An Adaptive Zolotarev Upper-Bound for the Singular Values of Loewner Matrices

VIRGINIA TECH

Mark Embree   Nilton Alan Garcia-Hilares

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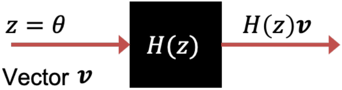
Outline

Data-Driven Model Reduction

Large scale dynamical system

$$\begin{aligned} \mathbf{E}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t). \end{aligned}$$

Transfer function: $H(z) = \mathbf{C}(z\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}$.



- Sample the transfer function (potentially by physical measurements).
- Recover a model for the system $(\mathbf{A}_r, \mathbf{B}_r, \mathbf{C}_r, \mathbf{E}_r)$.

“Data driven” because we can only sample $H(z)$, we do not know $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{E}$.

Loewner Framework for System Realization

Mayo-Antoulas show how to recover $\mathbf{A}, \mathbf{E} \in \mathbb{C}^{n \times n}$, $\mathbf{B} \in \mathbb{C}^{n \times m}$ and $\mathbf{C} \in \mathbb{C}^{p \times n}$ from measurements.

Left Data: $\mathbf{v}_i^* = \boldsymbol{\ell}_i^* H(\mu_i)$

- Points: $\mu_1, \ldots, \mu_\nu \in \mathbb{C}$.
- Directions: $\boldsymbol{\ell}_1, \ldots, \boldsymbol{\ell}_\nu \in \mathbb{C}^p$.
- Data: $\mathbf{v}_1, \ldots, \mathbf{v}_\nu \in \mathbb{C}^m$.

Right Data: $\mathbf{w}_j = H(\theta_j)\mathbf{r}_j$

- Points: $\theta_1, \ldots, \theta_\rho \in \mathbb{C}$.
- Directions: $\mathbf{r}_1, \ldots, \mathbf{r}_\rho \in \mathbb{C}^m$.
- Data: $\mathbf{w}_1, \ldots, \mathbf{w}_\rho \in \mathbb{C}^p$.

Construct $\mathbb{L}$ (Loewner matrix) and $\mathbb{L}_s$ (Shifted Loewner matrix) by

$$(\mathbb{L})_{i,j} = \frac{\mathbf{v}_i^* \mathbf{r}_j - \boldsymbol{\ell}_i^* \mathbf{w}_j}{\mu_i - \theta_j} \quad \text{and} \quad (\mathbb{L}_s)_{i,j} = \frac{\mu_i \mathbf{v}_i^* \mathbf{r}_j - \theta_j \boldsymbol{\ell}_i^* \mathbf{w}_j}{\mu_i - \theta_j}$$

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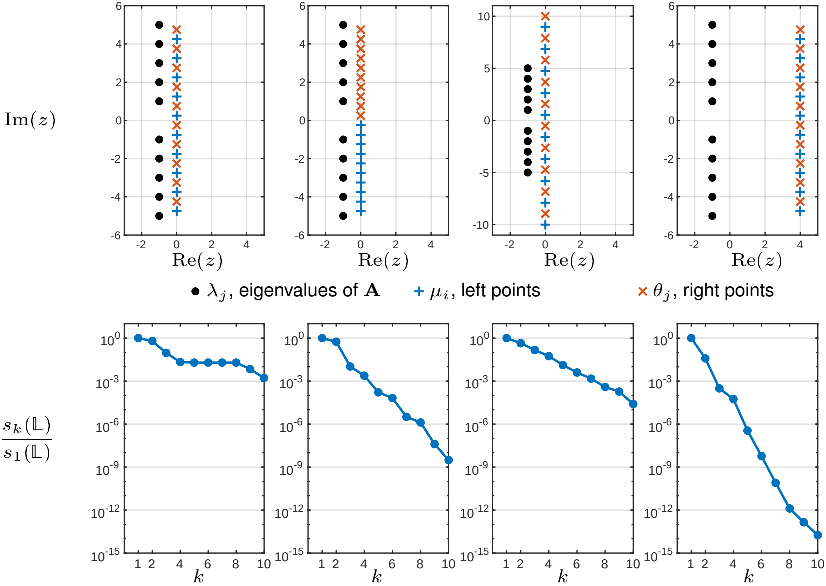
- If $\rho = \nu = n$, $\mathbf{A} = \mathbb{L}_s$, $\mathbf{E} = \mathbb{L}$ and $\sigma(\mathbf{A}, \mathbf{E}) = \sigma(\mathbb{L}_s, \mathbb{L})$.
- If $\rho, \nu > n$, is rank$(\mathbb{L}) = n$?
- rank$(\mathbb{L})$ usually reveals n (get $\mathbf{A}_r, \mathbf{E}_r$ from an SVD).

Interplay Between Data and Interpolation Points



Motivation

- We do not know n in most cases.
- We need to approximate n from $\mathsf{rank}(\mathbb{L})$.
- Singular values of $\mathbb{L}$ decay.
- Measurements could be noisy/inaccurate.
- $\mathsf{rank}(\mathbb{L})$ could be hard to determine.
- Singular values of $\mathbb{L}$ guide model reduction dimension.

Some questions

- How can we locate the interpolation points to make $\mathsf{rank}(\mathbb{L})$ as clear as possible?
- How does the location of the interpolation points affects the singular values of $\mathbb{L}$?
- Could poorly placed interpolation points cause us to underestimate $\mathsf{rank}(\mathbb{L})$?

Loewner framework

The Loewner matrix $\mathbb{L}$ and shifted Loewner matrix $\mathbb{L}_s$ of order $\nu \times \rho$ are defined entry-wise:

$$(\mathbb{L})_{i,j} = \frac{\mathbf{v}_i^* \mathbf{r}_j - \boldsymbol{\ell}_i^* \mathbf{w}_j}{\mu_i - \theta_j} \quad \text{and} \quad (\mathbb{L}_s)_{i,j} = \frac{\mu_i \mathbf{v}_i^* \mathbf{r}_j - \theta_j \boldsymbol{\ell}_i^* \mathbf{w}_j}{\mu_i - \theta_j}$$

The interpolation points and directions are collected in matrices:

$$\boldsymbol{\Theta} = \begin{bmatrix} \theta_1 & & \\ & \ddots & \\ & & \theta_\rho \end{bmatrix} \in \mathbb{C}^{\rho \times \rho}, \quad \mathbf{R} = \begin{bmatrix} | & & | \\ \mathbf{r}_1 & \cdots & \mathbf{r}_\rho \\ | & & | \end{bmatrix} \in \mathbb{C}^{m \times \rho}, \quad \mathbf{W} = \begin{bmatrix} | & & | \\ \mathbf{w}_1 & \cdots & \mathbf{w}_\rho \\ | & & | \end{bmatrix} \in \mathbb{C}^{p \times \rho},$$

$$\mathbf{M} = \begin{bmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_\nu \end{bmatrix} \in \mathbb{C}^{\nu \times \nu}, \quad \mathbf{L} = \begin{bmatrix} | & & | \\ \boldsymbol{\ell}_1 & \cdots & \boldsymbol{\ell}_\nu \\ | & & | \end{bmatrix} \in \mathbb{C}^{p \times \nu}, \quad \mathbf{V} = \begin{bmatrix} | & & | \\ \mathbf{v}_1 & \cdots & \mathbf{v}_\nu \\ | & & | \end{bmatrix} \in \mathbb{C}^{m \times \nu}.$$

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These matrices satisfy the Sylvester equations

$$\mathbb{L}\boldsymbol{\Theta} - \mathbf{M}\mathbb{L} = \mathbf{L}^*\mathbf{W} - \mathbf{V}^*\mathbf{R},$$

$$\mathbb{L}_s\boldsymbol{\Theta} - \mathbf{M}\mathbb{L}_s = \mathbf{L}^*\mathbf{W}\boldsymbol{\Theta} - \mathbf{M}\mathbf{V}^*\mathbf{R}.$$

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These matrices satisfy

$$\mathbb{L}_s = \mathbb{L}\boldsymbol{\Theta} + \mathbf{V}^*\mathbf{R},$$

$$\mathbb{L}_s = \mathbf{M}\mathbb{L} + \mathbf{L}^*\mathbf{W}.$$

Relating Poles to the Left Interpolation Points

Now let us follow Alex Townsend's suggestion.
Since the poles of $\mathbf{H}(z)$ are the eigenvalues of the matrix pencil $z\mathbb{L} - \mathbb{L}_s$ or the eigenvalues of $\mathbb{L}^{-1}\mathbb{L}_s$, provided $\mathbb{L}$ is invertible.

Assume $\nu = \rho = n$ (full rank).
Moreover, if $\mathbb{L}^{-1}\mathbb{L}_s$ is diagonalizable, there exists $\mathbf{\Lambda}, \mathbf{X} \in \mathbb{C}^{r \times r}$ such that

$$\mathbb{L}^{-1}\mathbb{L}_s = \mathbf{X}\mathbf{\Lambda}\mathbf{V}^{-1} \quad \text{or equivalently} \quad \mathbb{L}_s = \mathbb{L}\mathbf{X}\mathbf{\Lambda}\mathbf{X}^{-1} \quad \text{with} \quad \mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

From $\mathbb{L}_s = \mathbf{M}\mathbb{L} + \mathbf{L}^*\mathbf{W}$, we get

$$\mathbb{L}(\mathbf{X}\mathbf{\Lambda}\mathbf{X}^{-1}) - \mathbf{M}\mathbb{L} = \mathbf{L}^*\mathbf{W}.$$

If $\mathbb{L}^{-1}\mathbb{L}_s$ is not diagonalizable (rank deficient), then we use the Jordan canonical form to get a similar result.

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Relating Poles to the Right Interpolation Points

On the other hand, note that

$$\mathbb{L}^{-1}\mathbb{L}_s = \mathbb{L}^{-1}\mathbb{L}_s\left(\mathbb{L}^{-1}\mathbb{L}\right) = \mathbb{L}^{-1}\left(\mathbb{L}_s\mathbb{L}^{-1}\right)\mathbb{L},$$

Therefore there exists $\mathbf{\Lambda}, \mathbf{Y} \in \mathbb{C}^{n \times n}$ such that

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A Zolotarev Upper-Bound (Rational Approximation)

Let $\mathbf{S} \in \mathbb{C}^{m \times n}$ with $m \geqslant n$, $\mathbf{A} \in \mathbb{C}^{m \times m}$, $\mathbf{B} \in \mathbb{C}^{n \times n}$, $\mathbf{M} \in \mathbb{C}^{m \times \nu}$, and $\mathbf{N} \in \mathbb{C}^{n \times \nu}$ such that $\mathbf{S}$ satisfies the Sylvester matrix equation

$$\mathbf{AS} - \mathbf{SB} = \mathbf{MN}^*. \tag{1}$$

Theorem 1 [Beckermann and Townsend 2017]

Let $\mathbf{A} \in \mathbb{C}^{m \times m}$ and $\mathbf{B} \in \mathbb{C}^{n \times n}$ be normal matrices with $m \geqslant n$ and let $\mathscr{E}$ and $\mathscr{F}$ be complex sets such that $\sigma(\mathbf{A}) \subseteq \mathscr{E}$ and $\sigma(\mathbf{B}) \subseteq \mathscr{F}$. Suppose that the matrix $\mathbf{S} \in \mathbb{C}^{m \times n}$ satisfies (2), and $\gamma = \mathsf{rank}(\mathbf{MN}^*)$. Then, the singular values of $\mathbf{S}$ satisfy

$$s_{j+\gamma k}(\mathbf{S}) \leqslant Z_k(\mathscr{E}, \mathscr{F}) s_j(\mathbf{S}) \quad \text{for} \quad 1 \leqslant j + \gamma k \leqslant n,$$

where $Z_k(\mathscr{E}, \mathscr{F})$ is the Zolotarev number defined on next slide.

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Can we apply this Theorem to our Sylvester equations?

$$\mathbb{L}(\mathbf{X\Lambda X}^{-1}) - \mathbf{M}\mathbb{L} = \mathbf{L}^*\mathbf{W} \quad \text{and} \quad (\mathbf{Y\Lambda Y}^{-1})\mathbb{L} - \mathbb{L}\mathbf{\Theta} \ = \ \mathbf{V}^*\mathbf{R}.$$

Third Zolotarev Problem (Rational Approximation)

Given two disjoint closed complex sets $\mathscr{E}$ and $\mathscr{F}$, the third Zolotarev problem is to find the rational function

$$\phi(x) = \frac{p(x)}{q(x)},$$

where p and q polynomials of at most degree k, such that

- $|\phi(x)| \geqslant 1$ for $x \in \mathscr{F}$,
- $|\phi(x)|$ as a small as possible for $x \in \mathscr{E}$.

The number $Z_k(\mathscr{E}, \mathscr{F})$ (following infimum) is referred to as the Zolotarev number:

$$Z_k(\mathscr{E}, \mathscr{F}) = \inf_{\phi \in \mathscr{R}_{k,k}} \frac{\sup\limits_{z \in \mathscr{E}} |\phi(z)|}{\inf\limits_{z \in \mathscr{F}} |\phi(z)|},$$

where $\mathscr{R}_{k,k}$ denotes the space of rational functions of degree at most (k, k).

Facts about $Z_k(\mathscr{E}, \mathscr{F})$

Zolotarev number:

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The Zolotarev number satisfy:

- Bounded: $0 \leqslant Z_k(\mathscr{E}, \mathscr{F}) \leqslant 1$,
- Symmetric: $Z_k(\mathscr{E}, \mathscr{F}) = Z_k(\mathscr{F}, \mathscr{E})$,
- Monotonic: $Z_{k+1}(\mathscr{E}, \mathscr{F}) \leqslant Z_k(\mathscr{E}, \mathscr{F})$.

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Key Intuition

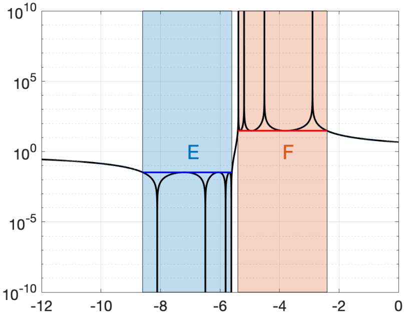
- When $\mathscr{E}$ and $\mathscr{F}$ are close,

  $Z_k(\mathscr{E}, \mathscr{F})$ is large.
- When $\mathscr{E}$ and $\mathscr{F}$ are well separated,

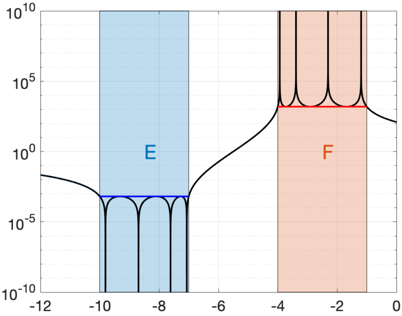
  $Z_k(\mathscr{E}, \mathscr{F})$ is small.

Example

Examples computed using the Rational Krylov Toolbox for Matlab



$$Z_4(E, F) = 1.1 \times 10^{-3}$$



$$Z_4(E, F) = 4.1733 \times 10^{-7}$$

See “The third and fourth Zolotarev problems” [Townsend 2016]

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General Idea

Next, we define the *observability* and *reachability* matrices associated with the sampled data:

$$\mathscr{O}_n := \begin{bmatrix} \boldsymbol{\ell}_1^* \mathbf{C}(\mu_1 \mathbf{E} - \mathbf{A})^{-1} \\ \vdots \\ \boldsymbol{\ell}_n^* \mathbf{C}(\mu_n \mathbf{E} - \mathbf{A})^{-1} \end{bmatrix}, \quad \mathscr{R}_n := \begin{bmatrix} (\theta_1 \mathbf{E} - \mathbf{A})^{-1} \mathbf{B} \mathbf{r}_1 & \cdots (\theta_n \mathbf{E} - \mathbf{A})^{-1} \mathbf{B} \mathbf{r}_n \end{bmatrix}.$$

Then the Loewner and shifted Loewner matrices can factor out as

$$\mathbb{L} = -\mathscr{O} \mathbf{E} \mathscr{R}, \qquad \mathbb{L}_s = -\mathscr{O} \mathbf{A} \mathscr{R}.$$

See [A. C. Antoulas, C. A. Beattie and S. Gügercin 2020].

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Therefore

$$\mathbb{L}^{-1} \mathbb{L}_s = \mathscr{R}_n^{-1} \mathbf{E}^{-1} \mathbf{A} \mathscr{R}_n \quad \text{and} \quad \mathbb{L}_s \mathbb{L}^{-1} = \mathscr{O}_n \mathbf{A} \mathbf{E}^{-1} \mathscr{O}_n^{-1}$$

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See [A. C. Antoulas, C. A. Beattie and S. Gügercin 2020].

Therefore

$$\mathbb{L}^{-1} \mathbb{L}_s = \mathscr{R}_n^{-1} \mathbf{E}^{-1} \mathbf{A} \mathscr{R}_n \quad \text{and} \quad \mathbb{L}_s \mathbb{L}^{-1} = \mathscr{O}_n \mathbf{A} \mathbf{E}^{-1} \mathscr{O}_n^{-1}$$

$$\mathbb{L}(\mathscr{R}_n^{-1} \mathbf{E}^{-1} \mathbf{A} \mathscr{R}_n) - \mathbf{M} \mathbb{L} = \mathbf{L}^* \mathbf{W} \quad \text{and} \quad (\mathscr{O}_n \mathbf{A} \mathbf{E}^{-1} \mathscr{O}_n^{-1}) \mathbb{L} - \mathbb{L} \boldsymbol{\Theta} \ = \ \mathbf{V}^* \mathbf{R}.$$

General Idea

Next, we define the *observability* and *reachability* matrices associated with the sampled data:

$$\mathscr{O}_n := \begin{bmatrix} \boldsymbol{\ell}_1^*\mathbf{C}(\mu_1\mathbf{E}-\mathbf{A})^{-1} \\ \vdots \\ \boldsymbol{\ell}_n^*\mathbf{C}(\mu_n\mathbf{E}-\mathbf{A})^{-1} \end{bmatrix}, \quad \mathscr{R}_n := \begin{bmatrix} (\theta_1\mathbf{E}-\mathbf{A})^{-1}\mathbf{B}\mathbf{r}_1 & \cdots (\theta_n\mathbf{E}-\mathbf{A})^{-1}\mathbf{B}\mathbf{r}_n \end{bmatrix}.$$

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See [A. C. Antoulas, C. A. Beattie and S. Gügercin 2020].

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$$\mathbb{L}^{-1}\mathbb{L}_s = \mathscr{R}_n^{-1}\mathbf{E}^{-1}\mathbf{A}\mathscr{R}_n \quad \text{and} \quad \mathbb{L}_s\mathbb{L}^{-1} = \mathscr{O}_n\mathbf{A}\mathbf{E}^{-1}\mathscr{O}_n^{-1}$$

$$(\mathbb{L}\mathscr{R}_n^{-1})(\mathbf{E}^{-1}\mathbf{A}) - \mathbf{M}(\mathbb{L}\mathscr{R}_n^{-1}) = \mathbf{L}^*\mathbf{W}\mathscr{R}_n^{-1} \quad \text{and} \quad (\mathbf{A}\mathbf{E}^{-1})(\mathscr{O}_n^{-1}\mathbb{L}) - (\mathscr{O}_n^{-1}\mathbb{L})\boldsymbol{\Theta} \;=\; \mathscr{O}_n^{-1}\mathbf{V}^*\mathbf{R}.$$

Key Singular Value Upper-Bound

Theorem 3 [R. A. Horn and C. R. Johnson 1991]

For $\mathbf{X}, \mathbf{Y} \in \mathbb{C}^{m \times n}$ and $q = \min\{m, n\}$,

$$s_{i+j-1}(\mathbf{X}\mathbf{Y}^*) \leqslant s_i(\mathbf{X}) s_j(\mathbf{Y}) \quad \text{for } 1 \leqslant i, j \leqslant q \text{ and } i + j \leqslant q + 1.$$

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Let $\mathbf{A}$ be a normal matrix and $\mathbf{E}$ the identity matrix, then

$$(\mathbb{L}\mathscr{R}_n^{-1})(\mathbf{A}) - \mathbf{M}(\mathbb{L}\mathscr{R}_n^{-1}) = \mathbf{L}^*\mathbf{W}\mathscr{R}_n^{-1}. \tag{3}$$

With $\gamma := \text{rank}(\mathbf{L}^*\mathbf{W}\mathscr{R}^{-1}) = \text{rank}(\mathbf{L}^*\mathbf{W}) \leqslant p$.

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$$s_{1+\gamma k}(\mathbb{L}) = s_{1+\gamma k}(\mathbb{L}\mathscr{R}_n^{-1}\mathscr{R}_n) \leqslant s_1(\mathscr{R}_n)\, s_{1+\gamma k}(\mathbb{L}\mathscr{R}_n^{-1}).$$

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Now apply Theorem 2 [B&T] to (3), taking the sets $\mathscr{E} = \sigma(\mathbf{M}) = M$ and letting $\mathscr{F} = \sigma(\mathbf{A}) = \Lambda$

$$s_{1+\gamma k}(\mathbb{L}) \leqslant s_1(\mathscr{R}_n) Z_k(\Lambda, M)\, s_1(\mathbb{L}\mathscr{R}^{-1}).$$

Note $\mathbb{L}\mathscr{R}_n^{-1} = -\mathscr{O}_n\mathbf{E}\mathscr{R}_n\mathscr{R}_n^{-1} = -\mathscr{O}_n$.

A Zolotarev Upper-Bound

Recall

$$\mathbb{L}_s = \mathbb{L}\mathbf{\Theta} + \mathbf{V}^*\mathbf{R} \qquad \text{and} \qquad \mathbb{L}_s = \mathbf{M}\mathbb{L} + \mathbf{L}^*\mathbf{W}.$$

with

$$\gamma_1 = \text{rank}(\mathbf{V}^*\mathbf{R}) \qquad \text{and} \qquad \gamma_2 = \text{rank}(\mathbf{L}^*\mathbf{W}).$$

Proposition ($\mathbf{A}$ normal and $\mathbf{E}$ identity)

Given $\mathbf{A}, \mathbf{E} \in \mathbb{C}^{n\times n}$, such that $\mathbf{A}$ is a normal matrix and $\mathbf{E}$ is the identity matrix. Define $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$, $\Theta = \{\theta_1, \ldots, \theta_n\}$, and $M = \{\mu_1, \ldots, \mu_n\}$. Then

- $\sigma_{1+\gamma_1 k}(\mathbb{L}) \leqslant \sigma_1(\mathscr{O}_n)\sigma_1(\mathscr{R}_n)Z_k(\Lambda, \Theta)$, for $1 \leqslant 1 + \gamma_1 k \leqslant n$
- $\sigma_{1+\gamma_2 k}(\mathbb{L}) \leqslant \sigma_1(\mathscr{O}_n)\sigma_1(\mathscr{R}_n)Z_k(\Lambda, M)$, for $1 \leqslant 1 + \gamma_2 k \leqslant n$

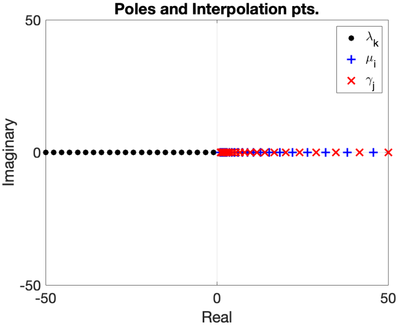
Example

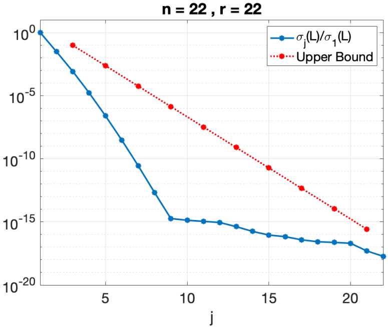
$\mathbf{A}, \mathbf{E} \in \mathbb{C}^{22 \times 22}$, $r = 22$

$\mathbf{A} = -\text{diag}(1:22)$, $\mathbf{E} = \mathbf{I}$

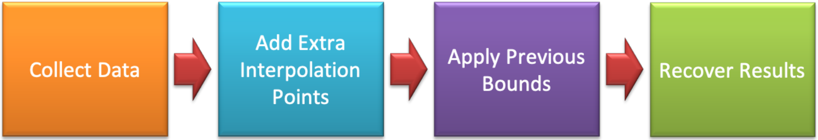
$a = 1$, $b = 50$;

`points = logspace(log10(a),log10(b),2*r);`





Main Result 1 $(\nu = \rho = r < n)$



Collect Data

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Theorem 4 (Model Reduction)

Given $\mathbf{A}, \mathbf{E} \in \mathbb{C}^{n \times n}$, such that $\mathbf{A}$ is a normal matrix and $\mathbf{E}$ is the identity matrix. Define $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$, $\mathit{\Theta} = \{\theta_1, \ldots, \theta_r\}$, and $M = \{\mu_1, \ldots, \mu_r\}$. Then

- $\sigma_{1+pk}(\mathbb{L}) \leqslant \sigma_1(\mathscr{O}_r)\sigma_1(\mathscr{R}_r)Z_{k-1}(\Lambda, \mathit{\Theta})$, for $1 \leqslant 1 + pk \leqslant r$,
- $\sigma_{1+mk}(\mathbb{L}) \leqslant \sigma_1(\mathscr{O}_r)\sigma_1(\mathscr{R}_r)Z_{k-1}(\Lambda, M)$, for $1 \leqslant 1 + mk \leqslant r$.

An Adaptive Zolotarev Upper-Bound

Theorem 3 [R. A. Horn and C. R. Johnson 1991]

For $\mathbf{X}, \mathbf{Y} \in \mathbb{C}^{m\times n}$ and $q = \min\{m, n\}$,

$$s_{i+j-1}(\mathbf{X}\mathbf{Y}^*) \leqslant s_i(\mathbf{X})s_j(\mathbf{Y}) \quad \text{for } 1 \leqslant i, j \leqslant q \text{ and } i + j \leqslant q + 1.$$

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Let $\mathbf{A}$ be a normal matrix and $\mathbf{E}$ the identity matrix. By Theorem 3 [H&J]

$$s_{1+\gamma k+i+j}(\mathbb{L}) = s_{1+\gamma k+i+j}(\mathbb{L}\mathscr{R}_n^{-1}\mathscr{R}_n) \leqslant s_{1+i}(\mathscr{R}_n)\, s_{1+\gamma k+j}(\mathbb{L}\mathscr{R}_n^{-1})$$

for any integers $i, j \geqslant 0$ such that $1 \leqslant 1 + \gamma k + i + j \leqslant n$.

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$$s_{1+\gamma k+j}(\mathbb{L}\mathscr{R}^{-1}) \leqslant Z_k(\Lambda, M)s_{1+j}(\mathbb{L}\mathscr{R}^{-1}),$$

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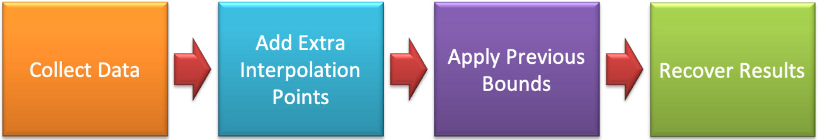
for any integers $i, j \geqslant 0$ such that $1 \leqslant 1 + \gamma k + i + j \leqslant n$. From Theorem 2 [B&T], with sets $\mathscr{E} = \sigma(\mathbf{M}) = M$ and $\mathscr{F} = \sigma(\mathbf{A}) = \Lambda$, we get

$$s_{1+\gamma k+j}(\mathbb{L}\mathscr{R}^{-1}) \leqslant Z_k(\Lambda, M) s_{1+j}(\mathbb{L}\mathscr{R}^{-1}),$$

and, via Theorem 3 [H&J] and $\mathbb{L} = -\mathscr{O}_n \mathbf{E} \mathscr{R}_n$, we get

$$s_{1+j}(\mathbb{L}\mathscr{R}_n^{-1}) = s_{1+j}(\mathscr{O}_n\mathbf{E}) \leqslant s_{1+j}(\mathscr{O}_n)\sigma_1(\mathbf{E}) = s_{1+j}(\mathscr{O}_n).$$

Main Result 2 ($\nu = \rho = r < n$)



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Corollary 5 (Model Reduction)

Given $\mathbf{A}, \mathbf{E} \in \mathbb{C}^{n \times n}$,such that $\mathbf{A}$ is a normal matrix and $\mathbf{E}$ is the identity matrix. Define $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$, $\Gamma = \{\theta_1, \ldots, \theta_r\}$, and $M = \{\mu_1, \ldots, \mu_r\}$. Then

- For $1 \leqslant 1 + pk + i + j \leqslant r$

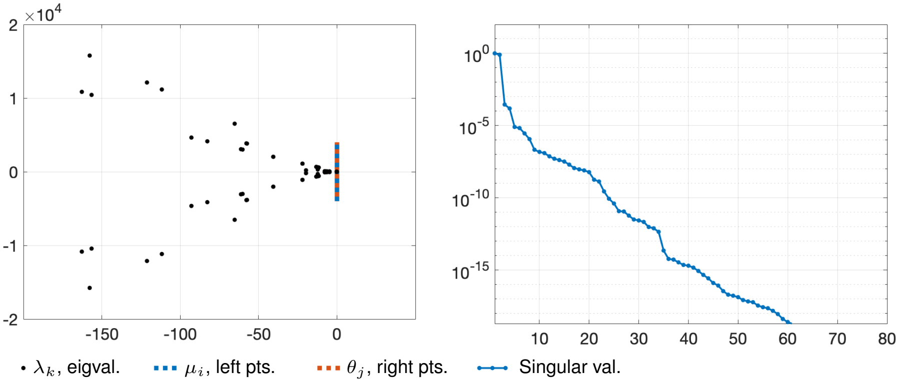
$$\sigma_{1+pk+i+j}(\mathbb{L}) \leqslant \sigma_{1+i}(\mathscr{O}_r)\sigma_{1+j}(\mathscr{R}_r) Z_{k-1}(\Lambda, \Gamma),$$

- For $1 \leqslant 1 + mk + i + j \leqslant r$

$$\sigma_{1+mk+i+j}(\mathbb{L}) \leqslant \sigma_{1+i}(\mathscr{O}_r)\sigma_{1+j}(\mathscr{R}_r) Z_{k-1}(\Lambda, M).$$

Example

Consider the CD player bench mark problem, with one input and 1 output.



For this example we used $r = 80$ interpolation points taken logarithmically spaced over $[-4000,\ -10] \cup [10,\ 4000]$.

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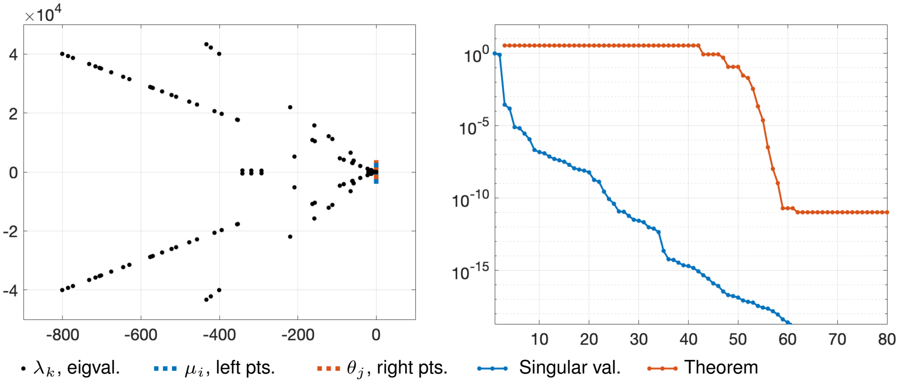

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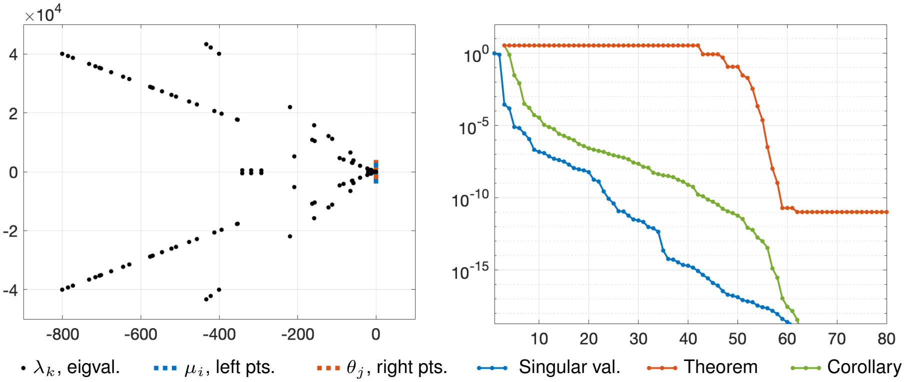

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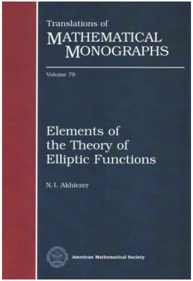
Conclusions

- New bounds inform how location of interpolation points and poles of system influence singular values of $\mathbb{L}$.

- In the MOR setting, the upper-bound seem to be less tight since we used a larger Zolotarev number $Z_{k-1}(\mathcal{E},\mathcal{F})$, but allowing to use smaller singular values of the observability and reachability matrices we improve the results.



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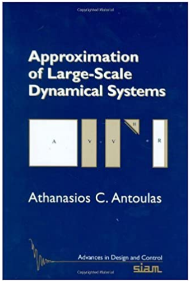

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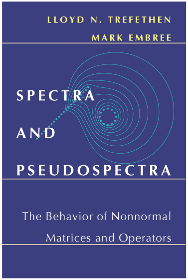
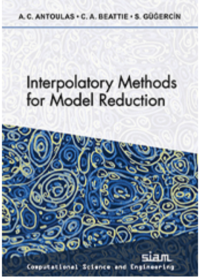


LLOYD N. TREFETHEN
MARK EMBREE

SPECTRA
AND
PSEUDOSPECTRA

The Behavior of Nonnormal
Matrices and Operators



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Interpolatory Methods
for Model Reduction

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