.612 \pm 5.683i$
		$-0.995 \pm 6.304i$
		$-1.236 \pm 6.345i$
		$-1.143 \pm 6.609i$
		$-0.995 \pm 7.024i$
		$-1.617 \pm 7.483i$

Table 3.8 Closed-loop eigenvalues for
membrane controlled using
discrete measurements

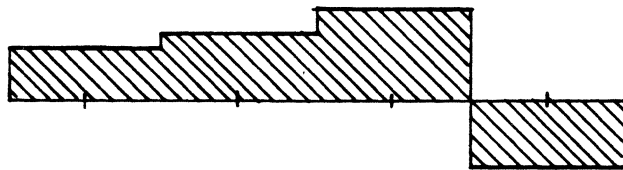
$-0.943825 \pm 1.888518i$
$-0.876501 \pm 2.625764i$
$-0.875206 \pm 3.310571i$
$-0.780590 \pm 3.559491i$
$-0.814189 \pm 3.780836i$
$-0.727226 \pm 4.487074i$
$-0.773937 \pm 4.848867i$
$-0.721109 \pm 5.179843i$
$-0.642197 \pm 5.714294i$



(a)



(b)



(c)

Figure 3.1 Natural control of a simply supported beam. (a) distributed control force; (b) discrete control force; (c) piecewise constant control force

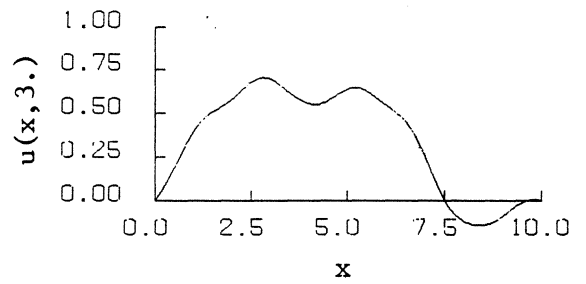
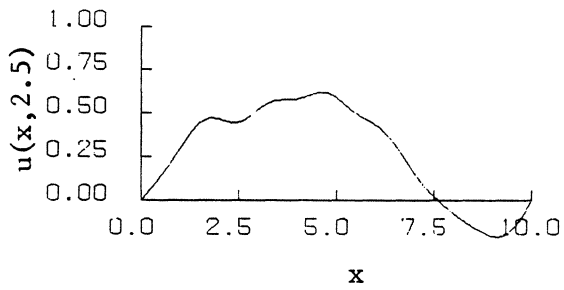
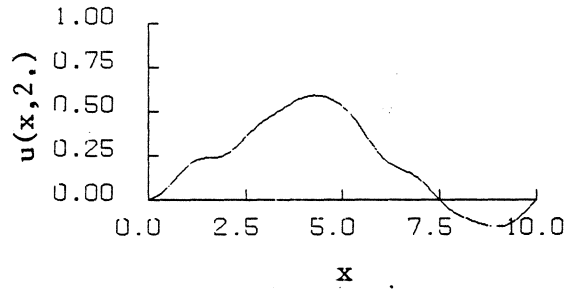
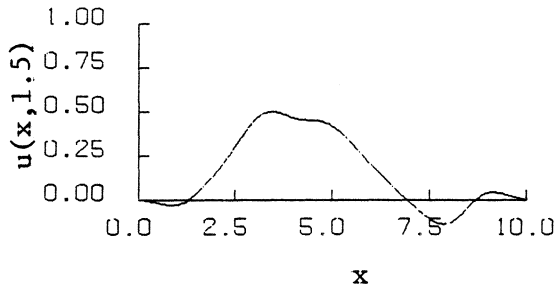
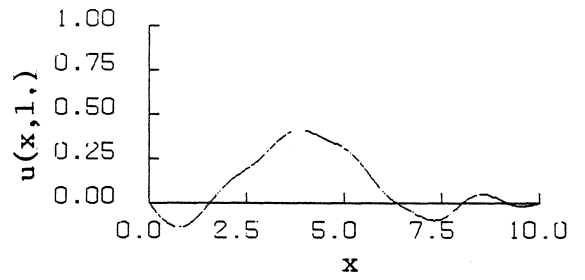
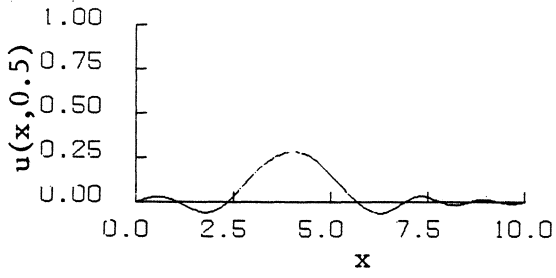


Figure 3.2 Free response of simply supported beam

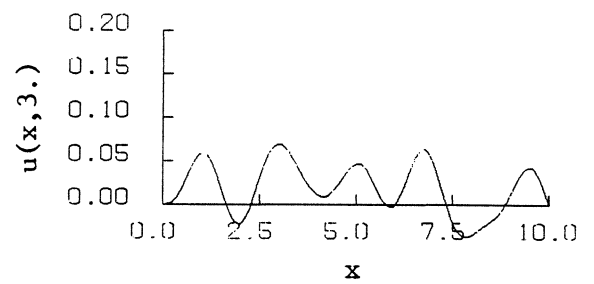
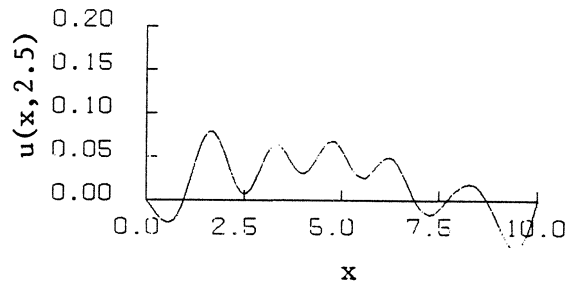
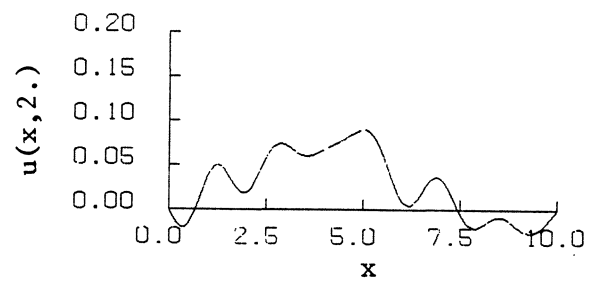
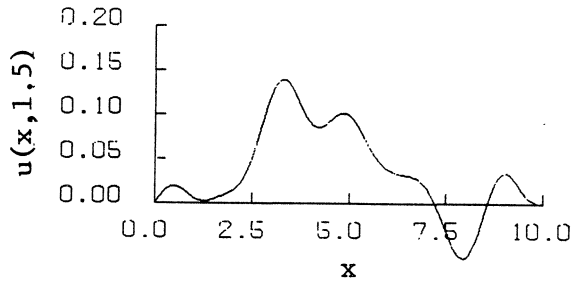
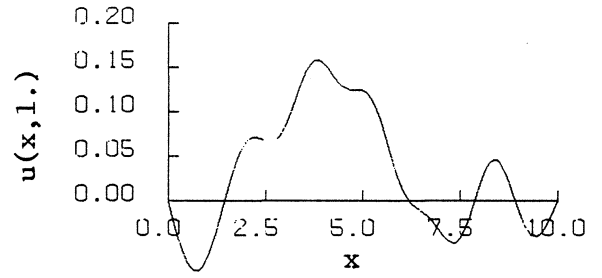
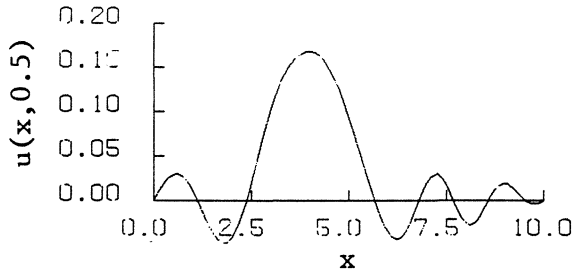


Figure 3.3 Response of simply supported beam using a distributed actuator

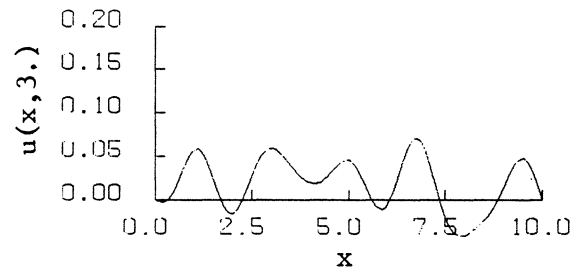
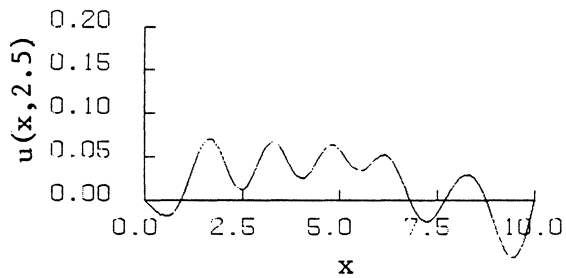
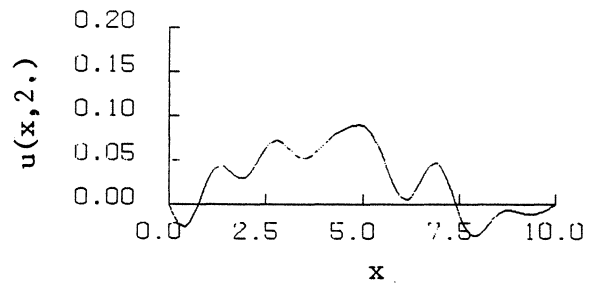
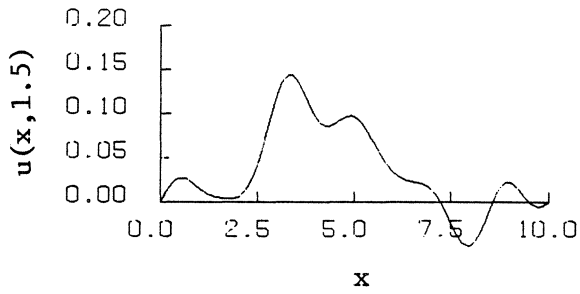
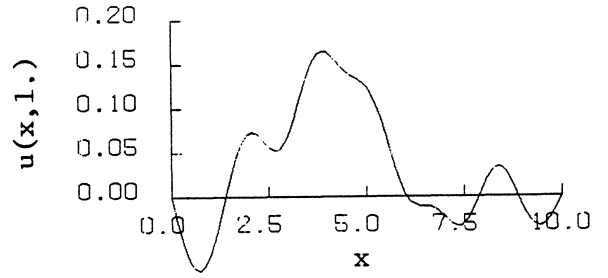
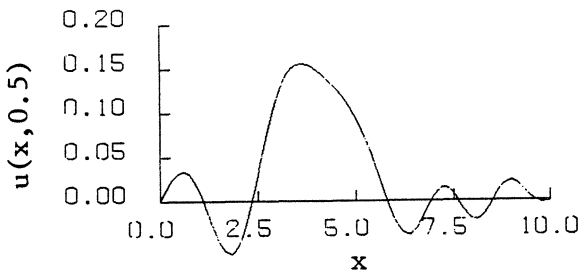


Figure 3.4 Response of simply supported beam using 6 discrete actuators

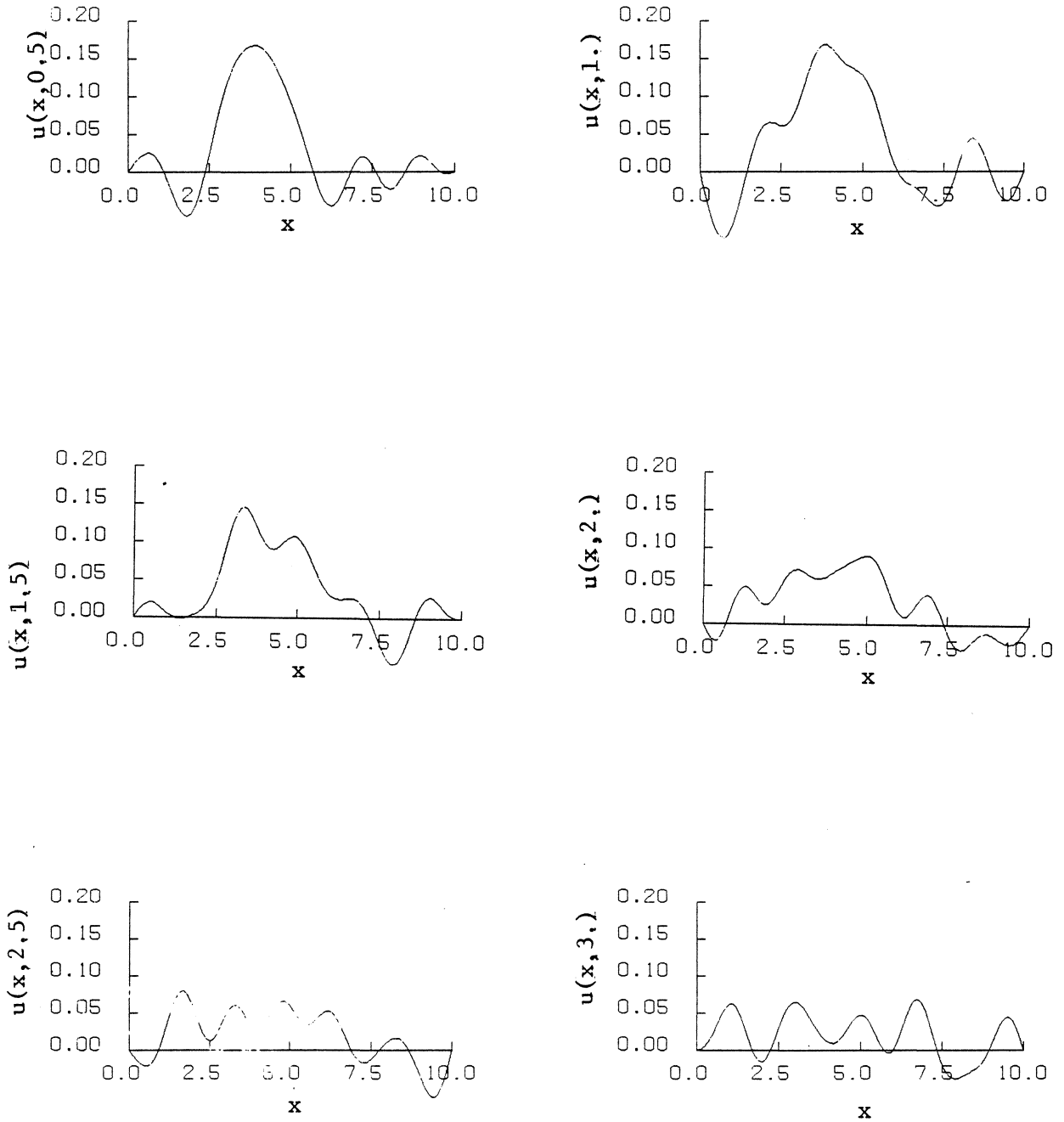
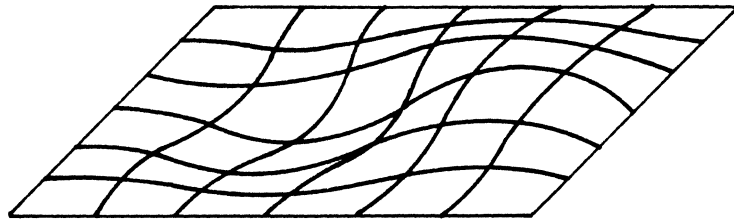
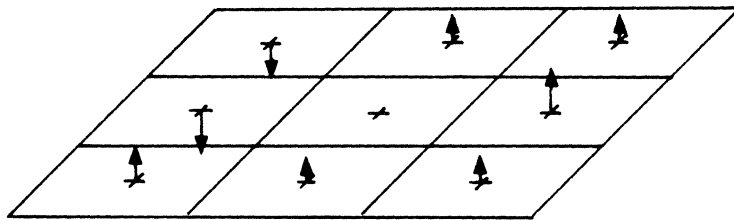


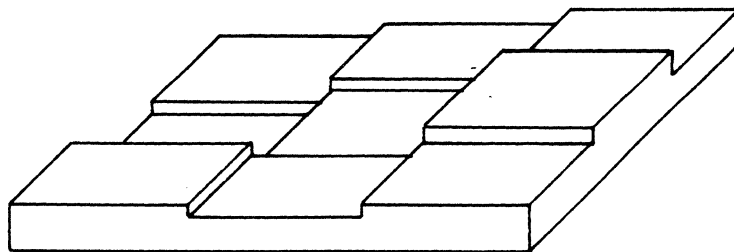
Figure 3.5 Response of simply supported beam using 6 piecewise constant actuators



(a)



(b)



(c)

Figure 3.6 Natural control of a rectangular membrane. (a) distributed control force; (b) discrete control forces; (c) piecewise constant control forces

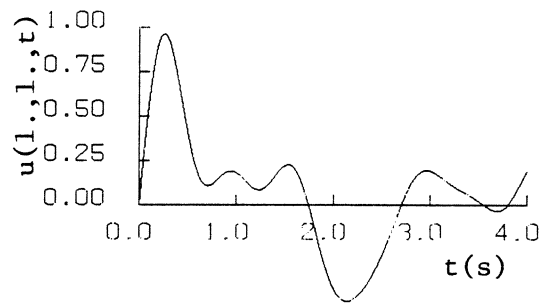


Figure 3.7 Free response of simply supported rectangular membrane

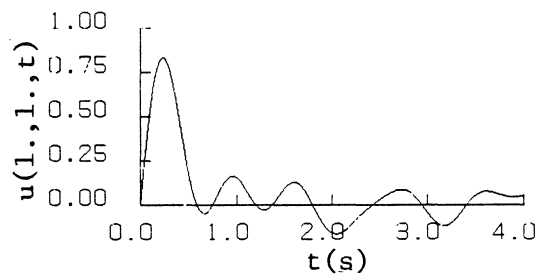


Figure 3.8 Response of simply supported rectangular membrane using a distributed actuator

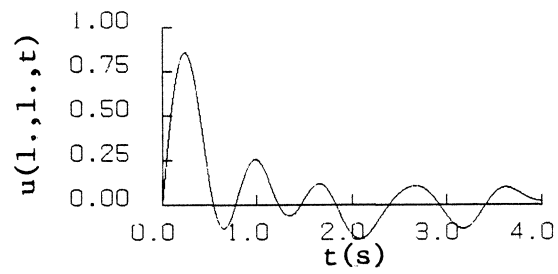


Figure 3.9 Response of simply supported rectangular membrane 16 discrete actuators

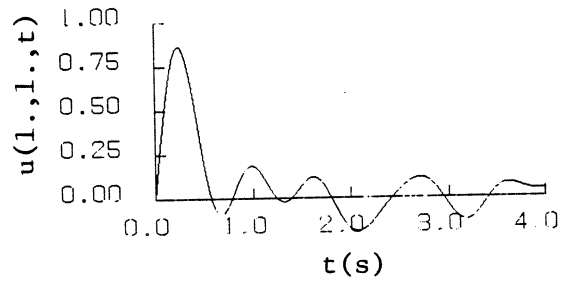


Figure 3.10 Response of simply supported rectangular membrane using 16 piecewise constant actuators

Chapter IV

CONTROL OF NON-SELF-ADJOINT DISTRIBUTED STRUCTURES

4.1 INTRODUCTION

A large class of distributed systems is characterized by a stiffness operator possessing the self-adjointness property. This implies a certain symmetry of the operator and of the boundary conditions, which implies further that the system eigenfunctions are mutually orthogonal. The problem of control of self-adjoint distributed systems has been treated elsewhere (Ref. 16).

Damping tends to destroy the self-adjointness of distributed systems. Damping can arise from various sources such as viscous damping associated with shearing forces in fluids, aerodynamic forces or internal hysteresis in cyclic stress. One finds nonconservative forces arising in power transmitting devices such as cranks and shafts and known as circulatory forces (Refs. 17,18), as well as forces arising in dual-spin satellites and known as constraint damping forces (Refs. 19,20). Note that hysteretic damping is commonly known as structural damping (Ref. 15).

In this chapter, the treatment of control of self-adjoint systems is extended to non-self-adjoint distributed systems. Formulation of the differential eigenvalue problem associated with the distributed system and its adjoint permits computation of the system eigenvalues and associated system eigenfunctions and adjoint system eigenfunctions also known as right and left eigenfunctions, respectively.

Based on these quantities, a modal operator transforms the equations of motion into an infinite set of modal equations.

Formulation of the differential eigenvalue problem associated with the linear control gain operator for linear feedback control permits computation of the eigenvalues and left and right eigenfunctions associated with the control gain operator. The modal equations of motion in the absence of feedback control represent an infinite set of independent first-order complex ordinary differential equations. Feedback control in general recouples the modal equations. Control of this form is commonly known as coupled control. There exists, however, a particular choice of eigenfunctions associated with the control gain operator in which the modal equations remain decoupled. Control of this form is known as independent modal-space control.

For independent modal-space control, the eigenfunctions associated with the open-loop system and those associated with the closed-loop system are identical. Moreover, formulation of the linear optimal control problem leads to an infinite set of independent complex scalar Riccati equations. The transient solution can be found with relative ease and the steady-state solution can be found in closed-form.

In coupled control, the distributed system is discretized in space resulting in a discrete model of the distributed system. It is possible to control only a small subset of the modes since the remaining modes can be grossly in error due to spatial discretization. In contrast to coupled control, the independent modal-space control method is capable of controlling the distributed system, i.e., the entire

infinite set of modes, at least in theory. The formulation is distributed in nature and ideally admits distributed forces. However, implementation often dictates the need for discrete actuators and sensors. To this end, it is shown how the independent modal-space control method can be implemented using discrete actuators and discrete sensors. As a numerical example, a whirling circular shaft is controlled using discrete actuators.

4.2 EQUATIONS OF MOTION OF NON-SELF-ADJOINT DISTRIBUTED STRUCTURES

The equation of motion for a non-self-adjoint distributed structure can be expressed in the form of the partial differential equation

$$m\ddot{u}(P,t) + (c+g)\dot{u}(P,t) + (k+h)u(P,t) = \underline{f}(P,t) \quad (4.1)$$

where $\underline{u}(P,t)$ is the vector of displacements at any point P in the open spatial domain D of the system and $\underline{f}(P,t)$ represents a distributed control force. We denote the mass matrix operator by m , the damping matrix operator by c and the stiffness matrix operator by k . The gyroscopic matrix operator is denoted by g and the circulator matrix operator is denoted by h . In addition to Eq. (4.1), the displacement $\underline{u}(P,t)$ is subject to the boundary conditions

$$B\underline{u}(P,t) = \underline{0} \quad , \quad P \in S \quad (4.2)$$

to be satisfied at every point P in S bounding D in which B represents a matrix operator expressing the homogeneous boundary conditions. The displacement $\underline{u}(P,t)$ and the distributed forces $\underline{f}(P,t)$ are elements of some Hilbert space Z with given norm $\| \cdot \|$ and inner product (\cdot , \cdot) .

The mass matrix operator m is self-adjoint, i.e., for any two admissible functions $\psi_i(P,t)$ and $\psi_j(P,t)$ in Z satisfying all the system boundary conditions (4.2), we have

$$(\psi_i, m\psi_j) = (\psi_j, m\psi_i) \quad (4.3)$$

Likewise the operators c and k are self-adjoint. The gyroscopic operator g is skew symmetric, i.e.,

$$(\psi_i, g\psi_j) = -(\psi_j, g\psi_i) \quad (4.4)$$

where again $\psi_i(P,t)$ and $\psi_j(P,t)$ are admissible functions. The circulatory operator h is also skew symmetric. We shall further assume that both the mass matrix operator and the stiffness matrix operator are positive definite. It will prove convenient to rewrite the equations of motion (4.1) in the state space. To this end, we introduce the state vector $\underline{w}(P,t) = [\dot{\underline{u}}^T(P,t) \underline{u}^T(P,t)]^T$ in $X = Z \times Z$ and write the state equations of motion

$$M\dot{\underline{w}}(P,t) = L\underline{w}(P,t) + \underline{F}(P,t) \quad (4.5)$$

where

$$M = \begin{bmatrix} m & 0 \\ 0 & k \end{bmatrix}, \quad L = \begin{bmatrix} -(c+g) & -(k+h) \\ k & 0 \end{bmatrix} \quad (4.6a,b)$$

and

$$\underline{F}(P,t) = [\underline{f}^T(P,t) \underline{0}^T]^T \quad (4.7)$$

It is of interest to control Eq. (4.5). In particular, we wish to control the system in an optimal manner, i.e., we shall minimize some criterion for performance leading to the linear control law

$$\underline{F}(P,t) = - G\underline{w}(P,t) \quad (4.8)$$

where G is a linear control gain operator. Before addressing the optimal control problem, it is necessary to consider some characteristics of non-self-adjoint distributed structures.

4.3 CHARACTERIZATION OF NON-SELF-ADJOINT DISTRIBUTED STRUCTURES

The state vector $\underline{w}(P,t)$ in Eq. (4.5) is an element of the Hilbert space X with inner product

$$\langle \underline{w}, \underline{w} \rangle = (\dot{\underline{y}}, \dot{\underline{y}}) + (\underline{u}, \underline{u}) \quad (4.9)$$

It is easy to verify that the matrix operator M is positive definite and self-adjoint. Moreover, the operator L is assumed to admit a discrete spectrum and thus admits a countable set of eigenvalues and left and right eigenfunctions. The distributed eigenvalue problems associated with the distributed system and its adjoint are given by

$$\lambda_r M \underline{\phi}_r(P) = L \underline{\phi}_r(P) \quad , \quad \lambda_s M \underline{\psi}_s(P) = L^* \underline{\psi}_s(P) \\ r, s = 1, 2, \dots \quad (4.10a, b)$$

where L^* is the adjoint of L . The left and right eigenfunctions are biorthogonal and can be normalized so as to satisfy

$$\langle \underline{\psi}_r, M \underline{\phi}_s \rangle = \delta_{rs} \quad , \quad \langle \underline{\psi}_r, L \underline{\phi}_s \rangle = \lambda_r \delta_{rs} \quad (4.11a, b)$$

where δ_{rs} is the Kronecker-delta. Consider the modal expansion of $\underline{w}(P,t)$ in $X = \text{span} \{ \underline{\phi}_r \}$,

$$\underline{w} = \sum_{r=1}^{\infty} \underline{\phi}_r q_r \quad , \quad q_r = \langle \underline{\psi}_r, M \underline{w} \rangle \quad , \quad r = 1, 2, \dots \quad (4.12a, b)$$

as well as the modal expansion of $\underline{F}(P,t)$,

$$\tilde{F} = \sum_{r=1}^{\infty} M\phi_r Q_r \quad , \quad Q_r = \langle \psi_r, \tilde{F} \rangle \quad (4.13a,b)$$

We refer to $q_r(t)$ and $Q_r(t)$ as modal coordinates and modal forces, respectively. Next, introduce the linear transformation $\theta \in L(X,Y)$ mapping elements in X onto the sequence space Y in the form

$$\theta w = (\langle \psi_1, Mw \rangle, \langle \psi_2, Mw \rangle, \dots) \quad (4.14)$$

The linear transformation θ is called the modal transformation and Y is called the modal space. The modal transformation θ has inverse

$$\theta^{-1}(q_1, q_2, \dots) = \sum_{r=1}^{\infty} \phi_r q_r = \tilde{w} \quad (4.15)$$

Preoperating on the equations of motion by θ and considering Eqs. (4.12) and (4.13), we obtain the sequence of modal equations of motion,

$$\dot{q}_r(t) = \lambda_r q_r(t) + Q_r(t) \quad , \quad r = 1, 2, \dots \quad (4.16)$$

Let us also transform the control law (4.8) into the modal space.

Preoperating on Eq. (4.8) by θ , we obtain the modal control law,

$$Q_r(t) = - \sum_{s=1}^{\infty} g_{rs} q_s(t) \quad , \quad g_{rs} = \langle \psi_r, G\phi_s \rangle \quad (4.17a,b)$$

The control gain operator G is assumed to admit a discrete spectrum and hence the distributed eigenvalue problems associated with the control gain operator G and its adjoint are written

$$g_r M\phi_r^*(P) = G\phi_r^*(P) \quad , \quad g_r M\psi_r^*(P) = G^*\psi_r^*(P) \\ r, s = 1, 2, \dots \quad (4.18a,b)$$

where G^* is the adjoint of G . The right and left eigenfunctions of

G , denoted by $\phi_r^*(P)$ and $\psi_r^*(P)$, are biorthogonal and can be normalized so that

$$\langle \psi_r^*, M\phi_s^* \rangle = \delta_{rs} \quad , \quad \langle \psi_r^*, G\phi_s^* \rangle = g_r \delta_{rs} \quad (4.19a,b)$$

where g_r are the control gain eigenvalues. By introducing the gain projection operator $\langle \psi_r^*, M \rangle M\phi_r^*$ of G , we can resolve G , such that

$$G = \sum_{r=1}^{\infty} g_r \langle \psi_r^*, M \rangle M\phi_r^* \quad (4.20)$$

Substituting Eq. (4.17) into (4.16), we obtain the closed-loop modal equations of motion

$$\dot{q}_r(t) - \lambda_r q_r(t) + \sum_{s=1}^{\infty} g_{rs} q_s(t) = 0 \quad , \quad r = 1, 2, \dots \quad (4.21)$$

The control problem amounts to determining the eigenfunctions $\phi_r^*(P)$ of G and associated eigenvalues g_r ($r = 1, 2, \dots$).

4.4 INDEPENDENT MODAL-SPACE CONTROL

The modal equations of motion (4.16) have the appearance of an infinite set of independent first-order complex ordinary differential equations and in the absence of control this is the case, in fact. Decoupling of this type is referred to as internal. However, the feedback forces in general recouple the modal equations of motion as can be seen from Eq. (4.21). This recoupling is referred to as external. There exists a control in which the modal equations remain decoupled. In particular, we consider control gain operators having eigenfunctions identical to the distributed system eigenfunctions, so

that

$$\tilde{\phi}_r^* = \tilde{\phi}_r \quad , \quad \tilde{\psi}_r^* = \tilde{\psi}_r \quad , \quad r = 1, 2, \dots \quad (4.22)$$

Control of this form is known as independent modal-space control.

Substituting Eq. (4.22) and Eq. (4.20) into (4.17b), we obtain

$$g_{rs} = \langle \tilde{\psi}_r, \sum_{t=1}^{\infty} g_t \langle \tilde{\psi}_t, M \tilde{\phi}_s \rangle \tilde{\phi}_t \rangle = g_r \delta_{rs} \quad (4.23)$$

Introducing Eq. (4.23) into (4.17a) leads to the independent modal-space control law

$$Q_r(t) = -g_r q_r(t) \quad , \quad r = 1, 2, \dots \quad (4.24)$$

Substituting Eq. (4.24) into (4.21) we obtain the closed-loop modal equations

$$\dot{q}_r(t) + (g_r - \lambda_r) q_r(t) = 0 \quad , \quad r = 1, 2, \dots \quad (4.25)$$

Equations (4.25) are now externally decoupled. Next, we consider three theorems pertinent to the independent modal-space control method.

Theorem 1.--The eigenfunctions associated with the open-loop system and the closed-loop system are identical if and only if the control is independent modal-space control.

For independent modal-space control it has been shown in Eq. (4.25) that the closed-loop equations are decoupled. Therefore, the closed-loop eigenfunctions are identical to the open-loop eigenfunctions. The differential eigenvalue problem associated with the closed-loop system, in which the closed-loop eigenfunctions are

identical to the open-loop eigenfunctions, is written

$$s_r M \phi_r(P) = (L + G) \phi_r(P) \quad , \quad r = 1, 2, \dots \quad (4.26)$$

Subtracting Eq. (4.12) from Eq. (4.26), we obtain the differential eigenvalue problem associated with the gain operator G , Eq. (4.18), in which the closed-loop eigenvalues s_r and the control gain eigenvalues g_r are related by $s_r = \lambda_r - g_r$ ($r = 1, 2, \dots$).

Theorem 2.--The closed-loop eigenvalues can be arbitrarily placed in the complex plane.

It follows from Eq. (4.25) that the control gain eigenvalues corresponding to arbitrarily placed closed-loop poles s_r are given by

$$g_r = \lambda_r - s_r \quad , \quad r = 1, 2, \dots \quad (4.27)$$

Theorem 3.--Independent modal-space control, in which the performance functional

$$J = \sum_{r=1}^{\infty} \left\{ h_r (q_r(t_f) - \hat{q}_r) (\bar{q}_r(t_f) - \bar{\hat{q}}_r) + \int_0^{t_f} (q_r \bar{q}_r + R_r Q_r \bar{Q}_r) dt \right\} \quad (4.28)$$

is minimized is unique and globally optimal where \hat{q}_r is a reference value to be driven at final time t_f and R_r and h_r are real positive modal weights. Note that each term in the summation represents a real quadratic functional corresponding to a mode of vibration. It is well known that the linear optimal solution is unique (Ref. 14). We seek stationary values of the augmented quadratic functionals

$$J_r = h_r(q_r(t_f) - \hat{q}_r)(\bar{q}_r(t_f) - \bar{\hat{q}}_r) + \int_0^{t_f} [q_r \bar{q}_r + R_r Q_r \bar{Q}_r + \Lambda_r(\dot{q}_r - \lambda_r q_r - Q_r)] dt \quad (4.29)$$

in which Λ_r are Lagrange multipliers. Interpreting Eqs. (4.16), (4.18) and Eq. (4.29) for unconstrained control forces, we have

$$\min J = \min \sum_{r=1}^{\infty} J_r = \sum_{r=1}^{\infty} \min J_r \quad (4.30)$$

Performing the minimization of Eq. (4.29) and assuming a linear control law in the form of Eq. (4.24), we obtain an infinite set of complex scalar Riccati equations

$$0 = -1 + (\lambda_r + \bar{\lambda}_r) R_r g_r + R_r g_r^2 + R_r \dot{g}_r, \quad r = 1, 2, \dots \quad (4.31)$$

a. Steady-State Solution

In the case of steady-state gains, we consider $\dot{g}_r = 0$. Then, Eqs. (4.31) reduce to simple complex scalar quadratics having the single closed-form solution with positive real part

$$g_r = \alpha_r + (1/R_r + \alpha_r^2)^{1/2} \quad (4.32)$$

where $\lambda_r = \alpha_r + i\omega_r$. Note that the steady-state gains are real. Therefore the closed-loop frequencies and the open-loop frequencies are identical.

b. Transient Solution

Let us write the transient solution of the ricatti equation in the form

$$g_r(t_r) = e_r(t_r)/\ell_r(t_r) \quad (4.33)$$

in which $t_r = t_f - t$ is the remaining time. Equation (4.33) is subject to the initial conditions

$$e_r(t_r=0) = h_r \quad , \quad l_r(t_r=0) = 1 \quad , \quad r = 1, 2, \dots \quad (4.34)$$

Introducing Eq. (4.33) into Eq. (4.31) and considering Eqs. (4.34), the nonlinear Riccati equation can be replaced by a linear equation of order two of the form

$$\begin{Bmatrix} e_r'(t_r) \\ l_r'(t_r) \end{Bmatrix} = M_r \begin{Bmatrix} e_r(t_r) \\ l_r(t_r) \end{Bmatrix} \quad (4.35)$$

where primes denote differentiation with respect to t_r and in which

$$M_r = \begin{bmatrix} \alpha_r & -R_r^{-1} \\ -1 & -\alpha_r \end{bmatrix} \quad (4.36)$$

4.5 CONTROL IMPLEMENTATION

In sec. 4.4, it was demonstrated how the independent modal-space control method can generate distributed controls $\tilde{F}(P,t)$ for distributed-parameter systems. In practice, infinite-dimensional controls may not be possible, in which case the interest lies in implementing independent modal-space control with a finite number of actuators and sensors. The problem reduces to how to approximate best distributed controls and measurements by discrete controls and measurements.

Let us write the actual control force as a projection of the computed distributed control force

$$\tilde{F}(P,t) = S\tilde{F}^*(P,t) \quad (4.37)$$

where S denotes the projection operator under consideration and $\tilde{F}^*(P,t)$ is the computed distributed control force. Similarly, the computed distributed displacement is a projection of the actual displacement, written

$$\tilde{w}^*(P,t) = T\tilde{w}(P,t) \quad (4.38)$$

where T represents the projection operator at hand and $u^*(P,t)$ denotes the computed distributed displacement. Given the projections S and T , the closed-loop eigenvalues can be found. Substituting Eq. (4.37) into (4.13), we obtain

$$Q_r = \langle \tilde{\psi}_r, S\tilde{F}^* \rangle = \sum_{s=1}^{\infty} S_{rs} Q_s^* \quad (4.39)$$

where

$$S_{rs} = \langle \tilde{\psi}_r, S\tilde{M}\tilde{\phi}_s \rangle \quad r,s = 1,2,\dots \quad (4.40)$$

and Q_s^* ($s = 1,2,\dots$) are computed modal controls. Moreover, substituting Eq. (4.38) into (4.12b), we obtain

$$q_s^* = \langle \tilde{\psi}_s, M\tilde{T}\tilde{w} \rangle = \sum_{t=1}^{\infty} T_{st} q_t \quad , \quad s,t = 1,2,\dots \quad (4.41)$$

where

$$T_{st} = \langle \tilde{\psi}_s, M\tilde{T}\tilde{\phi}_t \rangle \quad (4.42)$$

and q_s^* are computed modal coordinates. The modal controls are generated in the form

$$Q_r^*(t) = -g_r q_r^*(t) \quad , \quad r = 1,2,\dots \quad (4.43)$$

Substituting Eq. (4.43) into (4.39) and considering Eqs. (4.41) and

(4.16), we obtain the closed-loop modal equations

$$\dot{q}_r(t) - \lambda_r q_r(t) + \sum_{s=1}^{\infty} S_{rs} g_s \sum_{t=1}^{\infty} T_{st} q_t(t) = 0 \quad ,$$

$$r = 1, 2, \dots \quad (4.44)$$

Next, let us generate discrete controls in another manner. We propose to control n modes with $2n$ discrete actuators. To this end, we construct the $2n$ -dimensional vector of discrete control forces denoted by $\tilde{F}(t)$ and write

$$\tilde{F}(P, t) = \Delta \tilde{F}(t) \quad (4.45)$$

in which Δ is a rectangular matrix of spatial delta functions. Substituting Eq. (4.45) into (4.13b) we obtain

$$Q_c(t) = B_c \tilde{F}(t) \quad , \quad B_c = \int_D \psi^T \Delta dD \quad (4.46)$$

in which ψ is made up of n columns of the left eigenfunctions $\psi_r(P)$ and where $Q_c(t)$ is an n -dimensional vector of controlled modes.

Equation (4.46) can be inverted by separating real and imaginary parts as follows:

$$Q_c(t) = Q_{c1}(t) + iQ_{c2}(t) \quad , \quad B_c = B_{c1} + iB_{c2} \quad (4.47a, b)$$

From Eqs. (4.46) and (4.47), we obtain the discrete control vector in terms of the modal controls

$$\tilde{F}(t) = \begin{bmatrix} B_{c1} \\ B_{c2} \end{bmatrix}^{-1} \begin{Bmatrix} Q_{c1}(t) \\ Q_{c2}(t) \end{Bmatrix} \quad (4.48)$$

4.6 NUMERICAL EXAMPLE

As a simple example, we shall consider a uniform shaft whirling about its axis at constant angular velocity Ω . The shaft extends 10 units in the z direction and undergoes displacements $u_x(z,t)$ and $u_y(z,t)$ in the directions of the x and y body fixed axes, respectively, and shown in fig. 4.1. The distributed displacement vector in Eq. (4.1) is written $\underline{u}(P,t) = [u_x(z,t) \ u_y(z,t)]^T$. The specific quantities in Eq. (4.1) are given by (Ref. 2)

$$m = \begin{bmatrix} m_0 & 0 \\ 0 & m_0 \end{bmatrix}, \quad c = \begin{bmatrix} c_0+h_0 & 0 \\ 0 & c_0+h_0 \end{bmatrix}, \quad g = \begin{bmatrix} 0 & -2m_0\Omega \\ 2m_0\Omega & 0 \end{bmatrix} \quad (4.49a,b,c)$$

$$k = \begin{bmatrix} -m_0\Omega^2+EI()'''' & 0 \\ 0 & -m_0\Omega^2+EI()'''' \end{bmatrix}, \quad h = \begin{bmatrix} 1 & -h_0\Omega \\ h_0\Omega & 0 \end{bmatrix} \quad (4.49d,e)$$

where m_0 denotes the mass per unit length, c_0 and h_0 denote the internal damping per unit length and the external damping per unit length, respectively, and EI denotes the bending stiffness.

The boundary conditions, Eq. (4.2), are given by

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} EI()'' & 0 \\ 0 & EI()'' \end{bmatrix} \quad \text{at } z = 0,10 \quad (4.50a,b)$$

We shall consider linear optimal independent modal-space control. Measurement errors are neglected. The parameters of concern are given in Table 4.1. The shaft is given an initial impulse of 10 units at $z = 4$. The open-loop eigenvalues are given in Table 4.2. As seen from Table 4.1, the lowest two eigenvalues are unstable. Therefore, we

choose to control these two modes using discrete controls. Discrete control forces are generated in the x and y directions at $z = 10/3$ and at $z = 20/3$. Equation (4.48) is used to compute the discrete control forces. The controlled response is shown in Fig. 4.2. The closed-loop eigenvalues are given in Table 4.3.

4.7 CONCLUSIONS

The problem of control of non-self adjoint distributed-parameter systems has been formulated. The steady-state solution for the linear optimal independent modal-space control problem was found in closed-form. The transient solution can be found with relative ease. The formulation is distributed in nature and, not surprisingly, yields distributed control forces as a solution. However, practical considerations often preclude the use of distributed control forces. Therefore, the analysis is extended so to include discrete-point controls. Also, the modal coordinates needed for feedback are extracted from measurements taken at discrete points. As a numerical example, a uniform shaft rotating about its axis with constant angular velocity was considered. For a specific angular velocity, the lowest two modes are unstable. These two modes were controlled successfully using two pairs of discrete-point actuators.

Table 4.1 System parameters and control parameters for a whirling shaft

$m_o = 0.10$	$\Omega = 10.0$
$c_o = 2.00$	$g_1 = 2.75$
$h_o = 0.10$	$g_2 = 4.40$
$EI = 40.0$	

Table 4.2 Open-loop eigenvalues
for a whirling shaft

r	λ_r
1	$0.759 \pm 1.118i$
2	$2.401 \pm 2.248i$
3	$-4.124 \pm 5.684i$
4	$-23.401 \pm 17.751i$
5	$-21.759 \pm 18.881i$
6	$-16.875 \pm 25.684i$

Table 4.3 Closed-loop eigenvalues
for a whirling shaft

r	λ_r
1	- 1.999 ± 1.118i
2	- 1.999 ± 2.248i
3	- 4.124 ± 5.684i
4	-23.401 ± 17.751i
5	-21.759 ± 18.881i
6	-16.879 ± 25.684i

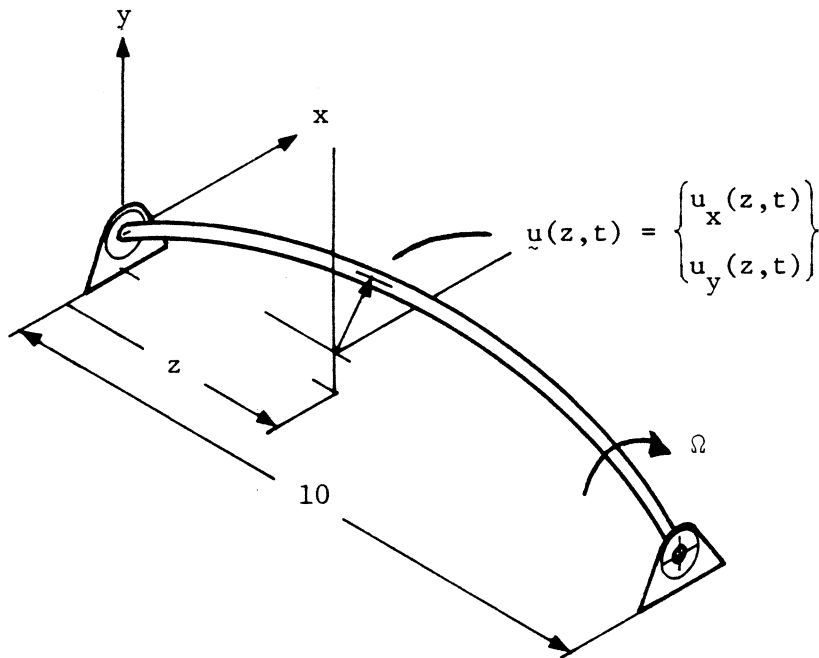


Figure 4.1 Uniform shaft whirling at constant angular velocity Ω

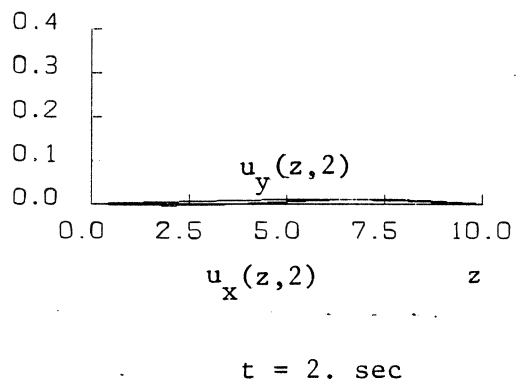
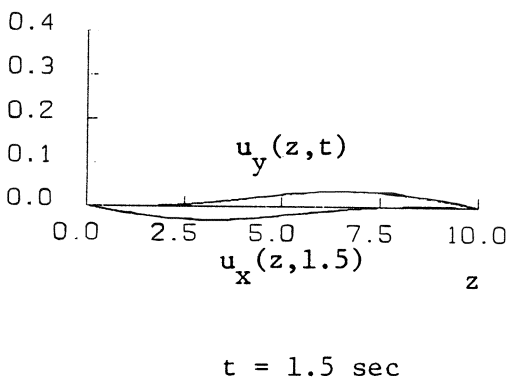
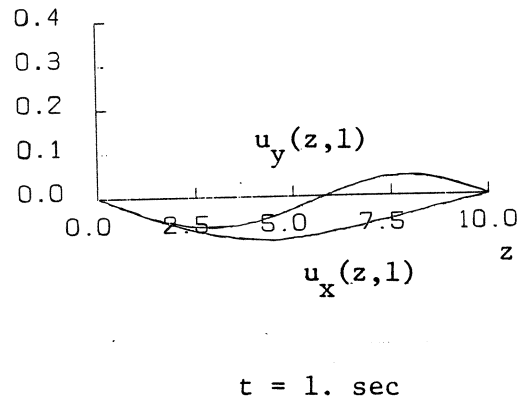
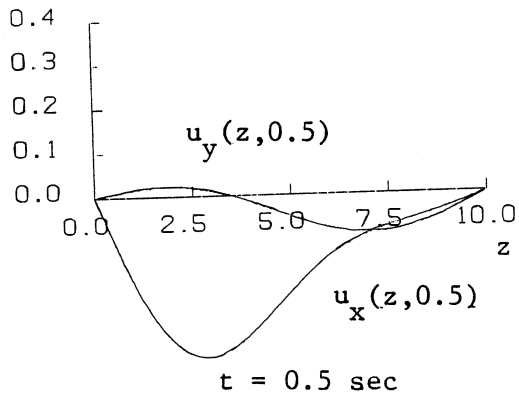


Figure 4.2 Controlled responses of whirling shaft

Chapter V

BLOCK-INDEPENDENT CONTROL OF DISTRIBUTED STRUCTURES

5.1 INTRODUCTION

In the absence of control feedback, the modal equations of motion are independent. In general, feedback controls depend on all of the modal coordinates, so that the feedback forces recouple the modal equations. This is known as coupled control (Refs. 23,24). A different approach to control is known as independent modal-space control (Refs. 25,26). This method consists of designing feedback forces such that the modal equations remain decoupled, thus reducing the control problem to the design of modal feedback control forces for a set of independent second-order differential equations. Then, the actual controls are synthesized from the modal controls by a simple linear transformation.

In coupled controls, any number of modes can be controlled using a single actuator, provided the system is controllable. However, even if the system is controllable, many implementation difficulties arise. The method is often faced with serious problems of sensitivity to the actuators and sensors locations and to system parameter errors. Furthermore, the solution of the Riccati equations for optimal control of large-order systems (> 100) can only be computed off-line, at best (Refs. 27,28).

In contrast with coupled controls, the independent modal-space control method is virtually free of computational difficulties. Because

the second-order differential equations are controlled independently, the method permits a wider choice of control techniques. Indeed, it can accommodate both linear and nonlinear controls. Moreover, the system is guaranteed to be controllable and the feedback control tends to be insensitive to system parameter errors (see Ref. 29). On the other hand, the method requires as many actuators as controlled modes, which may be regarded as a drawback in certain circumstances.

Naturally, the question arises whether some compromise between coupled controls and independent modal-space control is possible. Such an approach should require fewer actuators and at the same time it should be computationally efficient and relatively insensitive to parameter errors.

In coupled controls, the matrix of coefficients is fully populated. On the other hand, in the independent modal-space control method, the matrix of coefficients is block-diagonal, with each block being a 2×2 matrix. Hence, a compromise is proposed in which the matrix of coefficients is still block-diagonal, but each block is in general of larger order than two. The method consists of designing control forces so that blocks of modal equations remain independent. As a consequence, the control design for each block of modes is carried out independently of any other block, where the feedback control for each block is dyadic. Then, the actual controls are synthesized from the dyadic modal controls.

With the proposed approach, controllability is guaranteed, provided each block is controllable. For optimal control, an independent set of Riccati equations is solved, where each Riccati equation is

associated with a block of modes of manageable order. The number of actuators required is also shown to be less than the number required for the independent modal-space control method. On the other hand, the performance is not as good as that of a system designed by the independent modal-space control method.

A numerical example illustrating the block-independent control approach is presented.

5.2 EQUATIONS OF MOTION FOR SELF-ADJOINT DISTRIBUTED-PARAMETER SYSTEMS

The equations of motion of a self-adjoint distributed-parameter system (DPS) can be written in the form of the partial differential equation (Ref. 2)

$$Lu(P,t) + m(P)\ddot{u}(P,t) = f(P,t) + n(P,t), \quad P \in D \quad (5.1)$$

where $u(P,t)$ is the displacement of an arbitrary point P in the domain D , L is a linear differential operator of order $2p$ expressing the system stiffness and $m(P)$ is the distributed mass. Moreover, $f(P,t)$ is the distributed control force and $n(P,t)$ represents all other external forces, including noise. The displacement $u(P,t)$ is subject to the boundary conditions

$$B_i u(P,t) = 0, \quad P \in S, \quad i = 1, 2, \dots, p \quad (5.2)$$

where B_i are linear differential operators of orders ranging from zero to $2p-1$ and S represents the set of all boundary points of D .

Whereas $n(P,t)$ is an explicit function of time, the control force $f(P,t)$ depends on time only implicitly. Indeed, the force is governed by a control law having the general functional form

$$f(P,t) = f(u(P,t), \dot{u}(P,t)) \quad (5.3)$$

5.3 MODAL EQUATIONS FOR THE DPS

Let us assume that the operator L is self-adjoint and positive semidefinite and that $m(P)$ is positive definite. Then, all the eigenvalues are real and nonnegative; they are related to the natural frequencies ω_r of undamped oscillation by $\lambda_r = \omega_r^2$ ($r = 1, 2, \dots$). Moreover, the eigenfunctions are orthogonal and can be normalized so as to satisfy

$$\int_D m(P) \phi_r(P) \phi_s(P) dD = \delta_{rs} \quad r, s = 1, 2, \dots \quad (5.4)$$

$$\int_D \phi_r(P) L \phi_s(P) dD = \lambda_r \delta_{rs}$$

The partial differential equations of motion can be transformed into an infinite set of second-order ordinary differential equations. Indeed, according to the expansion theorem (Ref. 2), the displacement $u(P,t)$ can be represented by an infinite series of the space-dependent eigenfunctions $\phi_s(P)$ multiplied by time-dependent modal coordinates $u_s(t)$ of the form

$$u(P,t) = \sum_{s=1}^{\infty} \phi_s(P) u_s(t) \quad (5.5)$$

Introducing Eq. (5.5) into Eq. (5.1), premultiplying the results by $\phi_r(P)$ and integrating over the domain D , we obtain

$$\ddot{u}_r(t) + \omega_r^2 u_r(t) = f_r(t) + n_r(t) \quad , \quad r = 1, 2, \dots \quad (5.6)$$

where

$$f_r(t) = \int_D \phi_r(P) f(P, t) dD \quad (5.7a)$$

$$n_r(t) = \int_D \phi_r(P) n(P, t) dD \quad (5.7b)$$

are modal control forces and modal noises, respectively. In controlling the DPS, we control only a subset of modes, called controlled modes. The remaining modes are called residual modes.

5.4 CONTROL IMPLEMENTATION

In practice one must often use discrete actuators and sensors to implement the control. Let us consider N discrete actuators and M discrete sensors measuring displacements and velocities. The actuators are located at points P_j ($j = 1, 2, \dots, N$) and the sensors are located at points P'_j ($j = 1, 2, \dots, M$). The object is to control n modes with N actuators. The actuator forces can be treated as distributed by writing

$$f(P, t) = \sum_{j=1}^N F_j(t) \delta(P - P_j) \quad (5.8)$$

where $\delta(P - P_j)$ is a spatial Dirac-delta function. Introducing Eq.

(5.8) into (5.7a), we can express the modal control forces in terms of the actuator forces in the compact form

$$\underline{f}_C(t) = B_C \underline{F}(t) \quad (5.9)$$

where $\underline{f}_C(t) = [f_1(t) \ f_2(t) \ \dots \ f_n(t)]$ is an n -dimensional vector of modal forces, $\underline{F}(t) = [F_1(t) \ F_2(t) \ \dots \ F_N(t)]^T$ is an N -dimensional

vector of actuator forces and B_C is an $n \times N$ modal participation matrix with entries $B_{Cij} = \phi_i(P_j)$.

Assume that M modes are modelled and that the remaining unmodelled modes do not contribute significantly to the motion of the system. This amounts to truncating Eq. (5.5), so that

$$u(P,t) = \sum_{s=1}^M \phi_s(P) u_s(t) \quad (5.10)$$

Denoting the displacement measurement at $P = P'_j$ by $u_{Dj}(t) = u(P'_j, t)$ and the velocity measurement at P'_j by $u_{Vj}(t) = \dot{u}(P'_j, t)$ and using Eq. (5.10) we can write

$$\underline{U}_D(t) = B_M \underline{u}(t) \quad , \quad \underline{U}_V(t) = B_M \dot{\underline{u}}(t) \quad (5.11a,b)$$

where $\underline{U}_D(t) = [U_{D1}(t) \ U_{D2}(t) \ \dots \ U_{Dm}(t)]^T$ is an m -dimensional vector of displacement measurements, $\underline{U}_V(t) = [U_{V1}(t) \ U_{V2}(t) \ \dots \ U_{Vm}(t)]^T$ is an m -dimensional vector of velocity measurements, $\underline{u}(t) = [u_1(t) \ u_2(t) \ \dots \ u_m(t)]$ is an m -dimensional vector of modal displacements and B_M is an $m \times m$ matrix with entries $B_{Mij} = \phi_j(P'_i)$. The matrix B_M can be inverted off-line, so that using Eqs. (5.11) the modal coordinates and modal velocities can be computed from discrete measurements as follows:

$$\underline{u}(t) = B_M^{-1} \underline{U}_D(t) \quad , \quad \dot{\underline{u}}(t) = B_M^{-1} \underline{U}_V(t) \quad (5.12a,b)$$

5.5 BLOCK-INDEPENDENT CONTROL OF THE DPS

Let us consider L blocks of controlled modes. The interest lies in determining the number of actuators required to carry out independent

control of each block of modes. To this end, Eqs. (5.6) are rewritten for each block of modes in the compact form

$$\ddot{\underline{u}}_r(t) + \Lambda_r \underline{u}_r(t) = \underline{f}_r(t) = \underline{n}_r(t) \quad , \quad r = 1, 2, \dots, L \quad (5.13)$$

where $\underline{u}_r(t)$ are n_r -dimensional vectors of modal displacements associated with the r th block of modes. Likewise, $\underline{f}_r(t)$ and $\underline{n}_r(t)$ represent the n_r -dimensional vectors with the associated modal control forces and the modal noises, respectively, and the matrix Λ_r is an $n_r \times n_r$ diagonal matrix of the corresponding frequencies squared.

Each block of modes is controlled by dyadic control, which can be written as

$$\underline{f}_r(t) = \underline{c}_r z_r(t) \quad , \quad r = 1, 2, \dots, L \quad (5.14)$$

where \underline{c}_r is a constant n_r -dimensional vector to be determined and $z_r(t)$ is a scalar function of the modal coordinates associated with the r th block of modes having the general functional form

$$z_r(t) = z_r(\underline{u}_r(t), \dot{\underline{u}}_r(t)) \quad , \quad r = 1, 2, \dots, L \quad (5.15)$$

Equations (5.14) can be written in the compact matrix form

$$\underline{f}_C(t) = C \underline{z}(t) \quad (5.16)$$

where

$$\underline{f}_C(t) = [\underline{f}_1^T(t) \quad \underline{f}_2^T(t) \quad \dots \quad \underline{f}_L^T(t)]^T \quad (5.17a)$$

$$\underline{z}(t) = [z_1(t) \quad z_2(t) \quad \dots \quad z_L(t)]^T \quad (5.17b)$$

$$C = [I_1^* \underline{c}_1 \quad I_2^* \underline{c}_2 \quad \dots \quad I_L^* \underline{c}_L] \quad (5.17c)$$

$$I_r^* = [I_{r1} \quad I_{r2} \quad \dots \quad I_{rL}]^T \quad , \quad r = 1, 2, \dots, L \quad (5.17d)$$

in which I_{rs} is the $n_r \times n_s$ null matrix for $s \neq r$ and the $n_r \times n_r$ identity matrix for $s=r$. Substituting Eq. (5.9) into (5.16), we obtain

$$B_C \underline{F}(t) = C \underline{z}(t) \quad (5.18)$$

A solution to Eq. (5.18) exists provided the vector of actuator forces can be written in the form

$$\underline{F}(t) = S \underline{z}(t) \quad (5.19)$$

for some $N \times L$ matrix S to be determined. Substituting Eq. (5.19) into (5.18), it follows that

$$C = B_C S \quad (5.20)$$

for some matrices C and S . Equation (5.20) represents a linear homogeneous system of algebraic equations and can be rewritten, from Eq. (5.17c), as

$$I_{rC}^* - B_C s_r = 0 \quad , \quad r = 1, 2, \dots, L \quad (5.21)$$

where s_r ($r = 1, 2, \dots, L$) are the columns of S . A nontrivial solution of Eqs. (5.21) exists only if

$$\text{Rank } A_r < N = n_r \quad , \quad A_r = [-I_r^* \ B_C] \quad , \quad r = 1, 2, \dots, L \quad (5.22)$$

Note that in general the $n \times (N+n_r)$ matrix A_r is full rank. Only in contrived examples can one encounter a matrix A_r of deficient rank. These cases are not discussed here. Assuming A_r has full rank, it follows from Eq. (5.22) that

$$n < N + n_r \quad , \quad r = 1, 2, \dots, L \quad (5.23)$$

We now consider a number of possible cases:

A. Coupled Control

In this case, we consider one block of modes, $L = 1$, $n_r = n$. Then, according to Eq. (5.23) the control can be implemented using a single actuator, $N = 1$, provided the system is controllable.

B. Independent Modal-Space Control

According to the independent modal-space control method (Ref. 16), we control each mode independent of the other, so that $n_r = 1$ ($r = 1, 2, \dots, L$) and $L = n$. From Eqs. (23), at least n actuators are required to realize the control. Note that each mode is guaranteed to be controllable.

C. Block-Independent Control

Let us consider a system in which each block consists of K modes, so that $n_r = K$ ($r = 1, 2, \dots, L$). Then, $n = KL$ and from Eqs. (5.23) the control is realizable provided

$$N \geq K(L-1)+1 = n+1-K \quad (5.24)$$

actuators are used. A block-diagram of the control is given in Fig.

5.1.

5.6 LINEAR OPTIMAL CONTROL

Let us control a typical block of modes using linear optimal control theory. First, it is standard practice to rewrite Eqs. (5.13) in the state space. Introducing the state vector $w_r(t) =$

$[\Lambda_r^{1/2} \tilde{u}_r^T(t) \quad \dot{\tilde{u}}_r^T(t)]^T$ and letting $\tilde{u}_r(t) = \tilde{0}$, Eqs. (5.13) and (5.14) become

$$\dot{\tilde{w}}_r(t) = A_r \tilde{w}_r(t) + B_r \tilde{z}_r(t) \quad (5.25)$$

where

$$A_r = \begin{bmatrix} 0 & \Lambda_r^{1/2} \\ -\Lambda_r^{1/2} & 0 \end{bmatrix}, \quad B_r = \begin{bmatrix} 0 \\ c_r \end{bmatrix} \quad (5.26a,b)$$

We propose to minimize the quadratic performance index

$$P_r = \frac{1}{2} (\underline{w}_r^T(t_f) - \hat{\underline{w}}_r^T) H_r (\underline{w}_r(t_f) - \hat{\underline{w}}_r) + \frac{1}{2} \int_0^{t_f} (\underline{w}_r^T(t) \underline{w}_r(t) + R_r z_r^2(t)) dt \quad (5.27)$$

in which the first term in the integrand represents the energy associated with the r th mode and the second represents the control effort associated with the r th mode. Minimization of Eq. (5.27), regarding Eq. (5.25) as a constraint, yields a solution in the form of the algebraic Riccati equation (Ref. 30),

$$\dot{K}_r = -K_r A_r - A_r^T K_r - I - K_r B_r B_r^T K_r R_r^{-1} \quad (5.28)$$

The solution of Eq. (5.28) is used to produce the control

$$z_r(t) = -\frac{1}{R_r} B_r^T K_r \underline{w}_r(t) \quad (5.29)$$

In the steady-state case the Riccati equation reduces to an algebraic matrix equation.

5.7 NUMERICAL EXAMPLE

As a simple illustration, we consider controlling the bending vibration of a uniform beam hinged at both ends, such that

$$L = \partial^4 / \partial x^4, \quad m = 1 \quad 0 < x < 10 \quad (5.30a,b)$$

$$B_1 = 1 \text{ at } x = 0, \quad B_2 = \partial^2 / \partial x^2 \text{ at } x = 0, 10 \quad (5.31a,b)$$

The eigensolution is well known (Ref. 2) and consists of the eigenfunctions and natural frequencies

$$\phi_r(x) = \sqrt{\frac{1}{5}} \sin\left(\frac{r\pi x}{10}\right), \quad \omega_r = \left(\frac{r\pi}{10}\right)^2 \quad (5.32a,b)$$

We propose to control the four lowest modes of vibration using linear optimal control. For comparison, the beam is controlled by:

- a) coupled control using a single actuator,
- b) block-independent control using three actuators, and
- c) independent modal-space control using four actuators.

For coupled control, one actuator is placed at $x = 6$. For block-independent control, we consider two blocks of modes. The first block comprises of the first two modes and the second block comprises of the third and fourth modes. We consider for block-independent control three actuators placed at $x = 2, 4, 6$ and for independent modal-space control, the actuators are placed at $x = 2, 4, 6, 8$. In each case the performance indices, Eqs. (5.27), are minimized by solving a steady-state Riccati equation. In the case of independent modal-space control, the Riccati equations admit closed-form solutions (Ref. 25). The closed-loop poles are given in Table 5.1. For purposes of comparison, we define the control effort

$$C(t) = \int_0^t \sum_{r=1}^L |F_r(\tau)| d\tau \quad (5.33)$$

which is computed for each case. The free response at $x = 6$, the controlled response at $x = 6$ for each case and the associated control efforts are given in Fig. 5.2. All three methods control the beam,

but coupled control requires more control effort than independent modal-space control. As expected, block-independent control requires more control effort than independent modal-space control and less control effort than coupled control.

5.8 CONCLUSIONS

Block-independent control is designed to combine the computational advantages of independent modal-space control with the requirement of a relatively small number of actuators for coupled control. The intended compromise is achieved, but it is not as favorable as hoped. In particular, the number of actuators required is not substantially lower than the number of actuators required for independent modal-space control. Note that for block-independent control, each block of modes receives a single dyadic input and the number of actuators required exceeds the number of blocks.

A numerical example is presented in which the performances of independent modal-space control, block-independent control and coupled control are compared. As expected, the independent modal-space control requires the least amount of control effort and the block-independent control requires less control effort than coupled control.

Table 5.1 Closed-loop poles

Independent modal-space control	
$R_1 = R_2 = R_3 = R_4 = 30.$	
	$-0.117067 \pm 0.082334i$
	$-0.127527 \pm 0.394273i$
	$-0.128766 \pm 0.888215i$
	$-0.128992 \pm 1.579125i$
Coupled control $R_1 = 0.1$	
	$-0.57412 \pm 1.36557i$
	$-0.24616 \pm 0.85280i$
	$-0.13118 \pm 0.34704i$
	-1.64390
	-0.10598
Block-independent control $R_1 = 50, R_2 = 100.$	
	$-0.095276 \pm 0.385224i$
	$-0.161254 \pm 0.047725i$
	$-0.133439 \pm 1.578752i$
	$-0.081448 \pm 0.888581i$

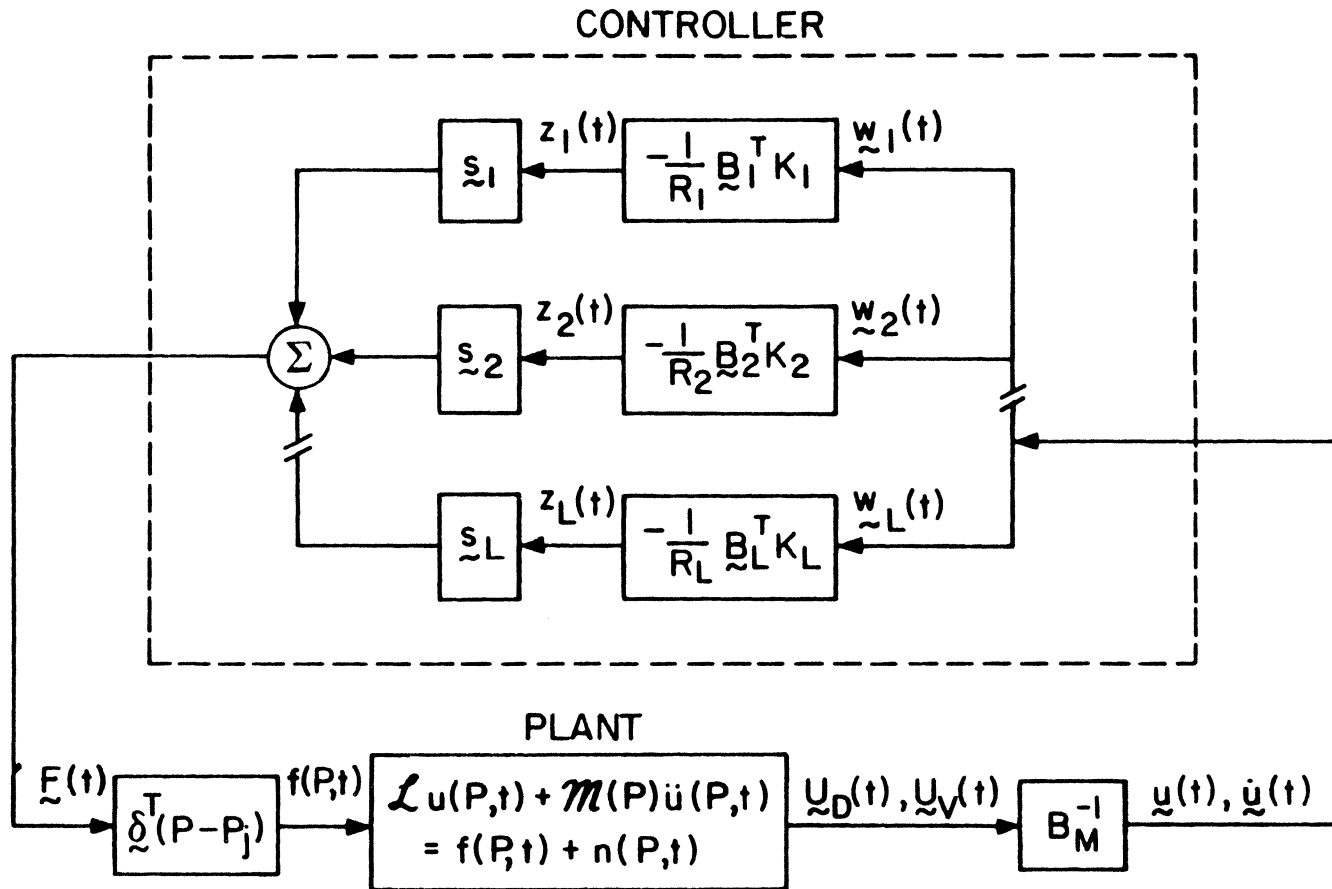


Figure 5.1 Block diagram for block-independent control

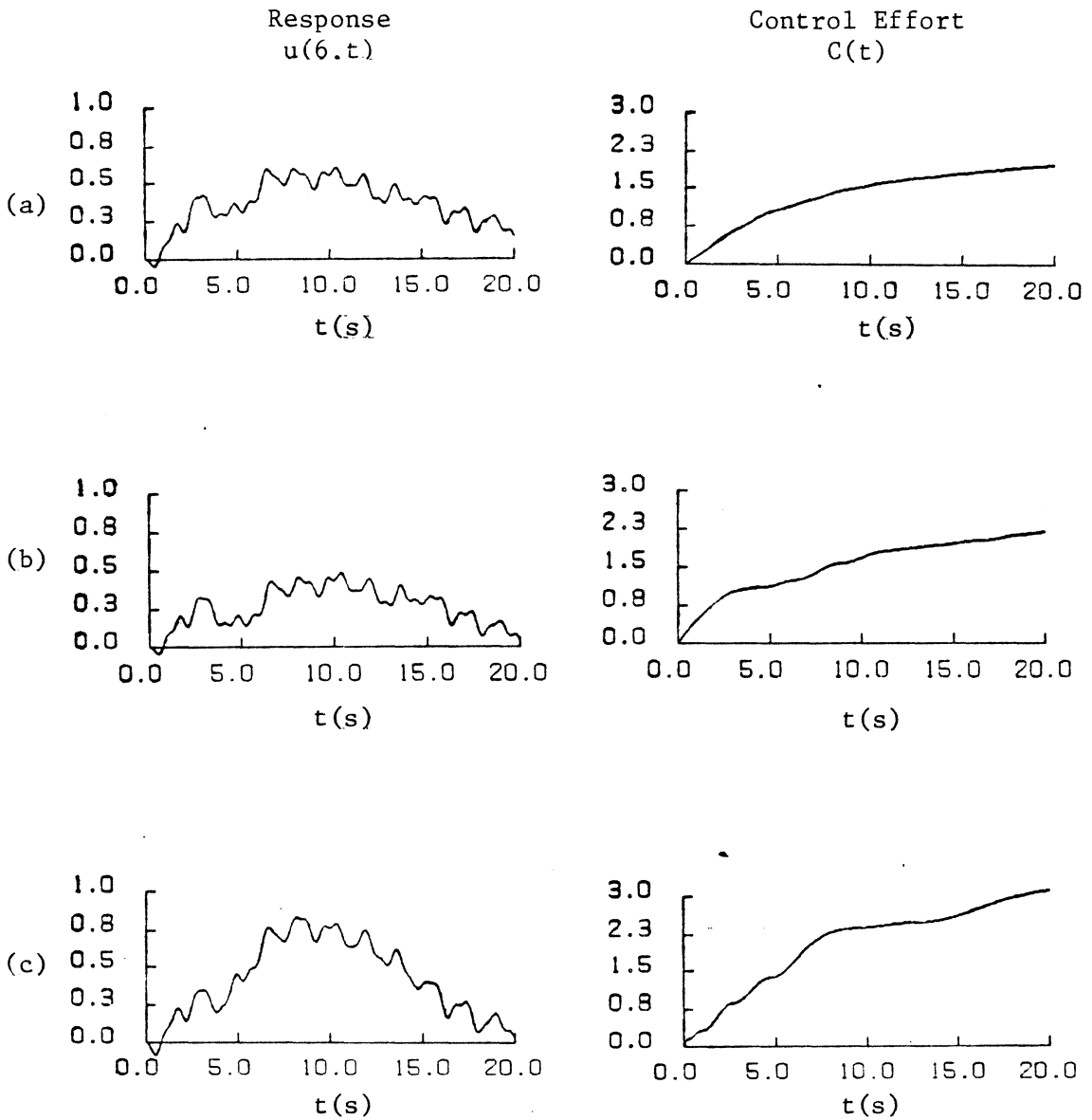


Figure 5.2 Response of beam and control effort. (a) independent modal-space control; (b) block-independent control; (c) coupled control

REFERENCES

1. G. Strang and G. J. Fix, An Analysis of the Finite Element Method, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1973.
2. L. Meirovitch, Computational Methods in Structural Dynamics, Sijthoff-Noordhoff, The Netherlands, 1980.
3. R. H. Gallagher, Finite Element Analysis, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1975.
4. K. H. Huebner, The Finite Element Method for Engineers, John Wiley & Sons, Inc., New York, 1975.
5. J. H. Wilkinson, The Algebraic Eigenvalue Problem, Oxford University Press, London, 1965.
6. J. N. Franklin, Matrix Theory, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1968.
7. B. A. Szabo, "Some recent developments in the finite element method", Comp. and Maths. with Appls., 5, 99-115, 1979.
8. I. Babuska, B. A. Szabo and I. N. Katz, "The p-version of the finite element method", SIAM J. Numer. Anal., 18, 515-545, 1981.
9. I. Babuska and M. R. Dorr, "Error estimated for the combined h and p versions of the finite element method", Numerische Mathematik, 37, 257-277, 1981.
10. O. C. Zienkiewicz, B. M. Irons, F. E. Scott and J. S. Campbell, "Three dimensional stress analysis", Proceedings of the IUTAM Symposium on High Speed Computing of Elastic Structures, Liège, 1970.
11. A. Peano, "Hierarchies of conforming finite elements for plane elasticity and plate bending", Comp. and Maths. with Appls., 2, 211-224, 1976.
12. O. C. Zienkiewicz, D. W. Kelly, J. Gago and I. Babuska, "Hierarchical finite element approaches, error estimates and adaptive refinement", Report C/R/382/81, Inst. Numerical Methods in Engineering, University College of Swansea, U.K., 1981.
13. L. Meirovitch and H. Baruh, "On the inclusion principle for the hierarchical finite element method", International Journal for Numerical Methods in Engineering, (to appear).

14. R. F. Curtain and A. J. Pritchard, Functional Analysis in Modern Applied Mathematics, Academic Press, New York, 1977.
15. L. Meirovitch, Analytical Methods in Vibration, The Macmillan Co., New York, N.Y., 1967.
16. L. Meirovitch and H. Baruh, "Control of self-adjoint distributed-parameter systems", Journal of Guidance, Control, and Dynamics, 5, 1, Jan.-Feb. 1982, pp. 60-66.
17. H. Ziegler, Principles of Structural Stability, Blaisdell Publishing Company, Waltham, Mass., 1968.
18. V. V. Bolotin, The Dynamic Stability of Elastic Systems, Holden-day, Inc., San Francisco, Calif., 1964.
19. K. Huseyin, Vibrations and Stability of Multiple Parameter Systems, Sijthoff & Noordhoff International Publishers, The Netherlands, 1978.
20. D. L. Mingori, "A stability theorem for mechanical systems with constraint damping", Journal of Applied Mechanics, 37, 1970, pp. 253-258.
21. P. W. Likins, "Stability analysis of mechanical systems with constraint damping", AIAA Journal, 5, 11, 1967, pp. 2091-2094.
22. L. Meirovitch and H. Baruh, "Optimal control of damped flexible gyroscopic systems", Journal of Guidance and Control, 4, 2, March-April 1981, pp. 157-163.
23. W. L. Brogan, Modern Control Theory, QPI Publisher, Inc., New York, 1974.
24. L. H. Meirovitch and H. Öz, "A comparison of control techniques for large flexible systems", Paper No. 81-195, AAS/AIAA Astrodynamics Specialists Conference, Lake Tahoe, Nevada, August 3-5, 1981.
25. L. Meirovitch and H. Baruh, "Control of self-adjoint distributed-parameter systems", Journal of Guidance, Control, and Dynamics, 5, 1, 1982, pp. 60-66.
26. H. Öz and L. Meirovitch, "Optimal modal-space control of flexible gyroscopic systems", Journal of Guidance and Control, 3, 3, 1980, pp. 218-226.
27. A. J. Laub, "A Schur method of solving algebraic Riccati equations", IEEE Transactions on Automatic Control, Vol. AC-24, No. 6, 1979, pp. 913-921.

28. H. Kwackernaak and R. Sivan, Linear Optimal Control Systems, Wiley-Interscience, New York, 1972.
29. L. Meirovitch and H. Baruh, "On the robustness of the independent modal-space control method", Journal of Guidance, Control, and Dynamics, 6, 1, 1983, pp. 20-25.
30. D. E. Kirk, Optimal Control Theory, Prentice-Hall, Inc., Englewood Cliffs, NJ, 1970.

**The vita has been removed from
the scanned document**

ACTIVE CONTROL OF DISTRIBUTED STRUCTURES

by

Lawrence M. Silverberg

(ABSTRACT)

The partial differential equations of motion for an uncontrolled distributed structure can be transformed into a set of independent modal equations by means of the system eigenfunctions.

In vibration analysis, the modal coordinates are referred to as natural coordinates. Active control forces generally recouple the modal equations so that the natural coordinates for the open-loop (uncontrolled) system cease to be natural coordinates for the closed-loop (controlled) system. Control of this form is known as coupled control. In contrast, it is shown that a method known as the independent modal-space control method is a natural control method; i.e., the natural coordinates of the open-loop system and of the closed-loop system are identical. Furthermore, it is shown that natural control provides a unique and globally optimal closed-form solution to the linear optimal control problem for the distributed structure. The optimal control forces are ideally distributed. If implementation of distributed control is not feasible, then the distributed control forces can be approximated by finite-dimensional control forces. The class of self-adjoint systems are first considered following a treatment of non-self-adjoint systems. Numerical examples of a beam, a membrane and a whirling shaft are presented.

In general, the eigenquantities for a distributed structure cannot be computed in closed-form, so that spatial discretization of the differential eigenvalue problem is necessary. A common discretization method is the finite element method leading to a discrete eigenvalue problem. Two bracketing theorems characterizing convergence of the discrete eigenvalue problem derived by the finite element method to the differential eigenvalue problem are formulated.

The independent modal-space control method requires as many actuators as controlled modes. In contrast, coupled control is capable of controlling any number of modes using a single actuator, provided controllability is ensured. However, coupled control is sensitive to errors in the system parameters. As a compromise between coupled control and independent modal-space control, a block-independent control method is developed in which blocks of modes are controlled independently. The performances of independent modal-space control, coupled control and block-independent control are compared.