Error Patterns:

What Do They Tell Us?

by

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(Abstract)

An analysis of computer diagnostic systems shows that most systems use answer data (product) for their analyses. This process of determining an error pattern , in addition, does little in the way of telling a teacher what should be done to help the child. This two-fold problem, extant in all computerized arithmetic diagnostic systems to date, prompted this study which sought other data sources in order to bring about more accurate computer analyses. A cognitive orientation suggested that the use of clinical diagnostic techniques should be explored as an alternative to error analysis. Essentially, these two approaches were compared. That is, to what extent does error pattern diagnosis (an essentially product oriented approach) and clinical mathematical diagnosis (a process oriented approach) interrelate?

Participants for this study were five, eight year olds from southwest Virginia. These children completed a test that was developed by Van Lehn (1982). This test was analyzed for error patterns and the children were selected on the basis of their error patterns. These children were then tested in a clinical setting using a measure developed for this study in cooperation with a clinical mathematics diagnostician.

The analysis was done on the results of these two measures and the protocols collected during the clinical interviews. The results indicated that there was no clear connection between the two types of diagnosis, but the analysis did yield a broader description of each individual participant. That is, error analysis or clinical mathematics alone does not completely describe an individual's knowledge of mathematics.

The conclusion of this study is that future computer diagnostic systems ought to take both approaches and build them into the system. That is, the best system would be a system that analyzes the child from multiple perspectives to arrive at a more complete picture of the child's capabilities.

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Introduction

The purpose of this study is to determine how error patterns and a clinical mathematics analysis interrelate for the long term purposes of designing a more appropriate computer diagnostic system for subtraction. The advent of the microcomputer in the classroom has prompted researchers to seek unique applications for the computer as a tool in the classroom (e.g. Computer-Managed Instruction, Computer-Based Instruction, Computer-Assisted Instruction, Interactive Video, LOGO, etc.) One such application that has evoked considerable attention recently is that of computer diagnostic systems. Diagnostic systems have been developed in a wide variety of areas such as medicine (e.g. Miller, Pople & Myers, 1982), math (e.g. Brown & Burton, 1978; Ohlsson & Langley, 1985; Young & O'Shea, 1981), and programming (e.g. Johnson, 1985). Of particular importance to this discussion are those systems specifically concerned with diagnosis in arithmetic.

Recently, several mathematical diagnostic systems have been developed for subtraction. These systems use answer (product) data for analysis (Blando, Carlsen & Sleeman, 1986; Brown & Burton, 1978; Ohlsson & Langley, 1985; Sleeman & Brown, 1982; Young & O'Shea, 1982). Orey and Miller (in press) evaluated two arithmetic diagnostic systems, Diagnostic Path Finder (DPF) by Ohlsson and Langley (1985) and Debuggy by Brown and Burton (1978). Resnick (1982) also analyzed Debuggy as well as the Production System (PS) by Young and O'Shea (1981) and attempted to project where diagnostic systems should evolve from these current systems.

Each of these systems uses answer (product) data only. Outside the realm of mathematics, however, Johnson (1985) has created a diagnostic computer system, Proust, to determine semantic (conceptual) errors in students' Pascal programs. The advantage Proust has over the arithmetic diagnostic systems discussed, is that a Pascal program represents a step-by-step solution of a problem. This step-by-step solution is essentially a protocol written in a specialized problem solving language. Therefore, Proust has a great deal more information about a student's solution process than just the product answer (Orey & Miller, in press).

The success of this program in its ability to closely analyze and "understand" a student's solution, has led researchers to investigate other types of information which can be supplied to an arithmetic diagnostic system (Resnick, 1982). For example, analytic tests designed to assess subcomponential understanding, suggested by Underhill, Uprichard and Heddens (1980), might be utilized to aid error pattern diagnosis. According to this approach, analytic tests can be constructed to assess the subcomponents represented in a checklist developed at Kent State University. In addition, these tests can be administered in a clinical setting. That is, the child solves the problems from the analytic test in a one-on-one setting with a clinician. The clinical approach uses semantic tasks as described by Resnick, uses questions that are designed to assess subcomponential understanding and are conducted on an individual basis. Error analysis examines syntactic knowledge, only examines procedural knowledge and the data is collected in group settings. Both of these analyses can potentially be used to build a framework of the child's understanding. However, a necessary part of the clinical process is to collect protocols which guide the analysis, a process that is difficult to automate (computerize).

Theoretical Framework

Computer Diagnostic Systems

The attempts that have been made to develop computer programs that diagnose student errors in the subtraction environment (e.g. Brown & Burton, 1978; Young & O'Shea, 1981; and Ohlsson & Langley, 1985) have been tentative. Each of the programs that have been developed reflect the pragmatic problems of operationalizing psychological theories in a computer environment as well as the limits of the theories which have been applied to the specific domains.

Debuggy. Brown and Burton (1978) created one of the first computer programs that was designed to specifically diagnose children's erroneous or "buggy" behavior in subtraction. The theoretical basis for Debuggy is that children are good at following procedures , "but that they often follow the wrong procedures" (Burton & Brown, 1978, p. 157). In order to determine the faulty algorithms that a particular child might be using, Debuggy makes use of an extensive "bug" library. This "bug" library was created by Brown and Van Lehn (1980) through the examination of the performance of 1325 students on a 15 item subtraction test. The analysis of these tests resulted in a library consisting of 110 primitive bugs and 20 compound bugs. (A primitive bug consists of one erroneous step in the subtraction algorithm as opposed to a compound bug which consists of two or more errors). The Debuggy system uses this library to analyze a child's erroneous behavior, that is, the child's erroneous responses are matched to the expected responses for each of the buggy procedures. This allows the system to create a list of "buggy" solutions that have the same solutions as that of the child. If one of these "buggy" procedures matches the child's solutions to each problem, then the child is said to exhibit that "bug". Depending on the number of solutions that are matched to a specific "bug," the child is classified as being consistent, fairly consistent or inconsistent.

There are two major criticisms of this system. First , if children do in fact "learn by doing," they may be classified as inconsistent in the overall task because they exhibited some procedural changes during the task (Orey & Miller, in press). The procedural changes that the children exhibit are a result of the process of learning, and a system that does not "know" that the children are going through this change may incorrectly diagnose the children as being random with respect to their solutions.

The second major criticism of Debuggy (and for that matter of all the computer diagnostic systems that have been developed to date) is that the child's performance is analyzed solely in terms of error patterns. The overall understanding that the child may have for the subtraction task environment is not considered. Resnick (1982) argues that Debuggy (as well as the Production System by Young and O'Shea, 1981) analyzes only the syntactic failure of the child's subtraction algorithm. The understanding the child possesses can only be inferred from such systems and is never considered in the system's development.

Production System (PS). Young and O'Shea (1981) attempted to make a computer diagnostic system that is more in line with current psychological theories for arithmetic by looking at the complete set of a child's errors on a subtraction task, much the same as Debuggy (Brown & Burton, 1978). In Young and O'Shea's theory of erroneous subtraction, the child has either forgotten, or never learned or mis-learned, some step in the subtraction algorithm. Therefore, any error can be explained in terms of the student's failure to apply one or more steps in the algorithm.

Two important points arose from this research. First, Young and O'Shea (1981) reported that they found two students that had apparently modified their behavior without intervention. That is, "...the children were consistently making the error, and thereafter got it right" (p.166). The second point is that when Young and O'Shea analyzed the students' errors, they found it "crucial" to have the children's actual worksheets. The reason for this was so that they would have more information at their disposal for determining student errors. They needed to see such scratch work as borrowing and crossing out because more information about student performance (process) in addition to answers (product) was necessary. This information, as yet, cannot be supplied to a computer.

Diagnostic Path Finder (DPF). Ohlsson and Langley (1985) have developed a general diagnostic system called the Diagnostic Path Finder (DPF), which has been tested in the subtraction domain in comparison with Debuggy. This diagnostic system becomes more specific through the inputs to the system which consist of both the erroneous answers to the problems and the "problem space" (Newell & Simon, 1972). For subtraction, Ohlsson and Langley defined operators to act on the problem, a column notation that could have empty positions (i.e. 342-47 in column form would have an empty hundreds place), and the termination criterion (all columns had been processed).

Beyond the inputs, Ohlsson and Langley designed this system based on two premises. First, people learn by doing. This assumption is the major weakness of DPF (Orey & Miller, in press). Because DPF treats each answer that a child gives in a subtraction task separately, many common misconceptions that the child has throughout the task are not directly determined. Each of the subtraction errors are considered independently by the DPF system. Ohlsson and Langley rationalize this approach to diagnosis through the idea that children learn by doing. Therefore, the child is continually modifying his or her algorithm throughout a task, and her or his algorithm for one problem may not be the same for any other problem. However, a child that is consistently following an erroneous algorithm may have more than one diagnosis for his behavior. Ohlsson and Langley's approach seems extreme, considering that only two test papers out of 51 tests showed a procedural change in Young and O'Shea's (1981) study. Furthermore, this theoretical basis takes a step backward for computer diagnostic systems in that it further limits the computer's ability to model a child's understanding of arithmetic (i.e. it only considers the child's answers to subtraction problems).

The second premise is that current psychological tenets can be employed to make a rating function for their best first search through the solution space. That is, by using psychological tenets, such as the notion that the human processing system is limited to five to nine pieces of information at one time (Miller, 1956), and that people automatize certain steps in the solution process (e.g. Anderson, 1983), the optimal diagnosis can be found quickly.

Proust. Johnson (1985) created a diagnostic computer system (Proust) to determine semantic errors in students' Pascal programs. As stated earlier, an advantage Proust has over arithmetic diagnostic systems is that because a Pascal program represents a step-by-step solution of a problem, Proust has a great deal more information about a student's solution process than just the product answer. The success of this program and its ability to closely analyze and "understand" a student's solution has led researchers to the task of finding other types of information to supply to an arithmetic diagnostic system (Resnick, 1982). In other words, tests and analyses that better describe a child's understanding of arithmetic can be used to determine where, in the solution process, the child is following an algorithm and where he is creating a solution process. This information might come from the methodology of clinical mathematics, in which the child's protocols are as rich a data source as a Pascal program. The integration of this approach into computer diagnosis is my long term goal and will be discussed further in coming sections of this paper.

Human Memory and Human Learning

Because of its relevance to computer application, the information processing model will serve as the basis for the analysis presented in this study. Information processing models direct us to examine children's understanding of subtraction from two separate perspectives suggested by Chomsky (1957,1959,1965,1968); these are syntactic versus semantic understanding. Chomsky initially drew the distinction by using the terms surface structure of the language (syntax) and the deep structure of the language (semantics). The surface structure is the grammatical rules for writing and speaking a language. The deep structure is the way the language is represented in memory. Shneiderman and Mayer (1979) use the terms syntax and semantics in describing the behavior of computer programmers. Syntax, for computer programmers, is dependent on the specific machine and language the programmer is using and is usually acquired via rote memorization. Semantics constitute programming concepts. Evidence for this distinction comes from the many chunking studies which find that experts tend to recall things in conceptual chunks (semantics) while novices recall things according to syntactical units (e.g. chess - Chase & Simon, 1973; Simon & Barenfeld, 1969; Simon & Gilmartin, 1973; computer programming - Adelson, 1981; Magliaro & Burton, in press; McKeithen, Reitman, Reuter, & Hirtle, 1981; or, physics - Larkin, McDermott, Simon & Simon, 1980). Resnick (1982) first articulated the syntactic/semantic distinction in mathematics. As she defines them, syntax is knowledge for the language of mathematics and semantics is knowledge for the underlying meaning beneath the language.

A seemingly worthwhile operationalization of these definitions would be to analyze mathematically written and spoken responses for syntactic knowledge and analyze explanations of the linguistic responses through the use of concrete materials for the study of semantic knowledge. Implicit in this discussion is the idea that children's understanding can be represented on a kind of two dimensional continuum (see Figure 1). As a child learns subtraction, portions of the algorithm are understood and used. Individuals do not learn arithmetic in the same way (e.g. Resnick & Ford, 1981). Some children may do well on mathematical language (paper and pencil tasks), but poorly on understanding tasks. Others may have a deep understanding of subtraction, but cannot perform well on paper and pencil tasks. In order to develop this two dimensional continuum, I will first discuss anthropological research that suggests the existence of semantic understanding without any understanding of arithmetic syntax. I will then discuss the role of syntax in arithmetic performance in terms of the notion of learning by doing. Learning by doing manifests itself in the existence of procedural changes, therefore, the discussion of syntax will be presented in terms of two hypothesized conditions for procedural change.

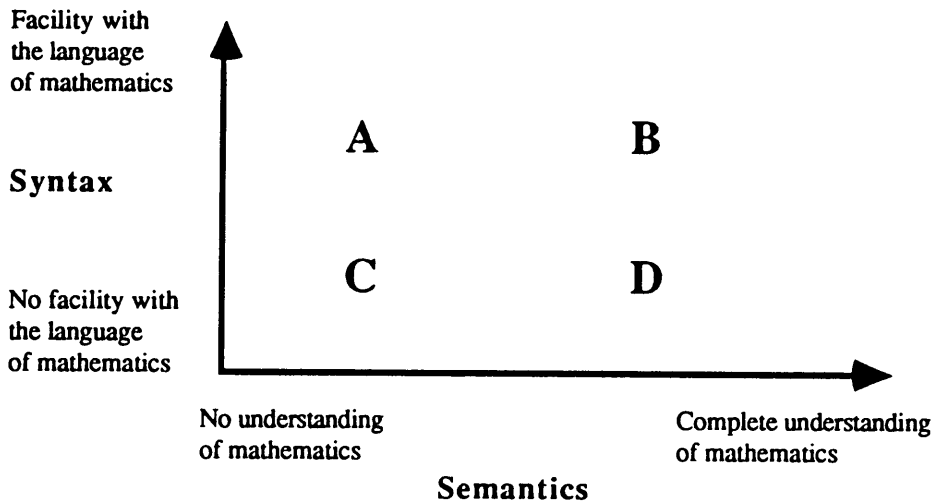

Figure 1. Two-dimensional continuum of learning.

The anthropological examples, to be discussed here, show that there are some children who have an understanding of subtraction, but have no facility for paper tasks

(Area D of Figure 1). That is, these children have developed along the horizontal axis of Figure 1. There are other children who can do calculations on paper (facility), but who have very little understanding of subtraction (Area A of Figure 1). That is, development along the vertical axis of Figure 1 (i.e. syntactical development). The last type of development deals with the horizontal growth relative to vertical growth. That is, the interaction that occurs between the two types of knowledge (Area B of Figure 1).

The first notion of development, that people can develop understanding for mathematics and have little or no facility with mathematics is documented in educational anthropology. Examples of people who have an understanding of arithmetic, but are unable to exhibit their knowledge on a paper and pencil task (i.e. strong semantic understanding, but little or no syntactic understanding) are found in much of the educational anthropology literature regarding context specific understanding. Herndon (1971) found a student who could not do arithmetic in class very well, but who worked in a bowling alley as a paid scorer, requiring fairly complex mental calculations. This prompted Herndon to give the class problems that were bowling scoring tasks, but the child was unable to perform them. There seemingly was no connection between the paper task and the mental calculation. (For more examples of context specific learning (or semantic knowledge of arithmetic) see Carraher, Carraher & Schliemann, 1983; Lave, Murtaugh & de la Rocha,1984).

I will present two hypothesized scenarios for explaining procedural changes which relate to students' apparent invention of algorithms or portions of algorithms. The importance of this analysis is that any attempt to diagnose children's misconceptions requires a detailed understanding of misconceptions. In order to discuss the understanding of syntax, it might serve to define the subtraction algorithm in a flowchart form (see Figure 2, reproduced from Resnick, 1982, p. 137). In order to illustrate the first hypothesis of procedural change, an example will be employed. A common error pattern exhibited by some children has been termed the "Smaller-From-Larger" bug (Brown & Burton, 1978). Here are some examples following this algorithm:

432	275	571
-157	-167	-498
325	112	127

The hypothesized algorithm that a child with this bug may have prior to any unexpected change in procedure is depicted in Figure 3. According to this figure, the child does not check to see if the bottom digit is less than the top digit. Therefore, the part of the algorithm at this branch (node B) is not integrated into the subtraction process (this is what is meant by the "fuzzy area of the algorithm").

Perhaps, for a procedural change to occur, a conflict must arise, either external or internal to the system, that suddenly strengthens the fuzziness depicted in Figure 3. An example of an external cue might be a teacher intervening in the solution process. More importantly, an example of an internal conflict might be a familiar problem for which the student knows the approximate value. The semantic feature model developed by Collins and Loftus (1975) accounts for this type of cue. That is, in the semantic feature model of human memory some occurrences in our experience are closer to the concept of interest. So a particular problem on a subtraction task might be more typical of borrowing for a particular child, thereby allowing the child to recall the borrowing procedure. This hypothesis assumes that the child has been given instruction on the entire subtraction algorithm, but some of the pointers of the internal structure have not been strengthened enough for the system to operate on it (i.e., the child has not assimilated the complete algorithm into his existing structure).

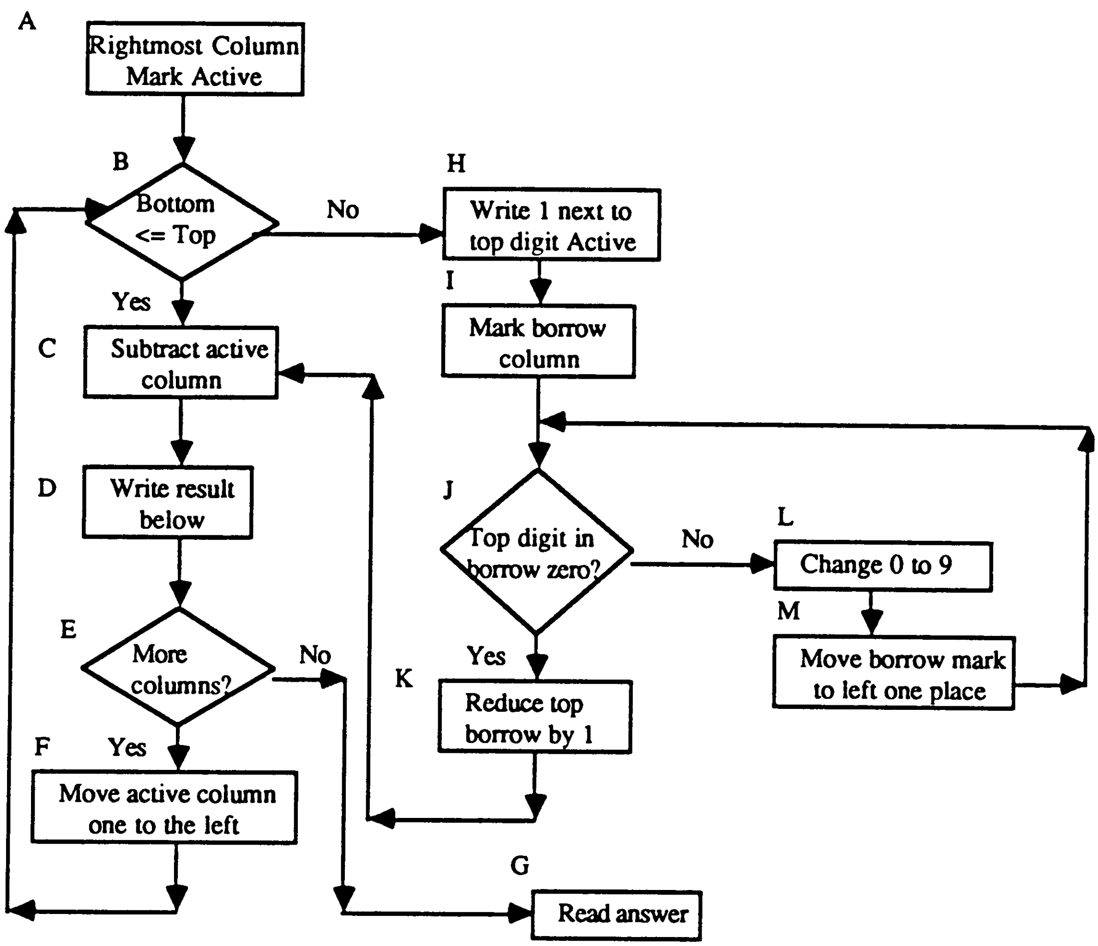

Figure 2. Flowchart representation of a subtraction algorithm.

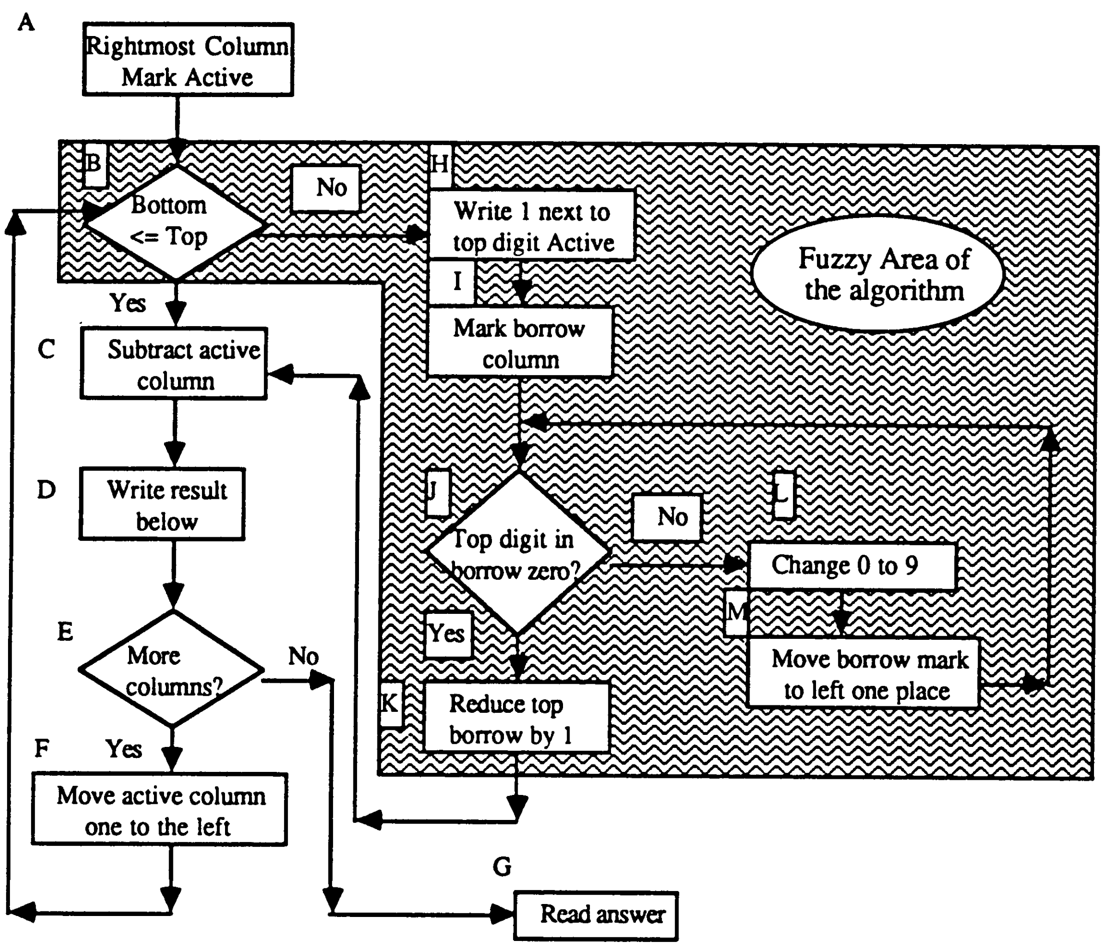

Figure 3. Hypothesized algorithm.

The above discussion is related to the vertical development of knowledge in Figure 1 (syntax). Horizontal development in Figure 1 (semantics) pertains to work done by such researchers as Herndon (1971), Ginsberg (1977) and Resnick (1982). The example Herndon gives us (above) seems to be the extreme case. However, Ginsberg and Resnick both relate children's performance on arithmetic to the children's understanding. Ginsberg speaks in terms of the child's mental framework. Resnick, on the other hand, refers to the child's semantic and syntactic memory.

Ginsberg (1977) presents numerous examples of children's unique solutions. Many of these unique solutions relate to concrete examples of arithmetic with which the child has had experience. Ginsberg refers to the differences between an individual's paper and pencil tasks and her informal knowledge as "gaps." Because of the active information processing system, the child will, at times, draw a connection between the two types of knowledge. For example, Ginsberg presents a child's performance on a verbal problem and then asks the child to represent that solution as a written problem. The child was able to quickly determine the solution to the verbal problem, but when she was asked to write it as a problem, she arrived at a faulty solution (see Figure 4).

The dialogue of this problem sequence shows that the child could easily determine the solution if the task environment were one of her own creation. However, the child did not make use of her informal knowledge in a standard subtraction algorithm. Eventually, the child did realize that her written solution was not correct. Therefore, the child recognized the gap. This could be the goal for every instructor (human or computer) - to get the child to close the gap.

The verbal problem: "You have twenty-four flowers. Six of these flowers are tulips. How many are Daffodils?"

The written solution:

$$\begin{array}{r} 6 \\ \underline{-24} \\ 22 \end{array}$$

(Ginsberg, 1977, p. 122)

Figure 4. An erroneous solution.

In Resnick's (1982) terms, the child would need to make mappings between her semantic memory (the understanding the child has for subtraction) and her syntactic memory (the written solution). Hence, a second explanation for procedural changes without teacher intervention is that at some point a conflict arises and the child must adapt her algorithm to accommodate the conflict.

The modification of a procedure by a child, given a certain problem, is what Brown and Van Lehn (1980) refer to as "repair theory." In this scenario the child is presented with a problem that can not be calculated using her procedure. For example, consider the child that does not know the process of regrouping. According to the theory of repair, when the child needs to borrow, the child will invent something to allow the solution to be calculated. In this case the most common solution strategy is to take the Smaller-From-Larger (e.g. 57 - 39 = 22). The repairs are not always faulty solutions, but according to the theory, they do take place and are quite common.

Random errors. The importance of a discussion of random errors is because every diagnostic computer system that has been written has classified a large percentage of children as being random. In the above discussion on cognitive frameworks, I have tried to show that a more in-depth analysis should be conducted in order to limit the large numbers of children who are being classified as being random.

Returning to Figure 1, many of the errors that have been classified as random can be explained in terms of Areas C and D. The children that are operating in these areas of the figure may exhibit many inventions based on their incomplete syntactical knowledge and its interaction with their semantic knowledge. The erroneous solutions of these children are usually unique and only meaningful if the child's semantic memory is understood.

Consider the flowchart in Figure 2. The role that semantic memory plays in the interrelationship of the flowchart can be anything from the entire structure (such as Herndon's student) to a particular box. For example, box H (of Figure 2) might relate to Dienes' blocks (Dienes, 1963). This child might make use of an internal Dienes' blocks representation to carry out the borrowing process. An error may occur because of a misunderstanding of Dienes' blocks rather than because of a faulty step in the algorithm.

Subcomponential understanding. It is very important to be sure that global skills are learned sequentially (Resnick & Ford, 1981). That is, addition is a prerequisite to subtraction, counting is a prerequisite to addition, and so on. The subcomponents of a given skill such as subtraction, however, are not necessarily learned in the same order by each child (Underhill et al., 1980). What does matter is that a child must understand all the subcomponents of subtraction in order to be completely proficient in subtraction (Area B of Figure 1). For example, understanding of place-value is a prerequisite for the understanding of the borrowing process. Knowing that the child does not understand place-value, for example, would explain borrowing error patterns exhibited by the child on a subtraction task. A form of diagnosis was developed for the assessment of subcomponents by Heddens (Underhill et al., 1980) and could be used for building a more complete model of the child's understanding of arithmetic with implications for error patterns. This will be discussed further in the next section.

Diagnostic Mathematics

Mathematics education has long been concerned with understanding student errors in arithmetic. Buswell (1926), for example, spent a great deal of time analyzing, chronicling and classifying error patterns. Roberts (1968) identified four general categories of arithmetic failure based on 766 third grade students' performance on a Stanford Achievement Test. They are:

1. Wrong operation: The pupil attempts to respond by performing an operation other than the one that is required to solve the problem.
2. Obvious computation error: The pupil applies the correct operation, but his response is based on an error in recalling basic number facts.
3. Defective algorithm: The pupil attempts to apply the correct operation, but makes errors other than number fact errors in carrying through the necessary steps.
4. Random response: The response shows no discernible relationship to the given problem (p. 442).

The final category which Roberts calls "random response" is found in much of the literature on classification of erroneous performance in arithmetic. Random errors are understood to be many different things. For example, a solution that can't be explained by any of the procedures in the "bug" library by Brown and Van Lehn would be considered random. Ohlsson and Langley might define the "random child" as a child that makes no use of any part of a standard subtraction algorithm. However, Ginsberg (1977) finds that children are seldom random in their solution. That is, "Children are seldom capricious or random" (Ginsberg, 1977, p. 128). The difference between those researchers who believe that there is randomness in errors and those that do not, seems to be in the methodology employed for collecting the data. Ginsberg uses a "think aloud" protocol for determining what processes a child uses while the computer diagnostic systems only use the answers to the problems. Quite often children will make use of procedures of their own invention, therefore, given only the answers, it is difficult to determine the processes employed by the child.

Ginsberg shows, through his analyses of many protocol reports, that children make use of varying amounts of informal knowledge of mathematics. This is what Resnick (1982) calls mappings between arithmetic syntax and semantics. Ginsberg finds that children may exhibit strengths in some cognitive components of the subtraction task, but extraordinary weaknesses in others. Therefore, it is believed that an analysis of the child's understanding of the cognitive components of subtraction (Ginsberg, 1977), as well as the informal understanding of mathematical principles (Resnick, 1982), may give a computer diagnostic system the ability to determine the child's unique error patterns.

The above discussion of how a child may build an erroneous process for performing subtraction may account for some of the random classification of children by diagnosticians. An intelligent tutoring system may access a child's informal knowledge of mathematics by making use of concrete examples such as Dienes' blocks. Such a computer system could give instruction on the use of Dienes' blocks and purposefully connect this process with the subtraction algorithm, as proposed by Brown (1983) and Resnick (1982) and demonstrated by Champagne and Rogalska-Saz (1982).

An intelligent diagnostician should not be too quick to declare that a child is performing arithmetic in a random fashion. It can be argued that mathematicians are inherently lazy, that is, that mathematicians are constantly seeking to overcome the drudgery of simple computation. This is the rationale for mathematicians finding short cuts or formulae for complex operations. This idea implies that great mathematical ability and understanding is involved in finding such short cuts. It would seem that some children exhibit these short cut methods or simplifications to problems. These short cuts sometimes lead to the correct solutions to problems; at other times they lead to what would be termed a random response. After all, if the short cut had no relation to a standard subtraction algorithm, a small error in the process may yield an incomprehensible solution. Thus, it is necessary to understand the framework from which the child is operating. This information can then be used to guide the child to an understanding of his error. This would appear to be the preferred action because it does not destroy an important mental framework for mathematical reasoning.

There is an old story told in many mathematics classes about Gauss. The story explains how an instructor gave, as punishment to the class, an assignment to calculate the sum $1+ 2+ ...+ 100$. After a very short while, Gauss exclaimed the answer to be 5050. Gauss had discovered that there was an underlying pattern to the sums and found the formula for the series, $1 + 2 + ... + n$, to be $n (n + 1) / 2$. Hence, he found a short cut for the solution of the task. If he had made an error and did not receive guidance about any underlying misconception, he may have lost a valuable learning experience.

Reisman (1978) has pointed out the problem of teaching arithmetic without respect to the mental framework of the child. Mathematics, as it is taught today, is developmental. Therefore, in order to understand subtraction, one must understand its inverse operation, addition. Addition is understood on the basis of counting, and so on. In other words, there are components of mathematical understanding and each component builds on previous components. Thus, the teaching of mathematics can be pictured as a spiral curriculum (Underhill et al., 1980).

Underhill et al. (1980) describe a variety of methods for determining subcomponential understanding in a hierarchy based on the curriculum spiral. The methodology for diagnosis follows a prescribed order. The first phase of such a diagnostic activity is to collect information on the child's computational and abstract abilities. This can be done by making use of a standardized test such as Key Math or a Stanford Achievement test. However, such a test only assesses global abilities. An analytical test must be constructed to analyze the non-hierarchical subcomponents of subtraction.

In order to adapt and organize such a test, Orey and Underhill (1987) analyzed the types of knowledge that may affect subtraction performance. This analysis yielded a matrix which is depicted in Figure 5. This model serves as a construct for distinguishing between knowledge types. It does not imply actual structures in memory. The model appears to be discrete, which is at odds with the two dimensional continuum of Figure 1. Further, the model may imply that there is interaction between the types of knowledge that are depicted. Because this has not been resolved, the model should not be used as evidence for this conclusion. The model only serves to enable a researcher to distinguish among knowledge types for their separate analysis.

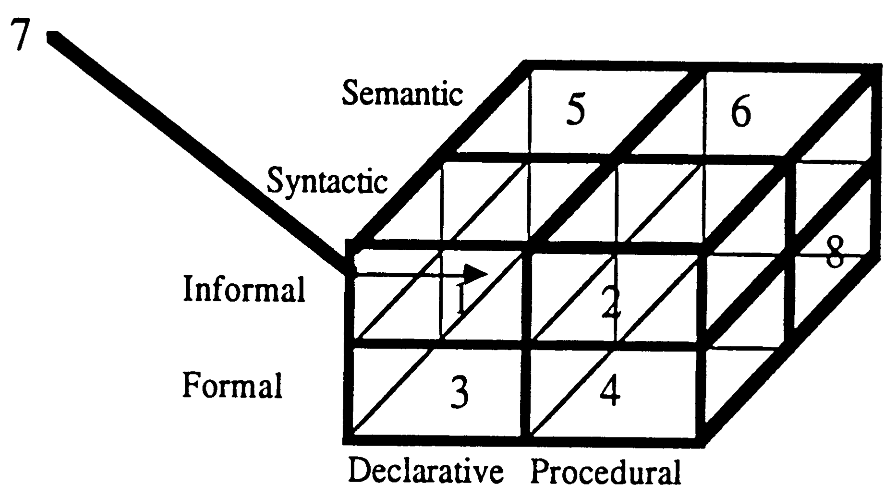

Figure 5. A three-dimensional model for exploring understanding of whole number subtraction

Semantic understanding, as described by Resnick (1987), is knowledge that can be elicited from the child through concrete materials. She has generally used Dienes' blocks, but I have also mentioned U.S. currency. Orey and Underhill (1987) draw the distinction between these types of knowledge by using the terms formal (e.g., Dienes' blocks) and informal (e.g., U.S. currency). Therefore, Resnick's semantic understanding accounts for cells five through eight. Her syntactic understanding, being defined as subtraction notation, accounts for cells one through four. Similarly, error pattern analysis only analyzes the knowledge depicted in cell four. The only approach that can account for one full plane of Figure 5 is a subcomponential analysis conducted in a clinical setting, which is termed clinical mathematics. However, procedural knowledge is usually assessed on a formal/syntactic level and declarative (e.g., place value) knowledge is either assessed in a semantic or syntactic context, not both.

Informal/Formal Dimension. Perhaps the best way of distinguishing between these two types of knowledge would be in terms of encoding specificity (Tulving & Thompson, 1973). The manner in which knowledge is retrieved from memory is directly related to the situation in which that information was stored. Therefore, to study formal knowledge, one must recreate the environment in which that information was learned. This is fairly straight forward since formal knowledge is defined as the knowledge that children learn in a school setting. The more difficult analysis is that of informal knowledge. Informal knowledge is learned in the children's home environment and attempts can be made to recreate those settings, but those recreations are truly difficult (or impossible, from the ethnographers point of view). The point is that attempts at studying the differences between formal and informal knowledge should make every effort at recreating the environment in which that knowledge was learned (through all the senses).

Orey and Underhill (1987) define informal knowledge as that which is learned simply by being a child in our culture. Operationalized, this definition includes knowledge of money and its properties (money has its own visual and tactile qualities). Formal knowledge, on the other hand, is knowledge which has been derived from direct instruction in schools. Formal knowledge might include concepts such as ones, tens, subtract, and include the subtraction algorithm as it is typically taught. In addition, Dienes blocks (among other materials used by teachers) are usually only found in the classroom experience. Therefore, formal knowledge, with few exceptions, has been extensively studied and used for diagnosis.

Informal knowledge, on the other hand, has not been extensively studied, although researchers have begun to explore this realm (Carraher, et al., 1983; Lave, et al., 1984). These researchers describe subjects who have extensive informal knowledge, but who are unable to apply this knowledge in formal settings, even when the same problems are presented in the classroom. The informal dimension was included in our analysis (Orey & Underhill, 1987) because we felt that some people are not able to make metacognitive connections between informal and formal knowledge, but that does not mean that no one can. For example, Papert (1980) describes an early childhood fascination with gears. This fascination allowed him to picture many mathematical principles in his mind, thus aiding his understanding of those principles.

There are other problems with the informal knowledge dimension - Can informal knowledge be assessed in a clinical setting, even if role playing and setting replications are employed? Or, is money an important aspect of informal knowledge for everyone? - to name a few. However, since this aspect of knowledge is relatively unexplored, it was considered important for analysis.

Syntactic/Semantic Dimension. As noted earlier, syntactic and semantic knowledge have been widely studied and both types of knowledge are highly recommended as part of formal instruction. To reiterate, syntax is knowledge of the language of mathematics. Semantics is the knowledge of what that language means. Syntactic knowledge is operationally defined as knowledge explored by tasks that are presented and whose responses are in a written or symbolic format. Semantic knowledge is operationally defined as knowledge exhibited by children in their explanations of their solutions using concrete materials (e.g., Dienes' blocks and US currency).

Declarative/Procedural Dimension. Procedural knowledge is essentially a step-by-step production for determining the solution to a problem (often depicted as a flowchart, Figure 2). In this case, the procedure for determining the solution to two-digit subtraction problems is the child's subtraction procedure. Declarative knowledge is a loose collection of concepts and sub-skills that may be prerequisite to understanding the algorithm. In the domain of subtraction, declarative concepts include representation of a number, tens and ones, and regrouping, among others. Gagne' (1985) makes the distinction between declarative and procedural knowledge as knowing that as opposed to knowing how, respectively. These two types of knowledge are used to develop protocol questions for all 8 cells in the matrix. The other dimensions of the matrix define the context and task environment on which these concepts are explored.

Summary

In order to develop a practical application for computer diagnosis in arithmetic, one must make use of the information available and develop pertinent information that is lacking. For this application one must look at existing programs and pool information from mathematics education and psychology. Psychology forms the basis from which to determine appropriate information to supply a diagnostic system. It appears that an analysis of a child's knowledge of each cell in Figure 5 would enhance an error pattern diagnostic system. It is hoped that by analyzing the matrix in combination with specific error patterns, a model of the child's understanding of subtraction can be constructed for that specific error pattern. This analysis would both give meaning to error patterns and have implications for future diagnostic and teaching computer systems.

Methodology

Participants

Participants for this study were third graders (approximately 8 years old) from a small, southwest Virginia, elementary school. Children were selected on the basis of their performance on a 20 item multi-digit subtraction test (Van Lehn, 1982). Four subjects were selected because their error patterns were generally the same and a fifth subject was selected because he had completed the Van Lehn test without error.

Materials

Because there are eight cells in the matrix in Figure 5, the inclusion of three digit numbers would have been overwhelming (given that there would have been at least eight questions for each cell, yeilding 64 questions). Therefore, the questions were limited to two-digit subtraction. Bundles of toothpicks (ones and tens) were utilized for the formal/semantic tasks. Dimes (10¢) and pennies (1¢) were used for the informal/semantic tasks (nickels [5¢] and quarters [25¢] were not included so that the base ten system could be more clearly analyzed). All syntactic questions were presented in both a verbal and written format. Verbal responses were recorded on audio tape. Written responses were recorded by the subject on paper that was supplied.

A total of 24 questions were asked in the interview. These questions were derived from the declarative/procedural axis of Figure 5. That is, three questions each were developed for declarative and procedural knowledge. There are four cells on the declarative plane and there are four cells on the procedural plane. These cells determined the context for the question. All 24 questions are listed in Appendix A with alternative verbiage for children who misunderstood the way the question was originally phrased.

Procedure

Participants were administered the Van Lehn exam on the first day. These exams were evaluated and participants were selected on the basis of error patterns. An attempt was made to group two participants with one error pattern and two other participants with another. The final participant was selected because he completed the Van Lehn test without error and was intended to serve as a point of comparison.

During the next four days, structured clinical interviews were conducted using the test that appears in Appendix A. The subjects were instructed to use "concurrent verbal reports" (Erikson & Simon, 1984) for their solutions to each problem. If the subjects failed to do so, they were instructed to use "retrospective reports" (Erikson & Simon, 1984) after they completed the problem. Erikson and Simon have written a book on the use of verbal reports as data in which they describe, concurrent and retrospective reports. Concurrent reports are what is typically thought of as think-aloud protocols. That is, the child verbalizes his or her thinking processes as he or she solves the problem. The major threat to validity for this approach is that the child's verbalizations limits the amount of information that can be held in short term memory. Therefore, concurrent reports interefere with the solution process. Retrospective reports, on the other hand, require the student to recall the processes he or she has just used to solve the problem. The major weakness of this approach is that the verbal report causes retrocactive interference, a type of forgetting. With this in mind, these sessions were recorded on audio tape and were transcribed for analysis. In addition, specific notes were taken as to the subjects' physical solution strategies to the semantic tasks.

Analysis

The data for this study came from three sources: 1) error patterns that were exhibited in the Van Lehn test; 2) correct and incorrect responses to the protocol questions that were determined from the notes taken during the interviews and the transcripts of the protocols; and, 3) the protocol data transcripts that served to explain how the children arrived at their solutions. These three data points were used to triangulate (Denzin, 1978) the results to get a fairly complete description of each participant. The latter two data points constitute the normal procedure found in clinical mathematics diagnosis (Underhill, et al., 1980). The former data point is typically used by computer diagnostic systems. A comparison was made between the error pattern analysis and the children's correct and incorrect responses to the protocol questions. The protocol transcripts served as the basis for resolving conflicts between the other two data points.

After reporting the analysis of each child's knowledge of mathematics from the three data points, an attempt was made to determine the extent to which the children's correct and incorrect responses to the protocol questions relate to error pattern diagnosis across subjects. In order to do this, an analysis of each subject's strengths within each cell of Figure 5 was completed. These strengths were then compared to the error patterns of the subjects to see if the subjects' knowledge of mathematics was related to their error pattern.

Results and Discussion

The children's performance on the Van Lehn test yielded no completely common error patterns among the children. Furthermore, no child was entirely consistent in the error pattern which he or she exhibited. Four of the children who were selected for the study were selected on the basis of being "mostly" consistent and "mostly" similar with respect to their error patterns. Also, these four children could be paired up so that there were two groups of two. This partitioning of the children was due to their error patterns being somewhat the same. The fifth student was selected for the interview portion of the study because he was characterized as being verbal by his teacher and because he completed the Van Lehn test without error. Each child's solutions to the Van Lehn test appear in Appendix B.

The reason the children were not consistent in their errors could be due to the fact that their mathematics lessons were, at the time of testing, multiplication and division. Because they were not currently working on subtraction, they were required to recall all the conditions of their subtraction algorithm. Therefore, their inconsistancies may be due to the fact that they could not completely recall their algorithm. The Van Lehn test was designed to elicit error patterns and has many special cases in the set of problems.

There are two viable explanations from repair theory (Brown & Van Lehn, 1980) to account for the subjects' performances. First, the subjects had learned the correct procedures or had created repairs at the time of learning subtraction that would have elicited consistency in their performance, but the subjects were unable to recall these procedures at test time. Because the Van Lehn test has a variety of special problems, the children were forced to make repairs to their algorithm that varied from problem to problem. Second, the subjects did not have consistent algorithms and exhibited a performance that would have been similar at the time when they were learning subtraction. Regardless of the correct explanation, clinical interviews were conducted with these children to determine the similarities and differences between a cognitive (or error) diagnosis and a curricular (or clinical mathematics) diagnosis. An analysis of each interview follows.

Kate

Kate's mother worked as a respiratory therapist, having received her certification from a special (post high school) program. Her father was a full time Ph.D. student in Geology. His master's degree had already been completed. Based on these data, Kate was categorized as a middle class child.

Table 1. Results of the diagnostic interview for Kate.

	Declarative		Procedural	
	Syntactic	Semantic	Syntactic	Semantic
Formal	3a √	7a √	4a √	8a X
	b √√	b √√	b √	b X
	c X	c X	c X	c X
Informal	1a √	5a √	2a √	6a √
	b XX	b √X	b √	b X
	c X	c X	c X	c X

Note: 'X' indicates an incorrect response and '√' indicates a correct response. In addition, the numbers indicate the cell and the progression of letters are in increasing complexity (i.e., 'a' is easier than 'c'). A double response indicates a two part question.

Kate completed the Van Lehn test, and her responses indicated some inconsistent error patterns: O-N=N (3 errors), borrows from leftmost top digit when borrowing is required in the unit's place and do not decrement the tens place (3 errors), borrows from the left most place, but does not distribute 10 until a second borrow is needed and decrement every place (1 error), fact errors (2 errors) and borrows from zero (1 error). In total, she was incorrect in 10 out of 20 problems. Generally, Kate's error patterns indicate a misunderstanding of what to do with zeros in the subtrahend and misconceptions of place value and regrouping, which is supported by her performance on declarative tasks in the protocol data.

Kate's performance on the interview portion of the study is summarized in Table 1. Her performance on these tasks, in general, indicates that her best performance was on those tasks created to tap her formal/syntactic knowledge. Her performance on items designed to explore meaning (semantics) is not as strong. She had little meaning to relate to what she was doing nor did she have strong connections between her formal and informal knowledge.

Before discussing her specific performance, it should be recalled that three question types were created for the declarative dimension of Figure 5 (the odd numbered questions of Appendix A). In addition, 3 question types were created for the procedural dimension of Figure 5 (the even numbered questions of Appendix A). I will discuss each question of the declarative dimension in terms of the context provided by the cells of that plane, then, I will do the same for the procedural plane.

Declarative. The first question from the declarative plane deals with representation of a number. Kate's knowledge of representation is consistent across contexts. Whether she performs syntactic or semantic tasks, Kate is able to represent numbers easily. Her knowledge of the convenience of grouping things in tens is indicated in these tasks. When asked to represent some value concretely, she consistently counted out the tens, then the ones. Also, she is able to represent a number abstractly given the number of tens and ones. However, her ability to extend this knowledge beyond representation is very limited.

Kate's problems arise when she needs to make more use of her concept of ten and one than just counting. For example, in question 1B:

E: ....I have several dimes and pennies in my pocket. All I have is dimes and pennies. The total amount is 76 cents. How many dimes do you think that is?

Long Pause
E: That's 76 cents in total ... there are only dimes and pennies.
Long Pause
E: Can you guess how many dimes that might be?
K: 40
E: 40? If there are 40 dimes, how many pennies would there be?
Long Pause
K: 30

Although Kate could quickly determine that 2 dimes and 4 pennies was 24 cents (question 1A), she was unable to extract the number of dimes and pennies from a value. In fact, she seems to be focusing on the value of 70 (or the ten's place) when she gives the consecutive responses 40 and 30. Another interesting aspect of the declarative plane is her response to question 3B. This question presents the number 52, and she is asked how many tens and ones are in 52? She quickly responds, "5 tens and 2 ones." For this question, she apparently makes use of her syntactic rules that she has learned in class. That is, the number of tens in a number is the digit in the tens place. She answers this question based on the syntactic structure of a number. However, she is unable to apply this rule to an "informal" context (such as, 1B).

The remaining 2 cells (5 and 7, semantic/formal and informal) at the first level of complexity deal with the relative values of 10 and 1 using concrete materials. The first question is - which is more, a dime or penny (cell 5), or - which is more, a bundle of 10 toothpicks or a single toothpick (cell 7). Kate was able to answer both these questions immediately and correctly. However, the interesting part of this question deals with how much more the ten is than the one. For the dime and penny, Kate responded "ten." For the bundle of 10 and the single toothpick, she respond "nine." Apparently, Kate perceives the bundle of toothpicks (a bundle of 10 put together with a rubber band) as a set of 10 objects which can be broken down, while she perceives a dime as one "thing" that is not made up of 10 other "things."

Based on the second level question of the declarative plane (the "B" questions on the odd numbered problems in Appendix A), it is apparent that Kate has some fundamental misconceptions of the numeration system. Because she can syntactically decompose a number into its number of tens and ones does not mean that she understands the concepts of tens and ones. As was shown, she has some facility with these concepts (both the syntactic problem and her tendency to count by tens and then ones when representing numbers), but her understanding is not good enough to allow for transfer to other contexts. This misunderstanding or limited understanding of the concepts of ten and one might account for the flawed repair she made on the regrouping problems (problems 11, 16, 19 and 20) on the Van Lehn test. As will be shown, she has even more difficulty with the last problem type, termed regrouping, of the declarative plane.

As is shown in Table 1, Kate did not answer any of the regrouping questions correctly (1C, 3C, 5C and 7C). However, she came close to answering problems 1C and 7C correctly, but her response was contingent upon the researcher's interjections. For example:

E: Can you buy more things with 5 dimes or 45 pennies?
K: 5 dimes
E: OK, why is that?
K: Well, a penny isn't much and a dime is more than a penny.
E: But there is only 5 dimes and there are 45 pennies.
Long pause.
K: I think it would be 45 pennies.
E: And why is that, because there are more of them?
K: Yeah.

Granted the experimenter is an authority figure, but Kate's conviction in the relative values of a dime and a penny is shaken when the experimenter points out the quantity of coins.

The second case of obvious experimenter influence is in problem 7C.

1 E: Represent 32 toothpicks

Long pause (counting out 3 bundles and 2 single toothpicks)
2 K: Here's 32 toothpicks.
3 E: OK, now is there another way of doing that without using 3 bundles and 2 toothpicks? Is there a way you can have 32 toothpicks?
4 K: Without this much?
5 E: Without using these bundles in this pattern, is there a way of having 32 toothpicks?
6 E: NO? You have to have bundles here to have 32 toothpicks
7 K: You don't have to, but counting this you do
8 E: If I took these away, all these bundles, could you still have 32 toothpicks using these toothpicks?
9 K: (yes)
10 E: How?
11 K: Just take out 32.

There are 2 issues of experimenter influence in this question. First, the initial question should have been phrased, "Without using 3 bundles and 2 single toothpicks, is there a way of having 32 toothpicks?" In lines 6 and 7, she begins to understand the question, but in line 8 the question posed leads to an answer. Hence, there is the second experimenter influence. It should be noted, however, that a more proficient student easily understood and answered the question. Kate's answer was considered incorrect because the experimenter's question in line 8 led to an answer.

The interesting aspect of this result is that Kate does not possess the ability to be flexible in her representation. Once the problem is solved, it is solved and to think of an alternative solution is ludicrous. As a further example of this, Kate responds to question 5C (Appendix A) with similar steadfastness.

E: OK, next question, represent 34 cents.
Long pause (counting)
E: OK, what do you have there?
K: What do you mean?
E: How much money is that?
K: 34 cents
E: and how many dimes do you have?
K: 3
E: and how many pennies?
K: 4
E: Is there another way to have 34 cents without having 3 dimes and 4 pennies?
Long pause

K: I don't think so.

Here, Kate clearly showed her inflexibility to alternative representations of the same number (i.e., 2 dimes and 14 pennies). These problems are essentially regrouping problems and Kate, out of the context of the subtraction algorithm, has difficulty using her knowledge. In addition, it is apparent from her performance on the Van Lehn test that her knowledge of the regrouping process in a subtraction algorithm context is tentative.

The last problem in the declarative plane (question 3C) lends further support to the notion that Kate's knowledge of the arabic numeration system is purely syntactic. That is, in response to the question "what number has 5 tens and 16 ones?; she writes 5 in the tens place and 16 in the ones place. Furthermore, when asked what this number is, she responds, "five-hundred sixteen." She does not see this as a conflict and is actually committed to this value as the true and correct answer. This response fits in well with her response and my interpretation of her response to question 3B.

Procedural. There are 3 question types for the procedural plane: A) Subtract without regrouping; B) Subtract with regrouping; and, C) Subtract with regrouping into a zero. These represent the 3 levels of the procedural plane and the cells 2, 4, 6 and 8 provide the context for these problems. For Kate, cells 2 and 4 assess essentially the same knowledge (syntactic) and is indicated in her performance (see Table 1). For cell 2, Kate immediately converts the word problem into a column subtraction task. This process differentiates her performance between cells 2 and 4, but because she does this correctly, her net performance is the same. For the first two problems in cells 2 and 4, Kate applies the subtraction algorithm correctly, as indicated here:

E: 87 take away 59
K: I have to rename because you can't take 7 from 9
E: OK
K: and you take away one which is 7 and then this is 17

Long pause
E: How did you figure out 17 minus 9, how did you do that
Long pause
E: It's right but how did you figure it out?
K: Well, I counted down from 16
E: From 16?
K: (yes)
E: Why 16?
Long pause
E: Can you say it out loud how you counted it?
K: 16, 15, 14, 13, 12, 11, 10, 9, 8 and wrote it down
E: Did you increment to what?
K: What do you mean?
E: How many times did you go down to get to 8?
Long pause
E: In other words you counted backwards, do you count backwards until you get to 8 all the time?
K: (no)
E: How far back did you count
K: I counted to 8, no, I counted to 9.

There are 2 interesting aspects to this protocol. First, it is apparent that she has a fairly well developed algorithm to perform subtraction. Second, large differences require her to use a somewhat inefficient counting strategy. This inefficient counting strategy may account for the fact that Kate made 2 fact errors (mentioned earlier) on the Van Lehn test (although those fact errors were 12-9 = 4 and 8-6 = 3, not very large differences).

An interesting thing happens on both 2C and 4C. As noted earlier, Kate made 3 errors of the form 0-N = N on the Van Lehn test. On both 2C and 4C, Kate makes the error 0-N=0. Using Brown and Van Lehn's terminology, this error would be called an "unstable bug" and is due to the fact that Kate is unsure of what to do with a 0 in the minuend. Therefore, she makes "repairs" to her algorithm to fix this problem. Moreover, she is "stable" over a given task (is able to recall a "repair" made earlier on the task), but is "unstable" across tasks (is unable to recall a "repair" made on a task a few days earlier). It should be noted that this explanation is not as "clean" as it appears, in that, Kate did not exhibit the O-N=N on every problem of that form on the Van Lehn task.

Cells 6 and 8 (semantic informal and formal, respectively) required the use of manipulatives, dimes and pennies and bundles of 10 toothpicks and single toothpicks, respectively. Kate shows rather quickly that she has no process, nor can she develop one, for the solution of subtraction problems using manipulatives. For the problems 6A and 8A, Kate represents the minuend and takes away the subtrahend. However, in both problems, she is unsure of what the answer is and only decides after experimenter guidance. This procedure, of course, fails in the last two problem types because regrouping is necessary.

Kate receives only minimal guidance from the experimenter with problems 6B and 6C. In 6B, she apparently focuses on the differences in the tens place (i.e., 43 cents for a 39 cents item yields a dime for change). This idea is reinforced in Kate's solution to 6C:

E: I give you 80 cents and I'm going to buy the stapler
K: And this is 57 cents
E: (yes)
Long pause (counting out 3 dimes)
E: That's a fair deal?
K: (yes)
E: I give you 80 cents and I buy the stapler which is 57 cents and I get 30 cents back?
K: (yes)

The results of the interview for cells 6 and 8 may be misleading, because it shows only that Kate was inexperienced with manipulatives. In the money tasks (cell 6), the experimenter provided no guidance, but guidance was provided for the last set of problems (cell 8).

Kate had a tremendous amount of difficulty on the last set of problems involving the use of toothpicks, but it may be instructive to look at 8C. Question 8C is 50-13 which has Kate's troublesome zero in the minuend.

E: 50 take away 13

Long pause
E: How many toothpicks do you have on the table?
K: 50
E: 50?
K: 50 right here
E: and how many are you supposed to have?
K: 50
E: OK
K: take away 13
E: How many do you have on the table?
K: right here?
E: all in total
K: 60
E: and how many are you supposed to have?
K: 50
E: then why don't you make it 50 first
K: OK
Long pause
E: now how are you going to take away 13?
Long pause (counting)
E: that's 9 isn't it?
K: so, take away these
Long pause
K: I don't understand this
E: How are you going, how do you represent 13?
K: 10 and these 3
E: and you have 3 more
K: yeah
E: but that wouldn't be 50 anymore, if you included those three in there wouldn't be 50 anymore? How many would it be?
K: 60 53
E: 53, so you can't put those 3 in there. Is there a way of representing 13 using bundles. How?
K: put, take 3 of these (bundles)
E: 3 of those? How much is 3 of those?
K: Um, I don't think there is a way
E: There isn't a way using bundles? OK
E: how many do you have left
K: Here?
E: No that's 13
K: 13 of 40
E: OK very good, how many tens are there in 13
K: 1
E: 1, so isn't that 1 ten and 3 ones?
K: yes
E: how many is that
K: 13
E: can you represent 13 using bundles?
K: Yes

The experimenter eventually shows her how she can do the problem using toothpicks. It is important to note that what knowledge Kate has for subtraction is syntactic notwithstanding the fact that she does not have much experience with manipulations, she is unable to apply her syntactic knowledge to a concrete or semantic task. Granted, it is not so important that we teach our children to work with bundles of toothpicks, but it is a fairly important task to be able to manage money. The knowledge that Kate has acquired in school is, at the time of interview, not used in her "real" life experiences. As will be described later, this is not the case for a child who is considered proficient in mathematics (see analysis of Ted).

Jan

Jan's parents both worked. Her mother worked in production and her father worked as a meat cutter. Her mother completed the tenth grade and her father the ninth. These data imply that Jan comes from a lower class family.

Jan completed the Van Lehn test with 13 errors out of 20 problems. 12 of the 13 errors showed the error pattern borrow-do-not-decrement from the last column. In addition, there were 3 fact errors, 2 cases of treating a one like a zero and one case where she skips over the first zero when regrouping into the unit's place. These additional errors were in combination with the 12 borrow-do-not-decrement errors, as well as the one error where she incremented instead of decrementing the digit. Generally, Jan's error patterns show a misconception of place value or declarative knowledge, contrary to the protocol results.

Jan's performance on the interview tasks is summarized in Table 2. As indicated in Table 2, Jan performed best on declarative concepts in the semantic cells (5 and 7) and performed best on procedural tasks in the formal/syntactic cell (4). Based on this cursory analysis, it would seem that Jan's semantic/declarative knowledge could be enhanced to alleviate her subtraction errors (i.e., on the Van Lehn test).

Table 2. Results of the diagnostic interview for Jan.

	Declarative		Procedural	
	Syntactic	Semantic	Syntactic	Semantic
Formal	3a √	7a √	4a √	8a X
	b XX	b √√	b √	b X
	c X	c X	c √	c X
Informal	1a √	5a √	2a X	6a √
	b XX	b √X	b X	b X
	c √	c √	c X	c X

Declarative. As with Kate, Jan is competent in representing a number in any context. Also, Jan uses an efficient counting strategy that makes use of tens. However, her true ability with counting strategies became more apparent on more difficult tasks.

Just like Kate, Jan is able to determine that a ten is greater than a one with both coins and toothpicks (5B and 7B, respectively). Jan, also, knows that a bundle of 10 toothpicks is 9 more than a single toothpick, but thinks that a dime is 10 more than a penny. This result parallels that of Kate. However, unlike Kate, Jan is unable to determine the number of tens and ones from the presentation of an arabic number (question 3B). This would indicate that Jan has somewhat less knowledge of the syntax of arithmetic.

E: OK, how many tens are there in the number 52
J: 52 how many tens?
E: (yes)
J: maybe 2
E: 2?
J: Yeah
E: OK, if there are 2 tens in 52 how many ones are there?
J: I don't know, there would be a lot of ones in 52, probably 10

What is apparent in this excerpt of the protocol is that Jan has minimal knowledge of the syntax of a number and that she is willing to guess. It would also appear that she may have guessed 2 as the number of tens because of the 2 in the unit's place. It is difficult to tell from this transcript, but apparently she does not know how to determine the answer to the question.

The excerpt from problem 1B shows that she does a certain amount of guessing when it comes to determining the number of tens and ones.

E: I have a bunch of change in my pocket and its just dimes and pennies and I have a total of 76 cents, how many dimes do you think that is?
J: 76 cents?
E: (yes)
(counting)
J: I think maybe 6
E: 6 dimes, if there are 6 dimes and there is a total of 76 cents how many pennies is that?
Long pause (subvocal)
E: 2 pennies?
J: I think so

This result is even more interesting because the preceding question, 1A, asks how much money is 2 dimes and 4 pennies? She quickly determines, via a counting strategy, the value 24 cents. Her ability to determine a value from a given number of dimes and pennies has no inverse operation for determining a number of dimes and pennies from a value. This is apparently due to the fact that she used a counting strategy to determine the value of 24 cents. The inverse counting strategy would be quite difficult to perform. Hence, she had no way of making use of her strategy in question 1A to find a solution to 1B.

As shown in Table 2, it was determined that Jan got 1C and 5C correct. Some interesting things transpired in these problems. In 1C, Jan seems to focus on the relative values of a dime and a penny and, after questioning by the experimenter, verifies her initial response by counting.

E: Can you buy more things with 5 dimes or 45 pennies?
J: 5 dimes
E: and why is that
J: because 5 dimes is more than 40 pennies, cause pennies is worth only one cent
E: (yes) and so how much money is that?
J: 4 pennies?
E: 45 pennies
J: 45
E: (yes) and how much money is that?
J: 5 dimes is (counting) 50 cents
E: how did you figure that out? I hear you kind of mumbling something over there.
J: I counted up
E: how did you count? I could have sworn I heard 10 20 is that how you did it?
J: (yes)
E: you counted 10 20 30 40 50
J: (yes)

Jan seems to be satisfied with her response after verification by counting. However, her counting verification procedure is time consuming as well as not allowing for additional knowledge to intervene. This is apparent in 5C.

E: represent 34 cents
J: 34, 34, 10 20 30...31 32 33 34
E: How many dimes do you have there
J: 3
E: how many pennies?
J: 4
E: is there a way to represent 34 cents without having 3 dimes and 4 pennies is there a different way you can have 34 cents
J: count out a whole bunch of pennies
E: how many pennies? How many pennies would you need to have 34 cents?
J: 1 2 3 4 ...34 (counts cents)
E: how many pennies are there?
J: 1 2 3 4 ...34 (counts coins)
E: how many pennies are there?
J: 34

Here, Jan shows her facility with counting by tens, then ones, to represent a number. She also knows that you can have 34 "cents" all in pennies. The interesting thing is that Jan does not have a one to one correspondence between the number of cents and the number of coins. Counting a group of objects is a tedious task, but Jan feels impelled to count both the number of cents (to verify the value) and the number of coins (to verify the quantity). Although it was determined that Jan got questions 1C and 5C correct, it is interesting to note the inefficient strategies she used to determine these answers.

In question 3C, Jan performs somewhat more poorly. For this question, Jan makes use of her limited syntactical knowledge, as well as her knowledge of the value of tens and ones. What number has 5 tens and 16 ones was determined by writing the value of 16 ones in the unit's place (i.e. 16). She then writes the value of 5 dimes (i.e. 50) in the tens place, yielding the number 5016. Her ability to estimate approximate values, if she has that ability, obviously does not play a role in her solution to this problem.

The last question in this plane may have been affected by Jan's inability to recall that the bundles of toothpicks contained 10 toothpicks. What is not apparent in the following excerpt is the fact that Jan begins to include bundles of toothpicks after she counts to 16 and counts by ones in clusters of five. That is, she considers each bundle as consisting of five toothpicks. Therefore, when you see 16...21, 22...26...32, what Jan is doing is counting more rapidly from 16 to 21 and including a bundle in the pile of toothpicks that, in the end, she calls 32 (the actual number of toothpicks is actually 3 bundles and 17 singles, or 47 toothpicks).

E: Alright show me 32, represent 32 toothpicks
J: 32
E: (yes)
J: 1 2 3 4 5, ... 1 2 3 4 5 6 7 8 ... 16-21, 21-26, 26-31, 32
E: OK, is there another way you can represent 32?
J: Yea you could use all them round toothpicks, its kind of an easier way so you don't have to put out 32
E: OK, and how many are in each of those bundles?
J: 10

There are two other interesting aspects to this transcript. First, she realizes that there is an easier way to represent 32 (i.e. 3 bundles and 2 singles). Why she did not elect to use this strategy first is unclear. The second point is that the experimenter asked at the end of the transcript how many toothpicks there are in each bundle, and she knows there are 10. Why she counted in increments of 5 is unclear, as well.

Jan uses some unconventional strategies to determine answers to questions in the declarative plane, but her declarative knowledge is not devoid. Therefore, it would appear that using this knowledge (as well as extending it) may serve to enhance her skills in using the subtraction algorithm.

Procedural. In Table 2, it would appear that Jan's richest context for using a subtraction algorithm is in the formal/syntactic cell (cell 4). In fact, of the other 9 questions in this plane, Jan was correct only once. Therefore, her purely syntactic algorithm cannot be either helped by other algorithms, nor (because of the weakness of the syntactic algorithm) can it be transferred to other contexts.

Because Jan performed best in the formal/syntactic cell (cell 4), an examination of the algorithm she uses (which is probably the algorithm she has learned in school) might be an appropriate starting point. First, it should be pointed out that the interview questions were limited to 2-digit numbers and the error pattern that Jan exhibited on the Van Lehn test requires at least 3-digits. Therefore, in the interview, Jan's error pattern was not elicited. It is instructive, however, to see Jan's verbalization of her algorithm, as shown here in problem 4B.

E: 87 take away 59
J: 87 take away fifty, here you're going to have to cross out, we are going to have to make that a 17, and then you say 9 take away 17 which would be (counting)
E: If you want to count, go ahead and count out loud
J: 9 10 11, 12 13 14 15 16 ... 8
E: OK

J: OK, and then you are going to cross out that and you are going to make it a 7. 1 less than that, ok, and 7 take away 5 is 2, 28

Here we see that Jan's algorithm is fairly well developed. We see that (other than the condition at J, Figure 2) Jan's algorithm is analogous to that which is depicted as a flowchart in Figure 2. Another point to be made about this excerpt is that Jan needed to use a counting strategy to construct the fact 17-9 = 8. The fact that she needed to use this somewhat "inefficient" strategy may account for the result that there were 2 fact errors on the Van Lehn test (18-9=8 and 10-8=7).

Jan has a well developed algorithm for determining solutions to 2-digit subtraction problems. Why then does she do so poorly in the other cells (2, 6 and 8) of the interview? To explain this, we need to examine each of these cells. In cell 2, informal/syntactic, the questions are essentially money, word problems. In order to answer these questions using her algorithm, Jan needs to convert the questions into a column format and apply the appropriate operation. As we shall see, she does convert the question into a column format and applies an operation, but not appropriately. For example, in 2A Jan writes the problem in column format correctly, but selects the wrong operation.

1 E: ... a gum ball costs 10 cents and you give the clerk a quarter which is 25 cents, how much change do you get back?
2 J: You give the clerk a quarter
3 E: which is 25 cents
4 J: 25 cents and you give
5 E: and the gum ball costs 10 cents
Long pause
6 J: 30 cents
7 E: 30 cents, how did you figure that out?
8 J: Well I was sort of saying like if say 5 plus 0 is 5 and 2 and 1 is 3
9 E: (yes) and that is what?
10 J: 30 cents

Not only does Jan select addition as the operation, but when she performs the operation, she does so incorrectly. That is, in line 8 she adds the units and then the tens, but forgets about the units when stating the result. Subtraction is obviously the correct operation for this problem and what makes her selection of addition as the operation more perplexing is the fact that she could have inferred that it was subtraction, since that was the purpose of the study. Another point to be made about this problem, as well as 2B and 2C, is that the amount of change she determines from the problem is more than the amount given (i.e. she gives the clerk 25 cents and gets 30 cents back). This discrepancy is never considered in the task. It appears that Jan's goal is to convert the words of the problem into a task that she can solve using her formal/syntactic knowledge. She goes from natural language syntax to arithmetic syntax without considering the semantics of the verbal problem. This is important to note here, because the experimenter goes on to the next problem and Jan assumes her procedure is correct and uses it on problems 2B and 2C.

There is a slight difference between how Jan solves 2A and how she solves both 2B and 2C. In 2A, Jan writes the problem in column form with 25 on the top and 10 on the bottom. However, in 2B and 2C where her solutions were $1.15 and $1.17, respectively, Jan writes what would be the minuend on the bottom. Therefore, if she had selected the correct operation, she would have had problems.

The remaining two cells (6 and 8) are procedural tasks involving manipulations. As will become apparent, Jan is unable to either explain her syntactic algorithm using manipulatives, nor use manipulatives to do subtraction.

The role playing activity, wherein Jan plays a clerk in a store and the experimenter plays a young child that has come into the store to make a purchase, is the task domain of cell 6. In the first problem, Jan is apparently willing to make a guess.

1 E: I just reach into my pocket and I hand you all the money that I have,

which happens to be 6 dimes which is 60 cents and 4 pennies which is 64 cents, a total of 64 cents. I hand you 64 cents and I want to buy that pen which is 52 cents, how much change do I get back.

2 J: you give me
3 E: 64 cents
4 J: 64 cents
5 E: 64 cents and the pen costs 52 cents
6 J: ok, so 4 pennies back
7 E: 4 pennies back, well here's 64 cents and it costs 52 cents, you have to make change.
8 J: so you are only supposed, let me see, 64 cents
9 E: That's what I gave you
10 J: OK, the pen only costs 52 cents
11 E: right, so I get some change back, how much
12 J: OK, (counting) so get 2 pennies and a dime back
13 E: OK, very good, and thats how much money
14 J: (counting) 12 cents

In line 7, the experimenter makes it clear that 4 cents is not an appropriate response. After the initial (apparent) guess and the correction in line 7, Jan reanalyzes the problem and finds the correct solution. Without the intervention, it is unclear whether Jan would have attempted another solution strategy or not. The strategy she uses is to count out the price of the pen from the coins that were given to her by the experimenter. This strategy will, of course, fail in the other problems (6B and 6C) in this cell. As a conjecture, her solution strategy at first seems to return the 4 pennies of the 64 cents that is handed to her. She knows that 64 cents is too much and returns the 4 pennies. She cannot, however, resolve the discrepancy between the price (52 cents) and the remaining amount of money (6 dimes or 60 cents). She does not think to use the cash register to resolve the conflict and initially accepts her 4 cent response as the correct response.

In 6A, Jan finds a correct solution strategy and also knows an incorrect solution strategy. This is important to note because the correct solution strategy does not work in 6B and the incorrect strategy yields an answer that is very close to the correct answer.

1 E: OK, next one, this time I am going to buy the key chain, how much is the key chain?
2 J: 39 cents
3 E: 39 cents, I again hand you all the money in my pocket which happens to be 4 dimes and 3 pennies which is 43 cents, how much change do I get back?
4 J: 43 (counting)
5 Long pause
6 J: 6 pennies
7 E: 6 pennies, OK, how did you figure that out?
8 J: Oh, you give me
9 E: 43 cents
10 J: 43 cents and I give you back 6 pennies
11 E: And how did you get that 6
12 J: Because I give you back 3 pennies and 2 more it would be the right amount of change

The long pause at line 5 (above) is where Jan determines her solution. In line 14, she gives a *post hoc* and unclear explanation of her solution. Because she does not verbalize her solution strategy while she is solving the problem and she gives an ambiguous explanation of her solution, it is necessary to analyze her solution strategies to the previous problem (6A).

The strategy that worked for Jan in 6A was to count out the price of the item from coins presented for its purchase. However, in 6B, it is impossible to count out 39 cents from 4 dimes and 3 pennies. She also knows that returning the extra pennies was not the appropriate strategy in 6A, so it probably would not work in 6B. There is the possibility that she made use of her syntactic/procedural knowledge in this problem. As described above, in problems 2B and 2C, Jan wrote the subtrahend in the top and the minuend in the bottom. Also, by the time she gets to problem 6B, she realized that this interview was related to subtraction. Therefore, she transforms this concrete task into 39-43 and determines the result of 6 pennies (9-3=6) without resolving the tens place. This explanation makes more sense than Jan's explanation in line 14 (above), but it does not resolve the problem of how she derived her explanation.

There are 8 lines of dialogue between line 5 (where she determines the result) and line 14 (where she explains her solution). By that time, much of her solution strategy would have been lost from short term memory. Therefore, her explanation of her solution strategy may have little to do with her actual solution strategy.

In 6C, Jan shows the strategy that she had used to correctly solve 6A. That is, she attempts to count out 57 cents from the 8 dimes that she was given for the purchase. She realizes (in line 6) that the extra 2 dimes are not needed, but there is a discrepancy between 60 and 57 cents. To resolve this conflict, she puts a penny with the 2 dimes and declares 30 cents as the amount of change (even though the actual coins are 2 dimes and 1 penny).

1 E: OK, once again I am the little kid and I hand you all the money that I got for that stapler, how much is the stapler?
2 J: 57 cents
3 E: 57 cents, this time I give you 1 2 3 4 5 6 7 8 dimes which is 80 cents, how much change do I get back
4 J: OK, let me see here, can I count out the 57 cents
5 E: (yes)
6 J: 10 20 30 40 50 ... 50 60 70 80, OK, 50 cents and one more makes 60, 80 OK, so you get back a penny ... 30 cents
7 E: How much is that?
8 J: 10 20 21 cents
9 E: 21 cents, OK, now how did you figure that out?
10 J: Well, because there is 80 cents, OK, and you only needed to give me a penny to make 57, so I give you back a penny and 2 dimes

Figure 5 was developed to show a variety of knowledge structures that might be accessed to help children understand subtraction and help diagnosticians determine how well children understand subtraction. It is obvious from Jan's protocol that assessing her knowledge of money will not aid her understanding of subtraction on a test (e.g. Van Lehn's test) because she has more syntactic knowledge than informal/semantic knowledge. However, we do gain information about how well Jan understands subtraction because she is unable to transfer her procedural knowledge to a different context.

In cell 8 (formal/semantic/procedural), the experimenter makes use of groups of toothpicks because Jan's teacher would use them from time to time in class. However, Jan had never used them. This fact may account for the result that she did not get any of these problems correct. It should be noted, however, that the experimenter initially gave Jan the option of using toothpicks or using paper and pencil to solve these problems. Jan elected paper and pencil as the context for determining the solution to each problem. She found the correct solution, using this strategy, to 8A and 8B. She makes a fact error (4-1=4) on 8C. After Jan found the paper and pencil solution to each problem, the experimenter told Jan to teach him how to find her solutions using toothpicks. This was where Jan exhibited a lack of knowledge in this context.

The first problem in this cell, based on the transcription, appears to be solved correctly. However, what actually transpires is that Jan removes 1 toothpick for the one in 21 and removes 2 toothpicks for the 2 in 21. That is, she takes the tens of the subtrahend from the ones of the minuend.

1 E: 47 take away 21
2 J: 7 take away 1 is 6, 4 take away 2 is 2
3 E: OK, now let's pretend like I'm a first grader and show me how you would do that problem using toothpicks.
4 J: Well, you would
5 E: teach me with the toothpicks
6 J: OK, you need to get our 47 toothpicks, 10 20 30 31 wait let's see here there are 10 in each 10 20 30 40 41 ... 47 OK, now you would take 2 from 4 so it would leave ... 4, take, take 2 from 4 leaves 2
7 E: and so where is the 26
8 J: over ... I think its over here (pointing at the paper solution)

She states the correct solution, 26, but this is found on the paper solution. In fact, when the experimenter asks Jan to show him where the 26 is, Jan points at the paper solution (lines 7

and 8). The actual toothpicks representation of the solution is 44, 4 tens and 4 ones.

Because Jan has so many problems with concrete representations, the experimenter uses a questioning technique to get Jan through the solution of 8B. In addition, because the intent of this study is to examine what each child knows about subtraction, it would be ludicrous to accept Jan's solution to 8B as being correct. Therefore, let it suffice to say that 8B was used as an instructional problem to see how this instruction would effect Jan's solution to 8C. The instruction attempted to show Jan the correspondence between her paper solution and process and her toothpicks solution and process. However, she does not use this newly "learned" solution process for problem 8C.

As noted earlier, Jan's paper solution to problem 8C has fact error (4-1=4). Therefore, her paper solution to 8C is 50-13=47. The lack of knowledge that Jan has for the correspondence between the solution on paper and the solution from the toothpicks is again shown in her solution to problem 8C.

E: Once again I want you to teach me and its going to be 50 take away 13.
J: OK, ok, 50 take away 13 ok 10...50, now you are going to have to take away 13, I think we are going to have to trade for these because you are going to have to take 1 ... 50. 50 toothpicks. OK and you have to take away OK, you cross out at the top and make that a 10, so you say 10 from 3. OK, let me count them 1 ... 10 ok, you count out 10 toothpicks and you are going to take 3 from it, that would leave 1 ... 7. 7 ok you are going over here and hmmmmm. 4 so you just have 4 left. Take away 4 and you still have all these left.
E: OK, and how much is all those?
J: (counting) 44

Here Jan shows that she has learned something from the instruction given on problem 8B. She initially counts out the minuend in bundles of ten. She realizes that a trade is necessary, but instead of trading 1 ten for 10 ones, Jan trades all 5 tens of the minuend for 50 ones. She then returns to the paper solution and notices the regrouping in the ones place has resulted in the condition 10-3. Therefore, Jan counts out 10 toothpicks from her pile of 50 and performs this calculation correctly. She then re-examines her paper solution and focuses on the tens place. However, she focused on the 4 tens in the minuend rather than the 1 ten in the subtrahend. She also forgets (possibly because her toothpicks representation is in ones) that the numbers in the tens place are representations of ten. She then removes 4 toothpicks for the 4 in the tens place of the minuend. Not withstanding her error for mistaking the 4 in the minuend for the 1 in the subtrahend, this solution process is analogous to Jan's solution to 8A. In addition, Jan makes a counting error at the end of the problem and counts the 43 remaining toothpicks as 44.

Throughout Jan's solution to 8C, she makes decisions and follows procedures based on her paper solution. However, Jan counts out her toothpick solution and states the result as 44. Her corresponding incorrect paper solution was written as 47. She never seemed to recognize this conflict.

As with Kate, Jan is unable to use her syntactic algorithm in different contexts. This inability to transfer her knowledge from context to context has two major implications. First, it indicates that Jan does not have a "strong" algorithm. That is, she has not had enough experience with the subtraction algorithm so that knowledge from the syntactic domain can be utilized in the semantic domain. Second, Jan's knowledge and experience from the semantic domain is so weak that she is unable to utilize this knowledge to aid the development and understanding of her syntactic knowledge. It may be appropriate for Jan's parents to allow her to gain more experience outside of school with money, and for Jan's teacher to work more extensively to develop her syntactic algorithm in conjunction with her increasing semantic knowledge.

Fred

Fred's parents both have some college experience (Mother, one year of college; father, two years of college). In addition, both of his parents work (mother, secretary; father, section chief). Based on these data, Fred's family is characterized as lower middle class.

Fred completed the Van Lehn test with twelve errors. These errors were not consistent in that he exhibited five different error patterns and did not exhibit the five error patterns in every problem that the error pattern could have been exhibited. His error patterns were: four (0-N=N), four (borrow from the left most column when borrowing into the unit's place and rewrite unit's place as 10 regardless of the original unit's place value), two (smaller-from-larger), one (fact error) and one (alignment, borrow from left most column when borrowing into unit's place and add one to left most place so you do not have to write a leading zero in the solution). In comparison, Fred's errors are more like Kate's than Jan's. That is, Kate had three (O-N=N) errors and Fred had four. In addition, as part of a compound error pattern, Kate had three errors of the form borrow from left most column and Fred had five errors where this error pattern was part of the faculty solution process. If we group like error patterns (0-N=N is also smaller-from-larger), the result is six errors that are smaller-from-larger and five errors where the regrouping process is performed incorrectly. This regrouping misconception was not brought out in the protocol data.

Fred's performance on the interview portion of the study is summarized in Table 3. As is indicated, Fred has a good understanding of declarative knowledge, only answering one question (5C) incorrectly out of a total of twelve questions. However, his declarative knowledge does not seem to influence his ability to perform subtraction. That is, except for cell 6 (working with dimes and pennies), Fred makes some errors in his solutions to problems that assess his procedural knowledge (in total, he makes five errors out of the twelve problems on the procedural dimension).

Table 3. Results of the diagnostic interview for Fred.

	Declarative		Procedural	
	Syntactic	Semantic	Syntactic	Semantic
Formal	3a √	7a √	4a √	8a √
	b √√	b √√	b √	b X
	c √	c √	c X	c X
Informal	1a √	5a √	2a √	6a √
	b √√	b √√	b X	b √
	c √	c X	c X	c √

Declarative . The first question in the declarative plane deals with representation of a number and Fred had no difficulty with these problems. Like Kate and Jan, Fred exhibits a tendency to make use of groupings of ten. That is, on the semantic tasks, Fred counts out the tens and ones. This indicates a mature counting strategy and an understanding of the structure of the base ten system.

Fred, unlike Kate and Jan, exhibited no difficulty on the second problem type in the declarative plane. Fred cannot only pull apart the syntax of a number (problems 1b and 3b), but understands that a representation of ten is made up of ten units. Hence, his ability to not only know that 10 is greater than 1, but that 10 is 9 more than 1. This second part of the problem (in 5b and 7b) proved to be difficult for Jan and Kate.

The final question type in this plane proved to be challenging for Fred. However, the difficulty may have arisen from the ambiguity in which the question is phrased.

1 E: OK, if you were to represent 34 cents how would you represent it
2 F: 3 dimes, 3 dimes, 4 pennies
3 E: You'd have 3 dimes and 4 pennies. Is there a way using dimes and pennies to also have 34 cents but not have 3 dimes and 4 pennies
4 F: What.
5 E: Can you have 34 cents in a different combination other than 3 dimes and four pennies
6 F: Um,
7 E: In other words, you have 3 dimes and 4 pennies there, that's 34 cents.

8 F: Yea.
9 E: Is there a way that you can have 34 cents a different way?
10 F: (no)
11 E: No?
12 F: Wait, let's see. No.

Line 3, where the question is first phrased, seems to indicate a clearly stated problem (i.e. it parallels the phrasing for other subjects who answered 5C correctly). In addition, the phrasing is similar to that of problem 7C (which Fred answered correctly). If we assume that Fred understands the question, then what occurs here is that Fred becomes locked into one specific representation. As was indicated above, Fred used a counting strategy where he counted out the tens then the ones. In this problem, he is unable to break down the components of the number to use an alternative representation (say 34 pennies, a somewhat less mature counting strategy).

Fred performed better than Kate and Jan on the declarative tasks. However, as indicated in Fred's performance on the Van Lehn test, Fred has difficulty with subtraction problems. We will now examine Fred's performance on the procedural plane keeping in mind his knowledge of declarative concepts.

Procedural. Fred is able to solve all three problems in cell 6 (semantic/informal), but has problems in the other three cells in this plane. The error that Fred makes in cell 2 (syntactic/informal) consists of writing the subtrahend on the top and performing the smaller-from-larger error pattern. In problem 2A, Fred apparently solves the problem in his head, but because problems 2B and 2C require borrowing, Fred writes the problems down (therefore, accounting for the fact that he answered 2A correctly). He does not attempt to determine if the written form of the problem makes sense, instead he writes the subtrahend on the top in 2B and continues the faulty process in 2C. Because Fred had made two smaller-from-larger errors on the Van Lehn test, he had solved problems in this manner before. Therefore, he did not perceive an erroneous conflict.

In cell 4 (syntactic/formal), Fred incorrectly solves one of the problems (problem 4C). In this solution, he makes use of his aforementioned error pattern, 0-N=N. This would indicate that this error pattern has some consistency across tasks for Fred. The other error patterns that Fred had exhibited on the Van Lehn test did not arise in the interview because the problems were limited to two-digits.

Fred correctly answers all three questions in cell 6 using a counting-up strategy. Apparently, Fred can make use of this strategy given the coins, whereas he makes use of an erroneous algorithm when given similar problems which are stated verbally (cell 2). In the problems in cell 6, Fred begins with the price of the item to be purchased and counts out change until he gets to the amount tendered. This strategy, although inefficient, allows Fred to correctly solve each of the problems in cell 6. However, he does not follow this strategy when he uses toothpicks instead of coins.

In cell 8 (semantic/formal), Fred correctly answers the first question (problem 8A) by counting out the minuend (47 toothpicks), then removing the subtrahend (21 toothpicks) from the pile of 47. The more interesting results come from the remaining two problems (8B and 8C). In problem 8B, Fred finds a solution that would indicate a smaller-from-larger strategy, but the dialogue indicates that it was not simply a smaller-from-larger solution.

E: 87 take away 59
F: 87, what is it 59?
E: (yes)
F: OK, take away, wait, you have to regroup
E: OK, how do you regroup using toothpicks?
F: Uh, you take a 10 and 7, wait, ok, you take 10, ah minus 9. 9, 9 will be 2. So it would be 2, 8 minus 5, 8 minus 5 would be 3, 32. Yikes. 32.

Here we see that Fred actually attempts a regrouping process, albeit mentally.

He borrows without decrementing and treats the 17-9 problem like a 7-9 problem with a smaller-from-larger strategy. This result is quite interesting because in a strict error pattern analysis the solution would be categorized as smaller-from-larger. This type of error indicates a fairly immature solution strategy because there is no attempt to regroup. However, as indicated above, regrouping was attempted and the result of an error pattern analysis would be misleading.

The result of 8C is interesting for two reasons: 1) Fred does not exhibit the 0-N=N error pattern; and, 2) Fred makes a borrow-do-not-decrement error similar to 8B.

E: OK. The last one. 50 take away 13.
F: 2, let's regroup.
Long Pause
F: minus three. Can't be. OK, I have right here. Five minus 13. OK
E: 50 take away 13.
F: Oh, darn, ok, let's, mean to take away 10, whether it would be 13 or (unclear) 10 minus 3 is 7 so I, 7 would be, it would be 47.

Here, Fred apparently borrows a ten so he can perform subtraction in the unit's place. He correctly calculates that result as 7. However, he forgets to decrement the 5 in the subtrahend's ten's place and incorrectly states the result as 47.

Fred seems to have a strong understanding of place value, but violates these concepts when he attempts to solve subtraction problems. In addition, when the context of subtraction changes, Fred determines a solution strategy for that context. He basically uses a different method for each of the four cells that were studied. The only successful strategy (cell 6, semantic/informal) happened to be the most immature strategy. It may be appropriate for Fred's teacher to step through a subtraction problem with Fred and access his place value knowledge when he attempts to use an erroneous solution strategy.

<u>Ann</u>

Ann comes from a single parent household. Her mother, the parent, is a "housewife" with a tenth grade education. Based on this data, Ann's socio-economic status is characterized as lower class.

Table 4. Results of the diagnostic interview for Ann.

	Declarative		Procedural	
	Syntactic	Semantic	Syntactic	Semantic
Formal	3a √	7a √	4a √	8a X
	b √ √	b √ √	b √	b X
	c √	c √	c √	c X
Informal	1a √	5a √	2a X	6a √
	b √ √	b √ √	b X	b √
	c X	c √	c X	c √

Ann completed the Van Lehn test with twelve errors. The breakdown of these errors are: 4 borrow-do-not-decrement; 4 borrow-do-not-decrement, borrow unnecessarily and confusion when difference is greater than 9; 2 borrow unnecessarily; 1 borrow-do-not-decrement and decrement wrong position; and, 1 borrow-do-not-decrement and borrow unnecessarily. It would appear that there is some consistency in the error patterns that Ann uses in the Van Lehn test, but again, like the other participants, Ann does not exhibit the error patterns on every problem where the error should have occurred. Generally, Ann's error patterns indicate a misunderstanding of place value. This misunderstanding is indicated in the difficulty she has with regrouping, and yet Ann correctly complete four problems that required regrouping. Ann's understanding of place value might best be described as tentative.

Ann's performance on the protocol activities is summarized in Table 4. As indicated in the table, Ann performed quite well on the declarative questions in the interview, contrary to the apparent cause of her errors on the Van Lehn test. On the procedural portion of the interview, Ann did well on problems from cells 4 and 6 (syntactic/formal and semantic/informal, respectively), but could not correctly solve problems from cells 2 and 8 (syntactic/formal and semantic/formal, respectively). This may be the result of her experience with doing problems with paper and pencil in class and using money at home. In addition, her inability to do verbally presented money problems and concrete problems using bundles of toothpicks may be a direct result of her inexperience with these types of problems.

Declarative. Ann answered all declarative questions, except 1C, correctly. The source of her regrouping difficulties may be explained in terms of how she answered some of these questions. For example, in problem 1B Ann quickly responds that the number of dimes in the researchers pocket is seven and that the number of pennies is six. This appears to be a purely syntactic analysis. In fact, it would appear that Ann's knowledge of syntax is aided by her knowledge of ten and one. This is indicated by her almost automatic response of 66 to problem 3C (what number has 5 tens and 16 ones).

Ann's problems with regrouping may be indicated by her responses to questions 5C and 7C. These two questions are parallel and Ann's responses are mathematically identical.

E: Alright, represent 34 cents - OK - now you have how many dimes here?
A: 3
E: And how many pennies
A: 4
E: Is there another way you could have 34 cents and not have 3 dimes and 4 pennies. Can you have 34 cents some other way?
A: Yes
E: How
A: a quarter and
E: Only with dimes and pennies
A: 3, 4 - 34 dimes not dimes 34 pennies
E: Is there another way
A: No

In this response, and Ann's response to 7C, we see that Ann thinks that there are 2 possible ways to represent a number, a collection of its units and its corresponding tens and ones. She is unable to spontaneously think of 34 as 2 tens and 14 ones, for example. However, she can spontaneously know that 5 tens and 16 ones is 66 (problem 3C). Therefore, she can put parts together to make a number, but she is unable to break a number up into its parts (a necessary process for the typical subtraction algorithm).

Ann's response to 1C complicates this analysis and provides more evidence for the tentative nature of Ann's knowledge of regrouping. Ann's response, that you can buy more things with 45 pennies than 5 dimes, indicates that Ann focused only on the quantity of coins rather than the coins relative value. This is particularly perplexing, given that Ann answered 3C correctly, while Jan and Kate, who had more difficulty with declarative questions, answered 1C correctly and 3C incorrectly. One explanation for this discrepancy is that Ann's protocol for the first two questions is very short, and therefore she may not have taken enough time to fully process the question.

Procedural. In the procedural plane, Ann reveals that her formal algorithm is useless in other contexts. She performs the subtraction problems in cell 4 (syntactic/formal) correctly on paper, but she does not use this algorithm in other contexts. She correctly answers the questions in cell 6 (semantic/informal) using a counting up strategy. In cells 2 and 8 (syntactic/informal and semantic/formal, respectively), Ann becomes thoroughly confused and does not know what to do. This inability to use her procedural knowledge in other contexts is a reflection of of what limited usefulness this skill is in her mind.

As described above, it would appear that Ann initially perceived the protocol tasks as being easy, reflected by how fast she answered the first few questions in the task (hence, the brevity of the protocol). This might explain her response to question 2A, that the change for a 10 cents purchase, given that the clerk receives a quarter,is 5 cents. This is the difference in the ones place,and Ann appears to forget about the tens place. The increase in the length of the protocol, and therefore the increase in processing time begins with problem 2B.

E: A candy bar costs 39 cents, you give the clerk 3 quarters which is 75 cents, how much change do you get back?
A: 15, 25 cents or 20 I don't know
E: 25 cents?
E: Would it help to write it down? No. The candy bar costs 39 cents and you give him 75 cents.
A: 50?
E: You are the one that's in charge, you tell me. How did you figure out 50? (recommends writing again)
A: 67 cents
E: 67 cents. OK. The next question.

Because regrouping is necessary for the solution of this problem, Ann slows down and gets a little confused. Once she decides that writing the problem on paper would help, she inverts the subtrahend and minuend. To handle the problem of a subtrahend being less than the minuend, Ann makes a borrow from some unknown quantity, thereby violating the conservation of value that is so important for proper regrouping. There appears to be no connection between Ann's declarative knowledge and her procedural knowledge. In addition, her limited knowledge of the regrouping process, and the reason for its existence, is exhibited in this problem.

Because the questions in cell 2 were designed to assess informal knowledge, question 2C was rephrased in an attempt to ellicit that knowledge. This rephrasing apparently did allow Ann to access this knowledge.

E: OK. next question. A slurpy costs 47 cents and you give the clerk 2 quarters and two dimes which is 70 cents, how much money do you get back?
A: 20 (focuses on dimes)
E: 20. OK I have another question for you. If you were buying a slurpy

and you had 2 quarters and 2 dimes how much would you really give them, give the clerk? It cost 47 cents.

A: 2 quarters

E: 2 quarters - why?

A: Because - was it 47 - there's tax and you give 50 and

E: No, tax is included - 47 cents. So if you gave them 50, and the total cost is 47 cents, how much change do you get back if you give them 50 cents.

A: 3

E: 3 cents - ok. So if you gave them 70 cents instead of 50 cents, how much more is 70 than 50?

A: 33

In fact, she brings to the problem the knowledge that there ought to be sales tax on the purchase. There are two important points to make about Ann's result. First, she apparently makes an addition or subtraction error when she states the result as 33 rather than 23. And second, Ann does not make use of her formal algorithm in this context. In fact, in problem 2B, she resists the researcher's suggestion that she write the problem down. In problem 2C (under the guidance of the researcher), Ann solves the simpler problem of 50-47 by counting up. She then solves the problem 70-50 (apparently as 30) and adds the solutions to the 2 smaller problems to get the solution of the larger problem.

In cell 4, Ann correctly uses the formal algorithm that she has learned in school. She also solves the problems in cell 6 correctly, but she does not use her formal algorithm in problems 6B and 6C. Ann responded so quickly to problem 6A that it is difficult to determine how she solved it. In problem 6B, however, she apparently initially attempts to use her algorithm, but has problems due to the fact that regrouping is necessary. This would indicate that she used her algorithm in the first problem (6A).

E: Next one - now I am going to buy that key chain. How much is that key chain.

A: 39

E: 39 - OK I throw up on the counter everything that I have as my worldly possessions and it happens to be 43 cents and the key chain is 39 cents.

This is your cash register, you can make change in the cash register.
A: That's hard - I'm not sure
E: Hum what did you put all this money down for - how much is that - 7 cents - 8 cents - why did you put that down?
A: It would be four cents back - right?
E: (yes), how did you figure that out?
A: I just added it up.
E: (LP) from 39 to 43 - very good. Next one.

When Ann's algorithm fails her, she falls back on alternative strategies such as this counting strategy.

We see that Ann uses a unique indexing system for counting in the next problem (6C). She was somewhat timid about counting out loud (as were the other children), a strategy which is typically discouraged in the classroom. It is a mental indexing system, where she mentally checks off the tens she counts. In addition, she only uses one hand to count, apparently a strategy developed because of the necessity for concealment of counting strategies in the classroom. She initially determines the answer based on the difference of the units place, an answer similar to 2A. Rather than let this answer pass, the researcher attempts to probe her solution strategy more deeply (a difference of 3 on her first solution to this problem closely parallels the solution strategy of the previous problem). After determining the nature of the error, Ann explains her unique counting strategy.

E: 57 cents - I am going to give you all the money in the world that I have which happens to be 80 cents.
A: 80 cents - so I put
E: 3 cents is my change
A: yea
E: How did you figure that out?
A: My fingers
E: And how did you do - what did you do when you used your fingers.
A: I counted it.
E: How did you count it - let me hear you count it.
A: I went 77, 78
E: No, it is 57.
A: Oh, so I did it wrong.

E: Want to start all over again. Yea? Now how would you do it then. Do it out loud so I can hear you.
A: Count by your fingers.
E: (yes), then do that - do it out loud
A: 7 - 58 - 9 60 61 62 63 64 65 66 67 68 69 70 71 72 73 74 75, 76, 77 78 79 80 - 23 cents
E: 23 cents - how did you keep track of that. You counted on the same fingers over and over again - how did you know it was 23 instead of 13 or 33
A: I had a ten like that so I put
Long Pause
A: Ten more and I got a 3 over here - so I knew it was 23.

Although the teacher used bundles of toothpicks in the classroom, she did not use them with all her students, Ann was one of those students who had never used these materials. Therefore, after the researcher determined that Ann knew the correct answer to problem 8A, the researcher demonstrated the use of toothpicks in the solution of subtraction problems (problem 8A, however, did not require regrouping). Ann attempts to follow this procedure on problem 8B, but gets stumped on the regrouping process. The researcher tries to guide her to the correct regrouping process, but Ann does not quite follow. She removes two toothpicks from one of the remaining bundles and declares the answer as 27. It is apparent from this demonstration that Ann is unable to explain the regrouping process using concrete materials (she used a counting strategy with the money tasks, and did not explain regrouping with these materials). Ann solves 8C mentally and does not use the materials at all.

E: Now you teach me on this one. 87 take away 59.
A: 7 - take away 9 - 5 - (LP) only 7
E: Now how many do you have to take away.
A: 9
E: ok, if you took away the 7 what would need to be taken away
A: 9 more
E: How many are left right now?
A: 37
E: And you have to take nine more away and how many single ones do you have.
A: 7
E: Is there some way of getting the other 2
A: no

E: How many are in here.
A: 10
E: Are there at least 2 in there.
A: yea
E: So what could you do?
A: Take 2 out of 20 and put on to 7
E: And take those away and what is left
A: 27
E: Is there a way you could have done it without taking the two out of here.
A: no

Ann had a variety of error patterns and might be classified as a random problem solver in subtraction. However, based on her protocol data, we have learned that she has a good understanding of declarative knowledge, both semantically and syntactically. The problem arises in her ability to regroup, a subprocess in the subtraction algorithm. Based on the error pattern data we can say that Ann is random; based on the protocol data we can say that Ann needs to develop her regrouping process. In order to develop this process, we can rely on her declarative knowledge. It would appear that prescription based on a diagnosis is better revealed in the protocol process than in the error pattern analysis.

Ted

Both of Ted's parents have college educations. In fact, Ted's father has a Ph.D. in Education and is employed in a local university. Ted's mother has a BA in English and is a "homemaker." Based on these data, Ted was classified as coming from a middle class family.

Ted was selected for this study because he had completed the Van Lehn test without error. The following transcript gives is a good self description of the type of student that Ted was.

E: You eat this stuff up. Do you always do pretty well in math.
T: Yea, I love it. It is my favorite. I do extra pages in math when I get my work done for the fun of it.

It would appear that Ted is the model math student , as is indicated by the fact that Ted completed the protocol tasks with only one error (6C and this error was pointed out to him upon completion of the interview, whereupon he corrected himself). The importance of including Ted in this study is to use his data as a basis for what a skilled student is. To say that Ted got all the problems correct does not help, the important aspect of Ted's interview is how he solved the problems.

Declarative . Ted got all the questions in this plane correct without any additional prompting from the researcher. In fact, the regrouping difficulties Ann had with problems 5C and 7C did not pose a problem to Ted. Ted found every possible grouping of dimes and pennies to make 34 cents in problem 5C, for example. The problems in this plane appeared to be trivial for Ted and he answered them more fully than was necessary.

Procedural. The more interesting results are those that were found in the procedural plane. Each child had the opportunity to solve the problems in cell 2 and 4 (syntactic/informal and formal respectively) in any manner that they wished. Ted chose to write the numerical problem on paper for the problems in cell 2. That is, he translated the word problem into a syntactical arithmetic problem. Therefore, his solutions to the problems in cell 2 were simply syntactical after the translation. It appears that Ted's short term memory was somewhat overloaded, and he needed to utilize secondary external storage (i.e. write the problem on paper).

This was not the case for the problems in cell 4 since the problem was syntactically presented in a column format. He simply performed mental calculations, without error, to determine the solutions to those problems. He utilized a similar strategy for the problems in cell 6 (semantic/informal). That is, in the role playing environment, he performed the mental calculation and simply returned the change due. This is where he made his only error, he incorrectly calculated 80-57=33. The important aspect to note here, however, is that Ted used his formal algorithm in every context. He had become so proficient in its use that it simply served as a tool for his problem solving processes.

However, the results of the interview from cell 8 (semantic/formal) indicate some caution about giving children lots of drill and practice. Ted, as well as the others, were asked to teach the researcher subtraction using the toothpick bundles. Unlike the others, Ted was able to explain the procedure, not just use it. This is what should give us caution about drill and practice. Proficiency is important, but understanding the procedure goes hand-in-hand with proficiency. Here is the transcript of Ted learning to teach a first grader how to do subtraction by using the toothpick bundles.

E: Very good. Teach me again except this time 87 take away 59.
T: It looked like 37 take away 59 now that was going to be ...
E: Yea, that typewriter doesn't work very well.
T: That's OK.
E: Again
T: Yea, again. Maybe you - its going to work, its going to work. Lets see this is 87 take away 59 straight away.
E: Why is that
T: Take away. Well you could take away 60 and add 1.
E: Is that fair
T: Yea, I guess. 17 minus 9 - I've got 8 left and then 70 minus 50 is 20
E: 28
T: 28
E: Now is that a good way to explain subtraction to a first grader. In other words
T: I guess so.
E: Lets go over here and show me how you would do that on paper and pencil. Do this problem.
T: 87 - I'm not used to this stuff. My hands have developed their own way of writing certain numbers.
E: What would be the first thing you would do in that problem?
T: I would see if I could subtract 9 from 7 and if I couldn't I would regroup. Take 10 from 80 which would make that 70 and add it to 7 which would make this 17.
E: Now how can you show me that regrouping that you just did, that borrowing using these toothpicks. What would be the same thing over here with the toothpicks that you did over here with the numbers.
T: Oh yea, cross out 7 borrow a 10 from over here.
E: This is a 10 and 7 ones
T: This is 17

E: Is there a better way than just having it all grouped together like this?
T: I am going to take the rubber band off.
E: You could take the rubber band off which I spent hours doing or you could put ten in there - right.
T: Right, take 10 out of here, 8, 9, 10 so this is 17 and this is 70.
E: How much is over here
T: 59
E: (yes) and what is left
T: 28

Granted, the researcher led the way, but the other subjects could not have followed that far. This indicates that Ted not only was proficient with his subtraction algorithm, but could explain it. In addition, he understood all the components of the algorithm by showing no difficulty with the declarative portion of the interview.

Summary

If we look at Tables 1-4, we can determine the knowledge types (cells) that each subject has a strong (arbitrarily set at 2 or greater) understanding and weak (again, arbitrarily set at 0 or 1 right) understanding. We can conclude that Kate has a good understanding of formal declarative concepts and a poor understanding (assuming 5b is incorrect) for informal declarative concepts. This conclusion was not evident in the protocol data, but I will use this "number right" criterion for this summary.

Referring to Table 5, Kate appears to have the least amount of knowledge for mathematics in the context of subtraction. However, her performance on the Van Lehn test, in comparison with the other erroneous subjects (disregarding Ted for the moment), was the best. The conclusion based on the protocol data and the number right criterion is that Kate has a good understanding of syntax with regard to her subtraction algorithm. This is true of Kate and not the other 3 erroneous subjects. This knowledge apparently allows her to perform the subtraction algorithm better than the other three, but her lack of understanding (semantics) for her algorithm prevents her from becoming proficient.

Table 5. Comparison of number right for interview tasks and the Van Lehn test.

Subject	(out of 24) Protocol Data	(out of 48) *Weighted Score	(out of 20) Van Lehn
Kate	11	15	10
Jan	11	25	7
Ann	17	33	8
Fred	18	32	8
Ted	23	45	20

* Note: The weighted score was calculated as follows: 1 point for each correct response at level 1; 2 points for each correct response at level 2; and, 3 points for each correct response at level 3 (i.e. 1X8 problems at level 1 + 2X8 problems at level 2 + 3X8 problems at level 3 = 48 total points)

Employing the number right criterion with Jan's protocol data (see Table 2), Jan has a "good" understanding of knowledge in cells 1, 4, 5 and 7 while she has a "poor" understanding for knowledge in cells 2, 3, 6 and 8. In fact, Jan got all 3 levels correct in cell 4 (the paper and pencil subtraction problems) and all 3 wrong in cells 2 and 8 (both also in the procedural plane). If we look at the declarative and procedural planes separately, Jan has a good understanding of declarative knowledge and a poor understanding of procedural knowledge (even though she got all 3 correct in cell 4 of the procedural plane).

In Table 5, the weighted scores indicate, in general, that Jan has more understanding of subtraction than Kate. However, Kate performed better on the Van Lehn test than Jan. To account for this discrepancy, it may be instructive to examine the error patterns of Kate and Jan. Kate had fundamental difficulties dealing with zeros in the subtrahend as well as difficulties with regrouping and place value (as indicated in the declarative plane of the protocol data). Jan, on the other hand, only had difficulties with place value. Perhaps this indicates that Jan's errors on the Van Lehn test represent a more "mature" understanding of mathematics than Kate's error patterns. In addition, Kate's protocol data shows a better understanding of mathematical syntax than Jan, perhaps allowing Kate to perform better on the purely syntactical task of the Van Lehn test. This, however, does not hold true in comparison with Fred and Ann.

Looking at the correct responses in Tables 3, 4 and 5, Fred and Ann performed surprisingly similarly. As indicated in Tables 3 and 4, Fred and Ann only performed poorly in cells 2 and 8. In addition, Fred and Ann both got eight correct on the Van Lehn test. Using a correct response criteria, Fred and Ann might be characterized as understanding mathematics equally well. However, their error patterns do not coincide and they performed the protocol tasks much better than Kate, even though Kate performed better on the Van Lehn test (10 correct).

Let us first examine the error pattern discrepancy. Both Fred and Ann had regrouping difficulties, but Fred also exhibited the smaller-from-larger error pattern (six times if you include the 0-N=N errors). Perhaps Fred became lazy when performing his subtraction algorithm (the regrouping process he used is rather complex) and opted for a shorter solution path (the smaller-from-larger error is a much easier algorithm to perform). In short, the smaller-from-larger error pattern is an error with the absence of regrouping. In which case, Fred and Ann both had regrouping errors. However, Kate (who exhibited the least amount of knowledge in the protocol tasks) had three (0-N=N) errors and Fred had four. Therefore, the error patterns of Kate and Fred are similar, yet their protocol performances are not. This calls into question the notion that Kate's error patterns are less "mature" than the other subjects'.

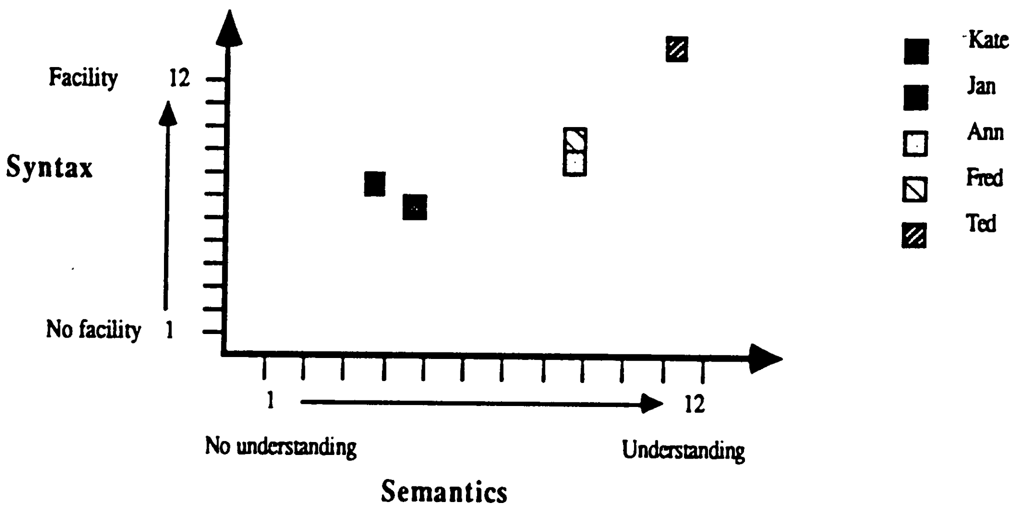

Figure 6. Where the subjects fit into the two-dimensional continuum of Figure 1.

The second issue is that Kate better performed better on the Van Lehn test than the others, while performing the worst on the protocol questions. An explanation for this discrepancy might be that Kate's strengths seem to be centered on syntax. In fact, her performance on procedural syntax is much better than her performance on procedural semantics (see Table 1). Perhaps Fred and Ann's knowledge of semantics interferes with their syntactical performance. Jan performs as well as Ann in the formal/syntactic cell (4), but her performance in the informal/syntactic cell (2) is poor. Further, Ted seems to solve all of the procedural problems syntactically. Therefore, the best approach to learning mathematics <u>may</u> be to develop children's syntactic procedures through a process of practice with feedback, rather than attempt to apply them to other situations.

Perhaps the best way of examining the differences among the five subjects in this study is to place them into the two-dimensional continuum of Figure 1. This is done in Figure 6. There are twelve questions each on the syntactic and semantic dimensions of Figure 5. A simple count of correct responses was utilized for the placement of each subject in Figure 6. Ted, of course, was placed in the upper right hand corner, representing proficiency with subtraction. Fred and Ann were placed in the middle of Area B (see Figure 1). Kate was placed in the lower middle of Area A, and Jan in the upper middle of Area D. From this graphical representation of the subjects' knowledge, we can say that Fred and Ann are "closer" to proficiency than Kate and Jan.

Conclusion

To reiterate, the purpose of this study was to determine how error patterns and a clinical approach to assessing the knowledge that a child has of mathematics interrelate for the long term purposes of designing a more comprehensive computer diagnostic system for subtraction. The analysis of the data produced from the five participants in this study yielded a very detailed account of their individual knowledge structures. In part, this was due to the fact that more than one data source was used (Denzin's triangulation). Essentially, an error pattern analysis and a clinical diagnostic interview were completed on each child. The clinical diagnosis consisted of creating subcomponential questions based on the preconceived categories of Figure 5. In addition, protocol data were collected during the diagnostic interview which generated additional data for assessing each subject's knowledge. The fact that these assessments yielded virtually non-overlapping data sets with respect to each child's knowledge of mathematics may not be surprising to clinical diagnosticians, but surely developers of diagnostic systems based on error patterns ought take heed.

Furthermore, this study did not discern any correspondence between error and clinical diagnoses, given the level of analyses employed. Another interesting outcome was also elicited from this study. No child was classified as a random solver, although just using the product (answer) data from each of the problems in the Van Lehn test might have indicated that they were random. Instead, the error analysis in combination with the clinical diagnosis yielded a detailed description of each individual's knowledge, each portion of which was attributable to only one of the assessments.

Three-dimensional Model

Semantic/Syntactic. The distinction between types of knowledge that was first brought to the content area of mathematics by Resnick (1982) proved to be an important demarcation. This distinction not only aids the development of questions (the intent of Figure 5), but also is an important way of catagorizing an individual's knowledge. Too many people learn the mechanics of mathematics without ever learning what the mechanics mean. For example, ask any undergraduate Calculus student the derivative of SIN X and then ask that same undergraduate the purpose of such a process. This provides a clear example of facility with (syntax) mathematics without understanding (semantics) mathematics.

There is a debate among mathematics teachers about the utility of teaching skills such as subtraction given that a calculator can perform this operation quickly and accurately. If there is an alternative benefit to learning a subtraction algorithm, it must be for the development of mathematical thinking. Kate and Jan have facility with some mathematical syntax without understanding its meaning. Ann and Fred have developed mathematical knowledge and have the potential for developing it further (even though both did poorer on the Van Lehn test than did Kate). In Figure 6, Ted has "got" it; Ann and Fred are "getting" it; and, Kate and Jan have the furthest to go.

Declarative/Procedural. Of course, the debate about the intimacy between these types of knowledge is not resolved, but I believe that declarative knowledge can be used to help develop procedural knowledge. For example, the "fuzzy area" of

Figure 3 might be made more clear by deferring to the child's declarative knowledge to prevent illegal regrouping. In addition, declarative knowledge of mathematics is an important component for the understanding of our numeration system, and therefore, is a measure of a child's mathematical knowledge.

Informal/Formal. U.S. currency was used to assess informal knowledge in this study. The questions were developed with the children's experience in mind and the role playing activities were thought to be a useful approach for the assessment of informal knowledge. However, I believe that because the protocol procedures were conducted at the school, the informal knowledge of the subjects was not fully brought out. Perhaps a better approach would have been to observe the children in their natural environments (certainly an approach supported by ethnographers). The potential does exist to use this knowledge, but further work in this area is needed. For example, questions such as these need to be answered: 1) Where is mathematics used in the children's daily lives?; 2) What are the conditions under which these mathematical problems arise?; 3) How are they solved?

One final comment about the three-dimensional model. In order to fully utilize this model, a developmental study ought to be undertaken. That is, over time, how does the knowledge develop and in what sequence? What are the important cells of Figure 5? Once this is established, we can determine where a child is at any given moment and where the child ought to go. This will aid in both diagnosis and prescription.

Computer Diagnosis

The important result for computer diagnosis is the fact that no participant was considered random. It appears that an error analysis performed in conjunction with subcomponent diagnosis is essential for subtraction diagnostic systems. The emphasis for computer diagnosis has been to develop computer algorithms that will account for more students' responses. The problem is, what do you do with an error diagnosis? Perhaps, if the purpose of diagnosis is for prescription, the addition of subcomponent diagnosis will aid both diagnostic and prescriptive capabilities. The development of such a program will provide greater insight into what knowledge is important for the development of mathematical thinking. Certainly, a strictly subcomponential approach to diagnosis can allow an subtraction diagnostic system to have prescriptive capabilities that an error pattern system is less able to do. The combination of the two approaches could allow a diagnostic system to closely analyze a child's algorithm (error pattern diagnosis) as well as determine the fundamental mathematical knowledge that the child has (subcomponent diagnosis). The dual approach might be better because where one approach fails, the other may be able to perform well.

The two approaches that were used in this study are clinical mathematics and error pattern diagnosis. Although the two approaches seem somewhat incommensurable, I feel that combining the two approaches will alleviate some of the problems that other diagnostic systems have encountered (i.e. the inability to account for about half of the children's solution strategies). Clinical mathematics education has been well formalized. The potential exists to use this process and it may improve future subtraction diagnostic systems.

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Appendix A. Protocol questions derived from the three-dimensional model of knowledge structures in mathematics.

1. Informal/declarative/syntactic

   A. Representation
      1. How much money is 2 dimes and 4 pennies?
      2. How much is 2 dimes and 4 pennies worth?
      3. What is the value of 2 dimes and 4 pennies?

   B. Tens and ones
      1. I have several dimes and pennies in my pocket and the total amount of money is 76¢, how many dimes do I have? Given that many dimes, how many pennies do I have?
      2. I have ONLY dimes and pennies. There is a total of 76¢. Can you guess how many dimes you think that is?
      3. Ask for the number of pennies, then dimes.

   C. Regrouping
      1. Can buy more things with 5 dimes or 45 pennies?
      2. If your mom said that you can have either 5 dimes or 45 pennies, which would you take?
      3. What is worth more, 5 dimes or 45 pennies?

2. Informal/procedural/syntactic

   A. A gum ball costs 10¢ and you give the clerk a quarter which is 25¢, how much money do you get back, or is left over?
   B. A candy bar costs 39¢ and you give the clerk three quarters which is 75¢, how much money do you get back, or is left over?
   C. A Slurpee costs 47¢ and a little kid gives the clerk all the money in his pocket, two quarters and two dimes which is 70¢, how much money does she get back or is left over?

3. Formal/declarative/syntactic

   A. Representation
      1. Write the number seventy-one.
      2. Write seventy-one as a number

   B. Tens and ones
      1. How many tens and ones are there in fifty-two?
      2. Can you guess?

   C. Regrouping
      1. What number has 5 tens and sixteen ones?
      2. What number is 16 ones?
      3. What number is 5 tens?

4. Formal/procedural/syntactic

   A. Subtract without regrouping
      1. 47 take away 21
      2. Find the difference 47 - 21

   B. Subtract with carrying

1. 87 take away 59
2. Find the difference 87 - 59

C. Subtract with regrouping into zero
   1. 50 take away 13
   2. Find the difference 50 - 13

5. Informal/declarative/semantic

   A. Represent or show 79¢
   B. Show a penny and a dime and ask, which is worth more? How much more?
   C. 1. Represent 34¢. Now represent it again another way
      2. Using pennies and dimes can you have 34¢ and not have 3 dimes and 4 pennies, How?

6. Informal/procedural/semantic (Do this in a store like, role playing kind of activity That is, I'm the customer. You are the clerk. And this is your cash register)

   A. Present 64¢ for a 52¢ pen. Then ask for appropriate change.
   B. Present 43¢ for a 39¢ key chain. Then ask for appropriate change.
   C. Present 80¢ for a 57¢ stapler. Then ask for appropriate change.

7. Formal/declarative/semantic

   A. Represent 56
   B. Show a long and a unit and ask which is worth more. (Call the longs tens and the units ones except in this question)
   C. Show me 72 using the blocks. Now show me 72 a different way.

8. Formal/procedural/semantic (If they choose to do the problem on paper, ask them to teach me using the blocks, like I am in first grade)

   A. 47 Take away 21.
   B. 87 Take away 59.
   C. 50 Take away 13.

Appendix B

The subjects' Van Lehn tests.

NAME: KATE

647 − 45 = 602	885 − 205 = 680	83 − 44 = 38
8305 − 3 = 8302	50 − 23 = 33	562 − 3 = 559
742 − 136 = 606	106 − 70 = 116	716 − 598 = 118
1564 − 887 = 0677	6591 − 2697 = 2783	311 − 214 = 097

KATE

1813 - 215 1598	102 - 39 0614	9007 - 6880 3127
4015 - 607 3318	702 - 108 604	2006 - 42 2077
10012 - 214 01706	8001 - 43 776.	

NAME: JAN

647 − 45	602
885 − 205	[illegible]
83 − 44	39
8305 − 3	[illegible]
50 − 23	27
562 − 3	559
742 − 136	606
106 − 70	116
716 − 598	218
1564 − 887	1777
6597 − 2697	5884
311 − 214	197

JAN

1813 - 215	102 - 39	9007 - 6880
4015 - 607	702 - 108	2006 - 42
10012 - 214	8001 - 43	

NAME: FRED

647 - 45 602	885 - 205 680	83 - 44 58
8305 - 3 8302	50 - 23 33	562 - 3 561
742 - 136 614	106 - 70 176	716 - 598 118
1564 - 837 0,677	6591 - 2697 3,894	311 - 214 096

FRED

```
  1813        102        9007
-  215      -  39      - 6880

  4015        702        2006
-  607      - 108      -   42

 10012       8001
-  214      -  43
```

NAME ANN

647 − 45 = 602	885 − 205 = 680	83 − 44 = 39
8305 − 3 = 8302	50 − 23 = 27	562 − 3 = 599
742 − 136 = 606	106 − 70 = 036	716 − 598 = 127
1564 − 887 = 0687	6591 − 2697 = 3904	311 − 214 = 107

ANN

```
  1813
- 215
  1608
```

```
  702
- 39
  063
```

```
  9007
- 6880
  2117
```

```
  4015
- 607
  3418
```

```
  702
- 108
  594
```

```
  2006
-   42
  1854
```

```
  10012
-   214
  08798
```

```
  8001
-   43
  7968
```

NAME: TED

647 − 45	885 − 205	83 − 44
8305 − 3	50 − 23	562 − 3
742 − 136	106 − 70	716 − 598 = 118
1564 − 887 = 0677	6591 − 2697 = 3844	311 − 214 = 0 [illegible] 7

TED

1813 - 215	102 - 39	9007 - 6880
4015 - 607	702 - 108	2006 - 42
10012 - 214	8001 - 43	

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