SOME MIXED AND ASSOCIATED BOUNDARY VALUE PROBLEMS IN THE THEORY OF THIN PLATES

by

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h - thickness of a plate

x, y - coordinates of a point

Γ - curve in the x, y plane

$\bar{n}$ - outer normal to a curve

$\bar{s}$ - unit tangent to a curve

$q(x,)$ - intensity of load

V_n - applied vertical shear force per unit length

M_n - applied bending moment per unit length

D - flexural rigidity of a plate $\dfrac{Eh^3}{12(1-\nu^2)}$

E - Young's Modulus

ν - Poisson's Ratio

$w(x, y)$ - deflection of the middle surface of a plate

R - region of (x, y) plane occupied by the middle plane of the plate

ϕ, ψ - functions

P - concentrated load

ξ, η - coordinates in the (x, y) plane

a, b - measurements of length

r, θ - polar coordinates of a point

$z = x + iy$

$\bar{z} = x - iy$

CHAPTER I

SOLUTION OF THE PLATE EQUATION

Boundary value problems in the field of mechanics are attacked by a variety of procedures. Outstanding among these are the method of series, the method of singularities, and the method of functional equations. Mixed boundary problems are most naturally treated by the latter of these three methods. In order, then, to lay a firm foundation for the treatment of such problems, we begin with a brief development of the plate equation, the associated boundary conditions, and the manner of treatment of such problems by these two latter approaches.

1.1) Differential Equation and Boundary Conditions.

We begin, then, by considering a plate of thickness $\underline{h}$ oriented as shown in fig. (1.). The middle surface is in the (x, y) plane and is bounded by the curve Γ . The outward normal to Γ is denoted by $\bar{n}$ and positive direction of arc length by $\bar{s}$. It is assumed that the upper surface of the plate is subjected to a load intensity $q(x, y)$. The vertical bounding surface is subjected to a resultant vertical shear force V_n per unit of length and a bending moment M_n per unit of length, positive as shown in fig. (1.). The total energy of the plate is then given by

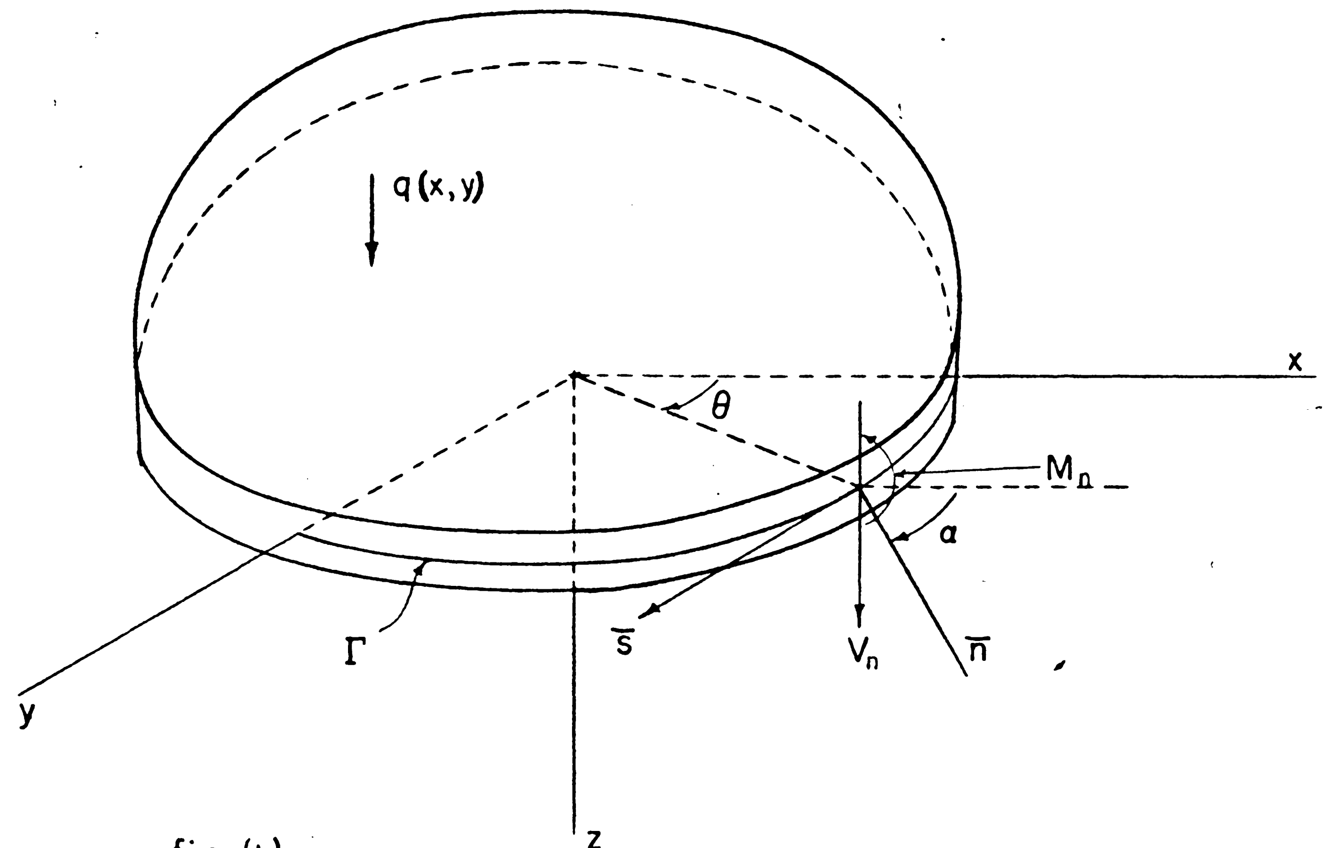

fig. (1.)

$$V = {}^{D}/_{2} \iint_R \left\{ (\nabla^2 w)^2 - 2(1-v)(w_{xx}w_{yy} - w_{xy}^2) \right\} dxdy$$

$$- \iint_R qw\, dxdy + \int_\Gamma \left\{ M_n \frac{\partial w}{\partial n} - V_n w \right\} ds \qquad (1.1.1)$$

where $\underline{w}$ is the deflection of the middle surface of the plate, and $\underline{R}$ is the region in the (x, y) plane bounded by Γ . The sign conventions used here are those in [1] *.

The following derivation is due to Kirchhoff [2] and is included here only for the sake of completeness.

Invoking the principle of minimum total energy, we find that the equilibrium position of the plate under the action of the applied loading is characterized by the function w (x, y) which gives $\underline{V}$ its minimum value. We have then

$$\delta V = {}^{D}/_{2}\, \delta \iint_R \left\{ (\nabla^2 w)^2 - 2(1-v)(w_{xx}w_{yy} - w_{xy}^2) - \frac{2}{D} qw \right\} dxdy$$

$$+ \delta \int_\Gamma \left\{ M_n \frac{\partial w}{\partial n} - V_n w \right\} ds = 0 . \qquad (1.1.2)$$

Since the δ operation and differentiation with respect to x, y are interchangeable, we obtain the following

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* Numbers in square brackets refer to the bibliography at the end of the text.

$$D \iint_R \delta w \left[\nabla^4 w - q/D\right] dxdy - D \int_\Gamma \left[\delta w \frac{\partial \nabla^2 w}{\partial n} - \nabla^2 w \frac{\partial \delta w}{\partial n}\right] ds$$

$$+ \int_\Gamma \left[M_n \frac{\partial \delta w}{\partial n} - V_n \delta w\right] ds - D(1-v) \iint_R \left[w_{xx} (\delta w)_{yy}\right.$$

$$\left. + w_{yy}(\delta w)_{xx} - 2 w_{xy}(\delta w_{xy})\right] dxdy = 0 \qquad (1.1.3)$$

from an application of Green's theorem for the plane.

If now, we introduce the functional

$$K(\phi, \psi) = -(1-v) \left[\phi_{xx} \psi_{yy} + \phi_{yy} \psi_{xx} - 2\phi_{xy} \psi_{xy}\right] \qquad (1.1.4)$$

then another application of Green's theorem produces

$$\iint_R K(\phi, \psi) dxdy = -(1-v) \int_\Gamma \frac{\partial \phi}{\partial n} \left[\psi_{yy} \cos^2\alpha + \psi_{xx} \sin^2\alpha - 2\psi_{xy} \sin\alpha \cos\alpha\right] ds$$

$$+ (1-v) \int_\Gamma \phi \frac{\partial}{\partial s} \left[(\psi_{xx} - \psi_{yy}) \sin\alpha \cos\alpha - \psi_{xy}(\cos^2\alpha - \sin^2\alpha)\right] ds$$

$$-(1-v) \sum_i \phi_i \left. (\psi_{xx} - \psi_{yy}) \sin\alpha \cos\alpha - \psi_{xy} (\cos^2\alpha - \sin^2\alpha)\right]_i \qquad (1.1.5)$$

It is to be noted that a finite summation occurs from the application of this theorem. However, these terms only arise when discontinuities in the slope of the boundary curve occur. If the bounding curve Γ has a continuously turning tangent, then this finite summation vanishes. If the bounding curve does not have a tangent line with this property, this term will still vanish provided the deflection vanishes. This term will have to be considered wherever a free corner appears.

Applying this expression to (1.1.3), we finally obtain

$$D\iint_R \delta w\left[\nabla^4 w - q/D\right]dxdy + \int_\Gamma \frac{\partial\,\delta w}{\partial n}\Big[M_n + D\Big\{\nabla^2 w - (1-\nu)$$

$$(w_{yy}\cos^2\alpha + w_{xx}\sin^2\alpha - 2w_{xy}\sin\alpha\cos\alpha)\Big\}\Big]\,ds$$

$$-\int_\Gamma \delta w\Big[V_n + D\Big\{\frac{\partial\nabla^2 w}{\partial n} - (1-\nu)\frac{\partial}{\partial s}\Big[(w_{xx} - w_{yy})\sin\alpha\cos\alpha$$

$$- w_{xy}(\cos^2\alpha - \sin^2\alpha)\Big]\Big\}\Big]\,ds$$

$$- D(1-\nu)\sum_i \delta w_i\left[(w_{xx}-w_{yy})\sin\alpha\cos\alpha - w_{xy}(\cos^2\alpha - \sin^2\alpha)\right]_i = 0 \qquad (1.1.6)$$

From this, there results the following differential equation

$$\nabla^4 w = \frac{q(x,y)}{D} \qquad (1.1.7)$$

From the form of the line integrals appearing in (1.1.6) it becomes apparent that these line integrals can be made to vanish in a variety of ways. Consequently, there arrises a multiplicity of boundary conditions. Indeed, we have the following possibilities:

i.) Along a clamped portion of the edge

$$w = 0 \quad , \quad \frac{\partial w}{\partial n} = 0 \tag{1.1.8}$$

ii.) Along a simply-supported portion of the edge

$$w = 0 \, , \quad \nabla^2 w - (1-v)\left[w_{yy}\cos^2\alpha + w_{xx}\sin^2\alpha - 2w_{xy}\sin\alpha\cos\alpha \right] = 0 \tag{1.1.9}$$

iii.) Along a free portion of the edge

$$\nabla^2 w = (1-v)\left[w_{yy}\cos^2\alpha + w_{xx}\sin^2\alpha - 2w_{xy}\sin\alpha\cos\alpha \right]$$

$$\frac{\partial \nabla^2 w}{\partial n} = (1-v)\frac{\partial}{\partial s}\left[(w_{xx} - w_{yy})\sin\alpha\cos\alpha - w_{xy}(\cos^2\alpha - \sin^2\alpha) \right]$$

$$\left[(w_{xx} - w_{yy})\sin\alpha\cos\alpha - w_{xy}(\cos^2\alpha - \sin^2\alpha) \right]_i = 0 \, , \; i = 1, 2, ---K \tag{1.1.10}$$

The finite set of conditions in the last line, of course, does not occur if there are no corners along the free arc.

It is to be noted that a fourth possibility exists; namely, the elastically supported edge. Since we shall not concern ourselves with such problems, we shall not consider the statement of this set of boundary conditions.

1.2) Basic Integral Theorem.*

In order to exploit the method of singularities, it will now be necessary to develop an integral theorem akin to the type which

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* This, and paragraphs (1.3) and (1.4) were developed independently by the author. The motivation obviously stems from the close link between these problems and those of potential theory.

proves so fruitful in potential theory. As would be expected, the structure of this relationship is obtained by the use of Green's theorem.

For any two functions u, v possessed of sufficient differentiability, Green's theorem for the plane yields

$$\iint_R \left[u\nabla^2 v - v\nabla^2 u \right] dxdy = \int_\Gamma \left[u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right] ds \qquad (1.2.1)$$

If we let $v = \nabla^2\phi$ and $u = \psi$, we obtain

$$\iint_R \left[\psi\nabla^4\phi - \nabla^2\phi\nabla^2\psi \right] dxdy = \int_\Gamma \left[\psi\frac{\partial\nabla^2\phi}{\partial n} - \nabla^2\phi\frac{\partial\psi}{\partial n} \right] ds \qquad (1.2.2)$$

and; alternately, letting $u = \nabla^2\phi$ and $v = \psi$, there results

$$\iint_R \left[\nabla^2\psi\nabla^2\phi - \phi\nabla^4\psi \right] dxdy = \int_\Gamma \left[\nabla^2\psi\frac{\partial\phi}{\partial n} - \phi\frac{\partial\nabla^2\psi}{\partial n} \right] ds \qquad (1.2.3)$$

On adding (1.2.2) to (1.2.3), we obtain the integral relation of paramount importance in the method of singularities for the biharmonic equation

$$\iint_R \left[\psi\nabla^4\phi - \phi\nabla^4\psi \right] dxdy = \int_\Gamma \left[\psi\frac{\partial\nabla^2\phi}{\partial n} - \nabla^2\phi\frac{\partial\psi}{\partial n} + \nabla^2\psi\frac{\partial\phi}{\partial n} - \phi\frac{\partial\nabla^2\psi}{\partial n} \right] ds \qquad (1.2.4)$$

The right-hand side of this expression involves a line integral whose integrand does not contain the precise kind of physical quantities which are specified at the boundary. Thus, we note that from (1.1.6) it follows that

$$\frac{-M_n}{D} - \nabla^2 w = -(1-v)\left[w_{yy}\cos^2\alpha + w_{xx}\sin^2\alpha - 2w_{xy}\sin\alpha\cos\alpha\right]$$

$$\frac{V_n}{D} + \frac{\partial\nabla^2 w}{\partial n} = (1-v)\frac{\partial}{\partial s}\left[(w_{xx} - w_{yy})\sin\alpha\cos\alpha - w_{xy}(\cos^2\alpha - \sin^2\alpha)\right] \tag{1.2.5}$$

where M_n, V_n are the applied bending moment and resultant vertical shear, respectively. We have implicitly assumed that the boundary under consideration has no corners. For any specific situation in which this might be the case, we shall make a special consideration.

With (1.2.5), we may then write (1.2.4) as

$$D \iint_R \left[\psi\nabla^4\phi - \phi\nabla^4\psi\right] dxdy =$$

$$\int_\Gamma \left\{\frac{\partial\psi}{\partial n} M_n(\phi) - \quad V_n(\phi) - \frac{\partial\phi}{\partial n} M_n(\psi) + \phi V_n(\psi)\right\} ds \tag{1.2.6}$$

Before a utilization of (1.2.6) is possible, it is necessary to introduce the concept of a Green's function. This function, which is the representation of a solution for a concentrated load, is the topic of the next section.

1.3) Green's Functions for the Plate Equation.

The homogeneous form of (1.1.7) has an infinity of solutions that are analytic* everywhere in R with the exception of a single point. If the distance from (x, y) to (ξ, η) is denoted by R_1, then it is easy to verify that one such solution is

$$R_1^2 \log R_1 \tag{1.3.1}$$

when considered as a function of (x, y). For ease of interpretation, we consider, momentarily, that the point (ξ, η) in fig. (2.) is the point (0, 0) and consider

$$g(x, y, \xi, \eta) = \frac{P}{8\pi D} R_1^2 \log R_1 \tag{1.3.2}$$

If we surround the origin by a small circle of radius ϵ, then to interpret g, we compute the vertical shear on a cylindrical surface for which this circle is the directrix and whose generators are parallel to the z-axis. (See fig. (3.).) We have

$$\nabla^2 g = \frac{\partial^2 g}{\partial r^2} + \frac{1}{r}\frac{\partial g}{\partial r} = \frac{P}{2\pi D}(\log r + 1) \tag{1.3.3}$$

On this surface $\frac{\partial}{\partial n} = \frac{\partial}{\partial r}$, hence $\frac{\partial \nabla^2 g}{\partial n} = \frac{P}{2\pi D r}$ and

$$Q_n = -D \frac{\partial \nabla^2 g}{\partial n} \tag{1.3.4}$$

so that the vertical shear on this lateral surface of the cylinder is

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* The word "analytic" when used in connection with functions of real variables is used in the sense of E. Goursat, A Course in Mathematical Analysis, vol. I, page 437.

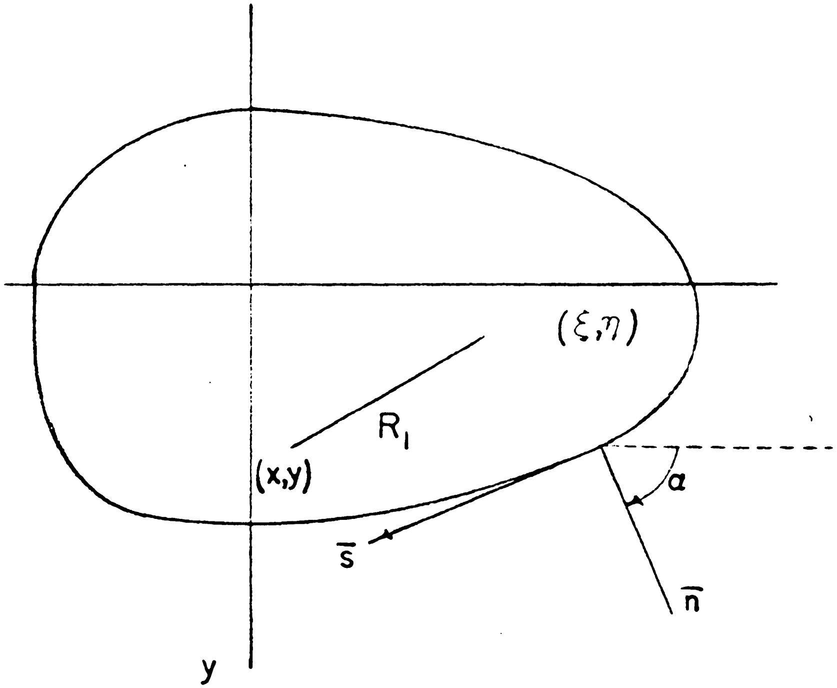


fig. (2.)

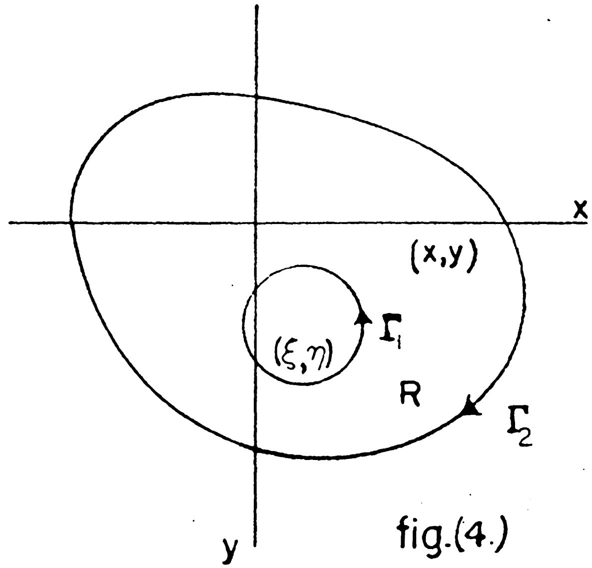


fig.(4.)

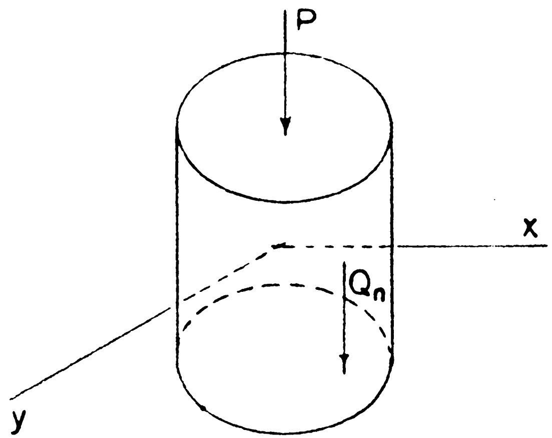


fig. (3.)

$$Q_n = -\frac{P}{2\pi}\frac{1}{\epsilon} \tag{1.3.5}$$

Since this value is uniform over this surface, the total vertical force on this face is

$$2\pi\epsilon Q_n = -P \tag{1.3.6}$$

Since, by definition, vertical shear is positive when it acts in the positive direction of the z-axis, we see that such a deflection function $g(x, y, \xi, \eta)$ resists a positive load $\underline{P}$. Thus, if we disregard the boundary conditions, we have the deflection due to a concentrated load $\underline{P}$. Unfortunately, circumstances are not as simple as this. For in order to obtain a complete solution to any problem, we must also take into account the conditions that prevail at the boundary.

In view of these statements, we pose the following problem: Determine two functions $G_1(x, y, \xi, \eta)$ and $G_2(x, y, \xi, \eta)$ such that

$$G_1(x, y, \xi, \eta) = \frac{P}{8\pi D}\left[R_1^2 \log R_1 + H_1(x, y, \xi, \eta)\right] \tag{1.3.7}$$

and

$$G_2(x, y, \xi, \eta) = \frac{P}{8\pi D}\left[R_1^2 \log R_1 + H_2(x, y, \xi, \eta)\right] \tag{1.3.8}$$

where $H_1(x, y, \xi, \eta)$ and $H_2(x, y, \xi, \eta)$ are analytic everywhere in $\underline{R}$ and on the boundary Γ, G_1 satisfies

$$G_1 = 0 \quad , \quad \frac{\partial G_1}{\partial n} = 0 \quad ; \quad x, y \in \Gamma \tag{1.3.9}$$

and G_2 satisfies

$$G_2 = 0, \ \nabla^2 G_2 = (1-\nu)\left[G_{2xx}\sin^2\alpha + G_{2yy}\cos^2\alpha - 2G_{2xy}\sin\alpha\cos\alpha\right]; x, y \in \Gamma \tag{1.3.10}$$

Furthermore, both G_1 and G_2 are to satisfy the homogeneous form of (1.1.7).

The existence and uniqueness of such functions for regions $\underline{R}$ which fall within the scope of engineering applications will be ventured as a conjecture.

The function G_1 shall be referred to as Green's function of the first kind. Physically, it is interpreted as the deflection of a plate clamped along its entire boundary and subjected to a concentrated load $\underline{P}$ at the point (ξ, η). The function G_2 will be referred to as Green's function of the second kind, and it is interpreted as the deflection of a plate simply-supported at its boundary and subjected to a concentrated load $\underline{P}$ at the point (ξ, η).

We remark now that these are not the only possible Green's functions for the plate equation. Indeed, a multiplicity of such functions exists as will be shown in later chapters. Our purpose for introducing these two functions at this stage is to prepare for the computation of other kinds in the later sections.

1.4) Solution of the Boundary Value Problems by Green's Theorem.

If now in (1.2.6), we associate $\psi \backsim w$ and $\phi \backsim G$ and consider fig. (4.), we obtain

$$\oint_{\Gamma_1} \{ \quad \} ds = \oint_{\Gamma_2} \{ \quad \} ds + D \iint_R G \nabla^4 w \, dxdy \qquad (1.4.1)$$

For convenience, we take Γ_1 to be a small circle of radius ϵ and center at (ξ, η). Then the only significant term in the integrand of

$$\oint \{ \quad \} ds \qquad (1.4.2)$$

$$-w \, V_n (G) \qquad (1.4.3)$$

For either G

$$-\lim_{\epsilon \to 0} \int_{\Gamma_1} w V_n ds = w(\xi, \eta) P \;, \; \lim_{\epsilon \to 0} D \iint_R G \nabla^4 w dxdy = \iint_R Gq dxdy \qquad (1.4.4)$$

By passing to the limit in (1.4.1), we obtain

$$P w(x, y) = \iint_R G q \, d\xi d\eta$$

$$\int_{\Gamma_2} \left\{ \frac{\partial w}{\partial n} M_n (G) - w V_n (G) - \frac{\partial G}{\partial n} M_n (w) + G V_n (w) \right\} ds \qquad (1.4.5)$$

The interchange of (x, y) with (ξ, η) is valid. This follows from the self-adjointness of the homogeneous form of (1.1.7).

If now, the normal derivative, β (s), and the deflection, α(s), are assigned at the boundary, we have

$$w(x,y) = \frac{1}{P}\int_{\Gamma_2}\left\{\beta(s)\, M_n(G_1) - \alpha(s)\, V_n(G_1)\right\}ds + \frac{1}{P}\iint_R G_1 q\, d\xi d\eta \tag{1.4.6}$$

On the other hand, if the deflection, $\alpha(s)$, and the bending moment, $\gamma(s)$, are assigned at the boundary

$$w(x,y) = -\frac{1}{P}\int_{\Gamma_2}\left\{\alpha(s)\, V_n(G_2) + \gamma(s)\frac{\partial G_2}{\partial n}\right\}ds + \frac{1}{P}\iint_R G_2 q\, d\xi d\eta \tag{1.4.7}$$

Expressions (1.4.6) and (1.4.7) now provide the necessary representations to attack the types of problems suggested in the introduction. To proceed, we must obtain explicit representations for G_1 and G_2 for the regions involved. This will be the subject matter of Chapter II.

It should be remarked that certain restrictions must be placed on the behavior of w and its derivatives if any of the regions under consideration are such that part of the boundaries extend to infinity. Indeed, there should be no contributions from the line integral over that part of the boundary that is at infinity. However, instead of investigating the type of behavior w must exhibit at infinity at this point, we shall relegate this topic to the chapter in which we present the Green's functions for these regions.

1.5) Reduction of the Problem to Functional Equations.

Certain of the problems which we shall solve will be most easily treated by the method of complex variables as developed for the biharmonic equation by Mushkelisvili 3 . Consequently, we shall briefly outline this procedure, also, for future use. All notation concerned with complex variables will be that of 3 .

If we introduce the transformation

$$z = x + iy \qquad \overline{z} = x - iy \tag{1.5.1}$$

then the differential equation (1.1.7) may be written in the form

$$\frac{\partial^4 w}{\partial z^2 \partial \overline{z}^2} = \frac{q(z, \overline{z})}{16 D} \tag{1.5.2}$$

The general solution of (1.5.2) may now be written as a sum of a complementary part plus a particular integral. Hence, we set

$$2w = \overline{z}\phi(z) + z\overline{\phi}(\overline{z}) + \psi(z) + \overline{\psi}(\overline{z}) + \frac{1}{8D} Q(z, \overline{z}) \tag{1.5.3}$$

where $\phi(z)$, $\psi(z)$ are arbitrary functions of the complex variable $\underline{z}$. The remaining term in this expression is a particular solution of the differential equation.

The notation in (1.5.3) is to be interpreted as follows. In general

$$f(z) = \sum_{n=0}^{\infty} F_n z^n \qquad \text{or} \qquad f(z) = \sum_{n=0}^{\infty} f_n z^{-n}$$

where one uses the expansion appropriate to the region containing $\underline{z}$. We have then the following

$$\overline{f}(\overline{z}) = \sum_{n=0}^{\infty} \overline{F}_n \overline{z}^n \quad , \qquad \overline{f}(\overline{z}) = \sum_{n=0}^{\infty} \overline{f}_n \overline{z}^{-n} \quad ,$$

$$\bar{f}(z) = \sum_{n=0}^{\infty} \bar{F}_n z^n \quad , \quad \bar{f}(z) = \sum_{n=0}^{\infty} \bar{f}_n z^{-n} \ .$$

It can be shown, see [3] page 109, that given any function $\underline{w}$ as a solution of (1.1.7), there always exist two functions ϕ, ψ which validate the representation in (1.5.3). We shall denote the interior of the plate by S^+ and the region exterior to the plate by S^-. If the plate is subjected to a concentrated load, it will be assumed that the singular part of $\underline{w}$ is representable in Q, and hence ϕ and ψ will be analytic in S^+. Thus for a concentrated load acting at z_o, $(z_o \in S^+)$, it follows that

$$Q_P = \frac{1}{\pi} (z - z_o)(\bar{z} - \bar{z}_o) \log (z - z_o)(\bar{z} - \bar{z}_o) \qquad (1.5.4)$$

We may write, quite generally,

$$2w = \bar{z}\phi_1(z) + z\bar{\phi}_1(\bar{z}) + \psi_1(z) + \bar{\psi}_1(\bar{z}) \qquad (1.5.5)$$

where $\underline{w}$ so defined satisfies the homogeneous form of (1.5.2) everywhere except $z = z_o$.

Since we already have the general solution for the differential equation represented, all that is needed in this approach is a satisfaction of the boundary conditions. For any given problem, if we demand that the functions ϕ and ψ are so chosen that the boundary conditions are satisfied, we are led to a system of functional equations which must hold along the boundary of the plate. We shall find this approach very useful in the solution of certain problems in the future chapters.

CHAPTER II

COMPUTATIONS OF GREEN'S FUNCTIONS

For a given differential equation, the associated Green's functions are dependent on the domain and boundary conditions alone. In the theory of the potential, the great success with which explicit solutions are obtained for the first boundary value problem occurs because of the invariance of Laplace's equation under conformal transformation.

Unfortunately, no such procedure, as simple as that mentioned above, exists for the biharmonic equation, although P. R. Garabedian [4] has recently given a formal method for the construction of Green's function of the first kind for the biharmonic equation. The method, for arbitrary regions, is rather cumbersome from a computational point of view. Indeed, very few explicit expressions occur for either the first or second Green's functions. The great success in the field of explicit presentation occurs with domains that extend to infinity in at least one direction. It should be remarked that the circle is an exception to this case. This, however, is not startling in view of the "quasi-invariance" of solutions to the biharmonic equation under inversion transformations.

Thus, we proceed to catalogue the Green's functions associated with such domains. In particular, we shall discuss the half-infinite, quarter-infinite regions, and the circle.

2.1) Half-Infinite Plate.

We shall consider the half-infinite plate which occupies the region $\underline{R}$ of the (x, y) plane defined by $y > 0$. This particular region enjoys the feature of possessing both of the Green's functions mentioned, in addition to a number of others, in elementary, closed forms.

Green's function of the first kind is well-known in the literature on plates. It was first obtained by J. H. Michell [5]. For computational purposes, it is more convenient to introduce the complex variable notation as in fig. (5.). With this, there results

$$G_1(z, \bar{z}, z_o, \bar{z}_o) = \frac{P}{16\pi D}\left[(z-z_o)(\bar{z}-\bar{z}_o)\log\frac{(z-z_o)(\bar{z}-\bar{z}_o)}{(\bar{z}-z_o)(z-\bar{z}_o)} - (z-\bar{z})(z_o-\bar{z}_o)\right] \tag{2.1.1}$$

A straightforward differentiation of (2.1.1) produces the following expression for the bending moment along the edge of the plate

$$M_n(G_1) = \frac{-P(\bar{z}_o-z_o)^2}{4\pi(z-z_o)(\bar{z}-\bar{z}_o)} = \frac{P}{\pi}\,\frac{\eta^2}{(x-\xi)^2+\eta^2} \tag{2.1.2}$$

Since we shall not have any immediate use for V_n, we shall not present it here.

Green's function of the second kind for this region is somewhat simpler to compute. Indeed, we examine

$$G_2 = \frac{P}{16\pi D}\left[(z-z_o)(\bar{z}-\bar{z}_o)\log(z-z_o)(\bar{z}-\bar{z}_o) - (z-\bar{z}_o)(\bar{z}-z_o)\log(z-\bar{z}_o)(\bar{z}-z_o)\right] \tag{2.1.3}$$

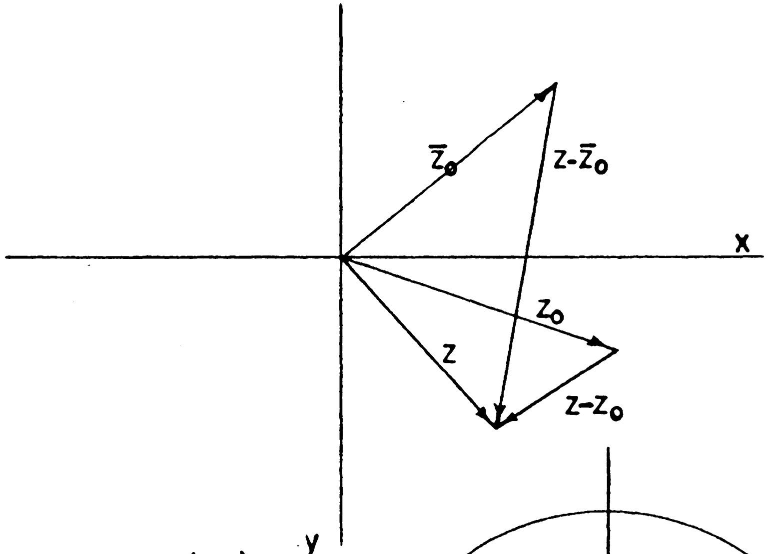


fig. (5.)

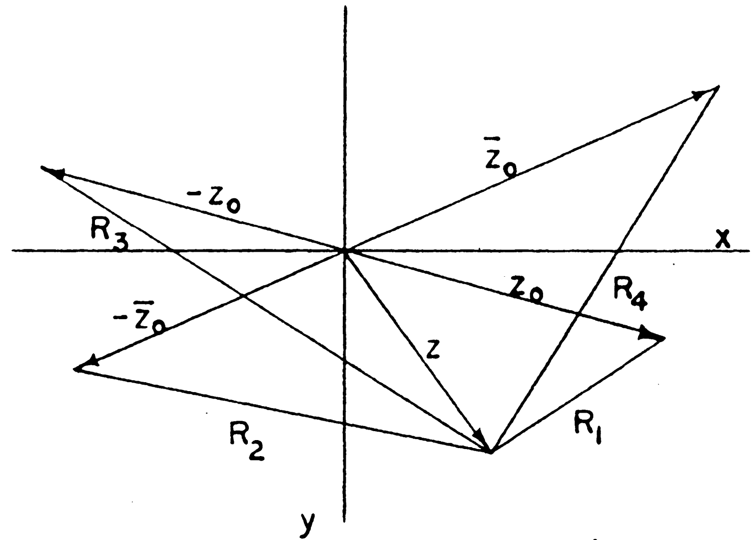


fig. (6.)

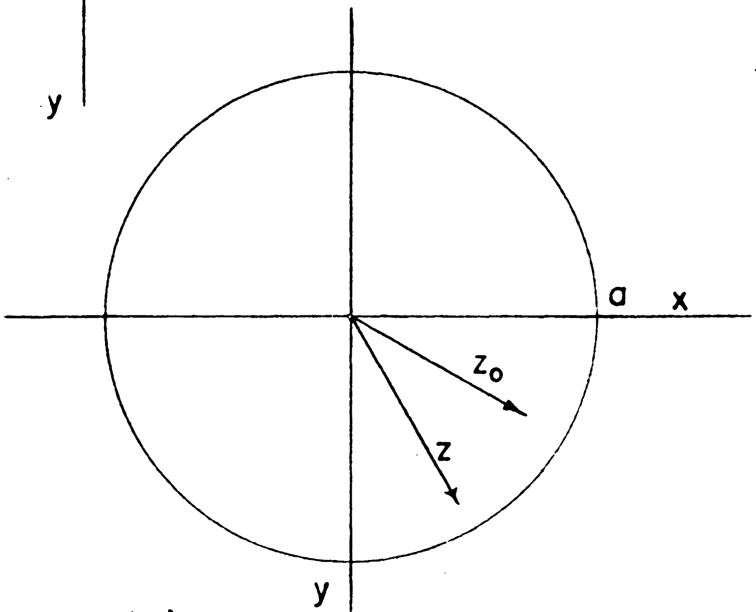


fig. (7.)

The motivation for consideration of this function stems directly from the method of images used in potential theory. Since $z = \bar{z}$ along the real axis, it follows, immediately, that

$$G_2 (x, x, z_o, \bar{z}_o) = 0 \tag{2.1.4}$$

Computing the Laplacian of G_2 considered as a function of z, $\bar{z}$, we obtain

$$\nabla^2 G_2 = \frac{P}{4\pi D} \log \frac{(z-z_o)(\bar{z}-\bar{z}_o)}{(z-\bar{z}_o)(\bar{z}-z_o)} \tag{2.1.5}$$

Evidently, this quantity vanishes for $z = \bar{z}$. Hence (2.1.3) has the desired boundary values along the real axis. Furthermore, $\nabla^2 G_2$ is a harmonic function. Thus (2.1.3) satisfies the biharmonic equation.

It is worthwhile noting that this function is such as to offer no resistance to rotation at the edge along the real axis. If one were to expand the appropriate quantities for large $|z|$, it would become apparent that the moments and shears at infinity contribute to the support of the plate. In any future considerations involving this function, we shall always demand that the finite edge always supports the plate.

Later computations will only require $\frac{\partial G_2}{\partial n}$. When this operation is carried out, there results

$$\frac{\partial G_2}{\partial n} = \frac{P\eta}{4\pi D} \left\{ \log \left[(x - \xi)^2 + \eta^2 \right] + 1 \right\} \tag{2.1.6}$$

This may also be expressed in terms of the complex representation.

2.2) Quarter-Infinite Plate.

Let the quarter-infinite plate occupy the region of the plane defined by $x > 0$, $y > 0$ as shown in fig. (6.). We are able, once more, to utilize the method of images to obtain the Green's function of the second kind for this region. Indeed, if we utilize the notation of fig. (6) where we have

$$R_1^2 = (z-z_o)(\bar{z}-\bar{z}_o) \quad , \qquad R_2^2 = (z+\bar{z}_o)(\bar{z}+z_o)$$
$$R_3^2 = (z+z_o)(\bar{z}+\bar{z}_o) \quad , \qquad R_4^2 = (z-\bar{z}_o)(\bar{z}-z_o) \quad , \tag{2.2.1}$$

then G_2 for this region is given by

$$G_2 = \frac{P}{16\pi D}\left[R_1^2 \log R_1^2 - R_4^2 \log R_4^2 - R_2^2 \log R_2^2 + R_3^2 \log R_3^2\right] \tag{2.2.2}$$

It is interesting to note that the edges of the plate contribute completely to the support of the plate. That this is the case, is easily seen if one computes the moments and shears connected with (2.2.2) and expands the resulting expressions for large $|z|$. When this is done, there results

$$M_x \, , M_y \, , M_{xy} = 0\left(\frac{1}{|z|^2}\right), \quad V_x \, , V_y = 0\left(\frac{1}{|z|^3}\right) \tag{2.2.3}$$

From this it follows that the shear forces and the bending moments at infinity vanish to such an order that they contribute nothing to the support of the plate.

Since there is a corner in the plate at the origin, we have the supports at this point offering a concentrated load of magnitude

$$R = 2D(1 - v)\frac{\partial^2 w}{\partial x \partial y}\Bigg|_{\substack{x=0\\y=0}} = \frac{1-v}{\pi} P \sin 2\theta_o \qquad (2.2.4)$$

and it tends to hold the corner down.

2.3) The Circular Plate.

Both Green's functions for the circular plate appear in the literature. The Green's function for the clamped edge was first given by J. H. Michell [5]. Our only use for this result will be its complex representation, hence we reproduce it at this point

$$G_1 = \frac{P}{16\pi D}\left[-(z-z_o)(\bar{z}-\bar{z}_o)\log\frac{(a^2-\bar{z}_o z)(a^2-z_o\bar{z})}{a^2(z-z_o)(\bar{z}-\bar{z}_o)} + \frac{(a^2-\bar{z}_o z)(a^2-z_o\bar{z})}{a^2} - (z-z_o)(\bar{z}-\bar{z}_o)\right] \qquad (2.3.1)$$

The notation used in this expression is the same as in the previous paragraphs and is shown in fig. (7.).

The Green's function for the simply-supported case has been given by a number of people. See, for example, [9, 10]. We reproduce the result given in [9] as

$$G_2 = r^2 \log r - (\frac{\rho'}{a} r_1)^2 \log (\frac{\rho'}{a} r_1)$$

$$+ \frac{(\rho^2-a^2)(\rho'^2-a^2)}{a^2}\left[\frac{2\lambda-\mu}{2(\lambda-\mu)} + \log a + \frac{\lambda-\mu}{\lambda}\psi(\frac{\rho\rho'}{a^2}, \gamma)\right],$$

$$\psi(t,\gamma) = \frac{1}{t^{1-\mu/\lambda}} \int_0^t t^{-\frac{\mu}{\lambda}} \log\sqrt{t^2 - 2t\cos\gamma + 1}\, dt \quad .$$

$$\rho'^2 = z_o \bar{z}_o \quad , \quad r^2 = (z - z_o)(\bar{z} - \bar{z}_o) \quad , \quad r_1^2 = (z - \frac{a^2}{z_o})(\bar{z} - \frac{a^2}{\bar{\bar{z}}_o}) \quad .$$

CHAPTER III

PROBLEMS FOR THE HALF-PLANE

We begin by considering the region occupied by the upper half of the complex plane. This region affords an investigator the most fertile grounds for the solution of mixed boundary value problems for two reasons: first, the finite part of the boundary extends to infinity along a straight line; and, secondly, the appropriate boundary conditions always lead to singular integral equations rather than integro-differential equations. In general, this permits the inversion of the equations involved.

3.1) Clamped Edge.

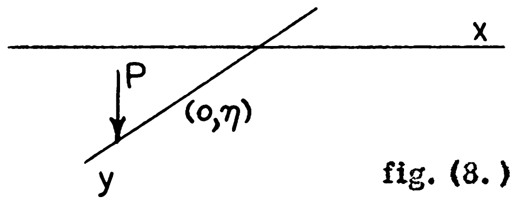
The solution of this problem is well-known in the literature on thin plates. Our only purpose here for considering this problem is to present a new derivation of the result based on the concept of a singular integral equation. This novel approach to the solution of an old problem will then serve as an introduction to the spirit of this thesis.

From (1.4.7), we have

$$w(x,y) = \frac{1}{P}\int_{-\infty}^{+\infty}\left[-\gamma(s)\frac{\partial G_2}{\partial n}\right]ds + \frac{1}{P}\int_{0}^{+\infty}\int_{-\infty}^{+\infty} G_2\, q\, d\xi d\eta \qquad (3.1.1)$$

where G_2 is given by (2.1.3) and $\frac{\partial G_2}{\partial n}$ by (2.1.6). This relation gives the deflection at any interior point if $\gamma(s)$ is assigned along the boundary and q is assigned in the interior of the region. In particular, if q is the loading shown in fig. (8.), then from the properties of G_2, it follows that w is zero along the boundary $y = 0$. Furthermore, since G_2 is continuous in the neighborhood of $(0, \eta)$, we have

$$\int_0^{+\infty}\int_{-\infty}^{+\infty} G_2\, q\, d\xi d\eta = P\, G_2\,(x, y, 0, \eta) \tag{3.1.2}$$



Hence

fig. (8.)

$$w(x, y) = \frac{-y}{4\pi D}\int_{-\infty}^{+\infty} \gamma(s)\left\{\log\left[(s-x)^2 + y^2\right] + 1\right\} ds + G_2(x, y, 0, \eta) \tag{3.1.3}$$

(It is to be noted that any time G_2 appears in this chapter, it will be understood to be defined by (2.1.3).)

Is it now possible to determine $\gamma(s)$ so that the normal derivative of w is zero along the boundary? With a suitable $\gamma(s)$ determined, we have the distribution along the edge of the plate of the bending moment necessary to straighten the normal. Since $w(x, 0) = 0$, we have

$$\frac{\partial w}{\partial n} = -\frac{\partial w}{\partial y} = 0 \text{ along } y = 0 \tag{3.1.4}$$

Now then

$$\frac{\partial w}{\partial y} = -\frac{1}{4\pi D}\int_{-\infty}^{+\infty} \gamma(s)\left\{\log\left[(s-x)^2+y^2\right]+1\right\}ds - \frac{y^2}{2\pi D}\int_{-\infty}^{+\infty}\frac{\gamma(s)\,ds}{(s-x)^2+y^2} + \frac{\partial G_2}{\partial y} \tag{3.1.5}$$

Substituting this into (3.1.4), we obtain

$$\frac{1}{4\pi D}\int_{-\infty}^{+\infty}\gamma(s)\left[\log(s-x)^2+1\right]ds + \frac{P\eta}{4\pi D}\left[\log(x^2+\eta^2)+1\right] = 0 \tag{3.1.6}$$

In view of the nature of G_2, it should be expected that the function $\gamma(s)$ must be subjected to the following restriction

$$\int_{-\infty}^{+\infty}\gamma(s)\,ds + P\eta = 0 \tag{3.1.7}$$

Actually, this amounts to a requirement of the fact that the edge of the plate must support the load. This condition will enforce the satisfaction of the moment equation for overall statical equilibrium of the plate.

With this assumption (3.1.6) becomes

$$2\int_{-\infty}^{+\infty}\gamma(s)\log|s-x|\,ds + P\eta\log(x^2+\eta^2) = 0 \tag{3.1.8}$$

We may now differentiate this expression with respect to x.

To perform this operation, we employ the usual algorithm for the differentiation of an integral containing a parameter. (That the interchange of integration and differentiation is legitimate follows from the fact that this is an extension of the case of the existence of the tangential derivative of the potential of a simple layer. See [6] page 30.) Thus if we differentiate (3.1.8) we obtain

$$\int_{-\infty}^{+\infty} \frac{\gamma(s)\, ds}{x-s} + \frac{P\eta x}{(x^2+\eta^2)} = 0 \qquad (3.1.9)$$

The solution of (3.1.9) can be effected in a number of ways. Indeed, if we write

$$-\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\gamma(s)\, ds}{s-x} = -\frac{P\eta x}{\pi(x^2+\eta^2)} \qquad (3.1.10)$$

then

$$\gamma(s) = -\frac{P\eta}{\pi^2}\int_{-\infty}^{+\infty} \frac{x\, dx}{(x^2+\eta^2)(x-s)} \qquad (3.1.11)$$

The necessary steps required to invert this integral equation will not be dwelled upon here since it is essentially a special case of a more general result which will be amplified upon in the next section. (See [6] page 374.) To effect the integration, split the integrand by partial fractions and utilize the fact that the integral is understood to be the principal value wherever this concept is needed. After some tedious integrations, there results

$$\gamma(s) = -\frac{P\eta^2}{\pi(s^2+\eta^2)} \qquad (3.1.12)$$

and we note that

$$\int_{-\infty}^{+\infty} \gamma(s)\, ds + P\eta = -P\eta + P\eta = 0$$

thus verifying (3.1.7).

With γ so determined, we may now write

$$w(x,y) = \frac{Py\eta^2}{4\pi^2 D} \int_{-\infty}^{+\infty} \frac{\log\left[(s-x)^2+y^2\right]+1}{(s^2+\eta^2)}\, ds + G_2(x,y,0,\eta) \qquad (3.1.13)$$

This is an integral representation of G_1 for the half-plane. It can be verified by direct differentiation that this function has the desired boundary values. In addition to this, it can be shown that this function also has the desired behavior at infinity.

Since the integration can be carried through, we proceed to effect it.

Let

$$F(x,y,\eta) = \int_{-\infty}^{+\infty} \frac{\log\left[(s-x)^2+y^2\right]}{s^2+\eta^2}\, ds \qquad (3.1.14)$$

so that

$$\frac{\partial F}{\partial x} = -2 \int_{-\infty}^{+\infty} \frac{(s-x)\, ds}{\left[(s-x)^2+y^2\right](s^2+\eta^2)} \qquad (3.1.14')$$

The integral in (3.1.14') is most easily accomplished by considering the following contour integration

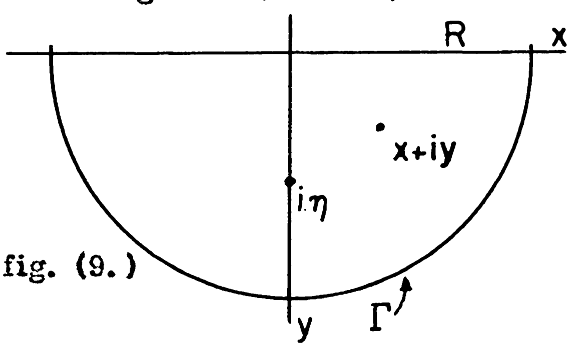


fig. (9.)

$$\int_{\Gamma} \frac{(z-x)\, dz}{\left[(z-x)^2+y^2\right]\left[z^2+\eta^2\right]}$$

where the contour Γ is indicated in fig. (9.). Since the integral over the arc vanishes as $R \to \infty$, we have

$$\int_{-\infty}^{+\infty} \frac{(z - x)\, dz}{\left[(z-x)^2 + y^2\right](z^2+\eta^2)} = 2\pi i \left[\frac{i\eta - x}{2i\eta\left[(i\eta-x)^2+y^2\right]} + \frac{i y}{2 iy\left[(x+iy)^2+\eta^2\right]}\right]$$

Consequently

$$\frac{\partial F}{\partial x} = -\frac{2\pi}{\eta}\,\frac{i\eta - x}{(i\eta-x)^2+y^2} - 2\pi i\,\frac{1}{(x+iy)^2+\eta^2}$$

and

$$F = \frac{\pi}{\eta}\log\left[(i\eta - x)^2 + y^2\right] - \frac{2\pi i}{\eta}\arctan\frac{x+iy}{\eta} + F(y,\eta)\ .$$

Now

$$2i\arctan x = -\log\frac{i+x}{i-x}\ ,$$

so that

$$F(x,y,\eta) = \frac{\pi}{\eta}\log\left[(i\eta - x)^2 + y^2\right] + \frac{\pi}{\eta}\log\frac{i\eta+x+iy}{i\eta-x-iy} + F(y,\eta)\ .$$

Hence

$$F(0,y,\eta) = \frac{\pi}{\eta}\log(y^2-\eta^2) + \frac{\pi}{\eta}\log\frac{n+y}{n-y} + F(y,\eta)\ .$$

From (3.1.14)

$$F(0,y,\eta) = \int_{-\infty}^{+\infty}\frac{\log(s^2+y^2)}{s^2+\eta^2}\,ds = \frac{\pi}{\eta}\log(\eta+y)^2\ .$$

Thus there results

$$F(x,y,\eta) = \frac{\pi}{\eta} \log\left[x^2 + (\eta + y)^2\right] ,$$

and we finally obtain, after performing the translation $x = x'-\xi$, $y = y'$,

$$G_1 = \frac{P}{16\pi D}\left[(x-\xi)^2 + (y-\eta)^2\right]\log\frac{(x-\xi)^2+(y-\eta)^2}{(x-\xi)^2+(y+\eta)^2} + \frac{Py\eta}{4\pi D} \qquad (3.1.15)$$

which is precisely the result given by (2.1.1) except for the fact that it is presented here in real form.

Thus we have answered the original question in the affirmative.

3.2) Moment Applied along a Portion of the Edge.

In this section, we shall consider the half-plane as shown in fig. (10.). To demonstrate the flexibility of solution of mixed boundary value problems, we shall attack this particular problem by the method of functional equations.

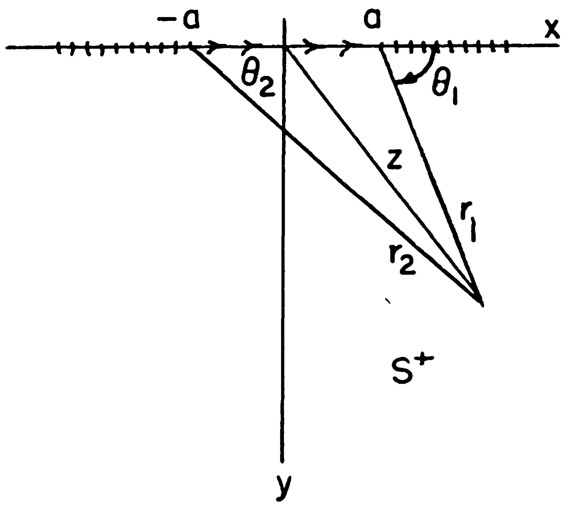


fig. (10.)

In accordance with previous agreement, we shall denote the region occupied by the plate as S^+. The definition of this problem is as follows: Determine a function w which is biharmonic everywhere in S^+ and, on the boundary $y = 0$ of the region S^+, w satisfies the following conditions

$$w = 0, \ |x| \leq \infty \ ; \ \frac{\partial w}{\partial n} = 0, \ |x| \geq a \ ; \ \nabla^2 w = -\frac{M_n}{D} \ |x| < a \qquad (3.2.1)$$

where $\underline{M}$ is constant. From (1.5.3), we have that the most general expression for w may be written as follows

$$2w = \bar{z}\,\phi(z) + z\,\bar{\phi}(\bar{z}) + \psi(z) + \bar{\psi}(\bar{z}) \qquad (3.2.2)$$

where $\phi(z)$, (z) are analytic functions of $\underline{z}$ in S^+ (Note: S^+ is the region defined by imaginary $z > 0$.) The structure of ϕ, ψ for large $|z|$ is assumed to be

$$\phi(z) = \frac{\phi_1}{z} + 0\left(\frac{1}{z^2}\right) \quad , \quad \psi(z) = \psi_0 + 0\left(\frac{1}{z}\right) \qquad (3.2.3)$$

It is to be noted that the assumed form (3.2.3), in view of (3.2.2), does not imply that $w \to 0$ for large $|z|$. Indeed, the best one can hope for in this situation is to produce a solution which remains bounded at infinity, thus eliminating any contribution to the support of the plate by any of its remote portions.

Since from (3.2.1), w vanishes on the real axis, we have the following

$$\sigma\,\phi(\sigma) + \sigma\,\bar{\phi}(\sigma) + \psi(\sigma) + \bar{\psi}(\sigma) = 0 \quad \sigma \in \text{ real axis} \qquad (3.2.4)$$

Hence, since $\sigma = \bar{\sigma}$ along the real axis, the notation in (3.2.4) has meaning in view of (3.2.2). If we now multiply through in (3.2.4) by $\frac{1}{2\pi i} \frac{d\sigma}{\sigma - z}$ and integrate over the real axis, there results

$$\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\sigma\phi(\sigma)\,d\sigma}{\sigma - z} + \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\sigma\bar{\phi}(\sigma)\,d\sigma}{\sigma - z} + \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\psi(\sigma)\,d\sigma}{\sigma - z} + \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\bar{\psi}(\sigma)\,d\sigma}{\sigma - z} = 0 \qquad (3.2.5)$$

Since the variable $\underline{z}$ is contained in the half-plane defined by $y > 0$, we obtain the following relation

$$z\phi(z) - \frac{1}{2}\phi_1 + \frac{1}{2}\bar{\phi}_1 + \psi(z) - \frac{1}{2}\psi_o + \frac{1}{2}\bar{\psi}_o = 0 \qquad (3.2.6)^*$$

Thus

$$\psi(z) = -z\phi(z) + \frac{1}{2}(\phi_1 - \bar{\phi}_1) + \frac{1}{2}(\psi_o - \bar{\psi}_o) \qquad (3.2.7)$$

The values ϕ_1, ψ_o survive the integration because of the assumed structure of ϕ, ψ at infinity. It should also be remarked that integrals of the type term 2 and term 4 in (3.2.5) are evaluated by observing that $\bar{\psi}(\sigma)$, $\sigma\bar{\phi}(\sigma)$ are the boundary values of functions $z\bar{\phi}(z)$, $\bar{\psi}(z)$ which are analytic in S^-; i.e., $\mathrm{Im}(z) < 0$. Substituting (3.2.9) back into (3.2.2), we obtain

$$2w = (z - \bar{z})\left[\phi(z) - \bar{\phi}(\bar{z})\right] \qquad (3.2.8)$$

Thus any function $\phi(z)$ analytic in S^+ and vanishing at infinity will produce a w which vanishes on the real axis and is bounded

* For the use of Cauchy's residue theorem, here, and throughout this thesis, reference should be made to [3] Chapters 12 and 18.

at infinity.

We now seek the impositions on $\phi(z)$ caused by the remaining boundary conditions in (3.2.1). Since

$$\frac{\partial w}{\partial n} = -\frac{\partial w}{\partial y} \quad \text{along} \quad y = 0 \tag{3.2.9}$$

we have

$$\frac{\partial w}{\partial n} = -i\left(\frac{\partial w}{\partial z} - \frac{\partial w}{\partial \bar{z}}\right) \quad \text{along} \quad y = 0 \tag{3.2.10}$$

From the conditions we have set, $\frac{\partial w}{\partial n}$ vanishes for $|x| \geqslant a$ and is unknown for $|x| < a$. If its value be $\beta(\sigma)$ for $\sigma \in |x| < a$, then we may write

$$i\left[\phi(\sigma) - \bar{\phi}(\sigma)\right] = \begin{cases} 0 & \sigma \in |x| \geqslant a \\ \beta(\sigma) & \sigma \in |x| \leqslant a \end{cases} \tag{3.2.11}$$

Again we obtain (as we did eq. (3.2.5))

$$i\,\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\phi(\sigma)\,d\sigma}{\sigma - z} - i\,\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\bar{\phi}(\sigma)\,d\sigma}{\sigma - z} = \frac{1}{2\pi i}\int_{-a}^{+a}\frac{\beta(\sigma)\,d\sigma}{\sigma - z} \tag{3.2.12}$$

and Cauchy's residue theorem produces

$$i\,\phi(z) = \frac{1}{2\pi i}\int_{-a}^{+a}\frac{\beta(\sigma)\,d\sigma}{\sigma - z}$$

or

$$\phi(z) = \frac{1}{2\pi i}\int_{-a}^{+a} \frac{(-i)\,\beta(\sigma)\,d\sigma}{\sigma - z} \tag{3.2.13}$$

Along the arc $-a \leqslant \sigma \leqslant +a$, the bending moment $M_n(\sigma)$ is given by

$$\nabla^2 w = 4\frac{\partial^2 w}{\partial z \partial \bar{z}} = -\frac{M_n}{D} \tag{3.2.14}$$

From this there results the following

$$2\left[\phi'(\sigma) + \bar{\phi}'(\sigma)\right] = -\frac{M_n}{D} \tag{3.2.15}$$

which holds along the arc $-a \leqslant \sigma \leqslant +a$.

If we differentiate (3.2.13), there results

$$\phi'(z) = \frac{1}{2\pi i}\int_{-a}^{+a} \frac{-i\beta(\sigma)\,d\sigma}{(\sigma - z)^2} \tag{3.2.16}$$

Now $\beta(\sigma)$ must be continuous along the edge of the plate, hence at $\sigma = \pm a$ we must have $\beta(\pm a) = 0$. Thus, if we integrate the right-hand side of (3.2.16) by parts, there results

$$\phi'(z) = \frac{1}{2\pi i}\int_{-a}^{+a} \frac{(-i)\,\beta'(\sigma)\,d\sigma}{(\sigma - z)} \tag{3.2.17}$$

Upon letting $z \to \sigma_o$, Plemelj's formulae give us

$$\phi'(\sigma_o) = -\frac{i}{2}\,\beta'(\sigma_o) + \frac{1}{2\pi i}\int_{-a}^{+a}\frac{(-i)\,\beta'(\sigma)\,d\sigma}{\sigma - \sigma_o}$$

$$\bar{\phi}'(\sigma_o) = \frac{i}{2}\,\beta'(\sigma_o) + \frac{1}{2\pi i}\int_{-a}^{+a}\frac{(-i)\,\beta'(\sigma)\,d\sigma}{\sigma - \sigma_o} \qquad (3.2.18)$$

where the integrals on the right-hand side are now understood to be principal values. It is to be noted that since $\beta(\sigma)$, σ, σ_o, $\beta'(\sigma_o)$ are all real, no conjugation bars appear on the right-hand side of the second of (3.2.18). Substituting back into (3.2.15), we obtain a singular integral equation of Cauchy type; namely,

$$\frac{1}{\pi i}\int_{-a}^{+a}\frac{\beta'(\sigma_o)\,d\sigma}{\sigma - \sigma_o} = \frac{M_n}{2Di} \qquad (3.2.19)$$

(Since equations of this type will have to be solved from time to time throughout this thesis, it seems worthwhile, once and for all, to indicate the general aspects of the inversion process involved in the solution. For a more detailed description see [6].)

To invert (3.2.19), we introduce the function

$$\Psi(z) = \frac{1}{2\pi i}\int_{-a}^{+a}\frac{\beta'(\sigma)\,d\sigma}{\sigma - z} \qquad (3.2.20)$$

This function $\Psi(z)$ is analytic in the entire plane slit from

-a to +a and vanishes at infinity. (See fig. (11.).) A choice now confronts us concerning the behavior of $\beta'(\sigma)$ near the endpoints of the slit. From the general theory concerning the solutions of equations of the type (3.2.19), $\beta'(\sigma)$ may be either zero or unbounded after the fashion $(\sigma^2-a^2)^{-\mu}$ where $\mu < 1$. If $\beta'(\sigma)$ were chosen to vanish at the endpoints of the slit, then it could be shown that in view of the choice for Mn, there exists no solution to (3.2.19).

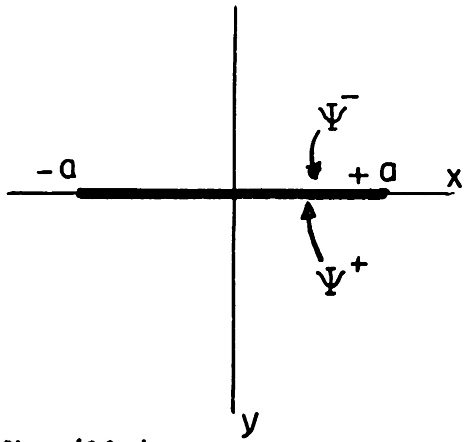


fig. (11.)

(See [6] page 251, eq. 88.10.) Therefore we may write

$$\Psi(z) = \frac{1}{2\pi i\sqrt{z^2-a^2}}\int_{-a}^{+a}\frac{\sqrt{\sigma^2-a^2}\,(\frac{M_n}{2Di})}{\sigma - z}\,d\sigma + \frac{K_1}{\sqrt{z^2-a^2}} \tag{3.2.21}$$

where K_1 is an arbitrary complex number. (See [6] page 251.)

By applying Cauchy's theorem, we obtain

$$\Psi(z) = \frac{1}{\sqrt{z^2-a^2}}\left(\frac{M_n}{4Di}\right)\left(\sqrt{z^2-a^2} - z\right) + \frac{K_1}{\sqrt{z^2-a^2}} \tag{3.2.22}$$

But Plemelj's formulae state that

$$\Psi^+(\sigma) + \Psi^-(\sigma) = \frac{M_n}{2iD} \quad , \quad \Psi^+(\sigma) - \Psi^-(\sigma) = \beta'(\sigma)$$

(The plus and minus signs on the function indicate the manner in which $\underline{z}$ approaches the slit. See fig. (11.).)

Now then, since $\sqrt{\sigma^2 - a^2}^{\,+} = -\sqrt{\sigma^2 - a^2}^{\,-}$, we obtain, immediately,

$$\frac{M_n/4Di}{\sqrt{\sigma^2 - a^2}}\left[\sqrt{\sigma^2 - a^2} - \sigma\right] + \frac{K_1 - \overline{K}_1}{\sqrt{\sigma^2 - a^2}} + \frac{M_n/4Di}{\sqrt{\sigma^2 - a^2}}\left[-\sqrt{\sigma^2 - a^2} - \sigma\right] = \beta'(\sigma) \tag{3.2.23}$$

Hence

$$\beta'(\sigma) = -\frac{M_n}{2Di}\frac{\sigma}{\sqrt{\sigma^2 - a^2}} + \frac{K_1 - \overline{K}_1}{\sqrt{\sigma^2 - a^2}} \tag{3.2.24}$$

We observe that since M_n is constant, $\beta(\sigma)$ is symmetric, and hence $\beta'(\sigma)$ is antisymmetric. Consequently $K_1 - \overline{K}_1 = 0$. Upon integration of (3.2.26), we obtain

$$\beta(\sigma) = -\frac{M_n}{2Di}\sqrt{\sigma^2 - a^2} \tag{3.2.25}$$

where the constant of integration has been dropped since $\beta(\pm a) = 0$ by continuity requirements on the slope. Placing this expression in (3.2.15) there results

$$\phi(z) = \frac{1}{2\pi i}\int_{-a}^{+a}\frac{M_n}{2D}\frac{\sqrt{\sigma^2 - a^2}}{\sigma - z}\,d\sigma \tag{3.2.26}$$

Again, an application of Cauchy's residue theorem produces

$$\phi(z) = \frac{M_n}{4D}\left[\sqrt{z^2 - a^2} - z\right]$$

Hence

$$w = -\frac{M_n}{8D}(z-\bar{z})\left\{\sqrt{z^2-a^2} - z - \overline{\sqrt{z^2-a^2} - z}\right\} \qquad (3.2.27)$$

It may be verified by direct substitution that this function satisfies the boundary conditions prescribed for it.

Since the bending moment for the edge of the plate is given by

$$4\frac{\partial^2 w}{\partial z \partial \bar{z}} = -\frac{M_n(\sigma)}{D}$$

it follows that

$$\frac{M_n(\sigma)}{D} = \frac{M}{2D}\left\{2 - \frac{z}{\sqrt{z^2-a^2}} - \frac{\bar{z}}{\sqrt{z^2-a^2}}\right\}_{z=\sigma} \qquad (3.2.28)$$

Evidently $M(\sigma)$ becomes infinite as $z \to a$. We may also compute the resultant vertical shear along the edge. When this is done there results

$$V_n(\sigma) = \frac{-i(1+v)M_n}{4}\left[\frac{-a^2}{(\sigma^2-a^2)^{3/2}} - \frac{-a^2}{\overline{(\sigma^2-a^2)^{3/2}}}\right]$$

For ease of handling this expression, it is now presented in real form as follows. If $r_1, r_2, \theta_1, \theta_2$ are four variables as depicted in fig. (10.), then we may write (3.2.27) as

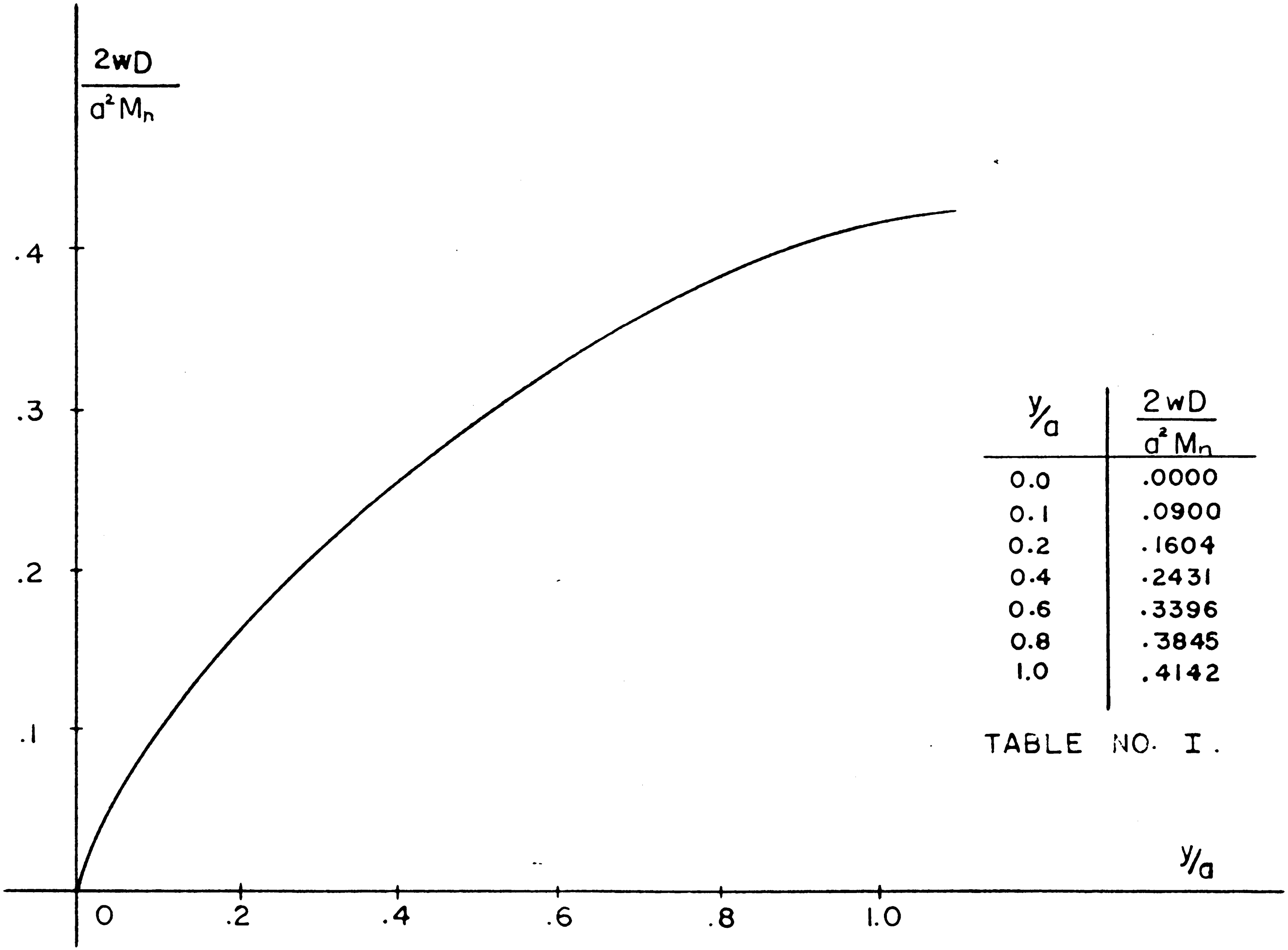
$$w = \frac{M_n y}{4D}\left[\sqrt{r_1 r_2}\sin\frac{\theta_1+\theta_2}{2} - y\right]$$

In conclusion, it might be worth while mentioning the behavior of w and its derivatives for large $|z|$. Indeed, if we expand the form for w in a series, valid in the neighborhood of infinity, we obtain

$$w = -\frac{M_n}{8D}(z-\bar{z})\left[-\frac{a^2}{2z}+\frac{a^2}{2\bar{z}}+0\left(\frac{1}{|z|^3}\right)\right]$$

We see, then, that w is bounded as $|z| \to \infty$. Furthermore, it follows from (3.2.27) that the first derivatives of w with respect to z or $\bar{z}$ are of the order $\frac{1}{|z|}$, all second derivatives are of the order $\frac{1}{|z|^2}$, and the third derivatives are of the order $\frac{1}{|z|^3}$. Consequently, the moments and shears vanish to such an order at infinity that they contribute nothing to the support of the plate. As previously stated, this will be the expected behavior of any of the solutions we obtain.

To lend some practicality to this solution, it seems worthwhile considering some numerical results and general aspects of this problem. The deflection along the y-axis and M_n and V_n along the edge are given in figs. (12) and (13). Because of the abrupt change in loading, a singularity develops at the edge of the applied load. The necessary force to be added at this junction (see [1] page 132 and [22],) is infinite. Had the transition in load been less abrupt this would not have occurred.



y/a	$\frac{2wD}{a^2 M_n}$
0.0	.0000
0.1	.0900
0.2	.1604
0.4	.2431
0.6	.3396
0.8	.3845
1.0	.4142

TABLE NO. I.

fig. (12.)

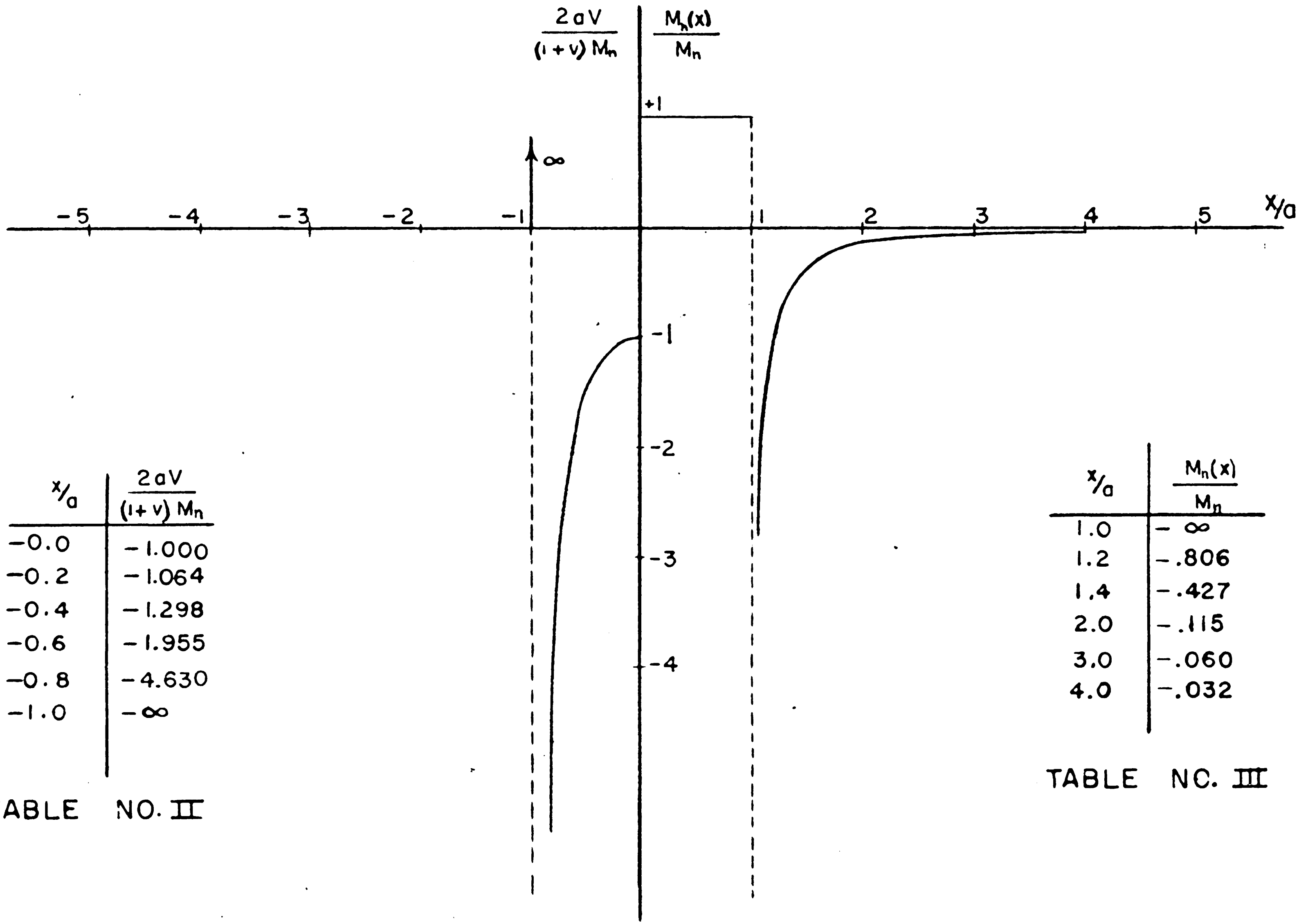


fig. (13)

x/a	$\frac{2aV}{(1+v)M_n}$
−0.0	−1.000
−0.2	−1.064
−0.4	−1.298
−0.6	−1.955
−0.8	−4.630
−1.0	−∞

TABLE NO. II

x/a	$\frac{M_n(x)}{M_n}$
1.0	− ∞
1.2	−.806
1.4	−.427
2.0	−.115
3.0	−.060
4.0	−.032

TABLE NO. III

3.3) Simply-Supported, Clamped, Simply-Supported.

We shall now seek a Green's function for a set of mixed boundary conditions for the half-plane. The boundary conditions will be such that for $|x| \leq a$, the plate is clamped, and for $|x| > a$, it is simply-supported. In the interior of the plate, a concentrated load will be placed at $z = z_o$ as in fig. (14.). In order that no contributions to the support of the plate arise from the function w in the neighborhood of infinity, we shall require then that the deflection in the neighborhood of infinity be such that all second and third derivatives of w behave like $\frac{1}{|z|^2}$ and $\frac{1}{|z|^3}$ respectively.

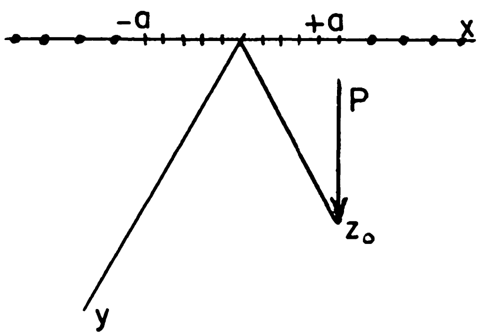


fig. (14.)

To begin then, we consider the function

$$2w = -(z-\bar{z})\left[\phi(z) - \bar{\phi}(\bar{z})\right]$$

$$+ \frac{P}{3\pi D}\left[(z-z_o)(\bar{z}-\bar{z}_o)\log(z-z_o)(\bar{z}-\bar{z}_o) - (z-\bar{z}_o)(\bar{z}-z_o)\log(z-\bar{z}_o)(\bar{z}-z_o)\right]. \tag{3.3.1}$$

Evidently, this function vanishes along the real axis. Let us now investigate the structure which $\phi(z)$ must have for large $|z|$ in order that the derivatives of w behave properly in the neighborhood of infinity. Since $2\frac{\partial \bar{w}}{\partial z} = 2\frac{\partial w}{\partial \bar{z}}$, it will be sufficient to consider only the derivative $\frac{\partial w}{\partial z}$. We have

$$2\,\frac{\partial G_2}{\partial z} = \frac{P}{8\pi D}\left[(z_o-\overline{z}_o)\log z\overline{z} + \frac{1}{z}(z_o-\overline{z}_o)(z-\overline{z}) + (z_o-\overline{z}_o)\right] + 0\left(\frac{1}{|z|}\right) \qquad (3.3.2)$$

Thus

$$2\,\frac{\partial w}{\partial z} = -\left[\phi(z) - \overline{\phi}(\overline{z})\right] - (z-\overline{z})\,\phi(z)$$

$$+\frac{P}{8\pi D}\left[(z_o-\overline{z}_o)\log z\overline{z} + \frac{1}{z}(z_o-\overline{z}_o)(z-\overline{z}) + (z_o-\overline{z}_o)\right] + 0\left(\frac{1}{|z|}\right)$$

Hence, if

$$\phi(z) = \frac{P}{8\pi D}(z_o-\overline{z}_o)\log z + F(z_o,\overline{z}_o) + 0\left(\frac{1}{|z|}\right)$$

and, if

$$\phi'(z) = \frac{P}{8\pi D}(z_o-\overline{z}_o)\,\frac{1}{z} + 0\left(\frac{1}{|z|^2}\right),$$

then

$$2\,\frac{\partial w}{\partial z} = -\frac{P}{8\pi D}(z_o-\overline{z}_o)\log z\overline{z} - F(z_o,\overline{z}_o) + \overline{F}(\overline{z}_o, z_o)$$

$$+\frac{P}{8\pi D}\left[(z_o-\overline{z}_o)\log z\overline{z} + \frac{1}{z}(z_o-\overline{z}_o)(z-\overline{z}) + (z_o-\overline{z}_o) - \frac{(z_o-\overline{z}_o)(z-\overline{z}_o)}{z}\right]$$

$$= F_1(z_o,\overline{z}_o) + 0\left(\frac{1}{|z|}\right) \qquad (3.3.3)$$

If we assume that for large

$$\phi(z) = \frac{P}{8\pi D}(z_o - \overline{z}_o)\log z + F(z_o, \overline{z}_o) + \phi_o(z) \qquad (3.3.4)$$

where $\phi_o(z)$ is analytic in the plane slit from $-a$ to $+a$ and vanishes at infinity, then $\underline{w}$ and its derivatives will have the proper form at infinity.

Thus in the evaluation of any future contour integrals, we shall assume (3.3.4) is a legitimate expansion for large $|z|$.

If now we compute the bending moment along the real axis, we obtain

$$2\left[\phi'(\sigma) + \overline{\phi}'(\sigma)\right] = \begin{cases} 0 & |\sigma| > a \\ -\dfrac{M_n(\sigma)}{D} & |\sigma| < a \end{cases} \qquad (3.3.5)$$

Because of the assumed form of $\phi(z)$, we have, on multiplying through (3.3.5) by $\frac{1}{2\pi i}\frac{d\sigma}{\sigma - z}$ and integrating from $-\infty$ to $+\infty$ along the real axis, the following result

$$\phi'(z) = \frac{1}{2\pi i}\int_{-a}^{+a} \frac{-\frac{1}{2D} M_n(\sigma)\, d\sigma}{\sigma - z} \qquad (3.3.6)$$

Now then, along the segment $|\sigma| < a$ of the real axis, the normal derivative, $\frac{\partial w}{\partial n}$, must vanish. Hence, the following expression

$$i\left[\phi(\sigma) - \overline{\phi}(\sigma)\right] = \frac{iP(z_o - \overline{z}_o)}{8\pi D}\left[\log(\sigma - z_o)(\sigma - \overline{z}_o) + 1\right]. \qquad (3.3.7)$$

is an identity along this segment of the real axis. Since this is the case, we may differentiate (3.3.7) with respect to σ. When this is done, there results

$$\phi'(\sigma) - \overline{\phi}'(\sigma) = \frac{P(z_o - \overline{z}_o)}{8\pi D}\left[\frac{1}{\sigma - z_o} + \frac{1}{\sigma - \overline{z}_o}\right] \qquad (3.3.8)$$

If one applies Plemelj's formulae to (3.3.6), there results the following singular integral equation of Cauchy type

$$\frac{1}{\pi i}\int_{-a}^{+a} \frac{M_n(\sigma_o)\,d\sigma_o}{\sigma_o - \sigma} = -\frac{P(z_o - \overline{z}_o)}{4\pi}\left[\frac{1}{\sigma - z_o} + \frac{1}{\sigma - \overline{z}_o}\right] \qquad (3.3.9)$$

To invert this equation, we introduce, as in section (3.2), the function $\Psi(z)$ defined by

$$\Psi(z) = \frac{1}{2\pi i\sqrt{z^2-a^2}}\int_{-a}^{+a} \frac{\sqrt{\sigma^2-a^2}\left\{\frac{-P(z_o-\overline{z}_o)}{4\pi}\left[\frac{1}{\sigma-z_o} + \frac{1}{\sigma-\overline{z}_o}\right]\right\}d\sigma}{\sigma - z} + \frac{K_1}{\sqrt{z^2-a^2}} \qquad (3.3.10)$$

(It is to be noted that $\Psi(z)$ has been so written as to make $M_n(\sigma)$ unbounded at the end-points $-a$, $+a$ of the clamped edge. The condition of bounded $M_n(\sigma)$ and the form of the right-hand side of (3.3.9) are untenable.)

Applying Cauchy's residue theorem, one obtains

$$\Psi(z) = -\frac{P(z_o-\bar{z}_o)}{8\pi}\left[\frac{1}{z-z_o} + \frac{1}{z-\bar{z}_o}\right] + \frac{K_1}{\sqrt{z^2-a^2}}$$

$$+\frac{P(z_o-\bar{z}_o)}{8\pi\sqrt{z^2-a^2}}\left[\frac{\sqrt{z_o^2-a^2}}{-z_o+z} + \frac{\overline{\sqrt{z_o^2-a^2}}}{z-\bar{z}} + 2\right] \qquad (3.3.11)$$

Accordingly, now, Plemelj's formulae when suitably applied to (3.3.11) enable us to compute $M_n(\sigma)$ as follows

$$M_n(\sigma) = \frac{P(z_o-\bar{z}_o)}{4\pi\sqrt{\sigma^2-a^2}}\left[\frac{\sqrt{z_o^2-a^2}}{\sigma-z_o} + \frac{\overline{\sqrt{z_o^2-a^2}}}{\sigma-\bar{z}_o} + 2\right] + \frac{2K_1}{\sqrt{\sigma^2-a^2}} \qquad (3.3.12)$$

To evaluate the constant K_1, we observe that

$$\int_{-a}^{+a} M_n(\sigma)\, d\sigma + \frac{P(z_o-\bar{z}_o)}{2i} = 0 \qquad (3.3.13)$$

This is nothing more than the statement of the fact that the bending moment along the edge which is clamped is equal and opposite in direction to the moment of $\underline{P}$ about this edge. In view of the order of magnitude of the moments and shears in the neighborhood of infinity, they contribute nothing to the support of the plate. Indeed, we obtain

$$\frac{P(z_o-\bar{z}_o)}{2i} + \frac{2K_1\pi}{i} - \frac{P(z_o-\bar{z}_o)}{2\pi i}\left[\arctan\frac{z_o-a}{\sqrt{z_o^2-a^2}} + \arctan\frac{z_o+a}{\overline{\sqrt{z_o^2-a^2}}}\right]$$

$$-\frac{P(z_o-\bar{z}_o)}{2\pi i}\left[\arc\tan\frac{\bar{z}_o-a}{\overline{\sqrt{z_o^2-a^2}}}+\arc\tan\frac{\bar{z}_o+a}{\overline{\sqrt{z_o^2-a^2}}}\right]=0$$

$$\therefore K_1=\frac{-P(z_o-\bar{z}_o)}{4\pi}.$$

So that finally, we have

$$M_n(\sigma)=\frac{P(z_o-\bar{z}_o)}{4\pi\sqrt{\sigma^2-a^2}}\left[\frac{\sqrt{z_o^2-a^2}}{\sigma-z_o}+\frac{\overline{\sqrt{z_o^2-a^2}}}{\sigma-\bar{z}_o}\right] \tag{3.3.14}$$

Consequently, we may write

$$\phi'(z)=\frac{1}{2\pi i}\int_{-a}^{+a}\frac{-P(z_o-\bar{z}_o)}{8\pi D\sqrt{\sigma^2-a^2}}\frac{\left[\frac{\sqrt{z_o^2-a^2}}{\sigma-z_o}+\frac{\overline{\sqrt{z_o^2-a^2}}}{\sigma-\bar{z}_o}\right]}{\sigma-z}d\sigma \tag{3.3.15}$$

Another application of Cauchy's residue theorem produces

$$\phi'(z)=\frac{-P(z_o-\bar{z}_o)}{16\pi D}\left[\frac{1}{\sqrt{z^2-a^2}}\left\{\frac{\sqrt{z_o^2-a^2}}{z-z_o}+\frac{\overline{\sqrt{z_o^2-a^2}}}{z-\bar{z}_o}\right\}-\frac{1}{z-z_o}-\frac{1}{z-\bar{z}_o}\right] \tag{3.3.16}$$

And, finally, an integration of (3.3.16) will produce

$$\phi(z)=\frac{-P(z_o-\bar{z}_o)}{16\pi D}\left\{\log\frac{\left[\bar{z}_oz-a^2-\overline{\sqrt{z_o^2-a^2}}\sqrt{z^2-a^2}\right]}{\left[z_oz-a^2+\sqrt{z_o^2-a^2}\sqrt{z^2-a^2}\right]}-\log(z_o-\bar{z}_o)^2+K_2\right\} \tag{3.3.17}$$

To evaluate the constant of integration K_2, we shall demand that the expression (3.3.17) be such that the boundary conditions imposed on w are satisfied along the edge of the plate. Indeed, if $|\sigma| < a$, then we obtain

$$\phi(\sigma) - \bar{\phi}(\sigma) = \frac{-P(z_o - \bar{z}_o)}{16\pi D}\left[\log \frac{1}{(\sigma - z_o)^2 (\sigma - \bar{z}_o)^2} + K_2 + \bar{K}_2\right] \tag{3.3.18}$$

Thus in order that (3.3.7) be satisfied, we must have

$$\frac{-P(z_o - \bar{z}_o)}{16\pi D}\left[-2 \log (\sigma - z_o)(\sigma - \bar{z}_o) + K_2 + \bar{K}_2\right]$$

$$= \frac{P(z_o - \bar{z}_o)}{8\pi D}\left[\log (\sigma - z_o)(\sigma - \bar{z}_o) + 1\right]$$

$$\therefore K_2 + \bar{K}_2 = -2 .$$

Since the imaginary part of K_2 is immaterial, we take $K_2 = -1$. Thus, we may finally write

$$2w = \frac{P(z - \bar{z})(z_o - \bar{z}_o)}{16\pi D} \log \frac{Q\bar{Q}}{R\bar{R}}$$

$$+ \frac{P(z-\bar{z})(z_o - \bar{z}_o)}{8\pi D} \log (z - \bar{z}_o)(\bar{z} - z_o) - \frac{P(z-\bar{z})(z_o - \bar{z}_o)}{8\pi D} + 2G_2$$

$$Q = \bar{z}_o z - a^2 - \overline{\sqrt{z_o^2 - a^2}}\sqrt{z^2 - a^2}, \quad R = z_o z - a^2 + \sqrt{z_o^2 - a^2}\sqrt{z^2 - a^2} \tag{3.3.19}$$

which evidently may be rewritten as

$$w = \frac{P(z-\bar{z})(z_o-\bar{z}_o)}{32\pi D} \log \frac{Q\bar{Q}}{R\bar{R}} + G_1 \qquad (3.3.20)$$

where G_1 is Green's function of the first kind for the half-plane.

It may be noted that as $a \to \infty$, the first term in (3.3.20) goes to zero, and we obtain G_1 as a limiting form which is certainly to be expected.

A detailed examination of (3.3.19) indicates that for large $|z|$ we have

$$w = \frac{P(z-\bar{z})(z_o-\bar{z}_o)}{32\pi D} \log \frac{\left[\bar{z}_o - \sqrt{\bar{z}_o^2-a^2}\right]\left[z_o - \sqrt{z_o^2-a^2}\right]}{\left[z_o + \sqrt{z_o^2-a^2}\right]\left[\bar{z}_o + \sqrt{\bar{z}_o^2-a^2}\right]} + \text{bounded terms} + 0\left(\frac{1}{|z|}\right) \qquad (3.3.21)$$

Similar considerations for the derivatives of <u>w</u> would show that this function has the desired behavior at infinity.

(3.4) <u>Clamped, Simply-Supported.</u>

We may now utilize the solution of sec. (3.3) to determine the Green's function for a region which is clamped from $-\infty$ to 0 and simply-supported from 0 to ∞. Indeed, if we shift the origin from 0 to a and permit $a \to \infty$, we obtain the following

$$w = \frac{P(z-\bar{z})(z_o-\bar{z}_o)}{16\pi D} \log \frac{\sqrt{z} - \sqrt{\bar{z}_o}}{\sqrt{z} + \sqrt{z_o}} \, \frac{\sqrt{\bar{z}} - \sqrt{z_o}}{\sqrt{\bar{z}} + \sqrt{\bar{z}_o}} + G_1 \qquad (3.4.1)$$

as the Green's function for this set of boundary conditions. It is to be noted that even though the extent of clamping is along an infinite segment, the deflection still remains unbounded as $|z|$ tends to infinity.

Most of the results in the chapter on half-planes involve a large number of parameters. This particular result involves a minimum of such quantities. In fig. (15.), we give the bending moment along the clamped portion of the edge.

x/η	$\frac{\sqrt{2}\,\pi M_n(x)}{P}$
0.0	$-\infty$
-.5	-1.365
-1.0	-.793
-2.0	-.342
-3.0	-.188
-4.0	-.123
-5.0	-.0873

TABLE NO. IV

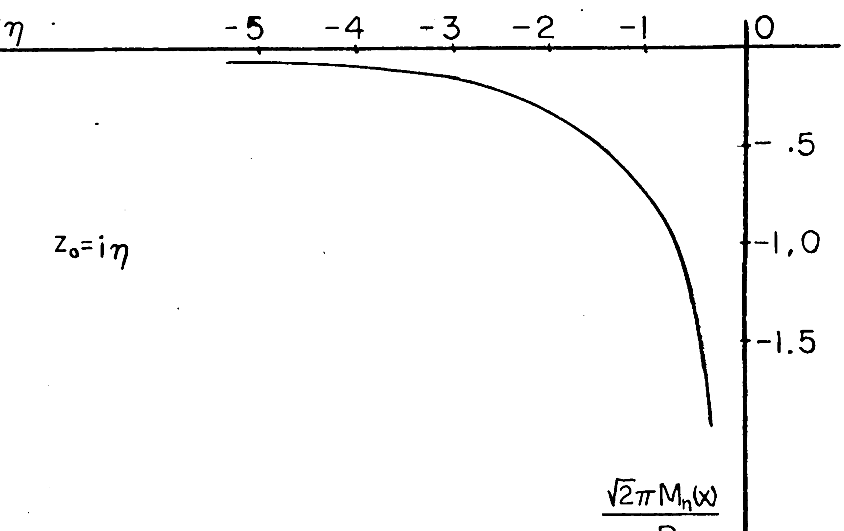


fig. (15.)

As is observed from this graph, the bending moment along the edge of the plate dies out quite rapidly; and, consequently, the effects are local.

(3.5) <u>Clamped, Simply-Supported, Clamped.</u>

Again we seek a Green's function for the half-plane shown in fig. (16.). In this case, we shall require that the plate be clamped for $|x| \geqslant a$ and simply-supported for $|x| < a$. Actually, this case is an extension to the case already treated in section (3.2). The difference is that the nature of the bending moment is more complex. In view of this, it is felt that a separate treatment will not be redundant.

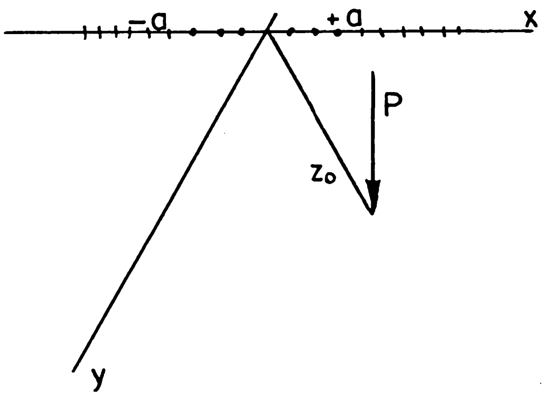


fig. (16.)

To begin, we consider the following expression:

$$2w = -(z-\bar{z})\left[\phi(z) - \bar{\phi}(\bar{z})\right] + 2G_1 \qquad (3.5.1)$$

where G_1 is the Green's function of the first kind for this region. The normal derivative along the real axis is given by

$$i\left[\phi(\sigma) - \bar{\phi}(\sigma)\right] = \begin{cases} 0 & |\sigma| > a \\ \beta(\sigma) & |\sigma| < a \end{cases} \qquad (3.5.2)$$

If now, we assume that $\phi(z) = \frac{\phi_1}{z}$ for large $|z|$, then on multiplying through by $\frac{1}{2\pi i}\,\frac{d\sigma}{\sigma - z}$ and integrating over a large semicircle, there results

$$\phi(z) = \frac{1}{2\pi i}\int_{-a}^{+a} \frac{-i\,\beta(\sigma)\,d\sigma}{\sigma - z} \qquad (3.5.3)$$

Since the moment $M_n(\sigma)$ is required to vanish along the segment

$-a < \sigma < +a$, we must have

$$\phi'(\sigma) + \bar{\phi}'(\sigma) = \frac{P(z_o - \bar{z}_o)}{8\pi D}\left[\frac{1}{\sigma - z_o} - \frac{1}{\sigma - \bar{z}_o}\right] \tag{3.5.4}$$

along the segment. If we differentiate (3.5.3) with respect to z and then integrate by parts, there results the following

$$\phi'(z) = \frac{1}{2\pi i}\int_{-a}^{+a} \frac{-i\,\beta'(\sigma)\,d\sigma}{\sigma - z} \tag{3.5.5}$$

Here, we have utilized the continuity requirement on the normal derivative along the edge of the plate. If we apply Plemelj's formulae to (3.5.5) and substitute these results in (3.5.4), we again obtain a singular integral equation of Cauchy type

$$\frac{1}{\pi i}\int_{-a}^{+a} \frac{\beta'(\sigma_o)\,d\sigma_o}{\sigma_o - \sigma} = -\frac{P(z_o - \bar{z}_o)}{8\pi D i}\left[\frac{1}{\sigma - z_o} - \frac{1}{\sigma - \bar{z}_o}\right] \tag{3.5.6}$$

Introducing the function

$$\Psi(z) = \frac{1}{2\pi i}\int_{-a}^{+a} \frac{\beta'(\sigma)\,d\sigma}{\sigma - z} \tag{3.5.7}$$

as before, there results

$$\Psi(z) = \frac{-P(z_o - \bar{z}_o)}{i16\pi D\sqrt{z^2 - a^2}}\left\{\sqrt{z^2 - a^2}\left[\frac{1}{z - z_o} - \frac{1}{z - \bar{z}_o}\right] - \frac{\sqrt{z_o^2 - a^2}}{z - z_o} + \frac{\overline{\sqrt{z_o^2 - a^2}}}{z - \bar{z}_o}\right\} + \frac{K_1}{\sqrt{z^2 - a^2}} \tag{3.5.8}$$

Again, we have demanded that $\beta'(\sigma)$ become infinite at the end-points of the slit. Consequently

$$\beta'(\sigma) = \frac{-P(z_o-\bar{z}_o)}{i\,8\pi D}\,\frac{1}{\sqrt{\sigma^2-a^2}}\left[-\frac{\sqrt{z_o^2-a^2}}{\sigma-z_o}+\frac{\overline{\sqrt{z_o^2-a^2}}}{\sigma-\bar{z}_o}\right]+\frac{2K_1}{\sqrt{\sigma^2-a^2}} \tag{3.5.9}$$

Placing (3.5.9) in (3.5.5) and applying Cauchy's residue theorem, we obtain

$$\phi'(z) = \frac{-P(z_o-\bar{z}_o)}{16\pi D}\left\{\frac{1}{\sqrt{z^2-a^2}}\left[\frac{\sqrt{z_o^2-a^2}}{z-z_o}-\frac{\overline{\sqrt{z_o^2-a^2}}}{z-\bar{z}_o}\right]-\frac{1}{z-z_o}+\frac{1}{z-\bar{z}_o}\right\}+\frac{2K_1}{\sqrt{z^2-a^2}} \tag{3.5.10}$$

An indefinite integration of (3.5.10) yields

$$\phi(z) = \frac{-P(z_o-\bar{z}_o)}{16\pi D}\log\left\{\frac{\left[z_o z-a^2-\sqrt{z_o^2-a^2}\,\sqrt{z^2-a^2}\right]}{\left[\bar{z}_o z-a^2-\overline{\sqrt{z_o^2-a^2}}\,\sqrt{z^2-a^2}\right]}\cdot\frac{(z-\bar{z}_o)^2}{(z-z_o)^2}\right\}$$

$$+\,2K_1\log\left[z+\sqrt{z^2-a^2}\right]+K_2 \tag{3.5.11}$$

From our previous assumptions concerning the nature of ϕ, we find that

$$K_2 = \frac{-P(z_o-\bar{z}_o)}{16\pi D}\log\frac{\bar{z}_o-\overline{\sqrt{z_o^2-a^2}}}{z_o-\sqrt{z_o^2-a^2}} \qquad K_1 \equiv 0 \tag{3.5.12}$$

This choice of values for the constant of integration, etc., insure the vanishing of ϕ in the neighborhood of infinity. Hence, ϕ is represented as

$$\phi(z) = \frac{-P(z_o-\bar{z}_o)}{16\pi D}\log\left\{\frac{\left[z_o z-a^2-\sqrt{z_o^2-a^2}\sqrt{z^2-a^2}\right](z-\bar{z}_o)^2\left[\bar{z}_o-\overline{\sqrt{z_o^2-a^2}}\right]}{\left[\bar{z}_o z-a^2-\overline{\sqrt{z_o^2-a^2}}\sqrt{z^2-a^2}\right](z-z_o)^2\left[z_o-\sqrt{z_o^2-a^2}\right]}\right\} \qquad (3.5.13)$$

So that Green's function for this region is given by

$$2w = \frac{P(z-\bar{z})(z_o-\bar{z}_o)}{16\pi D}\log\left\{\qquad\qquad\right\}$$

$$+\frac{P(z-\bar{z})(z_o-\bar{z}_o)}{16\pi D}\log\left\{\qquad\qquad\right\}+2G_1 \qquad (3.5.14)$$

We now compare this result with a combination of problems recently solved by W. R. Dean [7]. He considered the problem of finding the Green's function for an infinite plate clamped along the portion of the real axis for which $|x| > 1$ and an unsupported for $|x| < 1$. His plate supports a concentrated load at z_o as shown in fig. (17.). By mapping this region onto the unit circle by means of the transformation

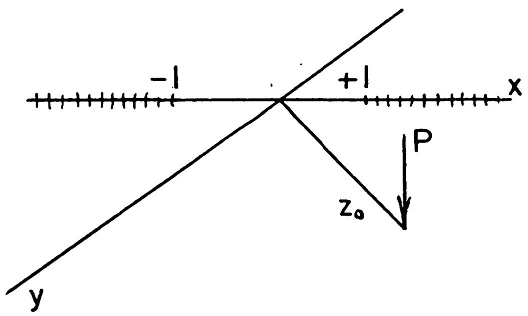


fig. (17.)

$$z = \frac{2\zeta}{1+\zeta^2},$$

he is able to piece together certain solutions in this domain, which satisfy the transformed problem. Thus, his solution is

$$w = -\frac{1}{2}(z-z_o)(\bar{z}-\bar{z}_o)\log\left[1+\frac{(1-\zeta\bar{\zeta})(1-\zeta_o\bar{\zeta}_o)}{(\zeta-\zeta_o)(\bar{\zeta}-\bar{\zeta}_o)}\right]$$

$$+\frac{(1-\zeta\bar{\zeta})(1-\zeta_o\bar{\zeta}_o)}{(1+\zeta^2)(1+\bar{\zeta}^2)(1+\zeta_o^2)(1+\bar{\zeta}_o^2)}\Big[(1+\zeta\bar{\zeta})(1+\zeta_o\bar{\zeta}_o)-(\zeta+\bar{\zeta})(\zeta_o+\bar{\zeta}_o)$$
$$-(\zeta-\bar{\zeta})(\zeta_o-\bar{\zeta}_o)\Big] \qquad (3.5.17)$$

If now, we place a concentrated load at $\bar{z}_o$ and use this solution, we obtain

$$w = -\frac{1}{2}(z-\bar{z}_o)(\bar{z}-z_o)\log\left[1+\frac{(1-\zeta\bar{\zeta})(1-\zeta_o\bar{\zeta}_o)}{(\zeta-\bar{\zeta}_o)(\bar{\zeta}-\zeta_o)}\right]$$

$$+\frac{(1-\zeta\bar{\zeta})(1-\zeta_o\bar{\zeta}_o)(1+\bar{\zeta}_o^2)^{-1}}{(1+\zeta^2)(1+\bar{\zeta}^2)(1+\zeta_o^2)}\Big[(1+\zeta\bar{\zeta})(1+\zeta_o\bar{\zeta}_o)-(\zeta+\bar{\zeta})(\zeta_o+\bar{\zeta}_o)$$
$$+(\zeta-\bar{\zeta})(\zeta_o-\bar{\zeta}_o)\Big] \qquad (3.5.18)$$

A subtraction of these two results gives, by the principle of superposition, the deflection of a semi-infinite plate $\operatorname{Im}(z)>0$, clamped from $|x|>1$ and simply-supported from $|x|<1$. This deflection is given by

$$W = \frac{1}{2}(z-z_o)(\bar{z}-\bar{z}_o)\log(z-z_o)(\bar{z}-\bar{z}_o) - \frac{1}{2}(z-\bar{z})(z_o-\bar{z}_o)$$
$$-\frac{1}{2}(z-\bar{z}_o)(\bar{z}-z_o)\log(z-\bar{z}_o)(\bar{z}-z_o)$$
$$-\frac{1}{2}(z-\bar{z})(z_o-\bar{z}_o)\log\frac{(\bar{\zeta}_o\zeta-1)(\zeta_o\bar{\zeta}-1)(\bar{\zeta}\bar{\zeta}_o-1)(\zeta\zeta_o-1)\,zz_o\bar{z}\bar{z}_o}{4\zeta\bar{\zeta}\zeta_o\bar{\zeta}_o} \qquad (3.5.19)$$

If now in (3.5.13), we replace certain z and z_o by

$$\frac{2\zeta}{1+\zeta^2}\quad,\quad\frac{2\zeta_o}{1+\zeta_o^2}\tag{3.5.20}$$

and $\underline{a}$ by 1 under the log sign, there results

$$\phi(z) - \overline{\phi}(\overline{z}) = \frac{-P(z_o-\overline{z}_o)}{16\pi D}\log\frac{(\zeta-\zeta_o)^2(\overline{\zeta}-\overline{\zeta}_o)^2(z-\overline{z}_o)^2(\overline{z}-z_o)^2}{(\zeta\overline{\zeta}_o-1)^2(\overline{\zeta}\zeta_o-1)^2(z-z_o)^2(\overline{z}-\overline{z}_o)^2}\,2\tag{3.5.21}$$

and, now, recomputing w, we have

$$2w = \frac{P(z-\overline{z})(z_o-\overline{z}_o)}{8\pi D}\log(z-\overline{z}_o)(\overline{z}-z_o) + 2G_1$$

$$\frac{-P(z-\overline{z})(z_o-\overline{z}_o)}{8\pi D}\log\frac{(\zeta\overline{\zeta}_o-1)(\overline{\zeta}\zeta_o-1)(\zeta_o\zeta-1)(\overline{\zeta}_o\overline{\zeta}-1)\,zz_o\overline{z}\overline{z}_o}{4\zeta\zeta_o\overline{\zeta}\overline{\zeta}_o}\tag{3.5.22}$$

which agrees, essentially, with (3.5.19) except for a constant multiplier. Hence, another check on the legitimacy of our procedures and assumptions.

3.6) Clamped, S.S., Clamped, S.S., Clamped.

From an inspection of the nature of solutions presented in the previous sections, it becomes apparent that in order to present a closed solution for a combination of clamping and simple-support, it is necessary to evaluate integrals of the type

$$\int_{\infty}^{z} \frac{\zeta^p d\zeta}{\eta(\zeta)} \quad , \quad \int_{\infty}^{z} \frac{\zeta^p d\zeta}{(\zeta - z_o)\eta(\zeta)} \tag{3.6.1}$$

where

$$G(\zeta) = \prod_{i=1}^{n} (\zeta^2 - a_i^2) \qquad \eta(\zeta) = +\sqrt{G(\zeta)} \tag{3.6.2}$$

and n is the number of strips over which the simple support occurs if the predominant condition is clamping, or it is the number of supports over which the clamping occurs if the predominant condition is a simply-supported edge. The general class of integrals in (3.6.1) is known as the hyperelliptic class. If $n = 1$, the integrals are elementary, as we have seen. If $n = 2$, the integrals are elliptic. Since no extensive tabulation of the cases for which $n \geqslant 3$ exists in the literature, we shall limit ourselves to a solution for $n = 2$ rather than present a general solution for arbitrary n. Actually, such a solution would be useless from an engineering point of view since, as already stated, functions defined by integrals of this type do not appear tabulated.

We begin to determine a Green's function for the case in which we have the following boundary conditions

$$\begin{aligned} &w = 0 \quad , \quad |x| < \infty \\ &\frac{\partial w}{\partial n} = 0 \ , \ |x| > b \ , \quad |x| < a \\ &M_n(w) = 0 \quad , \quad a < |x| < b \end{aligned} \tag{3.6.3}$$

along the edge $y = 0$. To effect an economy of writing, we shall denote the union of arcs over which the normal derivative vanishes by L_c, and the union of arcs for which the bending moment vanishes by L_s.

As has been done previously, we consider the function

$$2w = -(z-\overline{z})\left[\phi(z) - \overline{\phi}(\overline{z})\right] + 2G_1 \qquad (3.6.3)$$

with the usual assumption $\phi(z) = 0\left(\frac{1}{|z|}\right)$ for large $|z|$. The boundary conditions, (3.6.3) along the edge of the plate now take the form

$$i\left[\phi(\sigma) - \overline{\phi}(\sigma)\right] = \begin{cases} 0 & \text{on } L_c \\ \beta(\sigma) & \text{on } L_s \end{cases} \qquad (3.6.4)$$

$$M_n(w) = -2D\left[\phi'(\sigma) + \overline{\phi}'(\sigma)\right] + M_n(G_1) = 0 \text{ on } L_s \qquad (3.6.5)$$

The condition (3.6.4) along with the order assumption on $\phi(z)$ enables us to write

$$\phi(z) = \frac{1}{2\pi i}\int_{L_s} \frac{-i\beta(\sigma)\,d\sigma}{\sigma - z} \qquad (3.6.7)$$

This condition evidently states that $\phi(z)$ is analytic in the entire plane slit along the union of arcs L_s. Since $\beta(\sigma)$ must vanish at the end-points of L_s, we have

$$\phi'(z) = \frac{1}{2\pi i}\int_{L_s} \frac{-i\beta'(\sigma)\,d\sigma}{\sigma - z} \qquad (3.6.8)$$

From Plemelj's formulae and (3.6.5), we have the following singular integral equation

$$\frac{1}{\pi i}\int_{L_s} \frac{\beta'(\sigma_o)\, d\sigma_o}{\sigma_o - \sigma} = -\frac{P(z_o-\overline{z}_o)}{8\pi Di}\left[\frac{1}{\sigma - z_o} - \frac{1}{\sigma - \overline{z}_o}\right] \qquad (3.6.9)$$

If we let

$$\Psi(z) = \frac{1}{2\pi i}\int_{L_s} \frac{\beta'(\sigma_o)\, d\sigma_o}{\sigma_o - z} \qquad (3.6.10)$$

then with the condition that $\beta'(\sigma)$ is to be infinite at the end-points of L_s, we have

$$\Psi(z) = \frac{-P(z_o-\overline{z}_o)}{16\pi Di}\left\{\frac{1}{z-z_o} - \frac{1}{z-\overline{z}_o}\right\}$$

$$-\frac{P(z_o-\overline{z}_o)}{16\pi Di}\left\{-\frac{\sqrt{(z_o^2-a^2)(z_o^2-b^2)}}{z-z_o} + \frac{\overline{\sqrt{(z_o^2-a^2)(z_o^2-b^2)}}}{z-\overline{z}_o} - (z-\overline{z}_o)\right\}$$

$$\frac{1}{\sqrt{(z^2-a^2)(z^2-b^2)}} + (Az+B)\Big/\sqrt{(z^2-a^2)(z^2-b^2)} \qquad (3.6.11)$$

It becomes apparent now that the specification of $\phi(z)$ will involve elliptic integrals. In order then to conform with the notation for elliptic integrals in the complex plane as given in [8], we let

$$G(z) = (z^2-a^2)(z^2-b^2) = \eta^2(z) \quad ,$$

and

$$\eta(z) = +\sqrt{G(z)} \quad ;$$

that is, the branch such that $\lim_{|z|\to\infty} \frac{z^2}{\eta(z)} = +1$. Thus we may write, by application of Plemelj's formulae

$$\beta'(\sigma) = \frac{-P(z_o-\bar{z}_o)}{8\pi D i\,\eta(\sigma)} \left\{ \frac{-\eta(z_o)}{\sigma - z_o} + \frac{\overline{\eta(z_o)}}{\sigma - \bar{z}_o} \right\} + \frac{2A\sigma + 2B}{\eta(\sigma)} \qquad (3.6.12)$$

Placing this in (3. 6. 8) and applying Cauchy's residue theorem, there results

$$\phi'(z) = \frac{-P(z_o-\bar{z}_o)}{16\pi D} \left\{ \frac{\eta(z_o)}{(z-z_o)\eta(z)} - \frac{\overline{\eta(z_o)}}{(z-\bar{z}_o)\eta(z)} - \frac{1}{z-z_o} + \frac{1}{z-\bar{z}_o} \right\} - \frac{Bi}{\eta(z)}$$

(3. 6. 13)

It is to be noted that $\underline{A}$ has been taken as zero in order that $\phi(z)=0(\frac{1}{z})$ in the neighborhood of infinity. If we now write (see [8])

$$I_1(z) = \int_{\infty}^{z} \frac{d\xi}{\eta(\xi)} \quad , \quad I_3(z_1 z_o) \int_{\infty}^{z} \frac{d\xi}{(\xi - z_o)\eta(\xi)} \qquad (3.6.14)$$

then

$$\phi'(z) = -Bi\,I_1'(z)$$

$$\frac{-P(z_o-\bar{z}_o)}{16\pi D} \left\{ \eta(z_o)\, I_3'(z, z_o) - \overline{\eta(z_o)}\; I_3'(z, \bar{z}_o) - \frac{1}{z-z_o} + \frac{1}{z-\bar{z}_o} \right\}$$

(3. 6. 15)

It is apparent that $I_1(z)$ and $I_3(z, z_o)$ are elliptic integrals of the first and third kind respectively in the complex plane. From the definition (3.6.14), it follows that

$$\lim_{z \to \infty} I_1(z) = 0 \qquad \lim_{z \to \infty} I_3(z, z_o) = 0 \ .$$

Consequently

$$\phi(z) = -Bi\, I_1(z)$$
$$\frac{-P(z_o - \bar{z}_o)}{16\pi D}\left[\eta(z_o)\, I_3(z, z_o) - \overline{\eta(z_o)}\, I_3(z, \bar{z}_o) - \log \frac{z - z_o}{z - \bar{z}_o}\right] \tag{3.6.16}$$

and $\phi(z) = 0\left(\frac{1}{z}\right)$ for large $|z|$. It is also to be noted that since

$$I_3(z, z_o) = \frac{1}{\eta(z_o)} \log(z - z_o) + A(z - z_o)$$
$$I_3(z, \bar{z}_o) = \frac{1}{\overline{\eta(z_o)}} \log(z - \bar{z}_o) + A(z - \bar{z}_o) \tag{3.6.17}$$

where the expansions are legitimate for $|z - z_o| < \epsilon$ and $A(z - z_o)$ is analytic in this region, then $\phi(z)$ is analytic in the slit plane. It now remains to determine the constant B. For the previous problems this constant was evaluated on an order argument. If one examines (3.6.16) in light of (3.6.4) and the conditions $\beta(\sigma)$ must satisfy at the end-points of L_s, we find that B can be determined.

Before we proceed on this project, we remark that the form

(3.6.16) is unsuitable for computation of any sort, since the functions are not tabulated. It is more convenient to introduce, instead, Legendre's third normal form of elliptic integrals. This, as will be seen, can be accomplished by some elementary algebraic manipulations. Indeed, if we write

$$\eta(z_o) I_3(z, z_o) - \overline{\eta(z_o)}\, I_3(z, \overline{z}_o) =$$

$$\eta(z_o) \int_{\infty}^{z} \frac{\zeta\, d\zeta}{(\zeta^2 - z_o^2)\, \eta(\zeta)} - \overline{\eta(z_o)} \int_{\infty}^{z} \frac{\zeta\, d\zeta}{(\zeta^2 - \overline{z}_o^2)\, \eta(\zeta)}$$

$$+ z_o \eta(z_o) \int_{\infty}^{z} \frac{d\zeta}{(\zeta^2 - z_o^2)\, \eta(\zeta)} - \overline{z_o \eta(z_o)} \int_{\infty}^{z} \frac{d\zeta}{(\zeta^2 - z_o^2)\eta(\zeta)} ,$$

and consider

$$\Phi(z, z_o) = \frac{\eta(z_o)}{2} \int_{\infty}^{z^2} \frac{dx}{(x - z_o^2)\sqrt{(x-a^2)(x-b^2)}} ,$$

then

$$\Phi(z, z_o) = -\frac{1}{2} \log (z^2 - z_o^2)\left[2z_o^2 - (a^2 + b^2) - 2\eta(z_o)\right]$$

$$\frac{1}{2} \log\left\{ 2\eta^2(z_o) + (z^2 - z_o^2)\left[2z_o^2 - (a^2+b^2)\right] - 2\eta(z_o)\,\eta(z)\right\} \tag{3.6.18}$$

and consequently

$$\Phi(z, \overline{z}_o) = -\frac{1}{2} \log (z^2 - \overline{z}_o^2)\left[2\overline{z}_o^2 - (a^2 + b^2) - 2\,\overline{\eta(z_o)}\right]$$

$$+ \frac{1}{2} \log\left\{ 2\eta^2(\overline{z}_o) + (z^2 - \overline{z}_o^2)\left[2\overline{z}_o^2 - (a^2+b^2)\right] - 2\overline{\eta(z_o)}\,\eta(z)\right\} \tag{3.6.19}$$

Thus we may write

$$\phi(z) = \frac{-P(z_o-\overline{z}_o)}{32\pi D}\left[\Phi(z, z_o) - \Phi(z, \overline{z}_o)\right] + \frac{P(z_o-\overline{z}_o)}{16\pi D}\log\frac{z-z_o}{z-\overline{z}_o}$$

$$-\frac{P(z_o-\overline{z}_o)}{16\pi D}\left[z_o\eta(z_o)\int_{\infty}^{z}\frac{d\zeta}{(\zeta^2-z_o^2)\eta(\zeta)} - \overline{z_o\eta(z_o)}\int_{\infty}^{z}\frac{d\zeta}{(\zeta^2-\overline{z}_o^2)\eta(\zeta)}\right]$$

$$-BiI_1(z) \qquad (3.6.20)$$

With this form, $\phi(z)$ is now represented in terms of elementary functions and elliptic integrals of the first and third type in Legendre's classification.

If now $\beta(\sigma)$ is to vanish at the end-points of L_s, we must have

$$\phi(\pm b) - \overline{\phi}(\pm b) = 0$$

$$\phi(\pm a) - \overline{\phi}(\pm a) = 0 \qquad (3.6.21)$$

If we take the first of these conditions, we have, since

$$\eta(\sigma) = \overline{\eta(\overline{\sigma})} \qquad \text{for} \quad b<\sigma<\infty$$

the fact that

$$B + \overline{B} = 0 \qquad (3.6.22)$$

Taking the second of these conditions, and noting that

$$\eta(\sigma) = i\,\zeta(\sigma) = i\sqrt{(b^2-\sigma^2)(\sigma^2-a^2)} \qquad a<\sigma<b$$

$$\eta(\sigma) = -\zeta(\sigma) \qquad -a<\sigma<+a$$

$$\eta(\sigma) = -i\,\zeta(\sigma) \qquad -b<\sigma<-a,$$

we discover that this second condition is automatically satisfied if (3.6.22). The third condition implies that

$$\frac{P(z_o-\overline{z}_o)}{8\pi Di}\left[z_o\eta(z_o)\int_a^b \frac{d\sigma}{(\sigma^2-z_o^2)\,\zeta(\sigma)} - \overline{z_o\eta(z_o)}\int_a^b \frac{d\sigma}{(\sigma^2-\overline{z}_o^2)\,\zeta(\sigma)}\right]$$

$$+ (B-\overline{B})\int_a^b \frac{d\sigma}{\zeta(\sigma)} = 0 \qquad (3.6.23)$$

and this automatically enforces the fourth condition.

Thus we may finally write

$$\phi(z) = \frac{-P(z_o-\overline{z}_o)}{32\pi D}\left[\Phi(z,z_o) - \Phi(z,\overline{z}_o)\right] + \frac{P(z_o-\overline{z}_o)}{16\pi D}\log\frac{z-z_o}{z-\overline{z}_o}$$

$$-\frac{P(z_o-\overline{z}_o)}{16\pi D}\left[z_o\eta(z_o)\int_\infty^z \frac{d\zeta}{(\zeta^2-z_o^2)\eta(\zeta)} - \overline{z_o\eta(z_o)}\int_\infty^z \frac{d\zeta}{(\zeta^2-\overline{z}_o^2)\eta(\zeta)}\right]$$

$$+\frac{P(z_o-\overline{z}_o)}{16\pi D}\,\frac{I_1(z)}{K(k^1)}\left[\frac{z_o\eta(z_o)}{-z_o^2+b^2}\Pi\left(\frac{b^2-a^2}{b^2-z_o^2},\,k^1\right) - \frac{\overline{z_o\eta(z_o)}}{-\overline{z}_o^2+b^2}\Pi\left(\frac{b^2-a^2}{b^2-\overline{z}_o^2},\,k^1\right)\right]$$

$$k^{1\,2} = \frac{b^2-a^2}{b^2}, \quad 2w = -(z-\overline{z})\left[\phi(z) - \overline{\varphi}(\overline{z})\right] + 2\,G_1 \qquad (3.6.24)$$

When this result, along with its conjugate is placed in (3.6.3) which we repeat here, the solution of the problem is complete. Additionally, it may be pointed out that with $\underline{B}$ so determined, $\phi(z)$ is a single valued function.

This will complete our discussion of the half-plane.

CHAPTER IV

PROBLEMS FOR THE QUARTER-PLANE

After one leaves the half-plane, one would feel confident that the next problems, in degree of complexity, would be those connected with the quarter-plane. The type of singular integral equations obtained in these cases are; however, somewhat more complex. One can, however, formulate certain problems connected with this domain and avoid the solution of any singular integral equation by utilizing the method of images. This we proceed to do.

4.1) Simply-Supported, Clamped, Simply-Supported.

The solution for the boundary value problem shown in fig. (18.) is effected by utilizing the solution of sec. (3.3). Indeed, if in (3.3.20) we replace z_o by $-\bar{z}_o$ and P by $-P$, then we obtain

$$w_1 = \frac{-P(z-\bar{z})(z_o-\bar{z}_o)}{32\pi D} \log\left\{ z_o \to -\bar{z}_o \right\} + G_1\left(\begin{matrix} z_o \to -\bar{z}_o \\ P \to -P \end{matrix} \right) \quad (4.1.1)$$

If this solution is added to (3.3.20), then in view of the antisymmetric nature of the deflections, they will cancel out along the y-axis, and consequently so will the bending moment vanish. We have then as the solution to this problem

$$w = \text{R.H.S. } (3.3.20) + \text{R.H.S. } (4.1.1) \qquad (4.1.2)$$

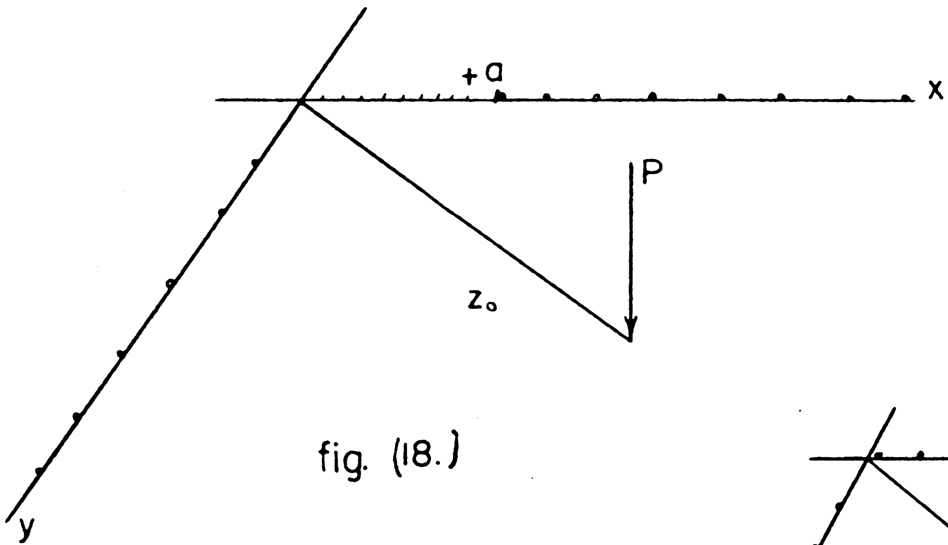


fig. (18.)

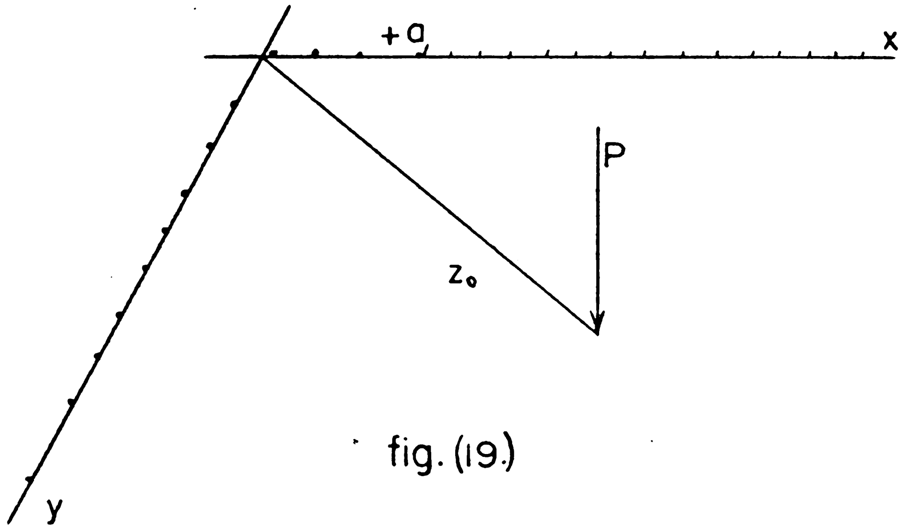


fig. (19.)

This solution has the property that along the y-axis the plate is simply-supported, along the x-axis from 0 to a it is clamped, and simply-supported along the x-axis from a to infinity.

With this Green's function, one could now formulate the problem of a plate simply-supported along the y-axis from infinity to a, clamped from a to 0, and satisfying the same conditions as (4.1.2) along the x-axis.

(4.2) Simply-Supported, Simply-Supported, Clamped.

Again a utilization of the method of images in connection with the problem shown in fig. (19.) will produce the solution to this problem. All that is necessary is to replace z_0 by $-\bar{z}_0$ and P by -P in (3.5.14) and add the resulting equation to (3.5.14). When this is done, there results

$$W = \text{R. H. S. } (3.5.14) + \text{R. H. S. } (3.5.14)_{\substack{z_0 \to -\bar{z}_0 \\ p \to -p}} \qquad (4.2.1).$$

This solution has the property that along the y-axis the plate is simply-supported, along the x-axis from 0 to a it is simply-supported, and along the x-axis from a to infinity it is clamped.

Again this Green's function could be utilized to solve a problem with conditions the same on each of the axes as in the previous problem. However, the final attainment of the solution would depend on one's ability to invert the singular integral equation involved.

CHAPTER V

PROBLEMS FOR THE CIRCULAR PLATE

In order of expected difficulty, the circular plate would seem to be associated with a number of mixed boundary value problems which would yield to analysis. Unfortunately, this does not appear to be the case. As the discussion for this region is developed, it will become evident that the nature of the singular integral equations will be such that they are actually singular integro-differential equations. This aspect of the problem, generally eliminates all hope of presenting any solutions for problems connected with this region in terms of functions known in classical analysis. We begin, then, with a discussion of the general aspects of such problems.

5.1) Formulation of Solution for a Class of Problems.

The Green's function for a clamped circular plate subjected to a concentrated load P at the point Z_0 was already given in (2.3). With this in mind, we now consider all problems involving circular plates for which the edge deflection is zero and subjected to a concentrated load $\underline{P}$ at the point z_0. Clearly, one boundary condition remains to be satisfied at the edge of the plate. Without being specific for the present, we shall label this condition as

$$B_2 \, [w] = 0 \qquad z \in \left\{ z\bar{z} = a^2 \right\} \tag{5.1.1}$$

The operator B_2 is assumed to be linear. Physically, it will amount to some requirement on the slope or bending moment or both. If now f(z) is some function of z which is analytic in the interior of the plate, then

$$w = (a^2 - z\bar{z}) \left[f(z) + \bar{f}(\bar{z}) \right] + G_1 \tag{5.1.2}$$

satisfies the condition

$$w = 0 \quad \text{for} \quad z \in \left\{ z\bar{z} = a^2 \right\} \tag{5.1.3}$$

Furthermore, since f(z) is analytic in the interior of the plate, the deflection given by (5.1.2) is such that the plate is subjected to a concentrated load at the point z_0. If now f(z) is so determined that

$$B_2 \left[(a^2 - z\bar{z})(f + \bar{f}) + G_1 \right] = 0 \quad \text{for } z \in \left\{ z\bar{z} = a^2 \right\} \tag{5.1.4}$$,

then w as given by (5.1.2) represents the Green's function for the conditions

$$w = 0, \; B_2 [w] = 0 \text{ on the boundary} \tag{5.1.5}$$

As in the case of the half-infinite plate, we shall begin by exhibiting the easy manner in which we can derive the solution of an old problem.

5.2) Simply-Supported Edge.

As a first application of the method, we consider the solution for the simply-supported edge. The solution of this problem was first given by J. Hadamard [9] and, more recently rediscovered by H. Reismann [10]. The solution presented here is novel in two respects. First, the ease of derivation, and, second, a closed form for the solution is obtained in terms of known functions. The method Hadamard presents is quite tedious by modern standards. His solution, furthermore, has one integral which he does not evaluate. We shall presently see that a closed form does, indeed, exist in terms of Gauss' hypergeometric function. Reismann, unaware of Hadamard's solution, obtains a series expansion for the result by the method of series. It should be remarked that our solution is of aesthetic value only, since in order to make any computations, it would be necessary to take the series form for the hypergeometric function.

For the problem at hand, then, the condition (5.1.1) becomes

$$B_2\,[w] = M_r(w) = -D\left[\nabla^2 w - (1-\nu)\left\{\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right\}\right]$$

$$= -D\left[4w_{z\bar{z}} - \frac{(1-\nu)}{a^2}\left(z w_z + \bar{z} w_{\bar{z}}\right)\right] = 0 \qquad z = ae^{i\theta} = \sigma \qquad (5.2.1).$$

(It is to be noted that $w_{\theta\theta}$ is zero along the boundary since the deflection vanishes.)

Now

$$B_2\,[G_1] = -\frac{P}{4\pi}\left[\frac{z_0(\bar{\sigma} - \bar{z}_0)}{a^2 - z_0\bar{\sigma}} + \frac{\bar{z}_0(\sigma - z_0)}{a^2 - \bar{z}_0\sigma} + 1 + \frac{z_0\bar{z}_0}{a^2}\right].$$

Thus the following functional equation must hold along the boundary if the bending moment is to vanish

$$\sigma f'(\sigma) + \bar{\sigma}\bar{f}'(\bar{\sigma}) + \frac{1+\nu}{2}\left[f(\sigma) + \bar{f}(\bar{\sigma})\right]$$

$$= \frac{P}{16\pi D}\left[\frac{z_0(a^2 - \bar{z}_0\sigma)}{a^2(\sigma - z_0)} + \frac{\bar{z}_0(\sigma - z_0)}{a^2 - \bar{z}_0\sigma} + 1 + \frac{z_0\bar{z}_0}{a^2}\right] \quad (5.2.2).$$

Applying Cauchy's residue theorem to (5.2.2) and recalling that f(z) is analytic inside the circle, there results

$$zf'(z) + \frac{1+\nu}{2}\left[f(z) + \bar{f}(0)\right] = \frac{P(a^2 - z_0\bar{z}_0)}{16\pi D(a^2 - \bar{z}_0 z)} \quad (5.2.3)$$

which is obviously an elementary type differential equation, the most general solution of which is

$$f(z) = \frac{P(a^2 - z_0\bar{z}_0)}{16\pi D} z^{-\frac{(1+\nu)}{2}} \int_0^z z^{\frac{1+\nu}{2} - 1}(a^2 - \bar{z}_0 z)^{-1} dz$$

$$- \bar{f}(0) + C_1 z^{-\frac{(1+\nu)}{2}} \quad (5.2.4).$$

Since f(z) is analytic in $z\bar{z} < a^2$, C_1 must be zero. The integral occurring is an Eulerian integral of the first kind and, consequently, may be expressed in terms of Gauss' hypergeometric function. When this is done, there results

$$f(z) = \frac{P(a^2 - z_0\bar{z}_0)}{8\pi D a^2(1+\nu)} F\left(\frac{1+\nu}{2}, 1, \frac{1+\nu}{2}; \frac{z\bar{z}_0}{a^2}\right) - \bar{f}(0) \quad (5.2.5).$$

With this, we may now express the Green's function for the simply-supported plate as

$$G_2 = \frac{P(a^2 - z_0\bar{z}_0)(a^2 - z\bar{z})}{8\pi D a^2 (1+\nu)} \left[F\left(\frac{1+\nu}{2}, 1, \frac{3+\nu}{2}; \frac{z\bar{z}_0}{a^2}\right) + F\left(\frac{1+\nu}{2}, 1, \frac{3+\nu}{2}; \frac{\bar{z}z_0}{a^2}\right) \right]$$

$$+ \frac{P(a^2 - z_0\bar{z}_0)(a^2 - z\bar{z})}{8\pi D a^2 (1+\nu)} (-1) + \frac{P}{16\pi D} \left[\frac{(a^2 - \bar{z}_0 z)(a^2 - z_0\bar{z})}{a^2} - (z - z_0)(\bar{z} - \bar{z}_0) \right]$$

$$- \frac{P(z - z_0)(\bar{z} - \bar{z}_0)}{16\pi D} \log \frac{(a^2 - \bar{z}_0 z)(a^2 - z_0\bar{z})}{a^2 (z - z_0)(\bar{z} - \bar{z}_0)} \qquad (5.2.6).$$

It is easy to verify by a straightforward number of differentiations that this function satisfies the boundary conditions imposed on it.

This solution also enables one to write down, almost immediately, the solution for the circular plate with zero deflection at the boundary and a normal which is elastically restrained against rotation. Indeed, this second condition takes the form

$$M_r \, [w] = \frac{kD}{a} \frac{\partial w}{\partial r}$$

which may be rewritten as

$$-D \left[4w_{z\bar{z}} - \frac{(1 - \nu - k)}{a^2} (z w_z + \bar{z} w_{\bar{z}}) \right] = 0 \qquad z = ae^{i\theta} .$$

The factor k is a dimensionless quantity which varies between zero and infinity. Consequently, Green's function for this case may be obtained from eq. (5.2.6) merely by replacing ν by $\nu + k$.

5.3) Simply-Supported, Clamped Edge.*

From the form of the left-hand side of (5.2.2), it becomes evident that any formulation of this mixed problem involving singular integral equations will terminate as a singular integro-differential equation. By mapping the circle onto the half-plane and considering the transformed boundary conditions, it is possible to set the problem in terms of a singular integro-differential equation for which the range of integration extends over the positive half of the x-axis. One is able to convert this equation into a second order difference equation in the Mellin transform of the unknown sought. This final equation is such; however, that the author is unable to determine its solution. Consequently, we shall abandon the previous method of attack on the problem and, instead, present a numerical procedure for the solution of such problems.

It is more convenient, in this approach, to work in terms of real variables.

Thus, if w_0 is a solution of the plate equation such that it corresponds to a load q over the surface of the plate, and if w_0 vanishes at the boundary of the plate, then

$$w(r, \theta) = (a^2 - r^2) \phi \ (r, \theta) + w_0 \tag{5.3.1}$$

*This problem was recently considered by D. I. Sherman from the point of view of solving a singular integral equation by an iterative procedure. His results contain no numerical solutions. See [13].

is a deflection function for a circular plate subjected to a load q and such that at the boundary of the plate

$$w(a, \theta) = 0$$

provided $\phi(r, \theta)$ is analytic and harmonic in the interior of the circle in fig. (20.). If the plate is clamped along the arc $-\alpha \leq \theta \leq +\alpha$ and is simply-supported along the remaining arc $\alpha \leq \theta \leq 2\pi - \alpha$, then

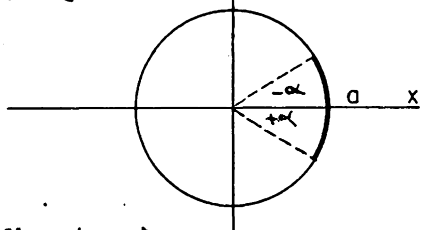


fig. (20.)

$$-2a\,\phi(a, \theta) + \left.\frac{\partial w_0}{\partial r}\right|_{r=a} = 0 \qquad (5.3.2)$$

along the clamped portion of the edge, and

$$4a \left.\frac{\partial \phi}{\partial r}\right|_{r=a} + 2(1+\nu)\,\phi + \left.\frac{M_r\,[w_0]}{D}\right|_{r=a} = 0 \qquad (5.3.3)$$

must hold along the simply-supported part of the boundary. In order to determine the harmonic function $\phi(r, \theta)$, we have a Dirichlet condition over a portion of the edge and a mixed condition over the remainder of the edge. This is an extremely complex type of problem to solve. Any attempts to treat this analytically always require some sort of numerical procedure. Let us then investigate the following possibility. We begin by defining two step functions

$$I_1(\theta) = \begin{cases} 0 \\ 1 \end{cases}, \quad I_2(\theta) = \begin{cases} 1 & -\alpha \leq \theta \leq +\alpha \\ 0 & \alpha \leq \theta \leq 2\pi - \alpha \end{cases} \qquad (5.3.4).$$

With these functions, the conditions (5.3.2) and (5.3.3) may be expressed as a single statement; namely,

$$\phi(a, \theta)\left[-2a\, I_2 + 2(1+\nu)\, I_1\right] + 4a\, I_1 \frac{\partial\phi}{\partial r}\Bigg|_{r=a} + I_2 \frac{\partial w_0}{\partial r}\Bigg|_{r=a} + I_1 \frac{M_r[w]}{D} = 0 \qquad (5.3.5).$$

Since ϕ is analytic and harmonic in the interior of the plate, we have

$$\phi(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta) \qquad (5.3.6).$$

With this, (5.3.5) is a function of θ which is identically zero on the boundary of the plate with the possible exception of the end-points of the clamped arc. From the nature of previous solutions, it might be expected that the moment would become infinite at the end-points of the clamped arc. The nature of this infinity is, however, such that it does not affect the solution obtained by the condition that the Fourier coefficients of (5.3.5) must all vanish.

Thus by multiplying through (5.3.5) first by $\cos m\theta$ and then by $\sin m\theta$ and integrating each of these expressions with respect to θ from 0 to 2π, there results

$$\int_0^{2\pi} \cos m\theta \left[-2aI_2 + 2(1+\nu)\, I_1\right] \phi\, d\theta + \int_0^{2\pi} 4aI_1 \frac{\partial\phi}{\partial r} \cos m\theta\, d\theta$$

$$= -\int_0^{2\pi} \left\{ \cos m\theta \, I_2 \frac{\partial w_0}{\partial r} + \cos m\theta \, I_1 \frac{M_r [w_0]}{D} \right\} d\theta \qquad m = 0, 1, 2, ---$$

$$\int_0^{2\pi} \sin m\theta \left[-2a \, I_2 + 2(1+\nu) I_1 \right] \phi \, d\theta + \int_0^{2\pi} 4a I_1 \frac{\partial \phi}{\partial r} \sin m\theta d\theta$$

$$= -\int_0^{2\pi} \left\{ \sin m\theta \, I_2 \frac{\partial w_0}{\partial r} + \sin m\theta \, I_1 \frac{M_r [w_0]}{D} \right\} d\theta \qquad m = 1, 2, 3 ---$$

(5.3.7).

This system of equations will serve for a determination of the constants a_0, $\{a_n, b_n\}$. It is evident that this is an infinite system of algebraic linear equations and thus to obtain the solution of any specific problem, one must specify the particular solution w_0 and then truncate the system of equations.

As an example, we consider the case of a uniformly loaded circular plate of radius a clamped from 0 to $\pi/2$, simply-supported from $\pi/2$ to $3\pi/2$, and clamped from $3\pi/2$ to 2π. In this case, we take

$$w_0 = \frac{q}{64D} (a^2 - r^2)^2$$

Since ϕ is harmonic and symmetric with respect to the x-axis, we have

$$\phi(r, \theta) = \sum_{n=0}^{\infty} A_n r^n \cos n\theta$$

If these forms are now substituted into eqs. (5.3.7), there results the following infinite system of linear algebraic equations

$$A_0 = \frac{qa^2}{32D(1+\nu)} \left[1 - \sum_1^{\infty} \frac{2B_m}{m} (-1)^m \sin \frac{m\pi}{2} \right]$$

$$\frac{\pi}{2} \left[\frac{1+\nu}{n} + B_n \right] - \sum_{m=1}^{\infty}{}' \ B_m \left[\frac{\sin \frac{\pi}{2}(m-n)}{m-n} + \frac{\sin \frac{\pi}{2}(m+n)}{m+n} \right] + \frac{2}{\pi n} \sin \frac{\pi n}{2} = 0$$

$$n = 1, 2, \ldots$$

where, for simplicity we have introduced

$$B_n = \frac{32Dna^{n-2}}{q\pi} A_n \qquad n = 1, 2, \ldots$$

An exact solution of this system of equations does not seem possible, hence we resort to solving a finite number of these equations. Even then, an appraisal of the error involved is somewhat difficult. In lieu of this, we first solved the first 12 of these equations for the unknowns, then we solved the first 24 of these equations, and finally, the first 36 of these equations. Although this is not to be construed as a proof of a sort, it will give some indication as to the variation in answers as caused by increasing the number of equations. These results are tabulated in Table (5) for $\nu = .3$.

---

* The accent on the summation implies that the term for m = n is absent. See for example [15] page 434.

For the solution based on the usage of the 36 equations, we have indeed that

$$A_0 = \frac{qa^2\pi}{32D}\ (.1176).$$

With this value for A_0 and the values for B_n tabulated in column 3 of Table (5), we are now in a position to compute the values of the slope and bending moment at the edge of the plate. The tabulated values of these quantities are given in Table (6) and a graphical representation occurs in Figs. (21) and (21'). We notice that where the slope should be zero; that is, the region $0 \leqslant \theta \leqslant 90^o$, it is very small as compared to the values on the simply-supported portion of the edge. As can be noticed, the results are such that the boundary conditions imposed on the deflection are approximately satisfied.

Tables (7a), (7b), (7c), (7d) give the values of the deflections and moments in the interior of the plate based on the solution of 12, 24, and 36 equations respectively. These results are plotted in Figs. (22a), (22b), (22c), (22d), (23a), (23b), (23c), (23d), (24a), (24b), (24c), (24d).

The maximum deflection of the plate based on 36 equations occurs at $\theta = 180^o$ and $r \doteq .18a$. Its value is

$$\frac{32w_{max}}{q\pi a^4} = .292$$

	12 eqs.	24 eqs.	36 eqs.
B_1	-.1676516760	-.1645904702	-.1632161386
B_2	..0445843578	.0539939686	.0578867212
B_3	..1120890729	.1067710903	.1041205832
B_4	-.0121788793	-.0377711501	-.0439349247
B_5	-.0918282440	-.0878531935	-.0847954584
B_6	.0026180990	.0246146204	.0320457526
B_7	.0790474307	.0776253713	.0747109103
B_8	.0157689413	-.0136955390	-.0239916414
B_9	-.0661340726	-.0704850760	-.0682317699
B_{10}	-.0346074292	.0040881228	.0165257896
B_{11}	.0441940637	.0645048075	.0634794611
B_{12}	.0505823970	.0047547247	-.0099488409
B_{13}		-.0507285160	-.0595133442
B_{14}		-.0131665567	.0039602850
B_{15}		.0524716000	.0560142318
B_{16}		.0213455724	.0016326487
B_{17}		-.0449994638	-.0516705216
B_{18}		-.0293430561	-.0069616549
B_{19}		.0551777547	.0491851272
B_{20}		.0368358595	.0121126611
B_{21}		-.0206110511	-.0456870846
B_{22}		-.0416843125	-.0171895655
B_{23}		-.0059339915	.0416989479
B_{24}		.0139829503	.0222237417
B_{25}			-.0371034192
B_{26}			-.0271845085
B_{27}			.0315941804
B_{28}			.0324626404
B_{29}			-.0146835833
B_{30}			-.0381041427
B_{31}			.0154779972
B_{32}			.0485080666
B_{33}			.0161418500
B_{34}			-.0154172680
B_{35}			-.0211199660
B_{36}			.0006121559

Solution for the
System of Eqs. (5.3.7)

For n = 12, 24, 36.
ν = .3

TABLE NO. V

θ/π	$-\frac{16D}{qa^3\pi}\frac{\partial w}{\partial r}$	$\frac{5M_r}{qa^2\pi}$
.0000	.00015	-.2415
.0625	.00031	-.2391
.1250	.00072	-.2323
.1875	.00129	-.2245
.2500	.00188	-.2195
.3125	.00238	-.2221
.3750	.00281	-.2409
.4375	.00381	-.2898
.5000	.07503	-.1406
.5625	.16807	-.0093
.6250	.20939	-.0059
.6875	.23513	-.0024
.7500	.25252	-.0036
.8125	.26447	-.0035
.8750	.27237	-.0012
.9375	.27690	.0013
.9922	.27831	.0014

TABLE NO. VI

Boundary Values of the Slope and Bending

Moment versus θ/π for n =36.

$32Dw(aP, \theta)/qa^4\pi$

n = 12

θ/P	0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
0°	.294	.274	.247	.215	.180	.144	.108	.073	.043	.019	0.00
45°		.278	.253	.221	.1[illegible]3	.142	.101	.064	.032	.009	0.00
90°		.289	.275	.251	.219	.181	.140	..098	.059	.028	0.00
135°		.301	.300	.289	.270	.242	.206	.163	.112	.057	0.00
180°		.307	.309	.301	.282	.252	.213	.165	.112	.056	0.00
n = 24											
0°	.279	.259	.233	.202	.169	.134	.100	.067	.038	.014	0
45°		.263	.239	.207	.171	.132	.093	.058	.029	.009	0
90°		.274	.259	.236	.205	.167	.125	.082	.043	.016	0
135°		.286	.284	.274	.256	.230	.196	.155	.107	.055	0
180°		.291	.294	.287	.269	.241	.203	.158	.106	.053	0
n = 36											
0°	.277	.257	.231	.201	.168	.134	.099	.067	.038	.015	0
45°		.261	.237	.206	.170	.131	.093	.058	.030	.010	0
90°		.272	.257	.234	.202	.164	.122	.079	.038	.009	0
135°		.284	.282	.272	.254	.228	.195	.154	.107	.054	0
180°		.289	.292	.285	.268	.240	.202	.157	.106	.053	0

Table No. VIIa

$5M_r/qa^2\pi$

n = 12

	0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
0°	.179	.157	.131	.100	.065	.026	-.016	-.064	-.117	-.170	----
45°	.184	.171	.150	.121	.084	.040	-.011	-.069	-.129	-.197	----
90°	.189	.185	.175	.156	.130	.097	.055	.004	-.052	-.107	----
135°	.184	.191	.192	.187	.178	.165	.145	.120	.089	.049	----
180°	.179	.196	.205	.208	.203	.189	.168	.138	.100	.054	----.

n = 24

	0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
0°	.173	.150	.124	.093	.060	.021	-.021	-.069↑	-.121	-.180	----
45°	.178	.164	.143	.114	.078	.033	-.017	-.073	-.133	-.196	----
90°	.184	.180	.169	.151	.124	.090	.047	-.006	-.071	▾.145	----
135°	.178	.185	.186	.181	.171	.157	.139	.113	.084	.046	----
180°	.173	.188	.198	.201	.197	.184	.163	.134	.097	.052	----

n = 36

	0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
0°	.171	.149	.123	.093	.059	.021	-.022	-.069	-.122	-.180	----
45°	.177	.163	.142	.113	.077	.033	-.016	-.073	-.132	-.195	----
90°	.174	.180	.169	.150	.124	.089	.046	-.008	-.076	-.163	----
135°	.177	.184	.185	.180	.170	.156	.138	.114	.084	.046	----
180°	.171	.187	.197	.200	.196	.184	.164	.135	.098	.053	----

Table No. VIIb

$5M_\theta / qa^2 \pi$

n = 12

	0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
0°	.189	.179	.167	.149	.129	.104	.076	.045	.019	-.022	----
45°	.184	.173	.160	.142	.122	.098	.071	.040	.003	-.040	----
90°	.179	.177	.171	.161	.146	.127	.105	.080	.053	.026	----
135°	.184	.191	.194	.194	.189	.180	.167	.150	.129	.104	----
180°	.189	.195	.197	.196	.191	.183	.171	.156	.136	.113	----

n = 24

	0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
0°	.184	.178	.161	.144	.123	.099	.071	.040	.005	-.033	----
45°	.178	.168	.154	.137	.116	.093	.067	.036	.002	-.038	----
90°	.172	.170	.163	.152	.137	.117	.092	.063	.031	.004	----
135°	.178	.185	.188	.187	.182	.174	.161	.144	.123	.100	----
180°	.184	.189	.191	.190	.185	.177	.165	.150	.130	.107	----

n = 36

	0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
0°	.183	.174	.161	.144	.123	.099	.071	.040	.005	-.031	----
45°	.177	.167	.153	.137	.117	.093	.067	.037	.002	-.034	----
90°	.171	.169	.162	.151	.135	.115	.089	.058	.022	-.010	----
135°	.177	.184	.187	.186	.182	.173	.160	.143	.123	.099	----
180°	.183	.189	.191	.190	.185	.176	.165	.149	.130	.106	----

Table No. VIIc

$5M_r\theta/qa^2\pi$

n = 12

	0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
0°	0	0	0	0	0	0	0	0	0	0	0
45°	-.0049	-.0091	-.0129	-.0162	-.0186	-.0197	-.0186	-.0149	-.0097	-.0045	----
90°	0	-.0012	-.0025	-.0039	-.0055	-.0076	-.0106	-.0159	-.0275	-.0554	----
135°	.0049	.0004	-.0043	-.0091	-.0137	-.0176	-.0202	-.0213	-.0208	-.0185	----
180°	0	0	0	0	0	0	0	0	0	0	0

n = 24

	0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
0°	0	0	0	0	0	0	0	0	0	0	0
45°	-.0059	-.0090	-.0135	-.0165	-.0185	-.0190	-.0127	-.0144	-.0099	-.0057	----
90°	0	-.0013	-.0026	-.0040	-.0057	-.0076	-.0102	-.0140	-.0218	-.0475	----
135°	.0059	.00153	-.0031	-.0080	-.0129	-.0171	-.0203	-.0217	-.0211	-.0187	----
180°	0	0	0	0	0	0	0	0	0	0	0

n = 36

	0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
0°	0	0	0	0	0	0	0	0	0	0	0
45°	-.0036	-.0102	-.0137	-.0166	-.0184	-.0187	-.0172	-.0139	-.0093	-.0044	----
90°	0	-.0013	-.0027	-.0041	-.0058	-.0078	-.0103	-.0140	-.0203	-.0414	----
135°	.0063	.0010	-.0027	-.0076	-.0125	-.0170	-.0202	-.0219	-.0215	-.0195	----
180°	0	0	0	0	0	0	0	0	0	0	0

Table No. VIId

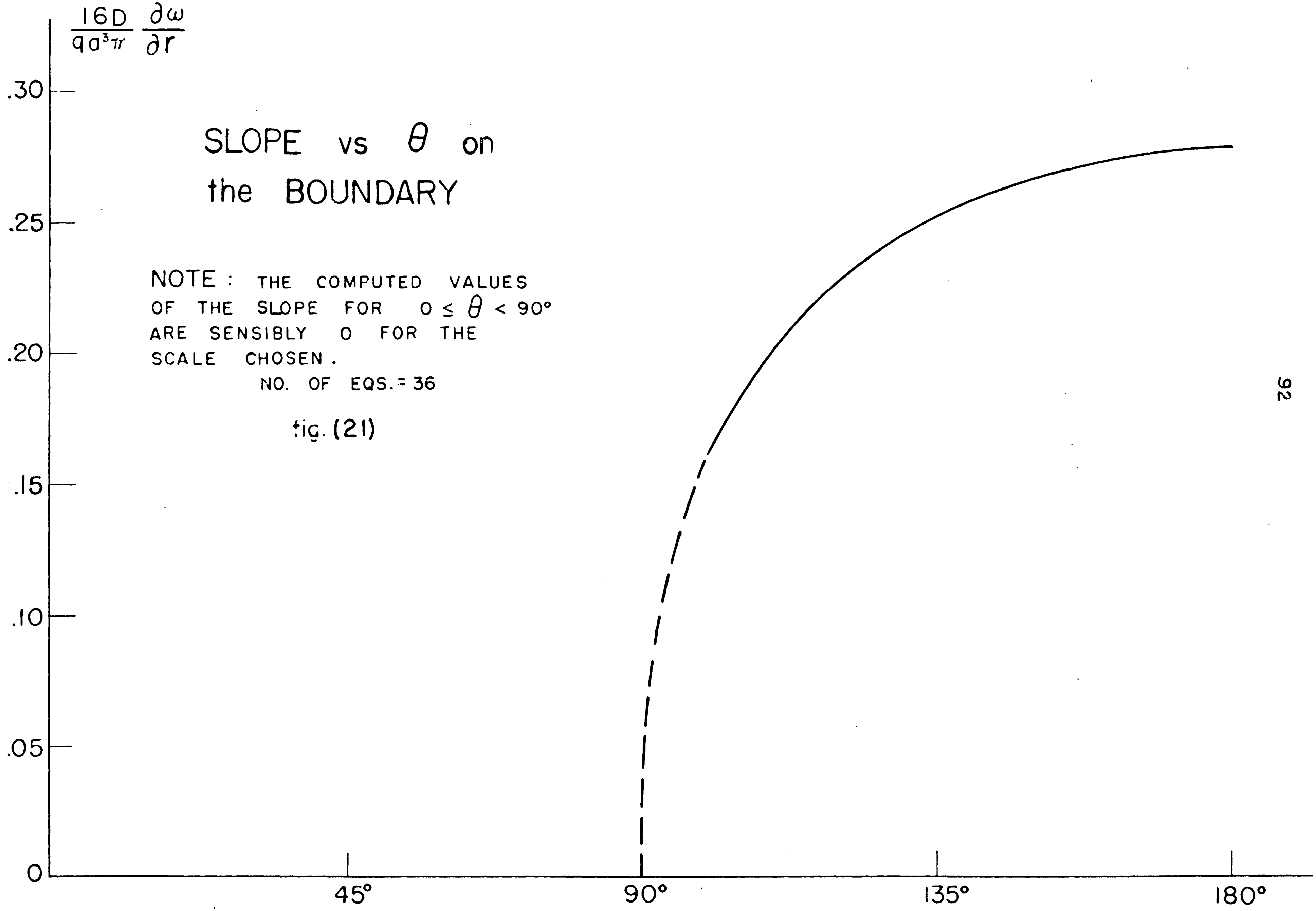
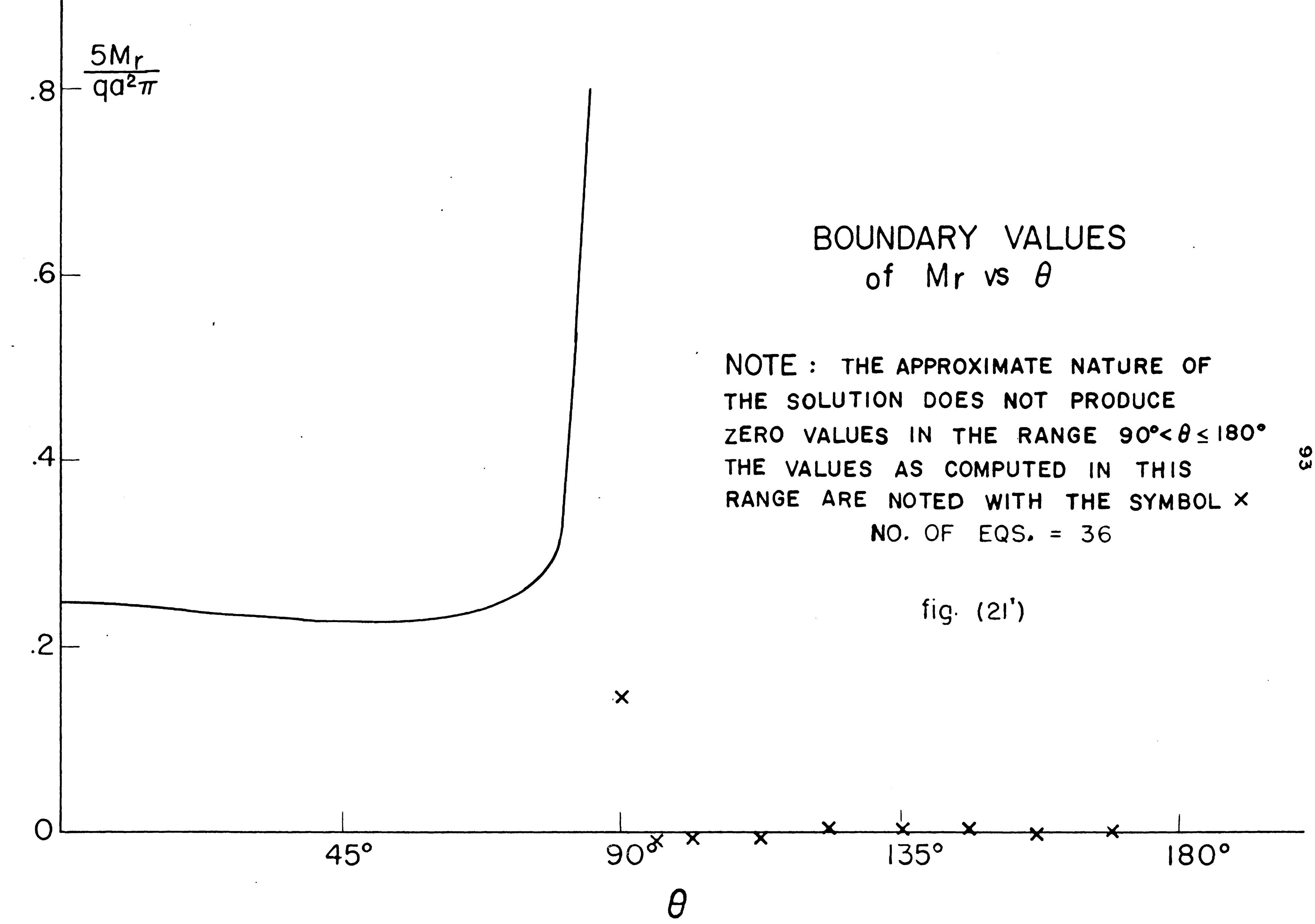
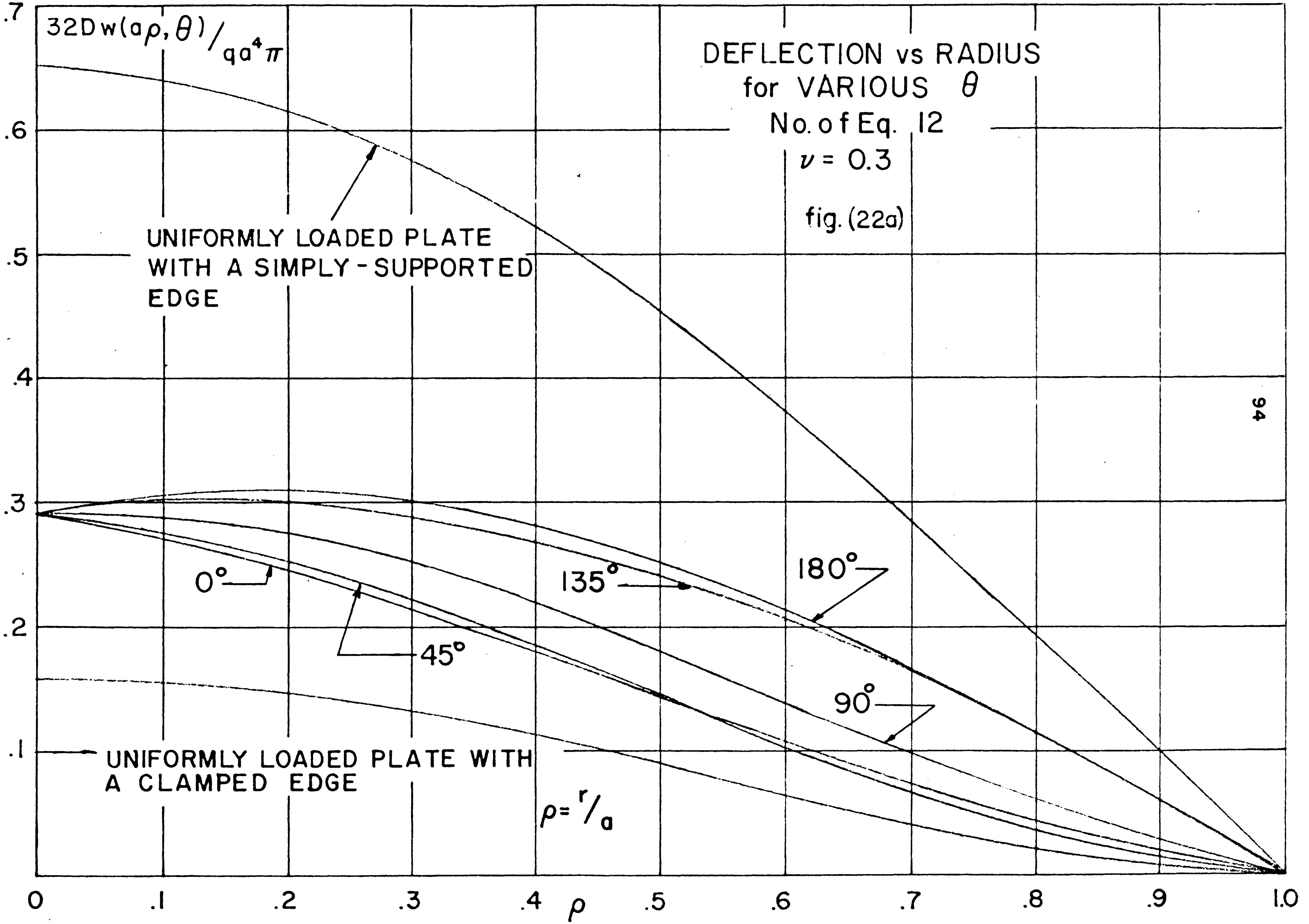

$\frac{16D}{qa^3\pi}\frac{\partial\omega}{\partial r}$
.30
.25
.20
.15
.10
.05
0
45°
90°
135°
180°
SLOPE vs θ on the BOUNDARY
NOTE : THE COMPUTED VALUES OF THE SLOPE FOR $0 \leq \theta < 90°$ ARE SENSIBLY 0 FOR THE SCALE CHOSEN.
NO. OF EQS. = 36

fig. (21)

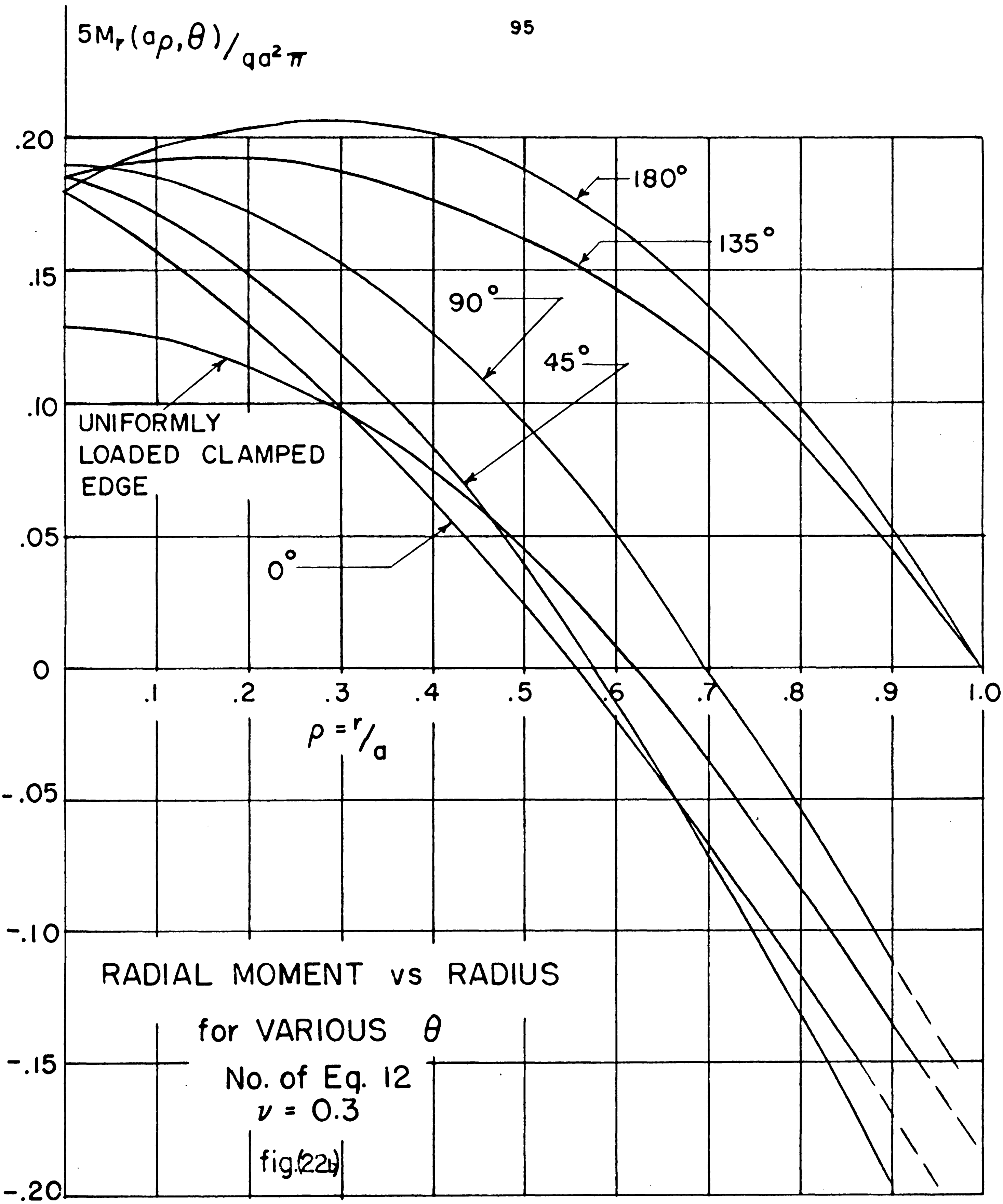


$\frac{5M_r}{qa^2\pi}$
.8
.6
.4
.2
0
45°
90°
135°
180°
θ
BOUNDARY VALUES of M_r vs θ
NOTE : THE APPROXIMATE NATURE OF THE SOLUTION DOES NOT PRODUCE ZERO VALUES IN THE RANGE $90° < \theta \leq 180°$ THE VALUES AS COMPUTED IN THIS RANGE ARE NOTED WITH THE SYMBOL ×
NO. OF EQS. = 36

fig. (21')

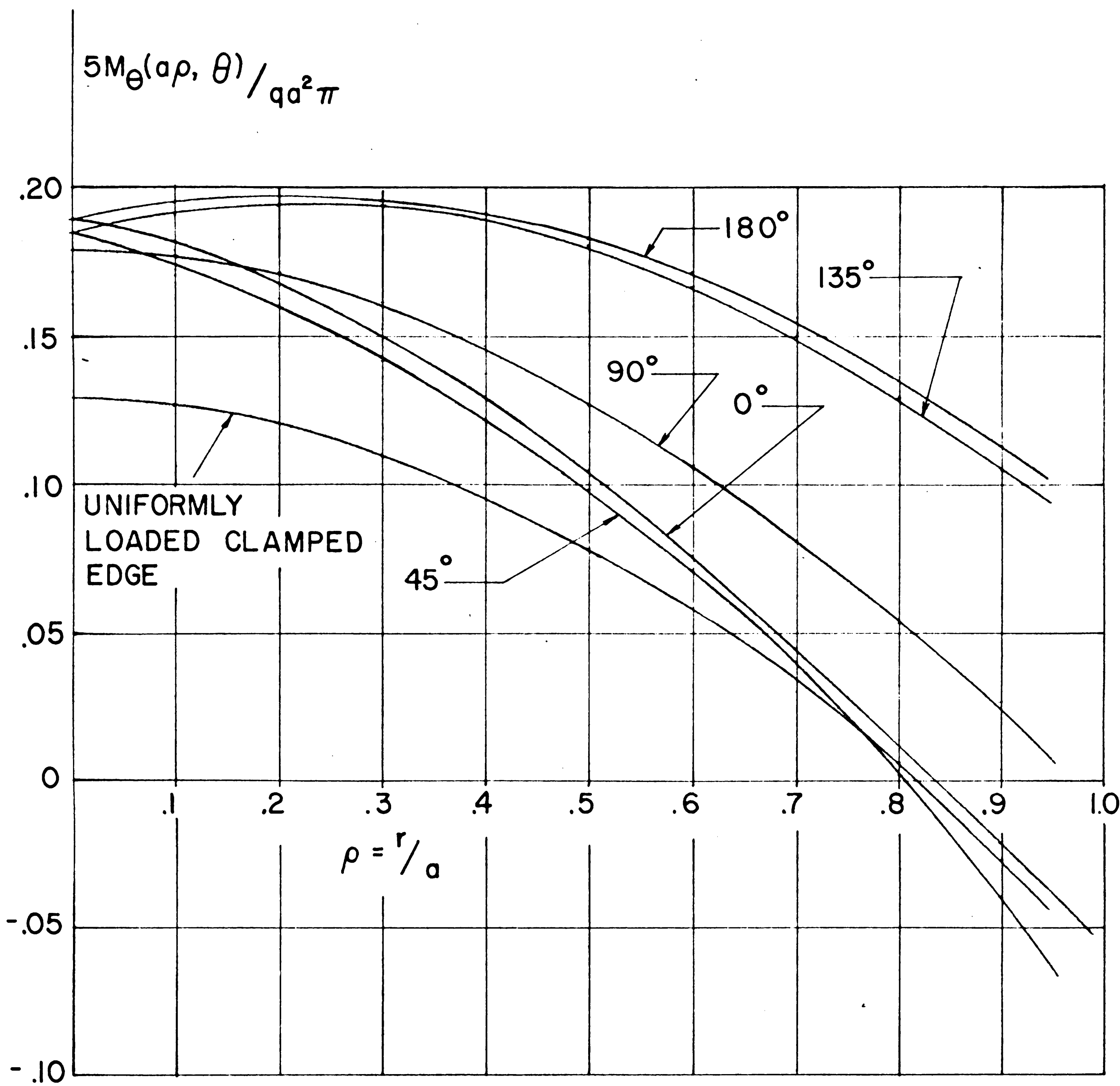


32Dw(aρ, θ)/qa⁴π
DEFLECTION vs RADIUS
for VARIOUS θ
No. of Eq. 12
ν = 0.3
fig. (22a)
UNIFORMLY LOADED PLATE WITH A SIMPLY-SUPPORTED EDGE
UNIFORMLY LOADED PLATE WITH A CLAMPED EDGE
0°
45°
90°
135°
180°
ρ = r/a
ρ
.7
.6
.5
.4
.3
.2
.1
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0



$5M_r(a\rho,\theta)/qa^2\pi$
.20
.15
.10
.05
0
-.05
-.10
-.15
-.20
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
$\rho = r/a$
180°
135°
90°
45°
0°
UNIFORMLY LOADED CLAMPED EDGE
RADIAL MOMENT vs RADIUS
for VARIOUS θ
No. of Eq. 12
ν = 0.3

fig.(22b)



TANGENTIAL MOMENT vs RADIUS

for VARIOUS θ

No. of Eq. 12

$\nu = 0.3$

fig.(22c)

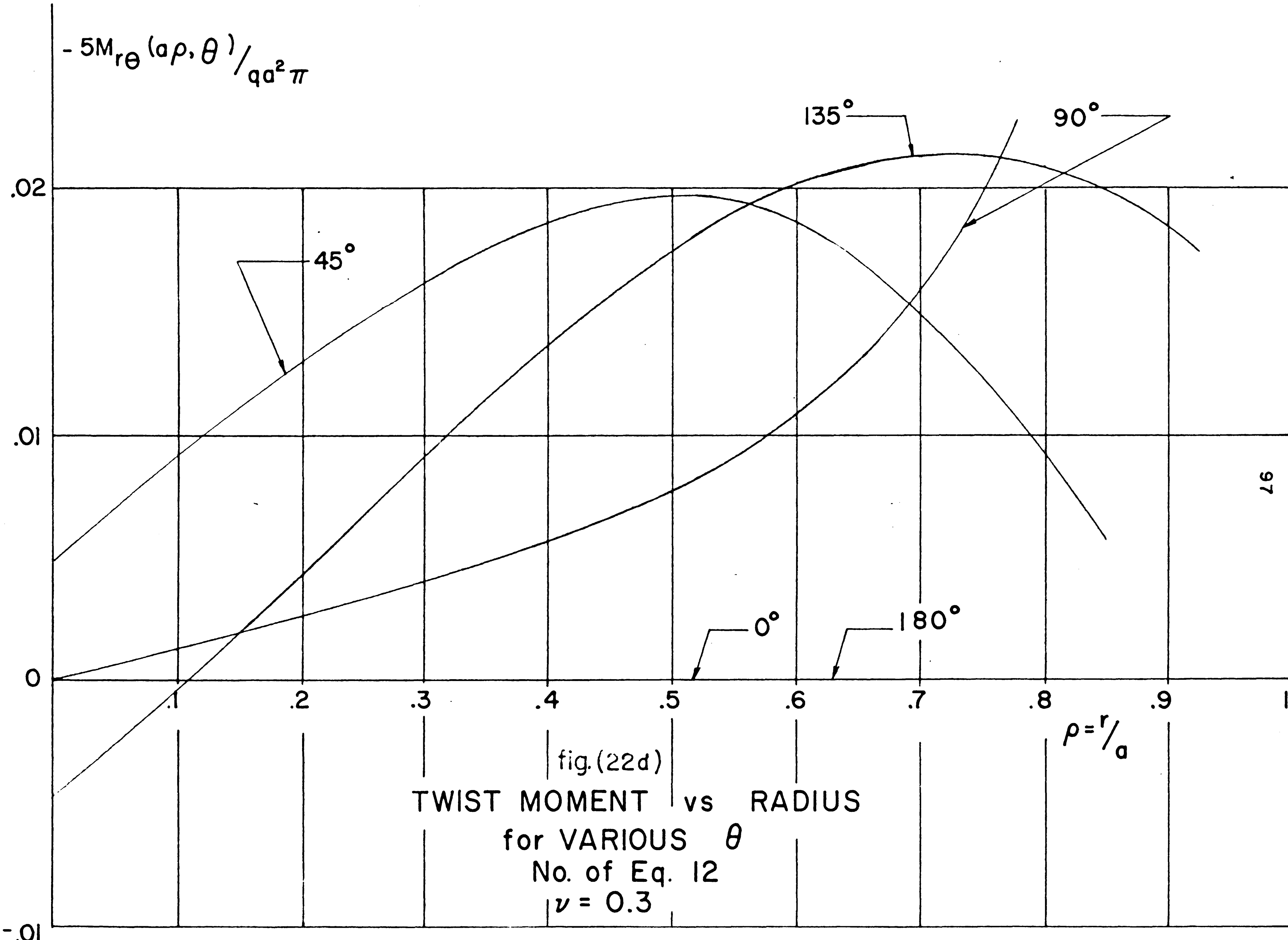
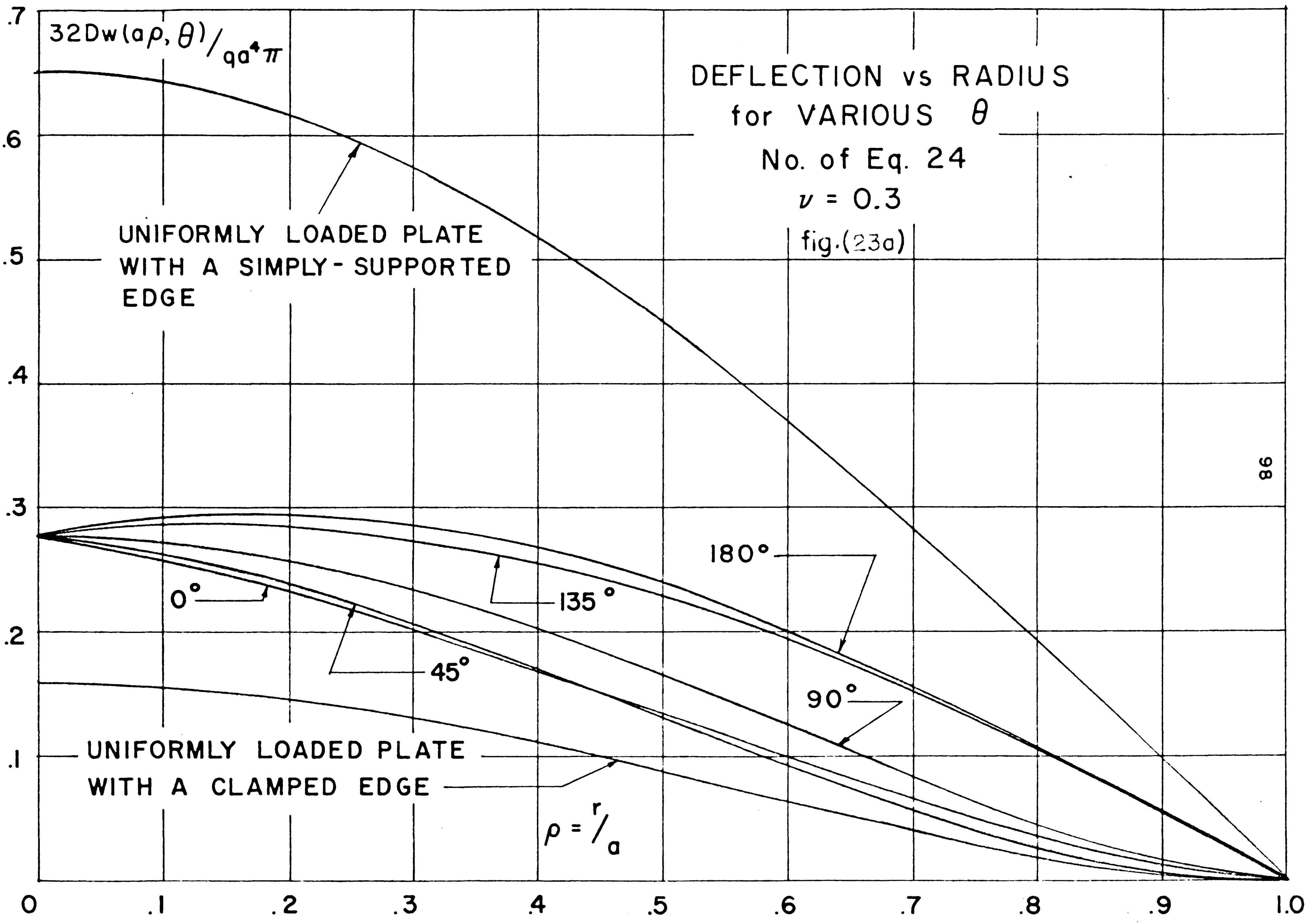
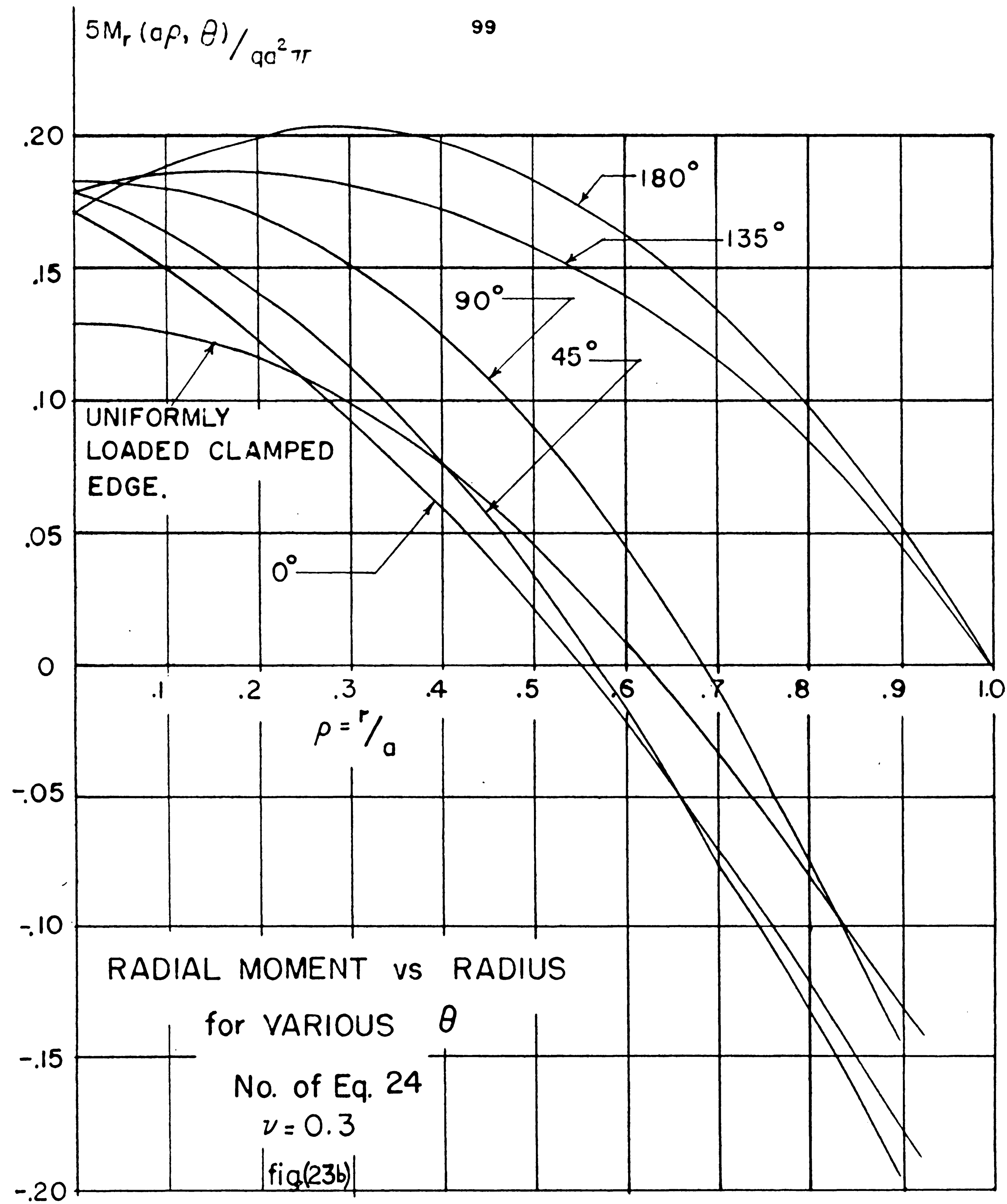


fig. (22d)

TWIST MOMENT vs RADIUS
for VARIOUS θ
No. of Eq. 12
$\nu = 0.3$



$32Dw(a\rho,\theta)/qa^4\pi$
DEFLECTION vs RADIUS
for VARIOUS θ
No. of Eq. 24
ν = 0.3
fig.(23a)
UNIFORMLY LOADED PLATE
WITH A SIMPLY - SUPPORTED
EDGE
180°
135°
0°
45°
90°
UNIFORMLY LOADED PLATE
WITH A CLAMPED EDGE
$\rho = r/a$
.7
.6
.5
.4
.3
.2
.1
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0

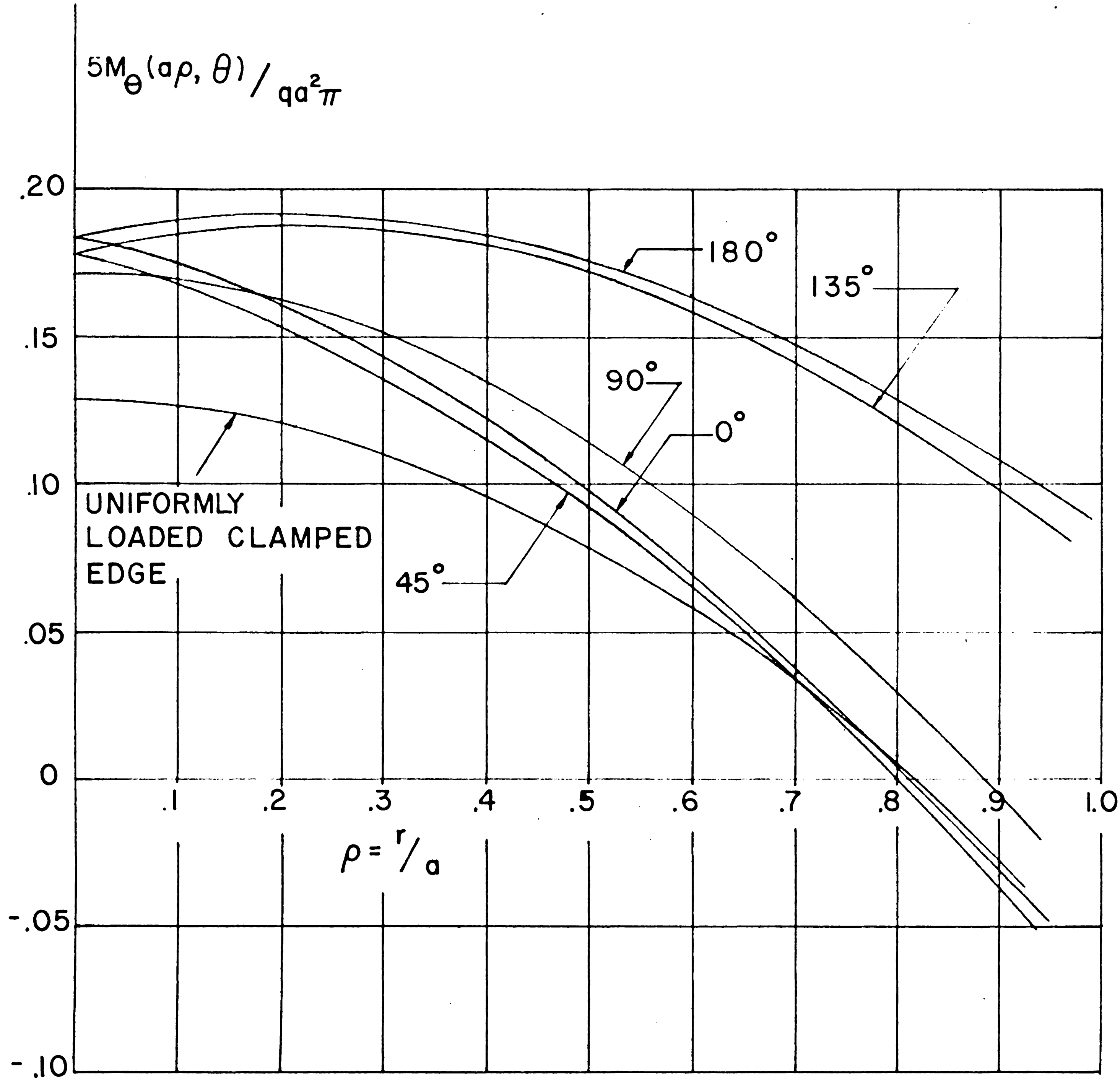


RADIAL MOMENT vs RADIUS for VARIOUS θ

No. of Eq. 24

$\nu = 0.3$

fig(23b)



TANGENTIAL MOMENT vs RADIUS

for VARIOUS θ

No. of Eq. 24

$\nu = 0.3$

fig.(23c)

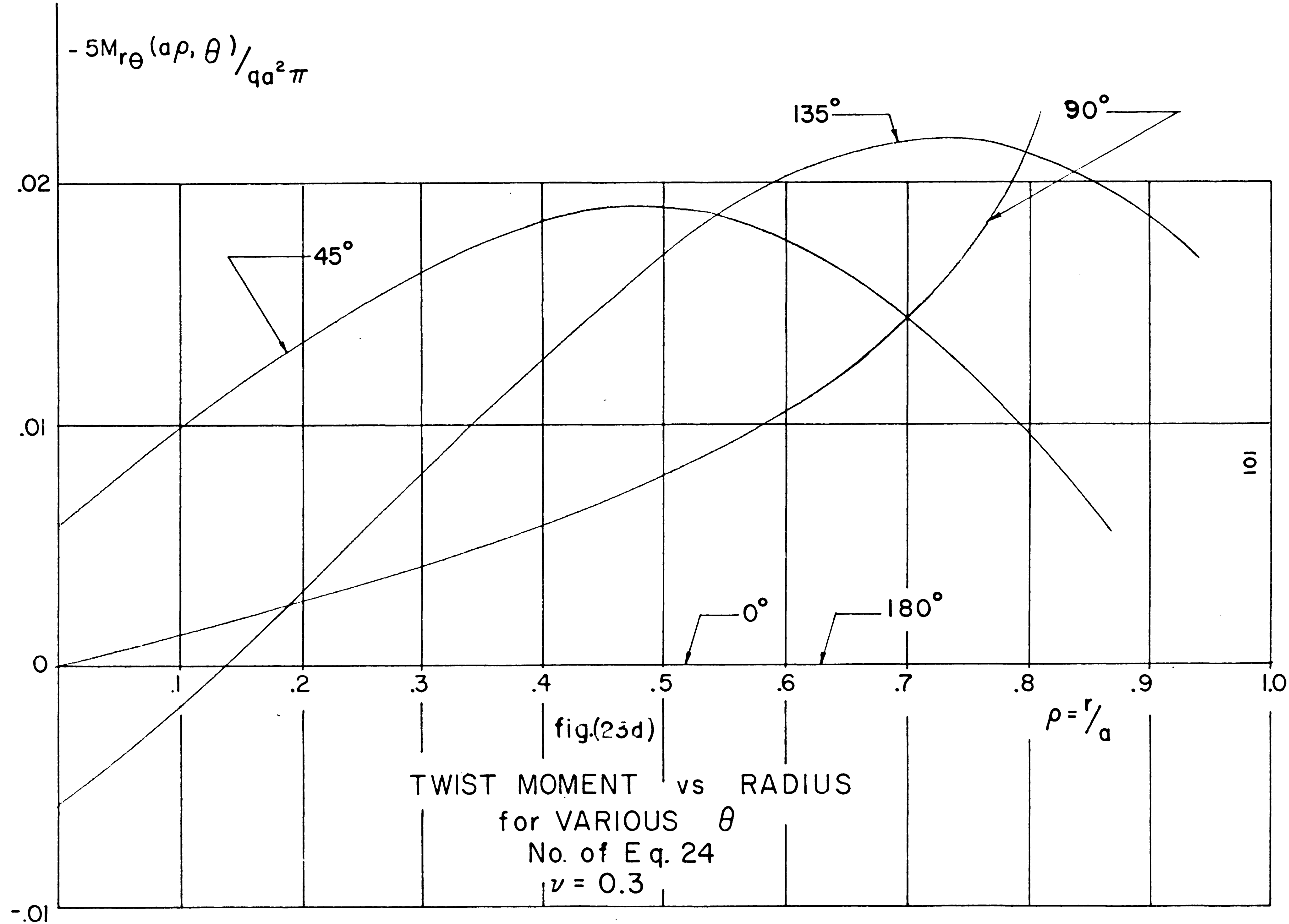
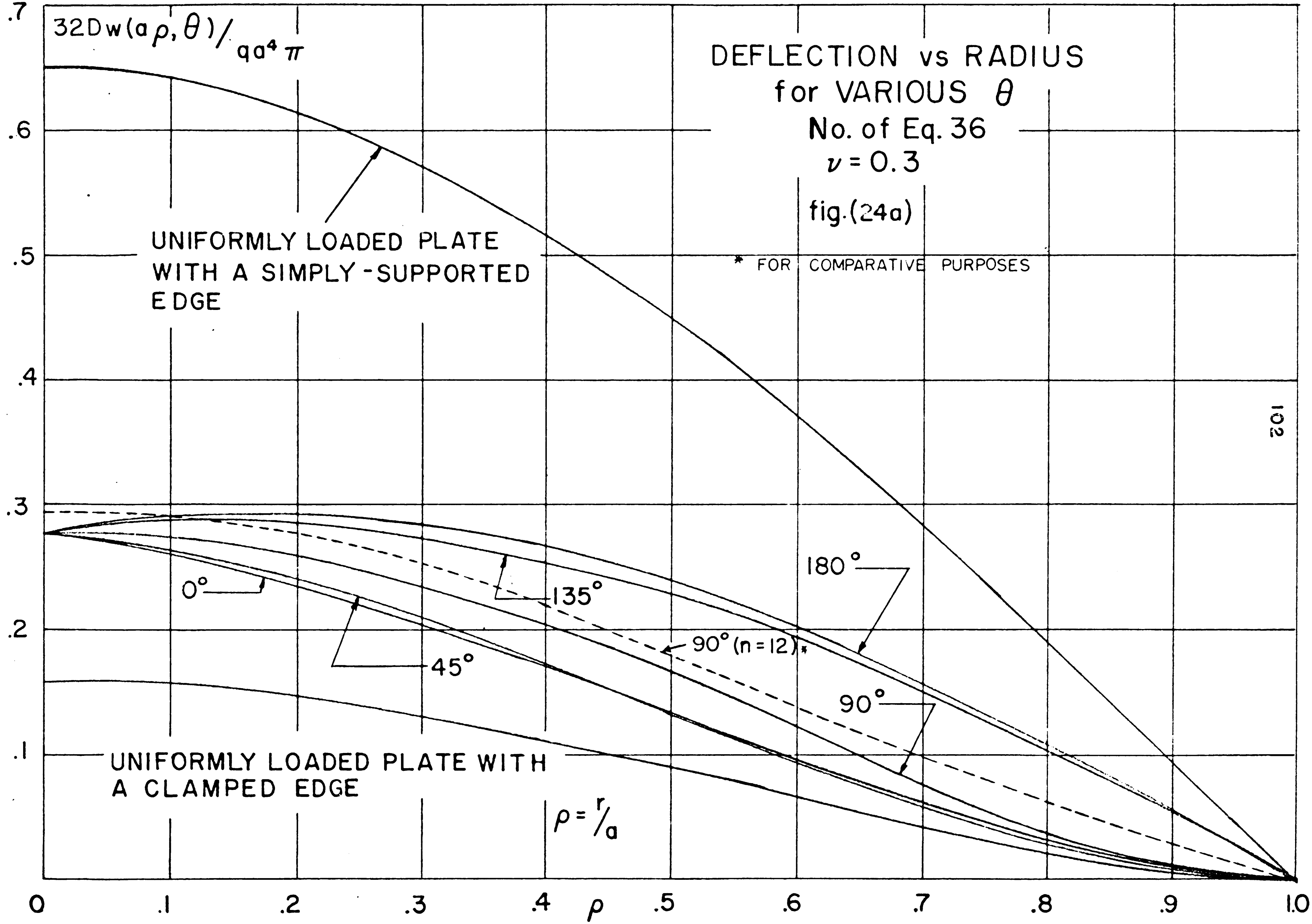
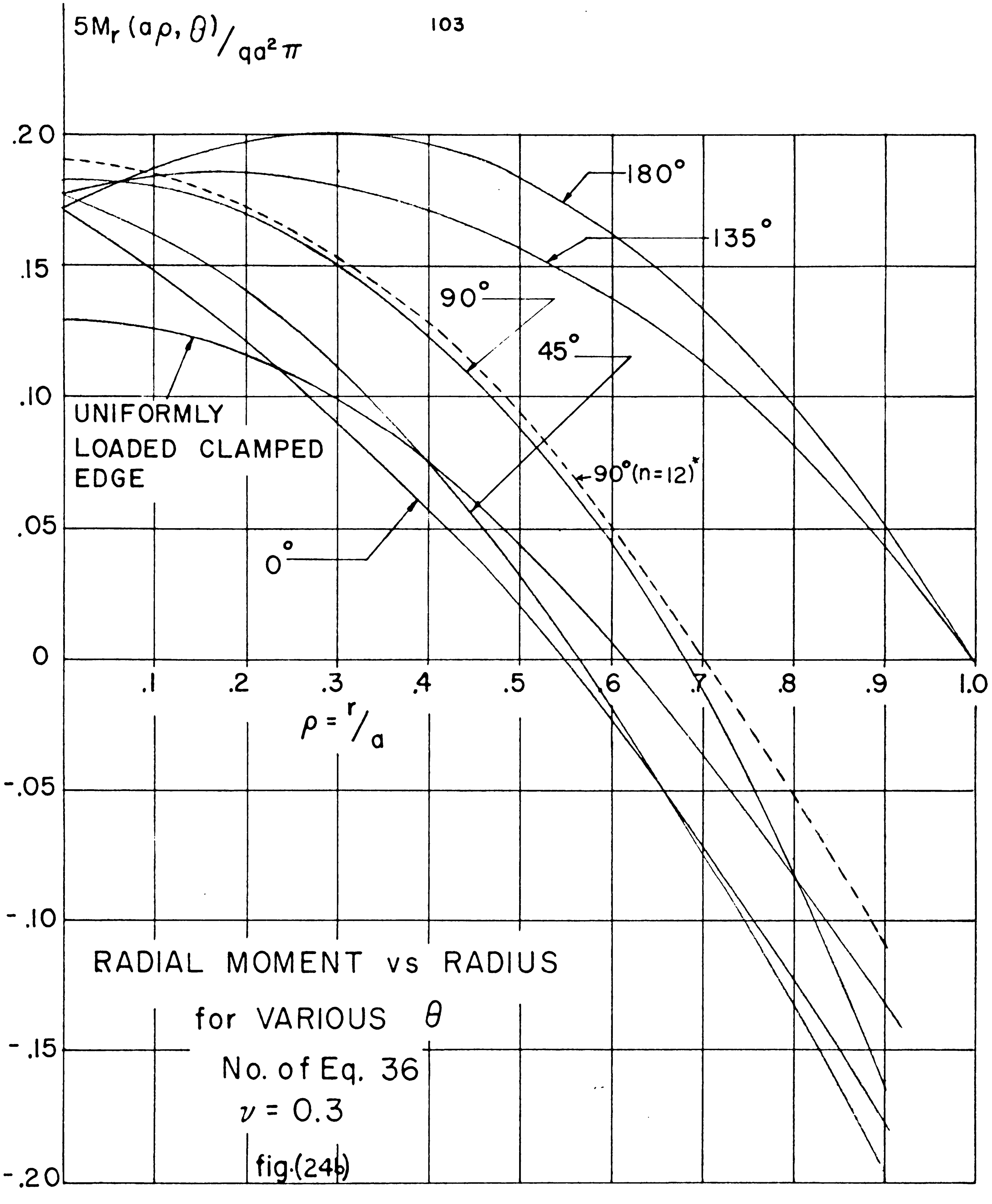


fig.(23d)

TWIST MOMENT vs RADIUS
for VARIOUS θ
No. of Eq. 24
$\nu = 0.3$


32Dw(aρ,θ)/qa⁴π
DEFLECTION vs RADIUS
for VARIOUS θ
No. of Eq. 36
ν = 0.3
fig.(24a)
* FOR COMPARATIVE PURPOSES
UNIFORMLY LOADED PLATE
WITH A SIMPLY-SUPPORTED
EDGE
0°
135°
180°
45°
90° (n=12)*
90°
UNIFORMLY LOADED PLATE WITH
A CLAMPED EDGE
ρ = r/a
ρ
.7
.6
.5
.4
.3
.2
.1
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0



RADIAL MOMENT vs RADIUS

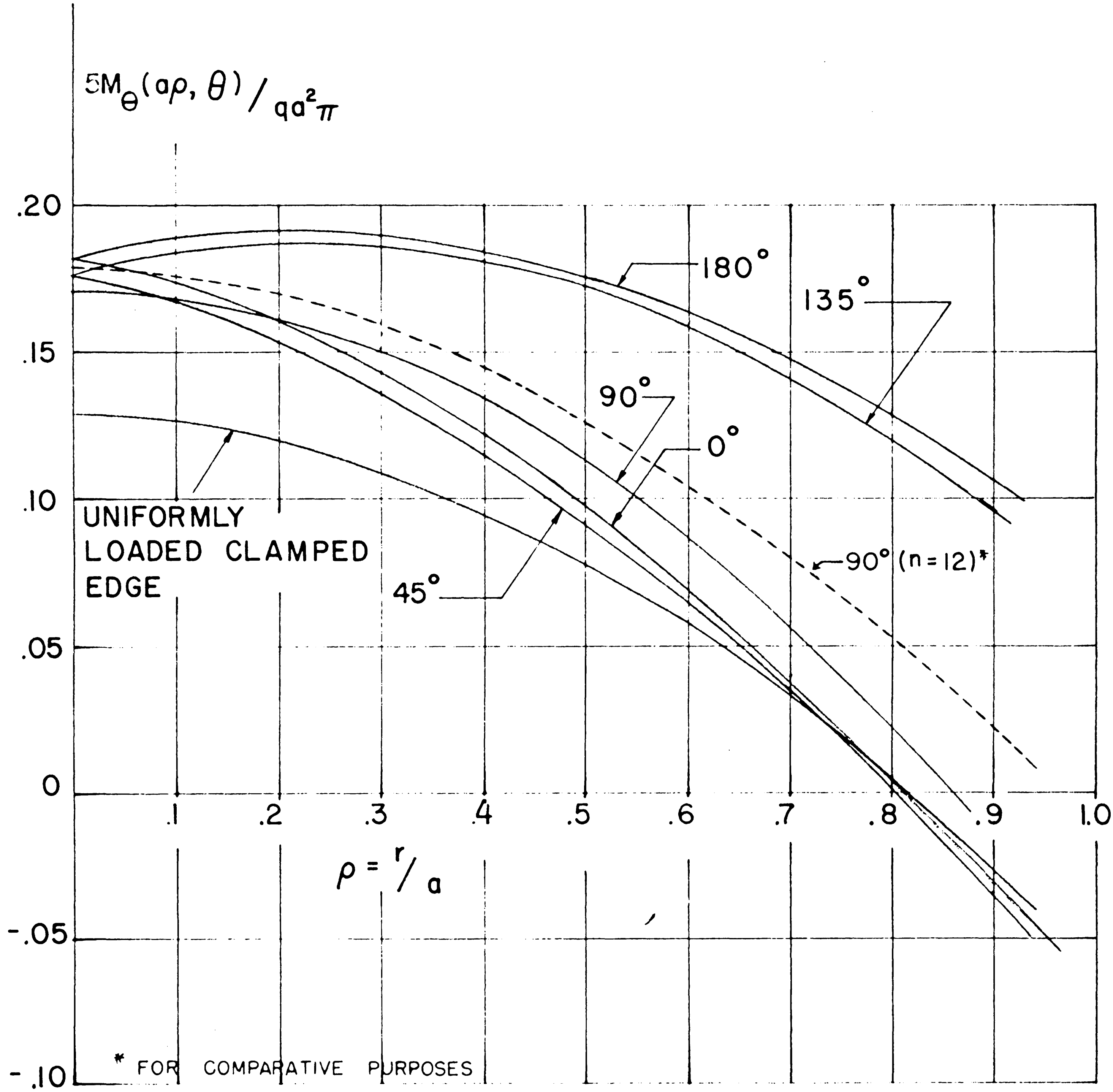
for VARIOUS θ

No. of Eq. 36

$\nu = 0.3$

fig.(24b)

* FOR COMPARATIVE PURPOSES



TANGENTIAL MOMENT vs RADIUS

for VARIOUS θ

No. of Eq. 36

$\nu = 0.3$

fig.(24c)

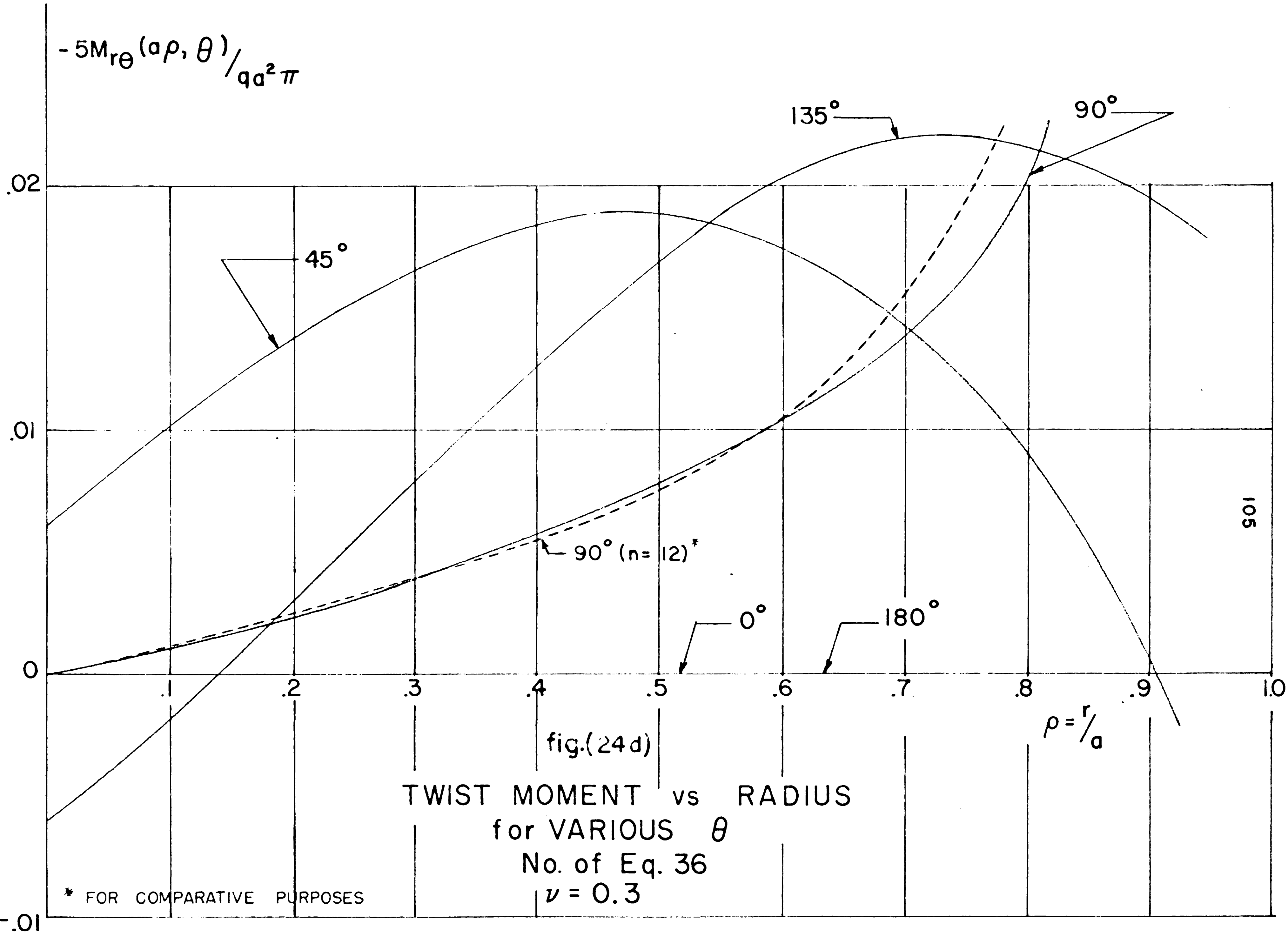


fig.(24d)

TWIST MOMENT vs RADIUS
for VARIOUS θ
No. of Eq. 36
$\nu = 0.3$

For comparative purposes, some of the corresponding results for the uniformly loaded circular plate with all of its edge clamped and all of its edge simply-supported are given in the figures. As is to be expected, the maximum values of the moments occur at the discontinuity in loading along the edge of the plate.

(5.4) <u>Circular Plate with a Reinforced Edge.</u>

In this section, we shall consider a circular plate with a reinforced edge. We shall assume that the resistance to rotation at the edge of the plate comes about from the resistance to twist of this built-up edge. If we assume that this built-up edge resists twisting according to

$$M_r + \frac{GJ_1}{a^3}\,\frac{\partial^3 w}{\partial r\,\partial\theta^2} = 0 \qquad (5.4.1)$$

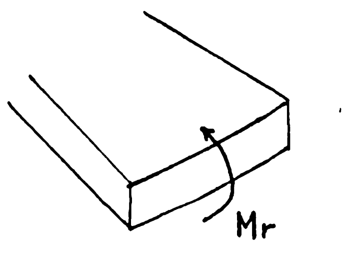


fig. (25)

where $\underline{a}$ is the radius of the circular plate, $\underline{G}$ the modulus in shear of the built-up edge, and GJ_1 is some equivalent torsional stiffness for the edge.

This boundary condition could be realized as a limiting case of a group of rigid links connected by torsional springs not capable of resisting bending.

circular plate. In addition, we shall take the edge deflection of the plate as zero, consequently, we have

$$w(a, \theta) = 0 \tag{5.4.2}$$

If the plate supports a concentrated load $\underline{P}$ at the point z_0, then we may take

$$w = (a^2 - z\bar{z}) \left[\phi(z) + \bar{\phi}(\bar{z}) \right] + G_1 \tag{5.4.3}$$

where G_1 is Green's function for the clamped circular plate. (See (2.3.1).) With this form for $\underline{w}$, ϕ is to be analytic inside and on the circle with center at the origin and radius $\underline{a}$. The singularity of the function G_1 takes care of the concentrated load requirement at z_0. The expression in (5.4.3) automatically satisfies (5.4.2).

Invoking the condition (5.4.1) on (5.4.3), we obtain the following functional equation

$$2(1 + \nu) D \left[\phi(\sigma) + \bar{\phi}(\bar{\sigma}) \right] + 4D \left[\sigma\phi'(\sigma) + \bar{\sigma}\bar{\phi}(\bar{\sigma}) \right]$$

$$- \frac{P}{4\pi} \left[\frac{z_0(\bar{\sigma} - \bar{z}_0)}{a^2 - z_0\bar{\sigma}} + \frac{\bar{z}_0(\sigma - z_0)}{a^2 - \bar{z}_0\sigma} + 1 + \frac{z_0\bar{z}_0}{a^2} \right]$$

$$+ 2GJa \left[\sigma^2\phi''(\sigma) + \bar{\sigma}^2\bar{\phi}''(\bar{\sigma}) + \sigma\phi'(\sigma) + \bar{\sigma}\bar{\phi}'(\bar{\sigma}) \right] = 0 \tag{5.4.4}$$

which holds along the entire boundary of the plate. We have in this expression $\sigma = ae^{i\theta}$. If now we multiply through (5.4.4) by $\frac{1}{2\pi i} \frac{d\sigma}{\sigma - z}$ and integrate around the circular boundary of the plate, there results

$$\frac{2GJ}{a} z^2\phi''(z) + (\frac{2GJ}{a} + 2D) z\phi'(z) + 2(1+\nu) D\phi(z)$$

$$= -2(1+\nu) D\overline{\phi}(0) + \frac{P}{4\pi} \frac{a^2 - z_0\overline{z}_0}{a^2 - \overline{z}_0 z} . \tag{5.4.5}$$

Letting $\alpha = \frac{Da}{GJ}$, we find that

$$\phi(z) = \phi_0 + \frac{P\alpha(a^2 - z_0\overline{z}_0)}{8\pi Da^2} \sum_{n=1}^{\infty} \frac{(\overline{z}_0 z/a^2)^n}{n^2 + 2\alpha n + (1+\nu)} . \tag{5.4.6}$$

By the ratio test, this series expansion is absolutely convergent for $|z| \leq a$, since $|z_0| < a$. The constant ϕ_0 is conditioned by

$$\phi_0 + \overline{\phi}_0 = \frac{P(a^2 - z_0\overline{z}_0)}{8\pi Da^2(1+\nu)} .$$

Thus we may finally write

$$w = \frac{\alpha P(a^2 - z_0\overline{z}_0)(a^2 - z\overline{z})}{8\pi Da^2} \left[\frac{1}{(1+\nu)\alpha} + \sum_{n=1}^{\infty} \frac{(\overline{z}_0 z/a^2)^n + (z_0\overline{z}/a^2)^n}{n^2 + 2\alpha n + (1+\nu)\alpha} \right]$$

$$+ G_1(z, \overline{z}, z_0, \overline{z}_0). \tag{5.4.7}$$

completing the analysis.

We note that if $z_0 = 0$, then the solution reduces to that of a simply-supported circular plate supporting a concentrated load P at its center. This, of course, was to be expected.

5.5) Slit Circular Plate.

This section introduces a method for treating the problem of multi-valued displacements in flat plates. Displacements of this type can be produced by cutting into a plate from the boundary, giving the two faces some relative displacement, and then welding the plate back together. The method introduced here will only be discussed in relation to a circular plate; however, it is easily extended to a plate of any shape.

Let us then imagine that we cut a circular plate from its periphery to its center along the line $\theta = -\pi$. Along this line, we shall assign an arbitrary relative displacement between the faces of the cut. Now across this cut, the quantities M_θ, $M_{r\theta}$, Q_θ are continuous. M_r may be, and in general, is discontinuous across the cut. Specifically we have

$$w = \bar{z}\,\phi(z) + z\overline{\phi}(\bar{z}) + \psi(z) + \overline{\psi}(\bar{z}) \qquad (5.5.1)$$

We choose

$$\phi(z) = A(z) \log z \qquad \psi(z) = B(z) \log z \qquad (5.5.2)$$

where $A(z)$ and $B(z)$ are single-valued and analytic inside the circle of radius $\underline{a}$ with center at the origin. For definiteness, we presume that the argument of $\log z$ varies from $-\pi$ to $+\pi$.

If M_θ, $M_{r\theta}$, Q_θ are to be continuous across the line $\theta = -\pi$, we must have that

$$M_\theta(r, \pi) - M_\theta(r, -\pi) = 0$$

$$M_{r\theta}(r, \pi) - M_{r\theta}(r, -\pi) = 0 \qquad (5.5.3)$$

$$Q_\theta(r, \pi) - Q_\theta(r, -\pi) = 0$$

Furthermore, if $16\pi i\, f(x)$ is the relative displacement across the cut, we have

$$w(r, +\pi) - w(r, -\pi) = 16\pi i\, f(x) \qquad (5.5.4)$$

$$x = re^{i\pi} = re^{-i\pi}$$

Placing (5.5.2) into (5.5.3) and (5.5.4), we obtain the following system of differential equations for the determination of $A(z)$ and $B(z)$

$$2(1 + \nu)(A' - \bar{A}') - x(1 - \nu)(A'' - \bar{A}'') - (1 - \nu)(B'' - \bar{B}'') = 0$$

$$x(A'' + \bar{A}'') + B'' + \bar{B}'' = 0 \qquad (5.5.5)$$

$$A'' + \bar{A}'' = 0$$

A solution of this system of equations is

$$A = p_0 + iq_0 + p_1 z + (1 - \nu) f'(z) \qquad (5.5.6$$

$$B = m_0 + (m_1 - iq_0)z - (1 - \nu) zf'(z) + 4f(z)$$

The quantities p_0, q_0, m_0, m_1 are constants.

As a specific application let us choose

$$f(z) = -\frac{\Delta i}{16\pi} \frac{z^4}{a^4}$$

and all of the constants of integration as zero. This choice eliminates the possibility of any dilemma concerning infinite deflections, moments, etc. at the origin. Substituting this choice of f(z) in (5. 5. 6), we obtain

$$\phi(z) = -\frac{\Delta i z^3(1-\nu)}{4\pi a^4}\log z \qquad \psi(z) = -\frac{\Delta i z^4 \nu}{4\pi a^4}\log z$$

In real form the deflection becomes

$$w = \frac{\Delta}{2\pi a^4} r^4 \log r \left[(1-\nu)\sin 2\theta + \nu \sin 4\theta \right]$$

$$+ \frac{\Delta}{2\pi}\frac{r^4}{a^4}\left[(1-\nu)\,\theta\cos 2\theta + \nu\,\theta\cos 4\theta \right].$$

With the deflection so determined, we still have edge moments and resultant shears. These may be removed by the addition of a single-valued deflection function so chosen that when it is added to the multi-valued deflection, the moment and shear at the boundary of the circle are zero. Since the stresses were assumed to be continuous across the cut, and the jump in displacement assigned, the change in slope across the cut is automatically determined.

As an alternate problem one could consider assigning values to the change in deflection and slope across the cut. Evidently the stresses in general no longer could be continuous. Indeed, all that could be specified now is that the bending moment and resultant vertical shear are continuous across the cut.

CHAPTER VI

THE RECTANGULAR PLATE

Of all shapes, the rectangular plate is probably the most frequent one occurring in engineering structures. Floor slabs in buildings, walls of buildings, and table tops for precision machinery are a few examples of the usage of rectangular plates in construction. The variety of edge conditions is as great as the variation in edge conditions from edge to edge.

The need for precise solutions for problems associated with this region is great. The number of precise solutions, on the other hand, which exist in the literature, is small. Even the problem of the completely clamped edge, to date, has not been treated by any methods except approximation procedures. With this in mind, one could not expect to attack any mixed problems associated with the rectangle, by any other means than the usual manner of solving plate problems in an approximate fashion. In this section, we will treat two problems. The first is connected with the determination of the derivatives of Green's function for the simply-supported rectangle plate, and the second is a numerical analysis of a rectangular plate with a partially clamped edge.

6.1) Green's Function for the Simply-Supported Plate.*

An application of the method presented in sec. (1.4) requires a knowledge of the Green's function for the region in question. The following development, while not presenting Green's function in a closed form for the simply-supported rectangular plate, represents the results of an attempt in this direction.

The deflection of a simply-supported plate subjected to a concentrated load at some interior point is well-known in the form of a bilinear series. In a recent paper, B. D. Aggarwala has given closed expressions for the vertical shears Q_x and Q_y in terms of Weierstrassian elliptic functions $\wp(z)$ and their derivatives. The purpose of this section is to show how his results can be extended directly to give closed expressions for the bending moments M_x, M_y and for the resultant vertical shears V_x, V_y at the boundary of the plate.

We begin by considering a rectangular plate the mid surface of which is oriented as shown in the Argand diagram of fig. (26). If the plate is subjected to a concentrated load P at the

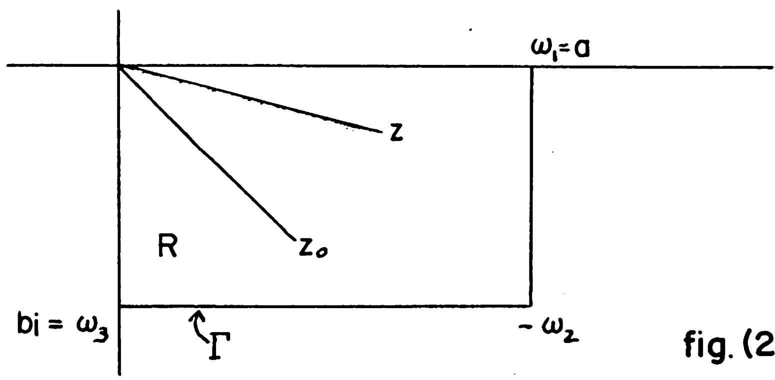


fig. (26.)

point z_0 and is simply-supported on the boundary Γ, then the transverse deflection, $w(x, y)$, is determined by the following conditions.

$$\nabla^4 w = 0 \text{ in } R; \quad w = 0, \quad \nabla^2 w = 0 \text{ on } \Gamma \quad , \tag{6.1.1}$$

and, additionally,

$$w - \frac{P}{16\pi D}(z - z_0)(\overline{z} - \overline{z}_0)\log(z - z_0)(\overline{z} - \overline{z}_0) = B_1(z, z_0, \overline{z}, \overline{z}_0), \tag{6.1.2}$$

where B_1 is analytic in R, $\overline{z}$ denoting the conjugate of z.

The physical interpretation of the condition expressed by Eq. (2) is nothing more than the fact that the singular part of w in R is such that the vertical shear caused by this deflection, when summed over the lateral surface of a cylinder of radius ϵ drawn about z_0 is equivalent to a load P. With w so determined, the bending moments, vertical shears, and resultant vertical shears are computed by the usual expressions

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \sigma\frac{\partial^2 w}{\partial y^2}\right) \qquad M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \sigma\frac{\partial^2 w}{\partial x^2}\right) \tag{a}$$

$$Q_x = -D\frac{\partial}{\partial x}\nabla^2 w \qquad Q_y = D\frac{\partial}{\partial y}\nabla^2 w \tag{b}$$

(6.1.3)

$$V_x = -D\frac{\partial}{\partial x}\left[\frac{\partial^2 w}{\partial x^2} + (2 - \sigma)\frac{\partial^2 w}{\partial y^2}\right] \tag{c}$$

$$V_y = -D\frac{\partial}{\partial y}\left[\frac{\partial^2 w}{\partial y^2} + (2 - \sigma)\frac{\partial^2 w}{\partial y^2}\right]. \tag{d}$$

Now in Eq. (6.1.1), we let $\nabla^2 w = M$ and find that

$$\nabla^2 M = 0 \text{ in } R; \quad M = 0 \text{ on } \Gamma$$

where M is subject to the condition that

$$M - \frac{P}{4\pi D}\log(z - z_0)(\overline{z} - \overline{z}_0)$$

is analytic in R. Now an elementary application of the Shwartz-Christoffel mapping theorem along with the method of images yields

$$\nabla^2 w = \frac{P}{4\pi D} \log \left\{ \frac{[\wp(z) - \wp(z_0)] \; [\wp(\overline{z}) - \wp(\overline{z}_0)]}{[\wp(\overline{z}) - \wp(z_0)] \; [\wp(z) - \wp(\overline{z}_0)]} \right\}. \tag{6.1.4}$$

The function $\wp(u)$ is the Weierstrassian elliptic function with periods $2\omega_1$, $2\omega_3$. (This result, of course, is well-known.)

From Eq. (4), it follows that w can be written in the form

$$w(x, y) = \frac{P}{16\pi D} (z - z_0)(\overline{z} - \overline{z}_0) \log \left\{ \frac{[\wp(z) - \wp(z_0)] [\wp(\overline{z}) - \wp(\overline{z}_0)]}{[\wp(\overline{z}) - \wp(z_0)] [\wp(z) - \wp(\overline{z}_0)]} \right\}$$

$$+ \frac{P}{16\pi D} B(z, z_0, \overline{z}, \overline{z}_0), \tag{6.1.5}$$

where the function B is analytic inside the rectangle, R, vanishes on the boundary Γ, and satisfies the following Poisson equation in R

$$\frac{\partial^2 B}{\partial z \, \partial \overline{z}} = -(\overline{z} - \overline{z}_0) \left[\frac{\wp'(\overline{z})}{\wp(\overline{z}) - \wp(\overline{z}_0)} - \frac{\wp'(\overline{z})}{\wp(\overline{z}) - \wp(z_0)} \right]$$

$$+ \text{ complex conjugate.} \tag{6.1.6}$$

Now Eq. (6.1.4) is one linear relation between the derivatives w_{xx} and w_{yy}. One more relation between these two quantities, independent of Eq. (4), will permit their explicit presentation. To this end, we note that in view of the definitions of Q_x and Q_y along with Eq. (6.1.4) both Q_x and Q_y are harmonic functions. Consequently, there results from Eq. (6.1.3b) the following relation

$$\nabla^2(\square^2 w) = -\frac{1}{D}\left(\frac{\partial Q_z}{\partial x} - \frac{\partial Q_y}{\partial y}\right), \quad \square^2 = \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}. \quad (6.1.7)$$

Furthermore, since

$$\nabla^2\left(\frac{x}{2} Q_x\right) = \frac{\partial Q_x}{\partial x}, \qquad \nabla^2\left(\frac{y}{2} Q_y\right) = \frac{\partial Q_y}{\partial y}, \quad (6.1.8)$$

Eq. (6.1.7) may be rewritten as

$$\nabla^2 \left\{ \square^2 w + \frac{1}{2D} (xQ_x - yQ_y) \right\} = 0. \quad (6.1.9)$$

It should be remarked that this is the key relationship for the determination of the desired quantities.

From Eq. (6.1.9), we have

$${}^2w + \frac{1}{2D} (xQ_x - yQ_y) = H_1(x, y), \quad (6.1.10)$$

where H_1 is a harmonic function. To determine H_1, uniquely, it is necessary to know its boundary values, and, additionally, the nature of its singularity in R. For a convenience in a future integration, we shall determine, instead of H_1, the function

$$I_1(x, y) = H_1(x, y) + \frac{\omega_3}{2Di} Q_y - \frac{\omega_1}{2D} Q_x. \quad (6.1.11)$$

Since the plate is simply-supported at the boundary, we have

$$w = 0, \qquad \nabla^2 w = 0 \text{ on } \Gamma \qquad \text{implying} \qquad \square^2 w = 0 \text{ on } \Gamma. \quad (6.1.12)$$

The boundary values of I_1 are then determined from Eq. (6.1.10) and Eq. (6.1.11). An investigation of Eq. (6.1.5) reveals the nature of the singular part of I_1.

From the addition theorem for the $\wp$ functions, it follows that $\wp(z) = \wp(\bar{z})$ anywhere on Γ. With this observation, the following three functions

$$R_1 = \frac{P}{8\pi D}\left[\frac{\bar{z}_0\ \wp'(z_0)}{\wp(z) - \wp(z_0)} - \frac{\bar{z}_0\wp'(z_0)}{\wp(\bar{z}) - \wp(z_0)}\right] + \text{complex conjugate}$$

$$R_2 = \frac{P}{4\pi D}\left[\frac{\wp'(z_0)}{\wp(z) - \wp(z_0)} - \frac{\wp'(z_0)}{\wp(\bar{z}) - \wp(z_0)}\right] + \text{complex conjugate} \qquad (6.1.13)$$

$$R_3 = R_2 i$$

(considered as functions of z with z_0 as a parameter) vanish on Γ. Additionally, each of these functions is a harmonic function with such type singularity in R that

$$J_1(x, y) = H_1(x, y) + R_1 + \frac{\omega_3}{2Di}(Q_y + R_3) - \frac{\omega_1}{2D}(Q_x + R_2) \qquad (6.1.14)$$

is analytic in R. Evidently J_1 assumes the same boundary values on Γ as does I_1.

Since Green's function for LaPlace's equation for a rectangular region is known, we have

$$J_1 = \int_{\Gamma} I_1 \frac{\partial G}{\partial n}\, ds. \qquad (6.1.15)$$

If now we perform the indicated integration in Eq. (6.1.16), then after some laborious computations there results

$$F \equiv \frac{1}{2} H_1(x, y) - \frac{1}{4D}(xQ_x - yQ_y)$$

$$= \frac{-i}{8\pi D}\left[4\omega_1\omega_3\zeta(z) - 2(\eta_1\omega_3 + \omega_1\eta_3)z - \pi i\bar{z}\right]\left[Q_z(z, z_0) -\right.$$

$$\left.- iQ_y(z, z_0)\right] - \frac{i}{8\pi D}\left[4\omega_1\omega_3\zeta(z_0) - 2(\eta_1\omega_3 + \omega_1\eta_3)z_0 - \pi i\bar{z}_0\right]$$

$$\left[Q_x(z_0, s) - iQ_y(z_0, z)\right] \qquad + \text{complex conjugate}, \qquad (6.1.16)$$

where by definition

$$\zeta(\omega_1) = \eta_1, \qquad \zeta(\omega_3) = \eta_3 .$$

(It is to be noted that $\zeta(z)$ is the Weierstrassian function associated with $\wp(z)$.)

This completes the analysis, for from Eqs. (6.1.4) and (6.1.10) it follows that

$$w_{xx} = \frac{1}{2}\nabla^2 w + F,$$

$$w_{yy} = \frac{1}{2}\nabla^2 w - F,$$

and, consequently, all of the expressions in Eqs. (6.1.3) can now be presented in closed form.

If we introduce Jacobi's theta functions, we can write

$$F = \frac{Pi}{8\pi^2 D}\left[\frac{\omega_3\theta_1'(\nu)}{\omega_1\theta_1(\nu)} + \pi i(\nu - \bar{\nu})\right]$$

$$\frac{\theta_1(\nu)\theta_2(\nu)\theta_3(\nu)\theta_4(\nu)\theta_1(\nu_0 + \bar{\nu}_0)\theta_1(\nu_0 - \bar{\nu}_0)}{\theta_1(\nu + \nu_0)\theta_1(\nu - \nu_0)\theta_1(\nu + \bar{\nu}_0)\theta_1(\nu - \bar{\nu}_0)}$$

$$+ \frac{Pi}{8\pi^2 D}\left[\frac{\omega_3\theta'_1(\nu_0)}{\omega_1\theta_1(\nu_0)} + \pi i(\nu_0 - \bar{\nu}_0)\right]$$

$$\frac{\theta_1(\nu_0)\theta_2(\nu_0)\theta_3(\nu_0)\theta_4(\nu_0)\theta_1(\nu+\bar{\nu})\theta_1(\nu-\bar{\nu})}{\theta_1(\nu_0+\nu)\theta_1(\nu_0-\nu)\theta_1(\nu_0+\bar{\nu})\theta_1(\nu_0-\nu)} + \text{complex conjugate},$$

where

$$\nu = \frac{z}{2\omega_1}, \qquad \nu_0 = \frac{z_0}{2\omega_1}.$$

By some arguments involving the properties of the real and complex parts of analytic functions z of a complex variable z, it is possible to show that

$$w_{xy} + \frac{1}{4D}(yQ_x + xQ_y) = -\frac{1}{2} K_1(z, z_0, \bar{z}, \bar{z}_0) + K(z_0, \bar{z}_0),$$

where K_1 is the harmonic conjugate of H_1. However, there seems to be no immediate way of evaluating the function K.

6.2) Partially Clamped Edge, Simply-Supported Edge.

We shall consider a rectangle oriented as shown in fig. (27) and subjected to a uniform load q over the surface of the plate. The shaded portion of the edge is clamped and the remaining portion of the edge is simply-supported.

To solve this problem, we shall use essentially the same procedure that S. Timoshenko [1] has introduced for the solution of the completely clamped plate.

The solution for a simply-supported rectangular plate subjected to a uniform load is given by

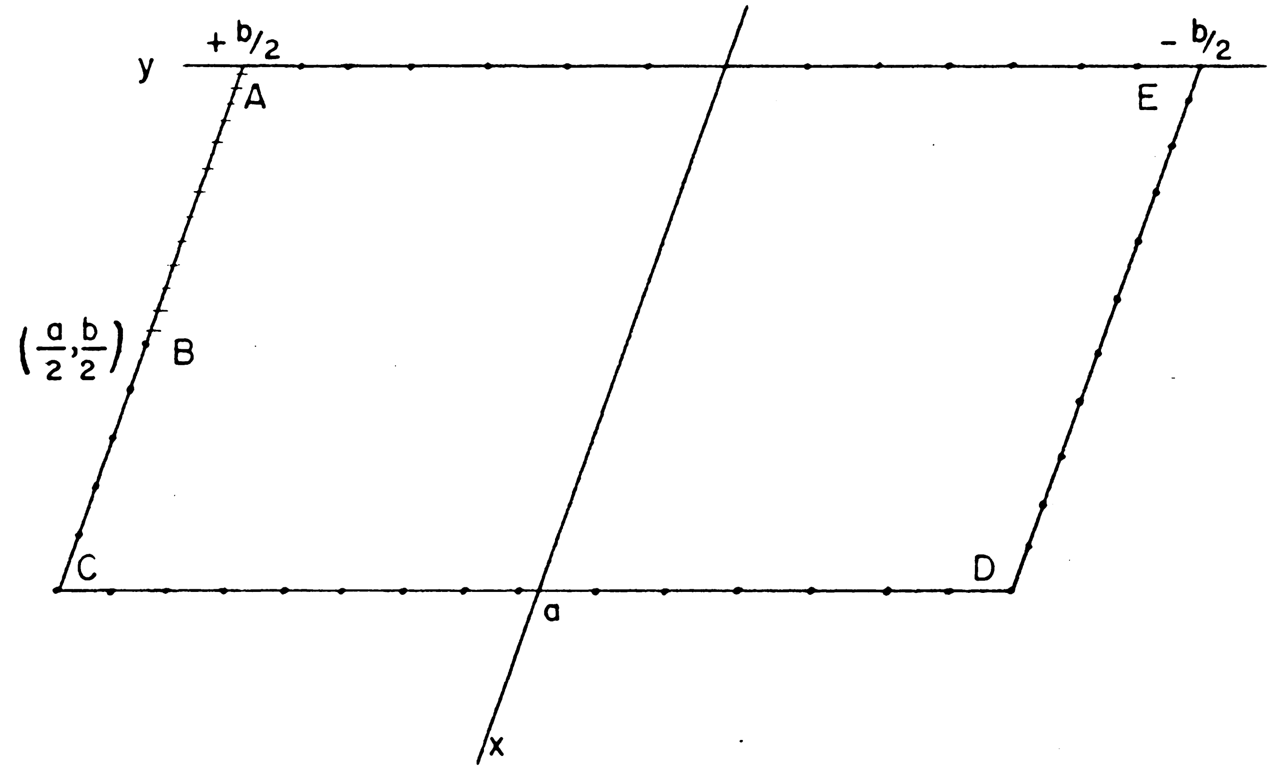


fig. (27.)

$$w_0(x,y) = \frac{qb^5}{8aD}\sum_{1,2,3}^{\infty} \frac{\sin^2\frac{m\pi}{2}}{\alpha_m^5}\left[1 + \frac{y\alpha_m}{b\cosh\alpha_m}\sinh\frac{2\alpha_m y}{b}\right]\sin\frac{m\pi x}{a}$$

$$-\frac{qb^5}{8aD}\sum_{1,2,3}^{\infty}\frac{\sin^2\frac{m\pi}{2}}{\alpha_m^5}\left[\frac{\alpha_m\tanh\alpha_m + 2}{2\cosh\alpha_m}\right]\cosh\frac{2\alpha_m y}{b}\sin\frac{m\pi x}{a}$$

(6.2.1.)

where $\alpha_m = \frac{m\pi b}{a}$. If we consider

$$w_1(x,y) = \sum_{1,2,3}\left[A_m\sinh\frac{2\alpha_m y}{b} + B_m\cosh\frac{2\alpha_m y}{b}\right]\sin\frac{m\pi x}{a}$$

$$+\sum_{1,2,3}\left[C_m\alpha_m\frac{2y}{b}\sinh\frac{2\alpha_m y}{b} + D_m\alpha_m\frac{2y}{b}\cosh\frac{2\alpha_m y}{b}\right]\sin\frac{m\pi x}{a}$$

(6.2.2.)

we find that this function satisfies

$$\nabla^4 w_1 = 0 \qquad (6.2.3),$$

hence

$$\nabla^4(w_0 + w_1) = q/D \qquad (6.2.4).$$

Thus the sum of these two functions is such that the plate is subjected to a uniform load $\underline{q}$. Since the deflection is to vanish over the entire boundary, w_1 must vanish on the boundary and consequently

$$B_m = -C_m\alpha_m\tanh\alpha_m \quad , \quad A_m = -D_m\alpha_m\coth\alpha_m \qquad (6.2.5).$$

Along the edge $y = -b/2$, $M_y = 0$. Therefore

$$C_m = D_m\tanh\alpha_m \qquad (6.2.6).$$

With these relations for the unknown constants in the sum $w_0 + w_1$, the plate is simply-supported over CD, DE, EA, and has zero deflection over the entire boundary. Furthermore, one set of constants $\{D_m\}$ remains to be determined. Since the edge ABC is clamped from A to B and simply-supported from B to C, there results

$$\sum_{1,2,3} X_m \sin m\xi = 0 \quad , \quad \frac{1}{2} < \xi < 1$$

$$\xi = \frac{\pi x}{a}$$

and

$$\sum_{1,2,3} \left\{ X_m \left[\frac{\coth 2\alpha_m}{\alpha_m} - 2\,\mathrm{csch}^2 2\alpha_m \right] + \frac{\sin^2 \frac{m\pi}{2}}{\alpha_m^3} \left[\mathrm{sech}^2 \alpha_m \; \frac{\tanh \alpha_m}{\alpha_m} \right] \right\} \sin m\xi = 0$$

$$0 < \xi < \frac{1}{2}$$

where (6.2.7.)

$$X_m = D_m \alpha_m^2 \sinh \alpha_m \left(\frac{32aD}{qb^5} \right) \qquad (6.2.8).$$

The solution of this problem is obtained once we have determined the $\{X_m\}$. As in the previous chapter on the circle, we again presume here that the singularity in the bending moment at the discontinuity in the edge conditions is not such as to disturb the Fourier expansion of the bending moment along the clamped portion of the edge.

Hence the uniqueness theorem for Fourier coefficients permits us to write

$$\frac{\pi}{2}\left\{\frac{\frac{\coth 2\alpha_n}{2\alpha_n} - \operatorname{csch}^2 2\alpha_n + \frac{1}{2}}{\frac{\coth 2\alpha_n}{2\alpha_n} - \operatorname{csch}^2 2\alpha_n - \frac{1}{2}} Q_n - \frac{\sin^2\frac{n\pi}{2}\left(\operatorname{sech}^2\alpha_n - \frac{\tanh\alpha_n}{\alpha_n}\right)}{\alpha_n^3\left(\frac{\coth 2\alpha_n}{2\alpha_n} - \operatorname{csch}^2 2\alpha_n - \frac{1}{2}\right)}\right\} + \sum_{m=1,2,\cdots}^{1} Q_m F(m, n) = 0 \qquad n = 1, 2, \cdots \qquad (6.2.9)$$

where

$$Q_n = X_n\left[\frac{\coth 2\alpha_n}{\alpha_n} - 2\operatorname{csch}^2 2\alpha_n - 1\right] + \frac{\sin^2\frac{n\pi}{2}}{\alpha_n^3}\left[\operatorname{sech}^2\alpha_n - \frac{\tanh\alpha_n}{\alpha_n}\right] \qquad (6.2.10),$$

and

$$F(m, n) = \int_0^{\frac{\pi}{2}} \sin m\xi \sin n\xi \, d\xi.$$

Taking twelve of these equations there results the following values for the Q's:

TABLE NO. VIII.

Q_1	-.155204439	Q_7	-.004518667
Q_2	-.066476326	Q_8	-.032105698
Q_3	.007375723	Q_9	.013475802
Q_4	.041922176	Q_{10}	-.024361605
Q_5	-.005817818	Q_{11}	-.026145939
Q_6	-.038222856	Q_{12}	-.002376705

The numerical results we shall give here are those connected with the edge of the plate over which the boundary conditions change. As one notices from the graph of the slope it is not identically zero where it should be zero. Indeed, it oscillates between bounds. This behavior is expected of any type of solution of the nature presented in this section. Furthermore, in the neighborhood of the discontinuity, the results are bad. The slope along the simply-supported edge is reasonable. The bending moment along the clamped portion is somewhat chaotic. This results, probably, from two different causes. First, the bending moment is associated with a higher derivative than the slope. Consequently, one can expect more variation in the function obtained by a numerical procedure. Second, some of this variation could be caused by the fact that a wrinkle of some sort forms across the corner containing the clamped portion of the edge and these wiggles could be perturbations in the boundary moments due to wrinkling.

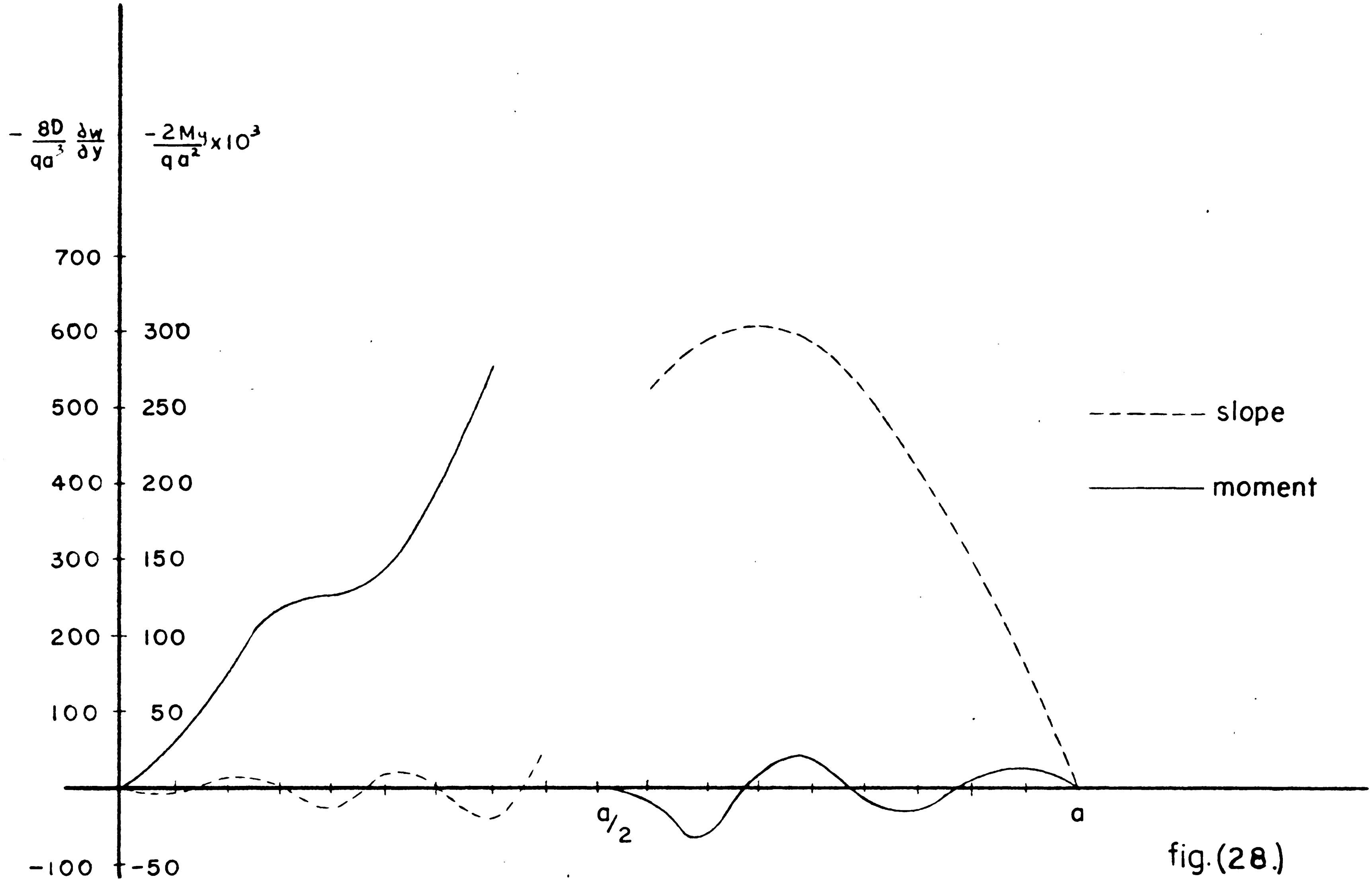
The deflection at the center of the plate, as compared to the simply-supported plate is given below. This, in some sense, validates the numerical computations.

S.S.	Our Case	C.E.
$w_c = .00406 \frac{qa^4}{D}$	$w_c = .00331\frac{qa^4}{D}$	$w_c = .0027\frac{qa^4}{D}$.

These results and the following graphs are for a square plate.

x/a	$-\frac{8D}{qa^3}\frac{\partial w}{\partial y}$	$-\frac{2M_y(x)}{qa^2}\ 10^3$
0	- 14.95	+ 28.25
1/18	+ 11.60	+ 78.7
2/18	2.37	119.8
3/18	- 20.60	125.16
4/18	18.59	133.68
5/18	11.40	202.3
6/18	- 49.71	283.4
7/18	- 49.90	269.27
8/18	320	---
9/18	527	- 4.296
10/18	594	- 38.055
11/18	609	6.585
12/18	588	24.738
13/18	517	- 2.899
14/18	421	- 22.191
15/18	302	1.358
16/18	155	20.29
17/18	0	0
1		

Table No. IX



$-\frac{8D}{qa^3}\frac{\partial w}{\partial y}$
$-\frac{2M_y}{qa^2} \times 10^3$
700
600
500
400
300
200
100
-100
300
250
200
150
100
50
-50
a/2
a
slope
moment

fig.(28.)

BIBLIOGRAPHY

1.) S. Timoshenko, Theory of Plates and Shells, McGraw-Hill Book Co., 1940.

2.) G. Kirchhoff, J. Crelle, vol. 40, 1850.

3.) N. I. Muschelishvili, Some Basic Problems of the Mathematical Theory of Elasticity, N. V. P. Groningen, Holland, 1953.

4.) P. R. Garabedian, A Partial Differential Equation Arising in Conformal Mapping, Tech. Report No. 8, Dept. of Math., Stanford Univ., Aug. 5, 1950.

5.) J. H. Michell, London Math. Soc. Proc., vol. 34, 1902.

6.) N. I. Muschelishvili, Singular Integral Equations, N. V. P. Groningen, Holland, 1953.

7.) W. R. Dean, Proc. Camb. Phil. Soc., vol. 49, 1953.

8.) A. Erdelyi, W. Magnus, F. Oberhettinger, F. G. Tricomi, Higher Transcendental Functions, vol. 2, McGraw-Hill Book Co., 1953.

9.) J. Hadamard, Ann. de l'Ec. Normale, series 3, vol. 18, 1901.

10.) H. Reismann, J. Applied Mech., vol. 19, no. 2, 1952.

11.) A. I. Kalandiya, Prikl. Mat. Mekh., vol. 16, no. 5, 1952.

12.) G. Mandzhavidze, Prikl. Mat. Mekh., vol. 15, no. 3, 1951.

13.) D. I. Sherman, Comptes Rendus Doklady, N. S., 1955.

14.) E. Goursat, A Course in Mathematical Analysis.

15.) E. T. Whittaker and G. N. Watson, A Course in Modern Analysis, MacMillan Co., New York, 1945.

16.) T. Fort, Infinite Series, Oxford Press, 1930.

17.) B. D. Aggarwala, Singularly Loaded Rectangular Plates, ZAMM, 1954.

18.) W. Nowacki, Rectangular Plates with Mixed Edge Conditions, Arch. Mech. Stos., 1951.

19.) H. Poritsky, Application of Analytic Functions to Two-Dimensional Biharmonic Analysis, Trans. Amer. Math. Soc., 1946.

20). J. J. Stoker, Math. Prob. Connected with the Bending and Buckling of Elastic Plates, Bul. Amer. Math. Soc., 1942.

21.) M. Stippes, A Note on the Simply-Supported Rectangular Plate, Quart. of Applied Math., Vol. XIV, no. 1, Apr. 1956.

22.) S. Woinowsky-Krieger, The Bending of a Wedge-Shaped Plate, JAM, vol. 20, no. 1, March, 1953.

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Abstract

The bending of thin flat plates has occupied the interests of mechanicians and applied mathematicians since J. L. Lagrange discovered the differential equation characterizing the behavior of such structural members.

One particular phase of investigation in this field concerns itself with the solution of the differential equation subject to given boundary conditions. Indeed, it may be safely stated that the bulk of the literature on the subject of flat plates is concerned with the solution of problems involving the specification of the transverse loading on the plate and the conditions at the boundary of the plate. Various mathematical techniques are available for the solution of such problems. Among these, the most prominent are, a) the method of series, b) the method of singularities, and c) the complex variable techniques.

A survey of the literature in this area has revealed a paucity of solutions of certain types of problems; notably, those problems in which boundary conditions are mixed along a portion of the edge of the plate which has a continuously turning tangent. By mixed boundary conditions, we mean a change in condition from prescription of bending moment and vertical shear to assignment of slope and deflection along a portion of the edge which has a continuously turning tangent.

In the first section of this thesis, a number of problems are considered for the half-plane. The attendant boundary conditions considered are combinations of clamping and simple support.

The second portion consists of a number of problems associated with the quarter-plane. Solutions for these problems are obtained by utilizing the method of images in conjunction with the solutions presented in the first section.

After this, we examine some problems connected with the circular plate. In particular, a numerical solution is given for a uniformly loaded circular plate simply-supported over half of its boundary and clamped over the remaining portion.

The last chapter is a brief discussion of plates in the form of rectangles. Here, a closed solution is presented for the bending moments in terms of Weierstrassian elliptic functions. Another numerical example is included for a uniformly loaded plate clamped over a portion of one edge and simply-supported over the remainder of its boundary.