Virginia Tech

VIRGINIA POLYTECHNIC INSTITUTE AND STATE UNIVERSITY

Grado Department of Industrial and Systems Engineering

Analytic Evaluation of the Expectation and Variance of Different Performance Measures of a Schedule under Processing Time Variability

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Thesis submitted to the Faculty of the
Virginia Polytechnic Institute and State University
in partial fulfillment of the requirements for the degree of

Master of Science
In
Industrial and Systems Engineering

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September 19, 2003
Blacksburg, Virginia

Keywords: Expectation-variance analysis, Processing time variability,
Schedule variance, Variance evaluation

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(ABSTRACT)

The realm of manufacturing is replete with instances of uncertainties in job processing times, machine statuses (up or down), demand fluctuations, due dates of jobs and job priorities. These uncertainties stem from the inability to gather accurate information about the various parameters (e.g., processing times, product demand) or to gain complete control over the different manufacturing processes that are involved. Hence, it becomes imperative on the part of a production manager to take into account the impact of uncertainty on the performance of the system on hand. This uncertainty, or variability, is of considerable importance in the scheduling of production tasks. A scheduling problem is primarily to allocate the jobs and determine their start times for processing on a single or multiple machines (resources) for the objective of optimizing a performance measure of interest. If the problem parameters of interest e.g., processing times, due dates, release dates are deterministic, the scheduling problem is relatively easier to solve than for the case when the information is uncertain about these parameters. From a practical point of view, the knowledge of these parameters is, most often than not, uncertain and it becomes necessary to develop a stochastic model of the scheduling system in order to analyze its performance.

Investigation of the stochastic scheduling literature reveals that the preponderance of the work reported has dealt with optimizing the expected value of the performance measure. By focusing only on the expected value and ignoring the variance of the measure used, the scheduling problem becomes purely deterministic and the significant ramifications of schedule variability are essentially neglected. In many a practical cases, a scheduler

would prefer to have a stable schedule with minimum variance than a schedule that has lower expected value and unknown (and possibly high) variance. Hence, it becomes apparent to define schedule efficiencies in terms of both the expectation and variance of the performance measure used. It could be easily perceived that the primary reasons for neglecting variance are the complications arising out of variance considerations and the difficulty of solving the underlying optimization problem. Moreover, research work to develop closed-form expressions or methodologies to determine the variance of the performance measures is very limited in the literature. However, conceivably, such an evaluation or analysis can only help a scheduler in making appropriate decisions in the face of uncertain environment. Additionally, these expressions and methodologies can be incorporated in various scheduling algorithms to determine efficient schedules in terms of both the expectation and variance.

In our research work, we develop such analytic expressions and methodologies to determine the expectation and variance of different performance measures of a schedule. The performance measures considered are both completion time and tardiness based measures. The scheduling environments considered in our analysis involve a single machine, parallel machines, flow shops and job shops. The processing times of the jobs are modeled as independent random variables with known probability density functions. With the schedule given *a priori*, we develop closed-form expressions or devise methodologies to determine the expectation and variance of the performance measures of interest. We also describe in detail the approaches that we used for the various scheduling environments mentioned earlier. The developed expressions and methodologies were programmed in MATLAB R12 and illustrated with a few sample problems. It is our understanding that knowing the variance of the performance measure in addition to its expected value would aid in determining the appropriate schedule to use in practice. A scheduler would be in a better position to base his/her decisions having known the variability of the schedules and, consequently, can strike a balance between the expected value and variance.

DEDICATION

To my beloved parents, wonderful brother and lovely sister

ACKNOWLEDGEMENTS

As true as it would be with any research effort, this endeavor would not have been possible without the guidance and support of a few people whom I stand to thank at this juncture. First and foremost, I express my sincere gratitude to my advisor and mentor, Dr. Subhash C. Sarin, who has been the backbone of this whole exercise. I am greatly indebted for all the things that I have learnt from him during the course of my research, academically and otherwise. His profound knowledge in this field and at the same time his down-to-earth demeanor goes unparalleled. He is an epitome of patience and his approach and reasoning while tackling complex research problems amazes me. In essence, I feel extremely privileged to have worked under his supervision.

I would also like to thank Dr. F. Frank Chen for agreeing to be on my committee and participate actively in my research. I am very grateful for the discussions that I had with him and his valuable insights during the course of my research. I am also grateful to Dr. Sanjay Jain for being in my committee and constantly participating in my research discussions. His ideas and suggestions have been of tremendous help to me in improving my research work. I was also fortunate enough to have worked with Dr. Chen and Dr. Jain as their Teaching Assistants earlier which were also wonderful learning experiences.

I would like to thank all my good friends and fellow graduate students for their friendship and support during my stay at Virginia Tech. I extend my appreciation to all the faculty and staff in the Grado Department of Industrial and Systems Engineering for making my sojourn here a pleasant and a memorable one.

Finally, this section would not be complete if I do not mention my family without whom I would not be what I am right now. For lack of words, I would just like to say that they are very special to me and will always be.

"*What is research, but a blind date with knowledge"*

– William Henry

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CHAPTER 1: INTRODUCTION

1.1 Uncertainty in Manufacturing

The realm of manufacturing is replete with instances of uncertainties in job processing times, machine statuses (up or down), demand fluctuations, due dates of jobs and job priorities. These uncertainties stem from the inability to predict with sufficient accuracy information about product demand, job processing times as well as occurrences like unexpected machine breakdowns, and arrival/cancellation of orders. Although the forecasting methods that are currently available are highly efficient and are based on state-of-the art techniques, product demand errors are invariably prone to occur in the production system. Process variability is also another significant factor that will always introduce variations and uncertainty into the manufacturing process.

Uncertainty is undoubtedly an undesirable factor in the manufacturing process as it does not give production managers a complete control over the manufacturing process. Some of the ill-effects of these uncertainties include system instability, excess inventory, customer dissatisfaction by not meeting the due dates, and more importantly, loss of revenue. Recent advances in manufacturing management techniques, such as agile manufacturing, have made variability an important design criterion in order to ensure predictability and dependability of production systems. Agility is the ability to thrive and prosper in an environment of constant and unpredictable change.

1.2 Uncertainty in Scheduling

As true as it would be with any other field within manufacturing, uncertainty factor is of considerable importance in production scheduling. Scheduling is a decision-making process that plays an important role in most manufacturing as well as in most information-processing environments. From a manufacturing perspective, a scheduling problem is primarily the determination of the starting times of the jobs, waiting to be processed, on a single or multiple machines (resources) for the objective of optimizing an appropriate performance measure of interest. The randomness in the scheduling system is due to varying processing times, machine breakdowns, incomplete information about customer due dates, among others.

Some of the major scheduling environments are:

1. Scheduling on a single machine,
2. Scheduling on identical machines in parallel,
3. Flow shops
4. Flexible flow shops
5. Job shops
6. Flexible job shops

Some of the important parameters that are involved in the scheduling process are job processing times, due dates, job release dates and job weights which denote their relative importance.

Some of the regular performance measures used to evaluate a schedule are:

1. Makespan
2. Maximum Lateness
3. Total Weighted Completion Time
4. Discounted Total Weighted Completion Time
5. Total Weighted Tardiness
6. Weighted Number of Tardy Jobs

There could also be objectives with a combination of two or more of the above-mentioned measures.

Deterministic Scheduling involves solving a scheduling problem with the objective of optimizing a performance measure of interest when the parameters of scheduling viz., job processing times, due dates, release dates, etc are known with certainty. On the other hand, *Stochastic Scheduling* involves dealing with problems when at least one of these parameters is not known with certainty. This branch of scheduling is relatively more complex and difficult than its deterministic counterpart.

1.3 Modeling Uncertainty in Scheduling

As stated earlier, uncertainty plays a major role in scheduling. A review of the stochastic scheduling literature reveals that the uncertain scheduling parameters are modeled as random variables. In majority of the work, the means and variances or the distributions of the random variables are assumed to be known *a priori*. The objective is then to optimize a function of the performance measure of interest. This performance measure is also a random variable as it is a function of the input variables which are random as well. Predominantly, this function of the output performance measure is its expectation, i.e., the objective is to minimize the expectation of the performance measure.

The reason for such an approach can be easily surmised from the fact that computing the expectation is easier and relatively less complex than any other function of the random variable, say for example, its variance. Additionally, optimization becomes arduous and may even be impossible with the incorporation of variance related measures. Moreover, determining the variance of the performance measure is highly complicated and laborious. Hence, preponderance of the work in stochastic scheduling has dealt with optimizing the expected value of a performance measure. For example, while scheduling jobs with random processing times on a single machine with completion time as the performance measure, the predominant motive is to minimize the total *expected* completion time of all the jobs. By focusing only on the expected value and ignoring the variance of the objective, the scheduling problem becomes purely deterministic and the significant ramifications of schedule variability are neglected. In many a practical cases, a scheduler would prefer to have a stable schedule with minimum variance than a schedule that has lower expected value and unknown (and possibly higher) variance. From our survey of the literature, simultaneous consideration of expectation and variance is very much an unexplored area and minimal work has been reported. Closed-form expressions for the expectation and variance of simpler performance measures like the total completion time on a single machine can be readily computed, but analytic expressions for other measures like the discounted completion time or any other tardiness based objective are complex and hard to evaluate. It is even harder for other complex scheduling environments like parallel machines, flow shops or job shops. Hence, the dearth of work in variance evaluation is the primary cause for neglecting variance considerations in schedule optimization. Consequently, it becomes imperative to understand and ascertain the variance of the different performance measures of interest.

1.4 Significance of Variance in Scheduling

As was mentioned earlier, it is vital to consider variance issues in scheduling problems. To illustrate the significance of variance, consider the following example where four jobs are waiting to be processed on a single machine with the objective of minimizing the total completion time (total flow time). The job processing times (in some specified units) are random with known means and variances as shown in Table 1.1.

N	***1***	***2***	***3***	***4***
μ	35	40	20	22
σ^2	8	5	20	0

Table 1.1. Total Completion Time Example

Conventionally, the approach in tackling the above problem would be to minimize the total expected completion time by sequencing the jobs using SEPT (Shortest Expected Processing Time) policy. Hence, the SEPT sequence is *3-4-1-2*. The expectation and the resulting variance of the total completion are, $E\left[\sum C_j\right]= 256$ and $Var\left[\sum C_j\right] = 356$.

However, by scheduling jobs in the 4-3-1-2 sequence, we have $E\left[\sum C_j\right]$ = 258 and $Var\left[\sum C_j\right]$ = 217. Schedule 2 possesses a slightly higher expected value but has a considerably lower variance than the SEPT schedule. This is illustrated in Figure 1.1 assuming that the expectations and variances of $\sum C_j$ for the two schedules are normally distributed.

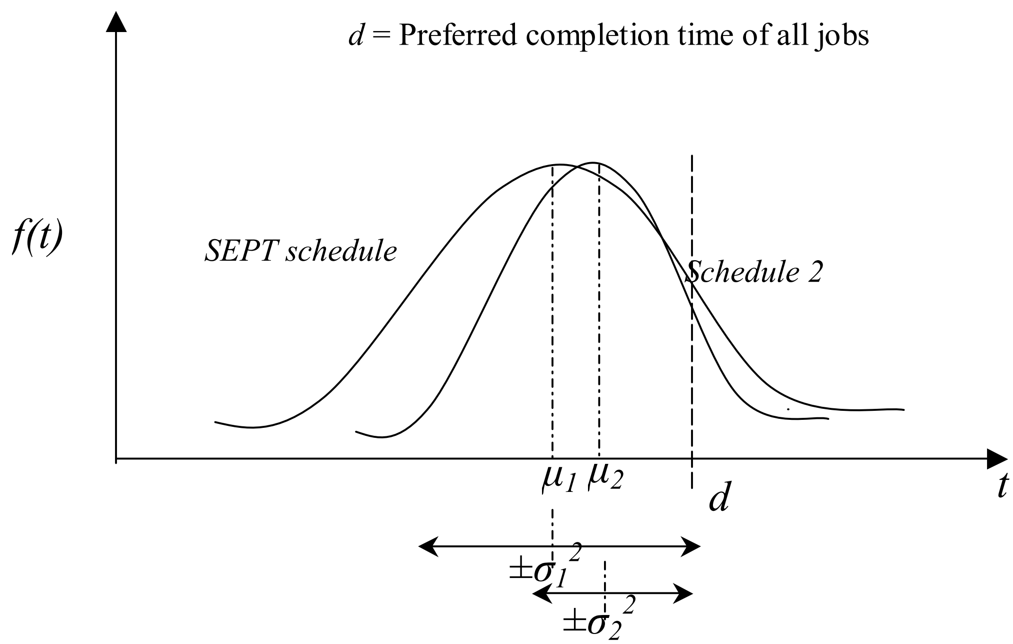


Figure 1.1. Representation of Normally Distributed Completion Time Variables

If the manufacturer prefers to deliver all the 4 jobs by d = 280, we can then analyze the probability with which the deadline will be met using the two schedules.

SEPT Schedule: $\mu_1 = 256$ and $\sigma_1^2 = 356$

$$\Pr\left[\sum C_j \le 280\right] \;=\; \Pr\left[Z_1 \le 1.272\right] = 0.898 = 89.8\%$$

Schedule 2: $\mu_2 = 258$ and $\sigma_2^2 = 217$

$$\Pr\left[\sum C_j \leq 280\right] = \Pr\left[Z_2 \leq 1.493\right] = 0.9324 = 93.24\%$$

(Z_i is the standard normal variable)

The probability of meeting the deadline is higher if the second schedule is employed. Hence, it is imperative to determine a sequence that is “good” in terms of both expectation and variance. This knowledge would not have been gained without determining the variance and by focusing the decision just on the expectation.

1.5 Problem Statement

As the importance of variance is understood from the earlier discussions and illustrations, it is inarguably a fruitful exercise to analytically evaluate the variance of different output performance measures in stochastic scheduling. Determining closed form expressions or methodologies to determine the expectations and variances is complex and at the same time challenging. We set out to develop closed-form expressions or methodologies for different performance measures for the following scheduling environments:

1. Scheduling on a single machine
2. Permutation flow shops with unlimited intermediate storage
3. Job shops with unlimited intermediate storage
4. Scheduling on identical machines in parallel

This analysis is contingent upon the fact that the schedule is given *a priori* and it is necessary to ascertain the expectation and variance of the given performance measure for that given schedule. Hence, the position of each job is known with certainty from the schedule. The randomness in the scheduling process is due to variable processing times with known means and variances. All other parameters like the job due dates and weights are assumed to be deterministic. In some cases, it might also be necessary to know the processing time distributions and those instances are appropriately cited. The interest is then to develop analytic expressions or methodologies to compute the parameters under consideration, viz., expectation and variance of the objective function value. The analysis does not involve optimization and is a vital exercise in modeling the performance of the scheduling system. However, it is worth mentioning that this endeavor is the starting point for variance considerations in schedule optimization. This knowledge would provide valuable insights in improving the performance of a schedule. A scheduler would be in a better position to base his decisions knowing the variability of the schedule and appropriately strike a balance between the expected value and variance. Additionally, these expressions and methodologies can be incorporated in various scheduling algorithms to determine efficient schedules in terms of both the expectation and variance.

The different models considered for our analysis were:

I. Single Machine Models

The different performance measures considered for the single machine case can be classified under two different categories, namely,

1. Completion time based measures and
2. Tardiness based measures

The various completion time based measures are:

1. Total Completion Time (Total Flow Time)
2. Total Weighted Completion Time
3. Total Weighted Discounted Completion Time

The various tardiness based measures are

1. Total Tardiness
2. Total Weighted Tardiness
3. Total Number of Tardy Jobs

4. Total Weighted Number of Tardy Jobs
5. Mean Lateness
6. Maximum Lateness

II. Parallel Machine Models

For parallel machines, both preemption and no preemption cases were considered. The performance measures were makespan and total completion time for the no preemption case and only makespan for the preemption case.

III. Flow shops

The objective was the makespan of a permutation flow shop with unlimited intermediate storage

IV. Job shops

The objective was to evaluate the makespan for a classical job shop with unlimited intermediate storage.

1.6 Outline of the Thesis

The organization of the thesis is as follows: Chapter 1 provides a brief introduction on the uncertainties involved in manufacturing, and especially in production scheduling. A concise overview of the modeling approaches of uncertainty in stochastic scheduling and the methodologies used in schedule optimization is also provided. The neglect of variance aspects in stochastic scheduling and its significance are elicited. Subsequently, motivation for the research work and problem statement are also outlined. Chapter 2 provides a review of the related research to the problem under consideration viz., variance issues in stochastic production systems. Significant work from the literature in the field of multiobjective stochastic scheduling and variance considerations in schedule optimization is reviewed. Subsequently, some of the research work done in performance modeling of production systems focusing on the variance of the output performance measure is also detailed. Chapter 3 consists of our work in determining closed-from expressions for the expectation and variance of various performance measures for the single machine models. Chapter 4 and Chapter 5 provide detailed analyses and methodologies for determining the expectation and variance of makespan in flow shops and job shops respectively. Chapter 6 deals with parallel machines and evaluation of makespan and completion time objectives for the no preemptions case. The underlying methodology for modeling and analyzing the preemptions case is also explained. Chapter 7 provides numerical illustrations of the developed expressions and methodologies for a few sample data sets. Finally, we summarize the overall thesis by stating the goal of this research and the significant results accomplished from our endeavor in Chapter 8. Extensions and directions for further research are also stated. Software programs that were written in MATLAB R12 for the implementation of the expressions and methodologies developed for each case are provided in the Appendices.

CHAPTER 2: REVIEW OF RELATED RESEARCH

In this chapter, a brief review is provided of research related to the problem on hand, viz., in the field of variance analysis of stochastic production systems. Initially, a survey of work done in the field of stochastic multiobjective or multicriterion scheduling is detailed. Finally, research work done in the area of performance modeling of production systems focusing on output variance is described.

2.1 Multiobjective or Multicriterion Stochastic Scheduling

Multiobjective or multicriterion optimization, especially in the field of scheduling, has always been an interesting and challenging topic for researchers. A scheduler's endeavor, from a practical point of view, is to optimize one or more objectives of interest simultaneously and achieve a trade-off solution, which is commonly referred as a *pareto-optimal* solution. A schedule is called *pareto-optimal* if it is not possible to decrease the value of one objective without increasing the other (or others).

Some of the important work reported in the field of deterministic multicriterion scheduling can be found in Wassenhove and Gilders (1980), Lin (1983), Nelson et al (1986), Daniels and Chambers (1990), Sarin and Hariharan (2000), among others.

On the stochastic front, Forst (1995) addressed the problem of minimizing the sum of the expected total weighted tardiness and the expected total weighted flowtime for the single machine and m-machine flow shop scheduling problems. He proved that an optimal sequence is obtained by sequencing the jobs in increasing stochastic order of their processing times. The job processing times are independent random variables and the jobs have a common, random due date. Lin and Lee (1995) consider a single-machine scheduling problem with known distributions of random processing times and due dates. The objective is to determine a schedule that minimizes a secondary criterion subject to a primary criterion being held at its best value. Three different models with completion times and lateness-related bicriterion objectives are formulated and algorithms for optimal solutions are provided.

The amount of information available on stochastic multicriterion objective scheduling is relatively limited when compared with its deterministic counterpart. Research on stochastic multicriterion objective scheduling is perceivably complex and is still a challenging field of research.

It would be germane at this juncture to mull over the fact that the multiple objectives that the researchers considered in stochastic multicriterion scheduling are predominantly completion time and tardiness related measures. From a problem modeling perspective, the different scheduling parameters, viz., job processing times, due dates etc, are primarily modeled as random variables with known distributions or if not, then possibly with other assumptions regarding their distributions. The performance measures of interest, like the total flow time or tardiness, are in turn random variables and a function of the random variables is optimized. Most often than not, this function is the expectation. This seemed valid enough because of the fact that consideration of other functions, for example the variance of completion time or lateness, would make the problem enormously complex and difficult to solve. However, it would be more appropriate and realistic to also consider the implications of the variance of the performance measures along with the expected value. The following section provides an adequate and elaborate review of the work in the field of stochastic scheduling wherein the objective is to optimize the performance measure of interest in terms of its expectation and variance. Surprisingly, literary work with specific focus in this area is exceedingly minimal and the area still remains vastly unexplored.

2.2 Variance Considerations in Stochastic Scheduling

2.2.1 Expectation-Variance Efficient Sequences

De et al (1992) emphasize the importance of considering both the expectation and variance of the performance measure in scheduling. It is mentioned that focusing on the expectation alone might not reflect the scheduler's risk attitude and to accurately account for risk, a scheduler must rather strive to maximize the expected utility function. As the utility function might be difficult to define, the other viable alternative is to determine stochastically optimal and efficient sequences. However, determining stochastically optimal or efficient sequences are technically complicated and are not of much relevance in practice. Hence, the authors suggest analyzing the schedules based on their efficiencies with respect to the expected value and variance of the performance measure. The problem considered is that of the scheduling of *n* jobs in a static environment with *random processing times* on a single machine under no preemption. The objective of interest is to minimize the sum of the flow times of all the jobs. The underlying motive in identifying efficient sequences is that it would strongly narrow down the choices for the scheduler and help in choosing a "good" schedule from this set for implementation in view of practical considerations.

A sequence *s* is termed as expectation-variance efficient (EV- efficient) if there exists no other sequence *s'* such that $EC(s') \leq EC(s)$ and $VC(s') \leq VC(s)$ with at least one of the inequalities holding strictly. EC and VC are the expected value and variance of the total flow time, respectively. An EV-efficient sequence *s* is termed as extreme or XEV-efficient if there exists an $\alpha \in [0, 1]$ for which *s* minimizes $Z_\alpha = \alpha EC + (1-\alpha)VC$.

Two different approaches are provided to determine the expectation-variance sequences for the flow time problem. One approach is by using dynamic programming while the other approach is to solve the problem as a linear assignment problem subject to a single side constraint (*bi-criteria assignment problem*). The XEV-sequences are determined by solving the problem as a bi-criteria transportation problem.

Possible extensions suggested are:

1. To apply the above concept could be extended to other single machine cases involving job weights, precedence relations and to also multiple machines.
2. To examine more complex environments, like general flow shops and job shops, in a similar manner.

2.2.2 Nondominating Schedules

Nagasawa et al (1998) also address the problem of solving for multiobjectives in the field of stochastic scheduling. The objective is to schedule jobs with *random processing times* on a single machine such that both the expected value and the variance of the total flow time are minimized. The idea is to assist the shop floor manager in selecting a preferred schedule from among a set of nondominating schedules (*N*) through an interactive system termed as the "interactive stochastic multiobjective scheduling system (*ISMSS*)". The work is a continuation of an earlier work done by Jung et al (1990) where an efficient algorithm based on pairwise-job-interchanges was proposed to determine this set *N*. The cardinality of this set *N* is normally quite large and it becomes imperative to develop an efficient way to select a preferred schedule from the set based on the user's requirements.

The primary stochastic multiobjective scheduling problem could be expressed as

$$P1 : \min \left\{ \begin{pmatrix} E[F(\pi)] \\ V[F(\pi)] \end{pmatrix} \middle| \pi \in \Pi \right\}$$

where $F(\pi)$, $E[F(\pi)]$ and $V[F(\pi)]$ denote, respectively, the total flowtime, the expected value and the variance of total flow time associated with schedule π belonging to the set of permutation schedules Π. Jung et al proposed a heuristic algorithm based on a job-interchange method to determine an approximate set of nondominated schedules to problem *P1*. The authors propose three different approaches to select the preferred schedule from this set. These approaches are also extended to the parallel machine case and results shown subsequently.

In the above two sections involving expectation-variance efficient sequences and non-dominating schedules, it is essential to compute the variance of the performance measure used. Noticeably, the only performance measure considered was the total completion time or total flow time. Closed-form expressions are easier to determine for the total completion time. However, closed-form expressions for other cases are hard to determine and do not exist. Our work is aimed at developing the relevant closed-form expressions for various performance measures.

2.2.3 Robust Scheduling

Another approach that has been employed to counter the effects of variance of the performance measure is based on the concept of robust scheduling. The notion of schedule robustness as presented by Daniels and Kouvelis (1995) is based on determining a schedule that minimizes the worst case deviation from optimality for the performance criterion of interest used with respect to all possible scenarios. The performance measures considered was the total flow time of all the jobs. The jobs realize a particular processing time under each scenario and there exists a processing time vector denoting the processing times of all the jobs to that particular scenario. An optimal schedule also exists for every possible scenario. An integer programming formulation is provided to determine the robust schedule.

The formulated Absolute Deviation Robust Scheduling Problem (*ADRSP*) is shown to be NP-hard and a branch-bound algorithm is used to solve the problem accurately. Two other heuristic based approaches are also illustrated.

In a subsequent paper, Daniels and Carrillo (1997) formulate a β-Robust scheduling problem (β –*RSP*) in a similar fashion where each possible scenario has an associated probability and a preferred target level '*T*' of the performance measure. The objective is to determine a sequence that maximizes the likelihood of achieving system performance no worse than the target level '*T*'. The means and variances of the processing time random variables are assumed to be known in formulating the problem.

It is evident from the problem formulation that β –*RSP* recognizes the effect of both expectation and variance of the performance criterion and seeks to optimize in both the dimensions. The β –*RSP* is shown to be *NP*-hard and a heuristic solution approach is also provided. The performance criterion of interest used is the total flow time in both of their works. Although this approach seems effective theoretically, it might not be practically feasible to enumerate all possible scenarios for the job processing times which are random.

2.2.4 Fluctuation Smoothing Policies for Re-entrant Lines

Lu et al (1994) address the problem of reducing the mean and variance of cycle-time in semiconductor manufacturing environments which feature the characteristic re-entrant process flows. In re-entrant flows, lots repeatedly return at different stages of their production to the same service stations for further processing. They introduce a new class of scheduling policies, Fluctuation Smoothing Policies, which achieve the best mean and variance of the cycle-time. The policies were tested by performing simulation experiments on two models of semiconductor manufacturing plants. Kumar and Kumar (1994) subsequently establish in their work that these policies are stable for all stochastic re-entrant lines under certain conditions.

2.3 Performance Modeling of Production Systems - Variance of an Output Performance Measure

As we strive to understand the importance of variance from the scheduling perspective, it is also apposite to consider the variance-related work in other production control systems. The following sections review the literature involved in computing the variance of the output in serial production lines operated using CONWIP, and others. There is an abundance of information in this domain and a review on some selective work is provided.

2.3.1 CONWIP Systems

CONWIP or CONstant Work–In-Process line is a pull-based production system proposed by Spearman et al (1990). The output parameters in a CONWIP line are, predominantly, throughput or time between departures (TBD) and flowtime. Considerable research has been done on CONWIP systems to study and analyze the mean and variance of these measures. Spearman and Hopp (1991) develop an expression to estimate the throughput of a constant work in process (CONWIP) manufacturing line subject to machine failures. They compute the throughput and average cycle time as a function of the WIP level.

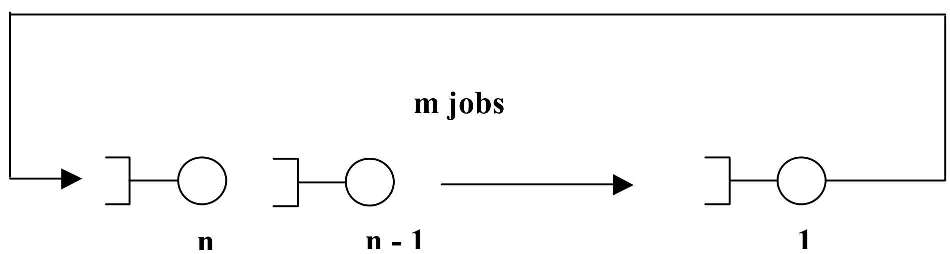


Figure 2.1. CONWIP Schematic

In a subsequent paper, Duenyas et al (1993) derive an approximation for the variance of the throughput of a CONWIP line with deterministic processing and random outages. Dar-el et al (1998) focus on CONWIP to develop estimates for four important performance measures: the means and variances of the time between departures (TBD) and flowtime. Mean TBD is the inverse of the mean throughput rate. The model assumes only finite means and variances of the processing time distributions.

2.3.2 Production Lines

It has been shown that the distribution of the output from a production line is asymptotically normal as a result of the central limit theorem. Hence, majority of the work dealing with uncertain production lines, due to random processing times or unreliable machines, strive only to determine the expectation and variance of the output performance measure of interest. Knowing the mean and variance of the output gives the asymptotic distribution of the throughput which could be used to derive other performance measures (for example, meeting a customer due date) based on the probability of other events.

Hendricks (1992) analyzes the mean and variance of the output process of a serial production line of N machines with exponential processing time distributions and finite buffer capacities. Analytic expressions for the interdeparture distribution and the correlation structure of the output process are developed using a continuous time Markov chain model.

Tan (1997) developed a closed-form expression for the variance of the throughput of an N-station production line with no intermediate buffers and time dependent failures. Time to failure and time to repair distributions are assumed to be exponential and the variance of the throughput is determined by modeling the process as an irreducible recurrent Markov process. In a subsequent paper, Tan (2002) determines the variance of the throughput of a production line with finite buffers by modeling the line as discrete time Markov chain. A thorough review of the existing literature on performance modeling of uncertain production lines is also provided.

2.4 Concluding Remarks

Thus, in this chapter, we have provided a brief review of the work done in the field of multiobjective stochastic scheduling and variance-related research in scheduling. Additionally, we reviewed some work on performance modeling of production systems related to output variance.

CHAPTER 3: SINGLE MACHINE MODELS

3.1 Introduction

In this chapter, we consider the single machine sequencing problem, and we develop closed-form expressions for the expectation and variance of various performance measures.

The case of the single machine is the simplest and easiest of all scheduling problems. It is relatively easier to deal with and is a good starting point for analyzing more complex scheduling environments like the flow shop or the job shop. The different performance measures considered for the single machine case can be classified into two categories, as follows:

1. Completion time based and
2. Tardiness based

The various Completion time based measures are:

1. Total Completion Time (Total Flow Time)
2. Total Weighted Completion Time
3. Total Weighted Discounted Completion Time

The various Tardiness based measures are

1. Total Tardiness
2. Total Weighted Tardiness
3. Total Number of Tardy Jobs
4. Total Weighted Number of Tardy jobs
5. Mean Lateness

6. Maximum Lateness

The processing time of a job is random and follows an arbitrary probability density function. We will use the following notations in the subsequent sections.

Notations Used:

n	-	Total number of jobs to be scheduled on the machine
$p_{[j]}$	-	Processing time of the job located at the 'j'th position of the given sequence (random variable)
$\mu_{[j]}$	-	Mean or expected value of the processing time of the job in the 'j'th position of the given sequence
$\sigma^2_{[j]}$	-	Variance of the processing time of the job in the 'j'th position of the given sequence
$C_{[j]}$	-	Completion time of the job in the 'j'th position of the given sequence (random variable)
$w_{[j]}$	-	Weight or importance factor associated with the job in the 'j'th position of the given sequence
$d_{[j]}$	-	Due date of the job in the 'j'th position of the given sequence
$L_{[j]}$	-	Lateness of the job in the 'j'th position of the given sequence
$T_{[j]}$	-	Tardiness of the job in the 'j'th position of the given sequence

$U_{[j]}$ - Unit penalty of the job in the 'j'th position of the given sequence

$C_{[j]}$ is defined as the time at which a job exits the system after getting processed on the machine (i.e., its completion time on the last machine (just one machine in this case) on which it requires processing). It is equal to the sum of the processing times of all the jobs that are processed prior to this job on that machine. Hence, it is primarily the sum of the waiting time of the job and its processing time.

$$C_{[j]} = \sum_{i=1}^{j} p_{[i]}$$

The lateness of a job in the 'j'th position is defined as $L_{[j]} = C_{[j]} - d_{[j]}$ which is positive when job [*j*] is completed late (after its due date) and negative when it is completed early (before its due date).

The tardiness of the job in the 'j'th position is defined as
$T_{[j]} = \max(0, L_{[j]}) = \max(0, C_{[j]} - d_{[j]})$.

The unit penalty of job in the 'j'th position is defined as

$$U_{[j]} = \left\{ \begin{array}{ll} 1, & if \ C_{[j]} > d_{[j]} \\ 0, & if \ C_{[j]} \le d_{[j]} \end{array} \right\}$$

3.2 Completion Time Based Objectives

3.2.1 Total Completion Time $\left(\sum_{j=1}^{n} C_{[j]}\right)$

Flow time of a job [*j*] is defined as the amount of time job [*j*] spends in the system and is given by $F_{[j]} = C_{[j]} - r_{[j]}$. In this analysis, all the jobs are available for processing at time zero and, hence, their ready times, $r_{[j]}$'s = 0. Hence, flow time and completion time of a job are synonymous and could be used interchangeably.

Flow time or Completion time measures the response of the system to individual demands of service and represents the interval the job waits between its arrival and departure. (This interval is sometimes called as the *turnaround time*.) The significance of this objective lies in that a minimal total completion time of the jobs will help maintain a low average in-process inventory (Baker, 1974).

For a given schedule, the total completion time is nothing but the sum of the completion times of all the jobs,

$$\sum_{j=1}^{n} C_{[j]} = \sum_{j=1}^{n}\sum_{i=1}^{j} p_{[i]}$$

Rearranging and summing up similar terms within the summation, we get,

$$\sum_{j=1}^{n} C_{[j]} = np_{[1]} + (n-1)p_{[2]} + ...+ 2p_{[n-1]} + p_{[n]}$$

$$= \sum_{j=1}^{n} (n+1-j)p_{[j]}$$

Expectation of the Total Completion Time $E\left[\sum_{j=1}^{n} C_{[j]}\right]$:

The expectation of $\sum_{j=1}^{n} C_{[j]}$ is given by:

$$E\left[\sum_{j=1}^{n} C_{[j]}\right] = E\left[\sum_{j=1}^{n}(n+1-j)p_{[j]}\right]$$

$$= E[np_{[1]} + (n-1)p_{[2]} + \dots\dots\dots\dots + 2p_{[n-1]} + p_{[n]}] \qquad \dots\dots\dots\dots(3.1)$$

Expectation and Variance of a linear combination of random variables:

Let $Y = a_0 + a_1 X_1 + a_2 X_2 + \dots\dots\dots\dots + a_{n-1} X_{n-1} + a_n X_n$

where Y , X_i 's are all random variables and a_i 's are all constants.

The expectation of a sum of random variables is the sum of the expectations of the individual random variables.

$$E[Y] = a_0 + \sum_{i=1}^{n} a_i E[X_i]$$

$$= a_0 + \sum_{i=1}^{n} a_i \mu_i \qquad \dots\dots\dots\dots (3.2)$$

The variance of the sum of random variables is given by

$$Var[Y] = \sum_{i=1}^{n} a_i^2 Var[X_i] + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ i \neq j}}^{n} a_i a_j Cov[X_i, X_j] \qquad \dots\dots\dots\dots (3.3)$$

Additionally, if the random variables are independent all the covariance terms are zero. In such a case, the variance is given by

$$Var\left[Y\right] = \sum_{i=1}^{n} a_i^2 Var\left[X_i\right] \qquad \text{....... (3.4)}$$

Applying (3.2) in (3.1),

$$E\left[\sum_{j=1}^{n} C_{[j]}\right] = n\mu_{[1]} + (n-1)\mu_{[2]} + \dots\dots\dots\dots + 2\mu_{[n-1]} + \mu_{[n]}$$

$$= \sum_{j=1}^{n} (\text{n} + 1 - \text{j})\mu_{[\text{j}]}$$

Variance of the Total Completion Time $Var\left[\sum_{j=1}^{n} C_{[j]}\right]$**:**

$$Var\left[\sum_{j=1}^{n} C_{[j]}\right] = Var\left[\sum_{j=1}^{n} (n+1-j) p_{[j]}\right]$$

$$= Var\left\lfloor np_{[1]} + (n-1)p_{[2]} + \dots\dots\dots + 2p_{[n-1]} + p_{[n]}\right\rfloor$$

As the processing times of the jobs are independent, applying (3.4) in the above expression we get,

$$Var\left[\sum_{j=1}^{n} C_{[j]}\right] = n^2\sigma^2_{[1]} + (n-1)^2\sigma^2_{[2]} + (n-2)^2\sigma^2_{[3]} + \dots\dots\dots\dots + 2^2\sigma^2_{[n-1]} + \sigma^2_{[n]}$$

$$= \sum_{j=1}^{n} (\text{ n} + 1 - \text{j})^2 * \sigma^2_{[j]}$$

3.2.2 Total Weighted Completion Time $\left(\sum_{j=1}^{n} w_{[j]} C_{[j]}\right)$

In this formulation, every job [j] has a corresponding weight associated with its completion time '$C_{[j]}$'. This weight could be considered as an importance factor or, alternately, may represent either a holding cost per unit time or the value added to job [j].

For a given schedule, the sum of weighted completion times of the jobs is given by,

$$\sum_{j=1}^{n} w_{[j]} C_{[j]} = \sum_{j=1}^{n} w_{[j]} \left(\sum_{i=1}^{j} p_{[i]} \right)$$

Rearranging and summing up similar terms within the summation, we get

$$= p_{[1]}(w_{[1]} + w_{[2]} + \ldots + w_{[n]}) + p_{[2]}(w_{[2]} + w_{[3]} + \ldots + w_{[n]}) + \quad p_{[n-1]}(w_{[n-1]} + w_{[n]}) + p_{[n]} w_{[n]}$$

$$= \sum_{j=1}^{n} p_{[j]} G_{[j]} \text{ where } G_{[j]} = \sum_{i=j}^{n} \mathrm{w}_{[\mathrm{i}]}$$

Expectation of the Total Weighted Completion Time $E\left[\sum_{j=1}^{n} w_{[j]} C_{[j]}\right]$:

The expectation of $\sum_{j=1}^{n} w_{[j]} C_{[j]}$ is determined using (3.1) and is given by,

$$E\left[\sum_{j=1}^{n} w_{[j]} C_{[j]}\right] = E\left[\sum_{j=1}^{n} p_{[j]} G_{[j]}\right] = \sum_{j=1}^{n} \mu_{[j]} G_{[j]}$$

Variance of the Total Weighted Completion Time $Var\left[\sum_{j=1}^{n} w_{[j]}C_{[j]}\right]$ **:**

The variance of $\sum_{j=1}^{n} w_{[j]}C_{[j]}$ is determined using (3.4) and is given by,

$$Var\left[\sum_{j=1}^{n} w_{[j]}C_{[j]}\right] = Var\left[\sum_{j=1}^{n} p_{[j]}G_{[j]}\right] = \sum_{j=1}^{n} \sigma_{[j]}^2 G_{[j]}^2$$

3.2.3 Total Weighted Discounted Completion Time $\left(\sum_{j=1}^{n} \mathrm{w}_{[j]}\left(1-e^{-rC_{[j]}}\right)\right)$

This is a more general cost function than the previous one. The costs are now discounted at a rate of '*r*', $0 < r < 1$ per unit time. That is, if the job [*j*] is not completed by time '*t*', an additional cost $\mathrm{w}_j e^{-rt}dt$ is incurred over the period [*t*, *t*+*dt*]. If job [*j*] is completed at time '*t*', the total cost incurred over the period [0,*t*] is $\mathrm{w}_j\left(1-e^{-rt}\right)$. The value of '*r*' is usually close to zero (Pinedo, 2002).

For a given schedule, the sum of weighted discounted completion times of the jobs is,

$$\sum_{j=1}^{n} w_{[j]}(1-e^{-rC_{[j]}}) = \sum_{j=1}^{n} w_{[j]} - \sum_{j=1}^{n} w_{[j]}e^{-rC_{[j]}} = W - \sum_{j=1}^{n} w_{[j]}e^{-rC_{[j]}},$$

where $W = \sum_{j=1}^{n} w_j$

Expectation of the Total Weighted Completion Time $E\left[\sum_{j=1}^{n} \mathrm{w}_{[\mathrm{j}]}\left(1-e^{-rC_{[j]}}\right)\right]$ **:**

$$E\left[\sum_{j=1}^{n} \mathrm{w}_{[\mathrm{j}]}\left(1-e^{-rC_{[j]}}\right)\right] = E\left[W-\sum_{j=1}^{n} w_{[j]}e^{-rC_{[j]}}\right]$$

$$= W-\sum_{j=1}^{n} E\left[w_{[j]}e^{-rC_{[j]}}\right]$$

$$= W-\sum_{j=1}^{n} w_{[j]}E\left[e^{-rC_{[j]}}\right]$$

$$= W-\sum_{j=1}^{n} w_{[j]}\mu_{DC[j]} \quad \text{where } \mu_{DC[j]} = E[e^{-rC_{[j]}}] \qquad \text{………. (3.5)}$$

Variance of the Total Weighted Completion Time $Var[\sum w_{[j]}(1-e^{-rC_{[j]}})]$ **:**

$$Var\left[\sum w_{[j]}(1-e^{-rC_{[j]}})\right] = Var\left[W-\sum_{j=1}^{n} w_{[j]}e^{-rC_{[j]}}\right]$$

$$= -Var\sum_{j=1}^{n}\left[w_{[j]}e^{-rC_{[j]}}\right] \text{ as } Var[W] = 0 \qquad \text{……….(3.6)}$$

Using the variance formula given in (3.3),

$$Var\sum_{j=1}^{n}\left[w_{[j]}e^{-rC_{[j]}}\right] = \sum_{j=1}^{n} w_{[j]}^{2}\,Var\left[e^{-rC_{[j]}}\right] + 2\sum_{\substack{i=1\\ i\neq j}}^{n}\sum_{j=i}^{n} w_{[i]}w_{[j]}Cov\left[e^{-rC_{[i]}},e^{-rC_{[j]}}\right]$$

$$= \sum_{j=1}^{n} w_{[j]}^{2}\sigma_{DC[j]}^{2} + 2\sum_{\substack{i=1\\ i\neq j}}^{n}\sum_{j=i}^{n} w_{[i]}w_{[j]}\sigma_{DC[ij]} \qquad \text{... ……. (3.7)}$$

$$\text{where } \sigma_{DC[j]}^{2} = Var[e^{-rC_{[j]}}] \text{ and } \sigma_{DC[ij]} = Cov\left[e^{-rC_{[i]}},e^{-rC_{[j]}}\right]$$

Using (3.7) in (3.6)

$$Var\left[\sum_{j=1}^{n} w_{[j]}(1-e^{-rC_{[j]}})\right] = -\left[\sum_{j=1}^{n} w_{[j]}^{2}\sigma_{DC[j]}^{2} + 2\sum_{\substack{i=1\\ i\neq j}}^{n}\sum_{j=i}^{n} w_{[i]}w_{[j]}\sigma_{DC[ij]}\right] \quad \text{……..(3.8)}$$

Methodology to determine $E\,[e^{-rC_{[j]}}]$ **:**

The moment-generating function of any random variable '*X*' is given by

$$M_X(t) = E(e^{tX}) \qquad \textit{for any real value of } t$$

$$= \sum_{all\ i} e^{t_i X_i} p_X(x_i) \qquad \textit{discrete } X$$

$$= \int_{-\infty}^{\infty} e^{tX} f_X(x)\, dx \qquad \textit{continous } X$$

Moreover, for sum of independent random variables, $Y = X_1 + X_2 + ...X_n$, the moment-generating function is given by

$$M_Y(t) = M_{X_1}(t).M_{X_2}(t)....................M_{X_n}(t)$$

$$= \prod_{j=1}^{n} M_{X_{[j]}}(t) \qquad \text{………. (3.9)}$$

Hence, $E\,[e^{-rC_{[j]}}]$ is the moment-generating function of the random variable $C_{[j]}$ with t = -r and 0 < r < 1. Additionally, $C_{[j]}$ is a random variable which is a sum of '*j*' independent random variables (processing times),

$$C_{[j]} = \sum_{i=1}^{j} p_{[i]} \text{ where } 1 \le j \le n$$

The moment-generating function $E\left[e^{-rC_{[j]}}\right]$ can, hence, be determined using (3.9), by the product of the moment generating functions of the individual processing times.

$$M_{C_{[j]}}(-r) = \prod_{i=1}^{j} M_{p_{[i]}}(-r) \qquad \text{……….} (3.10)$$

Methodology to determine $Var[e^{-rC_{[j]}}]$ **:**

$Var[e^{-rC_{[j]}}]$ can be determined using the formula $Var[X] = E[X^2] - (E[X])^2$

$$Var\left[e^{-rC_{[j]}}\right] = \sigma^2_{C[j]} = E\left[\left(e^{-rC_{[j]}}\right)^2\right] - \left(E\left[e^{-rC_{[j]}}\right]\right)^2$$

$$= E\left[e^{-2rC_{[j]}}\right] - \left(E\left[e^{-rC_{[j]}}\right]\right)^2$$

To determine $Cov[e^{-rC_{[i]}}, e^{-rC_{[j]}}]$**:**

$Cov[e^{-rC_{[i]}}, e^{-rC_{[j]}}]$ can be determined using the formula

$$Cov[X_1, X_2] = E[X_1.X_2] - (E[X_1].E[X_2])$$

$$Cov[e^{-rC_{[i]}}, e^{-rC_{[j]}}] = E[e^{-rC_{[i]}}.e^{-rC_{[j]}}] - \left(E[e^{-rC_{[i]}}].E[e^{-rC_{[j]}}]\right)$$

$E[e^{-rC_{[i]}}.e^{-rC_{[j]}}] = E[e^{-r(C_{[i]}+C_{[j]})}]$ and is the moment-generating function of the random variable $C_{[i]} + C_{[j]}$ which in turn is the sum of '*j*' independent random variables (job processing times)

$$C_{[i]} + C_{[j]} \quad = \quad 2p_{[1]} + 2p_{[2]} + ... + 2p_{[i]} + p_{[i+1]} + ... + p_{[j]} \qquad \text{where } i < j$$

Using (3.10),

$$M_{C_{[i]}+C_{[j]}}(-r) = M_{p_{[1]}}(-2r).M_{p_{[2]}}(-2r)....M_{p_{[i]}}(-2r).M_{p_{[i+1]}}(-r)......M_{p_{[j]}}(-r)$$

The moment generating functions can be determined by knowing the probability density functions of the random job processing times. Standard formulas for moment generating functions exist for all commonly used continuous distributions like exponential, normal, etc. For illustrative purposes, sample calculations are shown for exponentially and normally distributed processing times.

Illustration 1:

To compute the expectation and variance for exponentially distributed job processing times:

Assuming that the job processing times are exponentially distributed with rates $\lambda_{[j]}$'s, the moment generating function for the processing time of the job in the 'j' th position in the sequence is given by

$$M_{p_{[j]}}(-r) \quad = \quad E\,[e^{-rp_{[j]}}] \quad = \quad \frac{\lambda_{[j]}}{\lambda_{[j]} + r}$$

From (3.10),

$$M_{C_{[j]}}(-r) \quad = \quad E\left[e^{-rC_{[j]}}\right] \quad = \quad \left(\frac{\lambda_{[1]}}{\lambda_{[1]} + r}\right).\left(\frac{\lambda_{[2]}}{\lambda_{[2]} + r}\right)................\left(\frac{\lambda_{[j]}}{\lambda_{[j]} + r}\right)$$

$$= \quad \prod_{i=1}^{j}\left(\frac{\lambda_{[i]}}{\lambda_{[i]} + r}\right) \qquad \text{.......... (3.11)}$$

Applying (3.11) in (3.5), the expectation of the total weighted discounted completion time for exponentially distributed job processing times is given by

$$E\left[\sum w_{[j]}(1-e^{-rC_{[j]}})\right] = W - \sum_{j=1}^{n}\left(w_{[j]} \prod_{i=1}^{j}\left(\frac{\lambda_{[i]}}{\lambda_{[i]}+r}\right)\right)$$

The variance can be evaluated as follows:

$$Var\left[e^{-rC_{[j]}}\right] = \sigma^2_{DC[j]} = E\left[\left(e^{-rC_{[j]}}\right)^2\right] - \left(E\left[e^{-rC_{[j]}}\right]\right)^2$$

$$= E\left[e^{-2rC_{[j]}}\right] - \left(E\left[e^{-rC_{[j]}}\right]\right)^2$$

$$= \prod_{i=1}^{j}\left(\frac{\lambda_{[i]}}{\lambda_{[i]}+2r}\right) - \prod_{i=1}^{j}\left(\frac{\lambda_{[i]}^{\ 2}}{(\ \lambda_{[i]}\ +r\ )^2}\right) \quad \text{....... (3.12)}$$

$$Cov\ [e^{-rC_{[i]}}, e^{-rC_{[j]}}] = E[e^{-rC_{[i]}}.e^{-rC_{[j]}}] - \left(E[e^{-rC_{[i]}}].E[e^{-rC_{[j]}}]\right)$$

$$E[e^{-rC_{[i]}}.e^{-rC_{[j]}}] = \prod_{k=1}^{i}\left(\frac{\lambda_{[k]}}{\lambda_{[k]}+2r}\right)\prod_{k=i+1}^{j}\left(\frac{\lambda_{[k]}}{\lambda_{[k]}+r}\right)$$

$$\sigma_{DC[ij]} = \prod_{k=1}^{i}\left(\frac{\lambda_{[k]}}{\lambda_{[k]}+2r}\right)\prod_{k=i+1}^{j}\left(\frac{\lambda_{[k]}}{\lambda_{[k]}+r}\right) - \prod_{k=1}^{i}\left(\frac{\lambda_{[k]}}{\lambda_{[k]}+r}\right)\prod_{k=1}^{j}\left(\frac{\lambda_{[k]}}{\lambda_{[k]}+r}\right) \quad \text{.......(3.13)}$$

Expressions (3.12) and (3.13) can be substituted in (3.8) to obtain the variance of the total weighted discounted completion time, $Var[\sum w_{[j]}(1-e^{-rC_{[j]}})]$, for exponentially distributed job processing times.

Illustration 2:

To compute the expectation and variance for normally distributed job processing times:

Assuming that the job processing times are normally distributed with means $\mu_{[j]}$'s and variances $\sigma^2_{[j]}$'s, the moment generating function for the processing time of the job in the 'j' th position in the sequence is given by

$$M_{p_{[j]}}(-r) \quad = \quad E\,[e^{-rp_{[j]}}] \quad = \quad e^{\left(-r\mu_{[j]}+\sigma^2_{[j]}r^2/2\right)}$$

From (3.9),

$$M_{C_{[j]}}(-r) \quad = \quad E\left[e^{-rC_{[j]}}\right] \quad = \quad e^{\left(-r\mu_{[1]}+\sigma^2_{[1]}r^2/2\right)}.e^{\left(-r\mu_{[2]}+\sigma^2_{[2]}r^2/2\right)}.................e^{\left(-r\mu_{[j]}+\sigma^2_{[j]}r^2/2\right)}$$

$$= \quad e^{\left(-r(\mu_{[1]}+\mu_{[2]}...+\mu_{[j]})+(\sigma^2_{[1]}+\sigma^2_{[2]}+......+\sigma^2_{[j]})r^2/2\right)}$$

Denoting $\mu_{C[j]} \quad = \quad E\left[C_{[j]}\right] \quad = \quad \sum_{k=1}^{j}\mu_{[k]}$ and

$\sigma^2_{C[j]} \quad = \quad Var\left[C_{[j]}\right] \quad = \quad \sum_{k=1}^{j}\sigma^2_{[k]}$, we get

$$\mu_{DC[j]} \quad = \quad E[e^{-rC_{[j]}}] \quad = e^{\left(-r\mu_{C[j]}+\sigma^2_{C[j]}r^2/2\right)} \qquad \text{……….(3.14)}$$

Using (3.14) in (3.5) the expectation of the total weighted discounted completion time for exponentially distributed job processing times is given by

$$E\left[\sum w_{[j]}(1-e^{-rC_{[j]}})\right] \quad = \quad W - \sum_{j=1}^{n}\left(w_{[j]}\; e^{\left(-r\mu_{C[j]}+\sigma^2_{C[j]}r^2/2\right)}\right)$$

$$Var\left[e^{-rC_{[j]}}\right] = \sigma^2_{DC[j]} = E\left[\left(e^{-rC_{[j]}}\right)^2\right] - \left(E\left[e^{-rC_{[j]}}\right]\right)^2$$

$$= E\left[e^{-2rC_{[j]}}\right] - \left(E\left[e^{-rC_{[j]}}\right]\right)^2$$

$$= e^{\left(-2r\mu_{C[j]}+2\sigma^2_{C[j]}r^2\right)} - e^{\left(-2r\mu_{C[j]}+\sigma^2_{C[j]}r^2\right)} \qquad \ldots\ldots. (3.15)$$

$$Cov\,[e^{-rC_{[i]}}, e^{-rC_{[j]}}] = E[e^{-rC_{[i]}}.e^{-rC_{[j]}}] - \left(E[e^{-rC_{[i]}}].E[e^{-rC_{[j]}}]\right)$$
$$= E[e^{-r(C_{[i]}+C_{[j]})}] - \left(E[e^{-rC_{[i]}}].E[e^{-rC_{[j]}}]\right)$$

Denoting $\mu_{C[i]+C[j]} = E\left[C_{[i]} + C_{[j]}\right] = 2\sum_{k=1}^{i}\mu_{[k]} + \sum_{k=i+1}^{j}\mu_{[k]}$ and

$\sigma^2_{C[i]+C[j]} = Var\left[C_{[i]} + C_{[j]}\right] = 2\sum_{k=1}^{i}\sigma^2_{[k]} + \sum_{k=i+1}^{j}\sigma^2_{[k]}$, we get

$$\sigma_{DC[ij]} = = e^{\left(-r\mu_{C[i]+C[j]}+\sigma^2_{C[i]+C[j]}r^2/2\right)} - e^{\left(-r\mu_{C[i]}+\sigma^2_{C[i]}r^2/2\right)}.e^{\left(-r\mu_{C[j]}+\sigma^2_{C[j]}r^2/2\right)} \qquad \ldots\ldots\ldots(3.16)$$

(3.15) and (3.16) can be substituted in (3.8) to obtain the variance of the total weighted discounted completion time, $Var[\sum w_{[j]}(1-e^{-rC_{[j]}})]$, for normally distributed job processing times.

A similar approach could be adopted to compute the expectation and variance of the total weighted discounted completion time for jobs with other standard distributions for processing times.

3.3 Tardiness Based Objectives

3.3.1 Total Tardiness $\left(\sum_{j=1}^{n} T_{[j]}\right)$

Recall that tardiness $T_{[j]}$ for a job is given by

$$T_{[j]} = \max(L_{[j]},0) = \max(C_{[j]} - d_{[j]},0)$$

The expectation of total tardiness, using (3.2), is given by

$$E\left[\sum_{j=1}^{n} T_{[j]}\right] = \sum_{j=1}^{n} E\left[T_{[j]}\right] = \sum_{j=1}^{n} \mu_{T[j]} \quad \text{where} \quad \mu_{T[j]} = E[T_{[j]}] \qquad \text{.........(3.17)}$$

The variance of total tardiness, using (3.3), is similarly given by,

$$Var\left[\sum_{j=1}^{n} T_{[j]}\right] = \sum_{j=1}^{n} Var\left[T_{[j]}\right] + \sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{j=i}^{n} 2\, Cov\left[T_{[i]}, T_{[j]}\right]$$

$$= \sum_{j=1}^{n} \sigma_{T[j]}^{2} + \sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{j=i}^{n} 2\, \sigma_{T[ij]} \qquad \text{......... (3.18)}$$

where $\sigma_{T[j]}^{2} = Var\left[T_{[j]}\right]$ and $\sigma_{T[ij]} = Cov\left[T_{[i]}, T_{[j]}\right]$

From the definition of tardiness, it is evident that the tardiness random variables $(T_{[j]}'s)$ are not independent (as they are related by their individual completion times) and, hence, it is necessary to compute the covariance between every pairs of $T_{[j]}'s$.

Covariance analysis between the tardiness random variables:

Covariance, denoted by σ_{ij}, between any two random variables $[X_i, X_j]$ is given as follows,

$$Cov[X_i, X_j] = \sigma_{ij} = E[X_i.X_j] - [E[X_i]E[X_j]]$$

and the correlation coefficient is,

$$Cor[X_i, X_j] = \rho_{ij} = \frac{Cov[X_i, X_j]}{\sqrt{Var[X_i]}\sqrt{Var[X_j]}} = \frac{\sigma_{ij}}{\sigma_i \sigma_j}$$

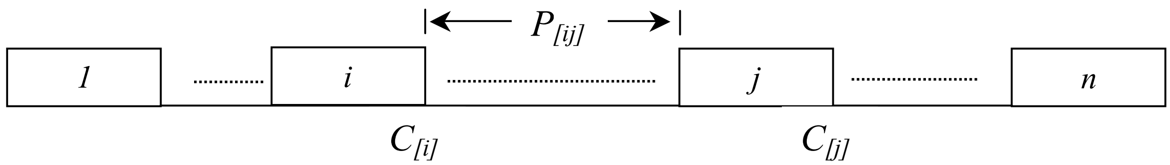


Figure 3.1. Representation of Completion Times of Jobs [*i*] and [*j*]

$$C_{[j]} = C_{[i]} + \{\textit{sum of processing times of all jobs between [i] and [j]}\} + p_{[j]}$$

$$C_{[j]} = C_{[i]} + P_{ij} + p_{[j]}$$

$$T_{[i]} = \max(L_{[i]}, 0) = \max(C_{[i]} - d_{[i]}, 0) \quad \text{.......(3.19)}$$

$$T_{[j]} = \max(L_{[j]}, 0) = \max(C_{[j]} - d_{[j]}, 0) = \max(C_{[i]} + P_{ij} + p_{[j]} - d_{[j]}, 0) \quad \text{.... (3.20)}$$

From expressions (3.19) and (3.20), it is evident that $T_{[i]}$ and $T_{[j]}$ are not independent. The completion time of the job early in the sequence ($C_{[i]}$) affects the completion time of the job later in the sequence ($C_{[j]}$).

To understand the correlation between the two random variables, $T_{[i]}$ and $T_{[j]}$, let's analyze the following three possible cases between the pair of jobs:

a) Both the jobs are tardy

In such a case, when both the jobs in the schedule are tardy, each job will have a positive tardiness value in that schedule. Considering two jobs [*i*] and [*j*], their tardiness values are

$$T_{[i]} \;=\; \max(L_{[i]},0) \;=\; \max(C_{[i]} - d_{[i]},0) \;=\; C_{[i]} - d_{[i]}$$

Similarly,

$$T_{[j]} \;=\; \max(L_{[j]},0) \;=\; \max(C_{[j]} - d_{[j]},0) \;=\; C_{[j]} - d_{[j]}$$

But $C_{[j]} - d_{[j]} \;=\; C_{[i]} + P_{ij} + p_{[j]} - d_{[j]} \;=\; T_{[i]} + P_{ij} + p_{[j]} - d_{[j]} + d_{[i]}$ and hence

$$T_{[j]} = \; T_{[i]} + P_{ij} + p_{[j]} - d_{[j]} + d_{[i]}$$

The plot between $T_{[i]}$ and $T_{[j]}$ is shown below in Figure 3.2. It is evident that there exists a linear positive correlation between $T_{[i]}$ and $T_{[j]}$ and the correlation coefficient, $\rho_{[ij]}$, is equal to one.

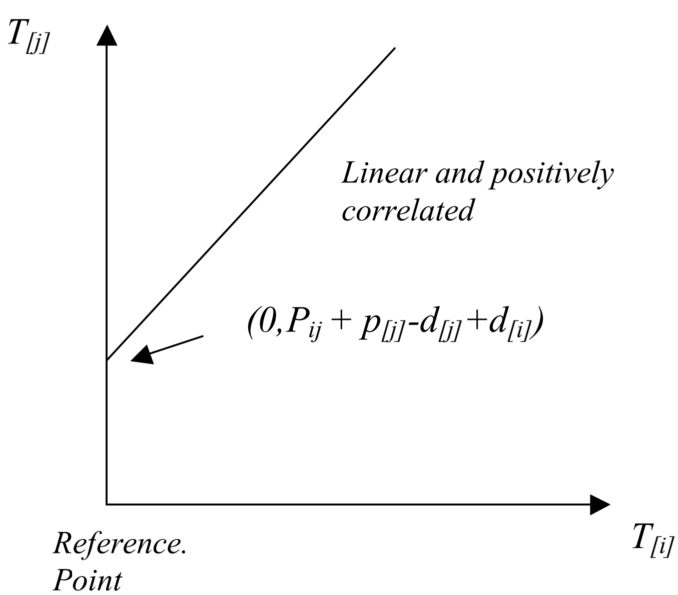


Figure 3.2. Plot between $T_{[i]}$ and $T_{[j]}$ when Both Jobs are Tardy

$$\rho_{T[ij]} \;=\; Cor\lfloor T_{[i]}, T_{[j]} \rfloor \;=\; 1$$

$$Cov\left[T_{[i]},T_{[j]}\right] = \sigma_{T[ij]} = Cor\left[T_{[i]},T_{[j]}\right]\sqrt{Var[T_{[i]}]}\sqrt{Var[T_{[j]}]}$$

$$= \rho_{T[ij]}\sigma_{T[i]}\sigma_{T[j]} = \sigma_{T[i]}\sigma_{T[j]}$$

b) Both the jobs are early

In such a case, when both the jobs are early, each job will have a tardiness value of zero. Considering two jobs [*i*] and [*j*],

$$T_{[i]} = \max(L_{[i]},0) = \max(C_{[i]}-d_{[i]},0) \quad \text{as } (C_{[i]}-d_{[i]}<0)$$
$$= \max(\textit{some negative value},0) = 0$$

$$\text{Similarly, } T_{[j]} = \max(C_{[j]}-d_{[j]},0) = 0$$

Note that, in this case, there exists no linear correlation between the tardiness parameters as there exists no relation between them.

$$\rho_{T[ij]} = Cor[T_{[i]},T_{[j]}] = 0$$

$$Cov\left[T_{[i]},T_{[j]}\right] = \sigma_{T[ij]} = Cor\left[T_{[i]},T_{[j]}\right]\sqrt{Var[T_{[i]}]}\sqrt{Var[T_{[j]}]}$$

$$= \rho_{T[ij]}\sigma_{T[i]}\sigma_{T[j]} = 0$$

c) One of the jobs is tardy while the other is early

Assuming the early job to be [*i*] and the late job to be [*j*], the tardiness values for the two random parameters are,

$$T_{[i]} = \max(L_{[i]},0) = \max(C_{[i]} - d_{[i]},0) = 0$$

$$T_{[j]} = \max(L_{[j]},0) = \max(C_{[j]} - d_{[j]},0)$$
$$= C_{[j]} - d_{[j]} = C_{[i]} + X_{ij} + p_{[j]} - d_{[j]}$$

In this case also, $T_{[i]}$ and $T_{[j]}$ are not linearly correlated as there exists no relation between them. Hence,

$$\rho_{T[ij]} = Cor[T_{[i]}, T_{[j]}] = 0$$

$$Cov\left[T_{[i]}, T_{[j]}\right] = \sigma_{T[ij]} = Cor\left[T_{[i]}, T_{[j]}\right]\sqrt{Var[T_{[i]}]}\sqrt{Var[T_{[j]}]}$$

$$= \rho_{T[ij]}\sigma_{T[i]}\sigma_{T[j]} = 0$$

The above three cases are summarized in the Table 3.1 below:

Case	**Correlation coefficient**	**Covariance**
Both jobs are tardy	1	$\sigma_{T[i]}\sigma_{T[j]}$
One of the jobs is tardy and the other is early	0	0
Both the jobs are early	0	0

Table 3.1. Correlation and Covariance between $T_{[i]}$ and $T_{[j]}$

Since the job processing times and, hence, the completion times are random, it would not be possible to know about the lateness or earliness of a job *a priori*. However, we can determine the probability of a job being late and the covariance expression can be accordingly modified as follows:

$$Cov\left[T_{[i]}, T_{[j]}\right] = \Pr[T_{[i]} > 0].\Pr[T_{[j]} > 0].Cor\left[T_{[i]}, T_{[j]}\right]\sqrt{Var[T_{[i]}]}\sqrt{Var[T_{[j]}]}$$

$$Cov\left[T_{[i]},T_{[j]}\right] = \Pr[T_{[i]}>0].\Pr[T_{[j]}>0].\rho_{T[ij]}.\sigma_{T[i]}.\sigma_{T[j]}$$

$$\Pr[T_{[i]}>0] = 1-\Pr[T_{[i]}\le 0] = 1-\Pr[C_{[i]}\le d_{[i]}] = 1-F_{C_{[i]}}(d_{[i]})$$

$$\Pr[T_{[j]}>0] = 1-\Pr[T_{[j]}\le 0] = 1-\Pr[C_{[j]}\le d_{[j]}] = 1-F_{C_{[j]}}(d_{[j]})$$

where $F_{C_{[i]}}(.)$ and $F_{C_{[j]}}(.)$ are the cumulative distribution function of the random variables $C_{[i]}$ and $C_{[j]}$, respectively.

$$\therefore \quad Cov\,[T_{[i]},T_{[j]}] = [1-F_{C_{[i]}}(d_{[i]})].[1-F_{C_{[j]}}(d_{[j]})].\rho_{T[ij]}.\sigma_{T[i]}.\sigma_{T[j]}$$

The variance expression (3.18) could then be rewritten as follows,

$$Var\left[\sum T_{[j]}\right] = \sum_{j=1}^{n}\sigma^2_{T[j]} + 2\sum_{i=1}^{n}\sum_{\substack{j=i \\ i\neq j}}^{n} [1-F_{C_{[i]}}(0)].[1-F_{C_{[j]}}(0)].\rho_{T[ij]}.\sigma_{T[i]}.\sigma_{T[j]} \qquad \text{......(3.21)}$$

Clark's Method to determine the greatest of a finite set of random variables:

The complexity in computing $\mu_{T[j]}$ and $\sigma^2_{T[j]}$ in expressions (3.17) and (3.21) arises from the fact that we need to compute the expectation and variance of a maximum function.

$$\mu_{T[j]} = E[\max(L_{[j]},0)] = E[\max(C_{[j]}-d_{[j]},0)]$$

$$\sigma^2_{T[j]} = Var[\max(L_{[j]},0)] = Var[\max(C_{[j]}-d_{[j]},0)]$$

There are no standard evaluations for the mean and expectation of a maximum operator available in the mathematical theory. However, in a pioneering work, Clark (1961) developed a method to recursively estimate the expectation and variance of the greatest of a finite set of random variables that are normally distributed. Accurate results can be obtained for maximum functions with two arguments while for higher number of arguments, close-to-accurate approximations can be obtained. Hence, for normally distributed processing times, Clark's method could be applied to find the desired expectation and variance expressions of $\sum T_j$. We proceed to describe our methodology using Clarke's equations in the following section for normally distributed job processing times.

Application of Clarke's equations for normally distributed processing times:

$C_{[i]}$ and $C_{[j]}$ are linear sums of job processing times which are assumed to be normally distributed. By the reproductive property of normal random variables, $C_{[i]}$ and $C_{[j]}$ are also normally distributed with means and variances as given below.

$$\mu_{C[j]} \quad = \quad E\left[C_{[j]}\right] \quad = \quad E\left[C_{[j]}\right] \quad = \quad \sum_{k=1}^{j} \mu_{[k]}$$

$$\sigma^2_{C[j]} \quad = \quad Var\left[C_{[j]}\right] \quad = \quad Var\left[C_{[j]}\right] \quad = \quad \sum_{k=1}^{j} \sigma^2_{[k]}$$

The cumulative distribution function *F* for a normal distribution is given by

$$F(x) \quad = \quad P\left(X \le x\right) \quad = \quad \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} \exp(-(1/2)[(t-\mu)/\sigma]^2)\, dt$$

Using the transformation $z = (x-\mu)/\sigma$, *F* could be evaluated as follows:

$$F(x) \quad = \quad P\left(Z \le \frac{x-\mu}{\sigma}\right) \quad = \quad \int_{-\infty}^{(x-\mu)/\sigma} \frac{1}{\sqrt{2\pi}} \exp(-(z^2/2))\, dt$$

$$= \int_{-\infty}^{(x-\mu)/\sigma} \varphi(z)\, dz \quad = \quad \Phi\left(\frac{x-\mu}{\sigma}\right)$$

where $\Phi(.)$ is unit normal cumulative distribution function.

Using these relations, $F_{C_{[i]}}(d_{[i]})$ and $F_{C_{[j]}}(d_{[j]})$ can be written as:

$$F_{C_{[i]}}\left(d_{[i]}\right) \quad = \quad \Phi\left(\frac{d_{[i]} - \mu_{C[i]}}{\sigma_{C[i]}}\right) \qquad \text{……….(3.22)}$$

$$F_{C_{[j]}}\left(d_{[j]}\right) \quad = \quad \Phi\left(\frac{d_{[j]} - \mu_{C[j]}}{\sigma_{C[j]}}\right) \qquad \text{……….(3.23)}$$

$$T_{[j]} \quad = \quad \max(L_{[j]}, 0) \quad = \quad \max(C_{[j]} - d_{[j]}, 0)$$

The second argument inside the maximum function viz., 0 can be assumed to be a random variable with mean μ = 0 and variance σ^2 = 0. The mean and variance of the first argument, $C_{[j]} - d_{[j]}$, are given below and is normally distributed by the reproductive property of normally distributed random variables. Hence, Clark's method could be applied to accurately determine the expectation and variance of $\max(C_{[j]} - d_{[j]}, 0)$

$$\mu_{[J]} \quad = \quad E\left[C_{[j]} - d_{[j]}\right] \quad = \quad E\left[C_{[j]}\right] - d_{[j]} \quad = \quad \sum_{k=1}^{j} \mu_{[k]} - d_{[j]}$$

$$\sigma_{[J]}^2 \quad = \quad Var\left[C_{[j]} - d_{[j]}\right] \quad = \quad Var\left[C_{[j]}\right] \quad = \quad \sum_{k=1}^{j} \sigma_{[k]}^2$$

The coefficient of linear correlation between the two arguments (0 and $T_{[j]}$) inside the maximum function is zero. $\mu_{T[j]}$ is then given by the first moment of the random variable $\max(C_{[j]} - d_{[j]}, 0)$:

$$\nu_{1[j]} = \mu_{T[j]} = \mu_{[J]}\Phi(\alpha_{[j]}) + a_{[j]}\varphi(\alpha_{[j]}) \qquad \text{…………(3.24)}$$

The second moment is given by

$$\nu_{2[j]} = (\mu_{[J]}^2 + \sigma_{[J]}^2)\Phi(\alpha_{[j]}) + \mu_{[J]}\, a_{[j]}\, \varphi(\alpha_{[j]})$$

$$\sigma_{T[j]}^2 = \nu_{2[j]} - \nu_{1[j]}^2$$

$$= \left[(\mu_{[J]}^2 + \sigma_{[J]}^2)\Phi(\alpha_{[j]}) + \mu_{[J]}\, a_{[j]}\, \varphi(\alpha_{[j]})\right] - \left[\mu_{[J]}\Phi(\alpha_{[j]}) + a_{[j]}\varphi(\alpha_{[j]})\right]^2 \qquad \text{……..(3.25)}$$

where, $a_{[j]}^2 = \sigma_{[J]}^2$, $\alpha_{[j]} = \mu_{[J]}/a_{[j]}$, $\varphi(z) = \dfrac{1}{\sqrt{2\pi}}\exp(-z^2/2)$

and $\Phi(x) = \int_{-\infty}^{x} \varphi(t)\, dt$

The expectation expression (3.17) can then be written using (3.24) as

$$E\left[\sum T_{[j]}\right] = E\left[T_{[1]}\right] + E\left[T_{[2]}\right] + \dots\dots\dots\dots + E\left[T_{[n-1]}\right] + E\left[T_{[n]}\right]$$

$$= \sum_{j=1}^{n} \mu_{[J]}\Phi(\alpha_{[j]}) + a_{[j]}\varphi(\alpha_{[j]})$$

The variance expression (3.21) can also be computed using expressions (3.22), (3.23) and (3.25)

3.3.2 Total Weighed Tardiness $\left(\sum_{j=1}^{n} w_{[j]}T_{[j]}\right)$

The expectation and variance for the total weighted tardiness measure are as shown below.

$$E\left[\sum_{j=1}^{n} w_{[j]}T_{[j]}\right] = \sum_{j=1}^{n} E\left[w_{[j]}T_{[j]}\right] = \sum_{j=1}^{n} w_{[j]}\mu_{T[j]} \quad \text{where} \quad \mu_{T[j]} = E\left[T_{[j]}\right]$$

$$Var\left[\sum_{j=1}^{n} w_{[j]}T_{[j]}\right] = \sum_{j=1}^{n} w_{[j]}^2 Var\left[T_{[j]}\right] + \sum_{i=1}^{n}\sum_{\substack{j=i \\ i\neq j}}^{n} 2\, w_{[i]}w_{[j]}Cov\left[T_{[i]},T_{[j]}\right]$$

$$= \sum_{j=1}^{n} w_{[j]}^2 \sigma_{T[j]}^2 + \sum_{\substack{i=1 \\ i\neq j}}^{n} \sum_{j=i}^{n} 2\, w_{[i]}w_{[j]}\sigma_{T[ij]}$$

where $\sigma_{T[j]}^2 = Var\left[T_{[j]}\right]$ and $\sigma_{T[ij]} = Cov\left[T_{[i]},T_{[j]}\right]$

Determining $\mu_{T[j]}, \sigma_{T[j]}^2$ *and* $\sigma_{T[ij]}$ for the above equations using Clark's equations have been described in the previous section. Hence, the expectation and variance of the total weighted tardiness can be determined.

3.3.3 Total Number of Tardy Jobs $\left(\sum_{i=1}^{n} U_{[j]}\right)$

Unit penalty of the job in the '*j*'th position, $U_{[j]}$, in the given schedule is given by

$$U_{[j]} = \begin{cases} 1, & if\ \ C_{[j]} > d_{[j]} \\ 0, & if\ \ C_{[j]} \leq d_{[j]} \end{cases} \qquad \text{.......(3.26)}$$

The expectation and variance of the total number of tardy jobs for a given sequence is,

$$E\left[\sum_{j=1}^{n} U_{[j]}\right] = \sum_{j=1}^{n} E\left[U_{[j]}\right] = \sum_{j=1}^{n} \mu_{U[j]}, \quad \text{where } \mu_{U[j]} = E\left[U_{[j]}\right] \qquad \text{..........(3.27)}$$

$$Var\left[\sum_{j=1}^{n} U_{[j]}\right] = \sum_{j=1}^{n} Var\left[U_{[j]}\right] + \sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{j=i}^{n} 2\, Cov\left[U_{[i]}, U_{[j]}\right]$$

However, there exists no linear correlation between the $U_{[j]}$'s and, hence, the covariance terms are all equal to zero.

$$\therefore \quad Var\left[\sum_{j=1}^{n} U_{[j]}\right] = \sum_{j=1}^{n} Var\left[U_{[j]}\right] = \sum_{j=1}^{n} \sigma^2_{U[j]} \text{ where } \sigma^2_{U[j]} = Var\left[U_{[j]}\right] \text{......... (3.28)}$$

Application of the Bernoulli's density function:

Bernoulli distribution can be applied at instances when there are only two possible outcomes for an individual trial. The discrete probability density function for a Bernoulli trial is given by,

$$p_j(x_j) = p(x_j) = \begin{cases} p & x_j = 1, \quad j = 1,2,\text{..........}n \\ (1-p) & x_j = 0, \quad j = 1,2,\text{..........}n \\ 0 & otherwise \end{cases}$$

The mean and variance are: $E\,[X_j] = p$ and $Var\,[X_j] = p(1-p)$

It could be noted from (3.26) that the unit penalty random variable $U_{[j]}$ is identical to a Bernoulli trial with 'p' being the probability that job 'j' will be late and 'q' not being late. Hence,

$$E[U_{[j]}] \quad = \quad \left(1.\Pr[C_{[j]} > d_{[j]}] + 0.\Pr[C_{[j]} \le d_{[j]}]\right)$$

$$= \quad 1 - \Pr[C_{[j]} \le d_{[j]}] \quad = \quad 1 - F_{C_{[j]}}[d_{[j]}] \quad = \quad \mu_{U[j]} \qquad \text{……….(3.29)}$$

where , $F_{C_{[j]}}(.)$ is the cumulative distribution function of the random variable $C_{[j]}$

$$Var\left[U_{[j]}\right] \quad = \quad \sigma^2_{U[j]} \quad = \quad \mu_{U[j]}\left(1 - \mu_{U[j]}\right) \qquad \text{……….(3.30)}$$

For illustrative purposes, we show the steps involved in determining $\mu_{U[j]}$ for normally distributed job processing times.

Illustration on determining $\mu_{U[j]}$ for normally distributed job processing times:

$C_{[i]}$ and $C_{[j]}$ are linear sums of job processing times which are normally distributed. By the reproductive property of normal random variables, $C_{[i]}$ and $C_{[j]}$ are also normally distributed with means and variances as given below:

$$\mu_{C[j]} \quad = \quad E\left[C_{[j]}\right] \quad = \quad \sum_{k=1}^{j} \mu_{[k]} \quad \text{and} \quad \sigma^2_{C[j]} \quad = \quad Var\left[C_{[j]}\right] \quad = \quad \sum_{k=1}^{j} \sigma^2_{[k]}$$

The cumulative distribution function F for a normal distribution is given by

$$F(x) \quad = \quad P\left(X \le x\right) \quad = \quad \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} \exp(-(1/2)[(t-\mu)/\sigma]^2)\, dt$$

Using the transformation $z = (x - \mu)/\sigma$, F could be evaluated as follows:

$$F(x) \quad = \quad P\left(Z \le \frac{x-\mu}{\sigma}\right) \quad = \quad \int_{-\infty}^{(x-\mu)/\sigma} \frac{1}{\sqrt{2\pi}} \exp(-(z^2/2)\, dt$$

$$= \quad \int_{-\infty}^{(x-\mu)/\sigma} \varphi(z)\, dz \quad = \quad \Phi\left(\frac{x-\mu}{\sigma}\right)$$

Using these relations, $F_{C_{[j]}}(d_{[j]})$ can be written as:

$$F_{C_{[j]}}\left(d_{[j]}\right) \quad = \quad \Phi\left(\frac{d_{[j]} - \mu_{C[j]}}{\sigma_{C[j]}}\right)$$

$$\mu_{U[j]} \quad = \quad 1 - F_{C_{[j]}}\left[d_{[j]}\right]$$

Using $\mu_{U[j]}$ from above and expressions (3.29) and (3.30) in (3.27) and (3.28), we get

$$E\left[\sum_{j=1}^{n} U_{[j]}\right] \quad = \quad \sum_{j=1}^{n}\left(1 - F_{C_{[j]}}\left[d_{[j]}\right]\right)$$

$$Var\left[\sum_{j=1}^{n} U_{[j]}\right] \quad = \quad \sum_{j=1}^{n}\left[\left(1 - F_{C_{[j]}}\left[d_{[j]}\right]\right).\, F_{C_{[j]}}\left[d_{[j]}\right]\right]$$

3.3.4 Total Weighted Number of Tardy Jobs $\left(\sum_{i=1}^{n} w_{[j]} U_{[j]}\right)$

The expectation and variance for the total weighted number of tardy jobs for a given sequence are as shown below.

$$E\left[\sum_{j=1}^{n} w_{[j]} U_{[j]}\right] \quad = \quad \sum_{j=1}^{n} w_{[j]} E\left[U_{[j]}\right] \quad = \quad \sum_{j=1}^{n} w_{[j]} \mu_{U[j]}, \qquad \text{where } \mu_{U[j]} = E\left[U_{[j]}\right]$$

$$Var\left[\sum_{j=1}^{n} w_{[j]}U_{[j]}\right] = \sum_{j=1}^{n} w_{[j]}^2 Var\left[U_{[j]}\right] = \sum_{j=1}^{n} w_{[j]}^2 \sigma_{U[j]}^2, \quad \text{where } \sigma_{U[j]}^2 = Var\left[U_{[j]}\right]$$

Similar to the analysis in the previous section for the performance measure of total number of tardy jobs, the expectation and variance expressions can be computed using Bernoulli's distribution.

3.3.5 Mean Lateness ($\overline{L}$)

The mean lateness for a given schedule is given by,

$$\overline{L} = \frac{1}{n}\left[\sum_{j=1}^{n} L_{[j]}\right]$$

$$= \frac{1}{n}\left[\sum_{j=1}^{n}\left(C_{[j]} - d_{[j]}\right)\right]$$

$$= \frac{1}{n}\left[\sum_{j=1}^{n} C_{[j]} - \sum_{j=1}^{n} d_{[j]}\right]$$

Expectation of the Mean Lateness $E\left[\overline{L}\right]$:

The expectation of $\overline{L}$ is given by

$$E\left[\overline{L}\right] = E\left[\frac{1}{n}\left(\sum_{j=1}^{n} C_{[j]} - \sum_{j=1}^{n} d_{[j]}\right)\right]$$

$$= \frac{1}{n}\left(E\left[\sum_{j=1}^{n} C_{[j]}\right] - \sum_{j=1}^{n} d_{[j]} \right)$$

$E\left[\sum_{j=1}^{n} C_{[j]}\right]$ has been already computed and is equal to $\sum_{j=1}^{n} (\mathrm{n}+1-\mathrm{j})\mu_{[\mathrm{j}]}$

$$\therefore \quad E[\overline{L}] = \frac{1}{n}\left[\sum_{j=1}^{n} (\mathrm{n}+1-\mathrm{j})\mu_{[\mathrm{j}]} - \sum_{j=1}^{n} d_{[j]}\right]$$

Variance of the Mean Lateness $Var[\overline{L}]$ **:**

$$Var[\overline{L}] = Var\left[\frac{1}{n}\left(\sum_{j=1}^{n} C_{[j]} - \sum_{j=1}^{n} d_{[j]}\right)\right]$$

$$= \frac{1}{n^2}\left(Var\left[\sum_{j=1}^{n} C_{[j]}\right]\right)$$

$Var\left[\sum_{j=1}^{n} C_{[j]}\right]$ has been already computed and is equal to $\sum_{j=1}^{n} (\mathrm{n}+1-\mathrm{j})^2 * \sigma_{[j]}^2$

$$\therefore \quad Var[\overline{L}] = \frac{1}{n^2}\left[\sum_{j=1}^{n} (\mathrm{n}+1-\mathrm{j})^2 \sigma_{[j]}^2 \right]$$

3.3.6 Maximum Lateness $(L_{\max})$

$$L_{\max} = \max\left(L_{[1]}, L_{[2]}, \ldots\ldots\ldots\ldots\ldots, L_{[n-1]}, L_{[n]}\right)$$

The parameter of concern, $L_{\max}$, is the maximum of '*n*' random variables and as stated earlier, Clark's equations can be recursively applied to approximately determine the expectation and variance under the assumption that the lateness variables are mutually related by the multivariate normal distribution.. Detailed computations involved in the determination of the expectation and variance of $L_{\max}$ are shown in the subsequent section.

The maximum function of '*N*' arguments can be recursively broken into '*N-1*' maximum functions as shown.

$$
\begin{aligned}
L_{\max} &= \max\left(\max\left(L_{[1]}, L_{[2]}, \ldots\ldots\ldots\ldots, L_{[n-1]}\right), L_{[n]}\right) \\
&= \max\left(\max\left(L_{[1]}, L_{[2]}, \ldots\ldots\ldots\ldots, L_{[n-1]}\right), L_{[n]}\right) \\
&= \max\left(\max\left(\max\left(L_{[1]}, L_{[2]}, \ldots\ldots, L_{[n-2]}\right), L_{[n-1]}\right), L_{[n]}\right) \\
&\vdots \\
&= \max(\ldots\ldots\max\left(\max\left(\max\left(L_{[1]}, L_{[2]}\right)L_{[3]}\right)L_{[4]}\right), \ldots\ldots\ldots, L_{[n-2]}\right), L_{[n-1]}\right), L_{[n]}\right)
\end{aligned}
$$

The mean and variance of the lateness of a job '*j*', $L_{[j]}$, are given by

$$\mu_{L[J]} = E\left[C_{[j]} - d_{[j]}\right] = E\left[C_{[j]}\right] - d_{[j]} = \sum_{k=1}^{j} \mu_{[k]} - d_{[j]}$$

$$\sigma^2_{L[J]} = Var\left[C_{[j]} - d_{[j]}\right] = Var[C_{[j]}] = \sum_{k=1}^{j} \sigma^2_{[k]}$$

Given that the processing times of the jobs are normally distributed, the random variable $L_{[j]}$ is also normally distributed with mean $\mu_{L[J]}$ and variance $\sigma^2_{L[J]}$.

Application of the Clark's Method:

The notations used are defined as follows,

$$r\,(L_{[i]}, L_{[j]}) \;=\; \textit{coefficient of linear correlation} = \;\rho_{[ij]}$$

$$a_{[ij]}^2 \;=\; \sigma_{L[i]}^2 + \sigma_{L[j]}^2 - 2\sigma_{L[i]}\sigma_{L[j]}\rho_{[ij]},$$

$$\alpha_{[ij]} \;=\; \frac{\mu_{L[i]} - \mu_{L[j]}}{a_{[ij]}},\; \varphi(z) \;=\; \frac{1}{\sqrt{2\pi}}\exp(-z^2/2) \text{ and } \Phi(x) \;=\; \int_{-\infty}^{x} \varphi(t)\,dt$$

Correlation analysis between the lateness random variables:

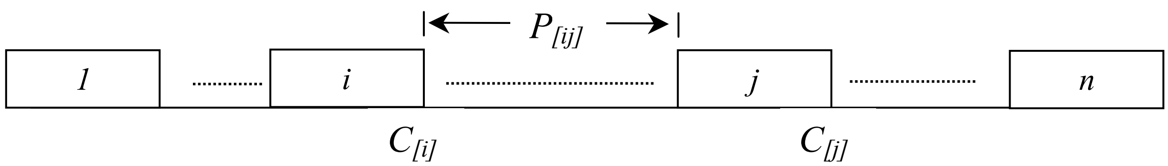


Figure 3.3. Representation of Completion Times of Jobs [*i*] and [*j*]

$$C_{[j]} \;=\; C_{[i]} + \{\, \textit{sum of processing times of all jobs between 'i' and 'j'} \,\} + p_{[j]}$$

$$C_{[j]} \;=\; C_{[i]} + P_{[ij]} + p_{[j]}$$

$$L_{[j]} \;=\; C_{[j]} - d_{[j]}$$

$$\;=\; C_{[i]} + P_{[ij]} + p_{[j]} - d_{[j]}$$

$$\;=\; L_{[i]} + P_{[ij]} + p_{[j]} - d_{[j]} - d_{[i]}$$

From this relation and as shown in Figure 3.4, it is evident that $L_{[i]}$ and $L_{[j]}$ are positively correlated with $\rho_{[ij]}$ = 1. Hence, the correlation coefficient is one for any pair of the lateness random variables.

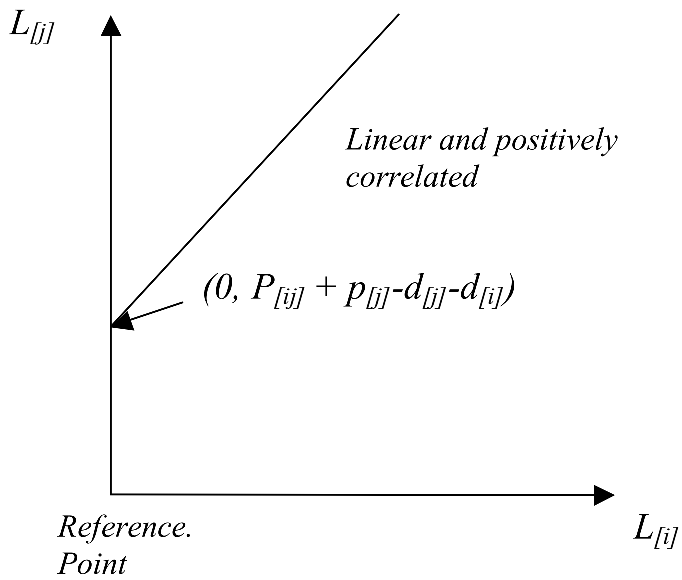


Figure 3.4. Plot between $L_{[i]}$ and $L_{[j]}$

Recursive method to determine the expectation and variance of $L_{\max}$ **:**

Expectation and Variance of $\max\,(L_{[1]}, L_{[2]})$ **:**

The expectation and variance of $\max\,(L_{[1]}, L_{[2]})$ are computed using Clark's equations as given below. The first moment, $E[\max\,(L_{[1]}, L_{[2]})]$, is given by,

$$\nu_{1[12]} \quad = \quad \mu_{[12]} \quad = \quad \mu_{L[1]}\Phi(\alpha_{[12]}) + \mu_{L[2]}\Phi(-\alpha_{[12]}) + a_{[12]}\varphi(\alpha_{[12]})$$

The second moment is given by

$$\nu_{2[12]} \quad = \quad \left(\mu_{L[1]}^2 + \sigma_{L[1]}^2\right)\Phi(\alpha_{[12]}) + \left(\mu_{L[2]}^2 + \sigma_{L[2]}^2\right)\Phi(-\alpha_{[12]}) + \left(\mu_{L[1]} + \mu_{L[2]}\right)a_{[12]}\varphi(\alpha_{[12]})$$

$$Var\left[\max\left(L_{[1]}, L_{[2]}\right)\right] \quad = \quad \nu_{2[12]} - \left(\nu_{1[12]}\right)^2$$

$$\sigma^2_{[12]} = \left(\mu^2_{L[1]} + \sigma^2_{L[1]}\right)\Phi(\alpha_{[12]}) + \left(\mu^2_{L[2]} + \sigma^2_{L[2]}\right)\Phi(-\alpha_{[12]}) + \left(\mu_{L[1]} + \mu_{L[2]}\right)a_{[12]}\varphi(\alpha_{[12]})$$
$$-\left(\mu_{L[1]}\Phi(\alpha_{[12]}) + \mu_{L[2]}\Phi(-\alpha_{[12]}) + a_{[12]}\varphi(\alpha_{[12]})\right)^2$$

$$\rho_{[123]} = r[L_{[3]}, \max(L_{[1]}, L_{[2]})] = [\sigma_{[1]}\rho_{[13]}\Phi(\alpha_{[12]}) + \sigma_{[2]}\rho_{[23]}\Phi(-\alpha_{[12]})]/\sigma_{[12]}$$

Expectation and Variance of $\max(L_{[1]}, L_{[2]}, L_{[3]})$**:**

$$\max(L_{[1]}, L_{[2]}, L_{[3]}) = \max(\max(L_{[1]}, L_{[2]}), L_{[3]})$$

$$\rho_{[123]} = r[L_{[3]}, \max(L_{[1]}, L_{[2]})]$$

$$a^2_{[123]} = \sigma^2_{[12]} + \sigma^2_{L[3]} - 2\sigma_{[12]}\sigma_{L[3]}\rho_{[123]}$$

$$\alpha_{[123]} = \frac{\mu_{[12]} - \mu_{L[3]}}{a_{[123]}}$$

The first moment of $E[\max(L_{[1]}, L_{[2]}, L_{[3]})]$ is given by,

$$\nu_{1[123]} = \mu_{[123]} = \mu_{L[12]}\Phi(\alpha_{[123]}) + \mu_{L[3]}\Phi(-\alpha_{[123]}) + a_{[123]}\varphi(\alpha_{[123]})$$

The second moment is given by

$$\nu_{2[123]} = \left(\mu^2_{L[12]} + \sigma^2_{L[12]}\right)\Phi(\alpha_{[123]}) + \left(\mu^2_{L[3]} + \sigma^2_{L[3]}\right)\Phi(-\alpha_{[123]}) +$$
$$\left(\mu_{L[12]} + \mu_{L[3]}\right)a_{[123]}\varphi(\alpha_{[123]})$$

$$Var[\max(L_{[1]}, L_{[2]}, L_{[3]})] = \nu_{2[123]} - (\nu_{1[123]})^2$$

$$\sigma^2_{[123]} = \left(\mu^2_{L[12]} + \sigma^2_{L[12]}\right)\Phi(\alpha_{[123]}) + \left(\mu^2_{L[3]} + \sigma^2_{L[3]}\right)\Phi(-\alpha_{[123]}) + \left(\mu_{L[12]} + \mu_{L[3]}\right)$$
$$a_{[123]}\varphi(\alpha_{[123]}) \quad - \left(\mu_{L[12]}\Phi(\alpha_{[123]}) + \mu_{L[3]}\Phi(-\alpha_{[123]}) + a_{[123]}\varphi(\alpha_{[123]})\right)^2$$

$$\begin{aligned} \rho_{[1234]} &= r[L_{[4]}, \max(L_{[1]}, L_{[2]}, L_{[3]})] \\ &= [\sigma_{[12]}\rho_{[124]}\Phi(\alpha_{[123]}) + \sigma_{[3]}\rho_{[34]}\Phi(-\alpha_{[123]})]/\sigma_{[123]} \end{aligned}$$

Thus, proceeding in a similar way and applying Clark's equations recursively in '*n-1*' steps, the expectation and variance of maximum lateness can be determined.

3.4 Concluding Remarks

In this chapter, we developed closed-from expressions for expectation and variance of various performance measures for sequencing on a single machine. The completion time based measures that we considered are the total completion time, total weighted completion time and total discounted weighted completion time. We developed generic expressions for each of the above measures and detailed computations were shown for the discounted weighted completion time measure when the job processing times follow exponential or normal distribution.

The tardiness based measures considered include total tardiness, total weighted tardiness, total number of tardy jobs, total weighted number of tardy jobs, mean lateness and maximum lateness. Generic expressions were developed for the mean lateness measure while for all the other measures the expectation and variance expressions were developed for normally distributed job processing times. Additionally, for the maximum lateness, approximate evaluations were obtained under the assumptions that the lateness random variables are mutually related by a multivariate normal distribution. Software programs developed for determining the various performance measures are provided in Appendix A. We now continue with our analysis for multi-machine models beginning with flow shops and proceed to job shops and parallel machine systems in the subsequent chapters.

CHAPTER 4: FLOW SHOP MODELS

4.1 Introduction

In Chapter 3, different single machine models were dealt with in detail and, now, we extend our expectation-variance analysis to multi-machine models. In this chapter, we consider a flow shop with 'm' machines and 'n' jobs waiting to be processed on all the machines. In a flow shop, the 'm' machines are arranged in series and all the jobs follow the same route. In our analysis, we study flow shops with unlimited intermediate storage (buffer capacity) and try to determine approximations for the expectation and variance of the makespan. Makespan is an important objective in a flow shop as a lower makespan value implies higher utilization of the machines. There are also other objectives related to due-dates and total completion time but they tend to be more complex and analytic evaluation of those models are much harder than the makespan models. We consider permutation flow shops in which the job sequence does not change between machines and also the jobs on each machine are processed on the *First Come First Served* principle. Understanding the working of a flow shop would aid in our evaluation of more complex systems like the job shop or parallel machines with preemptions.

The notations used in the analysis are as follows.

m	=	Number of machines
n	=	Number of jobs
$p_{[i,j]}$	-	Processing time of the job in the 'j'th position of the given permutation schedule on machine 'i' (random variable)
$\mu_{[i,j]}$	-	Mean or expected value of processing time of the job in the 'j'th position of the given permutation schedule on machine 'i'
$\sigma^2_{[i,j]}$	-	Variance of the processing time of the job in the 'j'th position of the given permutation schedule on machine 'i'
$C_{[i,j]}$	-	Completion time of the job in the 'j'th position of the given permutation

schedule on machine 'i' (random variable)

$S_{[i,j]}$ - Start time of the job in the 'j'th position of the given permutation schedule on machine 'i' (random variable)

4.2 Permutation Flow Shops with Unlimited Intermediate Storage (*Fm* | *prmu* | C_{max})

Given a permutation schedule for an 'm' machine flow shop, the completion time of the first job in the schedule on all the machines can be computed easily and is given by:

$$C_{[i,1]} = \sum_{l=1}^{l=i} p_{[l,1]} \qquad i = 1,2,\ldots\ldots\ldots\ldots m$$

Similarly, the completion time of all the jobs on the first machine,

$$C_{[1,j]} = \sum_{l=1}^{l=j} p_{[1,l]} \qquad j = 1,2,\ldots\ldots\ldots\ldots n$$

The completion times for all the other jobs on the machines can then be recursively found using the following expression:

$$C_{[i,j]} = \max(C_{[i-1,j]}, C_{[i,j-1]}) + p_{[i,j]} \qquad i = ,2\ldots\ \ldots m; \quad j = ,2\ldots\ldots\ldots n$$

where $\max(C_{[i-1,j]}, C_{[i,j-1]})$ is the possible starting time of job 'j' on machine 'i' and can be denoted as $S_{[i,j]}$. The starting time of a job 'j' on a machine 'i' is determined by the completion time of job 'j' on the previous machine '$i-1$' and the completion time of the previous job '$j-1$' in the sequence on machine 'i'.

$$C_{[i,j]} = S_{[i,j]} + p_{[i,j]} \qquad i = ,2\ldots\ \ldots m; \quad j = ,2\ldots\ldots\ldots n$$

The makespan of the given permutation schedule would then be the completion time of the last job on the last machine '*m*' in the given schedule.

$$C_{\max} = C_{[m,n]} = S_{[m,n]} + p_{[m,n]} = \max(C_{[m-1,n]}, C_{[m,n-1]}) + p_{[m,n]}$$

Network representation of a flow shop (*Fm* | *prmu* | C_{max}):

The value of the makespan under a given permutation schedule can also be computed by determining the critical path in a directed graph corresponding to the given schedule. A network can be nicely created for a given sequence of the flow shop problem as shown in Figure 4.1 below. Each node in the network represents an operation and carries a weight equal to the processing time of that operation. Node (*i,j*) , *i = 1,2……..,m-1*, and *j = 1,2,………,n-1*, has arcs going to nodes (*i+1,j*) and (*i,j+1*). Nodes corresponding to machine '*m*' have only one outgoing arc, as do the nodes corresponding to job '*n*'. Node (*m,n*) is the last operation and has no outgoing arcs. The total weight of the maximum weight path from node (*1,1*) to (*m,n*) corresponds to the makespan under the given permutation sequence *1,2,3,…………..,n*.

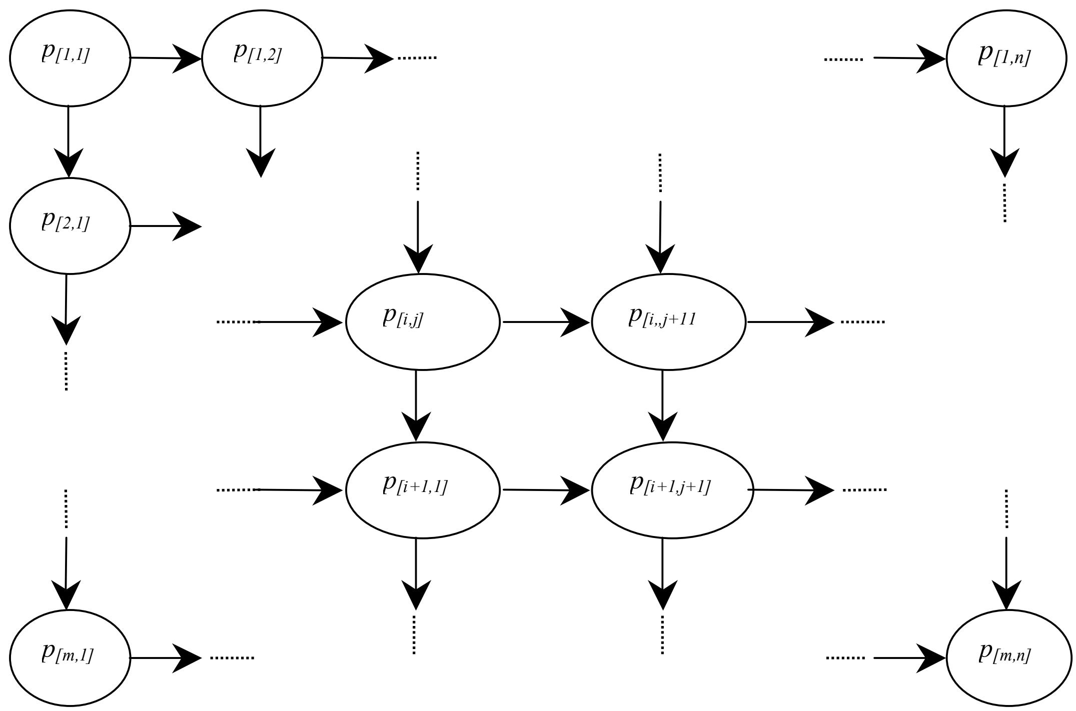


Figure 4.1. Directed Graph for the Computation of the Makespan in *Fm* | *prmu* | C_{max}

Expectation and variance of makespan:

As there are no standard evaluations available for the maximum operator, it becomes necessary to use Clark's equations for approximate evaluation of the makespan under the assumption that all the start and finish times are mutually related by a multivariate normal distribution. $C_{\max}$ can be found by iteratively computing the earlier completion times and their correlations. Wilhelm and Ahmadi-Marandi (1982) used a seven-step iterative procedure to determine the correlations necessary to evaluate the maximum function for their assembly system. They incorporated stochastic part arrival times in their assembly system model while the processing times were assumed to be deterministic. Flow shop is a special case of an assembly system and a similar approach can be adopted to determine the correlations and with stochastic processing times. We hereby show a recursive procedure to determine the correlations which are then used to evaluate the mean and variance of the completion times.

SRecursive procedure to determine the correlation, $r(C_{[i-1,j]}, C_{[i,j-1]})$, and hence evaluate the mean and variance of the completion time $C_{[i,j]}$:

The recursive procedure can be better understood from the following hierarchical structure of the completion times.

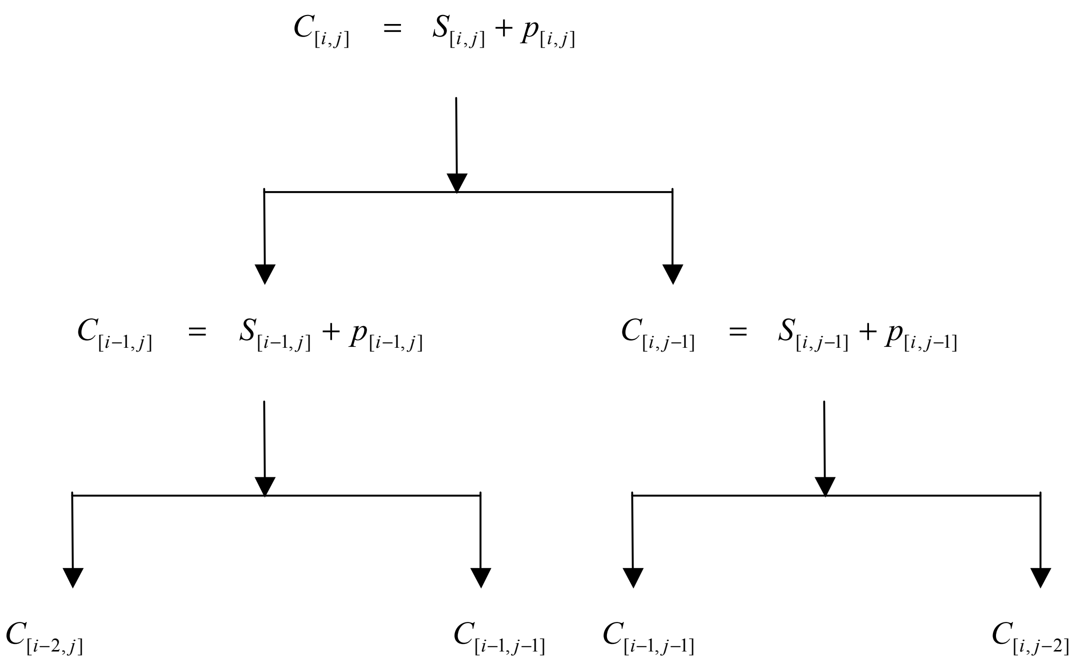


Figure 4.2. Hierarchical Structure Showing the Association between Completion Times

As expressed earlier, the completion time of the '*j*'th job on machine '*i*' is given by

$$C_{[i,j]} = S_{[i,j]} + p_{[ij]} = \max(C_{[i-1,j]}, C_{[i,j-1]}) + p_{[ij]}$$

We know the mean and variance of the completion times, $C_{[i-1,j]}$ and $C_{[i-1,j]}$ from the earlier iterations. The complexity lies in determining the crucial correlation factor between $C_{[i-1,j]}$ and $C_{[i-1,j]}$, denoted by $r(C_{[i-1,j]}, C_{[i,j-1]})$, which is required for use in the Clark's equations. The starting times for operations (*i-1,j*) and (*i,j-1*), $S_{[i-1,j]}$ and $S_{[i-1,j]}$, in turn depend on the previous completion times as shown in the Figure 4.2 above.

At the beginning of every iteration, the following parameters are known.

$r(C_{[i-1,j-1]}, C_{[i-1,j-1]}) \;=\; 1$, by definition

$r(C_{[i-1,j-1]}, C_{[i,j-2]})$ (known from the previous iterations)

$r(C_{[i-2,j]}, C_{[i-1,j-1]})$ (known from the previous iterations)

$r(C_{[i-2,j]}, C_{[i,j-2]}) \;=\; 0$ (assumption)

For the sake of computational simplicity, Wilhelm and Ahmadi-Marandi had assumed the correlation $r(C_{[i-2,j]}, C_{[i,j-2]})$ to be zero as those two operations are two machines and two jobs apart. Although it is possible to determine this correlation by proceeding further down the hierarchical, it would only make the computations cumbersome and time consuming for larger problems. It is also reasonable to surmise that the correlations get weaker as we go further down the hierarchical since the machines and jobs get further apart from each other. Wilhelm (1986) had, in fact, analyzed the importance of correlations between these random variables in a flow shop and identified a number of inherent relationships amongst them to compute better estimates of the transient performance of a flow shop with finite and infinite buffer capacities. However, we adopt the null correlation assumption for $r(C_{[i-2,j]}, C_{[i,j-2]})$ and provide a 4-step procedure to compute the mean and expectation for the completion times when the processing times are probabilistic.

Step 1:

Determine the correlation $\rho_1 = r(C_{[i-2,j]}, C_{[i,j-1]})$

$$\begin{aligned}
\rho_1 &= r(C_{[i-2,j]}, C_{[i,j-1]}) \\
&= r(C_{[i-2,j]}, \max(C_{[i-1,j-1]}, C_{[i,j-2]}) + p_{[i,j-1]}) \\
&= r(C_{[i-2,j]}, \max(C_{[i-1,j-1]} + p_{[i,j-1]}, C_{[i,j-2]} + p_{[i,j-1]}))
\end{aligned}$$

The processing time, $p_{[i,j-1]}$, is an independent random variables and, hence, the pair-wise correlations still hold true between the completion times to determine ρ_1.

Step 2:

Determine the correlation $\rho_2 = r(C_{[i-1,j-1]}, C_{[i,j-1]})$

$$\begin{aligned}\rho_2 &= r(C_{[i-1,j-1]}, C_{[i,j-1]}) \\ &= r(C_{[i-1,j-1]}, \max(C_{[i-1,j-1]}, C_{[i,j-2]}) + p_{[i,j-1]}) \\ &= r(C_{[i-1,j-1]}, \max(C_{[i-1,j-1]} + p_{[i,j-1]}, C_{[i,j-2]} + p_{[i,j-1]}))\end{aligned}$$

Step 3:

Determine the correlation $\rho = r(C_{[i-1,j]}, C_{[i,j-1]})$

$$\begin{aligned}\rho &= r(C_{[i-1,j]}, C_{[i,j-1]}) \\ &= r(\max(C_{[i-2,j]}, C_{[i-1,j-1}) + p_{[i-1,j]}, C_{[i,j-1]}) \\ &= r(\max(C_{[i-2,j]} + p_{[i-1,j]}, C_{[i-1,j-1} + p_{[i-1,j]}), C_{[i,j-1]})\end{aligned}$$

As stated earlier, processing times are independent random variables and, hence, the pairwise correlations still hold true between the completion times to determine ρ.

Step 4:

Determine the mean and variance of $C_{[i,j]}$

Once $\rho = r(C_{[i-1,j]}, C_{[i,j-1]})$ is known, the mean and variance of the starting time of operation (*i,j*), $S_{[i,j]}$, can then be found using the first two moments from Clark's work. The correlation ρ will then be used in the successive iterations for evaluating other completion times.

The mean and variance of the completion time, $C_{[i,j]}$, are then given by

$$E[C_{[i,j]}] \quad = \quad E[S_{[i,j]}] + E[p_{[i,j]}] = \quad E[S_{[i,j]}] + \mu_{[i,j]}$$

$$Var[C_{[i,j]}] \quad = \quad Var[S_{[i,j]}] + Var[p_{[i,j]}] = \quad Var[S_{[i,j]}] + \sigma^2_{[i,j]}$$

The iterative procedure begins for $C_{[2,2]}$ and the correlations $r(C_{[1,1]}, C_{[2,0]})$ and $r(C_{[0,2]}, C_{[1,1]})$ are assumed to be zero. The terms $C_{[0,j]}$ and $C_{[i,0]}$ can be construed as job arrival times and machine ready times respectively, which are deterministic and hence any correlation involving them is set to zero.

4.3 Concluding Remarks

In this chapter, we have analyzed the working of a permutation flow shop with unlimited buffer capacity and have detailed a 4-step recursive procedure for determining the correlation coefficients involved in computing the operation finishing times. Software program developed for determining the makespan for a flow shop is provided in Appendix B.

CHAPTER 5: JOB SHOP MODELS

5.1 Introduction

In this chapter, we analyze the performance of a job shop for the makespan objective. The classical job shop problem differs from the flow shop problem in the fact that the jobs do not follow the same route, i.e., the flow of work is not unidirectional. The system consists of 'm' machines and 'n' jobs and each job follows its own sequence of operations. It is not necessary for the jobs to visit all the machines. If the job has to visit a machine more than once before its completion, then it is said to recirculate. Recirculation is quite common in semiconductor manufacturing environments and is a complex phenomenon to analyze. In our analysis, we focus on the classical job shop problem with makespan objective and no recirculation. An identical approach is applicable for the recirculation case too.

The notations used in the analysis are as follows.

m - Number of machines

n - Number of jobs

j - Position index

k - Job index

i - Machine index

$p_{[i,j]}$ - Processing time of the job in the 'j'th position on machine 'i' of the given schedule (random variable)

$\mu_{[i,j]}$ - Mean or expected value of the processing time of the job in the 'j'th position on machine 'i' of the given schedule (random variable)

$\sigma^2_{[i,j]}$ - Variance of the processing time of the job in the 'j'th position on machine 'i' of the given schedule (random variable)

$C_{[i,j]}$ - Completion time of the processing time of the 'j'th job on machine 'i' (random variable)

$S_{[i,j]}$ - Start time of the processing time of the 'j'th job on machine 'i' (random variable)

S^i - {Processing sequence for jobs on machine 'i' in the given schedule} (Set of job indices)

N^i - $n(S^i)$ - Number of jobs processed on machine 'i'

R^k - {Processing sequence for job 'k'} (Set containing machine indices)

Job(i,j) - Function that returns the index of the job in the 'j'th position in machine 'i' sequence

Pos(i,k) - Function that returns the position of the job 'k' in the machine 'i' sequence

Pre(i,k) - Function that returns the index of the machine preceding machine 'i' in the processing sequence for job 'j'.

Subscripts written without brackets refer to the particular index and are not indicative of the positions. For example, C_{ik} represents the completion time of job 'k' on machine 'i' and does not refer to the position of 'k' in machine 'i' sequence.

5.2 Job Shops with Unlimited Intermediate Storage and No Recirculation *Jm* || C_{max}

The makespan, defined as $C_{\max} = \max(C^1_{\max}, C^2_{\max}, \ldots\ldots\ldots\ldots, C^m_{\max})$ is equivalent to the completion time of the last job to leave the system and $C^i_{\max}$ is the completion time of the last job on machine 'i'.

$$C^i_{\max} = C_{[i,N^i]} = S_{[i,N^i]} + p_{[i,N^i]}, \qquad \textit{for } i = 1,2,\ldots\ldots,m \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5.1)$$

However, the starting time of the last job on machine '*i*' depends on the completion time of the previous job in the machine sequence and also its completion time on the preceding machine in its sequence.

In general, the completion time of the job in the '*j*'th position on machine '*i*' is given by

$$C_{[i,j]} = S_{[i,j]} + p_{[i,j]}, \quad \text{for } i = 1,2,\ldots\ldots,m, \text{ and } j = 1,2,\ldots\ldots,n \quad \ldots\ldots\ldots\ldots\ldots (5.2)$$

However, $S_{[i,j]}$ is further given by,

$$S_{[i,j]} = \max(C_{[i,j-1]}, C_{[p,q]}), \text{ where } p = Pre(i,Job(i,j-1)) \text{ and } q = Pos(p,Job(i,j-1))$$

$$\text{for } i = 1,2,\ldots\ldots,m, \text{ and } j = 1,2,\ldots\ldots,n \quad \ldots\ldots\ldots(5.3)$$

The relations (5.2) and (5.3) have to be recursively used to determine (5.1).

The expectation and makespan of the completion time of the job in the '*j*'th position on machine '*i*' is then given by

$$E[C_{[i,j]}] = E[S_{[i,j]}] + E[p_{[i,j]}] = E[\max(C_{[i,j-1]}, C_{[p,q]})] + \mu_{[i,j]} \quad \ldots\ldots\ldots(5.4)$$

$$Var[C_{[i,j]}] = Var[S_{[i,j]}] + Var[p_{[i,j]}] = Var[\max(C_{[i,j-1]}, C_{[p,q]})] + \sigma^2_{[i,j]} \quad \ldots\ldots\ldots(5.5)$$

Expectation and variance of makespan:

As was noted earlier for the flow shops, the maximum operator has to be evaluated using Clark's equations for approximate evaluation of the makespan under the assumption that all the start and finish times are mutually related by a multivariate normal distribution. However, the computational procedure is much more tedious than it is for the flow shops. The correlation structure for the flow shops followed a generic pattern and it was easier to understand and evaluate the pair-wise correlations. Additionally, assumption with regards to the correlation $r(C_{[i-2,j]}, C_{[i,j-2]})$ (refer p.58) made the computations relatively simpler. On the other hand, the job shop correlation structure is much more complicated and involves complete enumeration of all the correlations to compute the means and variances of the completion times. An illustrative example is provided below to show the steps involved in the computation procedure.

Sample calculations for an example job shop problem:

Consider a *3*-machine-*3*-job job shop and with the following the job routings

Job ***1***: Machines: ***a – b – c***

Job ***2***: Machines: ***c – b – a***

Job ***3***: Machines: ***b – a – c***

The sequence of computations is shown for an arbitrary schedule given below.

Machine ***a***: Jobs ***1 -2 -3***

Machine ***b***: Jobs ***3-1-2***

Machine ***c***: Jobs ***2-1-3***

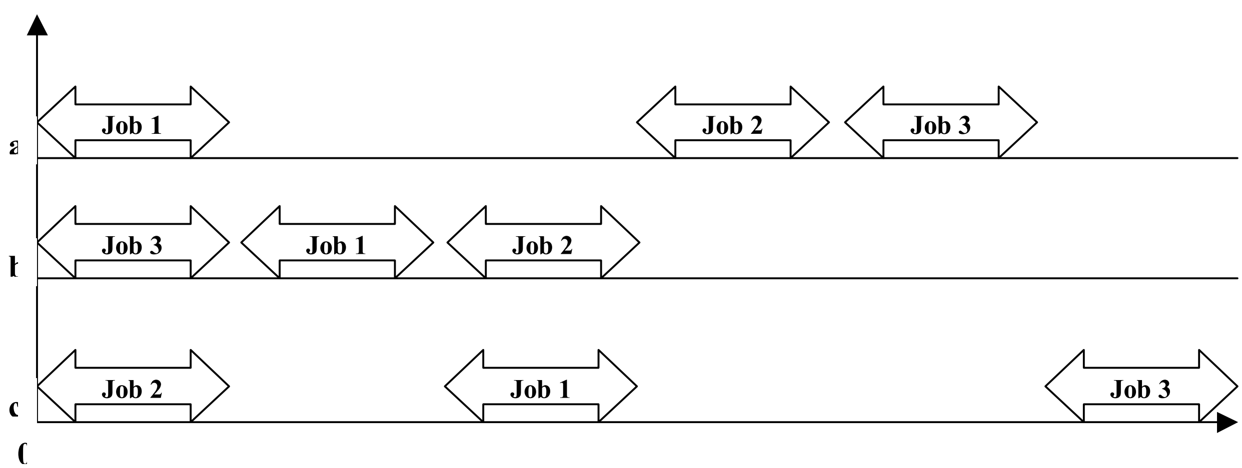


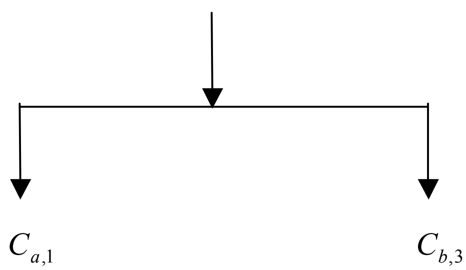
Figure 5.1. A Gantt Chart that Shows the Relative Positions of the Jobs for the Job Shop Example

The Gantt chart shown in Figure 5.1 above is drawn for the given schedule. It is not possible to ascertain the completion times as the processing times are probabilistic. Hence the relative positions of the jobs are schematically shown. For the given schedule, jobs *1*, *2* and *3* can start their first operations on jobs *a*, *c* and *b* respectively at time $t = 0$ and they can be referred to as the *initial operations*. These operations are independent of each other and hence there exists no correlation among them. However, for other subsequent operations it is necessary to recursively compute the correlations until the *initial operation*s are reached.

Sample Computations:

Operation (b,1):

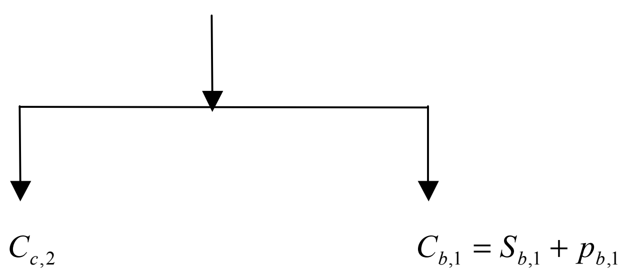
$$C_{b,1} \quad = \quad S_{b,1} + p_{b,1}$$

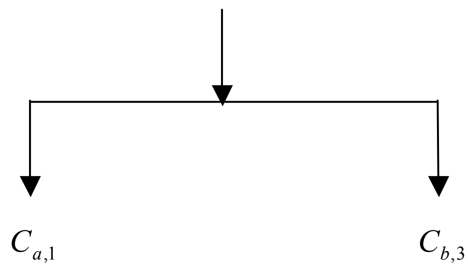


Since operations (*a,1*) and (*b,3*) are initial operations and independent, $r(C_{a,1}, C_{b,3}) = 0$

Operation (b,2):

$$C_{b,2} \quad = \quad S_{c,2} + p_{b,1}$$

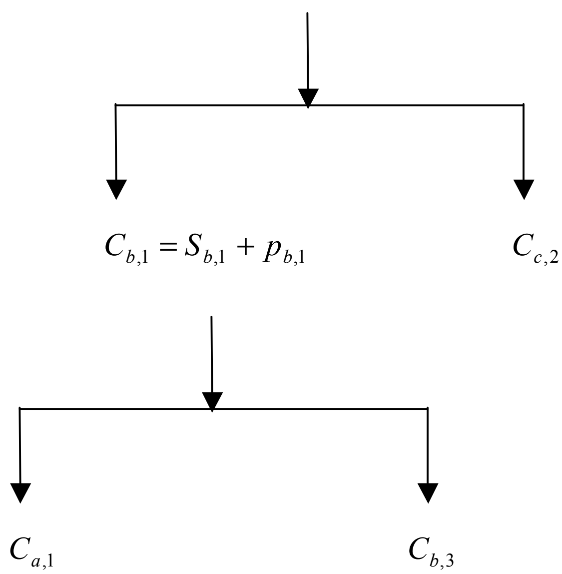




$r(C_{c,2}, C_{b,1})$ = $r(C_{c,2}, \max(C_{a,1}, C_{b,3}) + p_{b,1}$ = 0, as the three operations involved are all initial operations.

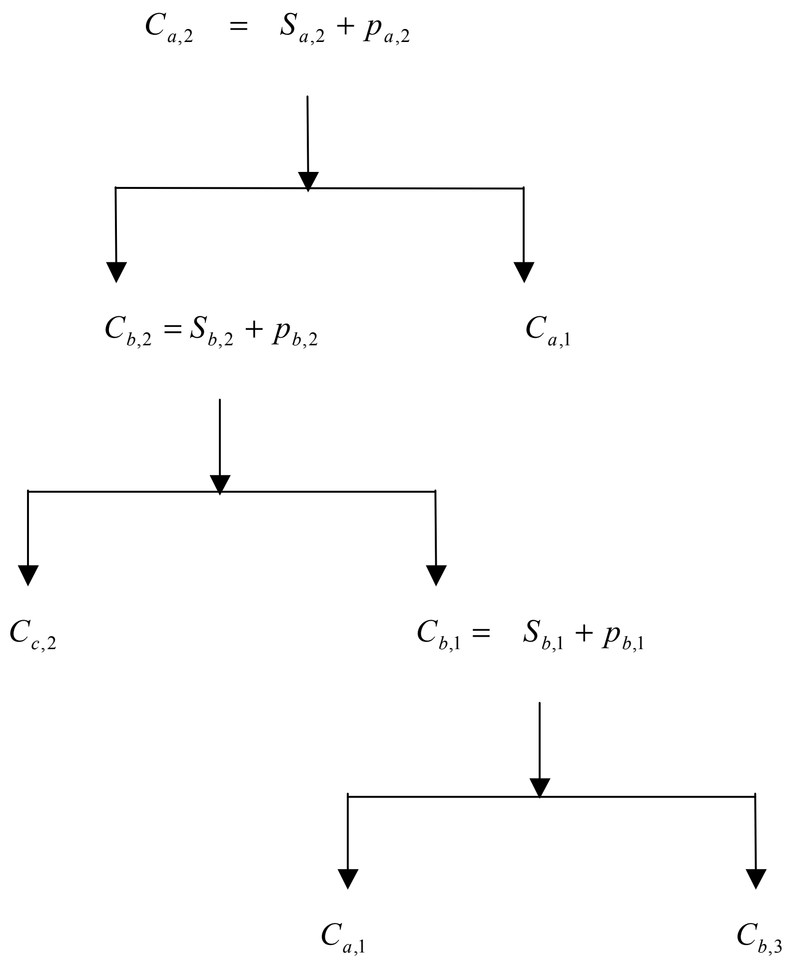
Operation (c,1):

$$C_{c,1} = S_{c,1} + p_{c,1}$$



$r(C_{b,1}, C_{c,2})$ = $r(\max(C_{a,1}, C_{b,3}) + p_{b,1}, C_{c,2})$ = 0, as computed from the earlier step.

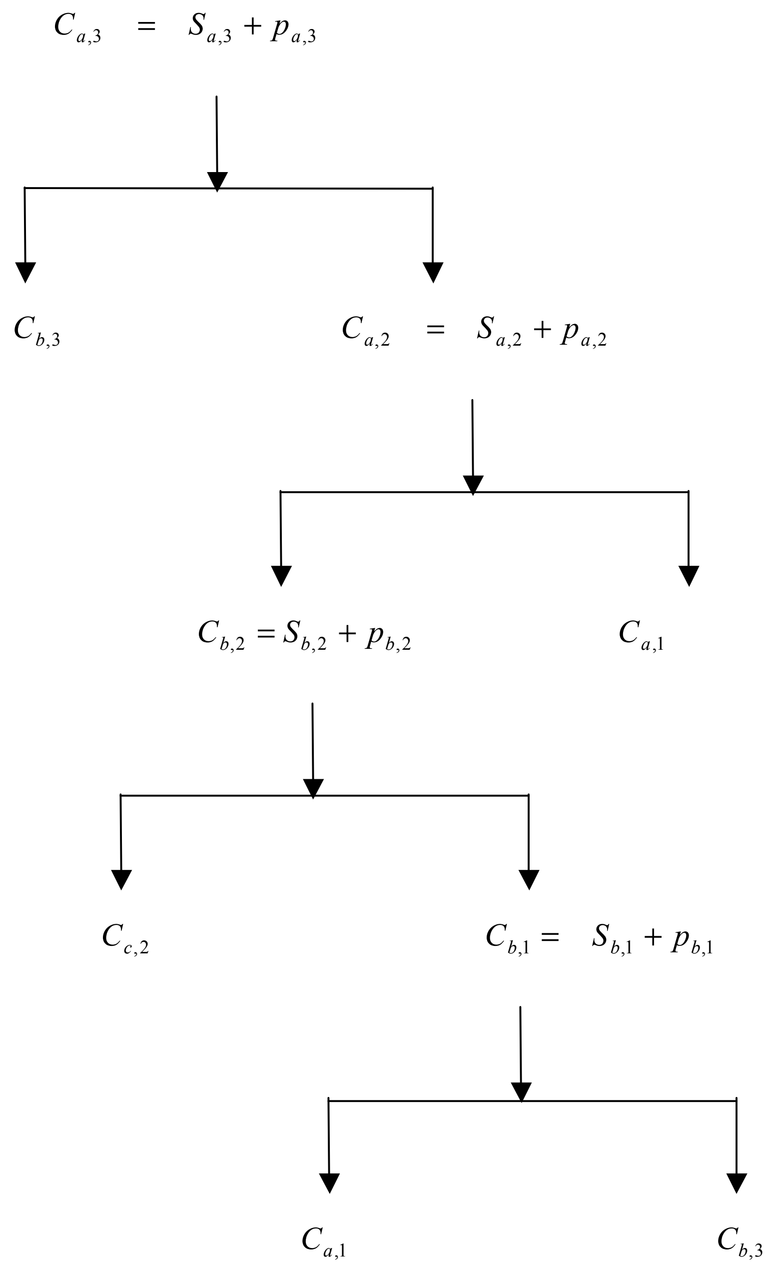
Operation (a,2):



The steps involved in computing $r(C_{b,2}, C_{a,1})$ are as follows:

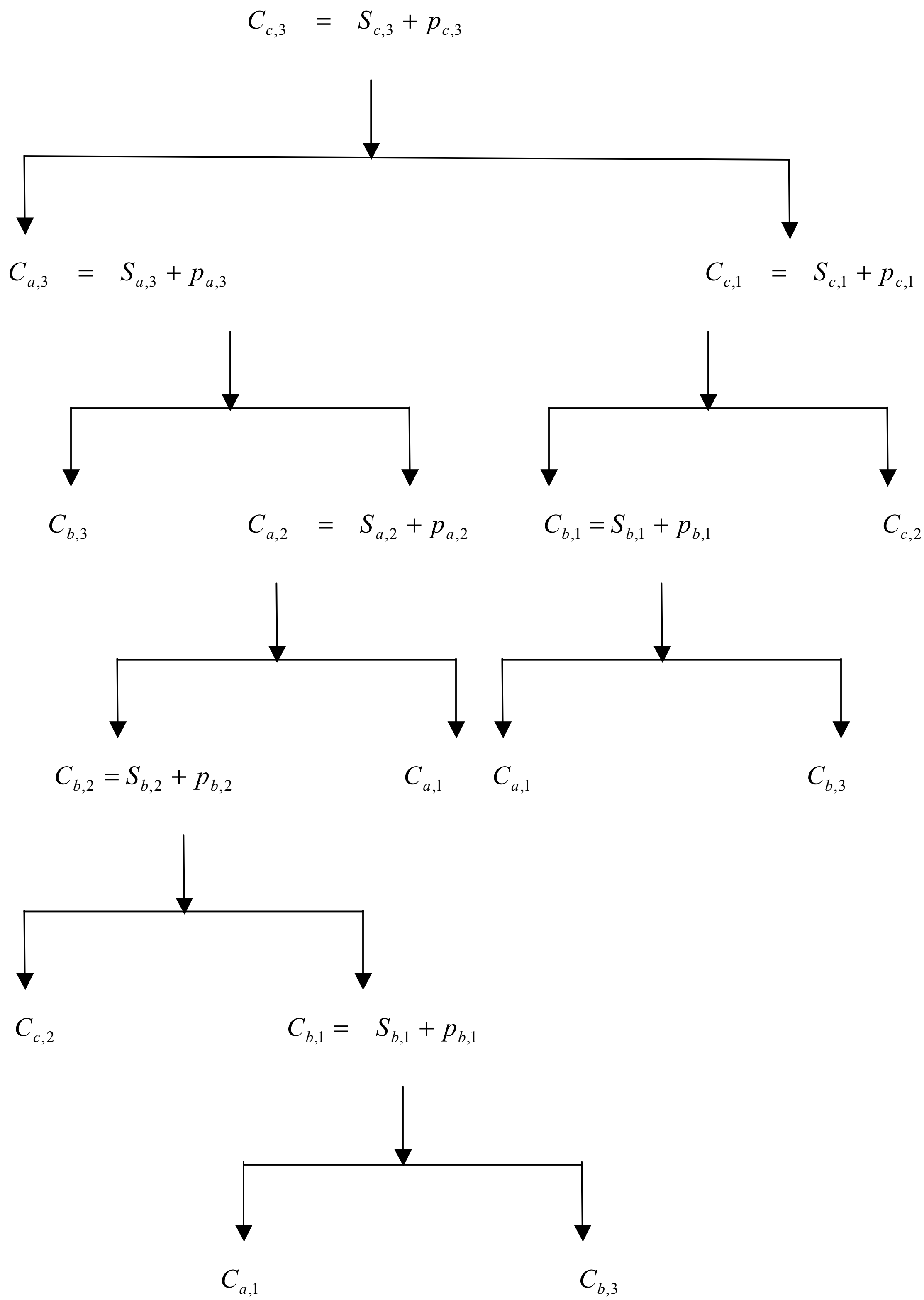
1. $r(C_{b,2}, C_{a,1}) = r(C_{b,2}, C_{a,1}) = r(\max(C_{c,2}, C_{b,1}) + p_{b,2}, C_{a,1})$
2. Considering the individual correlations $r(C_{c,2}, C_{a,1})$ and $r(C_{b,1}, C_{a,1})$,
   $r(C_{c,2}, C_{a,1})$ = 0, as they are initial operations and
   $r(C_{b,1}, C_{a,1}) = r(\max(C_{a,1}, C_{b,3}) + p_{b,1}, C_{a,1})$ = some positive number. It can be found from Clark's equations using $r(C_{a,1}, C_{a,1})$ = 1 and $r(C_{a,1}, C_{b,3})$ = 0.
3. Using these correlations in the earlier steps, $r(C_{b,2}, C_{a,1})$ can be finally found.

Operation (a,3):



The required correlation $r(C_{b,2}, C_{a,1})$ can be found by knowing the pair-wise correlations between $C_{b,3}, C_{b,2}$ and $C_{a,1}$. The correlations $r(C_{b,2}, C_{a,1})$ (*from the previous iteration*) and $r(C_{b,3}, C_{a,1})$ (*= 0, initial operation*) are known and the only unknown correlation is $r(C_{b,2}, C_{b.2})$. It can be found separately using a similar approach. Using these three correlations in Clarke's correlation equation would provide us with $r(C_{b,2}, C_{a,1})$.

Operation (c,3):



The final correlation $r(C_{a,3}, C_{c,1})$ can be found from this structure similar to earlier iterations.

In general, the recursive procedure to compute the completion times for a job shop can be summarized as follows:

Step 1:
Identify the jobs that can be started at time t = 0 and designate them as *initial operations*. Compute the mean and variance of their completion times. Set the correlation coefficients between them to be zero.

Step 2:
Identify the next schedulable job '*k*' for the given schedule and trace the preceding operations that affect its starting time and develop the hierarchical structure. The structure is expanded until the initial operations are reached.

Step 3:
Find the correlation coefficients recursively and determine the mean and expectation for the operation start time.

Step 4:
Compute the mean and variance of its completion time using the relation

$$C_{i,k} = S_{i,k} + p_{i,k}$$

Step 5:
Repeat steps 2-4 until all the jobs are scheduled.

Determining the makespan:

After computing the mean and variances of the completion times for all the operations, the makespan of the given schedule can be evaluated by using Clarke's equations recursively in a manner similar to finding the maximum lateness for the single machine model.

The makespan $C_{\max} \quad = \quad \max\,(C^1_{\max}, C^2_{\max}, \ldots\ldots\ldots\ldots, C^m_{\max})$ and

$C^i_{\max} \quad = \quad C_{[i,N^i]} \quad = \quad S_{[i,N^i]} + p_{[i,N^i]},$ *for i = 1,2,……,m*

The maximum function of '*m*' arguments can be recursively broken into '*m-1'* maximum functions as follows:

$$
\begin{aligned}
C_{\max} &= \max\left(C^1_{\max}, C^2_{\max}, \ldots\ldots\ldots\ldots, C^{m-1}_{\max}, C^m_{\max}\right) \\
&= \max\left(\max\left(C^1_{\max}, C^2_{\max}, \ldots\ldots\ldots\ldots, C^{m-1}_{\max}\right), C^m_{\max}\right) \\
&= \max\left(\max\left(\max\left(C^1_{\max}, C^2_{\max}, \ldots\ldots, C^{m-2}_{\max}\right), C^{m-1}_{\max}\right), C^m_{\max}\right) \\
&\vdots \\
&= \max(\ldots\ldots\max\left(\max\left(\max\left(C^1_{\max}, C^2_{\max}\right), C^3_{\max}\right), C^4_{\max}\right), \ldots\ldots\ldots C^{m-2}_{\max}\right), C^{m-1}_{\max}\right), C^m_{\max}\right)
\end{aligned}
$$

The mean and variance of $C^i_{\max}$ is the mean and variance of the completion time of the last job in its sequence which has been determined earlier.

$\mu_{C[i]} \quad = \quad E\left[C_{[i,N^i]}\right]$ and $\sigma^2_{C[i]} \quad = \quad Var[C_{[i,N^i]}]$

Application of the Clark's Method:

Given that the processing times of the jobs are normally distributed and $C^i_{\max}$'s are mutually related by a multivariate normal distribution, $C_{\max}$ can be found as follows.

The notations used are defined as follows,

$r\,(C^i_{\max}, C^j_{\max}) \quad = \quad$ *coefficient of linear correlation* $= \quad \rho_{[ij]}$

The correlation, $\rho_{[IJ]}$ is the correlation between the completion times of the last jobs on machine 'i' and machine 'j', $r(C_{[i,N^1]}, C_{[j,N^2]})$. This can be computed, if they do not exist from earlier computations, using the correlation procedure given in the previous section.

$$a_{[ij]}^2 = \sigma_{C[i]}^2 + \sigma_{C[j]}^2 - 2\sigma_{C[i]}\sigma_{C[j]}\rho_{[ij]},$$

$$\alpha_{[ij]} = \frac{\mu_{C[i]} - \mu_{C[i]}}{a_{[ij]}}, \ \varphi(z) = \frac{1}{\sqrt{2\pi}}\exp(-z^2/2) \text{ and } \Phi(x) = \int_{-\infty}^{x}\varphi(t)\,dt$$

Recursive method to determine the expectation and variance of $C_{\max}$:

Expectation and Variance of max $(C_{\max}^1, C_{\max}^2)$**:**

The expectation and variance of max $(C_{\max}^1, C_{\max}^2)$ are computed using Clark's equations as given below.

The first moment, $E[\max(C_{\max}^1, C_{\max}^2)]$, is given by,

$$\nu_{[12]} = \mu_{[12]} = \mu_{C[1]}\Phi(\alpha_{[12]}) + \mu_{C[2]}\Phi(-\alpha_{[12]}) + a_{[12]}\varphi(\alpha_{[12]})$$

The second moment is given by

$$\nu_{2[12]} = \left(\mu_{C[1]}^2 + \sigma_{C[1]}^2\right)\Phi(\alpha_{[12]}) + \left(\mu_{C[2]}^2 + \sigma_{C[2]}^2\right)\Phi(-\alpha_{[12]}) + \left(\mu_{C[1]} + \mu_{C[2]}\right)a_{[12]}\varphi(\alpha_{[12]})$$

$$Var\left[\max(C_{\max}^1, C_{\max}^2)\right] = \nu_{2[12]} - \left(\nu_{1[12]}\right)^2$$

$$\sigma_{[12]}^2 = \left(\mu_{C[1]}^2 + \sigma_{C[1]}^2\right)\Phi(\alpha_{[12]}) + \left(\mu_{C[2]}^2 + \sigma_{C[2]}^2\right)\Phi(-\alpha_{[12]}) + \left(\mu_{C[1]} + \mu_{C[2]}\right)a_{[12]}\varphi(\alpha_{[12]})$$
$$-\left(\mu_{C[1]}\Phi(\alpha_{[12]}) + \mu_{C[2]}\Phi(-\alpha_{[12]}) + a_{[12]}\varphi(\alpha_{[12]})\right)^2$$

$$\rho_{[123]} = r[C^3_{\max}, \max(C^1_{\max}, C^2_{\max})] = [\sigma_{C[1]}\rho_{[13]}\Phi(\alpha_{[12]}) + \sigma_{C[2]}\rho_{[23]}\Phi(-\alpha_{[12]})]/\sigma_{[12]}$$

Expectation and Variance of $\max(C^1_{\max}, C^2_{\max}, C^3_{\max})$**:**

$$\max(C^1_{\max}, C^2_{\max}, C^3_{\max}) = \max\left(\max\left(C^1_{\max}, C^2_{\max}\right), C^3_{\max}\right)$$

$$\rho_{[123]} = r[C^3_{\max}, \max(C^1_{\max}, C^2_{\max})]$$

$$a^2_{[123]} = \sigma^2_{[12]} + \sigma^2_{C[3]} - 2\sigma_{[12]}\sigma_{C[3]}\rho_{[123]}$$

$$\alpha_{[123]} = \frac{\mu_{[12]} - \mu_{L[3]}}{a_{[123]}}$$

The first moment of $E[\max(C^1_{\max}, C^2_{\max}, C^3_{\max})]$ is given by,

$$\nu_{1[123]} = \mu_{[123]} = \mu_{[12]}\Phi(\alpha_{[123]}) + \mu_{C[3]}\Phi(-\alpha_{[123]}) + a_{[123]}\varphi(\alpha_{[123]})$$

The second moment is given by

$$\nu_{2[123]} = \left(\mu^2_{[12]} + \sigma^2_{[12]}\right)\Phi(\alpha_{[123]}) + \left(\mu^2_{C[3]} + \sigma^2_{C[3]}\right)\Phi(-\alpha_{[123]}) + \left(\mu_{[12]} + \mu_{C[3]}\right)a_{[123]}\varphi(\alpha_{[123]})$$

$$Var\left[\max(C^1_{\max}, C^2_{\max}, C^3_{\max})\right] = \nu_{2[123]} - \left(\nu_{1[123]}\right)^2$$

$$\sigma^2_{[123]} = \left(\mu^2_{[12]} + \sigma^2_{[12]}\right)\Phi(\alpha_{[123]}) + \left(\mu^2_{C[3]} + \sigma^2_{C[3]}\right)\Phi(-\alpha_{[123]}) + \left(\mu_{[12]} + \mu_{C[3]}\right) a_{[123]}\varphi(\alpha_{[123]}) - \left(\mu_{[12]}\Phi(\alpha_{[123]}) + \mu_{C[3]}\Phi(-\alpha_{[123]}) + a_{[123]}\varphi(\alpha_{[123]})\right)^2$$

$$\rho_{[1234]} = r[C^4_{\max}, \max(C^1_{\max}, C^2_{\max}, C^3_{\max})] = [\sigma_{[12]}\rho_{[124]}\Phi(\alpha_{[123]}) + \sigma_{C[3]}\rho_{[34]}\Phi(-\alpha_{[123]})]/\sigma_{[123]}$$

Thus, proceeding in a similar way and applying Clark's equations recursively for '*m-1*' steps, the expectation and variance of the makespan can be determined.

5.3 Job Shops with Unlimited Intermediate Storage and with Recirculation $Jm \mid recrc \mid C_{max}$

Job shops with recirculation can be modeled in a fashion identical to that of the case with no recirculation. A recirculating job can be treated independently irrespective of the fact that it was processed on the same machine earlier. In the earlier job shop example, if job *'1'* visits '*b'* after completing its operation on *'c'* it is said to recirculate. However, the methodology explained earlier is still applicable as this operation is independent of its previous operation on the same machine. It only depends on the completion time of its predecessor on the same machine (*b,2*) and its completion time on the previous machine (*c,1*). Hence the solution methodology is the same irrespective of recirculation or not.

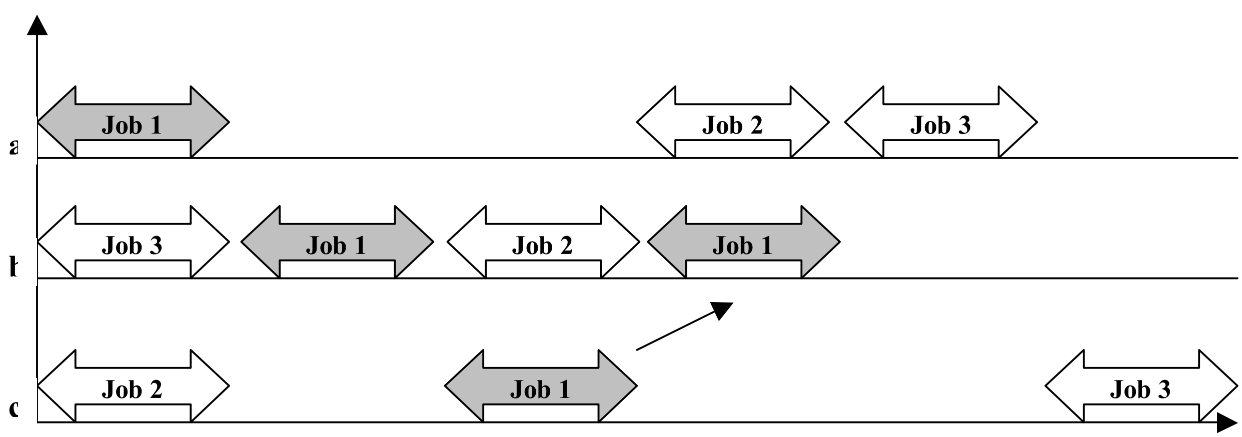


Figure 5.2. A Gantt Chart for the Job Shop Example with Recirculation

5.4 Concluding Remarks

In this chapter, we have analyzed the classical job shop problem with unlimited buffer capacity and have detailed a generic recursive procedure for determining the correlation coefficients involved in computing the operation completion times. This procedure is applicable for both the job recirculation and no recirculation cases. The makespan can then be found using Clark's equations recursively which is also outlined. Software program developed for determining the makespan for a job shop is provided in Appendix C.

CHAPTER 6: PARALLEL MACHINE MODELS

6.1 Introduction

In this setup, there are 'm' similar machines in parallel and 'n' jobs to be processed on these machines. When dealing with parallel machines, makespan becomes an objective of considerable interest as opposed to the single machine case where the makespan is invariant of job sequences. Preemptions also play a very important role in this case as compared to that in a single machine. Preemptions imply that it is not necessary to keep a job on a machine, once started, until completion. The processing of the job can be interrupted and the remaining work can be completed on the same machine or on any of the other machines. In our analysis, we consider preemption and no preemption cases separately.

6.2 Parallel Machines with No Preemptions

The two performance measures considered for the no preemption case are makespan and total completion time.

Some additional notations and modifications to the notations used in the earlier chapters are as follows.

m - number of parallel machines

$p_{[i,j]}$ - processing time of the job in the 'j'th position assigned to machine 'i' in the given schedule

$\mu_{[i,j]}$ - Mean or expected value of the processing time of the job in the 'j'th position assigned to machine 'i' in the given schedule

$\sigma^2_{[i,j]}$ - Variance of the processing time of the job in the 'j'th position assigned to machine 'i' in the given schedule

$C_{[i,j]}$ - Completion time of the job in the 'j'th position assigned to machine 'i' in the given schedule

$$C_{[i,j]} = \sum_{k=1}^{j} p_{[i,k]}$$

$$S^i = \{ Sequence\ of\ jobs\ assigned\ to\ machine\ 'i'\ in\ the\ given\ schedule \}$$

$$N^i = n(S^i) = Number\ of\ jobs\ assigned\ to\ machine\ 'i'$$

6.2.1 Makespan with No Preemptions $(C_{\max})$

The makespan, defined as $C_{\max} = \max(C^1_{\max}, C^2_{\max}, \ldots\ldots\ldots, C^m_{\max})$ is equivalent to the completion time of the last job to leave the system. Let $C^i_{\max}$ be the completion time of the last job on machine 'i'.

$$C^i_{\max} = \sum_{j \in S^i} p_j = \sum_{j=1}^{j=N^i} p_{[i,j]}$$

$$C_{\max} = \max(C^1_{\max}, C^2_{\max}, \ldots\ldots\ldots, C^m_{\max})$$

$C^1_{\max}, C^2_{\max}, \ldots\ldots\ldots, C^m_{\max}$ are linear functions of random variables of job processing times. $C_{\max}$ is the greatest of a finite set of random numbers. For normally distributed variables, Clark's equations could be recursively applied to determine the expectation and variance of $C_{\max}$. If the processing times of the '*n*' jobs are independent and normally distributed, then $C^1_{\max}, C^2_{\max}, \ldots\ldots\ldots\ldots, C^m_{\max}$ are also normally distributed by the reproductive property of normal random variables.

To determine $E[C_{\max}]$ and $Var[C_{\max}]$:

$$C_{\max} = \max\,(C^1_{\max}, C^2_{\max}, \ldots\ldots\ldots\ldots, C^m_{\max})$$
$$= \max\,(\max(C^1_{\max}, C^2_{\max}), \ldots\ldots\ldots\ldots, C^m_{\max})$$

The mean and variance of the makespan of machine '*i*' is given by,

$$E\,[C^i_{\max}] = \mu_{M[i]} = \sum_{j=1}^{j=N^i} \mu_{[i,j]}$$

$$Var\,[C^i_{\max}] = \sigma^2_{M[i]} = \sum_{j=1}^{j=N^i} \sigma^2{}_{[i,j]}$$

The makespan of machine '*i*' , $C^i_{\max}$, is normally distributed with mean $\mu_{M[i]}$ and $\sigma^2_{M[i]}$. Moreover, the random variables $C^1_{\max}, C^2_{\max}, \ldots\ldots\ldots\ldots, C^m_{\max}$ are all independent of each other as there are no preemptions allowed and, hence, the correlation coefficient is zero between any pair amongst them.

$$r\,(C^i_{\max}, C^k_{\max}) = \text{coefficient of linear correlation} = \rho_{[ik]} = 0 \quad \text{for all } i, k = 1,2,\ldots\ldots\ldots\ldots, m \text{ and } i \neq k \qquad \ldots\ldots(6.1)$$

$$a^2_{[ik]} = \sigma^2_{M[i]} + \sigma^2_{M[k]} - 2\sigma_{M[i]}\sigma_{M[k]}\rho_{[ik]},$$

$$\alpha_{[ik]} = \frac{\mu_{M[i]} - \mu_{M[k]}}{a_{[ik]}},$$

$$\varphi(z) = \frac{1}{\sqrt{2\pi}}\exp(-z^2/2) \text{ and } \Phi(x) = \int_{-\infty}^{x}\varphi(t)\,dt$$

Expectation and Variance of $\max\left(C_{\max}^{1}, C_{\max}^{2}\right)$**:**

The expectation and variance of $\max\left(C_{\max}^{1}, C_{\max}^{2}\right)$ are computed using Clark's equations as below:

The first moment, $E\left[\max\left(C_{\max}^{1}, C_{\max}^{2}\right)\right]$, is given by,

$$\nu_{1[12]} = \mu_{[12]} = \mu_{M[1]}\Phi(\alpha_{[12]}) + \mu_{M[2]}\Phi(-\alpha_{[12]}) + a_{[12]}\varphi(\alpha_{[12]})$$

where $a_{[12]}^2 = \sigma_{M[1]}^2 + \sigma_{M[2]}^2 - 2\sigma_{M[1]}\sigma_{M[2]}\rho_{[12]} = \sigma_{M[1]}^2 + \sigma_{M[2]}^2$ as $\rho_{[12]} = 0$

The second moment is given by

$$\nu_{2[12]} = \left(\mu_{M[1]}^2 + \sigma_{M[1]}^2\right)\Phi(\alpha_{[12]}) + \left(\mu_{M[2]}^2 + \sigma_{M[2]}^2\right)\Phi(-\alpha_{[12]}) + \left(\mu_{M[1]} + \mu_{M[2]}\right)a_{[12]}\varphi(\alpha_{[12]})$$

$$Var\left[\max\left(C_{\max}^{1}, C_{\max}^{2}\right)\right] = \nu_{2[12]} - \left(\nu_{1[12]}\right)^2$$

$$\sigma_{[12]}^2 = \left(\mu_{M[1]}^2 + \sigma_{M[1]}^2\right)\Phi(\alpha_{[12]}) + \left(\mu_{M[2]}^2 + \sigma_{M[2]}^2\right)\Phi(-\alpha_{[12]}) + \left(\mu_{M[1]} + \mu_{M[2]}\right)a_{[12]}\varphi(\alpha_{[12]}) - \left(\mu_{M[1]}\Phi(\alpha_{[12]}) + \mu_{M[2]}\Phi(-\alpha_{[12]}) + a_{[12]}\varphi(\alpha_{[12]})\right)^2$$

$$\rho_{[123]} = r\left[C_{\max}^{3}, \max\left(C_{\max}^{1}, C_{\max}^{2}\right)\right] = \left[\sigma_{M[1]}\rho_{[13]}\Phi\left(\alpha_{[12]}\right) + \sigma_{M[2]}\rho_{[23]}\Phi\left(-\alpha_{[12]}\right)\right]/\sigma_{M[12]} \quad \text{……..(6.2)}$$

$$= 0 \text{ as } \rho_{[13]} = 0 \text{ and } \rho_{[23]} = 0$$

Expectation and Variance of $\max\left(C_{\max}^{1}, C_{\max}^{2}, C_{\max}^{3}\right)$**:**

The expectation and variance of $\max\left(C_{\max}^1, C_{\max}^2, C_{\max}^3\right)$ are:

$$\max\left(C_{\max}^1, C_{\max}^2, C_{\max}^3\right) = \max\left(\max\left(C_{\max}^1, C_{\max}^2\right), C_{\max}^3\right)$$

$$r\left[\max\left(C_{\max}^1, C_{\max}^2\right), C_{\max}^3\right] = \rho_{[123]} = 0$$

$$a_{[123]}^2 = \sigma_{[12]}^2 + \sigma_{M[3]}^2 - 2\sigma_{[12]}\sigma_{M[3]}\rho_{[123]} = \sigma_{[12]}^2 + \sigma_{M[3]}^2$$

$$\alpha_{[123]} = \frac{\mu_{[12]} - \mu_{M[3]}}{a_{[123]}}$$

The first moment, $E\left[\max\left(C_{\max}^1, C_{\max}^2, C_{\max}^3\right)\right]$, is given by

$$\nu_{1[123]} = \mu_{[123]} = \mu_{M[12]}\Phi(\alpha_{[123]}) + \mu_{M[3]}\Phi(-\alpha_{[123]}) + a_{[123]}\varphi(\alpha_{[123]})$$

The second moment is given by

$$\nu_{2[123]} = \left(\mu_{M[12]}^2 + \sigma_{M[12]}^2\right)\Phi(\alpha_{[123]}) + \left(\mu_{M[3]}^2 + \sigma_{M[3]}^2\right)\Phi(-\alpha_{[123]}) + \left(\mu_{M[12]} + \mu_{M[3]}\right)a_{[123]}\varphi(\alpha_{[123]})$$

$$Var\left[\max\left(L_{[1]}, L_{[2]}, L_{[3]}\right)\right] = \nu_{2[123]} - \left(\nu_{1[123]}\right)^2$$

$$\sigma_{[123]}^2 = \left(\mu_{M[12]}^2 + \sigma_{M[12]}^2\right)\Phi(\alpha_{[123]}) + \left(\mu_{M[3]}^2 + \sigma_{M[3]}^2\right)\Phi(-\alpha_{[123]}) + \left(\mu_{M[12]} + \mu_{M[3]}\right)a_{[123]}\varphi(\alpha_{[123]} - \left(\mu_{M[12]}\Phi(\alpha_{[123]}) + \mu_{M[3]}\Phi(-\alpha_{[123]}) + a_{[123]}\varphi(\alpha_{[123]})\right)^2$$

$$\rho_{[1234]} = r\left[C_{\max}^3, \max\left(C_{\max}^1, C_{\max}^2, C_{\max}^3\right)\right]$$
$$= \left[\sigma_{[12]}\rho_{[124]}\Phi\left(\alpha_{[123]}\right) + \sigma_{[3]}\rho_{[34]}\Phi\left(-\alpha_{[123]}\right)\right] / \sigma_{[123]}$$

Using the correlation equations (4.1) and (4.2), $\rho_{[124]} = 0$ and $\rho_{[34]} = 0$ and, hence, $\rho_{[1234]} = 0$. Thus, in a similar way, Clark equations can be used recursively in '*m-1*' steps to determine the expectation and variance of $C_{\max}$.

6.2.2 Total Completion Time with No Preemptions $\left(\sum_{j=1}^{n} C_{[j]}\right)$

Let C^i be the total completion time of all the jobs assigned to machine '*i*'. C^i's are random variables being linear functions of random variables (*job processing times*) and are independent of each other as there are no preemptions allowed in the schedule.

$$C^i = \sum_{j \in S^i} C_{[i,j]} = \sum_{j=1}^{j=N^i} \sum_{i=1}^{j} p_{[i,j]}$$

Rearranging and summing similar terms within the summation, we get,

$$C^i = p_{[i,1]} + (p_{[i,1]} + p_{[i,2]}) + \ldots\ldots + (p_{[i,1]} + p_{[i,2]} + \ldots + p_{[i,N^i-1]}) + (p_{[i,1]} + p_{[i,2]} + \ldots + p_{[i,N^i]})$$

$$C^i = N^i p_{[i,1]} + (N^i - 1) p_{[i,2]} + (N^i - 2) p_{[i,3]} + \ldots\ldots\ldots\ldots + 2 p_{[i,N^i-1]} + p_{[i,N^i]}$$

$$= \sum_{j=1}^{N^i} (N^i + 1 - j) p_{[i,j]}$$

$$E[C^i] = E\left[\sum_{j=1}^{N^i} (N^i + 1 - j) p_{[i,j]}\right]$$

$$= \sum_{j=1}^{N^i} (N^i + 1 - j) \mu_{[i,j]}$$

$$Var\left[C^i\right] = Var\left[\sum_{j=1}^{N^i}(N^i+1-j)p_{[i,j]}\right]$$

$$= \sum_{j=1}^{N^i}(N^i+1-j)^2\sigma_{[i,j]}^2$$

Expectation of the Total Completion Time $E\left[\sum_{i=1}^{m}C^i\right]$**:**

The expectation of $\sum_{i=1}^{m}C^i$ is determined as follows:

$$E\left[\sum_{i=1}^{m}C^i\right] = \sum_{i=1}^{m}E\left[C^i\right] = \sum_{i=1}^{m}\sum_{j=1}^{N^i}(N^i+1-j)\mu_{[i,j]}$$

Variance of the Total Completion Time $Var\left[\sum_{i=1}^{m}C^i\right]$**:**

The variance of $\sum_{i=1}^{m}C^i$ is determined as follows:

$$Var\left[\sum_{i=1}^{m}C^i\right] = \sum_{i=1}^{m}Var\left[C^i\right] = \sum_{i=1}^{m}\sum_{j=1}^{N^i}(N^i+1-j)^2\sigma_{[i,j]}^2$$

6.3 Parallel Machines with Preemptions

A scheduling problem with preemptions is always a complex problem to model and is even harder for a stochastic environment. Preemptions allow the jobs to be interrupted during processing and the work remaining can be scheduled for processing later on the same machine or on any of the other parallel machines. In our expectation-variance analysis, we are given the sequence of jobs on each of the machines and, with preemptions being allowed, a job could be processed on multiple machines. An illustrative Gantt chart for a sample problem is shown in Figure 6.1 below.

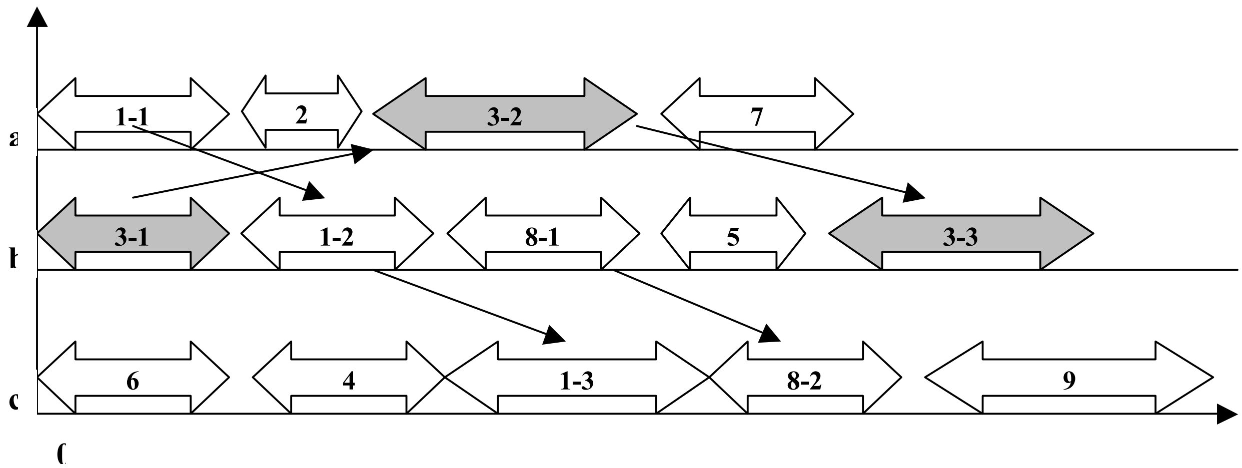


Figure 6.1. A Gantt Chart that Shows the Relative Positions of the Jobs for the Parallel Machine with Preemptions Example

A schematic representation of the relative positions of different jobs is shown as it is not possible to ascertain the completion times due to the probabilistic nature of the jobs. Jobs *1*,*3* and *8* are preempted while the other jobs are processed completely in one operation. For all the preempted jobs, the starting time of an operation depends on the completion time of its previous portion and also the completion time of its predecessor on the same machine.

For example, the completion time of the second portion of job '8-2*9*' on machine '*c*' is given by, $C_{c,8-2} = S_{c,8-2} + p_{c,8-2}$ where $S_{c,8-2}$ and $p_{c,8-2}$ represent the starting time and processing time of job '*8-2*' on machine '*c*' respectively. However, the starting time $S_{c,8-2}$ in turn depends on the completion time of job '*1-3*' on machine '*c*' and also the completion time of '8-*1'* on machine '*b*'.

The starting time $S_{c,8-2}$ can then be given as $S_{c,8-2} = \max(C_{b,8-1}, C_{c,1-3})$.

Recursively, the completion time of the second portion of job '*7*' on machine '*c*' is given by, $C_{c,1-3} = S_{c,1-3} + p_{c,1-3}$ which in turn depends on the completion times of its earlier portion, *1-2*, and its predecessor on machine '*c*', *4*, and so on. Other preempted jobs also require similar recursive computations.

As mathematical theory does not allow exact evaluation of the maximum operator for stochastic problems, it becomes necessary to use Clark's equations recursively for approximating the mean and variance of the completion times. However, in the recursion process, a number of correlation coefficients between the various parameters need to be determined. This recursive procedure is quite complex and some work has been done in this field for permutation flow shops and assembly systems (see Wilhelm (1986), Wilhelm & Ahmadi-Marandi (1982) and Srivastava & Sarin (1993)) as discussed in Chapter 4.

Analogy with the Job shop:

From the Gantt chart we can infer that this case of parallel machines with preemptions is analogous to a job shop with recirculation and, possibly, not all the jobs visit all the machines. A preempted job will have its own sequence for processing its preempted portions which is similar to the job processing routes in a job shop. Hence the solution methodology for job shops with recirculation is applicable in this case and is not repeated in this section.

6.4 Concluding Remarks

In this chapter, we determined closed-form expressions for the performance measures of makespan and total completion time for scheduling jobs on parallel machines with no preemptions. Generic expressions were developed for the case of total completion time while the makespan analysis is applicable to normally distributed job processing times.

The underlying solution methodology for the preemption case was found to be similar to that of a job shop with recirculation which has been dealt with in the previous chapter. Software programs developed for determining the various performance measures for the no preemptions case are provided in Appendix D.

7 NUMERICAL ILLUSTRATION AND ANALYSIS

7.1 Introduction

In this chapter, we illustrate the significance and applicability of the expressions and methodologies developed for determining the expected value and the variability of the performance measures for different scheduling environments. We analyze a few sample datasets using the programs developed in MATLAB R12 for all the cases that were described in Chapters 3, 4, 5 and 6. The MATLAB programs can be found in the Appendices.

7.2 Single Machine Models

7.2.1 Completion Time Based Objectives

7.2.1.1 Total Completion Time

Sample Data Set:

Total Number of Jobs = 4
Distribution Assumption, if any: No, applicable to any distribution

The mean and variance of the processing times of the 4 jobs are given in Table 7.1 below.

Job Index	**1**	**2**	**3**	**4**
Mean	40	20	60	25
Variance	15	15	20	5

Table 7.1. Data Set for Total Completion Time

Expectation and Variance of the Total Completion Time:

S No	**Sequence**	**E [TCT]**	**Var [TCT]**
1	1 2 3 4	365	460
2	1 2 4 3	330	415
3	1 3 2 4	405	485
4	1 3 4 2	410	455
5	1 4 3 2	375	380
6	1 4 2 3	335	365
7	2 1 3 4	345	460
8	2 1 4 3	310	415
9	2 3 1 4	365	485
10	2 3 4 1	350	455
11	2 4 3 1	315	380
12*	**2 4 1 3**	**295**	**365**
13	3 2 1 4	405	520
14	3 2 4 1	390	490
15	3 1 2 4	425	520
16	3 1 4 2	430	490
17	3 4 1 2	415	440
18	3 4 2 1	395	440
19	4 2 3 1	320	310
20**	**4 2 1 3**	**300**	**295**
21	4 3 2 1	360	335
22	4 3 1 2	380	335
23	4 1 3 2	360	310
24	4 1 2 3	320	295

Table 7.2. Expectation and Variance of the Total Completion Time

The expected value and variance of the total completion time for all possible sequences is tabulated above. It can be found that the *SEPT* (*Shortest Expected Processing Time*) sequence (*2-4-1-3*)* has a mean and variance of 295 and 365 respectively. However, the sequence (*4-2-1-3*)** has a slighter higher mean of 300 but a considerably lower variance of 295. This information is quite useful for any scheduler who would prefer lower variability in his system.

7.2.1.2 Total Weighted Completion Time

Sample Data Set:

Total Number of Jobs = 4
Distribution Assumption, if any: No, applicable to any distribution

The mean and variance of the processing times of the 5 jobs and their weights are given in Table 7.3 below.

Job Index	**1**	**2**	**3**	**4**
Mean	40	20	60	25
Variance	15	15	20	5
Weight	5	4	10	5

Table 7.3. Data Set for Total Weighted Completion Time

Expectation and Variance of the Total Weighted Completion Time:

S No	Sequence	E [TWCT]	Var [TWCT]
1	1 2 3 4	2365	18680
2	1 2 4 3	2315	17180
3	1 3 2 4	2405	17200
4	1 3 4 2	2405	16505
5	1 4 3 2	2355	14605
6	1 4 2 3	2315	15385
7	2 1 3 4	2305	19265
8	2 1 4 3	2255	17765
9	2 3 1 4	2205	18265
10	2 3 4 1	2130	17515
11*	**2 4 3 1**	**2080**	**15515**
12	2 4 1 3	2180	16015
13	3 2 1 4	2245	16085
14	3 2 4 1	2170	15335
15	3 1 2 4	2305	15800
16	3 1 4 2	2305	15105
17	3 4 1 2	2230	13955
18	3 4 2 1	2170	14090
19*	**4 2 3 1**	**2080**	**13170**
20	4 2 1 3	2180	13670
21**	**4 3 2 1**	**2120**	**11690**
22	4 3 1 2	2180	11555
23	4 1 3 2	2280	12455
24	4 1 2 3	2240	13235

Table 7.4. Expectation and Variance of the Total Weighted Completion Time

The expected value and variance of the total weighted completion time for all possible sequences is tabulated above. It can be found that the W*SEPT* (*Weighted Shortest Expected Processing Time*) sequences (*2-4-3-1*)* and (*4-2-3-1*)* have the same mean of 2080. However, the sequence (*4-2-3-1*)* is definitely better as its variability (13170) is less than the variability of (*2-4-3-1*)* (15515). It could also be witnessed that the sequence (*4-3-2-1*)** has a slighter higher mean of 2120 but has a considerably lower variability of 11690 which is much lesser than the other two sequences. So, this information would not have been gained if not for determining the variance of the performance measure.

7.2.1.3 Total Weighted Discounted Completion Time

Sample Data Set:

Total Number of Jobs = 4
Discount rate, r = 0.05;

The mean and variance of the processing times of the 5 jobs and their weights are given in Table 7.5 below.

Job Index	**1**	**2**	**3**	**4**
Mean	40	20	60	25
Variance	15	15	20	5
Weight	5	4	10	5

Table 7.5. Data Set for Total Weighted Discounted Completion Time

Normally Distributed Processing Times:

Distribution Assumption, if any: *Normal distribution*

Expectation and Variance of the Total Weighted Discounted Completion Time (*normal*):

S No	Sequence	E [TWDCT]	Var [TWDCT]
1	1 2 3 4	23.0736	0.0216
2	1 2 4 3	23.0216	0.0235
3	1 3 2 4	23.2258	0.0183
4	1 3 4 2	23.2269	0.0183
5	1 4 3 2	23.0884	0.0195
6	1 4 2 3	23.0445	0.0215
7	2 1 3 4	22.2120	0.1377
8	2 1 4 3	22.1601	0.1396
9	2 3 1 4	22.2923	0.1349
10	2 3 4 1	22.2779	0.1351
11	2 4 3 1	21.9013	0.1441
12*	**2 4 1 3**	**21.8782**	**0.1458**
13	3 2 1 4	23.3960	0.0040
14	3 2 4 1	23.3816	0.0042
15**	**3 1 2 4**	**23.4400**	**0.0035**
16**	**3 1 4 2**	**23.4411**	**0.0035**
17	3 4 1 2	23.4028	0.0036
18	3 4 2 1	23.3901	0.0039
19	4 2 3 1	22.0673	0.0360
20	4 2 1 3	22.0441	0.0378
21	4 3 2 1	22.3855	0.0267
22	4 3 1 2	22.3981	0.0264
23	4 1 3 2	22.3364	0.0275
24	4 1 2 3	22.2925	0.0295

Table 7.6. Expectation and Variance of the Total Weighted Discounted Completion Time (normal)

(*Schedules(s) that are optimal in terms of the expected value)
(**Other schedule(s) that is (are) efficient in terms of variance when compared with *schedule(s) and may have slightly higher expected value (s))

Exponentially Distributed Processing Times:

Distribution Assumption, if any: *Exponential distribution*

Expectation and Variance of the Total Weighted Discounted Completion Time (*exponential*):

S No	Sequence	E [TWDCT]	Var [TWDCT]
1	1 2 3 4	12.1524	29.0153
2	1 2 4 3	12.1524	29.5789
3	1 3 2 4	10.6857	27.5346
4	1 3 4 2	10.0571	26.1169
5	1 4 3 2	10.0571	24.4680
6	1 4 2 3	11.3143	26.4411
7	2 1 3 4	12.9524	35.1057
8	2 1 4 3	12.9524	35.9551
9	2 3 1 4	12.6190	35.4289
10	2 3 4 1	12.4762	36.1711
11	2 4 3 1	12.4762	35.6071
12	2 4 1 3	12.7619	35.7177
13	3 2 1 4	10.7857	31.5033
14	3 2 4 1	10.6429	32.1503
15	3 1 2 4	10.1857	29.6995
16	3 1 4 2	9.5571	28.4723
17*	**3 4 1 2**	**9.3429**	**27.8788**
18	3 4 2 1	9.8571	29.7421
19	4 2 3 1	11.4286	27.1633
20	4 2 1 3	11.7143	27.0867
21	4 3 2 1	9.8571	25.3224
22* & **	**4 3 1 2**	**9.3429**	**23.1837**
23	4 1 3 2	9.7714	22.1302
24	4 1 2 3	11.0286	24.1918

Table 7.7. Expectation and Variance of the Total Weighted Discounted Completion Time (exponential)

(*Schedules(s) that are optimal in terms of the expected value)

(**Other schedule(s) that is (are) efficient in terms of variance when compared with *schedule(s) and may have slightly higher expected value (s))

Note: For this sample data set, schedule *4-3-1-2* is the preferred schedule as it is efficient in terms of the variability too when compared with *3-4-1-2.*

7.2.2 Tardiness Based Objectives

7.2.2.1 Total Tardiness

Sample Data Set:

Total Number of Jobs = 4

Distribution Assumption, if any: *Normal distribution*

The mean and variance of the processing times of the 4 jobs and their due dates are given in Table 7.8 below.

Job Index	**1**	**2**	**3**	**4**
Mean	40	20	60	25
Variance	15	15	20	5
Due Dates	50	25	70	30

Table 7.8. Data Set for Total Tardiness

Expectation and Variance of the Total Tardiness:

S No	Sequence	E [TT]	Var [TT]
1	1 2 3 4	208.5988	73.9083
2	1 2 4 3	172.1219	76.5877
3	1 3 2 4	246.9841	8.7466
4	1 3 4 2	251.4670	170.0581
5	1 4 3 2	214.4227	82.5834
6	1 4 2 3	175.3922	22.5902
7	2 1 3 4	190.9712	159.8451
8	2 1 4 3	154.4943	142.7337
9	2 3 1 4	210.5408	134.7406
10	2 3 4 1	194.7552	9.6309
11	2 4 3 1	156.5824	131.9973
12*	**2 4 1 3**	**136.5367**	**184.6483**
13	3 2 1 4	247.0240	28.8840
14	3 2 4 1	231.2384	12.5092
15	3 1 2 4	266.0782	5.1747
16	3 1 4 2	270.5610	127.3928
17	3 4 1 2	254.8328	63.2140
18	3 4 2 1	235.3586	68.6629
19	4 2 3 1	157.2284	105.8939
20**	**4 2 1 3**	**137.1827**	**151.0990**
21	4 3 2 1	195.9139	104.0510
22	4 3 1 2	215.3881	74.8084
23	4 1 3 2	194.8723	130.5551
24**	**4 1 2 3**	**155.8418**	**32.2034**

Table 7.9. Expectation and Variance of the Total Tardiness

(*Schedules(s) that are optimal in terms of the expected value)
(**Other schedule(s) that is (are) efficient in terms of variance when compared with *schedule(s) and may have slightly higher expected value (s))

7.2.2.2 Total Weighted Tardiness

Sample Data Set:

Total Number of Jobs = 4
Distribution Assumption, if any: *Normal distribution*

The mean and variance of the processing times of the 5 jobs, their due dates and weights are given in Table 7.10 below.

Job Index	**1**	**2**	**3**	**4**
Mean	40	20	60	25
Variance	15	15	20	5
Due Dates	50	25	70	30
Weight	5	4	10	5

Table 7.10. Data Set for Total Weighted Tardiness

Expectation and Variance of the Total Weighted Tardiness:

S No	Sequence	E [TWT]	Var [TWT]
1	1 2 3 4	1277.0	6857.2
2	1 2 4 3	1209.7	2624.4
3	1 3 2 4	1308.4	4976.5
4	1 3 4 2	1306.1	341.60
5	1 4 3 2	1234.6	102.05
6	1 4 2 3	1202.3	727.7
7	2 1 3 4	1221.9	7321.9
8	2 1 4 3	1154.6	2840.6
9	2 3 1 4	1146.5	8187.3
10	2 3 4 1	1067.6	6858.9
11**	**2 4 3 1**	**975.1**	**454.78**
12	2 4 1 3	1064.8	4003.6
13	3 2 1 4	1199.0	3250.7
14	3 2 4 1	1120.1	4117.2
15	3 1 2 4	1254.6	3648.2
16	3 1 4 2	1252.3	1097.7
17	3 4 1 2	1173.7	2510.4
18	3 4 2 1	1116.2	2368.4
19*	**4 2 3 1**	**960.5**	**2331.8**
20	4 2 1 3	1050.2	1021.2
21	4 3 2 1	995.3	817.01
22	4 3 1 2	1052.7	1329.6
23	4 1 3 2	1136.8	1955.0
24	4 1 2 3	1104.5	4468.0

Table 7.11. Expectation and Variance of the Total Weighted Tardiness

(*Schedules(s) that are optimal in terms of the expected value)
(**Other schedule(s) that is (are) efficient in terms of variance when compared with *schedule(s) and may have slightly higher expected value (s))

7.2.2.3 Total Number of Tardy Jobs

Sample Data Set:

Total Number of Jobs = 4
Distribution Assumption, if any: *Normal distribution*

The mean and variance of the processing times of the 4 jobs and their due dates are given in Table 7.12 below.

Job Index	**1**	**2**	**3**	**4**
Mean	40	20	60	25
Variance	15	15	20	5
Due Dates	50	25	70	30

Table 7.12. Data Set for Total Number of Tardy Jobs

Expectation and Variance of the Total Number of Tardy Jobs:

S No	Sequence				E [TNT]	Var [TNT]
1* & **	**1**	**2**	**3**	**4**	**3.0049**	**0.0049**
2* & **	**1**	**2**	**4**	**3**	**3.0049**	**0.0049**
3* & **	**1**	**3**	**2**	**4**	**3.0049**	**0.0049**
4* & **	**1**	**3**	**4**	**2**	**3.0049**	**0.0049**
5* & **	**1**	**4**	**3**	**2**	**3.0049**	**0.0049**
6* & **	**1**	**4**	**2**	**3**	**3.0049**	**0.0049**
7	2	1	3	4	3.0644	0.1215
8	2	1	4	3	3.0644	0.1215
9	2	3	1	4	3.0529	0.1321
10	2	3	4	1	3.0529	0.1321
11	2	4	3	1	3.0980	0.0891
12	2	4	1	3	3.0980	0.0891
13	3	2	1	4	3.0127	0.0125
14	3	2	4	1	3.0127	0.0125
15	3	1	2	4	3.0127	0.0125
16	3	1	4	2	3.0127	0.0125
17	3	4	1	2	3.0127	0.0125
18	3	4	2	1	3.0127	0.0125
19	4	2	3	1	3.0127	0.0125
20	4	2	1	3	3.0127	0.0125
21	4	3	2	1	3.0113	0.0139
22	4	3	1	2	3.0113	0.0139
23	4	1	3	2	3.0123	0.0139
24	4	1	2	3	3.0123	0.0139

Table 7.13. Expectation and Variance of the Total Number of Tardy Jobs

(*Schedules(s) that are optimal in terms of the expected value)

(**Other schedule(s) that is (are) efficient in terms of variance when compared with *schedule(s) and may have slightly higher expected value (s))

Note: For this sample data set, there are multiple schedules which are efficient in terms of both the expected value and variance.

7.2.2.4 Total Weighted Number of Tardy Jobs

Sample Data Set:

Total Number of Jobs = 4

Distribution Assumption, if any: *Normal distribution*

The mean and variance of the processing times of the 4 jobs and their due dates are given in Table 7.14 below.

Job Index	**1**	**2**	**3**	**4**
Mean	40	20	60	25
Variance	15	15	20	5
Due Dates	50	25	70	30
Weight	5	4	10	5

Table 7.14. Data Set for Total Weighted Number of Tardy Jobs

Expectation and Variance of the Total Weighted Number of Tardy Jobs:

S No	Sequence				E [TWNT]	Var [TWNT]
1	1	2	3	4	19.025	0.122
2	1	2	4	3	19.025	0.122
3	1	3	2	4	19.025	0.122
4	1	3	4	2	19.025	0.122
5	1	4	3	2	19.025	0.122
6	1	4	2	3	19.025	0.122
7	2	1	3	4	20.224	2.238
8	2	1	4	3	20.224	2.238
9	2	3	1	4	19.939	5.760
10	2	3	4	1	19.939	5.7604
11	2	4	3	1	20.391	1.4288
12	2	4	1	3	20.391	1.428
13*	**3**	**2**	**1**	**4**	**14.127**	**1.251**
14*	**3**	**2**	**4**	**1**	**14.127**	**1.251**
15*	**3**	**1**	**2**	**4**	**14.127**	**1.251**
16*	**3**	**1**	**4**	**2**	**14.127**	**1.251**
17*	**3**	**4**	**1**	**2**	**14.127**	**1.251**
18*	**3**	**4**	**2**	**1**	**14.127**	**1.251**
19**	**4**	**2**	**3**	**1**	**19.063**	**0.312**
20**	**4**	**2**	**1**	**3**	**19.063**	**0.312**
21	4	3	2	1	19.050	0.447
22	4	3	1	2	19.050	0.447
23	4	1	3	2	19.061	0.322
24	4	1	2	3	19.061	0.322

Table 7.15. Expectation and Variance of the Total Weighted Number of Tardy Jobs

(*Schedules(s) that are optimal in terms of the expected value)

(**Other schedule(s) that is (are) efficient in terms of variance when compared with *schedule(s) and may have slightly higher expected value (s))

7.2.2.5 Mean Lateness

Sample Data Set:

Total Number of Jobs = 4

Distribution Assumption, if any: *No, applicable to any distribution*

The mean and variance of the processing times of the 4 jobs and their due dates are given in Table 7.16 below.

Job Index	**1**	**2**	**3**	**4**
Mean	40	20	60	25
Variance	15	15	20	5
Due Dates	50	25	70	30

Table 7.16. Data Set for Mean Lateness

Expectation and Variance of the Mean Lateness:

S No	Sequence	E [ML]	Var [ML]
1	1 2 3 4	47.500	22.813
2	1 2 4 3	38.750	20.625
3	1 3 2 4	57.500	25.313
4	1 3 4 2	58.750	25.625
5	1 4 3 2	50.000	23.438
6	1 4 2 3	40.000	20.938
7	2 1 3 4	42.500	21.563
8	2 1 4 3	33.750	19.375
9	2 3 1 4	47.500	22.813
10	2 3 4 1	43.750	21.875
11	2 4 3 1	35.000	19.688
12* & **	**2 4 1 3**	**30.000**	**18.438**
13	3 2 1 4	57.500	25.313
14	3 2 4 1	53.750	24.375
15	3 1 2 4	62.500	26.563
16	3 1 4 2	63.750	26.875
17	3 4 1 2	60.000	25.938
18	3 4 2 1	55.000	24.688
19	4 2 3 1	36.250	20.000
20	4 2 1 3	31.250	18.750
21	4 3 2 1	46.250	22.500
22	4 3 1 2	51.250	23.750
23	4 1 3 2	46.250	22.500
24	4 1 2 3	36.250	20.000

Table 7.17. Expectation and Variance of the Mean Lateness

(*Schedules(s) that are optimal in terms of the expected value)

(**Other schedule(s) that is (are) efficient in terms of variance when compared with *schedule(s) and may have slightly higher expected value (s))

Note: For this sample data set, schedule 2-4-1-3 is the optimal in terms of both the expectation and variance and hence it is the preferred schedule.

7.2.2.6 Maximum Lateness

Sample Data Set:

Total Number of Jobs = 4

Distribution Assumption, if any: *Normal distribution*

The mean and variance of the processing times of the 4 jobs and their due dates are given in Table 7.16 below.

Job Index	**1**	**2**	**3**	**4**
Mean	40	20	60	25
Variance	15	15	20	5
Due Dates	50	25	70	30

Table 7.18. Data Set for Maximum Lateness

Expectation and Variance of the Maximum Lateness:

S No	Sequence	E [MaxL]	Var [MaxL]
1	1 2 3 4	115	55
2* & **	**1 2 4 3**	**75**	**55**
3	1 3 2 4	115	55
4	1 3 4 2	120	55
5	1 4 3 2	120	55
6* & **	**1 4 2 3**	**75**	**55**
7	2 1 3 4	115	55
8* & **	**2 1 4 3**	**75**	**55**
9	2 3 1 4	115	55
10	2 3 4 1	95	55
11	2 4 3 1	95	55
12* & **	**2 4 1 3**	**75**	**55**
13	3 2 1 4	115	55
14	3 2 4 1	95	55
15	3 1 2 4	115	55
16	3 1 4 2	120	55
17	3 4 1 2	120	55
18	3 4 2 1	95	55
19	4 2 3 1	95	55
20* & **	**4 2 1 3**	**75**	**55**
21	4 3 2 1	95	55
22	4 3 1 2	120	55
23	4 1 3 2	120	55
24	**4 1 2 3**	**75**	**55**

Table 7.19. Expectation and Variance of the Maximum Lateness

(*Schedules(s) that are optimal in terms of the expected value)

(**Other schedule(s) that is (are) efficient in terms of variance when compared with *schedule(s) and may have slightly higher expected value (s))

Note: For this sample data set, all schedules have the same variance and hence all *schedules are optimal in terms of expectation and variance.

7.3 Permutation Flow Shops with Unlimited Intermediate Storage

Sample Data Set I:

Total Number of Jobs = 4

Total Number of Machines = 4

Distribution Assumption, if any: *Normal distribution*

The mean processing times of the 4 jobs on the 4 machines is given in Table 7.20 below.

Job Index	**1**	**2**	**3**	**4**
Machine 1	10	2	5	4
Machine 2	8	3	9	5
Machine 3	2	11	7	2
Machine 4	2	2	8	4

Table 7.20. Mean Processing Time Data Set I - Flow Shop

The variance of the processing times of the 4 jobs on the 4 machines is given in Table 7.21 below.

Job Index	**1**	**2**	**3**	**4**
Machine 1	3	1	2	2
Machine 2	1	2	2	1
Machine 3	2	1	2	1
Machine 4	2	2	5	3

Table 7.21: Variance of Processing Time Data Set I - Flow Shop

Expectation and Variance of the Makespan:

S No	Sequence	E [Makespan]	Var [Makespan]
1	1 2 3 4	38.0874	9.9050
2	1 2 4 3	33.0000	9.0000
3	1 3 2 4	42.5958	14.4535
4	1 3 4 2	34.7927	11.0346
5	1 4 3 2	39.0000	9.0000
6	1 4 2 3	37.0000	8.0000
7	2 1 3 4	36.0000	9.0000
8	2 1 4 3	27.0000	7.0000
9	2 3 1 4	26.4557	7.8738
10**	**2 3 4 1**	**25.6768**	**7.0281**
11	2 4 3 1	27.6717	7.3675
12*	**2 4 1 3**	**25.0585**	**8.5626**
13	3 2 1 4	33.0044	8.9701
14	3 2 4 1	34.0000	8.000
15	3 1 2 4	37.0335	8.8712
16	3 1 4 2	30.0335	7.8712
17	3 4 1 2	28.3928	8.6137
18	3 4 2 1	34.0000	8.0000
19	4 2 3 1	30.0000	8.0000
20	4 2 1 3	25.0716	8.7769
21	4 3 2 1	34.3029	12.4083
22	4 3 1 2	34.3029	12.4083
23	4 1 3 2	34.0000	26.0000
24	4 1 2 3	35.0000	10.0000

Table 7.22. Expectation and Variance of the Makespan - Flow Shop (Data Set I)

(*Schedules(s) that are optimal in terms of the expected value)

(**Other schedule(s) that is (are) efficient in terms of variance when compared with *schedule(s) and may have slightly higher expected value (s))

Sample Data Set II:

Total Number of Jobs = 10

Total Number of Machines = 10

Distribution Assumption, if any: *Normal distribution*

The mean processing times of the 10 jobs on the 10 machines is given in Table 7.23 below.

Job Index	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Machine 1	90	18	45	36	18	18	36	45	18	90
Machine 2	72	27	81	45	27	27	45	81	27	72
Machine 3	9	99	63	18	9	9	18	63	99	9
Machine 4	18	18	72	36	9	9	36	72	18	18
Machine 5	90	72	9	18	9	9	18	9	72	90
Machine 6	108	18	45	36	18	18	36	45	18	90
Machine 7	63	36	81	99	36	27	45	36	27	72
Machine 8	9	99	0	18	72	36	18	63	99	9
Machine 9	27	36	72	36	99	0	36	72	27	18
Machine 10	90	72	99	18	9	36	18	9	72	90

Table 7.23. Mean Processing Time Data Set II - Flow Shop

The variance of the processing times of the 4 jobs on the 4 machines is given in Table 7.24 below.

Job Index	1	2	3	4	5	6	7	8	9	10
Machine 1	9	3	6	6	3	3	6	6	3	9
Machine 2	3	6	6	3	0	0	3	6	6	3
Machine 3	6	3	6	3	0	0	3	6	3	6
Machine 4	6	6	15	9	6	6	9	15	6	6
Machine 5	9	3	6	6	3	3	6	6	3	9
Machine 6	3	3	6	3	0	0	3	6	3	6
Machine 7	0	6	6	3	0	0	3	6	6	3
Machine 8	9	3	6	6	3	3	6	6	3	9
Machine 9	6	6	15	9	6	6	9	15	6	6
Machine 10	3	3	6	6	3	3	6	6	3	12

Table 7.24. Variance of Processing Time Data Set II - Flow Shop

Expectation and Variance of the Makespan:

S No	Sequence	E [Makespan]	Var [Makespan]
1	10 9 8 2 3 4 6 5 1 7	1051.20	474.50
2	1 2 10 9 4 8 6 7 5 3	1064.80	231.60
3	4 3 1 6 2 10 9 7 5 8	1052.10	133.00
4	1 2 3 4 5 6 7 8 9 10	1017.60	72.80
5	10 9 8 7 6 5 4 3 2 1	999.28	103.14
6	2 1 4 6 8 10 5 7 9 3	1006.10	218.20
7	2 5 1 7 4 6 9 3 8 10	979.99	488.54
8	8 1 3 5 4 7 10 6 9 2	1062.30	60.00
9	6 1 9 7 3 8 4 2 10 5	952.33	136.08
10	5 9 3 10 1 6 8 2 7 4	947.07	567.95

Table 7.25. Expectation and Variance of the Makespan - Flow Shop (Data Set II)

Note: As there are numerous possible sequences, the applicability and significance of the program on larger problems is illustrated for a few sequences as shown above.

7.4 Job Shops with Unlimited Intermediate Storage

Sample Data Set I:

Total Number of Jobs = 3

Total Number of Machines = 3

Distribution Assumption, if any: *Normal distribution*

The mean processing times of the 3 jobs on the 3 machines is given in Table 7.26 below.

Job Index	**1**	**2**	**3**
M/c 1	28	35	7
M/c 2	42	11	34
M/c 3	16	80	43

Table 7.26. Mean Processing Time Data Set I - Job Shop

The variance of the processing times of the 3 jobs on the 3 machines is given in Table 7.27 below.

Job Index	**1**	**2**	**3**
M/c 1	10	22	3
M/c 2	25	6	15
M/c 3	5	35	12

Table 7.27. Variance of Processing Time Data Set I - Job Shop

The job routings for all the jobs is given in Table 7.28 below.

Operation Sequence	**Machine Index**		
Job 1	1	2	3
Job 2	3	2	1
Job 3	2	1	3

Table 7.28. Job Routing Data Set I - Job Shop

Given schedule:

The processing sequence for the 3 machines is given in Table 7.29 below.

Processing Sequence	**Job Index**		
M/c 1	1	2	3
M/c 2	3	1	2
M/c 3	2	1	3

Table 7.29. Machine Processing Sequence Data Set I - Job Shop

Expectation and Variance of the Completion Times:

The mean completion times of all the 3 jobs on all 3 machines for the given schedule are given in Table 7.30 below.

Job Index	**1**	**2**	**3**
M/c 1	28	127.86	134.86
M/c 2	76.2	92.86	34
M/c 3	97.86	80	177.85

Table 7.30. Mean Completion Time - Job Shop (Data Set I)

The variance of the completion times of all the 3 jobs on all 3 machines for the given schedule are given in Table 7.31 below.

Job Index	**1**	**2**	**3**
M/c 1	10	53.5	58.5
M/c 2	37.66	31.5	15
M/c 3	30.51	35	68.5

Table 7.31. Variance of Completion Time - Job Shop (Data Set I)

The mean and variance of the makespan for the given schedule is **177.86** and **68.5** respectively.

Note: As there are numerous possible schedules, the applicability and significance of the program is illustrated for one given schedule.

Sample Data Set II:

Total Number of Jobs = 15

Total Number of Machines = 11

Distribution Assumption, if any: *Normal distribution*

The mean processing times of the 15 jobs on the 11 machines is given in Table 7.32 below.

Job Index	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
M/c 1	72	23	55	91	43	49	75	72	39	5	24	68	80	95	47
M/c 2	26	85	50	68	70	69	90	9	80	88	49	18	60	23	28
M/c 3	38	67	61	72	34	9	91	91	63	68	18	77	40	92	24
M/c 4	19	43	93	96	12	27	17	31	52	50	91	60	56	6	77
M/c 5	30	39	7	99	30	59	8	38	35	53	33	18	51	10	48
M/c 6	98	13	80	68	7	72	37	98	25	24	19	15	89	23	29
M/c 7	74	19	57	60	74	45	71	49	35	20	38	90	91	46	77
M/c 8	8	9	46	43	84	99	50	90	47	57	99	10	72	72	8
M/c 9	43	74	72	12	40	61	65	62	22	53	9	60	86	41	55
M/c 10	75	26	42	11	69	63	98	72	74	58	35	72	79	34	49
M/c 11	43	73	72	11	40	60	65	62	21	52	9	60	85	40	55

Table 7.32. Mean Processing Time Data Set II - Job Shop

The variance of the processing times of the 15 jobs on the 11 machines is given in Table 7.33 below

Job Index	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
M/c 1	7	2	6	9	4	5	8	7	4	1	2	7	8	10	5
M/c 2	3	9	5	7	7	7	9	1	8	9	5	2	6	2	3
M/c 3	4	7	6	7	3	1	9	9	6	7	2	8	4	9	2
M/c 4	2	4	9	10	1	3	2	3	5	5	9	6	6	1	8
M/c 5	3	4	1	10	3	6	1	4	4	5	3	2	5	1	5
M/c 6	10	1	8	7	1	7	4	10	3	2	2	2	9	2	3
M/c 7	7	2	6	6	7	5	7	5	4	2	4	9	9	5	8
M/c 8	1	1	5	4	8	10	5	9	5	6	10	1	7	7	1
M/c 9	4	7	7	1	4	6	7	6	2	5	1	6	9	4	6
M/c 10	8	3	4	1	7	6	10	7	7	6	4	7	8	3	5
M/c 11	4	7	7	1	4	6	7	6	2	5	1	6	9	4	6

Table 7.33. Variance of Processing Time Data Set II - Job Shop

The job routings for all the jobs is given in Table 7.34 below.

Operation Sequence	Machine Index									
Job 1	8	10	1	7	5	9	3	6	2	4
Job 2	7	11	4	1	2	5	6	10	3	8
Job 3	2	4	6	5	1	3	7	9	10	8
Job 4	2	8	5	7	6	1	11	4	10	3
Job 5	8	3	9	6	2	7	4	1	10	5
Job 6	11	1	5	6	10	2	8	7	4	3
Job 7	7	3	9	2	10	5	8	1	6	4
Job 8	9	8	6	4	3	5	10	2	1	7
Job 9	5	1	10	6	8	4	3	11	7	2
Job 10	10	1	4	11	2	7	3	6	5	8
Job 11	8	4	5	6	3	7	1	10	2	9
Job 12	1	4	3	8	9	6	10	2	7	5
Job 13	10	2	4	7	3	11	8	1	6	5
Job 14	5	3	6	7	11	8	4	2	1	10
Job 15	3	6	10	9	1	7	4	8	2	5

Table 7.34. Job Routing Data Set II - Job Shop

Given schedule:

The processing sequence for the 11 machines is given in Table 7.35 below.

Processing Sequence		Job Index												
M/c 1	12	9	6	10	2	3	15	1	14	4	11	13	7	5	8
M/c 2	3	4	13	2	5	7	14	6	10	12	15	1	8	9	11
M/c 3	15	14	5	7	3	13	12	2	11	8	1	9	10	4	6
M/c 4	3	2	13	10	12	11	14	8	9	15	4	5	6	1	7
M/c 5	14	9	6	3	4	2	11	1	7	8	15	13	10	12	5
M/c 6	15	14	3	5	6	8	9	2	4	11	12	1	13	10	7
M/c 7	2	7	14	13	4	3	1	15	11	10	5	6	9	12	8
M/c 8	1	5	8	4	11	14	9	12	13	6	7	15	3	2	10
M/c 9	8	5	15	7	3	12	1	11	-	-	-	-	-	-	-
M/c 10	13	10	15	1	9	6	2	7	3	12	14	8	4	11	5
M/c 11	6	2	14	10	13	4	9	-	-	-	-	-	-	-	-

Table 7.35. Machine Processing Sequence Data Set II - Job Shop

Expectation and Variance of the Completion Times:

The mean completion times of all the 15 jobs on all 11 machines for the given schedule are given in Table 7.36 below.

Job Index	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
M/c 1	401.1	199	282.1	655.9	878	156	835	950	107	161	680	68	760	564.9	329.1
M/c 2	802.6	284	50	118	354	538.9	444	811.7	895.6	626.9	944.6	691.9	178	469.9	737.1
M/c 3	678.6	530.8	343.1	881.8	150	890.8	241	640.5	741.6	809.6	549.5	460.1	383.1	116	24
M/c 4	832.3	176	93	772.2	784.2	812.4	943.9	473.5	527.2	285.5	436.5	345.5	235.5	442.5	674.9
M/c 5	550.9	366.7	227.1	327.7	1018	215	567.5	678.5	45	969.9	469.5	987.9	916.9	10	785.1
M/c 6	776.6	434.1	219	502.1	226	298.1	926.9	396.1	421.1	889.9	521.1	593.9	865.9	139	29
M/c 7	520.9	19	446.9	389.9	730.9	775.9	90	999	811.3	656.9	635.9	901.3	326.5	185	597.9
M/c 8	8	763.9	754.9	225	92	650.9	700.9	182	468.1	1027	324	479.1	551.6	396	708.9
M/c 9	622	-	518.9	-	190	-	311.8	62	-	-	953.6	578.9	-	-	245.9
M/c 10	261	460.1	600.3	792.3	947	398	558.3	779.9	335	137	827.3	673.9	79	707.9	186
M/c 11	-	133	-	666.9	-	60	-	-	762.6	337.5	-	-	468.1	225	-

Table 7.36. Mean Completion Time - Job Shop (Data Set II)

The variance of the completion times of all the 15 jobs on all 11 machines for the given schedule are given in Table 7.37 below.

Job Index	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
M/c 1	34.73	18.86	22.73	49.84	70.21	16.00	66.21	77.21	11.00	16.99	50.21	7.00	58.21	40.84	27.73
M/c 2	57.15	27.85	5.00	12.00	34.85	37.84	43.85	56.68	44.46	46.84	49.46	49.07	18.00	30.84	50.39
M/c 3	44.15	35.85	28.73	60.56	14.00	61.56	23.00	41.59	50.15	57.15	32.59	40.13	32.13	11.00	2.00
M/c 4	38.19	17.00	9.00	50.25	51.25	43.19	66.87	42.27	36.57	23.41	38.27	29.41	18.41	39.27	51.97
M/c 5	38.97	26.89	16.73	22.89	73.08	22.00	39.67	45.59	5.00	68.12	41.27	70.12	63.87	1.00	55.39
M/c 6	54.15	41.51	21.00	48.51	22.00	27.51	64.87	37.51	40.51	60.87	50.48	43.97	58.87	13.00	3.00
M/c 7	35.97	2.00	28.97	22.97	47.49	52.49	9.00	82.20	50.03	40.49	47.97	59.03	27.41	18.00	43.97
M/c 8	1.00	56.58	55.58	22.00	9.00	44.99	49.99	18.00	45.34	74.12	32.00	36.46	38.85	39.00	50.58
M/c 9	45.91	-	35.97	-	18.00	-	22.50	6.00	-	-	50.46	41.97	-	-	19.65
M/c 10	27.00	44.50	55.78	43.97	77.21	40.00	51.78	56.96	34.00	14.00	47.97	47.07	8.00	50.07	19.00
M/c 11	-	13.00	-	50.84	-	6.00	-	-	52.15	28.41	-	-	41.13	22.00	-

Table 7.37. Variance of Completion Time - Job Shop (Data Set II)

The mean and variance of the makespan for the given schedule is 1028.5 and 55.84 respectively.

Note: As there are numerous possible schedules, the applicability and significance of the program on larger problems is illustrated for the one schedule as shown above.

7.5 Parallel Machines with No Preemptions

Sample Data Set:

Total Number of Jobs = 8
Total Number of Machines = 3

The mean and variance of the processing times of the 8 jobs are given in Table 7.29 below.

Job Index	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
Mean	2	4	6	8	7	10	12	3
Variance	2	3	2	2	3	2	4	3

Table 7.38. Data Set for Parallel Machines

7.5.1 Makespan with No Preemptions

Distribution Assumption, if any: *Normal distribution*

Solution Table:

S No	Sequence	E[Makespan]	Var[Makespan]
1	Machine 1: 1 – 2 - 3 Machine 2: 4 - 5 Machine 3: 6 – 7 - 8	25.0044	8.9382
2	Machine 1: 1 - 2 Machine 2: 3 - 4 - 5 Machine 3: 6 – 7 – 8	25.33	7.23
3* & **	**Machine 1: 1 – 2 -7 - 8** **Machine 2: 3 – 4 - 5** **Machine 3: 6**	**22.74**	**6.4760**
4	Machine 1: 1 - 8 Machine 2: 2 – 6 - 7 Machine 3: 4 – 5	26.002	8.91
5	Machine 1: 1 - 8 Machine 2: 2 – 6 - 7 Machine 3: 4 – 5	26.002	8.91

Table 7.39. Expectation and Variance of the Makespan – Parallel Machines

(*Schedules(s) that are optimal in terms of the expected value)

(**Other schedule(s) that is (are) efficient in terms of variance when compared with *schedule(s) and may have slightly higher expected value (s))

7.5.2 Total Completion Time with No Preemptions

Distribution Assumption, if any: *No, applicable to any distribution*

Solution Table:

S No	Sequence	E[TCT]	Var[TCT]
1	Machine 1: 1 – 2 - 3 Machine 2: 4 - 5 Machine 3: 6 – 7 - 8	100	80
2	Machine 1: 1 - 2 Machine 2: 3 - 4 - 5 Machine 3: 6 – 7 – 8	106	77
3	Machine 1: 1 – 2 -7 - 8 Machine 2: 3 – 4 - 5 Machine 3: 6	98	109
4* & **	**Machine 1: 1 - 8** **Machine 2: 2 – 6 - 7** **Machine 3: 4 – 5**	**74**	**61**
5	Machine 1: 1 - 8 Machine 2: 2 – 6 - 7 Machine 3: 4 – 5	89	106

Table 7.40. Expectation and Variance of the Total Completion Time – Parallel Machines

(*Schedules(s) that are optimal in terms of the expected value)
(**Other schedule(s) that is (are) efficient in terms of variance when compared with *schedule(s) and may have slightly higher expected value (s))

7.6 Concluding Remarks

Thus, in this chapter, we have numerically illustrated the significance and applicability of the expressions and methodologies developed for determining the expectation and variance of different performance measures for different scheduling environments. Larger problems were tested for flow shop and job shop cases to illustrate the efficiency of the programs in solving higher dimensionality instances. It can be witnessed that incorporating these expressions and methodologies in the various scheduling algorithms and search procedures would definitely aid in determining 'good' schedules that are efficient in terms of the expected value and variance of the performance measure involved.

8 THESIS SUMMARY AND FUTURE RESEARCH

8.1 Introduction

In this concluding chapter, we summarize the work done as part of our research endeavor and provide directions that would lead to future research in this interesting and challenging domain.

8.2 Thesis Summary

The primary motivation for this research was the inherent variability in the various parameters (job processing times, due dates among others) that are involved in production scheduling tasks and their significant impact on the performance of the system. Variability in scheduling has been modeled in the form of random variables and the majority of work undertaken in this field considered optimizing the expected value of the performance measure of interest. In doing so, very little importance was attributed to variability of the system and, consequently, very little significance was given to the variance of the performance measure. It is often witnessed that a scheduler's preference is not a schedule that has a lower expected value but the one that also minimizes the variability of the system. Hence schedule evaluation must be performed in terms of both the expectation and the variance of the performance measure. The inclusion of variance elements in schedule optimization leads to complexities that, apparently, have resulted in a dearth of reported work in the literature with regard to evaluating the variance of a performance measure. Our research was intended to address the problem of determining closed-form expressions and methodologies to determine the expectation and variance of the different performance measures and in diverse scheduling environments. This effort would lead the way in considering variance issues in schedule optimization and determining schedules that are both expectation and variance efficient.

We considered the following scheduling environments and analytically evaluated the expectation and variance of different performance measures:

1. Scheduling on a single machine,
2. Scheduling on identical machines in parallel,
3. Permutation flow shops with unlimited intermediate storage, and
4. Job shops with unlimited intermediate storage

The performance measures were both completion time based and due date based for the single machine while the makespan objective was analyzed for the flow shop and job shop cases. For parallel machine models, the measures used were total completion time and makespan for the case of no preemptions and only makespan for the preemptions case. The processing times were stochastic with known probability distributions and the schedule was given *a priori*. Computing the variance of the different performance measures was found to be highly complicated and not straightforward. For example, the analysis of tardiness based performance measures for the single machine involved the use of the maximum operator, and mathematical theory does not allow for exact evaluation of the maximum function. Hence, Clark's equations for normally distributed random variables were used in determining approximate evaluations of the performance measures. Similar approach was adopted for makespan related measures for parallel machines, flow shops and job shops. Additionally, studying and evaluating the correlations between the various random variables was a challenging and essential part of the research. The correlation structures became progressively complex as we shifted from a single machine to a multi-machine environment. The developed expressions and methodologies for all the cases were programmed in MATLAB R12 and their implementation was illustrated on some examples. Larger problems were also run for the flow shop and job shop cases to test the efficiencies of the developed programs.

8.3 Significance of the Research

Variance consideration in stochastic scheduling has been a very challenging field of research and it holds the interest of many researchers. The significance of our work lies in the fact that we have addressed a very serious issue that has got very limited attention in the literature. We have elicited an important point that schedules must be evaluated in terms of both the expectation and variance. By incorporating variance issues in scheduling, a scheduler will be in a position to make better decisions after knowing both the expected value and variability of a given schedule. This will in turn, importantly, increase the stability of the production system which is the ultimate goal of all production managers.

The software programs we developed can be easily incorporated in the existing scheduling packages to identify expectation-variance efficient schedules. Thus, our work is a major contribution to the field of research that involves determining schedules that are good in terms of both the expected value and variability of the performance measure.

8.4 Scope for Future Research

Variance consideration in stochastic scheduling is still a burgeoning field of research and our research effort can lead into quite a few directions for further exploration in this domain which are listed below:

1. Knowing the variability of the given schedule alone is not enough to improve the schedule. However, awareness of the variability is a good starting point in improving the schedule in terms of its expected value and variance. The next logical step in continuing this research would be in determining a schedule that is ‘good’ in terms of both of these parameters. This could be done by incorporating these expressions and methodologies in the various algorithms and search procedures that are available for scheduling problems.

2. Considering other processing time distributions for the expectation-variance analysis. Processing times were invariably assumed to be normally distributed for performance measures that involved the maximum operator and Clark’s equations were used for the analysis. However, other distributions can be analyzed and the errors in the approximation procedure can be investigated.

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APPENDICES

APPENDIX A – SOFTWARE PROGRAM FOR IMPLEMENTING THE SINGLE MACHINE MODELS

APPENDIX A.1 – Total Completion Time

```
% Objective: TOTAL COMPLETION TIME

% To estimate the expectation and variance of the total completion time
% Job sequence, mean and variance of the job processing times are given

% J - Set containing the job indices
% Seq - Set containing the job sequence
% Mu - Vector containing the mean processing times
% Sigma2 - Vector containing the variance of the processing times

function TCT()
        E=0;
        V=0;
        global J Seq Mu Sigma2
         n = length(J);             % Total number of jobs

% Compute the expectation
   for j = 1:n
      i = Seq(j);
      E = E + (n+1-j)*Mu(i,2);
   end

% Compute the individual variance
  for j = 1:n
```

```
        i = Seq(j);
        V = V + (n+1-j)^2*Sigma2(i,2);
    end

E           % Display the expected value of the total completion time
V           % Display the variance of the total completion time
```

APPENDIX A.2 – Total Weighted Completion Time

```
% Objective: TOTAL WEIGHTED COMPLETION TIME

% To estimate the expectation and variance of the total weighted completion time
% Job sequence, mean and variance of the job processing times, and job weights
(importance factors) are given

% J - Set containing the job indices
% Seq - Set containing the job sequence
% Mu - Vector containing the mean processing times
% Sigma2 - Vector containing the variance of the processing times
% Weight - Vector containing the job weights (importance factors)

function TWCT()
        E=0;
        V=0;
        global J Seq Sigma2 Mu Weight
        n = length(J);

% Compute the expectation
for j = 1:n
    G(j) = 0;
    for k = j:n
        i = Seq(k);
        G(j) = G(j) + Weight(i,2);
    end
    E = E + G(j)*Mu(Seq(j),2);
end
```

```
for j = 1:n
    G(j) = 0;
    for k = j:n
        i = Seq(k);
        G(j) = G(j) + Weight(i,2);
    end
    V = V + G(j)^2*Sigma2(Seq(j),2);
end

E       % Display the expected value of the total weighted completion time
V       % Display the variance of the total weighted completion time
```

APPENDIX A.3 – Total Weighted Discounted Completion Time

APPENDIX A.3.1– Normally Distributed Processing Times

% Objective: TOTAL WEIGHTED DISCOUNTED COMPLETION TIME FOR NORMALLY DISTRIBUTED PROCESSING TIMES

% To estimate the expectation and variance of the total weighted discounted completion time
% Job sequence, job processing time distributions, job weights and discount rate 'r' are given
% This program is developed for normally distributed job processing times

```
function TWDCTnor()
        E=0;
        V =0;
        global J Seq Weight r Mu Sigma2
        n = length (J);
        W = sum(Weight(:,end))        % Sum of all the weights

 % Compute the expectation
  for j = 1:n
    m = fEComp(j);
    v =  fVComp(j);
    M = exp(-r*m + v*r^2/2);
    E = E + Weight(Seq(j),2)*M;
  end

V1 = 0;  % Variance variable
V2 = 0;  % Covariance variable
```

```
% Compute the individual variance

    for j = 1:n
        m = fEComp(j);
        v =  fVComp(j);
        M1 = exp(-2*r*m + 2*v*r^2);
        M2 = exp(-2*r*m + v*r^2);
        V1 = V1 + (Weight(j,2)^2)*(M1 - M2);
    end

% Compute the pairwise covariance

    for i = 1:n
      for j = i:n
         m1 = fsumEComp(i,j);
         v1 = fsumVComp(i,j);
         m2 = fEComp(i);
         v2 =  fVComp(i);
         m3 = fEComp(j);
         v3 =  fVComp(j);

         M1 = exp(-r*m1 + v1*r^2/2);
         M2 = exp(-r*m2+ v2*r^2/2)*exp(-r*m3 + v3*r^2/2);
         V2 = V2 + Weight(i,2)*Weight(j,2)*(M1-M2);
     end
    end
   % Expectation and Variance of the total weighted discounted completion time

         E = W - E
         V =  -2*V2  - V1
```

```
%*********************************************************************
% To compute the expectation of the completion time of a job
% It is the sum of its mean processing time and all the jobs ahead of it in the schedule

function m = fEComp(u)
   global J Seq Mu Sigma2
   m = 0;
   for k = 1:u
      i = Seq(k);
      m = m + Mu(i,2);
   end

%*********************************************************************
% To compute the variance of the completion time of a job
% It is the sum of its processing time variance and all the jobs ahead of it in the schedule

function m = fVComp(u)
   global J Seq Mu Sigma2
   m = 0;
   for k = 1:u
      i = Seq(k);
      m = m + Sigma2(i,2);
   end

%********************************************************************
% To compute the expectation of the sum of the completion times of two jobs (Ci + Cj)

function m = fsumEComp(u,v)
   global J Seq Mu Sigma2
   m = 0;
   for k = 1:u
```

```
    i = Seq(k);
    m = m + 2*Mu(i,2);
  end

  for k = u+1:v
    i = Seq(k);
    m = m + Mu(i,2);
  end

%**********************************************************************
% To compute the variance of the sum of the completion times of two jobs (Ci + Cj)

function m = fsumVComp(u,v)
  global J Seq Mu Sigma2
  m = 0;
  for k = 1:u
    i = Seq(k);
    m = m + 2*Sigma2(i,2);
  end

  for k = u+1:v
    i = Seq(k);
    m = m + Sigma2(i,2);
  end
```

APPENDIX A.3.2– Exponentially Distributed Processing Times

% Objective: TOTAL WEIGHTED DISCOUNTED COMPLETION TIME

% To estimate the expectation and variance of the total weighted discounted completion time

% Job sequence, job processing time distributions, job weights and discount rate 'r' are given

% This program is developed for exponentially distributed job processing times with given 'lambdas'

```
function TWDCTexp()
        E=0;
        V=0;
        global J Seq Weight r lambda
        n = length (J);
        W = sum(Weight(:,end))      % Sum of all the weights

 % Compute the expectation

  for j = 1:n
    M = fprod(1,j,r);
    E = E + Weight(Seq(j),2)*M;
  end

        V1 = 0;  % Variance variable
        V2 = 0;  % Covariance variable

% Compute the individual variance
```

```
   for j = 1:n
        M1 = fprod(1,j,2*r);
        M2 = fprod(1,j,r)^2;
        V1 = V1 + (Weight(j,2)^2)*(M1 - M2);
   end

% Compute the pairwise covariance

   for i = 1:n
      for j = i:n
         M1 = fprod(1,i,2*r)*fprod(i+1,j,r);
         M2 = fprod(1,i,r)*fprod(1,j,r);
         V2 = V2 + Weight(i,2)*Weight(j,2)*(M1-M2);
      end
   end

% Variance of the total weighted discounted completion time

         E = W - E

         V =  -2*V2  - V1

%**********************************************************************
% Computing the moment generating functions for the job completion times when
% the individual job processing times are exponentially distributed
function t = fprod(l,u,s)
   M = 1;
   global Seq lambda
   for k = l:u
      i = Seq(k);
      m = lambda(i,2)/(lambda(i,2)+s);   % compute the m.g.f for individual processing
```

```
                                        times
    M = M*m;                            % compute the m.g.f for completion time for
                                          each job
  end
t = M;
```

APPENDIX A.4 – Total Tardiness

```
% Objective: TOTAL TARDINESS

% To estimate the expectation and variance of the total tardiness
% Job sequence, due dates, mean and variance of the job processing times are given
% Job processing times are assumed to be normally distributed
% Due - Vector containing the job due dates (deterministic)

function TT()
        global J Seq Mu Sigma2 Due n
        n = length(J);    % Total number of jobs

% Application of Clarke's equations to determine the expectation of the total tardiness

for j = 1: n
    i = Seq(j);
    MT(j) = fEComp(j)- Due(i,2);  % Mean of Tj, Tardiness of the jth job in the sequence
     VT(j) = fVComp(j);                % Variance of Tj
      a(j) = VT(j);
      alpha(j) = MT(j)/a(j);
      MMax(j) = MT(j)*normcdf(alpha(j)) + a(j)*fSi(alpha(j));
                                              % First Moment and Expectation of the max(Tj,0)
      SM(j) = (MT(j)^2 + VT(j))*normcdf(alpha(j)) + MT(j)*a(j)*fSi(alpha(j));
                                              % Second Moment of the max(Tj,0)
      VMax(j) = SM(j) - MMax(j)^2;   % Variance of the max(Tj,0)
end
```

```
% Determining the covariance between all pairs of Ti and Tj, Cov[Ti,Tj]

for i = 1: n
    for j = 1:n
        if i == j
            continue
        end
        Ro(i,j) = 1;
        Cov(i,j)=(1-fCDF(i))*(1-CDF(j))*Ro(i,j)*sqrt(abs(VMax(i)))*sqrt(abs(VMax(j)));
    end
end

% Expectation and Variance of the Total Tardiness

        E = sum(MMax)

        V = sum(VMax) + sum(sum(Cov))


%*****************************************************************
% To compute the cumulative distribution function for the completion time of a job at its due date
% It is the sum of its mean processing time and all the jobs ahead of it in the schedule

function m = fCDF(j)
    global Seq Due
    i = Seq(j);
    a = fEComp(j);
    b = fVComp(j);
    d = Due(i,2);
    m = normcdf(Due(i,2),a,sqrt(b));
```

```
%********************************************************************
% To compute the expectation of the completion time of a job
% It is the sum of its mean processing time and all the jobs ahead of it in the schedule

function m = fEComp(u)
   global J Seq Mu Sigma2
   m = 0;
   for k = 1:u
      i = Seq(k);
      m = m + Mu(i,2);
   end

%********************************************************************
% To compute the variance of the completion time of a job
% It is the sum of its processing time variance and all the jobs ahead of it in the schedule

function m = fVComp(u)
   global J Seq Mu Sigma2
   m = 0;
   for k = 1:u
      i = Seq(k);
      m = m + Sigma2(i,2);
   end

%********************************************************************
% To compute the normal distribution for a given number

function m = fSi(u)
   m = [1/sqrt(2*pi)]*exp(-u^2/2);
```

APPENDIX A.5 – Total Weighted Tardiness

```
% Objective: TOTAL WEIGHTED TARDINESS

% To estimate the expectation and variance of the total weighted tardiness
% Job sequence, due dates, weights, mean and variance of the job processing times are
given
% Job processing times are assumed to be normally distributed

function TWT()
        global J Seq Mu Sigma2 Weight Due n
        n = length(J);

% Application of Clarke's equations to determine the expectation of the total tardiness

for j = 1: n
    i = Seq(j);
    MT(j) = fEComp(j)- Due(i,2);  % Mean of Tj, Tardiness of the jth job in the sequence
    VT(j) = fVComp(j);                % Variance of Tj
    a(j) = VT(j);
   alpha(j) = MT(j)/a(j);
   MMax(j) = MT(j)*normcdf(alpha(j)) + a(j)*fSi(alpha(j));
                                        % First Moment and Expectation of the max(Tj,0)
  SM(j) = (MT(j)^2 + VT(j))*normcdf(alpha(j)) + MT(j)*a(j)*fSi(alpha(j));
                                        % Second Moment of the max(Tj,0)
   VMax(j) = SM(j) - MMax(j)^2;           % Variance of the max(Tj,0)
end

% Determining the covariance between all pairs of Ti and Tj, Cov[Ti,Tj]

for i = 1: n
```

```
    for j = 1:n
        if i == j
            continue
        end
        ro(i,j) = 1;
        Cov(i,j)=(1-fCDF(i))*(1-CDF(j))*ro(i,j)*sqrt(abs(VMax(i)))*sqrt(abs(VMax(j)));
    end
end

% Expectation and Variance of the Total Tardiness

        E = 0;
        V =0;

for j = 1:n
    k = Seq(j);
    E = E + Weight(k,2)*MMax(j);
    V = V + Weight(k,2)^2*VMax(j);
    for i = 1:n
        l = Seq(i);
        if i==j
            continue
        end
        V = V+ Weight(k,2)*Weight(l,2)*Cov(i,j);
    end
end

E
V
```

```
%**********************************************************************
% To compute the cumulative distribution function for the completion time of a job at its
due date
% It is the sum of its mean processing time and all the jobs ahead of it in the schedule

function m = fCDF(j)
   global Seq Due
   i = Seq(j);
   a = fEComp(j);
   b = fVComp(j);
   d = Due(i,2);
   m = normcdf(Due(i,2),a,sqrt(b));

%**********************************************************************
% To compute the expectation of the completion time of a job
% It is the sum of its mean processing time and all the jobs ahead of it in the schedule

function m = fEComp(u)
   global J Seq Mu Sigma2
   m = 0;
   for k = 1:u
      i = Seq(k);
      m = m + Mu(i,2);
   end

%**********************************************************************
% To compute the variance of the completion time of a job
% It is the sum of its processing time variance and all the jobs ahead of it in the schedule
function m = fVComp(u)
   global J Seq Mu Sigma2
   m = 0;
```

```
   for k = 1:u
      i = Seq(k);
      m = m + Sigma2(i,2);
   end

%*******************************************************************
% To compute the normal distribution for a given number

function m = fSi(u)
   m = [1/sqrt(2*pi)]*exp(-u^2/2);
```

APPENDIX A.6 – Total Number of Tardy Jobs

```
% Objective: TOTAL NUMBER OF TARDY JOBS

% To estimate the expectation and variance of the total number of tardy jobs
% Job sequence, due dates, mean and variance of the job processing times are given
% This program is developed for job processing times are assumed to be normally distributed
% Similar programs can be developed for other distributions too

function TNT()
        global J Seq Due Mu Sigma2 n
        n = length(J);
        E =0;
        V =0;
        for j =1:n
                E(j) = (1 - fCDF(j));
                V(j) = fCDF(j)*(1 - fCDF(j));
        end
        E, sum(E)
        V, sum(V)

%*******************************************************************
% To compute the cumulative distribution function for the completion time of a job at its due date
% It is the sum of its mean processing time and all the jobs ahead of it in the schedule


function m = fCDF(j)
   global Seq Due
   i = Seq(j);
```

```
    a = fEComp(j);
    b = fVComp(j);
    d = Due(i,2);
    m = normcdf(Due(i,2),a,sqrt(b));

%*****************************************************************
% To compute the expectation of the completion time of a job
% It is the sum of its mean processing time and all the jobs ahead of it in the schedule


function m = fEComp(u)
    global J Seq Mu Sigma2
    m = 0;
    for k = 1:u
        i = Seq(k);
        m = m + Mu(i,2);
    end
%*****************************************************************
% To compute the variance of the completion time of a job
% It is the sum of its processing time variance and all the jobs ahead of it in the schedule

function m = fVComp(u)
    global J Seq Mu Sigma2
    m = 0;
    for k = 1:u
        i = Seq(k);
        m = m + Sigma2(i,2);
   end
```

APPENDIX A.7 – Total Weighted Number of Tardy Jobs

```
% Objective: TOTAL WEIGHTED NUMBER OF TARDY JOBS

% To estimate the expectation and variance of the total weighted number of tardy jobs
% Job sequence, due dates, weights, mean and variance of the job processing times are
given
% Job processing times are assumed to be normally distributed
% Similar programs can be developed for other distributions too

function TWNT()
        global J Seq Due Mu Sigma2 Weight n
        n = length(J);
        E =0;
        V =0;
        for j =1:n
                 i = Seq(j);
                E(j) = Weight(i,2)*(1 - fCDF(j));
                V(j) = Weight(i,2)^2*fCDF(j)*(1 - fCDF(j));
        end
        E, sum(E)
        V, sum(V)

%*****************************************************************
% To compute the cumulative distribution function for the completion time of a job at its
due date
% It is the sum of its mean processing time and all the jobs ahead of it in the schedule

function m = fCDF(j)
   global Seq Due
   i = Seq(j);
```

```
   a = fEComp(j);
   b = fVComp(j);
   d = Due(i,2);
   m = normcdf(Due(i,2),a,sqrt(b));

%******************************************************************
% To compute the expectation of the completion time of a job
% It is the sum of its mean processing time and all the jobs ahead of it in the schedule

function m = fEComp(u)
   global J Seq Mu Sigma2
   m = 0;
   for k = 1:u
      i = Seq(k);
      m = m + Mu(i,2);
   end

%******************************************************************
% To compute the variance of the completion time of a job
% It is the sum of its processing time variance and all the jobs ahead of it in the schedule

function m = fVComp(u)
   global J Seq Mu Sigma2
   m = 0;
   for k = 1:u
      i = Seq(k);
      m = m + Sigma2(i,2);
   end
```

APPENDIX A.8 – Mean Lateness

```
% Objective: MEAN LATENESS

% To estimate the expectation and variance of mean lateness
% Job sequence, mean and variance of the job processing times and job due dates are
given

function ML()
        global J Seq Mu Sigma2 Due
        T=0;
        S=0;
        n = length(J);
        Duesum = sum(Due(:,end))   % Sum of the due dates

  for j = 1:n
     i = Seq(j);
     T = T + (n+1-j)*Mu(i,2);
  end
        E  = (T - Duesum)/n;

  for j = 1:n
     i = Seq(j);
     S =S + (n+1-j)*Sigma2(i,2);
  end
        V  = T/n^2;
E
V
```

APPENDIX A.9 – Maximum Lateness

```
% Objective: MAXIMUM LATENESS

% To estimate the expectation and variance of the maximum lateness
% Job sequence, due dates, weights, mean and variance of the job processing times are given
% Job processing times are assumed to be normally distributed
% The lateness random variables are mutually related by the multivariate distribution

function MaxL()
    global n J Due Seq MMax VMax Cor alpha Ro VL
    n = length(J);

    % Assigning a correlation value of 1 to all pairs of jobs
    for i = 1: n
        for j = 1:n
            if i==j
            continue
            end
            Ro(i,j) = 1;
        end
    end


% Assigning a correlation value of 1 between the 1st job and any other job in the sequence used in the recursive equations

for j = 2:n
  Cor(1,j) =1;
end
```

```
% Application of Clarke's equations

        i = Seq(1);
        ML(1) = fEComp(1) - Due(i,2);
        VL(1) = fVComp(1);
        MMax(1) = ML(1);
        VMax(1) = VL(1);

for j = 2: n
    i = Seq(j);
    ML(j) = fEComp(j)- Due(i,2);  % Mean of Lj, Lateness of the jth job in the sequence
    VL(j) = fVComp(j);            % Variance of Lj
    k = j-1;
    a(j)= sqrt(VMax(k) + VL(j) - 2*sqrt(VMax(k))*sqrt(VL(j))*fCor(k,j));
    alpha(j) = (MMax(k) - ML(j))/a(j);
    MMax(j) = MMax(k)*normcdf(alpha(j)) + ML(j)*normcdf(-alpha(j)) +
              a(j)*fSi(alpha(j));
                                   % First Moment or mean of Max(L1,L2,...Lj)
    SM(j) = (MMax(k)^2 + VMax(k))*normcdf(alpha(j)) + (ML(j)^2 +
            VL(j))*normcdf(-alpha(j)) + (MMax(k)+ ML(j))*a(j)*fSi(alpha(j));
                                   % Second Moment of Max(L1,L2,...Lj)

    VMax(j) = SM(j) - MMax(j)^2;    % Variance of Max(L1,L2,...Lj)
end

MMax(n)
VMax(n)

%******************************************************************
% To find the correlation Cor(k,j) = Cor(L1,L2,L3,...Lk...Lj)
```

```
function m = fCor(k,j)
   global MMax VMax Cor alpha Ro VL
   if k == 1
         m = 1;
   else
         m = (sqrt(VMax(k-1))*fCor(k-1,j)*normcdf(alpha(k)) +
         sqrt(VL(k))*Ro(k,j)*normcdf(-alpha(k)))/ sqrt(VMax(k));
   end


%*********************************************************************
% To compute the expectation of the completion time of a job
% It is the sum of its mean processing time and all the jobs ahead of it in the schedule

function m = fEComp(u)
   global Seq Mu
   m = 0;
   for k = 1:u
      i = Seq(k);
      m = m + Mu(i,2);
   end

%*********************************************************************
% To compute the variance of the completion time of a job
% It is the sum of its processing time variance and all the jobs ahead of it in the schedule

function m = fVComp(u)
   global Seq Sigma2
   m = 0;
   for k = 1:u
```

```
    i = Seq(k);
    m = m + Sigma2(i,2);
  end


%********************************************************************
% To compute the normal distribution for a given number

function m = fSi(u)
  m = [1/sqrt(2*pi)]*exp(-u^2/2);
```

APPENDIX B - SOFTWARE PROGRAM FOR IMPLEMENTING THE PERMUTATION FLOW SHOPS WITH UNLIMITED INTERMEDIATE STORAGE

```
% Objective: MAKESPAN

% To compute the expectation and variance of the makespan of a permutation flow shop
% Mean and variance of job processing times and the permutation sequence are known
% Job processing times are assumed to be normally distributed
% Operation start and finish time random variables are mutually related by a multivariate
distribution

function FS()
        global J M Seq Mu Sigma2 Ro MStart VStart MF VF
        n = length(J);
        m = length(M);

% Initializations of mean and variance of the finish times
% Finish times matrix contains a row and column for job '0' and machine '0'

for i=1:m+1
    for j=1:n+1
        MF(i,j) = 0;
        VF(i,j) = 0;
        MStart(i,j) = 0;
        VStart(i,j) = 0;
    end
end

% Finish times for all jobs on machine 1
```

```
for j = 2:n+1
        l = Seq(j-1);
        MStart(2,j) = MF(2,j-1);
        VStart(2,j) = VF(2,j-1);
        MF(2,j) = MStart(2,j) + Mu(1,l);
        VF(2,j) =  VStart(2,j) + Sigma2(1,l);
  end

% Finish times for job 1 on machines from 1 to m

 l = Seq(1);            % Index of the first job in the sequence

for i = 2:m+1
      MStart(i,2)=MF(i-1,2);
      VStart(i,2)=VF(i-1,2);
      MF(i,2) = MStart(i,2) + Mu(i-1,l);
      VF(i,2) = VStart(i,2) + Sigma2(i-1,l);
end

% Initialization of the correlations

  for p=1:m+1
     for q=1:n+1
        for r=1:m+1
           for s=1:n+1
                 if (p == 1) | (q == 1) | (r == 1) | (s ==1)
                    Ro(p,q,r,s) = 0;
                 end
           end
        end
     end
```

```
  end

% Computation of the mean and variance of the finishing times for all the jobs on all
machines

   for i = 3:m+1
      for j = 3:n+1
         l = Seq(j-1);
         Recursion(i,j);
         MF(i,j) =  MStart(i,j) + Mu(i-1,l);
         VF(i,j) =  VStart(i,j) + Sigma2(i-1,l);
      end
   end

MF
VF

%*****************************************************************
% To compute the mean and variance of the starting time, Sij, using a recursive
procedure

function Recursion(i,j)
global J M Seq Mu Sigma2 Ro MStart VStart MF VF

% Step 1
   % Finding the correlation r[Fi-2,j Si,j-1] = r[Fi-2,j max(Fi-1,j-1 Fi,j-2)]
   % Mean and variances of of Fi-2,j and Si,j-1 are known.
   % Ro(i-1,j-1,i,j-2) is known  - correlations inside the max function

      l = Seq(j-2);
      M1 = MF(i-1,j-1) + Mu(i-1,l);
```

```
      M2 = MF(i,j-2) + Mu(i-1,l);
      V1 = VF(i-1,j-1)+ Sigma2(i-1,l);
      V2 = VF(i,j-2)+ Sigma2(i-1,l);
      a = sqrt(V1+V2 - 2*sqrt(V1)*sqrt(V2)*Ro(i-1,j-1,i,j-2));
      if a < 0.0001
         a = 0;
         alpha = 0;
      else
         alpha = (M1-M2)/a;
      end

   % Ro(i-2,j,i-1,j-1) is known
   % Ro(i-2,j,i,j-2)  is assumed to be zero

      Ro(i-2,j,i,j-2) = 0;
      Cor1 = (sqrt(V1)*Ro(i-2,j,i-1,j-1)*normcdf(alpha) + sqrt(V2)*Ro(i-2,j,i,j-
               2)*normcdf(-alpha))/sqrt(VF(i,j-1));

% Step 2
   % Finding the correlation r[Fi-1,j-1 Si,j-1]        =        r[Fi-1,j-1 max(Fi-1,j-1 Fi,j-2)]
   % Ro(i-1,j-1,i-1,j-1) is equal to 1
   % Ro(i-1,j-1,i,j-2)  is known

      Ro(i-1,j-1,i-1,j-1)  = 1;
      Cor2 = (sqrt(V1)*Ro(i-1,j-1,i-1,j-1)*normcdf(alpha) + sqrt(V2)*Ro(i-1,j-1,i,j-
               2)*normcdf(-alpha))/sqrt(VF(i,j-1));

% Step 3

      l = Seq(j-1);
      M1 = MF(i-2,j) + Mu(i-2,l);
```

```
 M2 = MF(i-1,j-1)+ Mu(i-2,l);
 V1 = VF(i-2,j) + Sigma2(i-2,l);
 V2 = VF(i-1,j-1) + Sigma2(i-2,l);
a = sqrt(V1+V2 - 2*sqrt(V1)*sqrt(V2)*Ro(i-2,j,i-1,j-1));
 if a < 0.0001
    a = 0;
    alpha = 0;
 else
    alpha = (M1-M2)/a;
 end

Cor3 = (sqrt(V1)*Cor1*normcdf(alpha) + sqrt(V2)*Cor2*normcdf(-
           alpha))/sqrt(VF(i-1,j))
Ro(i-1,j,i,j-1) = Cor3;

% Final Step
  % Computing the mean and variance of the starting time, Sij

 M1 = MF(i-1,j);
 M2 = MF(i,j-1);
 V1 = VF(i-1,j);
 V2 = VF(i,j-1);
 a = sqrt(V1+V2 - 2*sqrt(V1)*sqrt(V2)*Ro(i-1,j,i,j-1));
 if a < 0.0001
    a = 0;
    alpha = 0;
 else
    alpha = (M1-M2)/a;
 end
MStart(i,j) = M1*normcdf(alpha) + M2*normcdf(-alpha) + a*fSi(alpha);
```

```
    SMStart(i,j) = (M1^2 + V1)*normcdf(alpha)+(M2^2 + V2)*normcdf(-
                    alpha)+(M1+M2)*a*fSi(alpha);
    VStart(i,j) = SMStart(i,j) - MStart(i,j)^2;


%********************************************************************
% To compute the normal distribution for a given number

function m = fSi(u)
   m = (1/sqrt(2*pi))*exp(-u^2/2);
```

APPENDIX C - SOFTWARE PROGRAM FOR IMPLEMENTING THE JOB SHOPS WITH UNLIMITED INTERMEDIATE STORAGE AND NO RECIRCULATION

```
% Objective: MAKESPAN
% To compute the expectation and variance of the makespan of a job shop
% Mean and variance of job processing times on each of the machines
% Job routings are given
% Sequence of operations on each machine is given

% JSeq is a "(nxm)" matrix and 'j'th row gives the route for job 'j'
% MSeq is a (mxn) matrix and 'i'th row gives the processing sequence for machine 'i'
% Mu is a (mxn) matrix and each Mu(i,j) gives the mean processing time of job 'j' on
machine 'i'
 % Sigma2 is a (mxn) matrix and each Sigma2(i,j) gives the variance of the processing
time of job 'j' on machine 'i'

function  JS()

    global J M JSeq MSeq Mu Sigma2 m n MStart VStart MF VF Ro InOps p N NofS
    global MC VC MMax VMax ALPHA

    n = length(J);  % Total number of jobs in the system
    m = length(M); % Total number of machines in the system

    % Determing the number of jobs assigned to each machine
     % Count until zero is reached in the sequence

     for i = 1:m
            N(i) = 0;
```

```
        NofS(i) = 0;       % To count on the number of jobs scheduled on machine 'i'
        for k = 1:n
                if MSeq(i,k)~=0   % JSeq changed to MSeq
                   N(i) = N(i) + 1 ;
                else
                   break
                end
        end
  end

 % Initializations of mean and variance of the finish times

  for i=1:m                    % i - machine index
    for k=1:n                  % k - machine index
          MStart(i,k) = 0;
          SMStart(i,k) = 0; %  This line newly added
          MF(i,k) = 0;
          VStart(i,k) = 0;
          VF(i,k) = 0;
          Sched(i,k) = 0;     % To keep track of the operations that have been scheduled
   end
 end

% Setting all possible correlations to be "Inf"

      for p=1:m
        for q=1:n
          for r=1:m
             for s=1:n
                   Ro(p,q,r,s) = Inf;
             end
```

```
        end
      end
    end


 % Determining the initial operations on each machine and their finish times
 % Operations that are independent and can start at time '0'

  p = 0;                    % To count the number of initial operations

  for i = 1:m
     k = MSeq(i,1);         % index of the job
     if  (JSeq(k,1) == i)
        p=p+1;
        InOps(p,1) = i;
                 % To store the index of the machine in the Initial Operations matrix
        InOps(p,2) = k;
                  % To store the index of the job in the Initial Operations matrix
        MStart(i,k) = 0;
        VStart(i,k) = 0;

        MF(i,k) = MStart(i,k) + Mu(i,k);
        VF(i,k) = VStart(i,k) + Sigma2(i,k);

        NofS(i) = 1;            % To count on the number of jobs scheduled on machine 'i'
        Sched(i,k) = 1;         % To keep track of the operations that have been scheduled
     end
   end


  % Setting zero correlation between all pairs of initial operations
```

```
 for x = 1:p
     for y = x:p
         a = InOps(x,1);
         b = InOps(x,2);
         c = InOps(y,1);
         d = InOps(y,2);

         if (a==c) & (b==d)
             Ro(a,b,c,d) = 1;
             Ro(c,d,a,b) = 1;
         else
             Ro(a,b,c,d) = 0;
             Ro(c,d,a,b) = 0;
         end
     end
 end

% Scheduling the next job on each machine and determining their finish times

loop =1;

while (loop == 1)

        for i = 1:m
            if (NofS(i)<N(i))
                 k = MSeq(i,NofS(i)+1);     % The next job to be scheduled on machine 'i'
            else
                continue
            end
```

```
l =  PreMac(i,k);                   % The machine required by 'k' before 'i'

if (l~=0)            % If the operation is the first operation for the job 'k'
     if Sched(l,k) == 1
         if (NofS(i) ==0)
                  % If the operation is the first operation on the machine 'i'
            MF(i,k) = MF(i,l)+Mu(i,k);
            VF(i,k) = VF(i,l)+Sigma2(i,k);
                  for x = 1:p
                              % To set a zero correlation with all the initial operations
                      a = InOps(x,1);
                      b = InOps(x,2);
                      Ro(i,k,a,b) = 0;
                      Ro(a,b,i,k) = 0;
                  end

            Ro(l,k,i,k) = 1;
                                         % Although it has a +1 correlation with the
                                        previous operation of the job on machine 'l'
            Ro(i,k,l,k) = 1;

            p = p+1;                      % To set this operation as an initial operation
            InOps(p,1) = i;
            InOps(p,2) = k;
            NofS(i) = NofS(i) +1;
            Sched(i,k) = 1;

        else
              Start(i,k);
              MF(i,k) =  MStart(i,k) + Mu(i,k);
              VF(i,k) =  VStart(i,k) + Sigma2(i,k);
```

```
            NofS(i) = NofS(i) +1;
            Sched(i,k) = 1;
          end

        end
    else
          temp = MSeq(i,NofS(i));
          MF(i,k) = MF(i,temp)+Mu(i,k);
          VF(i,k) = VF(i,temp)+Sigma2(i,k);

          for x = 1:p
                % To set a zero correlation with all the initial operations
             a = InOps(x,1);
             b = InOps(x,2);
             Ro(i,k,a,b) = 0;
             Ro(a,b,i,k) = 0;
          end

          Ro(i,temp,i,k) = 1;
                                          % Although it has a +1 correlation with the
                                   previous operation on the same machine
          Ro(i,k,i,temp) = 1;

          p = p+1;                 % To set this operation as an initial operation
          InOps(p,1) = i;
          InOps(p,2) = k;
          NofS(i) = NofS(i) +1;
          Sched(i,k) = 1;
     end

end
```

```
        loop = SchedOver;

    end

% To find the moments of the makespan

for i = 1:m
      k = MSeq(i,N(i));
      MC(i) = MF(i,k);
      VC(i) = VF(i,k);
end

% Application of Clark's equations

        MMax(1) = MC(1);
        VMax(1) = VC(1);

    for i = 2:m

      k = i-1;
      A(i)= sqrt(VMax(k) + VC(i) - 2*sqrt(VMax(k))*sqrt(VC(i))*fCor(k,i));
      ALPHA(i) = (MMax(k) - MC(i))/A(i);

      MMax(i) = MMax(k)*normcdf(ALPHA(i)) + MC(i)*normcdf(-ALPHA(i)) +
                  A(i)*fSi(ALPHA(i));
                            % First Moment or mean of Max(C1,C2,...Ci)

      SM(i) = (MMax(k)^2 + VMax(k))*normcdf(ALPHA(i)) + (MC(i)^2 +
               VC(i))*normcdf(-ALPHA(i)) + (MMax(k)+ MC(i))*A(i)*fSi(ALPHA(i));
                              % Second Moment of Max(C1,C2,...Ci)
```

```
    VMax(i) = SM(i) - MMax(i)^2;          % Variance of Max(C1,C2,...Ci)

    MMax;
    VMax;
 end

  MF, VF                              % Final Completion Times
  MMax(m),  VMax(m)                   % Final Makespan


%*****************************************************************
%  To find the correlation Cor(k,j) = Cor(L1,L2,L3,...Lk...Lj)

function t = fCor(k,i)
  global MC VC MMax VMax Ro ALPHA MSeq N

  x = MSeq(k,N(k));             % Finding the last job on machine 'k'
  y = MSeq(i,N(i));             % Finding the last job on machine 'i'

  if (Ro(k,x,i,y) == Inf)
       Correlation(k,x,i,y);
  end

  if (k==1)
    t = Ro(k,x,i,y);
    return
  end

  t = (VMax(k)*fCor(k-1,i)*normcdf(ALPHA(k)) + VC(i)*Ro(k,x,i,y)*normcdf(-
       ALPHA(k)))/ sqrt(VMax(k));
```

```
%*****************************************************************
% Function to determine the mean and variance of the start times of  job 'k'  on machine
'i'

   function  Start (i,k)
    global MStart VStart MF VF Ro

      a = PreMac(i,k);              % I Level   - Previous machine for job 'k' before 'i'
      b = PreJob(i,k);              % I Level - Previous job on 'i' before 'k'

   if (b~=0)
        Correlation(a,k,i,b);       % Call Correlation function to determine the correlation
        M1 = MF(a,k);
        M2 = MF(i,b);
        V1 = VF(a,k);
        V2 = VF(i,b);
        A = sqrt(V1+V2 - 2*sqrt(V1)*sqrt(V2)*Ro(a,k,i,b));
        Ro(a,k,i,b)
          if A == 0
              alpha = 0;
          else
              alpha = (M1-M2)/A;
          end

          MStart(i,k) = M1*normcdf(alpha) + M2*normcdf(-alpha) + A*fSi(alpha);
          SMStart(i,k) = (M1^2 + V1)*normcdf(alpha)+(M2^2 + V2)*normcdf(-
                         alpha)+(M1+M2)*A*fSi(alpha);
        VStart(i,k) = SMStart(i,k) - MStart(i,k)^2;
  else
```

```
      MStart(i,k) = MF(a,k);
      VStart(i,k) = VF(a,k);
      Ro(a,k,i,k)= 1;
      Ro(i,k,a,k)=1;
   end

%*****************************************************************
%  Function to determine the correlation Ro(a,k,i,b)

function Correlation(a,k,i,b)
global MStart VStart MF VF Ro

    if  (Ro(a,k,i,b) ~= Inf)
       return
    end

   if (a==i) & (k==b)
       Ro(a,k,i,b) = 1;
       Ro(i,b,a,k) = 1;
       return
    end

   % Check if (a,k) or (i,b) is an initial operation

        chk1 = CheckInOp(a,k);
        chk2 = CheckInOp(i,b);

   if (chk1==1)
      % (a,k) is an intial operation

      e = PreMac(i,b);
```

```
 f = PreJob(i,b);

 if (Ro(a,k,e,b) == Inf)
    Correlation(a,k,e,b);
 end

 if (Ro(a,k,i,f) == Inf)
    Correlation(a,k,i,f);
 end

 if (Ro(e,b,i,f) == Inf)
    Correlation(e,b,i,f);
 end

 M1 = MF(e,b);
 M2 = MF(i,f);
 V1 = VF(e,b);
 V2 = VF(i,f);

A = sqrt(V1+V2 - 2*sqrt(V1)*sqrt(V2)*Ro(e,b,i,f));

 if A == 0
    alpha = 0;
 else
    alpha = (M1-M2)/A;
 end

Ro(a,k,i,b) = (sqrt(V1)*Ro(a,k,e,b)*normcdf(alpha) +
                  sqrt(V2)*Ro(a,k,i,f)*normcdf(-alpha))/VF(i,b);

Ro(i,b,a,k) =  Ro(a,k,i,b);                     % Symmetricity
```

```
end

 if (chk2==1)
        % (i,b) is an intial operation

   c = PreMac(a,k);
   d = PreJob(a,k);

   if (Ro(i,b,c,k) == Inf)
      Correlation(i,b,c,k);
   end

   if (Ro(i,b,a,d) == Inf)
      Correlation(i,b,a,d);
   end

   if (Ro(c,k,a,d) == Inf)
      Correlation(c,k,a,d);
   end

   M1 = MF(c,k);
   M2 = MF(a,d);
   V1 = VF(c,k);
   V2 = VF(a,d);

  A = sqrt(V1+V2 - 2*sqrt(V1)*sqrt(V2)*Ro(c,k,a,d));

   if A ==0
      alpha = 0;
   else
```

```
        alpha = (M1-M2)/A;
      end

     Ro(a,k,i,b) = (sqrt(V1)*Ro(i,b,c,k)*normcdf(alpha) +
                    sqrt(V2)*Ro(i,b,a,d)*normcdf(-alpha))/VF(a,k);
     Ro(i,b,a,k) =  Ro(a,k,i,b);                  % Symmetricity
end

if (chk1==0) & (chk2==0)

        c = PreMac(a,k);
        d = PreJob(a,k);
        e = PreMac(i,b);
        f = PreJob(i,b);

        if (Ro(c,k,a,d) == Inf)
           Correlation(c,k,a,d);
        end

        if (Ro(c,k,e,b) == Inf)
           Correlation(c,k,e,b);
        end

        if (Ro(c,k,i,f) == Inf)
           Correlation(c,k,i,f);
        end

        if (Ro(a,d,e,b) == Inf)
           Correlation(a,d,e,b);
        end
```

```
if (Ro(a,d,i,f) == Inf)
   Correlation(a,d,i,f);
end

if (Ro(e,b,i,f) == Inf)
   Correlation(a,d,i,f);
end

 % Three way correlations between (e,b),(i,f) and F(i,b)

    M1 = MF(e,b);
    M2 = MF(i,f);
    V1 = VF(e,b);
    V2 = VF(i,f);
    A = sqrt(V1+V2 - 2*sqrt(V1)*sqrt(V2)*Ro(e,b,i,f));

    if A == 0
       alpha = 0;
    else
       alpha = (M1-M2)/A;
    end

   if (Ro(c,k,i,b) == Inf)
      Ro(c,k,i,b) = (sqrt(V1)*Ro(c,k,e,b)*normcdf(alpha) +
                     sqrt(V2)*Ro(c,k,i,f)*normcdf(-alpha))/VF(i,b);
      Ro(i,b,c,k) = Ro(c,k,i,b);
   end

   if (Ro(a,d,i,b) == Inf)
      Ro(a,d,i,b) = (sqrt(V1)*Ro(a,d,e,b)*normcdf(alpha) +
                     sqrt(V2)*Ro(a,d,i,f)*normcdf(-alpha))/VF(i,b);
```

```
            Ro(i,b,a,d) = Ro(a,d,i,b);
          end

% Three way correlations between (c,k),(a,d) and F(a,k)

          M1 = MF(c,k);
          M2 = MF(a,d);
          V1 = VF(c,k);
          V2 = VF(a,d);
           A = sqrt(V1+V2 - 2*sqrt(V1)*sqrt(V2)*Ro(c,k,a,d));

          if A == 0
             alpha = 0;
          else
             alpha = (M1-M2)/A;
          end

          Ro(a,k,i,b) = (sqrt(V1)*Ro(c,k,i,b)*normcdf(alpha) +
                          sqrt(V2)*Ro(a,d,i,b)*normcdf(-alpha))/VF(a,k);
           Ro(i,b,a,k) =  Ro(a,k,i,b);                    % Symmetricity
end

%**********************************************************************
%  Function to determine the check if (a,k) is one of the initial operations

  function  chk = CheckInOp(a,k)
  global InOps p

   for x = 1:p
      if (InOps(x,1) ==a) & (InOps(x,2) ==k)
         chk = 1;
```

```
            return
         end
         chk = 0;
      end

%**********************************************************************
%  Function to determine the previous machine for job 'k' before machine 'i'

      function l = PreMac(i,k)
         global JSeq n m

         for s = 1:m
            if (JSeq(k,s) == i)
               if s ==1
                  l =0;
               else
                  l = JSeq(k,s-1);
               end
            end
         end

%**********************************************************************
% Function to determine the previous job on machine 'i' before 'k'

      function l = PreJob(i,k)
         global MSeq n m

         for s = 1:n
            if (MSeq(i,s) == k)
               if s==1
                  l = 0;
```

```
        else
          l = MSeq(i,s-1);
        end
      end
    end

%*******************************************************************
% To compute the normal distribution for a given number

function m = fSi(u)
  m = (1/sqrt(2*pi))*exp(-u^2/2);


%*******************************************************************
% To check if all the jobs have been scheduled

function t = SchedOver()
  global NofS N m

      t=0;
      for i = 1:m
            if (NofS(i)~=N(i))
                  t =1;
             end
      end
```

APPENDIX D - SOFTWARE PROGRAM FOR IMPLEMENTING THE PARALLEL MACHINE MODELS

APPENDIX D.1 – Makespan with No Preemptions

```
% Objective: MAKESPAN WITH NO PREEMPTIONS

% To estimate the expectation and variance of the makespan with no preemptions
% Job sequence for each machine, mean and variance of the job processing times are given
% Job processing times are assumed to be normally distributed
% The makespan random variables are mutually related by the multivariate distribution
% Seq - Matrix containing the sequence of jobs assigned to each machine
% Row - machines, Columns -  jobs assigned ,zeros for remaining columns

function PMMS()
      global M J Seq MMax VMax Cor alpha Ro VMS
      m = length (M);
      n = length (J);

% Determing the number of jobs assigned to each machine
% Count until zero is reached in the sequence

for i = 1:m
    N(i) = 0;
    for k = 1:n
      if Seq(i,k)~=0
        N(i) = N(i) + 1 ;
      else
        break
```

```
        end
    end
end

% Assigning a correlation value of 1 to all pairs of machines

for i = 1: m
    for j = 1:m
        if i==j
            continue
        end
        Ro(i,j) = 0;
    end
end

% Assigning a correlation value of 1 between the 1st job and any other job in the sequence
% used in the recursive equations

for j = 2:m
    Cor(1,j) =0;
end

% Application of Clarke's equations

        MMS(1) = fEMS(1,N(1));
        VMS(1) = fVMS(1,N(1));
        MMax(1) = MMS(1);
        VMax(1) = VMS(1);

for j = 2:m
```

```
    MMS(j) = fEMS(j,N(j));  % Mean of CjMax, Makespan of the jth machine
    VMS(j) = fVMS(j,N(j));  % Variance of CjMax
    k = j-1;
    a(j)= sqrt(VMax(k) + VMS(j) - 2*sqrt(VMax(k))*sqrt(VMS(j))*fCor(k,j));
    alpha(j) = (MMax(k) - MMS(j))/a(j);
    MMax(j) = MMax(k)*normcdf(alpha(j)) + MMS(j)*normcdf(-alpha(j)) +
              a(j)*fSi(alpha(j));
                   % First Moment or mean of Max(Cmax1,Cmax2,Cmax3,...Cmaxj)
    SM(j) = (MMax(k)^2 + VMax(k))*normcdf(alpha(j)) + (MMS(j)^2 +
             VMS(j))*normcdf(-alpha(j)) + (MMax(k)+ MMS(j))*a(j)*fSi(alpha(j));
                   % Second Moment of Max(Cmax1,Cmax2,Cmax3,...Cmaxj)
    VMax(j) = SM(j) - MMax(j)^2;
                  % Variance of Max(Cmax1,Cmax2,Cmax3,...Cmaxj)
end

MMax(m),VMax(m)


%*******************************************************************
% To find the correlation Cor(k,j) = Cor(Cmax1,Cmax2,Cmax3,...Cmaxk...Cmaxj)

function m = fCor(k,j)
  global MMax VMax Cor alpha Ro VMS
  if k == 1
    m = 0;
  else
      m  =  (VMax(k-1)*fCor(k-1,j)*normcdf(alpha(k))  +  VMS(k)*Ro(k,j)*normcdf(-
      alpha(k)))/ sqrt(VMax(k));
  end
```

```
**********************************************************************
% To compute the expectation of the makespan of a machine
% It is the sum of the mean processing times of all the jobs assigned to that machine

function m = fEMS(i,u)
    global Seq Mu
    m = 0;
   for k = 1:u
       l = Seq(i,k);
       m = m + Mu(l,2);
    end


%**********************************************************************
% To compute the variance of the makespan of a machine
% It is the sum of processing time variances of all the jobs assigned to that machine

function m = fVMS(i,u)
    global Seq Sigma2
    m = 0;
    for k = 1:u
       l = Seq(i,k);
       m = m + Sigma2(l,2);
    end


%**********************************************************************
% To compute the normal distribution for a given number

function m = fSi(u)
    m = [1/sqrt(2*pi)]*exp(-u^2/2);
```

APPENDIX D.2 – Total Completion Time with No Preemptions

% Objective: TOTAL COMPLETION TIME WITH NO PREEMPTIONS

% To estimate the expectation and variance of the total completion time with no preemptions
% Job sequence for each machine, mean and variance of the job processing times are given

```
function PMTCT()
        global J M Seq Mu Sigma2
        E=0;
        V=0;
        m = length (M);
        n = length (J);

% Determing the number of jobs assigned to each machine

for i = 1:m
    N(i) = 0;
    for k = 1:n
        if Seq(i,k)~=0
            N(i) = N(i) + 1 ;
        else
            break
        end
    end
end
```

```
% To estimate the expectation of the total completion time

    for i = 1:m
        for j = 1:N(i)
            l = Seq(i,j);
            E = E + (N(i)+1-j)*Mu(l,2);
        end
    end

% To estimate the variance of the total completion time

    for i = 1:m
        for j = 1:N(i)
            l = Seq(i,j);
            V = V + (N(i)+1-j)^2*Sigma2(l,2);
        end
    end

E
V
```

VITA

BALAJI NAGARAJAN

Balaji Nagarajan hails from Chennai (Madras), capital city of the southern state of Tamil Nadu in India. He was born to Nagarajan Venkataraman and Indira Nagarajan on the 25th of December, 1978. He received his Bachelor's degree in Mechanical Engineering from College of Engineering, Anna University, Chennai in May 2000. To further pursue his interest in academia, he came to the United States and joined the graduate program at Virginia Polytechnic Institute and State University (Virginia Tech). He graduated with a Master's degree in Industrial and Systems Engineering from Virginia Tech in September 2003. During his stay at Virginia Tech, he worked as a Graduate Researcher for the Center for High Performance Manufacturing (CHPM), a center that works in collaboration with a number of manufacturing firms in and around Virginia to develop tools and technologies to improve their manufacturing practices. As a member of the Production & Information Systems group, he accomplished developing an advanced production planning and inventory control tool for the implementation and use in the member companies of the CHPM. He was also awarded the *Ingersoll-Rand Employee Scholarship* for the year 2002-03 for academic excellence during his graduate studies. His research interests are in the field of production planning and scheduling, mathematical modeling and optimization.