Thickness shear vibrations of a circular cylindrical piezoelectric shell

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Axial and tangential thickness shear vibrations of a circular cylindrical piezoelectric shell of monoclinic crystals are studied. The problems are solved analytically, and the frequency equations are derived. For a cylinder made of a rotated Y-cut quartz, the resonant frequencies are computed numerically, and it is shown that they approach that of a flat plate as the inner radius of the cylinder of finite thickness approaches infinity.

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INTRODUCTION

Thickness shear vibrations of crystals and piezoelectric plates have been studied either by using the three-dimensional equations of piezoelectricity or the two-dimensional equations of the plate theory.[1–3] The interest in these problems arises because of their applications as resonators. Vibrations of a circular cylindrical piezoelectric shell, with deformations assumed to be either axisymmetric or with the tangential displacement taken to be zero, and made of ceramics poled in various directions, have also been studied.[4–7] In this paper, axial and tangential thickness shear vibrations of a circular cylindrical piezoelectric shell made of a monoclinic crystal are studied. We derive exact solutions of the three-dimensional quasistatic piezoelectricity equations governing the free vibrations of a cylindrical shell with traction-free and electroded inner and outer surfaces. Frequency equations are also derived and solved numerically.

I. THICKNESS SHEAR VIBRATIONS OF A PLATE

Results for the thickness shear vibrations of a monoclinic piezoelectric plate[2] are summarized below for easy reference. Consider an infinite plate, shown in Fig. 1, of thickness $2h$ with traction-free and electroded boundaries at $x_2=\pm h$. Equations governing the deformations of the plate and the relevant boundary conditions are

$$T_{ji,j}=\rho\ddot{u}_i,\quad -h<x_2<h, \tag{1}$$

$$D_{i,i}=0,\quad -h<x_2<h, \tag{2}$$

$$T_{2i}=0,\quad \phi=0 \quad \text{at } x_2=\pm h, \tag{3}$$

where

$$T_{ij}=C_{ijkl}S_{kl}-e_{kij}E_k, \tag{4}$$

$$D_i=e_{ijk}S_{jk}+\epsilon_{ij}E_j, \tag{5}$$

$$S_{ij}=\tfrac{1}{2}(u_{i,j}+u_{j,i}), \tag{6}$$

$$E_i=-\phi_{,i}. \tag{7}$$

In Eqs. (1)–(7) and hereafter, T_{ij}, u_i, D_i, S_{ij}, and E_i are the components of the stress tensor, mechanical displacement, electric displacement, the strain tensor appropriate for infinitesimal deformations, and the electric field, respectively, ρ and ϕ the mass density and the electric potential, and C_{ijkl}, e_{ijk}, and ϵ_{ij} the elastic, piezoelectric, and dielectric constants, respectively. Furthermore, a comma followed by index j indicates partial differentiation with respect to x_j, a superimposed dot indicates differentiation with respect to time t, and a repeated index implies summation over the range of the index.

Equation (1) expresses the balance of linear momentum and Eq. (2) is the Gauss equation. The boundary conditions $(3)_1$ and $(3)_2$ imply that the bounding surfaces $x_2=\pm h$ are traction-free and have null electric potential prescribed there. Equations (4) and (5) are the constitutive relations and Eq. (6) is the strain-displacement relation; Eq. (7) is the electric-field-potential relation. Regarding $T_{ij}=T_{ji}$ and $S_{ij}=S_{ji}$ as vectors in a six-dimensional space with $\bar{T}_1=T_{11}$, $\bar{T}_2=T_{22}$, $\bar{T}_3=T_{33}$, $\bar{T}_4=T_{23}$, $\bar{T}_5=T_{31}$, and $\bar{T}_6=T_{12}$ etc., the material constants C_{ijkl} and e_{kij} may be written as 6×6 and 6×3 matrices. For monoclinic crystals,

$$\epsilon_{ij}=\epsilon_{ji},\quad \epsilon_{12}=\epsilon_{13}=0. \tag{8}$$

For the free-time-harmonic thickness shear vibrations in the x_1 direction, we seek solutions satisfying

$$u_1=\tilde{u}_1(x_2)e^{i\omega t},\quad u_2=0,\quad u_3=0,\qquad \phi=\tilde{\phi}(x_2)e^{i\omega t}. \tag{9}$$

Equations (1)–(7) become

$$T_{21,2}=-\rho\omega^2u_1,\quad D_{2,2}=0,\quad -h<x_2<h, \tag{10}$$

$$T_{21}=0,\quad \phi=0 \quad \text{at } x_2=\pm h, \tag{11}$$

$$T_{31}=2C_{56}S_{12}-e_{25}E_2,\quad T_{12}=2C_{66}S_{12}-e_{26}E_2, \tag{12}$$

$$D_2=2e_{26}S_{12}+\epsilon_{22}E_2,\quad D_3=2e_{36}S_{12}+\epsilon_{23}E_2, \tag{13}$$

$$S_{12}=\tfrac{1}{2}u_{1,2},\quad E_2=-\phi_{,2}, \tag{14}$$

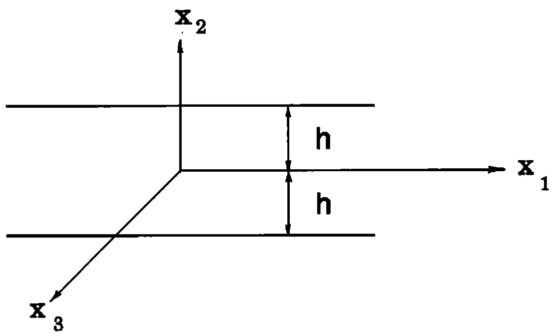

FIG. 1. Schematic sketch of a piezoelectric plate.

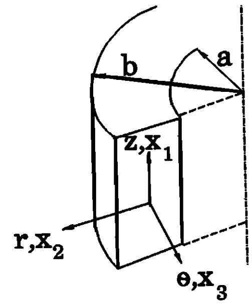

FIG. 2. A circular cylindrical shell and the choice of base vectors for studying its axial thickness shear vibrations.

wherein we have dropped the tildes superimposed upon u_1 and ϕ. The general solution of (10) is

$$u_1=a_1 \sin kx_2+a_2 \cos kx_2, \tag{15}$$

$$\phi=\frac{e_{26}}{\epsilon_{22}}(a_1 \sin kx_2+a_2 \cos kx_2)+a_3x_2+a_4, \tag{16}$$

where

$$k^2=\rho\omega^2/\bar{C}_{66}, \quad \bar{C}_{66}=C_{66}+e_{26}^2/\epsilon_{22}, \tag{17}$$

and a_1, a_2, a_3, and a_4 are arbitrary constants. Substitution from (15) and (16) into boundary conditions (11) gives the following set of homogeneous equations for the determination of a_1, a_2, a_3, and a_4:

$$k\left(C_{66}(a_1 \cos kh-a_2 \sin kh)+\frac{e_{26}^2}{\epsilon_{22}}(a_1 \cos kh - a_2 \sin kh)\right)+e_{26}a_3=0, \tag{18}$$

$$k\left(C_{66}(a_1 \cos kh+a_2 \sin kh)+\frac{e_{26}^2}{\epsilon_{22}}(a_1 \cos kh + a_2 \sin kh)\right)+e_{26}a_3=0, \tag{19}$$

$$\frac{e_{26}}{\epsilon_{22}}(a_1 \sin kh+a_2 \cos kh)+a_3h+a_4=0, \tag{20}$$

$$\frac{e_{26}}{\epsilon_{22}}(-a_1 \sin kh+a_2 \cos kh)-a_3h+a_4=0. \tag{21}$$

The vanishing of the determinant of the coefficient matrix of Eqs. (18)–(21) gives the following frequency equation:

$$\sin kh(\tan kh-kh/k_{26}^2)=0, \quad k_{26}^2=e_{26}^2/\bar{C}_{66}\epsilon_{22}. \tag{22}$$

II. AXIAL THICKNESS SHEAR VIBRATIONS OF A CIRCULAR CYLINDRICAL SHELL

We consider a cylindrical shell, shown in Fig. 2, made of a monoclinic crystal and with inner radius a and outer radius b. It is more convenient to use cylindrical coordinates, and we refer the reader to Love's book[8] for the governing equations (1) and (2), constitutive relations (4), and strain-displacement relations (6) written in cylindrical coordinates. It is preferable to work in terms of physical components of stresses, strains, and the electric displacement. The constitutive relations (4) and (5), and electric-field-potential relations (7) become

$$\begin{aligned}
T_{zz}&=C_{11}S_{zz}+C_{12}S_{rr}+C_{13}S_{\theta\theta}+2C_{14}S_{r\theta}-e_{11}E_z,\\
T_{rr}&=C_{12}S_{zz}+C_{22}S_{rr}+C_{23}S_{\theta\theta}+2C_{24}S_{r\theta}-e_{12}E_z,\\
T_{\theta\theta}&=C_{13}S_{zz}+C_{23}S_{rr}+C_{33}S_{\theta\theta}+2C_{34}S_{r\theta}-e_{13}E_z,\\
T_{r\theta}&=C_{14}S_{zz}+C_{24}S_{rr}+C_{34}S_{\theta\theta}+2C_{44}S_{r\theta}-e_{14}E_z,\\
T_{\theta z}&=2C_{55}S_{\theta z}+2C_{56}S_{zr}-e_{25}E_r-e_{35}E_\theta,\\
T_{zr}&=2C_{56}S_{\theta z}+2C_{66}S_{zr}-e_{26}E_r-e_{36}E_\theta,\\
D_z&=e_{11}S_{zz}+e_{12}S_{rr}+e_{13}S_{\theta\theta}+2e_{14}S_{r\theta}+\epsilon_{11}E_z,\\
D_r&=2e_{25}S_{\theta z}+2e_{26}S_{zr}+\epsilon_{22}E_r+\epsilon_{23}E_\theta,\\
D_\theta&=2e_{35}S_{\theta z}+2e_{36}S_{zr}+\epsilon_{23}E_r+\epsilon_{33}E_\theta,\\
E_r&=-\frac{\partial\phi}{\partial r}, \quad E_\theta=-\frac{1}{r}\frac{\partial\phi}{\partial\theta}, \quad E_z=-\frac{\partial\phi}{\partial z}.
\end{aligned} \tag{23}$$

For the free-time-harmonic thickness shear vibrations in the axial direction, we seek solutions of the form

$$u_z=\tilde{u}_z(r)e^{i\omega t}, \quad u_r=0, \quad u_\theta=0, \quad \phi=\tilde{\phi}(r)e^{i\omega t}. \tag{24}$$

With (24), and dropping the tildes superimposed upon u_z and ϕ, the governing equations and the boundary conditions simplify to

$$\frac{\partial T_{rz}}{\partial r}+\frac{T_{rz}}{r}=-\rho\omega^2u_z, \quad \frac{1}{r}\frac{\partial}{\partial r}(rD_r)=0, \quad a<r<b, \tag{25}$$

$$S_{zr}=\frac{1}{2}\frac{\partial u_z}{\partial r}, \quad E_r=-\frac{\partial\phi}{\partial r}, \tag{26}$$

$$T_{rz}=2C_{66}S_{rz}-e_{26}E_r, \quad D_r=2e_{26}S_{rz}+\epsilon_{22}E_r, \tag{27a}$$

$$T_{\theta z}=2C_{55}S_{\theta z}-e_{25}E_r, \quad D_\theta=2e_{36}S_{rz}+\epsilon_{23}E_r, \tag{27b}$$

$$T_{rz}=0, \quad \phi=0 \quad \text{at } r=a,b. \tag{28}$$

Substitution from (26) and (27) into (25) yields

$$\frac{\partial}{\partial r}(C_{66}u_{z,r}+e_{26}\phi_{,r})+\frac{1}{r}(C_{66}u_{z,r}+e_{26}\phi_{,r}) =-\rho\omega^2u_z, \tag{29}$$

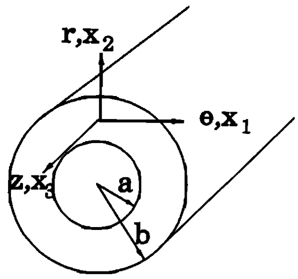

FIG. 3. A circular cylindrical shell and the choice of base vectors for studying its tangential thickness shear vibrations.

$$\frac{\partial}{\partial r}(e_{26}u_{z,r}-\epsilon_{22}\phi_{,r})+\frac{1}{r}(e_{26}u_{z,r}-\epsilon_{22}\phi_{,r})=0. \quad (30)$$

The elimination of ϕ from (29) and (30) gives

$$u_{z,rr}+\frac{1}{r}u_{z,r}+k^2u_z=0, \quad (31)$$

where

$$k^2=\rho\omega^2/\bar{C}_{66}, \quad \bar{C}_{66}=C_{66}+e_{26}^2/\epsilon_{22}. \quad (32)$$

Solving Eqs. (31) and (30) for u_z and ϕ, we obtain

$$u_z=C_1J_0(kr)+C_2Y_0(kr), \quad (33)$$

$$\phi=\frac{e_{26}}{\epsilon_{22}}u_z+\frac{e_{26}}{\epsilon_{22}}(C_3 \ln r+C_4), \quad (34)$$

where J_0 and Y_0 are zeroth-order Bessel's functions of the first and second kind, respectively, and constants C_1, C_2, C_3, and C_4 are to be determined from the boundary conditions. The requirement that functions u_z and ϕ satisfy boundary conditions (28) give four homogeneous equations for the determination of C_1, C_2, C_3, and C_4. These four equations have a nontrivial solution only if

$$\frac{kaJ_1(ka)\ln a/b+k_{26}^2[J_0(ka)-J_0(kb)]}{kbJ_1(kb)\ln a/b+k_{26}^2[J_0(ka)-J_0(kb)]}=\frac{kaY_1(ka)\ln a/b+k_{26}^2[Y_0(ka)-Y_0(kb)]}{kbY_1(kb)\ln a/b+k_{26}^2[Y_0(ka)-Y_0(kb)]}, \quad (35)$$

where relations $J_0' = -J_1$, $Y_0' = -Y_1$ have been used; J_1 and Y_1 are first-order Bessel's functions of the first and second kind, respectively. Equation (35) is the equation for the determination of the frequency k, and thence ω through Eq. $(32)_1$.

III. TANGENTIAL THICKNESS SHEAR VIBRATIONS OF A CIRCULAR CYLINDRICAL SHELL

We now study the tangential thickness shear vibrations of a cylindrical shell made of a monoclinic crystal aligned as shown in Fig. 3 wherein the coordinate system is also depicted. With respect to the coordinate axes shown, the constitutive relations take the form

$$\begin{aligned}
T_{\theta\theta}&=C_{11}S_{\theta\theta}+C_{12}S_{rr}+C_{13}S_{zz}+2C_{14}S_{rz}-e_{11}E_\theta,\\
T_{rr}&=C_{12}S_{\theta\theta}+C_{22}S_{rr}+C_{23}S_{zz}+2C_{24}S_{rz}-e_{12}E_\theta,\\
T_{zz}&=C_{13}S_{\theta\theta}+C_{23}S_{rr}+C_{33}S_{zz}+2C_{34}S_{rz}-e_{13}E_\theta,\\
T_{rz}&=C_{14}S_{\theta\theta}+C_{24}S_{rr}+C_{34}S_{zz}+2C_{44}S_{rz}-e_{14}E_\theta,\\
T_{z\theta}&=2C_{35}S_{z\theta}+2C_{56}S_{\theta r}-e_{25}E_r-e_{35}E_z, \qquad (36)\\
T_{\theta r}&=2C_{56}S_{z\theta}+2C_{66}S_{\theta r}-e_{26}E_r-e_{36}E_z,\\
D_\theta&=e_{11}S_{\theta\theta}+e_{12}S_{rr}+e_{13}S_{zz}+2e_{14}S_{rz}+\epsilon_{11}E_\theta,\\
D_r&=2e_{25}S_{\theta z}+2e_{26}S_{r\theta}+\epsilon_{22}E_r+\epsilon_{23}E_z,\\
D_z&=2e_{35}S_{\theta z}+2e_{36}S_{r\theta}+\epsilon_{23}E_r+\epsilon_{33}E_z.
\end{aligned}$$

We assume that the free-time-harmonic thickness shear vibrations in the tangential direction are given by

$$u_\theta=\tilde{u}_\theta(r)e^{i\omega t}, \quad u_r=0, \quad u_z=0, \quad \phi=\tilde{\phi}(r)e^{i\omega t}. \quad (37)$$

For motions of this type, the governing equations and boundary conditions simplify to

$$\frac{\partial T_{r\theta}}{\partial r}+\frac{2}{r}T_{r\theta}=-\rho\omega^2u_\theta, \quad \frac{1}{r}\frac{\partial}{\partial r}(rD_r)=0, \quad (38)$$

$$S_{r\theta}=\frac{1}{2}\left(u_{\theta,r}-\frac{u_\theta}{r}\right), \quad E_r=-\phi_{,r}, \quad (39)$$

$$T_{z\theta}=2C_{56}S_{r\theta}-e_{25}E_r, \quad T_{\theta r}=2C_{66}S_{r\theta}-e_{26}E_r, \quad (40)$$

$$D_r=2e_{26}S_{r\theta}+\epsilon_{22}E_r, \quad D_z=2e_{36}S_{r\theta}+\epsilon_{33}E_r, \quad (41)$$

$$T_{r\theta}=0, \quad \phi=0 \quad \text{at } r=a,b, \quad (42)$$

where we have dropped the tildes superimposed upon u_θ and ϕ. A solution of Eq. $(38)_2$ is

$$D_r=e_{26}C_3/r, \quad (43)$$

where C_3 is an arbitrary constant. Equations (43), $(41)_1$, and $(39)_2$ result in

$$\phi_{,r}=\frac{e_{26}}{\epsilon_{22}}\left(2S_{r\theta}-\frac{C_3}{r}\right). \quad (44)$$

Substitution from (40), $(39)_1$, and (44) into $(38)_1$ yields

$$u_{\theta,rr}+\frac{1}{r}u_{\theta,r}+\left(k^2-\frac{1}{r^2}\right)u_\theta=\frac{k_{26}^2C_3}{r^2}, \quad (45)$$

where k and k_{26} are defined by Eqs. $(32)_1$ and $(22)_2$, respectively. A solution of Eq. (45) is

$$u_\theta=C_1J_1(kr)+C_2Y_1(kr)+C_3P(kr), \quad (46)$$

where C_1 and C_2 are arbitrary constants, and

$$P(kr)=\frac{\pi}{2}k_{26}^2[Y_1(kr)F(kr)-J_1(kr)G(kr)], \quad (47)$$

$$F(kr)=\int_{ka}^{kr}\frac{J_1(\xi)}{\xi}d\xi, \quad G(kr)=\int_{ka}^{kr}\frac{Y_1(\xi)}{\xi}d\xi. \quad (48)$$

Substituting from $(39)_1$ and (46) into (44), and integrating the resulting equation, we obtain

$$\phi=\frac{e_{26}}{\epsilon_{22}}\{C_1[J_1(kr)-F(kr)]+C_2[Y_1(kr)-G(kr)]+C_3[P(kr)-Q(kr)-\ln r]+C_4\}, \quad (49)$$

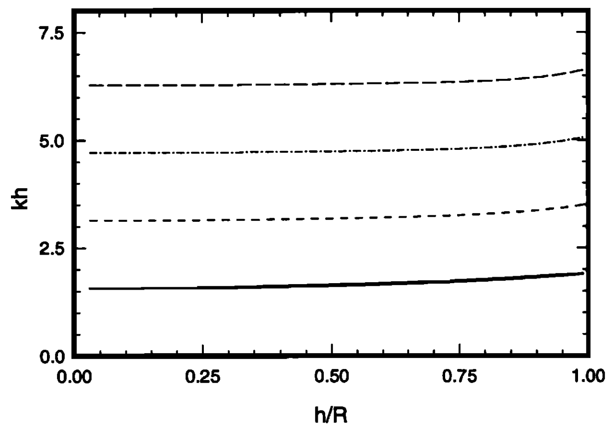

FIG. 4. First four resonant frequencies for axial thickness shear vibrations of a circular cylindrical shell (— 1st; ······· 2nd; ----- 3rd; — — — — 4th).

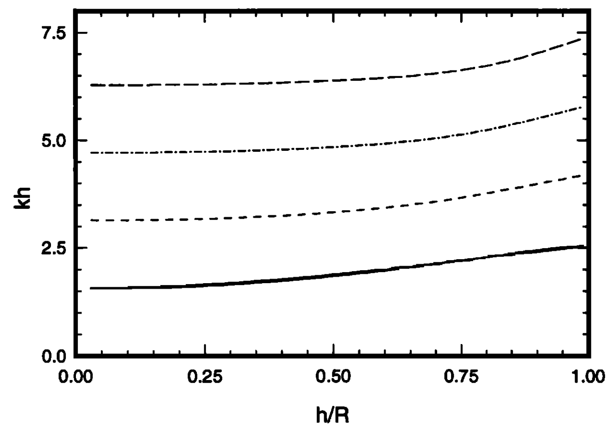

FIG. 5. First four resonant frequencies for tangential thickness shear vibrations of a circular cylindrical shell (— 1st; ······· 2nd; ----- 3rd; — — — — 4th).

where

$$Q(kr)=\int_{ka}^{kr}\frac{P(\xi)}{\xi}\,d\xi. \tag{50}$$

The four homogeneous algebraic equations for C_1, C_2, C_3, and C_4 obtained by substituting from (46) and (49) into the boundary conditions (42) will have a nontrivial solution only if

$$\frac{[P(kb)-Q(kb)+\ln a/b]kaJ_2(ka)+[J_1(ka)-J_1(kb)+F(kb)]k_{26}^2}{[P(kb)-Q(kb)+\ln a/b]kbJ_2(kb)+[J_1(ka)-J_1(kb)+F(kb)][P(kb)-kbP'(kb)+k_{26}^2]}$$
$$=\frac{[P(kb)-Q(kb)+\ln a/b]kaY_2(ka)+[Y_1(ka)-Y_1(kb)+G(kb)]k_{26}^2}{[P(kb)-Q(kb)+\ln a/b]kbY_2(kb)+[Y_1(ka)-Y_1(kb)+G(kb)][P(kb)-kbP'(kb)+k_{26}^2]}, \tag{51}$$

which is the desired equation for the determination of k and of ω via Eq. $(17)_1$.

IV. NUMERICAL RESULTS

For a rotated Y-cut quartz,[9]

$C_{66}=29.01$ GPa,

$e_{26}=-0.095$ C/m^2,

$\epsilon_{22}=39.82\times10^{-12}$ C/V m,

frequency equations (35) and (51) are solved numerically, and the first four frequencies are depicted in Figs. 4 and 5, respectively. In Figs. 4 and 5, $R=(a+b)/2$ is the average radius of the cylinder. Keeping $2h=b-a$ fixed and letting $a\to\infty$, we see that in each case the resonant frequencies of the cylindrical shell approach those of the flat plate given by (22).

V. CONCLUSIONS

We have studied analytically the axial and the tangential thickness shear vibrations of a circular cylindrical piezoelectric shell, and have computed the first four resonant frequencies for a rotated Y-cut quartz shell.

ACKNOWLEDGMENTS

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