Galerkin Approximations of General Delay Differential Equations with Multiple Discrete or Distributed Delays

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(ABSTRACT)

Delay differential equations (DDEs) are often used to model systems with time-delayed effects, and they have found applications in fields such as climate dynamics, biosciences, engineering, and control theory. In contrast to ordinary differential equations (ODEs), the phase space associated even with a scalar DDE is infinite-dimensional. Oftentimes, it is desirable to have low-dimensional ODE systems that capture qualitative features as well as approximate certain quantitative aspects of the DDE dynamics. In this thesis, we present a Galerkin scheme for a broad class of DDEs and derive convergence results for this scheme. In contrast to other Galerkin schemes devised in the DDE literature, the main new ingredient here is the use of the so called Koornwinder polynomials, which are orthogonal polynomials under an inner product with a point mass. A main advantage of using such polynomials is that they live in the domain of the underlying linear operator, which arguably simplifies the related numerical treatments. The obtained results generalize a previous work to the case of DDEs with multiply delays in the linear terms, either discrete or distributed, or both. We also consider the more challenging case of discrete delays in the nonlinearity and obtain a convergence result by assuming additional assumptions about the Galerkin approximations of the linearized systems.

Galerkin Approximations of General Delay Differential Equations with Multiple Discrete or Distributed Delays

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(GENERAL AUDIENCE ABSTRACT)

Delay differential equations (DDEs) are equations that are commonly used to model systems with time-delayed effects. DDEs have found applications in fields such as climate dynamics, biosciences, engineering, and control theory. However, the solutions to these equations can be difficult to approximate. In a previous paper, a method to approximate certain types of DDEs was described. In this thesis, it is shown that this method may also approximate more general types of DDEs.

Contents

Chapter 1

Delay Differential Equations

1.1 Examples of Delay Differential Equations

Delay Differential Equations (DDEs) are differential equations where the current derivative is affected by past values. We denote the delay by $\tau > 0$ and $C = C([-\tau, 0], \mathbb{R}^n)$. If $x \in C([\sigma - \tau, \sigma + A], \mathbb{R}^n)$ for $A > 0$ and $\sigma \in \mathbb{R}$, then we define $x_t \in C$ by $x_t(\theta) = x(t + \theta)$ for $-\tau \leq \theta \leq 0$. If $\Omega \subseteq \mathbb{R} \times C$, then the general form of a DDE is given by

$$\dot{x}(t) = f(t, x_t), \tag{1.1}$$

where $f : \Omega \to \mathbb{R}^n$. Here the "$\cdot$" represents the right-hand derivative. DDEs commonly arise in situations where there is a time-delayed effect. They have been applied to problems in the life sciences, engineering, climate science, economics, and control theory [DvGVLW95, GCS15, GZT08, HVL93, Kua93, LS10, Mac89, MN07, RCC$^+$14, Ste89]. Below, we give some examples where DDEs may be arise.

1.1.1 Logistic Equation with Discrete Delay

One way of modeling single species dynamics is by assuming the size of the population N satisfies the following logistic equation

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN(t)\left(1 - \frac{N(t)}{K}\right), \tag{1.2}$$

where $r > 0$ is the growth rate and $K > 0$ is the carrying capacity. Hutchison notes for Eq. (1.2) to hold, the mechanism by which the effective growth rate of $r(1 - \frac{N}{K})$ is derived must operate quickly after the population reaches a value of N [Hut48]. However, there may be a large time gap, $\tau > 0$, between when the population reaches a value of N and the corresponding effective growth rate is applied; e.g., after laying a clutch of eggs, the growth rate of a population of *Daphnia* will not be affected until some time later when the eggs hatch. In the cases when τ is large, Hutchinson suggests the following equation to model the population:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN(t)\left(1 - \frac{N(t-\tau)}{K}\right). \tag{1.3}$$

The above is commonly referred to as *Hutchinson's equation* and is used to model single species dynamics, and the term $N(t-\tau)$ is often referred to as a *discrete delay* term. Similar insights in regards to delayed effects have been made to predator-prey systems and disease dynamics, and these also give rise to DDEs which model these scenarios [L+23, WC57].

1.1.2 Logistic Equation with Distributed Delay

If we observe the population dynamics of a parasite which completes it life cycle in the same host and does not kill the host, then we get a distributed lag effect. Initially, the host presents an ideal environment for the parasite and grows exponentially. However, when the host becomes resistant the parasite population will begin to decrease rapidly. An appropriate model for the population is

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN(t)\left(1 - \frac{N(t)}{K} - \int_0^t N(s)G(t-s)\,\mathrm{d}s\right), \tag{1.4}$$

where G is a given memory function [McD78]. Here we call the term $\int_0^t N(s)G(t-s)\,\mathrm{d}s$ a *distributed delay* term.

1.1.3 Ship Stabilization

The next example concerns a simplified model of ship stabilization [Nei78]. One may describe the dynamics of a ship by

$$I\ddot{\varphi} + h\dot{\varphi} = -K\psi, \tag{1.5}$$

where $I > 0$, $K > 0$, $h \neq 0$, φ is the ship deviation angle and ψ the turning angle of the rudder. We also assume that ψ satisfies

$$T\dot{\psi} + \psi = \alpha\xi + \beta\xi, \tag{1.6}$$

where $T > 0$ and ξ is the measured value of the ship deviation angle. One might assume that the measuring of the ship deviation angle is delayed, or that

$$\xi(t) = \varphi(t-\tau) \tag{1.7}$$

for $\tau > 0$. From Eqs. (1.5) to (1.7), one can get the following DDE describing φ:

$$TI\,\dddot{\varphi}(t) + (Th - I)\ddot{\varphi}(t) - h\dot{\varphi}(t) + K\beta\dot{\varphi}(t-\tau) + K\alpha\varphi(t-\tau) = 0. \tag{1.8}$$

One can find many other examples of delayed actions resulting in DDEs from engineering [KN].

1.2 Existence of Solutions

A function x is a solution to Eq. (1.1) on $[\sigma - \tau, \sigma + A)$ for $\sigma \in \mathbb{R}$ and $A > 0$ if $x \in C([\sigma - \tau, \sigma + A), \mathbb{R}^n)$, $(t, x_t) \in D$, and $x(t)$ satisfies Eq. (1.1) for $t \in [\sigma, \sigma + A)$. Furthermore, we say that a solution has initial value ϕ at σ or passes through (σ, ϕ) if $x_\sigma = \phi$.

In [HVL93], a fixed point argument is used to prove the following existence theorem.

Theorem 1.1. *Suppose Ω is an open subset of $\mathbb{R} \times C$ and $f \in C(\Omega, \mathbb{R}^n)$. If $(\sigma, \phi) \in \Omega$, then there is a solution of Eq. (1.1) passing through (σ, ϕ).*

This guarantees some local continuous solution to the DDE. For uniqueness, we need further regularity on f. For instance, $f(t, \phi)$ Lipschitz in ϕ on compact sets of $\mathbb{R} \times C$ gives uniqueness of a local solution [HVL93, Thm. 2.3]. However, in this thesis we rely on the existence of *global* solutions to the DDE. Because of this, it is necessary to look at continuations of solutions. That is, if x is a solution of Eq. (1.1) defined on $[\sigma - \tau, a)$ for $a > \sigma$, then $\hat{x}$ is a continuation of x if $\hat{x}$ is defined on $[\sigma - \tau, b)$ for $b > a$, coincides with x on $[\sigma - \tau, a)$, and satisfies Eq. (1.1) on $[\sigma, b)$. A solution x is noncontinuable if it has no continuation. One may apply Zorn's lemma to prove the existence of noncontinuable solutions.

Again from [HVL93], we have the following result.

Theorem 1.2. *Suppose Ω is an open set in $\mathbb{R} \times C$ and $f \in C(\Omega, \mathbb{R}^n)$. If x is a noncontinuable solution of Eq. (1.1) on $[\sigma - \tau, b)$ and W is the closer of the set $\{(t, x_t) : \sigma \leq t < b\}$ in $\mathbb{R} \times C$, then W compact implies there is a sequence $\{t_k\}$ of real numbers, $t_k \to b^-$ as $k \to \infty$ such that (t_k, x_{t_k}) tends to $\partial\Omega$ as $k \to \infty$.*

One can apply this theorem to get existence of a global solution to Eq. (1.1). In particular, if $\Omega = \mathbb{R} \times C$, then one need only show that neither $x(t)$ nor $\dot{x}(t)$ blow up at finite values; this would imply that if x is defined on $[\sigma - \tau, b)$ then a continuation x_b would exist.

1.3 DDEs to be Considered in this Thesis

We consider systems of nonlinear DDEs involving multiple discrete or distributed delays, either in the linear term or in the nonlinearity. Such DDEs can be put into the following form:

$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t} &= \boldsymbol{A}\boldsymbol{x}(t) + \sum_{i=1}^{p} \boldsymbol{B}_i \boldsymbol{x}(t-\tau_i) + \sum_{i=1}^{p} \boldsymbol{C}_i \int_{t-\tau_i}^{t} \boldsymbol{x}(s)\,\mathrm{d}s \\ &\quad + \boldsymbol{F}\left(t, \boldsymbol{x}(t), \boldsymbol{x}(t-\tau_1), \cdots, \boldsymbol{x}(t-\tau_p), \int_{t-\tau_1}^{t} \boldsymbol{x}(s)\,\mathrm{d}s, \cdots, \int_{t-\tau_p}^{t} \boldsymbol{x}(s)\,\mathrm{d}s\right),\end{aligned} \tag{1.9}$$

where the unknown $\boldsymbol{x}$ is an $\mathbb{R}^d$-valued vector function with d representing the dimension of the DDE system; p is a positive integer, representing the total number of delays; the τ_i's are distinctive positive scalars arranged in ascending order; $\boldsymbol{A}$, $\boldsymbol{B}_i$, and $\boldsymbol{C}_i$ $(1 \le i \le p)$ are given $d \times d$ matrices; and $\boldsymbol{F} \colon \mathbb{R}^{2+2p} \to \mathbb{R}^d$ is a given continuous vector function.

When the nonlinearity in (1.9) does not contain the discrete delay terms, $\boldsymbol{x}(t-\tau_1), \cdots, \boldsymbol{x}(t-\tau_p)$, the convergence of many different Galerkin schemes of such systems has been investigated in the literature; see e.g. [CZ95, BB78, BK79, BRI84, CGLW16, KS78, Kap86, KS87, Kun82, FGG10, IT86] and references therein. In this case, a typical function space adopted for the approximation is the Hilbert space $\mathcal{H} := L^2([-\tau, 0); \mathbb{R}^d) \times \mathbb{R}^d$ where $\tau := \tau_p$.[1] The convergence is based on a reformulation of the system (1.9) into an abstract ODE defined on $\mathcal{H}$, which contains an autonomous linear part together with a nonlinear perturbation. The

[1] Possibly with a weighted norm for the $L^2([-\tau, 0); \mathbb{R}^d)$ part.

linear part is then shown to generate a C_0-semigroup. One then uses the linear semigroup in a variation of parameters implicit representation of the solution to the nonlinear system. Mathematical tools used then are the Trotter-Kato convergence theorem (for the linear semigroups) together with a Gronwall inequality argument; see e.g. [CGLW16].

A key contribution of [CGLW16] is the usage of a new type of orthogonal basis based on the Koornwinder polynomials, which allows for a relatively straightforward verification of all the assumptions required by the Trotter-Kato theorem and for the associated numerical treatment to be handled in a direct way.

For the type of DDEs considered here with Eq. (1.9), the discrete delay terms involved in the nonlinearity no longer allow for a reformulation of the nonlinearity as a mapping from $\mathcal{H}$ to $\mathcal{H}$ simply because it is not appropriate to evaluate a typical function in $L^2([-\tau, 0); \mathbb{R}^d)$ at a given point. Rigorous convergence results are much less investigated in the literature for this latter case. To the best of our knowledge, existing methods are based on either the theory of nonlinear dissipative semigroup [Kap82] or a nonlinear version of the Trotter-Kato theorem [Ban82]. The nonlinear dissipative semigroup approach avoids the use of Trotter-Kato theorem, but requires restrictive assumptions on the form of nonlinearities; on the other hand, the nonlinear Trotter-Kato approach makes use of heavy functional analysis tools. Can we still deal with this general case within the same framework as done in [CGLW16] by relying on the linear Trotter-Kato theorem and a basic Gronwall argument?

In this thesis, we delineate conditions on the linear semigroups associated with the Galerkin approximations that allow for a positive answer to this challenging question. In order to isolate the difficulites, we consider separately two cases. Firstly, in Chapter 4 we consider DDEs that have multiple discrete delays in the linearity but only distributed delays in the nonlinearity. By embedding the given problem into a higher-dimensional problem for which each constitutional equation involves only a single discrete delay, we obtain the same uniform convergence results as reported in [CGLW16] without requiring additional assumptions. Secondly, in Chapter 5 we consider DDEs with a single discrete delay in the linear and nonlinear parts. In this case, we make some assumptions about the convergence of our Galerkin scheme for the associated linearized equations to show a pointwise convergence result. Galerkin approximation of the general form given by Eq. (1.9) can then be addressed by combining the results given in Chapters 4 and 5.

Chapter 2

C_0-Semigroups

2.1 Introduction

Semigroups of linear operators – and in particular C_0-semigroups – are important tools in the study of partial differential equations and delay differential equations. Here we give a brief overview of the properties and theorems of C_0-semigroups, which will be used when proving the results. For a thorough look semigroups of linear operators, see [Paz83]; see also [RR04, Chap. 12] for applications to PDEs, and [CZ95, Chap. 2] for applications to DDEs.

2.2 Motivation and Definition

Suppose that X is a Banach space and let $\mathcal{L} : D(\mathcal{L}) \to X$ be a linear operator. If we are given $u_0 \in X$, then the abstract Cauchy problem for $\mathcal{L}$ with initial condition u_0 consists of finding a function $u(t)$ which solves

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = \mathcal{L}\,u(t), \quad u(0) = u_0. \tag{2.1}$$

If X is finite-dimensional, then $\mathcal{L}$ can be represented by a matrix and the solution to Eq. (2.1) is given explicitly by $u(t) = e^{\mathcal{L}\,t}u_0$. However, when $\mathcal{L}$ is not a bounded operator (e.g. $\mathcal{L}$ is a differential operator on $X = L^2([0,1])$), $e^{\mathcal{L}\,t}$ is no longer defined in the usual sense. To generalize the idea, we instead rely on the idea of a strongly continuous semigroup or a C_0-semigroup.

Definition 2.1. Let X be a Banach space. A family $\{T(t)\}$, $t \geq 0$, of bounded linear operators from X to X is a *strongly continuous semigroup* or a C_0*-semigroup* if it satisfies

i) $T(t+s) = T(t)T(s)$ for $t, s \geq 0$,

ii) $T(0) = I$, and

iii) $\lim_{t\to 0^+} T(t)x = x$, for every $x \in X$.

One can check that $e^{\mathcal{L}\,t}$ is a C_0-semigroup when X is finite dimensional (or $\mathcal{L}$ is bounded). One can also prove that Item iii) can be replaced with the equivalent property that $t \mapsto T(t)x$ is continuous for all $x \in X$, c.f. [Paz83, Cor. 2.3, p. 4].

We also have a useful bound on the norm of C_0-semigroups.

Theorem 2.2. *Let $T(t)$ be a C_0-semigroup on X. There exists constants $\omega \geq 0$ and $M \geq 1$ such that*

$$\|T(t)\|_X \leq Me^{\omega t}, \quad \forall t \geq 0. \tag{2.2}$$

A proof of this fact can be found in [Paz83, Thm. 2.2, p. 4].

2.3 Infinitesimal Generators and Approximations

Now to give meaning to $T(t)$ being the "exponential" of a linear operator $\mathcal{L}$, we recall the following definition.

Definition 2.3. A linear operator $\mathcal{A}$ defined by

$$D(\mathcal{A}) = \left\{ x \in X : \lim_{t \to 0^+} \frac{T(t)x - x}{t} \text{ exists} \right\} \tag{2.3}$$

and

$$\mathcal{A}x = \lim_{t \to 0^+} \frac{T(t)x - x}{t}, \quad \forall x \in D(\mathcal{A}) \tag{2.4}$$

is the *infinitesimal generator* of the semigroup $T(t)$ with $D(\mathcal{A})$ the domain of $\mathcal{A}$.

When X is finite-dimensional, the infinitesimal generator of $e^{\mathcal{L}t}$ is $\mathcal{L}$. If we again consider Eq. (2.1) and suppose that $\mathcal{L}$ generates a C_0-semigroup $T(t)$ and $u_0 \in D(\mathcal{L})$, then the abstract Cauchy problem may be solved by $u(t) = T(t)u_0$ (see [Paz83, Thm. 2.4, p. 4]). Hence, if one has that $\mathcal{L}$ is an infinitesimal generator for a C_0-semigroup, then one can begin to characterize the solution of Eq. (2.1). However, computation of $T(t)u_0$ is not always straightforward when $\mathcal{L}$ is unbounded, and there may be no simple analytic expression for the solution. One idea of circumventing this difficulty is by approximating $\mathcal{L}$ by "simpler" linear operator which generate known C_0-semigroups. Under certain assumptions, this approximation will hold for the corresponding C_0-semigroups as well.

Theorem 2.4 (Trotter-Kato)**.** *Suppose that $\mathcal{A}$ is the infinitesimal generator for the C_0-semigroup $T(t)$ and $\mathcal{A}_n$ (for $n \in \mathbb{N}$) is the infinitesimal generator for the C_0-semigroup $T_n(t)$. If there exist $M \geq 1$, $\omega \geq 0$ independent of n and t for which*

$$\|T_n(t)\|, \|T(t)\| \leq Me^{\omega t} \tag{2.5}$$

and

$$\lim_{n\to\infty} \mathcal{A}_n x = \mathcal{A}x, \quad \forall x \in D(\mathcal{A}), \tag{2.6}$$

then

$$\lim_{n\to\infty} T_n(t)x = T(t)x, \quad \forall t \geq 0, x \in X \tag{2.7}$$

and the convergence in Eq. (2.7) is uniform in t for t in bounded intervals.

Note that this is a specific case of [Paz83, Thm. 4.5, p. 88].

Chapter 3

Koornwinder Polynomials

3.1 Properties and Basic Results of Koornwinder Polynomials

In the study of DDEs, it is common to reformulate the problem into an abstract ODE with a functional phase space. To this end, we introduce the following Hilbert space used in [CGLW16]:

$$\mathcal{H} := L^2([-\tau, 0); \mathbb{R}^d) \times \mathbb{R}^d, \tag{3.1}$$

where the inner product is defined for (f_1, γ_1), $(f_2, \gamma_2) \in \mathcal{H}$, as:

$$\langle (f_1, \gamma_1), (f_2, \gamma_2) \rangle_{\mathcal{H}} := \frac{1}{\tau} \int_{-\tau}^{0} \langle f_1(\theta), f_2(\theta) \rangle \, \mathrm{d}\theta + \langle \gamma_1, \gamma_2 \rangle, \tag{3.2}$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product on $\mathbb{R}^d$. Now Eq. (3.2) is naturally associated with the measure

$$\nu(\mathrm{d}\theta) = \frac{1}{\tau} \mathrm{d}\theta + \delta_0, \tag{3.3}$$

with δ_0 the Dirac delta concentrated at $\theta = 0$. We wish to study orthogonal polynomials with respect to this measure.

Koornwinder in [Koo84] dealt with the case of orthogonal polynomials on $[-1, 1]$ associated with measures given by

$$\nu(\,\mathrm{d}x) = \frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}(1-x)^{\alpha}(1+x)^{\beta}\,\mathrm{d}x + M\delta_{-1} + N\delta_1,\ \alpha, \beta > -1, \tag{3.4}$$

i.e. associated with measures having a Jacobi weight on $[-1, 1]$ with two point-masses added to the extremities of the interval. We take $\alpha = \beta = M = 0$ and $N = 1$ to get polynomials on $[-1, 1]$ orthogonal with respect to the measure $\frac{1}{2}\,\mathrm{d}\theta + \delta_1$. We call these functions the *Koornwinder polynimals* and denote the nth degree polynomial by K_n.

From [Koo84, Eq. (2.1)], the sequence of Koornwinder polynomials $\{K_n\}$ can be built from the Legendre polynomials L_n by

$$K_n(s) := -(1+s)\frac{d}{\mathrm{d}s}L_n(s) + (n^2+n+1)L_n(s), \quad s \in [-1, 1],\ n \in \mathbb{N}_0. \tag{3.5}$$

Furthermore, we reproduce from [CGLW16, Prop. 3.1] some simple properties that $\{K_n\}$ satisfy.

Proposition 3.1. *The polynomial K_n defined in (3.5) admits the following expansion in terms of the Legendre polynomials:*

$$K_n(s) = -\sum_{j=0}^{n-1}(2j+1)L_j(s) + (n^2+1)L_n(s), \qquad n \in \mathbb{N}_0; \tag{3.6}$$

and the following normalization property holds:

$$K_n(1) = 1, \qquad n \in \mathbb{N}_0. \tag{3.7}$$

Moreover, the sequence given by

$$\{\mathcal{K}_n := (K_n, K_n(1)) : n \in \mathbb{N}_0\} \tag{3.8}$$

forms an orthogonal basis of the product space

$$\mathcal{E} := L^2([-1,1);\mathbb{R}) \times \mathbb{R}, \tag{3.9}$$

where $\mathcal{E}$ is endowed with the following inner product:

$$\langle (f,a),(g,b)\rangle_{\mathcal{E}} = \frac{1}{2}\int_{-1}^{1} f(s)g(s)\,\mathrm{d}s + ab, \quad (f,a),(g,b) \in \mathcal{E}. \tag{3.10}$$

Finally, $\left\{\frac{\mathcal{K}_n}{\|\mathcal{K}_n\|_{\mathcal{E}}}\right\}$ forms a Hilbert basis of $\mathcal{E}$ where the norm $\|\mathcal{K}_n\|_{\mathcal{E}}$ of $\mathcal{K}_n$ induced by $\langle\cdot,\cdot\rangle_{\mathcal{E}}$ possesses the following analytic expression:

$$\|\mathcal{K}_n\|_{\mathcal{E}} = \sqrt{\frac{(n^2+1)((n+1)^2+1)}{2n+1}}, \qquad n \in \mathbb{N}_0. \tag{3.11}$$

Applying a linear transformation to the orthogonal polynomials on $[-1,1]$ will give us a set of orthogonal polynomials on $[-\tau,0]$, from which we can construct an orthogonal basis on $\mathcal{H}$. We define a linear transformation $\mathcal{T}$ by

$$\mathcal{T}\colon [-\tau,0] \to [-1,1], \qquad \theta \mapsto 1 + \frac{2\theta}{\tau}. \tag{3.12}$$

We can now define the polynomial K_n^τ by

$$\begin{aligned} K_n^\tau\colon [-\tau,0] &\to \mathbb{R}, \\ \theta &\mapsto K_n\Big(1+\frac{2\theta}{\tau}\Big), \qquad n \in \mathbb{N}_0. \end{aligned} \tag{3.13}$$

Since the sequence $\{\mathcal{K}_n = (K_n, K_n(1)) : n \in \mathbb{N}_0\}$ forms an orthogonal basis for $\mathcal{E}$ (cf. Proposition 3.1), it follows then that the polynomial sequence $\{\mathcal{K}^\tau_n := (K^\tau_n, K^\tau_n(0)) : n \in \mathbb{N}_0\}$ forms an orthogonal basis for the space $\mathcal{H} = L^2([-\tau, 0); \mathbb{R}) \times \mathbb{R}$ endowed with the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ defined in Eq. (3.2) when $d = 1$.

3.2 Vectorized Koornwinder Polynomials

In order to deal with the case of multiple delays considered in Chapter 4, we need to vectorize the polynomials from the previous subsection so that they form an orthogonal basis for

$$\mathcal{H}^{\boldsymbol{\tau}} := L^2([-\tau_1, 0); \mathbb{R}) \times \cdots \times L^2([-\tau_p, 0); \mathbb{R}) \times \mathbb{R}^p, \tag{3.14}$$

where $\boldsymbol{\tau} = (\tau_1, \tau_2, \cdots, \tau_p)$ with $0 < \tau_1 \leq \tau_2 \leq \cdots \leq \tau_p$. The inner product of this space is given by

$$\langle \Psi, \Phi \rangle_{\mathcal{H}^{\boldsymbol{\tau}}} = \sum_{i=1}^{p} \frac{1}{\tau_i} \int_{-\tau_i}^{0} \Psi_i^D(\theta) \Phi_i^D(\theta) \, \mathrm{d}\theta + \langle \Psi^S, \Phi^S \rangle, \qquad \forall \Psi, \Phi \in \mathcal{H}^{\boldsymbol{\tau}}. \tag{3.15}$$

The construction will be similar to that in [CGLW16, Section 3.3]. For $j \in \mathbb{N}$, let

$$d_j = \left\lfloor \frac{j-1}{p} \right\rfloor \tag{3.16}$$

and let

$$r_j = \begin{cases} \mathrm{mod}(j, p), & \text{if} \quad \mathrm{mod}\ (j, p) \neq 0 \\ p, & \text{otherwise} \end{cases}.$$

Define

$$\mathbf{K}_j^{\boldsymbol{\tau}} := (\underbrace{0, \ldots, 0,}_{r_j \text{ - 1 entries}} K_{d_j}^{\tau_{r_j}}, \underbrace{0, \ldots, 0}_{p - r_j \text{ entries}}). \tag{3.17}$$

We shall also define

$$\mathbf{K}_j^{\boldsymbol{\tau}}(0) := (\underbrace{0, \ldots, 0,}_{r_j \text{ - 1 entries}} K_{d_j}^{\tau_{r_j}}(0), \underbrace{0, \ldots, 0}_{p - r_j \text{ entries}}). \tag{3.18}$$

We introduce

$$\mathbb{K}_j^{\boldsymbol{\tau}} := (\mathbf{K}_j^{\boldsymbol{\tau}}, \mathbf{K}_j^{\boldsymbol{\tau}}(0)), \qquad j \in \mathbb{N}. \tag{3.19}$$

One can check that $\{\mathbb{K}_j^{\boldsymbol{\tau}} : j \in \mathbb{N}\}$ forms an orthogonal basis for $\mathcal{H}^{\boldsymbol{\tau}}$. We define

$$\mathcal{X} := L^2([-\tau_1, 0); \mathbb{R}) \times \cdots \times L^2([-\tau_p, 0); \mathbb{R}) \tag{3.20}$$

and

$$\langle f, g \rangle_{\mathcal{X}} = \sum_{i=1}^{p} \frac{1}{\tau_i} \int_{-\tau_i}^{0} f(\theta) g(\theta) \, \mathrm{d}\theta, \tag{3.21}$$

i.e., $\mathcal{X}$ is the delay part of $\mathcal{H}^{\boldsymbol{\tau}}$.

Proposition 3.2. *The following convergence results hold for any $f \in \mathcal{X}$ and $\alpha \in \mathbb{R}^p$:*

$$\begin{aligned}
&\sum_{j=1}^{\infty} \frac{\langle \alpha, \mathbf{K}_j^{\boldsymbol{\tau}}(0) \rangle}{\|\mathbb{K}_j^{\boldsymbol{\tau}}\|_{\mathcal{H}}} \mathbf{K}_j^{\boldsymbol{\tau}} = 0 \quad \textit{with respect to } \|\cdot\|_{\mathcal{X}}; \\
&\sum_{j=1}^{\infty} \frac{\langle \alpha, \mathbf{K}_j^{\boldsymbol{\tau}}(0) \rangle}{\|\mathbb{K}_j^{\boldsymbol{\tau}}\|_{\mathcal{H}}} \mathbf{K}_j^{\boldsymbol{\tau}}(0) = \alpha; \\
&\sum_{j=1}^{\infty} \frac{\langle f, \mathbf{K}_j^{\boldsymbol{\tau}} \rangle_{\mathcal{X}}}{\|\mathbb{K}_j^{\boldsymbol{\tau}}\|_{\mathcal{H}}} \mathbf{K}_j^{\boldsymbol{\tau}} = f \quad \textit{with respect to } \|\cdot\|_{\mathcal{X}}; \textit{ and} \\
&\sum_{j=1}^{\infty} \frac{\langle f, \mathbf{K}_j^{\boldsymbol{\tau}} \rangle_{\mathcal{X}}}{\|\mathbb{K}_j^{\boldsymbol{\tau}}\|_{\mathcal{H}}} \mathbf{K}_j^{\boldsymbol{\tau}}(0) = 0.
\end{aligned} \tag{3.22}$$

Proof. Note that $\{\mathbb{K}_j^{\boldsymbol{\tau}} : j \in \mathbb{N}\}$ forms an orthogonal basis for $\mathcal{H}$. Then for $\Psi = (\Psi^D, \Psi^S) \in \mathcal{H}$, we have the following decomposition

$$\begin{aligned}\Psi &= \sum_{j=1}^{\infty} \frac{\langle \Psi, \mathbb{K}_j^{\boldsymbol{\tau}} \rangle_{\mathcal{H}^{\boldsymbol{\tau}}}}{\|\mathbb{K}_j^{\boldsymbol{\tau}}\|_{\mathcal{H}}^2} \mathbb{K}_j^{\boldsymbol{\tau}} \\ &= \sum_{j=1}^{\infty} \Big(\langle \Psi^D, \mathbf{K}_j^{\boldsymbol{\tau}} \rangle_{\mathcal{X}} + \langle \Psi^S, \mathbf{K}_j^{\boldsymbol{\tau}}(0) \rangle \Big) \frac{\mathbb{K}_j^{\boldsymbol{\tau}}}{\|\mathbb{K}_j^{\boldsymbol{\tau}}\|_{\mathcal{H}}^2}.\end{aligned} \tag{3.23}$$

If we set $\Psi = (\mathbf{0}, \alpha) \in \mathcal{H}_p$ and equalize the functional and numerical parts of each side of Eq. (3.23), we get the first two convergence results. Similarly, setting $\Psi = (f, \mathbf{0})$ and equalizing the functional and numerical parts of Eq. (3.23), we get the last two convergence results. □

For the case when $\boldsymbol{\tau} = (\tau, \tau, \cdots, \tau)$ for $\tau > 0$, we may write $\mathbb{K}_n^{\tau}$ instead of $\mathbb{K}_n^{\boldsymbol{\tau}}$. In this case, our construction corresponds exactly to the $\mathbb{K}_n^{\tau}$ described in [CGLW16, Section 3.3]

Chapter 4

Multiple Discrete Delays in the Linearity

4.1 One Dimensional Case with Multiple Linear Delay Terms

We now consider the following one-dimensional DDE given by

$$\begin{aligned} \frac{\mathrm{d}x(t)}{\mathrm{d}t} &= ax(t) + \sum_{i=1}^{p} b_i x(t-\tau_i) + F\left(x(t), \int_{t-\tau_1}^{t} x(s)\,\mathrm{d}s, \cdots, \int_{t-\tau_p}^{t} x(s)\,\mathrm{d}s\right), \quad t > 0 \\ x(0) &= \alpha, \\ x(t) &= \varphi(t), \quad t \in [-\tau_p, 0) \end{aligned} \tag{4.1}$$

where $x(t)$ is a function from $[-\tau_p, \infty)$ to $\mathbb{R}$ and $F : \mathbb{R}^p \to \mathbb{R}$ is Lipschitz continuous. We embed this into the following multidimensional problem:

$$\begin{aligned}
\frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t} &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\begin{bmatrix} x_1(t-\tau_1) \\ x_2(t-\tau_2) \\ \vdots \\ x_p(t-\tau_p) \end{bmatrix} + \boldsymbol{F}(\boldsymbol{x}_t), && t > 0 \\
\boldsymbol{x}(0) &= [\alpha, \alpha, \cdots, \alpha]^T, \\
x_1(\theta) &= \varphi(\theta), && -\tau_1 \le \theta < 0 \\
x_2(\theta) &= \varphi(\theta), && -\tau_2 \le \theta < 0 \\
&\vdots \\
x_p(\theta) &= \varphi(\theta), && -\tau_p \le \theta < 0
\end{aligned} \tag{4.2}$$

where $\boldsymbol{A} = a\boldsymbol{I}$ with $\boldsymbol{I}$ the identity matrix, $\boldsymbol{B} = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}^T \begin{bmatrix} b_1 & \cdots & b_p \end{bmatrix}$,

$$\boldsymbol{F}(\boldsymbol{x}_t) = \begin{bmatrix} F\left(x_1(t), \int_{t-\tau_1}^t x_1(s)\,\mathrm{d}s, \cdots, \int_{t-\tau_p}^t x_p(s)\,\mathrm{d}s\right) \\ \vdots \\ F\left(x_p(t), \int_{t-\tau_1}^t x_1(s)\,\mathrm{d}s, \cdots, \int_{t-\tau_p}^t x_p(s)\,\mathrm{d}s\right) \end{bmatrix}. \tag{4.3}$$

and x_i the ith component of $\boldsymbol{x}$. Note that x_p will satisfy the one-dimensional problem. Denote $\boldsymbol{\tau} = (\tau_1, \tau_2, \cdots, \tau_p)$. We may reference

$$\begin{bmatrix} x_1(t-\tau_1) & x_2(t-\tau_2) & \cdots & x_p(t-\tau_p) \end{bmatrix}^T \tag{4.4}$$

by an abuse of notation, $\boldsymbol{x}(t-\boldsymbol{\tau})$. We will prove results for the more general DDE given by

$$\begin{aligned}
\frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t} &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{x}(t-\boldsymbol{\tau}) + \boldsymbol{G}(\boldsymbol{x}_t, \boldsymbol{x}_t(0)), && t > 0 \\
\boldsymbol{x}(0) &= \boldsymbol{\gamma}, \\
x_1(\theta) &= f_1(\theta), && -\tau_1 \le \theta < 0 \\
x_2(\theta) &= f_2(\theta), && -\tau_2 \le \theta < 0 \\
&\vdots \\
x_p(\theta) &= f_p(\theta), && -\tau_p \le \theta < 0
\end{aligned} \tag{4.5}$$

where $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{p\times p}$, $\boldsymbol{G} : \mathcal{H}^{\boldsymbol{\tau}} \to \mathbb{R}^p$ Lipschitz, $\boldsymbol{\gamma} \in \mathbb{R}^p$ and $f_i \in L^2([-\tau_i, 0); \mathbb{R})$. Namely, we shall show that we can approximate the solution of the above by Galerkin problems using vectorized Koornwinder polynomials as defined in Section 3.2. We can reformulate the above DDE to be in the form of an abstract Cauchy problem on $\mathcal{H}^{\boldsymbol{\tau}}$, which was defined in Section 3.2. We define the linear operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \to \mathcal{H}^{\boldsymbol{\tau}}$ by

$$\mathcal{A}\begin{pmatrix} \Psi^D \\ \Psi^S \end{pmatrix} := \begin{pmatrix} \frac{d^+\Psi^D}{d\theta} \\ \boldsymbol{A}\Psi^S + \boldsymbol{B}\Psi^D(-\boldsymbol{\tau}) \end{pmatrix} \tag{4.6}$$

with the domain of $\mathcal{A}$ given by

$$\mathcal{D}(\mathcal{A}) = \{(\Psi^D, \Psi^S) \in \mathcal{H}^{\boldsymbol{\tau}} \ : \ \Psi_i^D \in H^1([-\tau_i, 0); \mathbb{R}), \ \lim_{\theta\to 0^-} \Psi_i^D(\theta) = \Psi_i^S, \ \text{ for } i = 1, \cdots, p\}. \tag{4.7}$$

The abstract ODE is then given by

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \mathcal{A}u(t) + \mathcal{G}(u(t)), \tag{4.8}$$

with

$$[\mathcal{G}(\Psi)](\theta) := \begin{cases} 0, & \theta \in [-\tau, 0), \\ \boldsymbol{G}\left(\Psi^D, \Psi^S\right), & \theta = 0, \end{cases} \qquad \forall\, \Psi = (\Psi^D, \Psi^S) \in \mathcal{H}^{\boldsymbol{\tau}}. \tag{4.9}$$

Since $\boldsymbol{G}$ was Lipschitz, we automatically have $\mathcal{G}$ Lipschitz. We also define the subspace

$$\mathcal{H}_N^{\boldsymbol{\tau}} := \operatorname{span}\{\mathbb{K}_1^{\boldsymbol{\tau}}, \cdots, \mathbb{K}_{Np}^{\boldsymbol{\tau}}\}, \tag{4.10}$$

i.e., it is the subspace of vectorized Koornwinder polynomials with degree less than or equal to N. To get the Np-dimensional Galerkin approximation, we define the following operator

$$\mathcal{A}_N := \Pi_N \mathcal{A} \Pi_N, \tag{4.11}$$

where

$$\Pi_N : \mathcal{H}^{\boldsymbol{\tau}} \to \mathcal{H}_N^{\boldsymbol{\tau}} \tag{4.12}$$

is the orthogonal projector into $\mathcal{H}_N^{\boldsymbol{\tau}}$. Note that $\mathbb{K}_j^{\tau} \in \mathcal{D}(\mathcal{A})$ for each $j \in \mathbb{N}$, so $\mathcal{H}_N^{\boldsymbol{\tau}} \subset \mathcal{D}(\mathcal{A})$ and the operator in (4.11) is well-defined. We have the Galerkin problem given by

$$\frac{\mathrm{d}u_N}{\mathrm{d}t} = \mathcal{A}_N u_N(t) + \Pi_N \mathcal{G}(u_N(t)). \tag{4.13}$$

We can extend $e^{\mathcal{A}_N t}$ to a C_0-semigroup $T_N(t)$ on $\mathcal{H}$ as follows:

$$T_N(t)u = e^{\mathcal{A}_N t}\Pi_N u + (I - \Pi_N)u, \qquad u \in \mathcal{H}. \tag{4.14}$$

To apply the results given in [CGLW16, Thm. 4.1], we need to prove the necessary assumptions about $\mathcal{A}$ and $\mathcal{A}_N$. That is, $\mathcal{A}$ generates a C_0 semigroup $T(t)$ and

(A1) The following uniform bound is satisfied by $\{T_N(t)\}_{N\geq 0, t\geq 0}$

$$\|T_N(t)\| \leq Me^{\omega t}, \qquad N \geq 0, \qquad t \geq 0 \tag{4.15}$$

where $\|T_N(t)\| = \sup\{\|T_N(t)u\|_{\mathcal{H}^{\boldsymbol{\tau}}} : \|u\|_{\mathcal{H}^{\boldsymbol{\tau}}} = 1, u \in \mathcal{H}^{\boldsymbol{\tau}}\}$.

(A2) The following convergence holds:

$$\lim_{N\to\infty} \|\mathcal{A}_N u - \mathcal{A}u\|_{\mathcal{H}^{\boldsymbol{\tau}}} = 0, \qquad \forall u \in \mathcal{H}^{\boldsymbol{\tau}}. \tag{4.16}$$

We first show that $\mathcal{A}$ is an infinitesimal generator of a C_0 semigroup. The proof will be similar to [CZ95, Thm 2.4.6]. We have the following result by slightly altering the proof from [CZ95, Thm 2.4.1]:

Theorem 4.1. *Consider the DDE Eq. (4.5). For every $\boldsymbol{\gamma} \in \mathbb{R}^p$ and for any choices of $f_i \in L^2([-\tau_i, 0); \mathbb{R})$ for each $i = 1, 2, \cdots, p$, there exists a unique function $\boldsymbol{x}(\cdot)$ on $[0, \infty)$ that is absolutely continuous and satisfies Eq. (4.5) almost everywhere. This function is called the solution of Eq. (4.5), and it satisfies*

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{\gamma} + \int_0^t e^{\boldsymbol{A}(t-s)}\boldsymbol{B}\boldsymbol{x}(s-\boldsymbol{\tau})\ \mathrm{d}s \qquad \text{for } t \geq 0. \tag{4.17}$$

We shall show that the following holds:

Lemma 4.2. *If $x(t)$ is the solution to Eq. (4.5), then the following inequalities hold:*

$$|\boldsymbol{x}(t)|^2 \leq C_t \left[|\boldsymbol{\gamma}|^2 + \sum_{i=1}^{p} \frac{1}{\tau_i} \|f_i\|^2_{L^2([-\tau_i,0);\mathbb{R})} \right], \quad \text{and} \tag{4.18}$$

$$\int_0^t |\boldsymbol{x}(s)|^2\ \mathrm{d}s \leq D_t \left[|\boldsymbol{\gamma}|^2 + \sum_{i=1}^{p} \frac{1}{\tau_i} \|f_i\|^2_{L^2([-\tau_i,0);\mathbb{R})} \right], \tag{4.19}$$

where C_t and D_t are constants that depend only on t.

Proof. We know that for $e^{\boldsymbol{A}t}$ there exists $M_0 > 0$ and $\omega > 0$ such that $|e^{\boldsymbol{A}t}| \leq M_0 e^{\omega t}$. Let

$$M := \max\{M_0, |\boldsymbol{B}|\}.$$

Then from Eq. (4.17) we have

$$\begin{aligned}
|\boldsymbol{x}(t)| &\leq |\boldsymbol{\gamma}||e^{\boldsymbol{A}t}| + \left|\int_0^t e^{\boldsymbol{A}t}\boldsymbol{B}\boldsymbol{x}(s-\boldsymbol{\tau})\,\mathrm{d}s\right| \\
&\leq M_0|\boldsymbol{\gamma}|e^{\omega t} + \int_0^t M_0 e^{\omega(t-s)}|\boldsymbol{B}||\boldsymbol{x}(s-\boldsymbol{\tau})|\,\mathrm{d}s \\
&\leq M|\boldsymbol{\gamma}|e^{\omega t} + M^2\int_0^t e^{\omega(t-s)}|\boldsymbol{x}(s-\boldsymbol{\tau})|\,\mathrm{d}s \\
&\leq M|\boldsymbol{\gamma}|e^{\omega t} + M^2\sum_{i=1}^{p}\int_0^t e^{\omega(t-s)}|x_i(s-\tau_i)|\,\mathrm{d}s \\
&= M|\boldsymbol{\gamma}|e^{\omega t} + M^2 e^{\omega t}\sum_{i=1}^{p}\int_{-\tau_i}^{t-\tau_i} e^{-\omega(\theta+\tau_i)}|x_i(\theta)|\,\mathrm{d}s.
\end{aligned} \tag{4.20}$$

We also have that for $i = 1, 2, \cdots, p$

$$\begin{aligned}
\int_{-\tau_i}^{t-\tau_i} e^{-\omega(\theta+\tau_i)}|x_i(\theta)|\,d\theta &\leq \int_{-\tau_i}^{0} e^{-\omega(\theta+\tau_i)}|x_i(\theta)|\,d\theta + \int_0^t e^{-|a|(\theta+\tau_i)}|x_i(\theta)|\,\mathrm{d}\theta \\
&= \int_{-\tau_i}^{0} e^{-\omega(\theta+\tau_i)}|f_i(\theta)|\,d\theta + \int_0^t e^{-|a|(\theta+\tau_i)}|x_i(\theta)|\,\mathrm{d}\theta \\
&\leq \int_{-\tau_i}^{0} |f_i(\theta)|\,d\theta + \int_0^t e^{-\omega(\theta+\tau_i)}|x_i(\theta)|\,\mathrm{d}\theta \\
&\leq \sqrt{\tau_i}\|f_i\|_{L^2([-\tau_i,0);\mathbb{R})} + \int_0^t e^{-\omega(\theta+\tau_i)}|x_i(\theta)|\,\mathrm{d}\theta \\
&\leq \sqrt{\tau_i}\|f_i\|_{L^2([-\tau_i,0);\mathbb{R})} + \int_0^t e^{-\omega\theta}|x_i(\theta)|\,\mathrm{d}\theta \\
&\leq \sqrt{\tau_i}\|f_i\|_{L^2([-\tau_i,0);\mathbb{R})} + \int_0^t e^{-\omega\theta}|x_i(\theta)|\,\mathrm{d}\theta.
\end{aligned} \tag{4.21}$$

From (4.20) and (4.21) we have

$$\begin{aligned}|\boldsymbol{x}(t)| &\leq M|\boldsymbol{\gamma}|e^{\omega t} + M^2 e^{\omega t}\sum_{i=1}^{p}\left[\sqrt{\tau_i}\|f_i\|_{L^2([-\tau_i,0);\mathbb{R})} + \int_0^t e^{-\omega\theta}|\boldsymbol{x}(\theta)|\,\mathrm{d}\theta\right] \\ &\leq M|\boldsymbol{\gamma}|e^{\omega t} + M^2 e^{\omega t}\sum_{i=1}^{p}\left[\sqrt{\tau_i}\|f_i\|_{L^2([-\tau_i,0);\mathbb{R})}\right] + M^2 e^{\omega t} p\int_0^t e^{-\omega\theta}|\boldsymbol{x}(\theta)|\,\mathrm{d}\theta.\end{aligned} \tag{4.22}$$

Set $\alpha = M|\boldsymbol{\gamma}| + M^2\sum_{i=1}^{p}\left[\sqrt{\tau_i}\|f_i\|_{L^2([-\tau_i,0);\mathbb{R})}\right]$, $\beta = M^2 p$ and $g(t) = e^{-\omega t}|\boldsymbol{x}(t)|$, we get the following inequality

$$g(t) \leq \alpha + \beta\int_0^t g(\theta)\,\mathrm{d}\theta. \tag{4.23}$$

Applying the integral form of Grönwall's inequality yields

$$g(t) \leq \alpha e^{\beta t}. \tag{4.24}$$

Thus

$$\begin{aligned}|\boldsymbol{x}(t)| &\leq e^{(M^2p+\omega)t}\left[M|\boldsymbol{\gamma}| + M^2\sum_{i=1}^{p}\sqrt{\tau_i}\|f_i\|_{L^2([-\tau_i,0);\mathbb{R})}\right] \\ &\leq e^{(M^2p+\omega)t}\left[M|\boldsymbol{\gamma}| + M^2\tau_p\sum_{i=1}^{p}\frac{1}{\sqrt{\tau_i}}\|f_i\|_{L^2([-\tau_i,0);\mathbb{R})}\right] \\ &\leq e^{(M^2p+\omega)t}\max\{M, M^2\tau_p\}\left[|\boldsymbol{\gamma}| + \sum_{i=1}^{p}\frac{1}{\sqrt{\tau_i}}\|f_i\|_{L^2([-\tau_i,0);\mathbb{R})}\right]\end{aligned} \tag{4.25}$$

and squaring both sides yields

$$|\boldsymbol{x}(t)|^2 \leq e^{2(M^2p+\omega)t}(\max\{M, M^2\tau_p\})^2\left[|\boldsymbol{\gamma}|^2 + \sum_{i=1}^{p}\frac{1}{\tau_i}\|f_i\|^2_{L^2([-\tau_i,0);\mathbb{R})}\right]. \tag{4.26}$$

This gives Eq. (4.18), and integrating gives Eq. (4.19). □

The following theorems are proven similarly to [CZ95, Thm 2.4.4] and [CZ95, Thm 2.4.6].

Theorem 4.3. *Let the operator $T(t)$ be defined by*

$$T(t)\begin{pmatrix} f(\cdot) \\ \boldsymbol{\gamma} \end{pmatrix} := \begin{pmatrix} \boldsymbol{x}(t+\cdot) \\ \boldsymbol{x}(t) \end{pmatrix}, \tag{4.27}$$

where $\boldsymbol{x}(\cdot)$ is the solution to Eq. (4.5). Then $T(t)$ for $t \geq 0$ satisfies:

i. $T(t) \in \mathcal{L}(\mathcal{H}^{\boldsymbol{\tau}})$ for all $t \geq 0$;

ii. $T(t)$ is a C_0-semigroup on $\mathcal{H}^{\boldsymbol{\tau}}$.

Theorem 4.4. *Consider the C_0-semigroup defined by (4.27). Its infinitesimal generator is given by Eq. (4.6) with domain Eq. (4.7).*

The assumption given by **(A2)** is proven nearly identically to [CGLW16, Lem. 4.1]. The assumption **(A1)** requires more work but follows a similar argument to those given in [CGLW16, Lem. 4.2, Lem. 4.3]. In the following, we provide the needed details for the verification of **(A1)**.

Proposition 4.5. *Let $\mathcal{A}$ be defined such as in Eq. (4.6). Then*

$$\langle \mathcal{A}\Psi, \Psi \rangle_{\mathcal{H}^{\boldsymbol{\tau}}} \leq \omega \|\Psi\|^2_{\mathcal{H}^{\boldsymbol{\tau}}}, \quad \forall \Psi \in \mathcal{D}(\mathcal{A}), \tag{4.28}$$

with

$$\omega = \left(\frac{1}{2\tau_1} + |\boldsymbol{A}| + \frac{\tau_p}{2}|\boldsymbol{B}|^2 \right). \tag{4.29}$$

Proof. Let $\Psi \in \mathcal{D}(\mathcal{A})$. By the definition of $\mathcal{A}$, we have

$$\langle \mathcal{A}\Psi, \Psi \rangle_{\mathcal{H}^{\boldsymbol{\tau}}} = \underbrace{\sum_{i=1}^{p} \frac{1}{\tau_i} \int_{-\tau_i}^{0} \frac{\mathrm{d}^+\Psi_i^D}{\mathrm{d}\theta}(\theta) \Psi_i^D(\theta)\, \mathrm{d}\theta}_{\text{(i)}} + \underbrace{\langle \boldsymbol{A}\Psi^S, \Psi^S \rangle}_{\text{(ii)}} + \underbrace{\langle \boldsymbol{B}\Psi^D(-\boldsymbol{\tau}), \Psi^S \rangle}_{\text{(iii)}}. \tag{4.30}$$

(i) For $i = 1, 2, \dots, p$, we have

$$\begin{aligned}\frac{1}{\tau_i}\int_{-\tau_i}^{0}\frac{\mathrm{d}^+\Psi_i^D}{\mathrm{d}\theta}(\theta)\Psi_i^D(\theta)\,\mathrm{d}\theta &= \frac{1}{2\tau_i}\left((\Psi_i^S)^2-(\Psi_i^D(-\tau_i))^2\right)\\ &\le \frac{1}{2\tau_1}(\Psi_i^S)^2-\frac{1}{2\tau_p}(\Psi_i^D(-\tau_i))^2.\end{aligned} \tag{4.31}$$

So

$$\sum_{i=1}^{p}\frac{1}{\tau_i}\int_{-\tau_i}^{0}\frac{\mathrm{d}^+\Psi_i^D}{\mathrm{d}\theta}(\theta)\Psi_i^D(\theta)\,\mathrm{d}\theta \le \frac{1}{\tau_1}|\Psi^S|^2-\frac{1}{2\tau_p}|\Psi^D(-\tau)|^2. \tag{4.32}$$

(ii) We have

$$\langle \boldsymbol{A}\Psi^S, \Psi^S\rangle \le |\boldsymbol{A}||\Psi^S|^2. \tag{4.33}$$

(iii) We have

$$\begin{aligned}\langle \boldsymbol{B}\Psi^D(-\tau), \Psi^S\rangle &\le |\boldsymbol{B}||\Psi^D(-\tau)||\Psi^S|\\ &= \left(\frac{1}{\sqrt{\tau_p}}|\Psi^D(-\tau)|\right)\left(\sqrt{\tau_p}|\boldsymbol{B}||\Psi^S|\right)\\ &\le \frac{|\Psi^D(-\tau)|^2}{2\tau_p}+\frac{\tau_p|\boldsymbol{B}|^2|\Psi^S|^2}{2}.\end{aligned} \tag{4.34}$$

Thus from (1), (2), and (3), we have that

$$\begin{aligned}\langle \mathcal{A}\Psi, \Psi\rangle_{\mathcal{H}^\tau} &= \left(\frac{1}{2\tau_1}+|\boldsymbol{A}|+\frac{\tau_p}{2}|\boldsymbol{B}|^2\right)|\Psi^S|^2\\ &\le \left(\frac{1}{2\tau_1}+|\boldsymbol{A}|+\frac{\tau_p}{2}|\boldsymbol{B}|^2\right)\|\Psi\|_{\mathcal{H}^\tau}^2,\end{aligned} \tag{4.35}$$

as desired. □

We now prove the following statement:

Proposition 4.6. *Let $\mathcal{A}$ be defined as in Eq. (4.6). Then, the linear semigroups $T(t)$ and*

$T_N(t)$ generated respectively by $\mathcal{A}$ and $\mathcal{A}_N$ defined in (4.11), satisfy

$$\|T(t)\| \leq e^{\omega t} \quad \text{and} \quad \|T_N(t)\| \leq e^{\omega t}, \qquad t \geq 0, \tag{4.36}$$

with ω given by Eq. (4.29).

Proof. We have that $T(t)$ is a C_0-semigroup with infinitesimal generator $\mathcal{A}$. By [Paz83, Thm. 2.4 c) p.5], it follows that $T(t)u_0 \in \mathcal{D}(\mathcal{A})$ for all $u_0 \in \mathcal{D}(\mathcal{A})$ and that

$$\frac{\mathrm{d}}{\mathrm{d}t} T(t)u_0 = \mathcal{A}T(t)u_0, \qquad \forall\, u_0 \in \mathcal{A},\ t \geq 0. \tag{4.37}$$

Thus

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\|T(t)u_0\|^2_{\mathcal{H}^\tau} &= 2\langle \mathcal{A}T(t)u_0, T(t)u_0\rangle_{\mathcal{H}^\tau} \\ &\leq 2\omega\|T(t)u_0\|^2_{\mathcal{H}^\tau}, \end{aligned} \tag{4.38}$$

for any $u_0 \in \mathcal{D}(\mathcal{A})$. Applying Gronwall's inequality and taking the square root of both sides gives

$$\|T(t)u_0\|_{\mathcal{H}} \leq e^{\omega t}\|u_0\|_{\mathcal{H}}, \tag{4.39}$$

for $u_0 \in \mathcal{D}(\mathcal{A})$. For $x \in \mathcal{H}$, since $\mathcal{D}(\mathcal{A})$ is dense in $\mathcal{H}$ we can pick $\{u_n\}_{n=1}^{\infty} \subset \mathcal{D}(\mathcal{A})$ where $u_n \to x$ in $\mathcal{H}^\tau$. Thus

$$\begin{aligned} \|T(t)x\|_{\mathcal{H}^\tau} &= \|T(t)(x - u_n + u_n)\|_{\mathcal{H}^\tau} \\ &\leq \|T(t)\|_{\mathcal{H}^\tau} \cdot \|x - u_n\|_{\mathcal{H}^\tau} + e^{\omega t}\|u_n\|_{\mathcal{H}^\tau}, \end{aligned} \tag{4.40}$$

where the first term on the right goes to 0 and the second term on the right goes to $e^{\omega t}\|x\|_{\mathcal{H}}$ as $n \to \infty$. Thus the inequality holds for all $x \in \mathcal{H}$ and

$$\|T(t)\|_{\mathcal{H}^\tau} \leq e^{\omega t}, \qquad t \geq 0. \tag{4.41}$$

For the estimate on T_N, we first note that for $u_0 \in \mathcal{H}$

$$\begin{aligned} \|T_N(t)u_0\|^2_{\mathcal{H}^\tau} &= \left\langle e^{\mathcal{A}_N t}\Pi_N u_0 + (I-\Pi_N)u_0, e^{\mathcal{A}_N t}\Pi_N u_0 + (I-\Pi_N)u_0 \right\rangle_{\mathcal{H}^\tau} \\ &= \left\langle e^{\mathcal{A}_N t}\Pi_N u_0, e^{\mathcal{A}_N t}\Pi_N u_0 \right\rangle_{\mathcal{H}^\tau} + \left\langle (I-\Pi_N)u_0, (I-\Pi_N)u_0 \right\rangle_{\mathcal{H}^\tau} \\ &= \|e^{\mathcal{A}_N t}\Pi_N u_0\|^2_{\mathcal{H}^\tau} + \|(I-\Pi_N)u_0\|^2_{\mathcal{H}^\tau}. \end{aligned} \tag{4.42}$$

Also note for $\varphi, \psi \in \mathcal{H}^\tau$ that

$$\begin{aligned} \langle \Pi_N\varphi, \psi\rangle_{\mathcal{H}^\tau} &= \langle \varphi, \Pi_N\psi\rangle_{\mathcal{H}^\tau} - \langle (I-\Pi_N)\varphi, \Pi_N\psi\rangle_{\mathcal{H}^\tau} + \langle \Pi_N\varphi, (I-\Pi_N)\psi\rangle_{\mathcal{H}^\tau} \\ &= \langle \varphi, \Pi_N\psi\rangle_{\mathcal{H}^\tau}, \end{aligned} \tag{4.43}$$

where $(I-\Pi_N)\varphi$ and $(I-\Pi_N)\psi$ are orthogonal to the space $\mathcal{H}_N$ and the terms

$$\langle (I-\Pi_N)\varphi, \Pi_N\psi\rangle_{\mathcal{H}^\tau} \text{ and } \langle \Pi_N\varphi, (I-\Pi_N)\psi\rangle_{\mathcal{H}^\tau}$$

evaluate to 0. We can thus justify moving Π_N between terms in $\langle\cdot,\cdot\rangle_{\mathcal{H}^\tau}$. Therefore

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\|T_N(t)u_0\|^2_{\mathcal{H}^\tau} &= \frac{\mathrm{d}}{\mathrm{d}t}\|e^{\mathcal{A}_N t}\Pi_N u_0\|^2_{\mathcal{H}^\tau} \\ &= 2\langle \mathcal{A}_N e^{\mathcal{A}_N t}\Pi_N u_0, e^{\mathcal{A}_N t}\Pi_N u_0\rangle_{\mathcal{H}^\tau} \\ &= 2\langle \Pi_N\mathcal{A}\Pi_N e^{\mathcal{A}_N t}\Pi_N u_0, e^{\mathcal{A}_N t}\Pi_N u_0\rangle_{\mathcal{H}^\tau} \\ &= 2\langle \mathcal{A}\Pi_N e^{\mathcal{A}_N t}\Pi_N u_0, \Pi_N e^{\mathcal{A}_N t}\Pi_N u_0\rangle_{\mathcal{H}^\tau} \\ &\le 2\omega\|e^{\mathcal{A}_N t}\Pi_N u_0\|^2_{\mathcal{H}^\tau} \\ &\le 2\omega\left(\|e^{\mathcal{A}_N t}\Pi_N u_0\|^2_{\mathcal{H}^\tau} + \|(I-\Pi_N)u_0\|^2_{\mathcal{H}^\tau}\right) \\ &= 2\omega\|T_N(t)u_0\|^2_{\mathcal{H}^\tau} \end{aligned} \tag{4.44}$$

Again applying Gronwall's inequality to Eq. (4.44) and taking the square root of each side gives implies the desired inequality. □

Thus we have **(A1)**. We can now apply the results of [CGLW16, Thm. 4.1].

Theorem 4.7. *For any* $u_0 \in \mathcal{H}^{\boldsymbol{\tau}}$*, Eq.* (4.8) *has a unique mild solution* $t \to u(t; u_0)$ *emanating from the initial data* u_0*. Moreover,* u *can be approximated uniformly on any bounded interval* $[0, T]$ *by the sequency* $\{t \to u_N(t; \Pi_N u_0)\}$ *of mild solutions for Eq.* (4.13) *subject to the initial condition* $u_N(0) = \Pi_N u_0$*. Namely,*

$$\lim_{N\to\infty} \sup_{t\in[0,T]} \|u(t; u_0) - u_N(t; \Pi_N u_0)\|_{\mathcal{H}^{\boldsymbol{\tau}}} = 0, \qquad \forall\, T > 0. \tag{4.45}$$

Proof. The result is a direct application of [CGLW16, Thm. 4.1] once we can verify Assumptions **(A1)** and **(A2)** given above. Note that **(A1)** is verified in Proposition 4.6. The condition **(A2)** follows almost identically to the proof of [CGLW16, Lem. 4.1]. Note also that the requirement of $\mathcal{G}$ being Lipschitz in [CGLW16, Thm. 4.1] is satisfied here as well since $\boldsymbol{G}$ is assumed to be Lipschitz. □

4.2 Multidimensional Case with Multiple Linear Delay Terms

We now consider the d-dimensional DDE of the form:

$$\frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t} = \boldsymbol{A}\boldsymbol{x}(t) + \sum_{i=1}^{p} \boldsymbol{B}_i \boldsymbol{x}(t - \tau_i) + \boldsymbol{F}\left(\boldsymbol{x}(t), \int_{t-\tau_1}^{t} \boldsymbol{x}(s)\,\mathrm{d}s, \cdots, \int_{t-\tau_p}^{t} \boldsymbol{x}(s)\,\mathrm{d}s\right). \tag{4.46}$$

We can imbed this into a $(d \times p)$-dimensional problem given by

$$\begin{bmatrix} \dot{\boldsymbol{x}}_1(t) \\ \dot{\boldsymbol{x}}_2(t) \\ \vdots \\ \dot{\boldsymbol{x}}_p(t) \end{bmatrix} = \widetilde{\boldsymbol{A}} \begin{bmatrix} \boldsymbol{x}_1(t) \\ \boldsymbol{x}_2(t) \\ \vdots \\ \boldsymbol{x}_p(t) \end{bmatrix} + \widetilde{\boldsymbol{B}} \begin{bmatrix} \boldsymbol{x}_1(t-\tau_1) \\ \boldsymbol{x}_2(t-\tau_2) \\ \vdots \\ \boldsymbol{x}_p(t-\tau_p) \end{bmatrix} + \widetilde{\boldsymbol{F}}(\boldsymbol{x}_t), \tag{4.47}$$

where

$$\widetilde{\boldsymbol{A}} := \begin{bmatrix} \boldsymbol{A} & & & \\ & \boldsymbol{A} & & \\ & & \ddots & \\ & & & \boldsymbol{A} \end{bmatrix}, \qquad \widetilde{\boldsymbol{B}} := \begin{bmatrix} \boldsymbol{B}_1 & \boldsymbol{B}_2 & \cdots & \boldsymbol{B}_p \\ \boldsymbol{B}_1 & \boldsymbol{B}_2 & \cdots & \boldsymbol{B}_p \\ \vdots & \vdots & & \vdots \\ \boldsymbol{B}_1 & \boldsymbol{B}_2 & \cdots & \boldsymbol{B}_p \end{bmatrix}, \tag{4.48}$$

and

$$\widetilde{\boldsymbol{F}}(\boldsymbol{x}_t) := \begin{bmatrix} \boldsymbol{F}\left(\boldsymbol{x}_1(t), \int_{t-\tau_1}^{t} \boldsymbol{x}_1(s)\,\mathrm{d}s, \cdots, \int_{t-\tau_p}^{t} \boldsymbol{x}_p(s)\,\mathrm{d}s\right) \\ \boldsymbol{F}\left(\boldsymbol{x}_2(t), \int_{t-\tau_1}^{t} \boldsymbol{x}_1(s)\,\mathrm{d}s, \cdots, \int_{t-\tau_p}^{t} \boldsymbol{x}_p(s)\,\mathrm{d}s\right) \\ \vdots \\ \boldsymbol{F}\left(\boldsymbol{x}_p(t), \int_{t-\tau_1}^{t} \boldsymbol{x}_1(s)\,\mathrm{d}s, \cdots, \int_{t-\tau_p}^{t} \boldsymbol{x}_p(s)\,\mathrm{d}s\right) \end{bmatrix} \tag{4.49}$$

This problem is of the form given in Eq. (4.5). We set

$$\boldsymbol{\tau} = (\underbrace{\tau_1, \cdots, \tau_1}_{d \text{ times}}, \underbrace{\tau_2, \cdots, \tau_2}_{d \text{ times}}, \cdots, \underbrace{\tau_p, \cdots, \tau_p}_{d \text{ times}}) \tag{4.50}$$

and reformulate the system Eq. (4.47) into an abstract ODE of the form Eq. (4.8) defined on the space $\mathcal{H}^{\boldsymbol{\tau}}$. As we have seen in Section 4.1, we can approximate the solution of this abstract ODE on $\mathcal{H}^{\boldsymbol{\tau}}$ via a Galerkin scheme.

Chapter 5

Discrete Delays in the Nonlinearity

5.1 Reformulation of the DDE into an Abstract ODE

In this chapter, we consider DDEs with discrete delays in the nonlinearity. In order to simplify the presentation, we examine the simple setting of a scalar DDE with a single discrete delay $\tau > 0$:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = ax(t) + bx(t-\tau) + F(x(t-\tau)), \tag{5.1}$$

where $a, b \in \mathbb{R}$, and $F : \mathbb{R} \to \mathbb{R}$ is a given scalar function. To construct a Galerkin scheme for this DDE, we must first reformulate it into an abstract ODE.

In the reformulation of Eq. (5.1), the linear part is dealt with in the usual way; whereas the treatment of the nonlinear part requires to use a more regular subspace of $L^2([-\tau, 0); \mathbb{R})$ due to the presence of the discrete delay term $x(t-\tau)$ in F. For this reason, we present the reformulation in two steps.

5.1.1 Reformulation of the linear part

The treatment of the linear part is classical; and we follow [CGLW16, Sect 2] for the presentation. Recall our definition of $\mathcal{H}$ and its inner product from Eq. (3.1) and Eq. (3.2). Since our DDE Eq. (5.1) is one-dimensional, we take $d = 1$ to get

$$\mathcal{H} := L^2([-\tau, 0); \mathbb{R}) \times \mathbb{R}, \tag{5.2}$$

where the inner product is defined for (f_1, γ_1), $(f_2, \gamma_2) \in \mathcal{H}$, as:

$$\langle (f_1, \gamma_1), (f_2, \gamma_2) \rangle_{\mathcal{H}} := \frac{1}{\tau} \int_{-\tau}^{0} f_1(\theta) f_2(\theta) \, \mathrm{d}\theta + \gamma_1 \gamma_2. \tag{5.3}$$

Define the linear operator $\mathcal{A} : D(\mathcal{A}) \to \mathcal{H}$ by

$$[\mathcal{A}\Psi](\theta) := \begin{cases} \dfrac{\mathrm{d}^+ \Psi^D}{\mathrm{d}\theta}, & \theta \in [-\tau, 0), \\ a\Psi^S + b\Psi^D(-\tau), & \theta = 0, \end{cases} \tag{5.4}$$

for any $\Psi = (\Psi^D, \Psi^S)$ that lives in the domain, $D(\mathcal{A})$, defined as

$$D(\mathcal{A}) := \left\{ \Psi \in \mathcal{H} : \Psi^D \in H^1([-\tau, 0); \mathbb{R}^d), \lim_{\theta \to 0^-} \Psi^D(\theta) = \Psi^S \right\}. \tag{5.5}$$

Note that the reformulation above implies that if $x : [-\tau, \infty)$ satisfies the linear DDE

$$\begin{aligned} &\frac{\mathrm{d}x(t)}{\mathrm{d}t} = ax(t) + bx(t - \tau), && t > 0, \\ &x(0) = \alpha, \\ &x(t) = f(t), && t \in [-\tau, 0), \end{aligned} \tag{5.6}$$

then $u(t) = (x_t, x_t(0))$, where $x_t(\theta) = x(t+\theta)$ for $\theta \in [-\tau, 0)$, satisfies the linear, abstract ODE

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}t} &= \mathcal{A}u, \quad t > 0 \\ u(0) &= u_0, \end{aligned} \tag{5.7}$$

where $u_0 = (f, \alpha)$. From [CZ95, Thm. 2.4.1], the DDE in Eq. (5.6) has a solution $x(t)$. Furthermore, if we define $T(t) : \mathcal{H} \to \mathcal{H}$ by

$$T(t)(f, \alpha) := (x_t, x_t(0)), \quad t \geq 0, \tag{5.8}$$

then $T(t)$ is a C_0-semigroup on $\mathcal{H}$ and $\mathcal{A}$ is its infinitesimal generator [CZ95, Thm. 2.4.4; Thm. 2.4.6]. With this, we know that the solution to Eq. (5.7) is $T(t)u_0$.

5.1.2 Reformulation of the nonlinear part.

As mentioned before, the discrete delay term involved in F no longer allows for a reformulation of the nonlinearity as a mapping from $\mathcal{H}$ to $\mathcal{H}$. Instead, one needs to restrict the domain to a more regular subspace of $\mathcal{H}$ for which it makes sense to talk about pointwise value of its member functions at the discrete delay $\theta = -\tau$. For our purpose, it turns out that it is sufficient to require the historical part of each element to be a bounded right-continuous function. Namely, we introduce the following subspace of $\mathcal{H}$:

$$X := \mathcal{C}^+([-\tau, 0); \mathbb{R}) \times \mathbb{R} \subseteq \mathcal{H}, \tag{5.9}$$

where $\mathcal{C}^+([-\tau,0);\mathbb{R})$ denotes the set of bounded right-continuous functions on the interval $[-\tau,0)$. The space X is endowed with the following inner product:

$$\begin{aligned}(\Phi,\Psi)_X &:= (\Phi,\Psi)_{\mathcal{H}} + \Phi^D(-\tau)\Psi^D(-\tau)\\ &= \frac{1}{\tau}\int_{-\tau}^{0}\Phi^D(\theta)\Psi^D(\theta)\,\mathrm{d}\theta + \Phi^S\Psi^S + \Phi^D(-\tau)\Psi^D(-\tau), \qquad \forall\,\Phi,\Psi\in X.\end{aligned} \tag{5.10}$$

We will also use $\|\cdot\|_X$ to denote the norm on X induced by the above inner product.

We can now reformulate the nonlinear term $F(x(t-\tau))$ in Eq. (5.1) as a mapping from X to X by defining $\mathcal{F}\colon X\to X\subseteq\mathcal{H}$ as follows:

$$[\mathcal{F}(\Psi)](\theta) := \begin{cases} 0, & \theta\in[-\tau,0),\\ F\left(\Psi^D(-\tau)\right), & \theta=0,\end{cases} \qquad \forall\,\Psi=(\Psi^D,\Psi^S)\in X. \tag{5.11}$$

Throughout we make the following working assumption on solutions of the DDE Eq. (5.1).

> We assume that for any initial data $\phi=(\phi^D,\phi^S)\in\mathcal{H}$ with $\phi^D\in\mathcal{C}^+([-\tau,0),\mathbb{R})$ the DDE Eq. (5.1) admits a unique continuous solution $x:[0,\infty)\to\mathbb{R}$ that satisfies $x(\theta)=\phi^D(\theta)$ over $[-\tau,0]$. **(H1)**

Note that thanks to this assumption, x_t is in $\mathcal{C}^+([-\tau,0))$ and $u(t)\in X$ for any $t\in[0,\infty)$. Of course, since X when endowed with the norm induced from the inner product defined by Eq. (5.10) is not a Banach space, it may thus not be appropriate to formulate an existence theory of solutions for the DDE Eq. (5.1) on X. However, once the existence is addressed, it serves well the purpose of Galerkin approximation as will be illustrated. For later use, we record below the variation of constants formula.

5.1.3 Variation of constants formula

Suppose that $t \mapsto x(t)$ is the unique everywhere continuous solution of the DDE Eq. (5.1) on $[-\tau, \infty)$ subject to $x(t) = \phi(t)$ on $[-\tau, 0]$ for some given continuous function ϕ. By setting

$$u(t) = (x_t, x_t(0)), \quad t \geq 0, \tag{5.12}$$

we have that u satisfies, in a pointwise sense, the following abstract ODE:

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}t} &= \mathcal{A}u(t) + \mathcal{F}(u(t)), \\ u(0) &= u_0, \end{aligned} \tag{5.13}$$

where $u_0 := (\phi, \phi(0))$. In particular, u thus defined satisfies the variation of constants formula:

$$u(t) = T(t)u_0 + \int_0^t T(t-s)\mathcal{F}(u(s))\,\mathrm{d}s. \tag{5.14}$$

We reproduce below a sketch of the derivation of this formula, and refer the interested readers to [Paz83, pg. 105] for more details. Note that $\mathcal{F}(u(t))$ is well-defined and u satisfies

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = \mathcal{A}u(t) + \mathcal{F}(u(t)), \quad t > 0. \tag{5.15}$$

Then for fixed t and $s \in (0, t)$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s}\left[T(t-s)u(s)\right] &= -\mathcal{A}T(t-s)u(s) + T(t-s)\mathcal{A}u(s) + T(t-s)\mathcal{F}(u(s)) \\ &= T(t-s)\mathcal{F}(u(s)). \end{aligned} \tag{5.16}$$

The function $\mathcal{F} \circ u : [0, \infty) \to \mathcal{H}$ is continuous and locally integrable. Integrating the above with respect to s from 0 to t gives

$$u(t) - T(t)u_0 = \int_0^t T(t-s)\mathcal{F}(u(s))\,\mathrm{d}s, \tag{5.17}$$

and Eq. (5.14) follows.

5.2 Galerkin Approximations based on the Koornwinder Basis

We recall the construction of a Galerkin approximation from [CGLW16, Sec. 4.2] to create a Galerkin approximation of Eq. (5.15). Let us now define

$$\mathcal{H}_N := \operatorname{span}\{\mathcal{K}_0^\tau, \dots, \mathcal{K}_{N-1}^\tau\}. \tag{5.18}$$

Let Π_N be the associated orthogonal projector of $\mathcal{H}_N$. By the construction of the orthogonal basis $\{\mathcal{K}_n^\tau\}$, we have that $\mathcal{H}_N \subset D(\mathcal{A})$. The N-dimensional Galerkin approximation of Eq. (5.13) is

$$\begin{aligned} \frac{\mathrm{d}u_N}{\mathrm{d}t} &= \mathcal{A}_N u_N + \Pi_N \mathcal{F}(u_N), \\ u_N(0) &= \Pi_N u_0, \end{aligned} \tag{5.19}$$

where $\mathcal{A}_N := \Pi_N \mathcal{A} \Pi_N$. The linear operator $\mathcal{A}_N$ on the finite dimensional space $\mathcal{H}_N$ defines the C_0-semigroup $e^{\mathcal{A}_N t}$. This can be extended to a C_0-semigroup on $\mathcal{H}$:

$$T_N(t)u = e^{\mathcal{A}_N t}\Pi_N u + (I - \Pi_N)u, \; u \in \mathcal{H}. \tag{5.20}$$

From Eq. (5.19), we derive the variation of constants formula for the Galerkin approximation:

$$u_N(t) = T_N(t)\Pi_N u_0 + \int_0^t T_N(t-s)\Pi_N \mathcal{F}(u_N(t))\,\mathrm{d}s. \tag{5.21}$$

We recall now the following results about $T(t)$ and $T_N(t)$ from [CGLW16, Lemma 4.3 and Thm. 4.1].

Proposition 5.1. *There exists $M \geq 1$ and $\omega \geq 0$ such that for any $t > 0$ and $N \in \mathbb{N}$,*

$$\|T_N(t)\|_{\mathcal{H}}, \|T(t)\|_{\mathcal{H}} \leq Me^{\omega t}. \tag{5.22}$$

Also, for any $T > 0$,

$$\lim_{N\to\infty} \sup_{t\in[0,T]} \|T(t)\Phi - T_N(t)\Pi_N\Phi\|_{\mathcal{H}} = 0, \qquad \forall\, \Phi \in \mathcal{H}. \tag{5.23}$$

The proof of Eq. (5.23) relies on Proposition 2.4.

5.3 Pointwise convergence in X

The main purpose of this section is to prove a pointwise convergence result for the Galerkin approximations of Eq. (5.15) as stated in Proposition 5.2. This convergence result relies on the following technical assumption about the Galerkin approximations of the linearized equation initialized at a particular initial data $\widehat{\Psi}$, as stated below.

Let $\widehat{\Psi} : [-\tau, 0] \to \mathbb{R}$ be defined by

$$\widehat{\Psi}(\theta) := \begin{cases} 0, & \theta \in [-\tau, 0), \\ 1, & \theta = 0. \end{cases} \tag{5.24}$$

We assume that

$$\left.\begin{array}{l} \text{For each } t_f > 0, \text{ there exists a constant } C := C(t_f) \text{ such that} \\ \qquad \left|\left[T_N(t)\Pi_N\widehat{\Psi}\right]^D(-\tau)\right| \le C, \quad \forall\, t \in [0, t_f],\ N \in \mathbb{N}. \\ \text{Furthermore, we assume that} \\ \qquad \left[T_N(t)\Pi_N\widehat{\Psi}\right]^D(-\tau) \underset{N\to\infty}{\longrightarrow} \left[T(t)\widehat{\Psi}\right]^D(-\tau), \quad \text{for a.e. } t \in [0, \infty). \end{array}\right. \tag{H2}$$

We also need to make a similar assumption about $u \in D(\mathcal{A})$:

$$\left.\begin{array}{l} \text{For each } t_f > 0 \text{ and } u \in D(\mathcal{A}), \text{ there exists a constant } C := C(t_f) \text{ such that} \\ \qquad \left|\left[T_N(t)\Pi_N u\right]^D(-\tau)\right| \le C, \quad \forall\, t \in [0, t_f],\ N \in \mathbb{N}. \\ \text{Furthermore, we assume that} \\ \qquad \left[T_N(t)\Pi_N u\right]^D(-\tau) \underset{N\to\infty}{\longrightarrow} \left[T(t)u\right]^D(-\tau), \quad \text{for } t \in [0, \infty). \end{array}\right. \tag{H3}$$

We assume stronger convergence in (**H3**) than in (**H2**) since we have more regularity for $u \in D(\mathcal{A})$, i.e., one may view u as a continuous function on $[-\tau, 0]$ with $u(0) = u^S$ but $\widehat{\Psi}$ would have a discontinuity.

Theorem 5.2. *Consider the DDE Eq. (5.1). Assume F is globally Lipschitz, and Assumptions (**H1**), (**H2**), and (**H3**) hold. For any $u_0 \in X$, let $u(\cdot; u_0)$ be the solution of the reformulated abstract ODE Eq. (5.13) with initial data u_0, and $u_N(\cdot; \Pi_N u_0)$ the solution of the corresponding N-dimensional Galerkin system Eq. (5.19). Assume furthermore that*

$u_0 \in D(\mathcal{A})$. *Then, for any* $t > 0$ *it holds that*

$$\lim_{N\to\infty} \|u(t; u_0) - u_N(t; \Pi_N u_0)\|_X = 0. \tag{5.25}$$

The proof of the above theorem follows the same strategy as the proof of [CGLW16, Theorem 4.1], although we cannot obtain uniform convergence via this strategy here. We carry out the steps in a few preparatory lemmas.

We first note that the linear solution semigroup $\{T(t)\}$ associated with the reformulation of the linear DDE Eq. (5.6) is invariant on the space X as summarized in Proposition 5.3. The solution operator $T_N(t)\Pi_N$ associated with the Galerkin approximation of the linear DDE has the same property. This result justifies the use of the norm $\|\cdot\|_X$ on certain functions later on.

Lemma 5.3. *For any* $t \geq 0$*, the solution operator* $T(t)$ *of the abstract ODE Eq.* (5.7) *and the solution operator* $T_N(t)\Pi_N$ *of the Galerkin approximation of Eq.* (5.7) *both map* X *into itself.*

Proof. From [CZ95, Thm. 2.4.1], if $u_0 \in X$, then the solution $x(t)$ of Eq. (5.6) with initial conditions u_0^D and u_0^S is continuous on $[0, \infty)$. This is sufficient to say that $T(t)u_0 = (x_t, x_t(0)) \in X$ and that $T(t)$ maps X into X. Similarly, we have that Π_N maps $X \subseteq \mathcal{H}$ into $\mathcal{H}_N$ and $T_N(t)$ maps $\mathcal{H}_N$ into $\mathcal{H}_N$. Hence $T_N(t)\Pi_N$ maps X into $\mathcal{H}_N$, which is a subset of X. □

We can now prove the following result.

Lemma 5.4. *Given any* $u_0 \in X$*, let* u *be the solution of Eq.* (5.13) *with initial data* u_0*, and* u_N *the solution of the corresponding* N*-dimensional Galerkin system Eq.* (5.19)*. Then, for* *any $t > 0$, there exists a positive constant $C > 0$ depending on u_0 and t such that*

$$\left\| T_N(t-s)\Pi_N\Big(\mathcal{F}(u(s)) - \mathcal{F}(u_N(s))\Big)\right\|_X \le C\|u(s) - u_N(s)\|_X, \quad \forall\, s \in [0,t],\ N \in \mathbb{N}. \tag{5.26}$$

Proof. Using the definition of the inner product on X given by Eq. (5.10), We get

$$\begin{aligned}\left\| T_N(t-s)\Pi_N\Big(\mathcal{F}(u(s)) - \mathcal{F}(u_N(s))\Big)\right\|_X^2 &= \left\| T_N(t-s)\Pi_N\Big(\mathcal{F}(u(s)) - \mathcal{F}(u_N(s))\Big)\right\|_{\mathcal{H}}^2 \\ &\quad + \left|\Big[T_N(t-s)\Pi_N\Big(\mathcal{F}(u(s)) - \mathcal{F}(u_N(s))\Big)\Big]^D(-\tau)\right|^2.\end{aligned} \tag{5.27}$$

The first term on the right-hand side of Eq. (5.27) can be bounded as follows:

$$\begin{aligned}\left\| T_N(t-s)\Pi_N\Big(\mathcal{F}(u(s)) - \mathcal{F}(u_N(s))\Big)\right\|_{\mathcal{H}} &\le Me^{\omega(t-s)}\big\|\mathcal{F}(u(s)) - \mathcal{F}(u_N(s))\big\|_{\mathcal{H}} \\ &\le Me^{\omega t}\|\mathcal{F}(u(s)) - \mathcal{F}(u_N(s))\|_{\mathcal{H}} \\ &= Me^{\omega t}\left|F([u(s)]^D(-\tau)) - F([u_N(s)]^D(-\tau))\right| \\ &\le \mathrm{Lip}(F)Me^{\omega t}\left|[u(s)]^D(-\tau) - [u_N(s)]^D(-\tau)\right| \\ &\le \mathrm{Lip}(F)Me^{\omega t}\|u(s) - u_N(s)\|_X.\end{aligned} \tag{5.28}$$

Here the M and ω are from Proposition 5.1. The second term on the right-hand side of Eq. (5.27) can be estimated as follows. By noting that

$$[\mathcal{F}(\Psi)](\theta) = F\left(\Psi^D(-\tau)\right)\widehat{\Psi}(\theta) \qquad \forall\, \Psi = (\Psi^D, \Psi^S) \in X,$$

where $\widehat{\Psi}$ is defined by Eq. (5.24), we get

$$\begin{aligned}
&\left|\left[T_N(t-s)\Pi_N\Big(\mathcal{F}(u(s))-\mathcal{F}(u_N(s))\Big)\right]^D(-\tau)\right| \\
&\qquad = \left|F([u(s)]^D(-\tau))-F([u_N(s)]^D(-\tau))\right|\cdot\left|\left[T_N(t-s)\Pi_N\widehat{\Psi}\right]^D(-\tau)\right| \\
&\qquad \leq \mathrm{Lip}(F)\left|[u(s)]^D(-\tau)-[u_N(s)]^D(-\tau)\right|\cdot\left|\left[T_N(t-s)\Pi_N\widehat{\Psi}\right]^D(-\tau)\right| \\
&\qquad \leq \mathrm{Lip}(F)\|u(s)-u_N(s)\|_X\left|\left[T_N(t-s)\Pi_N\widehat{\Psi}\right]^D(-\tau)\right| \\
&\qquad \leq \mathrm{Lip}(F)C(t)\|u(s)-u_N(s)\|_X,
\end{aligned} \tag{5.29}$$

where the last inequality follows by using the Assumption (**H2**).

By defining

$$C := \max\{\mathrm{Lip}(F)Me^{\omega t}, \mathrm{Lip}(F)C(t)\}, \tag{5.30}$$

and applying Eq. (5.28) and Eq. (5.29) to Eq. (5.27), we get the desired estimate

$$\|T_N(t-s)\Pi_N(\mathcal{F}(u(s))-\mathcal{F}(u_N(s)))\|_X \leq C\|u(s)-u_N(s)\|_X. \tag{5.31}$$

□

Let us now introduce the following definitions:

$$\begin{aligned}
r_N(t) &:= \|u(t)-u_N(t)\|_X, \\
\epsilon_N(t) &:= \|T(t)u_0 - T_N(t)\Pi_N u_0\|_X, \\
d_N(t,s) &:= \|\left(T(t-s)-T_N(t-s)\Pi_N\right)\mathcal{F}(u(s))\|_X.
\end{aligned} \tag{5.32}$$

One can apply the variation-of-constants formula Eq. (5.14) and the above definitions to get that

$$\begin{aligned} r_N(t) &\le \epsilon_N(t) + \int_0^t d_N(t,s)\,\mathrm{d}s + \int_0^t \|T_N(t-s)\Pi_N\big(\mathcal{F}(u(s)) - \mathcal{F}(u_N(s))\big)\|_X\,\mathrm{d}s \\ &\le \epsilon_N(t) + \int_0^t d_N(t,s)\,\mathrm{d}s + C\int_0^t r_N(s)\,\mathrm{d}s. \end{aligned} \tag{5.33}$$

Applying Gronwall's inequality to Eq. (5.33) gives

$$\begin{aligned} r_N(t) &\le \left[\epsilon_N(t) + \int_0^t d_N(t,s)\,\mathrm{d}s\right] + \int_0^t Ce^{C(t-s)}\left[\epsilon_N(s) + \int_0^s d_N(s,r)\,\mathrm{d}r\right]\,\mathrm{d}s \\ &\le \left[\epsilon_N(t) + \int_0^t d_N(t,s)\,\mathrm{d}s\right] + Ce^{CT}\int_0^t \left[\epsilon_N(s) + \int_0^s d_N(s,r)\,\mathrm{d}r\right]\,\mathrm{d}s. \end{aligned} \tag{5.34}$$

We wish to show that $r_N(t) \to 0$ as $N \to \infty$ for each fixed $t \in [0,T]$. To this end, we show that each term on the right-hand side of Eq. (5.34) converges to 0 as $N \to \infty$ and $t \in [0,T]$ fixed. This is achieved by Propositions 5.5 and 5.6 below.

Proposition 5.5. *Let ϵ_N be as defined in Eq. (5.32). Then, for each $t \ge 0$, it holds that*

$$\lim_{N\to\infty} \epsilon_N(t) = 0 \ \text{and} \ \lim_{N\to\infty}\int_0^t \epsilon_N(s)\,\mathrm{d}s = 0. \tag{5.35}$$

Proof. From the definition of the X-norm, we have that

$$\epsilon_N(t)^2 = \|T(t)u_0 - T_N(t)\Pi_N u_0\|_{\mathcal{H}}^2 + |[T(t)u_0]^D(-\tau) - [T_N(t)\Pi_N u_0]^D(-\tau)|^2. \tag{5.36}$$

The first term on the right-hand side converges uniformly to 0 by the Eq. (5.23). The second term converges to zero thanks to (**H3**). This gives that $\epsilon_N(t) \to 0$.

To show the other convergence, note that $\epsilon_N(s)$ converges pointwisely to 0 on $[0,t]$. Furthermore, we may uniformly bound $\epsilon_N(s)$ by again observing the equality Eq. (5.36) and applying the uniform bounds on $\|T_N(\cdot)\|_{\mathcal{H}}$ and on $|[T_N(\cdot)\Pi_N u_0]^D(-\tau)|$. Then by the Bounded

Convergence Theorem, we have $\int_0^t \epsilon_N(s)\,\mathrm{d}s \to 0$. $\square$

Proposition 5.6. *For each fixed $t > 0$, it holds that*

$$\lim_{N\to\infty} \int_0^t d_N(t,s)\,\mathrm{d}s = 0 \text{ and } \lim_{N\to\infty} \int_0^t \int_0^s d_N(s,r)\,\mathrm{d}r\,\mathrm{d}s = 0. \tag{5.37}$$

Proof. We can again apply the definition of the X-norm to get that

$$\begin{aligned} d_N^2(t,s) &= \|\left(T(t-s) - T_N(t-s)\Pi_N\right)\mathcal{F}(u(s))\|_{\mathcal{H}}^2 \\ &\quad + |[T(t-s)\mathcal{F}(u(s))]^D(-\tau) - [T_N(t-s)\Pi_N\mathcal{F}(u(s))]^D(-\tau)|^2. \end{aligned} \tag{5.38}$$

For fixed t and s, the first term of the right-hand side converges to zero by Eq. (5.23). For the second term, we have convergence a.e. to 0 by (**H2**). So for fixed t, $d_N(t,s)$ converges almost everywhere to 0 for $s \in [0,t]$. Furthermore, we can uniformly bound $d_N(t,s)$ by (**H2**). Thus by the Bounded Convergence Theorem, we have $\int_0^t d_N(t,s)\,\mathrm{d}s \to 0$ as $N \to \infty$.

The second convergence follows by the observations that $\int_0^{\cdot} d_N(\cdot,r)\,\mathrm{d}r$ converges pointwise to 0 by our earlier work and can uniformly bounded on $[0,t]$. This allows us to apply the Bounded Convergence Theorem to get that $\int_0^t \int_0^s d_N(s,r)\,\mathrm{d}r\,\mathrm{d}s \to 0$ as $N \to \infty$. $\square$

Proof of Proposition 5.2. Apply Propositions 5.5 and 5.6 to the inequality in Eq. (5.34).

$\square$

Bibliography

[Ban82] H. Banks. Identification of nonlinear delay systems using spline methods. In V. Lakshmikantham, editor, *Nonlinear Phenomena in Mathematical Sciences*, pages 47–55. Academic Press, New York, 1982.

[BB78] H. Banks and J. Burns. Hereditary control problems: Numerical methods based on averaging approximations. *SIAM J. Control Optim.*, 16:169–208, 1978.

[BK79] H. Banks and F. Kappel. Spline approximations for functional differential equations. *Journal of Differential Equations*, 34(3):496–522, 1979.

[BRI84] H. Banks, I. Rosen, and K. Ito. A spline based technique for computing Riccati operators and feedback controls in regulator problems for delay equations. *SIAM J. Sci. Stat. Comput.*, 5:830–855, 1984.

[CGLW16] M. Chekroun, M. Ghil, H. Liu, and S. Wang. Low-dimensional galerkin approximations of nonlinear delay differential equations. *Discrete and Continuous Dynamical Systems*, 36(8):4133–4177, 2016.

[CZ95] R. Curtain and H. Zwart. *An Introduction to Infinite-Dimensional Linear Systems Theory*, volume 21. Springer, 1995.

[DvGVLW95] O. Diekmann, S. van Gils, S. Verduyn Lunel, and H. Walther. *Delay Equations: Functional, Complex, and Nonlinear Analysis*, volume 110 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1995.

[FGG10] S. Federico, B. Goldys, and F. Gozzi. HJB equations for the optimal control of differential equations with delays and state constraints, I: regularity of viscosity solutions. *SIAM J. Control Optim.*, 48(8):4910–4937, 2010.

[GCS15] M. Ghil, M. Chekroun, and G. Stepan. A collection on 'Climate Dynamics: Multiple Scales and Memory Effects', Editorial. *R. Soc. Proc. A*, 471(20150097), 2015.

[GZT08] M. Ghil, I. Zaliapin, and S. Thompson. A delay differential model of ENSO variability: parametric instability and the distribution of extremes. *Nonlin. Processes Geophys.*, 15(3):417–433, 2008.

[Hut48] G. Hutchinson. Circular causal systems in ecology. *Ann. N. Y. Acad. Sci.*, 50:221–246, 1948.

[HVL93] J. K. Hale and S. M. Verduyn Lunel. *Introduction to Functional-Differential Equations*, volume 99 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1993.

[IT86] K. Ito and R. Teglas. Legendre-tau approximations for functional-differential equations. *SIAM J. Control Optim.*, 24(4):737–759, 1986.

[Kap82] F. Kappel. An approximation scheme for delay equations. In V. Lakshmikantham, editor, *Nonlinear Phenomena in Mathematical Sciences*, pages 585–595. Academic Press, New York, 1982.

[Kap86] F. Kappel. Semigroups and delay equations. In *Semigroups, Theory and Applications, Vol. II (Trieste, 1984)*, volume 152 of *Pitman Res. Notes Math. Ser.*, pages 136–176. Longman Sci. Tech., Harlow, 1986.

[KN] V. Kolmanovskii and V. Nosov. *Stability of functional differential equations.*

[Koo84] T. Koornwinder. Orthogonal polynomials with weight function $(1-x)^{\alpha}(1+x)^{\beta}+M\delta(x+1)+N\delta(x-1)$. *Canadian mathematical bulletin*, 27(2):205–214, 1984.

[KS78] F. Kappel and W. Schappacher. Autonomous nonlinear functional differential equations and averaging approximations. *Nonlinear Analysis: Theory, Methods & Applications*, 2(4):391–422, 1978.

[KS87] F. Kappel and D. Salamon. Spline approximation for retarded systems and the Riccati equation. *SIAM J.Control Optim.*, 25:1082–1117, 1987.

[Kua93] Y. Kuang. *Delay Differential Equations with Applications in Population Dynamics.* Academic Press, 1993.

[Kun82] K. Kunisch. Approximation schemes for the linear-quadratic optimal control problem associated with delay equations. *SIAM J. Control and Optimization*, 20(4):506–540, 1982.

[L$^+$23] A. Lotka et al. Contribution to the analysis of malaria epidemiology. i. general part. *American Journal of Hygiene*, 3(Supp), 1923.

[LS10] M. Lakshmanan and D. Senthilkumar. *Dynamics of Nonlinear Time-delay Systems.* Springer, Heidelberg, 2010.

[Mac89] N. MacDonald. *Biological Delay Systems: Linear Stability Theory*, volume 8 of *Cambridge Studies in Mathematical Biology*. Cambridge University Press, Cambridge, 1989.

[McD78] N. McDonald. Time lags in biological models, lect. notes in biomathematics, 1978.

[MN07] W. Michiels and S. Niculescu. *Stability and Stabilization of Time-Delay Systems: An Eigenvalue-based Approach*, volume 12 of *Advances in Design and Control*. SIAM, Philadelphia, PA, 2007.

[Nei78] Ju. I. Neimark. *Dynamical Systems and Controlled Processes*. Nauka, Moscow, 1978.

[Paz83] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44;44.;. Springer-Verlag, New York, 1983.

[RCC+14] L. Roques, M. Chekroun, M. Cristofol, S. Soubeyrand, and M. Ghil. Parameter estimation for energy balance models with memory. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Science*, 470(2169):20140349, 2014.

[RR04] M. Renardy and R. Rogers. *An introduction to partial differential equations*, volume 13. Springer Science & Business Media, 2004.

[Ste89] G. Stepan. *Retarded Dynamical Systems: Stability and Characteristic Functions*, volume 210 of *Pitman Research Notes in Mathematics Series*. Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1989.

[WC57] P. Wangersky and W. Cunningham. Time lag in prey-predator population models. *Ecology*, 38(1):136–139, 1957.