COGNITION AND INSTRUCTION
https://doi.org/10.1080/07370008.2021.1896521

Children's Spontaneous Additive Strategy Relates to Multiplicative Reasoning

Ron Tzur[a], Heather Lynn Johnson[a], Anderson Norton[b], Alan Davis[a], Xin Wang[c], Michael Ferrara[a], Cody Harrington[a], and Nicola Mercedes Hodkowski[d]

[a]University of Colorado Denver, Denver, CO, USA; [b]Virginia Tech, Blacksburg, VA, USA; [c]RMC Denver, Denver, CO, USA; [d]Digital Promise, Washington, DC, USA

ABSTRACT

We examine a hypothesis implied by Steffe's constructivist model of children's numerical reasoning: a child's *spontaneous* additive strategy may relate to a foundational form of multiplicative reasoning, termed multiplicative double counting (mDC). To this end, we mix quantitative and qualitative analyses of 31 fourth graders' responses during clinical, task-based interviews. All participants spontaneously used one of three additive strategies—counting-on, doubling, or break-apart-make-ten (BAMT)—to correctly solve an addition word problem (8 + 7). We found between-group differences, with asymmetric association of those ordinal variables. We found counting-on to be mainly related to premultiplicative reasoning and BAMT to mDC reasoning. We discuss the theoretical significance and implications of this corroboration of Steffe's model.

Children's spontaneous additive strategy relates to multiplicative reasoning

> Part of the attraction of qualitative, noncomparative methods … was that they promised to generate hypotheses that could then be tested. That promise has gone largely unmet. … The field needs to return more of its energies to conducting hypothesis-testing studies. (Kilpatrick, 2001, pp. 424–425)

Children's development of multiplicative reasoning has long been a major concern for mathematics educators (Clark & Kamii, 1996; Harel & Confrey, 1994; Steffe & Cobb, 1998; Verschaffel, Greer, & DeCorte, 2007). It reflects an awareness that advancing from additive to multiplicative reasoning involves construction of new cognitive structures (Boulet, 1998; Clark & Kamii, 1996; Lester & Steffe, 2013; Steffe, 1992; Ulrich, 2016), which are foundational for fractional and proportional reasoning (Hackenberg, 2007, 2013; Hackenberg & Tillema, 2009; Simon, 2006; Thompson & Saldanha, 2003) and ultimately for algebraic reasoning (Hackenberg & Lee, 2015; Steffe, Liss, & Lee, 2014). Relating multiplicative to additive reasoning, aside from juxtaposing them through "repeated addition" (Baroody, 2006; Kling & Bay-Williams, 2015; Whitacre & Nickerson, 2016), can help address the nature of this cognitive shift and the conceptual preparation leading to it. Our purpose in this study is to articulate possible relationships between additive strategies children may use and their multiplicative reasoning. Specifically, we address the research problem: How might a spontaneous strategy a child uses to solve an addition problem relate to the child's current ability to engage in a foundational form of multiplicative reasoning, called *multiplicative double counting* (mDC; see Tzur et al., 2013)?

CONTACT Ron Tzur ron.tzur@ucdenver.edu School of Education and Human Development, University of Colorado Denver, 1380 Lawrence St., Denver, CO 80204, USA.
This article has been republished with minor changes. These changes do not impact the academic content of the article.

We illustrate what we mean by mDC reasoning and then additive reasoning, followed by our use of s*trategy* and *spontaneous*. Consider two children who used individual cubes to build a set of equal-sized towers (e.g., 6 towers, 3 cubes each). Each child is then asked how many cubes there are in all six towers, and both correctly answer, “18.” At issue, for reasoning, is not just or mainly the correct answer, but rather the child’s goal (intention) and mental actions. One child, engaging in *pre*multiplicative reasoning (see more below), may count all the cubes one-by-one (e.g., 1-2-3- … −17-18). The other child, engaging in mDC, would *intentionally and simultaneously* keep track of the accruing amounts of cubes in each tower (triplets of ‘ones’, or 1 s) and of towers as displays of six units made of those triplets (e.g., “One-tower-is-three cubes, two-is-six, … 6-is-18”). Such a coordinated mental action provides a cognitive basis for multiplication-as-measurement (Davydov, 1992; Izsák & Beckmann, 2019; Simon, Kara, Norton, & Placa, 2018).

As for additive reasoning, in this study we focus on three strategies children who reason numerically (Steffe & Cobb, 1998; Wright, Martland, Stafford, & Stanger, 2006) may use spontaneously when solving a single-digit addition problem (8 + 7). In one strategy, known as *counting-on*, the child begins counting from a given addend and counts 1 s of the other number (e.g., 8; 9-10-11-12-13-14-15). In another strategy, known as *doubling ±1* (Verschaffel et al., 2007, called it “near-ties”), the child uses known addition facts to quickly add a number to itself while being aware of needed compensations (e.g., 7 + 7 = 14, then 1 more is 15). The third strategy is known as *break-apart-make-ten* (BAMT; Murata & Fuson, 2006), or adding through-10 (Wright et al., 2006), or decomposition-to-10 (Torbeyns, Verschaffel, & Ghesquiere, 2005). Here, the child decomposes one of the given units (addends) so that a combination with the other addend would amount to 10 (e.g., “Taking 2 from 7 and giving it to 8 makes 10; then 10 + 5 = 15).

Having depicted those three additive strategies, we now further explain what we mean by *strategy*, followed by our meaning for *spontaneous*. Our study draws on a constructivist theory (Piaget, 1985; von Glasersfeld, 1995) and corresponding lines of research about children’s construction of arithmetical knowledge (Lester & Steffe, 2013; Steffe, 1992, 1994, 2010; Steffe & Cobb, 1998; Ulrich, 2015, 2016; Wright et al., 2006). To situate this study more broadly, we also allude to literature about spontaneous strategies along with what we mean by this term when using a constructivist lens on children’s numerical reasoning.

Drawing on Bisanz and LeFevre (1990), Steffe and Cobb (1998), Stern (1992), Underwood (1978), and Wright et al. (2006), we define strategy as *a goal-directed cognitive process that one invokes and uses as a facilitative method for solving a problem (or task)*. This definition implies that a particular person, using a particular strategy, can do so because they have available integrated conceptual-procedural knowledge (Gaschler, Vaterrodt, Frensch, Eichler, & Haider, 2013; Rittle-Johnson, Siegler, & Alibali, 2001). Key here is that, *for that person*, the particular strategy is associated with the particular problem for which it is invoked (Allegra et al., 2020; Siegler & Stern, 1998; Verschaffel, Luwel, Torbeyns, & Van Dooren, 2009). For example, children using counting-on to add 8 + 7 as explained above are reasoning strategically in that for the goal of figuring out a total of 1 s in both addends they do not merely count all 1 s. Rather, they intentionally invoke a process to curtail the counting process (Steffe & Cobb, 1988; Wright et al., 2006). This cognitive process of curtailment facilitates a method of starting the count from one addend and counting just the 1 s of another addend. Importantly, our definition of strategy does not imply any level of deliberateness in selecting or using a strategy; it could occur completely (or partly) outside the problem solver’s awareness (Siegler, 2000; Simon, Kara, Place, and Avitzur, 2018; Simon, Tzur, Heinz, & Kinzel, 2004; Steffe, 1991; Verschaffel et al., 2009; von Glasersfeld, 1995). Thus, one way to infer a person’s available conceptual knowledge is to ascertain strategies they use and why, for them, it seems optimal and helpful to do so (Allegra et al., 2020;

McMullen, Chan, Mazzocco, & Hannula-Sormunen, 2019; Steffe & Cobb, 1998; Stern, 1992; Wright et al., 2006).

A person using a strategy to solve a problem may self-initiate it spontaneously or be impelled (e.g., cued) to do so (Gaschler et al., 2013; McMullen et al., 2019; Lester & Steffe, 2013). Studying spontaneous strategies is important, in that problem solving performance often differs significantly from guided strategy usage (Gaschler et al., 2013; McMullen et al., 2019; Stern, 1992; Steffe & Cobb, 1988). Additionally, spontaneous strategies may predict future performance on higher-level concepts (McMullen et al., 2019). Drawing on McMullen et al. (2019), Steffe and Tzur (1994), and Verschaffel et al. (2009), by spontaneous strategy we refer to the *self-initiated (undirected), unaided carrying out of a goal-directed process indicative of the person's aptitude in solving a problem at hand*. This definition seems consistent with cognitive psychology research, in which spontaneously used strategies indicate stronger linkage and greater familiarity compared to competing strategies linking task information and goal-directed actions (Bisanz & LeFevre, 1990; Gaschler et al., 2013; Grabner et al., 2007; McMullen et al., 2019; Siegler & Stern, 1998; Stern & Schneider, 2010). Thus, a spontaneous strategy is telling; it provides a 'window' into a person's own interpretation of and reasoning about the problem and its solution. For example, a child's spontaneous use of counting-on indicates *they reason* that recounting all 1 s in the first addend is unnecessary to find the total of 1 s in two addends (Secada, Fuson, & Hall, 1983; Siegler & Shipley, 1995; Tzur & Lambert, 2011; Wright et al., 2006).

Having defined spontaneous strategy, we turn to explicating its central role, from a constructivist perspective, in testing a hypothesis about relationships between a child's spontaneous additive strategy and their current mDC reasoning. Using this perspective, we explain a spontaneous strategy the child uses in terms of the conceptions into which they *readily assimilate* a given problem situation (Lester & Steffe, 2013; Piaget, 1985; Steffe, 1992; Steffe & Cobb, 1998; Thompson, 2008; Ulrich, 2015; von Glasersfeld, 1995). By focusing on the child's spontaneous strategy, we acknowledge availability of other strategies (Torbeyns et al., 2005; Wright et al., 2006). Nevertheless, we are interested in the additive strategy a child first uses spontaneously and then explains (in a clinical interview setting), because it allows inference into plausible conceptions by which the child also assimilates multiplicative situations (Lester & Steffe, 2013; Steffe, 1992; Ulrich, 2016). Moreover, for such a concept the child's spontaneous strategy can relate to the more advanced of two stages in constructing it, a stage marked by solving a problem without being prompted (Simon, Placa, & Avitzur, 2016; Tzur, 2019; Tzur & Simon, 2004). Simply put, a child's spontaneous strategy suggests the least we can infer to be readily available to them. For example, children who can self-initiate the build-up of units of 10 (i.e., spontaneously using BAMT) consider counting-on as an unnecessarily laborious strategy (Wright et al., 2006). To further theorize the relationship we set out to test in this study, we turn to Steffe's program of research that explains both additive and multiplicative reasoning in terms of units and mental operations on them.

Theoretical framework

The theory-driven hypothesis we set out to test in this study— that children's spontaneous additive strategies are related to their current mDC reasoning—is rooted in a framework that draws on Steffe's progressions in children's numerical reasoning with whole numbers (e.g., Steffe, 1992, 1994, 2010; Steffe & Cobb, 1988). We thus depict additive and multiplicative reasoning in terms of core distinctions about (a) units and (b) their organization in *number sequences*. We use these constructs to distinguish (a) different forms of additive reasoning and (b) mDC as a foundational form of multiplicative reasoning (Tzur et al., 2013). We illustrate these abstract constructs with items in our mDC assessment that our study participants solved.

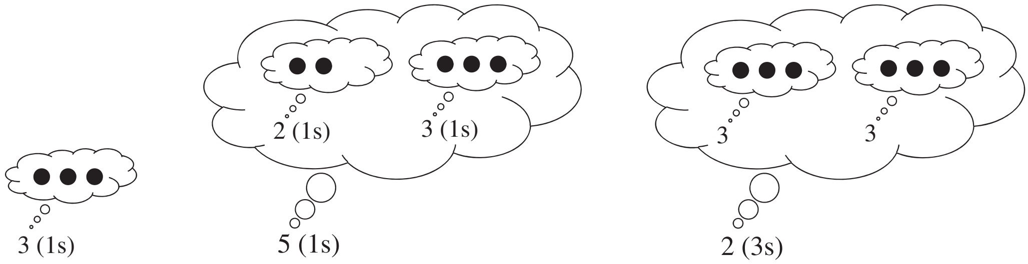


Figure 1. Examples of composite units and 1 s that compose them.

Number sequences: a model of children's numerical reasoning

Steffe (1992, 2010) and colleagues' (e.g., Steffe & Cobb, 1988) research with children yielded distinct types of numerical reasoning, each depicted as a number sequence. A defining criterion distinguishing those number sequences involves two types of units, *singletons* (units of 1, or "ones," symbolized as "1 s") and *composite units*—larger units a child's mental system composes of smaller units. We present Figure 1 to illustrate these terms. It helps 'seeing' a mental structure a child's mind can establish, in which 1 s can be contained within composite units in various ways. For example, a child may conceive of the number 3 as composed of three 1 s; the number 5 as composed of five 1 s, or of two 1 s and three 1 s; and the number 6 as composed of six 1 s, or of three 1 s combined with three 1 s, or as two 'copies' of the same composite unit of 3.

The sequences are more than just lists of numbers. They characterize students' conceptions, grounded in the kinds of units that students construct. To distinguish each sequence, Steffe and Cobb (1988) used the feature of how a child is inferred to conceive of smaller numbers as nested within larger numbers. Steffe (1992) and Lester and Steffe (2013) termed the first three number sequences as follows (from least to more advanced): *Initial Number Sequence* (INS), *Tacitly Nested Number Sequence* (TNS), and *Explicitly Nested Number Sequence* (ENS). Ulrich (2015, 2016) has articulated how those sequences underlie additive and multiplicative reasoning (further explained below). We draw on Ulrich's (personal communication) assertion that a child's spontaneous strategy can reveal their number sequence: counting-on reveals the least advanced sequence (INS), doubling reveals the intermediate sequence (TNS), and BAMT reveals the more advanced sequence (ENS). Importantly, this 'mapping' differs from a common expectation that doubling would be most closely related to multiplicative reasoning (Baroody, 2006; Kling & Bay-Williams, 2015; Verschaffel et al., 2007). In contrast to such an expectation, we hypothesize that children who spontaneously employ a BAMT strategy will reason differently and likely outperform students using a doubling strategy (and, of course, counting-on) when solving multiplicative tasks.

The INS operates on numbers as segments resulting from the child's counting activity, beginning from "one." For example, 8 represents the eight acts of counting from "one" to "eight," so that the child does not need to reengage in that counting activity (Steffe & Cobb, 1988; Steffe & von Glasersfeld, 1985). Thus, when adding two numbers, children operating with an INS can begin counting-on from one addend, while keeping track of additional counts through some activity with tangible or figural items (Fuson, 1992; Wright et al., 2006). For example, starting at eight, a child may use fingers to keep track of seven additional counts. Because children operating with INS work with units of 1 (Lester & Steffe, 2013; Ulrich, 2015), they neither decompose their number sequence into segments (e.g., decompose 7 into 2 and 5) nor re-compose it within 10 (e.g., "give 2 to the 8 to make a ten"). Thus, in assimilating an additive task, children operating with an INS could readily employ a counting-on strategy but not BAMT. Accordingly, in assimilating a multiplicative task they would likely operate on 1 s in some sequential way.

Establishing the INS is a conceptual foundation for the nesting of segments of one's number sequence within larger segments (Steffe, 1992; Ulrich, 2015). For example, a child may separately conceive of the first eight counting acts as a composite unit of eight, and of the first ten counting acts as ten. This may eventually lead them to conceive of eight as being nested within ten. Such nesting of eight within ten could enable conceiving of ten as being composed of eight and two (hence, $10 = 8 + 2$).

Advancing from INS, the TNS and ENS generate the nesting of numbers. In the TNS, children can operate on numbers nested within larger numbers (Steffe & Cobb, 1988; Ulrich, 2015). For example, a child could conceive of the number 8 as being composed of two smaller numbers, such as 7 and 1. This could be useful for solving a problem such as $7 + 8$, if the child knows their doubles (e.g., $7 + 7 = 14$; so $7 + 8$ is the same as 7 plus 7, plus 1 more, hence, 15). Thus, in assimilating an additive task, children operating with a TNS might spontaneously employ a doubling strategy, and consider it less laborious than counting-on (Wright et al., 2006). Critically, however, they *do not yet reverse the nesting operation* by decomposing larger numbers into smaller ones (Lester & Steffe, 2013; Steffe & Cobb, 1988; Ulrich, 2015). As for assimilating a multiplicative situation, they often use premultiplicative reasoning – operating on 1 s or, *sequentially*, on composite units and 1 s – though simultaneous operation on both types of units may be available to some.

Advancing from TNS, children operating with an ENS can reverse the nestedness of their number sequence by mentally decomposing any composite unit from a larger composite unit (Steffe, 2010; Steffe & Cobb, 1988; Ulrich, 2016). For example, with ENS a child can disembed 8 from 10, or 2 from 7, without losing track of the 8 within the 10 or the 2 within the 7. This opens the way for a child to coordinate two numbers (e.g., 8 within 10) with decomposing two other numbers (e.g., 7 into $5 + 2$), to create a 'friendly number sum' of $8 + 2 = 10$, while keeping track of the remaining 5 nested in 7 (Fuson, 1992; Murata & Fuson, 2006; Torbeyns et al., 2005; Wright et al., 2006). Accordingly, in assimilating an additive task, children operating with an ENS can readily employ a BAMT strategy. In assimilating a multiplicative task, the availability of composite units as entities in their own right can support their simultaneous count of composite units and 1 s that constitute those units (Lester & Steffe, 2013). The ways of reasoning characteristic of each number sequence (INS, TNS, ENS) underlie our distinction between additive and multiplicative reasoning.

Distinguishing additive and multiplicative reasoning

We use the term *additive reasoning* in reference to unit-preserving operations on 1 s and composite units (Schwartz, 1991). For example, consider the problem we used with our study participants: "For breakfast, Ana ate 8 grapes. For lunch, Ana ate 7 grapes. How many grapes did Ana eat in all?" Whether a child is spontaneously using a counting-on, a doubling, or a BAMT strategy – the child is operating on the same type of units (here, grapes as units of 1). We infer additive reasoning when we can infer that a child is operating on the same type of unit to obtain other amounts of that same unit.

In contrast, *multiplicative reasoning* refers to unit-transforming operations, in which a child's simultaneous operations on 1 s and composite units produce another type of unit (Confrey & Smith, 1995; Davydov, 1992; Schwartz, 1991; Simon, 2012; Simon, Kara, Norton, et al., 2018; Thompson & Saldanha, 2003; Ulrich, 2016). For example, consider the following item from our mDC assessment (Problem #3): "Alex put 6 towers in a box. Alex made each tower with 3 cubes. How many cubes in all did Alex use to make all 6 towers?" (Note: underlined words appear this way in the mDC assessment.) A child could solve this problem by a simultaneous count of both the composite units and the 1 s in each of them (e.g., "One-tower-is-3-cubes, two-is-6, … 6-is-18"). From such a simultaneous count, which involves unit-transformation, we infer the child is

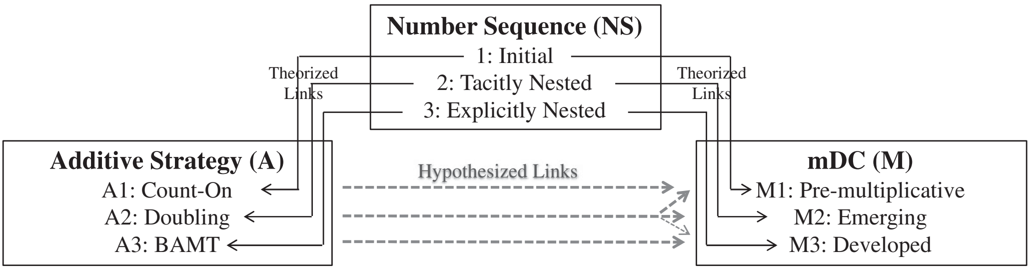


Figure 2. Relating the child's number sequence, additive strategy, and mDC.

distributing items of one composite unit (e.g., cubes per tower) over items of another composite unit (number of towers), to find the size of a third composite unit (total number of cubes).

Tzur et al. (2013) called this foundational form of multiplicative reasoning *multiplicative double counting* (mDC). The reason for expressly denoting the multiplicative aspect of double counting is that counting-on, an additive strategy, also involves double counting (e.g., adding $8+7$ by linking 9-is-1, 10-is-2, etc.). The crucial difference is that in counting-on the child double counts in a *1-for-1 manner*, whereas in mDC the child coordinates the accrual of both 1 s and composite units in a *many-for-1* manner (Clark & Kamii, 1996; Steffe & Cobb, 1998).

In summary, we have argued that forms of additive and multiplicative reasoning can be related in terms of the child's numerical operations on/with units (Lester & Steffe, 2013; Steffe, 1992; Steffe & Cobb, 1988; Ulrich, 2015, 2016). We explained this conceptually by linking each of the additive strategies on which we focus (counting-on, doubling, BAMT) to a number sequence that it indicates (INS, TNS, and ENS, respectively) and to a corresponding form of multiplicative reasoning (Hackenberg & Tillema, 2009; Ulrich, 2015, 2016). Figure 2 presents these relationships. In this study, we test the hypothesis that the additive strategies children use spontaneously should show statistically significant association with mDC due to being rooted in corresponding number sequences.

Impetus for the present study: validation (prior) study of the mDC assessment

The impetus for the present, exploratory study grew out of a prior, mixed-methods study we conducted to develop and validate a written assessment to measure children's mDC reasoning (Johnson et al., 2018). In this section, we briefly present aspects of the validation study that support our use of this instrument, including results that led us to conduct the present study. We present items of the mDC assessment later, as part of our Methods section. Here, we only note that those items included one additive task ($8+7$) used to elicit the child's spontaneous strategy and four items to assess their reasoning in multiplicative tasks.

We evaluated the construct validity of the mDC written assessment, as well as inter-rater reliability of coding and internal consistency reliability of scores (Anastasi & Urbina, 1997). Our goal was to establish validity and reliability of the assessment as an instrument for measuring *mDC reasoning*—not just of problem-solving performance. The validation study involved three phases: item development and refinement, followed by qualitative, video-recorded clinical interviews, followed by the use of Rasch analysis to test dimensionality, fit, and reliability.

The first phase involved four steps: (a) generating an initial set of items consisting of situated conceptual problems; (b) scrutinizing and revising the items for language content by two team members with expertise in English as a second language; (c) conducting trial clinical interviews with five students as they worked through the situated problems, leading to further revision; and (d) having three experts (not from the project team), highly familiar with mDC as a theoretical construct, examine the tasks for consistency with multiplicative reasoning. Based on the experts' feedback, we revised the mDC assessment to exclude irrelevant language that would affect difficulty for reasons unrelated to multiplicative reasoning.

The second phase allowed linking every child's written responses with records of the child's observable behaviors while first solving each problem on their own and then explaining their solutions. In one school, we used the final, revised version with each grade-4 student who agreed to work with us during a video-recorded clinical interview ($n = 28$, all consented). The researcher followed a protocol the team designed, so that students' work on the mDC assessment items would emulate later administration of the written assessment in real classrooms. Specifically, drawing on Tzur's (2007) notion of fine grain assessment, each of the four mDC items began with a version of the problem that gave no hints. This no-hint version was followed by at least two versions that provided gradually more explicit, visual and linguistic hints.

To distinguish a child's solution to no-hint and with-hint versions, the researcher gave the child no hints as to how a problem might be solved aside from those found in the written assessment and made sure the child would not go back to problems/versions solved previously. To enable linking the child's reasoning with their solutions to no-hint and with-hint versions of the mDC problems, he probed into their thought processes only after the child solved all problems on their own. Based on the child's responses, and in the child's presence, he then wrote down what the child explained. In a way consistent with member checking (Birt, Scott, Cavers, Campbell, & Walter, 2016; Carlson, 2010), the researcher asked if what he wrote was how the child had solved the problem (all children confirmed). Additional (invisible) sheets in the spreadsheet had already been programmed to recode and calculate scores for each mDC sub-question, problem, and the total of all 4 problems.

Before the start of an interview with the following child or at the end of the day, the researcher silently and privately entered one more score – his on-the-spot inference about the mDC level of the child who had just completed the interview. This inference was scored on a 6-point scale, using two sub-levels for each of the three stages in our conceptual model (0 or 1 for premultiplicative, 2 or 3 for hint-dependent, and 4 or 5 for independent mDC). Obviously, he knew the child's answers to each mDC item. Importantly, however, in reaching his inferences and scoring – he strove to avoid this information. Instead, he drew on his expertise in noticing and interpreting behavioral indicators of the child's coordination of the counting of composite units and 1 s. It was during this process that he began noticing (and ignored the best he could) a pattern of association between the child's spontaneous additive strategy and mDC reasoning.

In phase three, to further examine construct validity and reliability of the mDC assessment, we administered it in whole-class settings to a *much larger pool* of 3rd and 4th graders from two different schools ($N = 373$). To establish statistically the extent to which student written responses to the four mDC assessment items could serve as a good enough proxy of the latent trait (dimension) we call mDC reasoning, we used a partial-credit (2-1-0) scoring method. If a child correctly answered the no-hint version of an mDC item we scored it as 2; else, if the child correctly answered *either or both* versions with hints – we scored it as 1; if the child gave no correct responses we scored it as 0. A child could thus obtain a total score from 0–8.

We analyzed the dataset for conformity to the Rasch Model, which assumes a theoretically ordered set of items or tasks within a single conceptual dimension – such as multiplicative reasoning (Bond & Fox, 2015; Boone, Staver, & Yale, 2014). After establishing that the mDC measure satisfied Rasch criteria of unidimensionality and reliability (see below) we examined its properties with a larger participant pool ($N = 434$) that included participants of the validation study and the present study. This allowed creating, also for each child interviewed in those two studies, a Rasch person-ability logit score on an interval scale, which we linearly transformed to range from 0–100. As expected, we found a very high correlation between the transformed logit scores and the partial credit, 0–8 raw scores ($r_s = 0.95$, $p < .001$), as well as the no-hint 0–4 raw scores ($r_s = 0.95$, $p < .001$).

Table 1. No-hint correct responses linked to the stage in mDC reasoning ($r_s = 0.84$, $p < .001$).

mDC Stage	0-Correct	1-Correct	2-Correct	3-Correct	4-Correct	Total
PreMDC	7	3	0	0	0	**10**
Hint-Depend	1	3	2	0	0	**6**
Independent	0	1	4	3	4	**12**
Total	**8**	**7**	**6**	**3**	**4**	**28**

Bold indicates $p < .001$.

Four key Rasch analyses supported the validity and reliability of the written assessment as a measure of mDC reasoning as a single latent trait. First, Rasch factor analysis showed that the principal component explained 36.2% of the total variance, similar to the model-expected variance (35.9%) and supportive of unidimensionality (Linacre, 2011). Person mean (−0.4) and item mean (0.00) and standard deviations (person = 1.5, item = 0.33) were consistent with the high percentage of explained variance, further supporting unidimensionality. Second, eigenvalues of all secondary contrasts were at or below 1.4, indicating no additional significant dimensions (Brentari & Golia, 2007; Linacre, 2011; Smith & Miao, 1994). Third, mean-square values for infit (ranging between $0.99 - 1.11$) and outfit ($0.81 - 1.13$), along with positive point-measure correlations (all at least +0.69, compared to a minimum of +0.4), indicated that each item contributed to reliability in the placement of students along a conceptual dimension. Fourth, Cronbach internal consistency reliability ($\alpha = 0.7$) was acceptable (Nunnally & Bernstein, 1994). Further, the very high correlations between the equal-interval Rasch logit scores and the ordinal (0–8 or 0–4) scales ($r_s = .95$) lends support for cautiously using the latter in statistical analyses as an interval-like scale.

The clinical interview data from the validation study were consistent with the Rasch modeling results. The correlation between (a) the researcher's placement of students in respect to three stages of mDC reasoning (pre-mDC, hint-dependent, independent mDC) based on the interview, and (b) the Rasch-based (0–100) scale or the partial-credit (0–8) scale was $r_s = 0.84$, $p < .001$. This supported using a child's score on the mDC written assessment as a good proxy of their mDC reasoning as inferred by an expert interviewer. Specifically, correctly solving at least two no-hint problems was established as a threshold of independent mDC reasoning, as 93% of children who solved at most 1 no-hint problem were at the premultiplicative or hint-dependent stages (Table 1).

Finally, we examined the theory-supported pattern of association we had found between children's spontaneous additive strategy and mDC reasoning. Using the mDC measure (in logit scores) as the dependent variable, we found significant differences between categories of spontaneous additive strategy (Somer's $d = 0.58$, $p < .001$). These between-strategy differences (see Figure 3) were the impetus for the present study, which we designed specifically to examine this asymmetric pattern of association.

Methods

The present, mixed-methods, exploratory study involved a single set of fourth grade students ($n = 31$), not previously studied, whom we video-recorded as they worked to solve a series of problems in a clinical interview setting (Clement, 2000). Each child first solved the problems on their own and then articulated their thought processes in response to the researcher's (first author) follow-up questions. Our quantitative and qualitative analyses addressed two research questions:

1. Can a systematic statistical relationship be established between the children's mDC reasoning and their spontaneous strategy used to solve an unfamiliar addition problem?
2. Is the pattern of relationship between the spontaneous additive strategy and multiplicative reasoning consistent with children's underlying numerical reasoning?

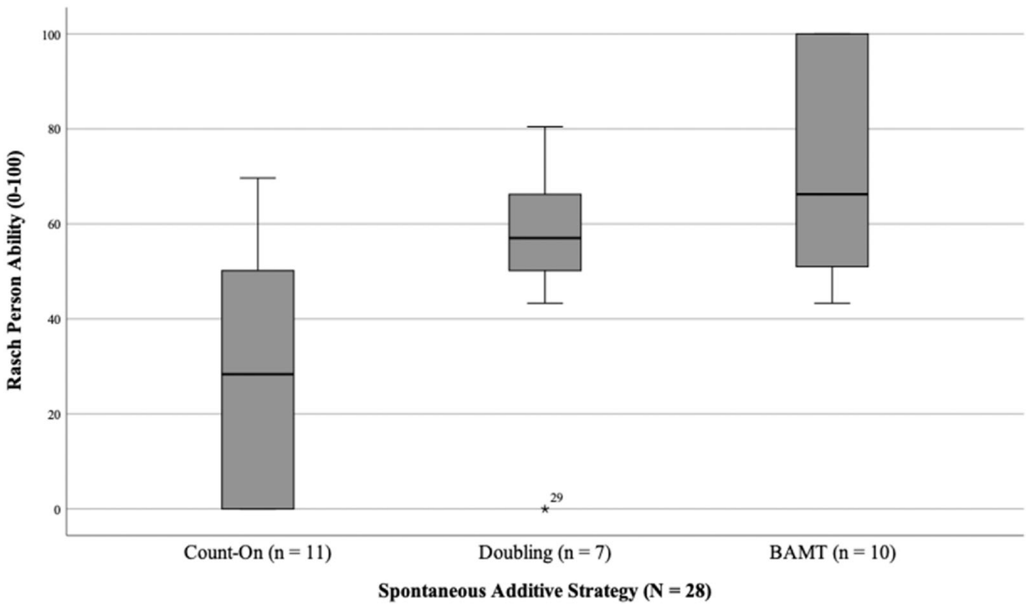


Figure 3. An emerging pattern of relationship between additive strategies and mDC reasoning (Rasch measure).

Our quantitative analysis focused on patterns of association (e.g., asymmetric) between the child's spontaneous additive strategy and mDC reasoning as measured on the assessment we had previously validated. Our qualitative analysis focused on children's numerical reasoning. In both, we tested a theory-driven hypothesis suggested by our earlier validation study: counting-on would be mainly related with premultiplicative reasoning, BAMT with developed mDC, and doubling with various levels of premultiplicative or multiplicative reasoning. Our null hypothesis was thus that there are no generalizable differences in mDC reasoning between categories. Next, we describe the setting and participants of the present study, the instrument (mDC assessment), the data collection (including interview), and the quantitative and qualitative methods of analysis.

Setting and participants

For the present study, an initial sample consisted of 41 fourth graders (age $\sim$10) at an elementary public school in a large urban school district in the United States, all mainstreamed for mathematics instruction. The study was part of a larger effort in which we co-taught those students with their homeroom teacher. Ethical and educational considerations required that we include all children in all activities, but we included data only from consented children. The students attended the same school and were demographically similar to fourth grade students who had participated in the validation study the previous year.

From the initial sample, we excluded students for whom data would be insufficient or not relevant to the purpose of this study. We excluded three students whose comprehension of the word problems during the clinical interviews seemed inadequate. We excluded five more students who did not appear to have even the initial number sequence (INS), which we inferred from their counting-all strategy when solving the additive problem ($8 + 7$). Finally, we excluded two students for whom our independent raters could not agree on the spontaneous strategy each used to solve the additive problem. The final sample consisted of 31 students (13 females), of whom 87% identified as people of color (58% Latinx and 29% African-American). Eighteen (58%) had been designated English Language Learners (ELL), and five (16%) had an official Individualized Educational Plan (IEP).

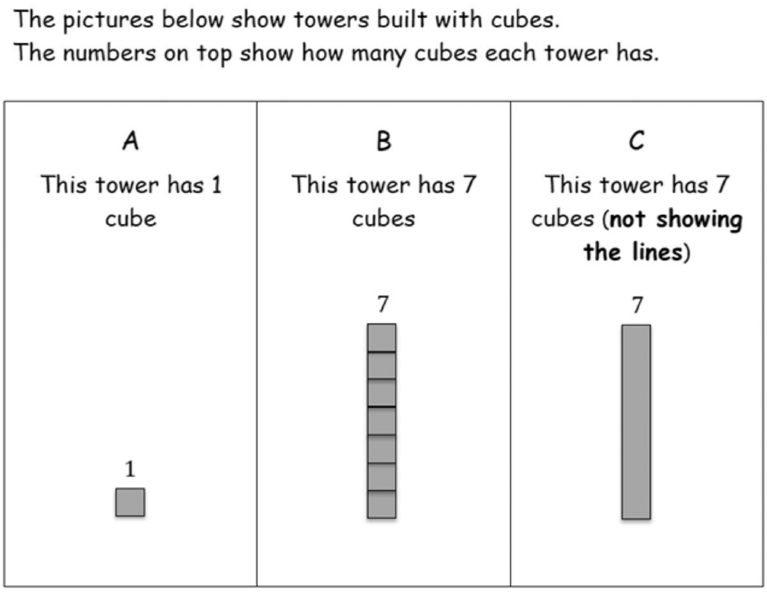


Figure 4. Auxiliary item to assess the child's recognition of towers/cubes pictorials.

Instrument: mDC written assessment

We used the mDC Written Assessment as a measure of students' mDC reasoning. Evidence of its construct validity, unidimensionality, and reliability have been presented above. The assessment consists of seven items: two auxiliary items, one item used to elicit the child's spontaneous additive strategy, and four items to assess the child's current capacity to engage in mDC. We describe each of those items below.

Auxiliary items on the mDC assessment

Because two of the four items to assess mDC reasoning were situated in a context of towers and cubes, we included in the mDC assessment two auxiliary items to ensure that a child could recognize this context. In the first item, which starts the assessment, the researcher places a handful of cubes ($\sim$10) on the desk and asks the child to build a tower with 7 cubes. In the other, which is sequenced *after* additive Problem #1, the researcher shows three pictures of (i) a single cube, (ii) a marked tower of 7 cubes, and (ii) an unmarked tower of 7 cubes (Figure 4). First for picture B, and then for picture C, the researcher asks the child to actually point to the respective picture and circle "Yes" or "No" for understanding that it shows a tower of 7 cubes. When administering the mDC assessment in classrooms, students would similarly be asked to build towers of 7 cubes and then point and circle their responses.

Additive screener item on the mDC assessment: problem #1

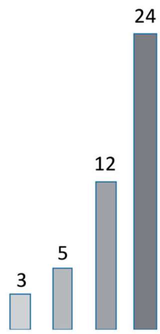
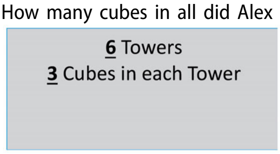
Aside from the two auxiliary items, we designed Problem #1 as a screener for inferring an additive strategy children use *spontaneously*. To recall, Problem #1 states: "For breakfast, Ana ate 8 grapes. For lunch, Ana ate 7 grapes. How many grapes did Ana eat in all?" We designed the context, numbers, and wording to meet two additional criteria (Hodkowski et al., 2016): (a) motivate the child to solve all other problems through initial success and (b) gauge the child's reading comprehension when solving a problem successfully.

Consistent with Wright et al. (2006) advice and with Torbeyns et al. (2005) findings, we chose two single-digit numbers ($8 + 7$) that can help elicit the three additive strategies. For children who may use counting-on, this choice increases the likelihood of indicating the starting number and their method of keeping track to know where to stop counting 1 s of the second addend (Tzur & Lambert, 2011). For children who may use doubling, this choice increases the likelihood they would do so, as these addends are just 1 unit apart. We chose not to use equal addends (e.g., $7+7$) to increase the likelihood children who opt for doubling also indicate compensation for the extra 1 (e.g., $8=7+1$). For children who may use BAMT, choosing $8+7$ as opposed to smaller numbers (e.g., $7+6$) increases the likelihood they use this strategy due to the proximity to 10. Yet, we chose not to use 9 (e.g., $8+9$) to increase the likelihood these children intentionally decompose the addends into nested numbers that are composite units (e.g., $7=5+2$, $10=8+2$).

Multiplicative double counting (mDC) items on the mDC assessment

We designed Problems #2 through #5 as items to measure children's current ability to engage in mDC reasoning. To recall, each of those mDC problems began with a no-hint version, followed

Table 2. The no-hint version of the four multiplicative reasoning word problems (items).

Item	
2	The picture shows towers made of ***3*** cubes, ***5*** cubes, ***12*** cubes, and ***24*** cubes. In this problem, Pat only has towers with ***3*** cubes. Pat cannot break apart any tower. Can Pat build a tower of ***24*** cubes using only towers of ***3*** cubes? (If yes – the child is asked to fill in how many.)  Rationale: Figure out if, to what extent, and how a child coordinates the number of composite units, and 1 s in each, while constituting a larger, known composite unit (24).
3	The picture shows a box. Alex put ***6*** towers in the box. Alex made each tower with ***3*** cubes. The numbers on the picture show this. How many cubes in all did Alex use to make all ***6*** towers?  Rationale: Figure out if, to what extent, and how a child coordinates the total of 1 s in a given number of composite units (6) constituted by a given number of 1 s (3) to find the total of 1 s (18).
4	There are ***4 teams*** in a school. Each team has ***5 players***. Joy said there are 35 players in all, because she skip-counted by 5 (Joy said, "5-10-15 … " and kept going to 35). Is Joy correct? If not, how many teams did Joy count? (Circle one option from 4, 5, 7, 20, 35; or write a number.) [This no-hint version included no image] Rationale: Figure out if a child recognizes, and negates, an error in "repeated addition" by keeping track (coordinating) the number of composite units (4 teams) with 1 s in each (5 players); then how) the child figures out the numerical error (stop the composite units count at 7 teams).
5	Sam baked ***28*** smiley ***cookies***. He put **all** the cookies in boxes. Sam put ***4 cookies in each box***. The picture shows only **one of the boxes**. How many **boxes** did Sam use for **all 28 cookies**? Rationale: Figure out if, to what extent, and how a child coordinates the number of composite units, and 1 s in each, while constituting a larger, known composite unit (28).

by versions of the same problem with visual hints. For each of those items, Table 2 presents the wording of its *no-hint version* and our conceptual rationale for it.

We note that *each version of each word problem* included initial sub-questions asking the child to fill in blanks with information given in the problem. For example, in Problem #4 we asked the child to fill in the blank the number of players (5) given in the problem: "In each team there are ___ players." We included these sub-questions to increase children's experience of success while providing evidence of their comprehension of the written problems.

Protocol and data collection

We collected the data about participants' spontaneous additive strategy and mDC reasoning through clinical, task-based interviews (Clement, 2000), following a protocol similar to the validation study. One deviation from that protocol, to better fit with the purpose of the present study, was not to use member checking by writing down the child's answer in their presence. Rather, the determination of the child's additive strategy was postponed to a later stage, in which two independent raters observed the video-recorded interviews.

The first author interviewed each student in a secluded office at the school. He had developed a trusting relationship with the children long before, through regularly co-teaching mathematics with their classroom teacher. In the interview room, he explained the interview process to the child, asking them to do their best to solve each problem. He asked that they use a pen to write or draw whatever helped them solve the problems and explicitly encouraged them to use their fingers if that helped. He stated that some problems might be more difficult than others, and that he was really interested in how they approached and solved each problem, not whether the answer was correct. He told the child that if after some effort they could not solve a problem, they should write "IDK" (for "I Don't Know") and move on. No other manipulatives (e.g., cubes) were provided so that a child's solution would be initiated and carried out while operating on numerical or figural (but not tangible) items. He further emphasized that there was no time limit, and that his questions intended to understand how they thought about the problems. He clarified that nobody in the school would see their work, and it would not be used for grading.

He also asked each child if they would like to read the problems aloud themselves or prefer that he read the problems to them. In both the validation and the present study roughly 50% of students chose each option. Once read, students began solving the problem silently. All students correctly determined that 8 grapes for breakfast and 7 grapes for lunch was a total of 15 grapes, and their actions and the time it took to utter and write down the answer indicated none had used fact retrieval.

When the child finished solving all four mDC problems, the researcher used probing questions following a set procedure: (a) return to the page that included the child's solution, (b) explicate to the child (out loud) specifics of *their* actions that he noticed, and (c) ask the child questions about thought processes that led to those specific actions. For example, if the child counted six fingers and responded "18" to Problem #3, the researcher might have asked the child to 'say out loud' how they reached the answer. If the child further explained they counted 3 s, he asked how they knew to stop (e.g., "I counted five fingers on this hand and one more; so 6") or what each finger represented (e.g., "a tower").

Quantitative analysis

For additive strategies, we had two independent raters coding each child's spontaneously used strategy by observing the video recording of just the initial part of every clinical interview. To control against possible expectation bias, the raters had neither knowledge of the child's performance on the mDC items nor of the interveiwer's rating. We trained two team members, graduate research

Table 3. Steps in independently determining the child's spontaneous additive strategy.

Step	Essence of Raters' Work
1	Prepare a trimmed video file for each child, containing data about their solution to the additive strategy (30–120 s); prepare a spreadsheet for the raters to enter their inferred strategy for each child.
2	Provide the two raters with the spreadsheet and access to all 43 trimmed video files; let them work independently on rating while using the categories: (a) counting-all, (b) counting-on, (c) doubling-7, (d) doubling-8, (e) BAMT, (f) other, and (9999) if unable to determine the strategy.
3	The first author received each rater's list separately and let an Excel spreadsheet identify agreements. He found 33 agreements and 10 disagreements. Those 10 consisted mainly of "9999" and "other."
4	Using email, he told them about the 10 video files in need of reconciliation, while providing: (a) brief descriptions to clarify the meaning of "spontaneous" (what the children used on their own, and supportive data from the child's explanation) and of what constitutes evidence for each strategy (e.g., starting to count from 1 and by 1 s for counting all, starting to count from one of the addends and following with count of 1 s for counting-on, mentioning the addition of the same addend twice for doubling, decomposing one or both addends into sub-units of 2 or larger for BAMT); and (b) an initial log of main events in the video to help interpret what the child does/says.
5	The two raters revisited the 10 video files separatetly, and sent their rating to the first author. He found that only four disagreements between the two raters remained. One of the raters sent her notes about further data she could glean that were not included in his initial logs. He thus went back to those four videos, and created a full transcript of language and actions the child's used. He then sent the list of the remaining four conflicted videos and the transcripts back to the raters. One of them agreed with the other that one child used BAMT; the other agreed that another child used counting-on; raters remained in disagreement about two children (counting-all vs. counting-on). Because in our study we did not include students who spontaneously used counting-all, we excluded those last two children to avoid biasing the results in favor of our hypothesis.
6	We entered the final, agreed-upon rating of each child's spontaneous additive strategy into the Excel spreadsheet with all coded data of the 4 mDC problems (and sub-questions), copied the file of data into SPSS, and ran the statistical analysis.

assistants, and experienced elementary school teachers to identify a child's spontaneous additive strategy through observation of children's work. We asked them to pay close attention to the child's language and bodily motions (fingers, hand under the table, lips moving), which were used to both find the solution (*before* any probing) and then to explain it to the interviewer. Our goal was not simply to establish a reasonable level of interrater reliability. Rather, we aimed to reach 100% agreement between the two raters, which they would anchor in clearly identified evidence about the strategy they inferred a child had used spontaneously. For example, in the last step of this multistep process (#6, Table 3) we excluded two of the five children that seemed to one rater to have been using counting-all. Although the other rater inferred counting-on, we excluded these two borderline cases in order to avoid falsely biasing the statistical findings in favor of our theory-driven hypothesis. After the two raters agreed on all children's additive strategies, we converted the ratings to a three-point, ordinal scale: counting-on = 1, doubling = 2, BAMT = 3.

Considering the child's spontaneous additive strategy as the independent variable and the child's mDC reasoning as the dependent variable, we conducted three statistical tests of association. In one test, the dependent variable consisted of the Rasch person-ability measure for all 4 mDC problems – a linear scale ranging 0–100 based upon nine values obtained through the 2-1-0 partial-credit analysis. We crosstabulated the two variables and used Somers' d to test for an asymmetric association between children's additive strategy (independent variable) and their current mDC reasoning (dependent variable). We used Somers' d because, as a proportional reduction in error (PRE) statistic (Agresti & Finlay, 1997), it provides the percentage improvement in predicting the ordinal rank of the dependent variable given the independent variable, thereby serving as a measure of effect size (Newson, 2002, 2006). To further test multiple pair-wise comparisons (e.g., counting-on vs. doubling), we used the Holm-Bonferroni (Holm, 1979) post-hoc test as a more accurate alternative to the Bonferroni method (Abdi, 2010; Olejnik, Li, Supattathum, & Huberty, 1997; Wright, 1992).

In a second test, building on the results of the Somers' d aymmetric association, we used Spearman's rho to further test correlation between the child's additive strategy (ordinal) and their

Rasch person-ability measure (interval). In a third analysis, we focused on the child's response to *each no-hint mDC problem* separately, recognizing that in school settings hints are not typically provided on tests, and Rasch logit scores are not familiar or available. The analysis aimed to determine the extent to which a child's additive strategy related to problem solving reflective of the independent stage of mDC reasoning. Use of the Mann-Whitney (MWz) nonparametric test allowed us to examine this relationship at the item level.

Qualitative analysis

We undertook the qualitative analysis to provide insight into the underlying conceptions of number (INS, TNS, ENS) of particular children. Analysis of the video recordings of clinical interviews provided insight into students' hypothesized conceptual processes that could not be inferred from their pencil-and-paper responses to the instruments alone. First, we reviewed all video-recorded interviews while creating logs of critical events in each (Powell, Francisco, & Maher, 2003). Using those logs, we selected three exemplars that best illustrate theoretical linkages between a child's spontaneous additive strategy and mDC reasoning. We transcribed those segments of the recordings and further analyzed details of those children's activity. In this detailed, line-by-line analysis, we paid special attention to ways in which each child coordinated (or not) the count of composite units with the accrual of 1 s. Zaro (all names are pseudonyms) is an exemplar of a child who used BAMT and engaged in mDC reasoning; Adel (counting-on) and Carl (doubling) demonstrate premultiplicative reasoning.

Our analysis includes all three additive strategies. While we particularly stress the difference between counting-on and BAMT, we also intend to account for issues surrounding doubling as a commonly-used strategy. Using Carl's example, we emphasize how this strategy, which is often linked with multiplication (Kling & Bay-Williams, 2015), may mask operations on composite units that we distinguish from mDC.

Results

In this section, we interweave quantitative and qualitative analyses to address the two complementary research questions about plausible relationships between children's spontaneous additive strategy and their mDC reasoning. Taken together, the two methods of analysis provide insight into children's thinking about numbers in additive and multiplicative situations. We first present qualitative analysis of one child (Zaro) serving as an exemplar of spontaneously using BAMT and the ENS (decomposing 7 into the nested 5 and 2, "then give 2 to 8; that's 10; then 5 more is 15") and an independent stage of mDC reasoning. Then, we present a quantitative analysis of all 31 children. Finally, we present two examplars of children whose spontaneous use of counting-on (Adel) or doubling (Carl) was related with premultiplicative reasoning. Indeed, those last two examplars do not exhaust all possible relationships. They do serve, however, to shed light on particular ways of conceptualizing (operating on) composite units and 1 s that may involve premultiplicative reasoning.

Relating BAMT to mDC reasoning

We begin with Zaro's work on Problem #3 because, to recap, it is an archetypal multiplicative task: it gives information about the number of composite units (here, 6 towers) and the number of 1 s in each of them (here, 3 cubes each), and asks the student to find the total of 1 s (here, 18 cubes). Excerpt 1a shows how Zaro solved Problem #3; Excerpt 1 b shows how he further explained to the researcher why the answer is 18 (Z stands for Zaro, R stands for researcher).

Figure 5. Zaro's drawing of 6 lines (for towers) and later three dots on each (for 3 cubes each).

Excerpt 1a: Zaro's solution to mDC Problem #3.

Z: Wait, hold on. (Counting with one finger at a time until he has 4 fingers extended on his left hand) 3, 6, 9, 12. Wait. (Whispers) 15. (Then counts 1 s) 16, 17 (Extends a finger on his right hand – apparently symbolizing the 6th tower).

R: Take your time, there's no pressure.

Z: Yeah. Wait, 18. (Fills in the blank for the answer) 18.

Excerpt 1 b: Zaro's explanation of his solution to mDC Problem #3.

R: [To solve Problem #3] you said 18. How did you get 18?

Z: Yeah; I was just, like, counting by 3's (extends three fingers on his left hand, one at a time).

R: Can you do it for me out loud so that I can see how you do it; and show your fingers?

Z: (Holds left hand up, then simultaneously raises single fingers for each unit of 3) So, like: Three (raises his index finger); Six (raises his middle finger); Nine (Raises his ring finger); Twelve (raises his pinky finger); Fifteen (raises his thumb); (Whispers) 16, 17, 18. And then I got 18 (raises his index finger on right hand). Because I was using, like: Six (puts both hands together to show 6 fingers standing for the 6 towers given in the problem).

R: [A little later] Could you draw it for me?

Z: So, like: You could use 6 towers of 3 (draws 6 lines on the page while counting them as he draws each) 1, 2, 3, 4, 5, 6. [Figure 5 shows Zaro's completed drawing; at the current point in the excerpt, he literally drew just six lines—no dots, yet.]

R: So, this is - what?

Z: Yeah, and then, you just can use, like your fingers, because then you can skip count so you can figure [pause]; So, like this, pretty much (he draws 3 dots on each line; see Figure 5).

R: So, how is your drawing related to your fingers? Is this similar?

Z: Um, yes; because it's like counting by 3's. Like, for example, I would just use my fingers to count by 3's (points at the lines he drew on the page). [Here, Zaro repeated the double-count on his fingers as done above.]

R: In this problem, what were the fingers showing? Towers? Cubes? The box?

Z: They were kind of like showing [both] the cubes *and* the towers (italics added by authors).

Our claim that Zaro's solution indicates mDC reasoning rests on four main aspects of his actions and language. First, he spontaneously used his fingers to supply figural units in executing the simultaneous double-count of composite units (6 towers) and accruing 3 s (cube triplets). That is, upon assimilating this problem *he* initiated a coordination of two different units, and thus the change to a third unit (total of eighteen 1 s), while anticipating when to stop the counting activity (Tzur et al., 2013). Second, when asked to show how he solved the problem Zaro first

Table 4. Percentages of children who correctly solved each mDC item (no-hint).

MR Problem	2	3	4	5	Total
Percentage of correct solutions	39%	33%	41%	45%	39%

drew just the six lines to represent the composite units (6 towers). The lines he drew were schematic, as opposed to, say, a full display of rectangles with marks to indicate towers made of 3 single cubes. Only after he drew all 6 lines, seemingly anticipating the next step in a plan, he also supplied 3 dots to each line. Third, to show how he figured out the total while all 1 s were shown being embedded within their respective composite units, he accounted for the accrual of 1 s by using multiples of three - not by counting every dot separately (which we shall later contrast with Carl's solution). Finally, Zaro identified fingers raised with the lines indicating towers, and claimed the fingers stood for *both* towers and cubes.

Those four aspects indicated Zaro's mDC reasoning: simultaneously counting items of two different composite units would yield the answer—a transformation into a quantity comprised of the total number of 1 s. Zaro is thus an exemplar of the relationship between a child's spontaneous use of BAMT and current ability to engage in mDC reasoning. Next, we present statistical analysis of all 31 children's responses to the four mDC problems.

Multiplicative double counting (mDC) – entire sample

Table 4 provides percentages of children ($n = 31$) who correctly solved each mDC problem. Despite *all* 31 children's correct solutions to the screener, addition Problem #1, at most 45% of them solved each mDC problem correctly. Furthermore, only 13 children (42%) correctly solved at least 2 no-hint problems – the threshold of independent mDC reasoning established during the validation study. We interpret their success on additive Problem #1 to indicate their use of additive reasoning and their difficulty with mDC Problems 2–5 to indicate constraints to their ability to engage in mDC. This contrast between additive and mDC reasoning lends support to researchers' *theoretical predictions* of a conceptual leap involved in shifting from additive to multiplicative reasoning (Lester & Steffe, 2013; Ulrich, 2015). Our hypothesis, that a child's spontaneous additive strategy would show asymmetric association with their mDC reasoning, led us to further examine these averages while using "strategy" as a category (ordinal variable).

Between-category differences in mDC reasoning

We present between-category analyses of children's mDC reasoning, with categories reflecting the strategy they spontaneously and successfully used to solve Problem #1 ($8 + 7 = 15$). Because we found no statistically significant difference between the two doubling groups ($7 + 7$ or $8 + 8$), we combined them into a single category, doubling. We first analyze between-category differences in child responses to all 4 mDC problems and then to each mDC problem separately. Importantly, a Mann-Whitney test found no statistically significant difference between English language learners (mean rank = 16.5) and their counterparts (mean rank = 15.59), $U = 112$, $p = .8$.

Figure 6 shows a boxplot of the children's Rasch person-ability measure (0–100 scale) as a dependent variable with the spontaneous additive category as an independent variable. The results, with one outlier (#931), support our hypothesis: BAMT ($Mdn = 72.6$) > doubling ($Mdn = 42.7$) > counting-on ($Mdn = 34.0$). We kept this outlier for a twofold reason. First, it illustrates the possibility a child would use doubling to solve the additive problem and then reason with mDC (gleaned from his interview) while correctly solving all four mDC items. Second, it demonstrates that we report on the full set of children. In the Discussion we return to this point, particularly the small sample limitation.

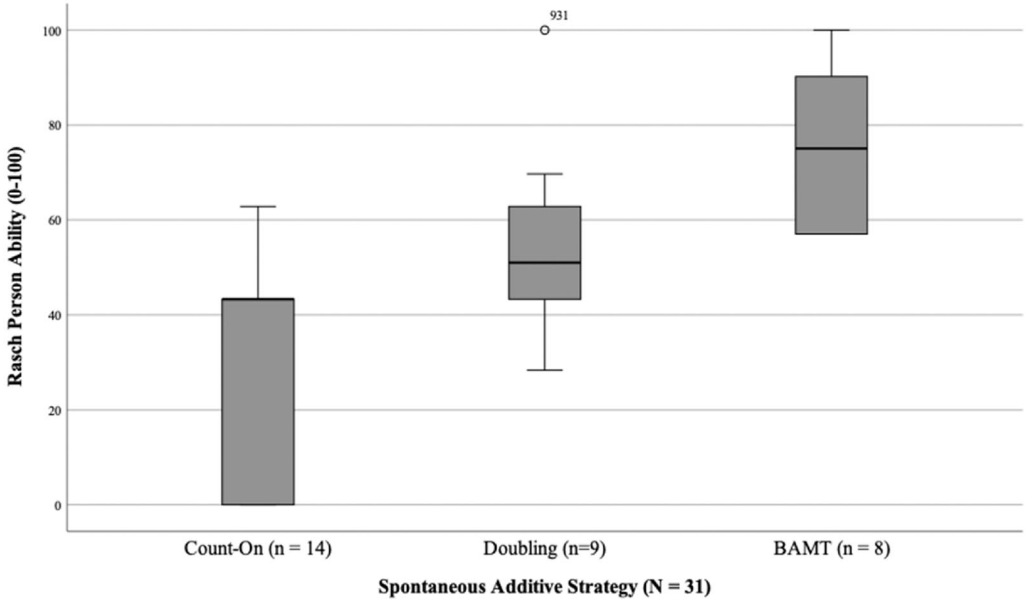


Figure 6. Rasch measure (mDC, 0–100) as a variable dependent on a child's additive strategy.

Table 5. By-category frequencies of the nine Rasch person-ability values ($d = 0.71$, $p < .001$).

Rasch Score →	0	28	43	51	57	63	70	80	100
Counting-on ($n = 14$)	**4**	**0**	**8**	**1**	0	1	0	0	0
Doubling ($n = 9$)	0	1	2	2	1	1	1	0	1
BAMT ($n = 8$)	0	0	0	0	**3**	**0**	**1**	**2**	**2**

Bold indicates $p < .001$.

To further test the asymmetric association between a child's spontaneous additive strategy and mDC (or premultiplicative) reasoning, we used crosstabulation and the Somers' d test. Table 5 shows that this association, with mDC considered dependent on additive strategy, is statistically significant ($d = 0.71$, $p < .001$; it corresponds to Goodman-Kruskal's $\Gamma = 0.78$). That is, by knowing a child's spontaneous strategy we make 71% fewer prediction errors about their mDC reasoning. Holm-Bonferroni post-hoc pair-wise comparisons showed that all three differences were statistically significant with medium-to-large Cohen's d effect size: counting-on vs. doubling ($d = 0.58$, $p = .006$), counting-on vs. BAMT ($d = 0.95$, $p < .001$), and doubling vs. BAMT ($d = 0.58$, $p = 0.007$).

As an additional measure of relationship, we used Spearman's rho to measure the correlation between additive strategy (ordinal) and mDC reasoning in Rasch logits (interval). The correlation was high and significant ($r_s = 0.73$, $p < .001$).

Finally, we used the Mann-Whitney nonparametric test (with standardized z) to examine differences between groups on each mDC problem separately (Table 6). Success of children who spontaneously used BAMT differed significantly from children who used counting-on on all problems (with $p = .054$ for Problem #5). Per-problem differences did not reach statistical significance between counting-on and doubling nor between doubling and BAMT.

Results presented in Table 6 emphasize two points that, combined, further support our claim about how the child's spontaneous additive strategy relates to the child's current capacity to engage in mDC reasoning. First, we focus on responses to mDC Problem #4. To solve it correctly, they needed to figure out the correct number of teams that Joy counted (35 players would

Table 6. Between-group differences on each mDC item: Counting-on vs. BAMT.

Problem No.	2	3	4	5
MWz	$z = 2.06$ $p = .039$	$z = 2.25$ $p = .024$	$z = 3.15$ $p = .002$	$z = 1.93$ $p = .054$

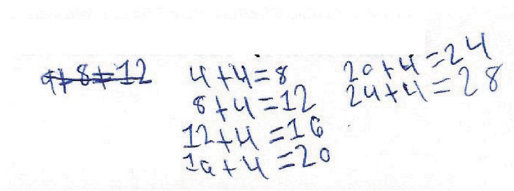


Figure 7. Adel's progressive summation of grouped 1 s.

make 7 teams of 5 players each). Only one child who used counting-on (7% of this category) could solve this problem correctly; the other thirteen (93%) were unsuccessful. Among those thirteen, ten children (71% of the entire counting-on category) incorrectly selected "35" as the number of teams that Joy counted. This error is telling. We interpret it as empirical support to Lester and Steffe, (2013) and Ulrich's (2015) claim that children's use of counting-on (and INS) reflects their reliance on operating on 1 s, and thus also premultiplicative reasoning.

Second, children's relative success on Problems #2 and #3 further point to a central element in Steffe's (1992) model on which we focus, namely, number as a composite unit. To successfully solve each of these problems, a child would have to *coordinate* the accrual of 1 s and composite units. Of the 14 children who spontaneously used counting-on, only three (∼21% of this category) correctly solved these problems. These data support our claim that a child's spontaneous use of counting-on relates to their current, premultiplicative reasoning. In contrast, among the eight children who spontaneously used BAMT, five (∼63%) correctly solved Problem #2, five (∼63%) correctly solved Problem #3, seven (∼88%) correctly solved Problem #4, and six (75%) correctly solved Problem #5. These data support our claim that children's spontaneous use of BAMT, which they use to solve addition tasks by decomposing the second addend into sub-units larger than 1, relates to their current ability to engage in mDC reasoning.

Further qualitative analysis: typical premultiplicative solutions to mDC problems

Unlike Zaro's (BAMT) example, those who spontaneously use counting-on, and to a lesser extent also doubling, are less likely to engage in mDC when solving multiplicative tasks. The two exemplars we present in this section show premultiplicative reasoning; the first (Adel) used counting-on and the second (Carl) used doubling, corresponding to the INS and TNS, respectively. We present these exemplars to further illuminate that counting-on (mostly) or doubling (to some extent) may be related to premultiplicative reasoning. We selected mDC items on which their work evidently exemplifies premultiplicative reasoning.

First exemplar (Adel): masked operation on 1 s. Figure 7 shows the final artifact of Adel's (counting-on) work when solving Problem #5. To arrive at this solution, she first correctly filled the numbers given in the realistic word problem into the screener items.

This indicated her (reading) comprehension of the quantities given in the problem. Then, to find how many boxes (4 cookies each) are needed to place all 28 cookies, she mentally added $4 + 4$ to get 8 and wrote down the next sum she would work on, "$4 + 8 = 12$." Adel then crossed that sum and started over in what indicated an intention to produce partial sums of groups of four 1 s each, until reaching the given total of cookies. Due to the interviewing protocol, which we designed to minimize interruption and/or prompting, the interviewer did not probe for her

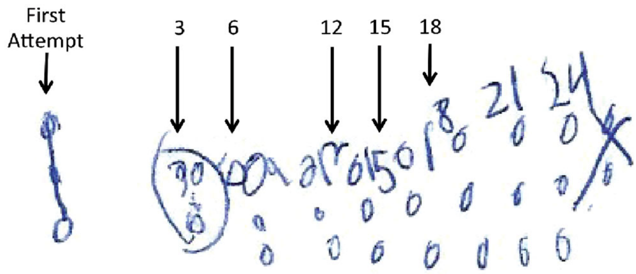


Figure 8. Carl's reconstitution of every composite unit from its 1 s (printed numerals/words/arrows were added by the authors to assist understanding of the child's work).

goal in doing so. Thus, we had limited data about the possibility that her planned, sequential activity included an intention to first produce and then count the number of "4 s" she would then write, as seen in the work of Carl below.

For each equation past the second sum (12), Adel first wrote the two addends (e.g., $12 + 4$). Then, she used her fingers to keep track as she added four more 1 s (e.g., 13-14-15-16) and wrote that last number on the right side of the equal sign. She proceeded in this sequential way until arriving at 28. Importantly, at that point Adel neither counted the number of 4 s she had used nor indicated any attempt to do so. Rather, she turned to the final step in her solution of this problem—writing her incorrect answer (28) for how many boxes were used for all 28 cookies.

Adel's final step in this process, as well as her counting of 1 s for partial sums and the lack of any observable attempt to count the 4 s she produced, indicate that she operated on units of 1. This operation on 1 s is a hallmark of the INS (Ulrich, 2015). She did group those singletons into 4 s in the partial sums, which is consistent with each "4" being a segment of her counting sequence (i.e., INS), and could thus be linked to a repeated addition notion of multiplication. We infer, however, that those numerical depictions served her intention to keep track of how many 1 s have been accrued until reaching the given total, not to produce a countable set of those composite units (groups of 4 dots). From Adel's way of repeatedly adding 4 s, we infer her intention was to sequentially amass groups of four 1 s, which is consistent with Ulrich's (2016) depiction of how a child reasoning with INS may approach such a situation. That is, although she organized partial sums by groups, she seemed focused on the total number of 1 s throughout the process. Said differently, a child reasoning with INS, and likely spontaneously using counting-on, seems to 'lose sight' of composite units potentially represented by her self-produced numerals (Steffe, 1992; Ulrich, 2015). For such a child it thus makes sense to respond with the total (of 1 s) when asked about the number of composite units that constitute that total.

Second exemplar (Carl): doubling may also mask premultiplicative reasoning. Carl's work when solving Problem #2 (see final artifact in Figure 8), serves to illustrate premultiplicative operations used by children whose spontaneous additive strategy is doubling (he said, "$7 + 7$ is 14, and 1 more is 15"). That is, Carl's example illustrates constraints common to children reasoning with TNS (i.e., not yet decomposing composite units), in spite of the more advanced additive strategy they use. We note that other children who use doubling may not be so constrained. Nevertheless, our purpose is to highlight the possibility that doubling is used along with such premultiplicative reasoning, in part to challenge the common view of doubling as a form of multiplicative reasoning.

To solve Problem #2 (how many towers of 3 produce a tower of 24), Carl first drew three dots and then linked them with a line (see "First Attempt," Figure 8, left side). This line appeared as an attempt to 'show' a figural composite unit (here, a schematic depiction of one tower). Consistent with TNS involving 'in action' use of composite units, however, he soon abandoned this attempt (which could have appeared a similar start to Zaro's). Instead, Carl started over by first drawing three single dots, writing the number 3 near the top one, and circling that triplet. He then proceeded with a sequential process of producing, 'in action', a composite unit (triplet) of single dots and writing the accruing number of 1 s, albeit not circling the triplets. Past the 12th dot, he figured out the next multiple of 3 by adding three 1 s (e.g., 13-14-15). Letting go of circling each triplet indicated he saw the triplet as a unit for subsequently counting the number of those units. Importantly for our claim about Carl's TNS ('in action'), he did not stop producing 1 s (in triplets) at 24. Rather, he continued to produce two more dots before realizing that he reached the goal of his activity as given in Problem #2. His final step in the process supported our inference about his premultiplicative reasoning—he counted the number of composite units by pointing to each of the partial sums he wrote down when producing the entire collection of 24 dots. To us, this last step indicated that his intention from the outset was to keep track, *sequentially*, of accrual of each type of unit (1 s and composite units).

From Carl's solution we inferred premultiplicative reasoning because it was sequential, that is, it did not include simultaneous coordination of 1 s and composite units. In Ulrich's (2016) depiction of the TNS, he could solve the multiplication problem "in the moment" (p. 38). Key to Carl's incorrect answer was that, at the final step, he counted (twice - to check himself) all the triplets *except for the first* (circled) group with the number 3 in it—thus obtaining and entering "7" as his answer. We emphasize that our choice of Carl's example is not due to his incorrect answer, but due to his TNS-like, in-the-moment solution – sequentially operating on 1 s and only then on composite units.

Combined, Adel and Carl's examples demonstrate challenges they faced when solving mDC items. Both manifested the nature of processing that Steffe's (1992) model predicts—premultiplicative operations only on 1 s, or sequentially on 1 s and composite units. Following Ulrich (2016), we link those with the Initial Number Sequence (INS). Thus, these two exemplars highlight the conceptual underpinnings of our central claim: the spontaneous additive strategy a child uses relates to their current capacity to engage in mDC as both are rooted in the child's concept of number.

Discussion

We demonstrated an important linkage between additive and multiplicative reasoning explained by focusing on a central element of Steffe's model—number as a composite unit. We found an asymmetric pattern of association between a child's spontaneous additive strategy (counting-on, doubling, or BAMT) and their current capacity to engage in mDC. In the Theoretical Framework, following Ulrich's (2015, 2016) elucidation of Steffe and colleagues' work, we explained why such a linkage is expected in terms of units the child operates on—1s and/or composite units. We acknowledge that a child's spontaneous additive strategy, used in a single trial, may not be the only or most advanced way of operating available to them. Nevertheless, it is a strategy that was readily available and made sense to the child in the moment (Gaschler et al., 2013; Steffe & Cobb, 1988; Verschaffel et al., 2009), which our findings suggest can serve as a "litmus test" of their current numerical reasoning. Hence, our study corroborates a twofold implication of Steffe's model. A child's spontaneous use of counting-on to solve an addition problem relates mostly to premultiplicative reasoning; and a child's spontaneous use of BAMT relates mostly to mDC reasoning. Next, we discuss the importance and implications of our study.

Theoretical importance: statistical corroboration of a conceptual model

Kilpatrick (2001) asserted that the field of mathematics education faces the challenge of testing and corroborating conceptually sound models produced through qualitative studies. This assertion pertains to models, developed by Steffe and others (Steffe, 1992; Steffe & Cobb, 1998; Steffe, von Glasersfeld, Richards, & Cobb, 1983), about children's numerical reasoning. To date, only one line of work—on fractions—has tested those models statistically (Norton & Wilkins, 2009, 2012). Consistent with Lamon's (2007) plea, our study offers an empirical link between children's additive strategies and an initial, foundational form of whole number multiplicative reasoning (mDC) by providing statistical corroboration of existing conceptual models.

Specifically, our mixed-method analyses corroborate the conceptual model of number as a composite unit in general and the three number sequences in particular—Initial Number Sequence (INS), Tacitly Nested Number Sequence (TNS), and Explicitly Nested Number Sequence (ENS) (Steffe & Cobb, 1988; Ulrich, 2015, 2016). Each of those number sequences is inferred to underlie, respectively, each of the three additive strategies of counting-on, doubling, and BAMT – as well as premultiplicative and mDC reasoning. Figure 2 depicted those relationships. Our qualitative analysis revealed plausible conceptual processes connecting the child's concept of number as a composite unit with the child's additive strategy and with the child's mDC reasoning. That is, to engage in mDC when solving a task, it is not enough for a child to perform the actions leading to a correct (or incorrect) answer. Rather, a child has to supply the conceptual structures (units) and mental actions (e.g., iteration of composite units) from which an observer might infer whether the child reasoned multiplicatively or otherwise. Prior research has explicated the difference in those types of reasoning in terms of whether children can work simultaneously with units of 1 and composite units (Hackenberg & Tillema, 2009; Tzur et al., 2013). Our study lended further warrant to Steffe's (1992) and Ulrich's (2015, 2016) articulation of linkages between those types of reasoning.

Implications for future research

We consider four key implications of this study for future research. First, our claim about children's spontaneous additive strategy being related to their ability to engage in mDC reasoning relies on a conceptual explanation corroborated by statistically significant correlations. Using the Somers' d test, we also showed it is consistent with a possibility that the child's spontaneous additive strategy foretells their mDC (or premultiplicative) reasoning. Of course, one limitation of these results is the small sample ($n = 31$). We believe that a future study (e.g., experimental or quasi-experimental) could further confirm this possibility.

Another implication of our study draws on having pointed out how a single additive problem a child uses spontaneously relates to mDC reasoning. This relationship opens the way for designing future studies that: (a) ascertain what additive strategies, other than the one a child uses spontaneously, are available to them and (b) quantify gradations in the child's construction of number as composite unit.

Third, this study implies the need to critically examine the design, and findings, of studies about children's success in solving problems (e.g., Russell & Ginsburg, 1984) and/or the impact of instructional interventions on children's learning and outcomes (e.g., Woodward, 2006). Often, lack of success or of impact may be rooted in children's lack of opportunities to develop a cognitive prerequisite necessary to afford the intended performance. Based on the present study, we stress the need for such examination particularly when teaching interventions focus on mathematical concepts that involve various forms of multiplicative reasoning (e.g., fractions). Specifically, learning concepts rooted in such reasoning may prove to be "a bridge too far" for children who spontaneously use counting-on (45% of our 4th graders sample) and to lesser extent – also doubling.

Finally, our study points to a need for studying how teachers may benefit from knowing the association between a child's spontaneous stratgegy and current ability to engage in mDC. Findings of the present study showed that, in spite of being taught multiplication for more than a year (grade 3), only 42% of students indicated ability to engage in mDC reasoning. To address this issue, in our PD work with teachers we have conducted an exploratory study of how teachers may use the single additive problem (7 + 8) to support students who struggled with mDC. Specifically, we intended for teachers to: (a) recognize the critical linkage between a child's additive strategy and their current capacity to engage in mDC, (b) learn how to conduct short, task-based interviews to elicit children's additive strategies (spontaneous and prompted), and (c) adapt subsequent instruction to meet the needs of children based on their spontaneous additive strategy. We stress that this does not imply directly teaching these additive strategies as procedures.

Teachers' initial responses to this assessment measure were encouraging. They reported a substantial shift in their view of children's performance and a related, gradual shift in how they adapted their instruction to children's conceptual understandings. Specifically, these teachers expressed appreciation of the main learning goal needed for children who use counting-on and doubling, namely to further promote their conception of number (e.g., learn to decompose addends into composite sub-units). Elsewhere (Tzur et al., 2018), we reported on encouraging preliminary findings of the impact of those teachers' work on their 3rd and 4th graders' growth in mDC reasoning (from an average of 14% success in year-start to 39% in year-end—a large Cohen's d effect size of 0.83). Our exploratory work points to future research to examine (a) teachers' understanding of the association between additive strategies and mDC reasoning and (b) how instruction informed by such an understanding may impact students' growth.

Concluding remarks

In our study, we focused on a foundational form of whole number multiplicative reasoning (mDC) that draws on the works of Steffe (1992, 1994), Lester and Steffe (2013) and Ulrich (2015, 2016). Specifically, we focused on a child's numerical reasoning that takes as input not just 1 s but also composite units, and transforms two types of given units into a third type. Based on this stance, we focused on multiplicative double counting (mDC), a way of reasoning characterized by simultaneous count of the accrual of composite units and 1 s to solve multiplicative problem situations. We demonstrated that children's spontaneous additive strategy relates, plausibly in an asymmetric pattern, to their current capacity for mDC reasoning. In our qualitative analysis, we further demonstrated the need to extend assessment beyond the correctness of a child's answer, or strategy, and to focus on the reasoning that may underlie it. In particular, when a child's spontaneous additive strategy is counting-on or doubling, our study points to the need to provide opportunities to conceive of the nested relationships between numbers and thus to also decompose numbers strategically (e.g., BAMT). Such opportunities can promote children's mDC and beyond, through further opportunities in which they learn to coordinate and keep track of different types of units.

Acknowledgments

This study was supported by the US National Science Foundation under grant #1503206. The opinions expressed do not necessarily reflect the views of the National Science Foundation.

Disclosure statement

No financial interest or benefits that create a conflict of interest are involved in this study.

Funding

This study was approved by the *Colorado Multiple Institutional Review Board* (COMIRB). We have no conflicts of interest to disclose. Our work was funded in part by the US National Science Foundation under grant #1503206. The opinions expressed do not necessarily reflect the views of the Foundation.

ORCID

Ron Tzur http://orcid.org/0000-0001-7140-8351
Alan Davis http://orcid.org/0000-0002-1044-1200
Michael Ferrara http://orcid.org/0000-0001-6526-5104

References

Abdi, H. (2010). Holm's sequential Bonferroni procedure. In N. Salkind (Ed.), *Encyclopedia of research design* (Vol. 1, pp. 574–577). Thousand Oaks, CA: SAGE. doi:10.4135/9781412961288.n178

Agresti, A., & Finlay, B. (1997). *Statistical methods for the social sciences*. Upper Saddle River, NJ: Prentice Hall.

Allegra, M., Seyed-Allaei, S., Schuck, N. W., Amati, D., Laio, A., & Reverberi, C. (2020). Brain network dynamics during spontaneous strategy shifts and incremental task optimization. *NeuroImage*, *217*, 116854. doi:10.1016/j.neuroimage.2020.116854

Anastasi, A., & Urbina, S. (1997). *Psychological testing* (7th ed.). Upper Saddle River, NJ: Prentice-Hall.

Baroody, A. J. (2006). Why children have difficulties mastering the basic number combinations and how to help them. *Teaching Children Mathematics*, *13*(1), 22–31. www.jstor.org/stable/41198838. doi:10.5951/TCM.13.1.0022

Birt, L., Scott, S., Cavers, D., Campbell, C., & Walter, F. (2016). Member checking: A tool to enhance trustworthiness or merely a nod to validation? *Qualitative Health Research*, *26*(13), 1802–1811. doi:10.1177/1049732316654870

Bisanz, J., & LeFevre, J.-A. (1990). Strategic and nonstrategic processing in the development of mathematical cognition. In D. F. Bjorklund (Ed.), *Children's strategies: Contemporary views of cognitive development* (pp. 213–244). Hillsdale, NJ: Lawrence Erlbaum.

Bond, T. G., & Fox, C. M. (2015). *Applying the Rasch model: Fundamental measurement in the human sciences* (3rd ed.). Mahwah, NJ: Erlbaum.

Boone, W. J., Staver, J. R., & Yale, M. S. (2014). *Rasch analysis in the human sciences*. New York, NY: Springer.

Boulet, G. (1998). Didactical implications of children's difficulties in learning the fraction concept. *Focus on Learning Problems in Mathematics*, *20*(4), 19–34.

Brentari, E., & Golia, S. (2007). Unidimensionality in the Rasch model: How to detect and interpret. *Statistica*, *62*(3), 253–261. doi:10.6092/issn.1973-2201/3508

Carlson, J. A. (2010). Avoding traps in member checking. *The Qualitative Report*, *15*(5), 1102–1113. https://nsuworks.nova.edu/cgi/viewcontent.cgi?article=1332&context=tqr.

Clark, F. B., & Kamii, C. (1996). Identification of multiplicative thinking in children in grades 1–5. *Journal for Research in Mathematics Education*, *27*(1), 41–51.

Clement, J. (2000). Analysis of clinical interviews: Foundations and model. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 547–590). Mahwah, NJ: Lawrence Erlbaum.

Confrey, J., & Smith, E. (1995). Splitting, covariation, and their role in the development of exponential functions. *Journal for Research in Mathematics Education*, *26*(1), 66–86. doi:10.2307/749228

Davydov, V. V. (1992). The psychological analysis of multiplication procedures. *Focus on Learning Problems in Mathematics*, *14*(1), 3–67.

Fuson, K. C. (1992). Research on whole number addition and subtraction. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 243–275). New York, NY: Macmillan.

Gaschler, R., Vaterrodt, B., Frensch, P. A., Eichler, A., & Haider, H. (2013). Spontaneous usage of different shortcuts based on the commutativity principle. *PLoS One*, *8*(9), e74972. doi:10.1371/journal.pone.0074972

Grabner, R. H., Ansari, D., Reishofer, G., Stern, E., Ebner, F., & Neuper, C. (2007). Individual differences in mathematical competence predict parietal brain activation during mental calculation. *NeuroImage*, *38*(2), 346–356. doi:10.1016/j.neuroimage.2007.07.041

Hackenberg, A. J. (2007). Units coordination and the construction of improper fractions: A revision of the splitting hypothesis. *The Journal of Mathematical Behavior*, *26*(1), 27–47. doi:10.1016/j.jmathb.2007.03.002

Hackenberg, A. J. (2013). The fractional knowledge and algebraic reasoning of students with the first multiplicative concept. *The Journal of Mathematical Behavior*, *32*(3), 538–563. doi:10.1016/j.jmathb.2013.06.007

Hackenberg, A. J., & Lee, M. Y. (2015). Relationships between students' fractional knowledge and equation writing. *Journal for Research in Mathematics Education*, *46*(2), 196–243. doi:10.5951/jresematheduc.46.2.0196

Hackenberg, A. J., & Tillema, E. S. (2009). Students' whole number multiplicative concepts: A critical constructive resource for fraction composition schemes. *The Journal of Mathematical Behavior*, *28*(1), 1–18. doi:10.1016/j.jmathb.2009.04.004

Harel, G., & Confrey, J. (Eds.). (1994). *The development of multiplicative reasoning in the learning of mathematics*. Albany, NY: State University of New York Press.

Hodkowski, N. M., Hornbein, P., Gardner, A., Johnson, H. L., Jorgensen, C., & Tzur, R. (2016). Designing a stage-sensitive written assessment of elementary students' scheme for multiplicative reasoning. In M. B. Wood, E. Turner, M. Civil, & J. A. Eli (Eds.), *Proceedings of the 38th Annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1581–1587). Tucson, AZ: The University of Arizona.

Holm, S. (1979). A simple sequentially rejective multiple test procedure. *Scandinavian Journal of Statistics*, *6*(2), 65–70. http://www.jstor.org/stable/4615733.

Izsák, A., & Beckmann, S. (2019). Developing a coherent approach to multiplication and measurement. *Educational Studies in Mathematics*, *101*(1), 83–103. doi:10.1007/s10649-019-09885-8

Johnson, H. L., Tzur, R., Hodkowski, N. M., Jorgensen, C., Wei, B., Wang, X., & Davis, A. (2018). A written, large-scale assessment measuring gradations in students' multiplicative reasoning. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.), *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 163–170). Umeå: PME.

Kilpatrick, J. (2001). Where's the evidence? *Journal for Research in Mathematics Education*, *32*(4), 421–427. doi:10.2307/749702

Kling, G., & Bay-Williams, J. M. (2015). Three steps to mastering multiplication facts. *Teaching Children Mathematics*, *21*(9), 548–559. doi:10.5951/teacchilmath.21.9.0548

Lester, F. K., & Steffe, L. P. (2013). Establishing mathematics education as an academic field: A constructive odyssey. *Journal for Research in Mathematics Education*, *44*(2), 353–371. doi:10.5951/jresematheduc.44.2.0353

Lamon, S. J. (2007). Rational numbers and proportional reasoning: Toward a theoretical framework. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 629–667). Charlotte, NC: Information Age. [Record #3510 is using a reference type undefined in this output style.]

Linacre, J. M. (2011). *Practical Rasch measurement - further topics* (3). https://www.winsteps.com/a/winsteps-tutorial-further-3.pdf

McMullen, J., Chan, J. Y.-C., Mazzocco, M. M. M., & Hannula-Sormunen, M. M. (2019). Spontaneous mathematical focusing tendencies in mathematical development and education. In A. Norton & M. W. Alibali (Eds.), *Constructing Number: Merging Perspectives from Psychology and Mathematics Education* (pp. 69–86). Cham: Springer International Publishing.

Murata, A., & Fuson, K. (2006). Teaching as assisting individual constructive paths within an interdependent class learning zone: Japanese first graders learning to add using 10. *Journal for Research in Mathematics Education*, *37*(5), 421–456.

Newson, R. (2002). Parameters behind "nonparametric" statistics: Kendall's tau, Somers' D and median differences. *The Stata Journal: Promoting Communications on Statistics and Stata*, *2*(1), 45–64. doi:10.1177/1536867X0200200103

Newson, R. (2006). Confidence intervals for rank statistics: Somers' d and extensions. *The Stata Journal: Promoting Communications on Statistics and Stata*, *6*(3), 309–334. doi:10.1177/1536867X0600600302

Norton, A., & Wilkins, J. L. M. (2009). A quantitative analysis of children's splitting operations and fraction schemes. *The Journal of Mathematical Behavior*, *28*(2–3), 150–161. doi:10.1016/j.jmathb.2009.06.002

Norton, A., & Wilkins, J. L. M. (2012). The splitting group. *Journal for Research in Mathematics Education*, *43*(5), 557–583. doi:10.5951/jresematheduc.43.5.0557

Nunnally, J. C., & Bernstein, I. H. (1994). *Psychometric theory*. New York, NY: McGraw Hill.

Olejnik, S., Li, J., Supattathum, S., & Huberty, C. J. (1997). Multiple testing and statistical power with modified Bonferroni procedures. *Journal of Educational and Behavioral Statistics*, *22*(4), 389–406. doi:10.3102/10769986022004389

Piaget, J. (1985). *The equilibration of cognitive structures: The central problem of intellectual development* (T. Brown & K. J. Thampy, Trans.). Chicago, IL: The University of Chicago. (1975 (French)

Powell, A. B., Francisco, J. M., & Maher, C. A. (2003). An analytical model for studying the development of learners' mathematical ideas and reasoning using videotape data. *The Journal of Mathematical Behavior*, *22*(4), 405–435. doi:10.1016/j.jmathb.2003.09.002

Rittle-Johnson, B., Siegler, R. S., & Alibali, M. W. (2001). Developing conceptual understanding and procedural skill in mathematics: An iterative process. *Journal of Educational Psychology*, *93*(2), 346–362. doi:10.1037/0022-0663.93.2.346

Russell, R. L., & Ginsburg, H. P. (1984). Cognitive analysis of children's mathematics difficulties. *Cognition and Instruction*, *1*(2), 217–244. doi:10.1207/s1532690xci0102_3

Schwartz, J. L. (1991). Intensive quantity and referent transforming arithmetic operations. In J. Hiebert & M. J. Behr (Eds.), *Number concepts and operations in the middle grades* (3rd ed., Vol. 2, pp. 41–52). Lawrence Erlbaum; National Council of Teachers of Mathematics. (1988)

Secada, W. G., Fuson, K. C., & Hall, J. W. (1983). The transition from counting-all to counting-on in addition. *Journal for Research in Mathematics Education*, *14*(1), 47–57. doi:10.2307/748796

Siegler, R. S. (2000). Unconscious insights. *Current Directions in Psychological Science*, *9*(3), 79–83. doi:10.1111/1467-8721.00065
Siegler, R. S. (2013). Implications of cognitive science research for mathematics education. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A research companion to principles and standards for school mathematics* (pp. 219–233). Reston, VA: National Council of Teachers of Mathematics.
Siegler, R. S., & Shipley, C. (1995). Variation, selection, and cognitive change. In T. Simon & G. Halford (Eds.), *Developing cognitive competence: New approaches to process modeling* (pp. 31–76). Hillsdale, NJ: Erlbaum.
Siegler, R. S., & Stern, E. (1998). Conscious and unconscious strategy discoveries: A microgenetic analysis. *Journal of Experimental Psychology: General*, *127*(4), 377–397. doi:10.1037/0096-3445.127.4.377
Simon, M. A. (2006). Key developmental understandings in mathematics: A direction for investigating and establishing learning goals. *Mathematical Thinking and Learning*, *8*(4), 359–371. doi:10.1207/s15327833mtl0804_1
Simon, M. A. (2012). Extending the coordination of cognitive and social perspectives. *PNA*, *6*(2), 43–49.
Simon, M. A., Kara, M., Norton, A., & Placa, N. (2018). Fostering construction of a meaning for multiplication that subsumes whole-number and fraction multiplication: A study of the Learning Through Activity research program. *The Journal of Mathematical Behavior*, *52*, 151–173. doi:10.1016/j.jmathb.2018.03.002
Simon, M. A., Kara, M., Placa, N., & Avitzur, A. (2018). Towards an integrated theory of mathematics conceptual learning and instructional design: The Learning Through Activity theoretical framework. *The Journal of Mathematical Behavior*, *52*, 95–112. doi:10.1016/j.jmathb.2018.04.002
Simon, M. A., Placa, N., & Avitzur, A. (2016). Participatory and anticipatory stages of mathematical concept learning: Further empirical and theoretical development. *Journal for Research in Mathematics Education*, *47*(1), 63–93. doi:10.5951/jresematheduc.47.1.0063
Simon, M. A., Tzur, R., Heinz, K., & Kinzel, M. (2004). Explicating a mechanism for conceptual learning: Elaborating the construct of reflective abstraction. *Journal for Research in Mathematics Education*, *35*(5), 305–329. doi:10.2307/30034818
Smith, R. M., & Miao, C. Y. (1994). Assessing unidimensionality for Rasch measurement. In M. Wilson (Ed.), *Objective measurement: Theory into practice* (Vol. 2, pp. 316–327). Norwood, NJ: Ablex.
Steffe, L. P. (1991). Children's construction of number sequences and multiplying schemes. In J. Hiebert & M. J. Behr (Eds.), *Number concepts and operations in the middle grades* (3rd ed., Vol. 2, pp. 119–140). Reston, VA: Lawrence Erlbaum; National Council of Teachers of Mathematics.
Steffe, L. P. (1992). Schemes of action and operation involving composite units. *Learning and Individual Differences*, *4*(3), 259–309. doi:10.1016/1041-6080(92)90005-Y
Steffe, L. P. (1994). Children's multiplying schemes. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (1 ed., pp. 3–39). Albany, NY: State University of New York Press.
Steffe, L. P. (2010). Operations that produce numerical counting schemes. In L. P. Steffe & J. Olive (Eds.), *Children's fractional knowledge* (pp. 27–47). New York, NY: Springer.
Steffe, L. P., & Cobb, P. (1988). *Construction of arithmetical meanings and strategies*. New York, NY: Springer-Verlag.
Steffe, L. P., & Cobb, P. (1998). Multiplicative and divisional schemes. *Focus on Learning Problems in Mathematics*, *20*(1), 45–62.
Steffe, L. P., Liss, D. R. I., & Lee, H. Y. (2014). On the operations that generate intensive quantity. In L. P. Steffe, K. C. Moore, & L. L. Hatfield (Eds.), *Epistemic algebraic students: Emerging models of students' algebraic knowing* (Vol. 4, pp. 49–79). Laramie, WY: Wyoming Institute for the Study and Development of Mathematical Education (WISDOMe).
Steffe, L. P., & Tzur, R. (1994). Interaction and children's mathematics. *Journal of Research in Childhood Education*, *8*(2), 99–116. doi:10.1080/02568549409594859
Steffe, L. P., & von Glasersfeld, E. (1985). Helping children to conceive of number. *Recherches en Didactique Des Mathematiques*, *6*(2/3), 269–303.
Steffe, L. P., von Glasersfeld, E., Richards, J., & Cobb, P. (1983). *Children's Counting Types* (1 ed.). New York, NY: Praeger.
Stern, E. (1992). Spontaneous use of conceptual mathematical knowledge in elementary school children. *Contemporary Educational Psychology*, *17*(3), 266–277. doi:10.1016/0361-476X(92)90065-7
Stern, E., & Schneider, M. (2010). A digital road map analogy of the relationship between neuroscience and educational research. *ZDM*, *42*(6), 511–514. doi:10.1007/s11858-010-0278-1
Thompson, P. W. (2008). *Conceptual analysis of mathematical ideas: Some spadework at the foundations of mathematics education*. Paper presented at Proceedings of the annual meeting of the International Group for the Psychology of Mathematics Education, Morélia, Mexico.
Thompson, P. W., & Saldanha, L. (2003). Fractions and multiplicative reasoning. In J. Kilpatrick & G. Martin (Eds.), *Research companion to the NCTM Standards* (pp. 95–113). Washington, DC: National Council of Teachers of Mathematics.

Torbeyns, J., Verschaffel, L., & Ghesquiere, P. (2005). Simple addition strategies in a first-grade class with multiple strategy instruction. *Cognition and Instruction*, *23*(1), 1–21. doi:10.1207/s1532690xci2301_1

Tzur, R. (2007). Fine grain assessment of students' mathematical understanding: Participatory and anticipatory stages in learning a new mathematical conception. *Educational Studies in Mathematics*, *66*(3), 273–291. doi:10.1007/s10649-007-9082-4

Tzur, R. (2019). Hypothetical learning trajectory (HLT): A lens on conceptual transition between mathematical "markers. In D. Siemon, T. Barkatsas, & R. Seah (Eds.), *Researching and using progressios (trajectories) in mathematics education* (pp. 56–74). Rotterdam: Sense.

Tzur, R., Johnson, H. L., Hodkowski, N. M., Jorgensen, C., Nathenson-Mejía, S., & Wei, B. (2018). Impact of a student-adaptive pedagogy PD program on students' multiplicative reasoning. In T. E. Hodges, G. J. Roy, & A. M. Tyminski (Eds.), *Proceedings of the 40th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1084–1090). Greenville, SC: University of South Carolina & Clemson University.

Tzur, R., Johnson, H. L., McClintock, E., Kenney, R. H., Xin, Y. P., Si, L., … Jin, X. (2013). Distinguishing schemes and tasks in children's development of multiplicative reasoning. *PNA*, *7*(3), 85–101.

Tzur, R., & Lambert, M. A. (2011). Intermediate participatory stages as Zone of Proximal Development correlate in constructing counting-on: A plausible conceptual source for children's transitory 'regress' to counting-all. *Journal for Research in Mathematics Education*, *42*(5), 418–450. doi:10.5951/jresematheduc.42.5.0418

Tzur, R., & Simon, M. A. (2004). Distinguishing two stages of mathematics conceptual learning. *International Journal of Science and Mathematics Education*, *2*(2), 287–304. doi:10.1007/s10763-004-7479-4

Ulrich, C. (2015). Stages in constructing and coordinating units additively and multiplicatively (part 1). *For the Learning of Mathematics*, *35*(3), 2–7.

Ulrich, C. (2016). Stages in constructing and coordinating units additively and multiplicatively (part 2). *For the Learning of Mathematics*, *36*(1), 34–39.

Underwood, G. (1978). Concepts in information processing theory. In G. Underwood (Ed.), *Strategies of information processing* (pp. 1–22). London: Academic Press.

Verschaffel, L., Greer, B., & DeCorte, E. (2007). Whole number concepts and operations. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 557–627). Charlotte, NC: Information Age.

Verschaffel, L., Luwel, K., Torbeyns, J., & Van Dooren, W. (2009). Conceptualizing, investigating, and enhancing adaptive expertise in elementary mathematics education. *European Journal of Psychology of Education*, *24*(3), 335–359. www.jstor.org/stable/23421697. doi:10.1007/BF03174765

von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning*. Washington, DC: Falmer.

Whitacre, I., & Nickerson, S. D. (2016). Prospective elementary teachers making sense of multidigit multiplication: Leveraging resources. *Journal for Research in Mathematics Education*, *47*(3), 270–307. doi:10.5951/jresematheduc.47.3.0270

Woodward, J. (2006). Developing automaticity in multiplication facts: Integrating strategy instruction with timed practice drills. *Learning Disability Quarterly*, *29*(4), 269–289. doi:10.2307/30035554

Wright, R. J., Martland, J., Stafford, A. K., & Stanger, G. (2006). *Teaching number: Advancing children's skills and strategies* (2nd ed.). Thousand Oaks, CA: Sage.

Wright, S. P. (1992). Adjusted p-values for simultaneous inference. *Biometrics*, *48*(4), 1005–1013. doi:10.2307/2532694