CHAPTER II.

SYSTEM MODELING AND FORMULATION

2.1 Basic Assumptions and Kinematics of the Model

The structural model used herein aiming at simulating the lifting surface of advanced flight vehicles is that of a cantilevered thin-walled closed-section beam. Herein, attention will be confined to uniform single cell beams. Two systems of coordinate, namely (s, z, n) and (x, y, z) are used to define points of the thin-walled beam (see Fig. 2.1). Notice that the z- axis is located as to coincide with the locus of symmetrical points of the cross-section along the wing span. The beam model incorporates the following nonclassical features:

- Anisotropy of constituent material layers.
- Transverse shear deformation.
- Nonuniform torsional model, in the sense that the rate of twist $d\varphi/dz$ is no longer assumed to be constant (as in the Saint-Venant torsional model) but a function of the spanwise coordinate.
- Primary and secondary warping effects.

As a result of the incorporation of transverse shear effects, the present beam model is capable of providing results also for thick-walled beams and/or when their constituent materials exhibit high flexibilities in transverse shear. We also postulate non-deformablility of beam cross-sections in their own planes but the possibility to warp out of their own planes. In addition, we assume that hoop stress resultants are negligibly small compared with the remaining ones. In accordance with the above assumptions and in order to reduce the 3-D problem to an equivalent 1-D, the components of the displacement vector are expressed as [2]

$$
\begin{aligned}
u(x,y,z,t) &= u_o(z,t) - y\,\varphi(z,t) \\
v(x,y,z,t) &= v_o(z,t) + x\,\varphi(z,t) \qquad\qquad (2.1.1\text{ a-c}) \\
w(x,y,z,t) &= w_o(z,t) + \theta_x(z,t)\left[y(s) - n\frac{dx}{ds}\right] \\
&\quad + \theta_y(z,t)\left[x(s) + n\frac{dy}{ds}\right] - \varphi'(z,t)\left[F_w(s) + n\,a(s)\right]
\end{aligned}
$$

where

$$
\begin{aligned}
\theta_x(z,t) &= \gamma_{yz}(z,t) - v_o'(z,t) \\
\theta_y(z,t) &= \gamma_{xz}(z,t) - u_o'(z,t) \qquad\qquad (2.1.2\text{ a-c})
\end{aligned}
$$

and

$$
a(s) = -y(s)\frac{dy}{ds} - x(s)\frac{dx}{ds}
$$

Here $\theta_x(z,t)$ and $\theta_y(z,t)$ denote the rotations about axes x and y respectively, while γ_{yz} and γ_{xz} denote the transverse shear strains in the planes yz and xz respectively and the primes denote derivatives with respect to the $z-$ coordinate. When the transverse shear effect is discarded, from Eq. (2.1.2) it is readily seen that in this case

$$\theta_x \rightarrow -v_o' \ , \quad \theta_y \rightarrow -u_o' \ . \tag{2.1.2d,e}$$

The primary warping function is expressed as

$$F_w = \int_0^s [r_n(s) - \Psi]\, ds \tag{2.1.3}$$

where the torsional function Ψ and the quantity $r_n(s)$ are

$$\Psi = \frac{\oint_C \frac{r_n(s)}{h(s)} ds}{\oint_C \frac{ds}{h(s)}} \tag{2.1.4a}$$

and

$$r_n = x(s)\frac{dy}{ds} - y(s)\frac{dx}{ds} \tag{2.1.4b}$$

respectively. Figure 2.2 displays the configuration of a cross-section of a thin-walled beam structure and reveals the geometrical meaning of $a(s)$ and $r_n(s)$ as well.

Equations (2.1.1) and (2.1.2) reveal that the kinematic variables, $v_o(z,t)$, $w_o(z,t)$, $\theta_x(z,t)$, $\theta_y(z,t)$ and $\varphi(z,t)$ representing three translations in the x, y, z directions and three rotations about the x, y, z directions, respectively are used to define the displacement components u, v and w. The quantity $h(s)$ denotes the integral around the entire periphery C of the mid-line contour of the cross-section of the beam; while $\int_0^s r_n(s)ds$ is referred to as the sectorial area. For the case of the uniform thickness h in the circumferential direction, Eq. (2.1.4a) reduces to $\Psi = 2A_C/\beta$ where A_C denotes the cross-sectional area bounded by the mid-line while β denotes the total length of the contour mid-line.

Considering the case of composite TWBs constituted by the superposition of a finite number N of individually homogeneous layers, it is assumed that the material of each constituent layer is linearly elastic and anisotropic and that the layers are perfectly bonded.

The 3-D constitutive equations of a generally orthotropic elastic material could be expressed in matrix form as:

$$\begin{Bmatrix} \sigma_{ss} \\ \sigma_{zz} \\ \sigma_{nn} \\ \sigma_{zn} \\ \sigma_{ns} \\ \sigma_{sz} \end{Bmatrix} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{13} & 0 & 0 & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{23} & 0 & 0 & \overline{Q}_{26} \\ \overline{Q}_{13} & \overline{Q}_{23} & \overline{Q}_{33} & 0 & 0 & \overline{Q}_{36} \\ 0 & 0 & 0 & \overline{Q}_{44} & \overline{Q}_{45} & 0 \\ 0 & 0 & 0 & \overline{Q}_{54} & \overline{Q}_{55} & 0 \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{36} & 0 & 0 & \overline{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{ss} \\ \varepsilon_{zz} \\ \varepsilon_{nn} \\ \gamma_{zn} \\ \gamma_{ns} \\ \gamma_{sz} \end{Bmatrix} \tag{2.1.5}$$

In Eq. (2.1.5) the $\overline{Q}_{ij}$ denote the transformed elastic coefficients associated with the kth layer in the global coordinate system of the structure while $\gamma_{pr} = 2\varepsilon_{pr}$ when $p \neq r$ and the ε_{ij} denote the components of the strain tensor.

Based on the kinematic representations, Eq. (2.1.1) and (2.1.2), one can obtain the strain measure as following [2]:

$$\varepsilon_{zz}(n,s,z,t) = \varepsilon_{zz}^{o}(s,z,t) + n\varepsilon_{zz}^{n}(s,z,t) \tag{2.1.6a}$$

where

$$\varepsilon_{zz}^{o}(s,z,t) = w_o'(z,t) + \underline{\theta_x'(z,t)y(s)} + \theta_y'(z,t)x(s) - \varphi''(z,t)F_w(s) \tag{2.1.6b}$$

and

$$\varepsilon_{zz}^{n}(s,z,t) = \theta_y'(z,t)\frac{dy}{ds} - \underline{\theta_x'(z,t)\frac{dx}{ds}} - \varphi''(z,t)a(s) \tag{2.1.6c}$$

are the axial strains associated with the primary and secondary warping, respectively. The membrane shear strain component can be expressed in the form

$$\gamma_{sz}(s,z,t) = \gamma_{sz}^{o}(s,z,t) + 2\frac{A_C}{\beta}\varphi'(z,t) \tag{2.1.7a}$$

where

$$\gamma_{sz}^{o}(s,z,t) = \left[u_o'(z,t) + \theta_y(z,t)\right]\frac{dx}{ds} + \underline{\left[v_o'(z,t) + \theta_x(z,t)\right]\frac{dy}{ds}} \tag{2.1.7b}$$

and the transverse shear strain component as

$$\gamma_{nz}(s,z,t) = \left[u_o'(z,t) + \theta_y(z,t)\right]\frac{dy}{ds} - \underline{\left[v_o'(z,t) + \theta_x(z,t)\right]\frac{dx}{ds}} \tag{2.1.8}$$

In the above equations, the underscored terms define that part belonging to the transverse bending, whereas the remaining ones are associated with the axial warping (via the terms related with w_o), lagging (related with u_o and θ_y) and with the twist (φ).

2.2 Constitutive Equations of Piezoactuator Patches

We assume that the master structure consists of m layers and the actuator of l piezoelectric layers. Along the circumferential s, spanwize z and transverse n directions, the piezoactuators are distributed according to (see Fig. 2.3)

$$\begin{aligned} R_k(n) &= H(n - n_{k^-}) - H(n - n_{k^+}), \\ R_k(s) &= H(s - s_{k^-}) - H(s - s_{k^+}), \\ R_k(z) &= H(z - z_{k^-}) - H(z - z_{k^+}) \end{aligned} \tag{2.2.1}$$

where R is a spatial function and $H(\,)$ denotes the Heaviside distribution, in which the subscript k in parenthesis identifies the kth layer. The linear constitutive equations for a three-dimensional piezoelectric continuum, expressed in Voigt's contracted notation, are [41]

$$\sigma_i = C_{ij}^{\xi} S_j - e_{ri}\xi_r \quad , \quad D_r = e_{rj}S_j + \varepsilon_{rl}^{S}\xi_l \tag{2.2.2 a,b}$$

in which summation over repeated indicies is implied. Moreover, σ_i and $S_j(i, j = 1,2, \quad 6)$ denote the stress and strain components respectively, where

$$S_j = \begin{cases} S_{pr}, & p = r, \quad j = 1,2,3 \\ 2S_{pr}, & p \neq r, \quad j = 4,5,6 \end{cases} \tag{2.2.3}$$

Moreover, C_{ij}^{ξ}, e_{ri} and ε_{rl}^{S} are the elastic (measured for conditions of a constant electric field), piezoelectric and dielectric constants (measured under constant strain) and ξ_r and $D_r (r = 1,2,3)$ denote the electric field intensity and electric displacement vector respectively. Whereas Eq. (2.2.2a) describes the converse piezoelectric effect, consisting of the generation of mechanical stress or strain in response to an electric field, Eq. (2.2.2b) describes the direct piezoelectric effect, consisting of the generation of an electrical charge under a mechanical force. In adaptive structures, the direct effect is used for sensing and converse effect is used for active control. Eq. (2.2.2) are valid for the most general case of anisotropy, i.e. for triclinic crystals. In the following, we restrict the piezolectric anisotropy to the case of hexagonal symmetry, the $n-$axis being an axis of rotatory symmetry coinciding with the direction of polarization (thickness polarization), as can be seen from Fig. 2.4 [39]. We also confine ourselves to an in-plane isotropic piezoelectric continuum. In this case, the piezoelectric continuum is characterized by five independent elasticcoefficients, $C_{11} = C_{22}$, $C_{13} = C_{23} = C_{31} = C_{32}$, C_{33}, $C_{44} = C_{55}$, $C_{66} (\equiv (C_{11} - C_{12})/2)$, three independent piezoelectric coefficients, $e_{15} = e_{24}$, $e_{31} = e_{32}$, and e_{33} and two independent dielectric constants, $\varepsilon_{11} = \varepsilon_{22}$, ε_{33} [41].

At this point, we assume that the master structure is made of anisotropic material layers, the anisotropy being of the monoclinic type. We also assume that the electric field vector ξ_i is represented in terms of the component ξ_3 only, implying $\xi_1 = \xi_2 = 0$. As a result of the uniform voltage distribution, ξ_3 depends on time alone and is independent of the spatial position. Invoking the stipulated distribution law of piezoactuators, Eq. (2.2.1) and the stipulated anisotropy properties, the three-dimensional constitutive equations for the actuator layers can be expressed as

$$\begin{bmatrix} \sigma_{ss} \\ \sigma_{zz} \\ \sigma_{sz} \end{bmatrix}_{(k)} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{11} & 0 \\ 0 & 0 & \dfrac{C_{11} - C_{12}}{2} \end{bmatrix}_{(k)} \begin{bmatrix} S_{ss} \\ S_{zz} \\ S_{sz} \end{bmatrix}_{(k)} - \begin{bmatrix} e_{31}^{(k)} \xi_3^{(k)} R_k(n) R_k(s) R_k(z) \\ e_{31}^{(k)} \xi_3^{(k)} R_k(n) R_k(s) R_k(z) \\ 0 \end{bmatrix} \tag{2.2.4a}$$

and

$$\sigma_{nz}^{(k)} = C_{44}^{(k)} S_{nz}^{(k)} \tag{2.2.4b}$$

The last terms in Eq. (2.2.4a) identify the actuation stresses induced by the applied electric field.

2.3 Global Constitutive Equations

Integrating the three-dimensional constitutive equations through the thickness of the master structure and actuators and postulating that the hoop stress resultant N_{ss} is negligibly small when compared with the remaining ones, two-dimensional constitutive equations, referred to also shell-constitutive equations, are obtained as:

the stress resultants:

$$\begin{Bmatrix} N_{zz} \\ N_{sz} \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \end{bmatrix} \begin{Bmatrix} \varepsilon_{zz}^{o} \\ \gamma_{sz}^{o} \\ \phi' \\ \varepsilon_{zz}^{n} \end{Bmatrix} - \begin{Bmatrix} N_{zz}^{a} \\ 0 \end{Bmatrix} \tag{2.3.1}$$

the transverse shear stress resultant:

$$N_{nz} = A_{44}\gamma_{nz} \tag{2.3.2}$$

the stress couples:

$$\begin{Bmatrix} L_{zz} \\ L_{sz} \end{Bmatrix} = \begin{bmatrix} K_{41} & K_{42} & K_{43} & K_{44} \\ K_{51} & K_{52} & K_{53} & K_{54} \end{bmatrix} \begin{Bmatrix} \varepsilon_{zz}^{o} \\ \gamma_{sz}^{o} \\ \phi' \\ \varepsilon_{zz}^{n} \end{Bmatrix} - \begin{Bmatrix} L_{zz}^{a} \\ 0 \end{Bmatrix} \tag{2.3.3}$$

where K_{ij} denote the modified local stiffness coefficients, listed in the Appendix A and N_{zz}^{a} and L_{zz}^{a} denote the modified piezoelectrically induced-stress resultant and stress couple respectively, expressed as

$$N_{zz}^{a}(s,z,t) = \left(1 - \frac{A_{12}}{A_{11}}\right) \sum_{k=1}^{l} \xi_3^{(k)}(t)(n_{k^+} - n_{k^-}) e_{31}^{(k)} R_k(s,z)$$

(2.3.4a,b)

$$L_{zz}^{a}(s,z,t) = \left(\frac{1}{2}(n_{k^+} + n_{k^-}) - \frac{B_{12}}{A_{11}}\right) \sum_{k=1}^{l} \xi_3^{(k)}(t)(n_{k^+} - n_{k^-}) e_{31}^{(k)} R_k(s,z)$$

where $$R_k(s,z) = R_k(s)R_k(z)$$

2.4 System Kinetic Energy

With the displacements given by Eq. (2.1.1), the systme kinetic energy takes the form

$$T = \frac{1}{2}\int_0^L \oint \sum_{k=1}^{N} \int_{h(k)} \rho^{(k)} \left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2 \right] dndsdz \tag{2.4.1}$$

$$= \frac{1}{2}\int_0^L \oint \sum_{k=1}^{N} \int_{h(k)} \rho^{(k)} \Big[(\dot{u}_o - y\dot{\phi})^2 + (\dot{v}_o + x\dot{\phi})^2 + \Big\{ (\dot{w}_o + y\dot{\theta}_x + x\dot{\theta}_y - F_w\dot{\phi}')$$

$$+ n\left(\frac{dy}{ds}\dot{\theta}_y - \frac{dx}{ds}\dot{\theta}_x - a\dot{\phi}'\right) \Big\}^2 \Big] dndsdz$$

where L denotes the beam length.

Carrying out the indicated integrations with respect to n and s, the kinetic energy can be expressed in the compact form as

$$T = \frac{1}{2}\int_0^L \Big[b_1 \dot{u}_o^2 + (b_4 + b_5)\dot{\varphi}^2 + b_1 \dot{v}_o^2 + b_1 \dot{w}_o^2 + (b_5 + b_{15})\dot{\theta}_y^2 + (b_4 + b_{14})\dot{\theta}_x^2 + (b_{10} + b_{18})\dot{\varphi}'^2 \Big] dz \tag{2.4.2}$$

In matrix form, T can be rewritten as

$$T = \frac{1}{2}\int_0^L \begin{Bmatrix} \dot{u}_o \\ \dot{v}_o \\ \dot{w}_o \\ \dot{\theta}_x \\ \dot{\theta}_y \\ \dot{\varphi} \\ \dot{\varphi}' \end{Bmatrix}^T \begin{bmatrix} b_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (b_4 + b_{14}) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (b_5 + b_{15}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (b_4 + b_5) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & (b_{10} + b_{18}) \end{bmatrix} \begin{Bmatrix} \dot{u}_o \\ \dot{v}_o \\ \dot{w}_o \\ \dot{\theta}_x \\ \dot{\theta}_y \\ \dot{\varphi} \\ \dot{\varphi}' \end{Bmatrix} dz \tag{2.4.3a}$$

where the various mass coefficients b_i are displayed in the Appendix A.

For the non-shearable beam, when the transverse shear effect is discarded, i.e., $\theta_x \to -v_o'$ and $\theta_y \to -u_o'$, T has the form

$$T = \frac{1}{2}\int_0^L \begin{Bmatrix} \dot{u}_o \\ \dot{v}_o \\ \dot{w}_o \\ \dot{v}_o' \\ \dot{u}_o' \\ \dot{\varphi} \\ \dot{\varphi}' \end{Bmatrix}^T \begin{bmatrix} b_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (b_4 + b_{14}) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (b_5 + b_{15}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (b_4 + b_5) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & (b_{10} + b_{18}) \end{bmatrix} \begin{Bmatrix} \dot{u}_o \\ \dot{v}_o \\ \dot{w}_o \\ \dot{v}_o' \\ \dot{u}_o' \\ \dot{\varphi} \\ \dot{\varphi}' \end{Bmatrix} dz \tag{2.4.3b}$$

2.5 System Potential Energy

Having in view the assumption of non-deformability of the cross-section of thin-walled beams implying that ε_{ss}, ε_{nn} and γ_{sn} should be zero, the potential energy can be shown to have the expression

$$
\begin{aligned}
V &= \frac{1}{2}\int_{\tau}\sigma_{ij}\varepsilon_{ij}\,d\tau \\
&= \frac{1}{2}\int_0^L \oint_C \sum_{k=1}^{N}\int_{h(k)}\left[\sigma_{zz}\varepsilon_{zz} + \sigma_{sz}\gamma_{sz} + \sigma_{nz}\gamma_{nz}\right]_{(k)} dndsdz \\
&= \frac{1}{2}\int_0^L \oint \sum_{k=1}^{N}\int_{h(k)}\Bigg\{ \sigma_{zz}^{(k)}\left[w_o' + x\theta_y' + y\theta_x' - F_w\varphi'' + n\left(\frac{dy}{ds}\theta_y' - \frac{dx}{ds}\theta_x' - a\varphi''\right)\right] \\
&\qquad + \sigma_{sz}^{(k)}\left[(u_o' + \theta_y)\frac{dx}{ds} + (v_o' + \theta_x)\frac{dy}{ds} + \frac{2A_C}{\beta}\varphi'\right] \\
&\qquad + \sigma_{nz}^{(k)}\left[(u_o' + \theta_y)\frac{dy}{ds} - (v_o' + \theta_x)\frac{dx}{ds}\right]\Bigg\} dndsdz \\
&= \frac{1}{2}\int_0^L \oint \Bigg[N_{zz}(w_o' + x\theta_y' + y\theta_x' - F_w\varphi'') + L_{zz}\left(\theta_y'\frac{dy}{ds} - \theta_x'\frac{dx}{ds} - a\varphi''\right) \\
&\qquad + N_{sz}\left((u_o' + \theta_y)\frac{dx}{ds} + (v_o' + \theta_x)\frac{dy}{ds}\right) + 2N_{sz}\frac{A_C}{\beta}\varphi' \\
&\qquad + N_{nz}\left((u_o' + \theta_y)\frac{dy}{ds} - (v_o' + \theta_x)\frac{dx}{ds}\right)\Bigg] dsdz
\end{aligned} \tag{2.5.1}
$$

where $d\tau \equiv dndsdz$ denotes the differential volume element.
The one-dimensional stress-resultants and stress couples in Eq. (2.5.1) are defined in the Ref. 3:

$$
\begin{aligned}
T_z(z,t) &= \oint N_{zz}\,ds \\
M_y(z,t) &= \oint \left(x\,N_{zz} + L_{zz}\frac{dy}{ds}\right) ds \\
M_x(z,t) &= \oint \left(y\,N_{zz} - L_{zz}\frac{dx}{ds}\right) ds \\
Q_x(z,t) &= \oint \left(N_{sz}\frac{dx}{ds} + L_{zn}\frac{dy}{ds}\right) ds \\
Q_y(z,t) &= \oint \left(N_{sz}\frac{dy}{ds} - L_{zn}\frac{dx}{ds}\right) ds \\
B_w(z,t) &= \oint \left(F_w(s)N_{zz} + a(s)L_{zz}\right) ds \\
M_z(z,t) &= 2\oint N_{sz}\,\Psi\,ds
\end{aligned} \tag{2.5.2}
$$

in which T_z, Q_x and Q_y denote the axial force and shear forces in the x- and y- directions, respectively, M_x, M_y and M_z are the bending and twist moments about the x-, y- and z-axes respectively, while B_w is the bimoment [43].

In view of Eqs. (2.3.1) and (2.3.3), the one-dimensional stress measures T_z, M_x, M_y and B_w can be cast in the more convenient form

$$T_z = \hat{T}_z - T_z^a \qquad M_x = \hat{M}_x - M_x^a$$
$$M_y = \hat{M}_y - M_y^a \qquad B_w = \hat{B}_w - B_w^a \tag{2.5.3}$$

where overcaret identify purely mechanical term and superscript a denotes piezoelectrically induced terms. The mechanical terms are expressed as

$$\begin{bmatrix} \hat{T}_z \\ \hat{M}_y \\ \hat{M}_x \\ Q_x \\ Q_y \\ \hat{B}_w \\ M_z \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & a_{47} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} & a_{57} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} & a_{67} \\ a_{71} & a_{72} & a_{73} & a_{74} & a_{75} & a_{76} & a_{77} \end{bmatrix} \begin{bmatrix} w_o' \\ \theta_y' \\ \theta_x' \\ (u_o' + \theta_y) \\ (v_o' + \theta_x) \\ \phi'' \\ \phi' \end{bmatrix} \tag{2.5.4}$$

while the piezoelectrically induced terms are

$$T_z^a = \oint N_{zz}^a ds = \oint \sum_{k=1}^{l} \left[\xi_3^{(k)} (n_{k^+} - n_{k^-}) e_{31}^{(k)} R_k(s,z) \right] ds$$

$$M_x^a = \oint \left(y N_{zz}^a - \frac{dx}{ds} L_{zz}^a \right) ds$$

$$= \oint \sum_{k=1}^{l} \xi_3^{(k)} (n_{k^+} - n_{k^-}) e_{31}^{(k)} R_k(s,z) \left[y \left(1 - \frac{A_{12}}{A_{11}} \right) + \frac{dx}{ds} \frac{B_{12}}{A_{11}} \right] ds$$

$$\underline{- \frac{1}{2} \oint \left[\frac{dx}{ds} \sum_{k=1}^{l} \xi_3^{(k)} (n_{k^+}^2 - n_{k^-}^2) e_{31}^{(k)} R_k(s,z) \right] ds}$$

(2.5.5 a-d)

$$M_y^a = \oint \left(x N_{zz}^a + \frac{dy}{ds} L_{zz}^a \right) ds$$

$$= \oint \sum_{k=1}^{l} \xi_3^{(k)} (n_{k^+} - n_{k^-}) e_{31}^{(k)} R_k(s,z) \left[x \left(1 - \frac{A_{12}}{A_{11}} \right) + \frac{dy}{ds} \frac{B_{12}}{A_{11}} \right] ds$$

$$\underline{+ \frac{1}{2} \oint \left[\frac{dy}{ds} \sum_{k=1}^{l} \xi_3^{(k)} (n_{k^+}^2 - n_{k^-}^2) e_{31}^{(k)} R_k(s,z) \right] ds}$$

$$B_w^a = \oint \left(F_w N_{zz}^a + a L_{zz}^a \right) ds$$

$$= \oint \sum_{k=1}^{l} \xi_3^{(k)} (n_{k^+} - n_{k^-}) e_{31}^{(k)} R_k(s,z) \left[F_w \left(1 - \frac{A_{12}}{A_{11}} \right) - a \frac{B_{12}}{A_{11}} \right] ds$$

$$\underline{+ \frac{1}{2} \oint \left[a \sum_{k=1}^{l} \xi_3^{(k)} (n_{k^+}^2 - n_{k^-}^2) e_{31}^{(k)} R_k(s,z) \right] ds}$$

In these equations, $a_{ij} (= a_{ji})$ are stiffness coefficients displayed in the Appendix A.

It is noticed that piezoelectrically induced terms are proportional to the applied electric field ξ_3. In the case of actuators placed symmetrically throughout the thickness of the beam, the underlined terms in Eq. (2.5.5) vanish.

The unshearable counterpart of Eq. (2.5.4a) becomes:

$$\begin{bmatrix} \hat{T}_z \\ \hat{M}_y \\ \hat{M}_x \\ \hat{B}_w \\ \hat{M}_z \end{bmatrix} = \begin{bmatrix} a_{11} & -a_{12} & -a_{13} & a_{16} & a_{17} \\ -a_{21} & a_{22} & a_{23} & -a_{26} & -a_{27} \\ -a_{31} & a_{32} & a_{33} & -a_{36} & -a_{37} \\ a_{61} & -a_{62} & -a_{63} & a_{66} & a_{67} \\ a_{71} & -a_{72} & -a_{73} & a_{76} & a_{77} \end{bmatrix} \begin{bmatrix} w_o' \\ u_o'' \\ v_o'' \\ \varphi'' \\ \varphi' \end{bmatrix} dz \tag{2.5.5e}$$

Similarly, using Eq. (2.4.6) as well as the two-dimensional constitutive equations Eqs. (2.3.1)-(2.3.3) and integrating with respect to n and s, the potential energy can be shown to have the form

$$V = \frac{1}{2} \int_0^L \left[T_z w_o' + M_y \theta_y' + M_x \theta_x' + Q_x (u_o' + \theta_y) + Q_y (v_o' + \theta_x) - B_w \varphi'' + M_z \varphi' \right] dz \tag{2.5.6a}$$

$$= \frac{1}{2} \int_0^L \Big[a_{11} (w_o')^2 + a_{22} (\theta_y')^2 + a_{33} (\theta_x')^2 + a_{44} (u_o' + \theta_y)^2 + a_{55} (v_o' + \theta_x)^2$$

$$+ a_{66} (\varphi'')^2 + a_{77} (\varphi')^2 + 2 a_{12} w_o' \theta_y' + 2 a_{13} w_o' \theta_x' + 2 a_{14} w_o' (u_o' + \theta_y)$$

$$+ 2 a_{15} w_o' (v_o' + \theta_x) + 2 a_{16} w_o' \varphi'' + 2 a_{17} w_o' \varphi' + 2 a_{23} \theta_y' \theta_x' + 2 a_{24} \theta_y' (u_o' + \theta_y)$$

$$+ 2 a_{25} \theta_y' (v_o' + \theta_x) + 2 a_{26} \theta_y' \varphi'' + 2 a_{27} \theta_y' \varphi' + 2 a_{34} \theta_x' (u_o' + \theta_y)$$

$$+ 2 a_{35} \theta_x' (v_o' + \theta_x) + 2 a_{36} \theta_x' \varphi'' + 2 a_{37} \theta_x' \varphi' + 2 a_{45} (u_o' + \theta_y)(v_o' + \theta_x)$$

$$+ 2 a_{46} (u_o' + \theta_y) \varphi'' + 2 a_{47} (u_o' + \theta_y) \varphi' + 2 a_{56} (v_o' + \theta_x) \varphi'' + 2 a_{57} (v_o' + \theta_x) \varphi'$$

$$+ 2 a_{67} \varphi'' \varphi' \Big] dz \tag{2.5.6b}$$

The potential energy V can be cast in matrix form as

$$V = \frac{1}{2}\int_0^L \begin{Bmatrix} w_o' \\ \theta_y' \\ \theta_x' \\ (u_o'+\theta_y) \\ (v_o'+\theta_x) \\ \varphi'' \\ \varphi' \end{Bmatrix}^T \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & a_{47} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} & a_{57} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} & a_{67} \\ a_{71} & a_{72} & a_{73} & a_{74} & a_{75} & a_{76} & a_{77} \end{bmatrix} \begin{Bmatrix} w_o' \\ \theta_y' \\ \theta_x' \\ (u_o'+\theta_y) \\ (v_o'+\theta_x) \\ \varphi'' \\ \varphi' \end{Bmatrix} dz \qquad (2.5.6c)$$

As in the formulation of the kinetic energy, for the non-shearable beam, V can be expressed as:

$$\begin{aligned} V = \frac{1}{2}\int_0^L [\, & a_{11}(w_o')^2 + a_{22}(u_o'')^2 + a_{33}(v_o'')^2 + a_{66}(\varphi'')^2 + a_{77}(\varphi')^2 \\ & -2a_{12}w_o'\,u_o'' - 2a_{13}w_o'\,v_o'' + 2a_{16}w_o'\,\varphi'' + 2a_{17}w_o'\,\varphi' + 2a_{23}u_o''\,v_o'' \\ & -2a_{26}u_o''\varphi'' - 2a_{27}u_o''\varphi' - 2a_{36}v_o''\,\varphi'' - 2a_{37}v_o''\,\varphi' + 2a_{67}\varphi''\varphi' \,]\, dz \end{aligned} \qquad (2.5.7a)$$

In matrix form, the potential energy can be represented as

$$V = \frac{1}{2}\int_0^L \begin{Bmatrix} w_o' \\ u_o'' \\ v_o'' \\ \varphi'' \\ \varphi' \end{Bmatrix}^T \begin{bmatrix} a_{11} & -a_{12} & -a_{13} & a_{16} & a_{17} \\ -a_{21} & a_{22} & a_{23} & -a_{26} & -a_{27} \\ -a_{31} & a_{32} & a_{33} & -a_{36} & -a_{37} \\ a_{61} & -a_{62} & -a_{63} & a_{66} & a_{67} \\ a_{71} & -a_{72} & -a_{73} & a_{76} & a_{77} \end{bmatrix} \begin{Bmatrix} w_o' \\ u_o'' \\ v_o'' \\ \varphi'' \\ \varphi' \end{Bmatrix} dz \qquad (2.5.7b)$$

2.6 Extended Hamilton's Principle

The boundary value problem, consisting of the governing systems and the boundary conditions, can be derived conveniently by means of the extended Hamilton's principle [44], which can be stated in the form

$$\begin{gathered} \int_{t_1}^{t_2} (\delta T - \delta V + \delta W)\, dt = 0, \\ \delta u_o = \delta v_o = \delta w_o = \delta\theta_x = \delta\theta_y = \delta\varphi = 0 \quad \text{at } t = t_1, t_2 \end{gathered} \qquad (2.6.1)$$

Herein T is the kinetic energy, V the potential energy, δW the virtual work of the nonconservative forces, which can be written as

$$\delta W = \int_0^L \left[\begin{aligned} & p_x(z,t)\delta u_o(z,t) + p_y(z,t)\delta v_o(z,t) + p_z(z,t)\delta w_o(z,t) \\ & + m_x(z,t)\delta\theta_x(z,t) + m_y(z,t)\delta\theta_y(z,t) + (m_z + b_w')\delta\varphi(z,t) \end{aligned}\right] dz$$

$$+\int_0^L \left(T_z^a \delta w_o' - M_y^a \delta\theta_y' - M_x^a \delta\theta_x' + B_w^a \varphi''\right) dz \tag{2.6.2}$$

where p_x, p_y and p_z denote the external force per unit length and m_x, m_y and m_z twist moment about x, y and z axis respectively, while b_w is the bimoment of the external loads. Their expressions can be found in Ref. 43. In addition, piezoelectrically induced terms denoted by superscript a are defined previously.

Carrying out the usual steps [44], we can write

$$\int_{t_o}^{t_1} \delta T dt = -\int_{t_o}^{t_1}\Bigg[\int_0^L \Big(b_1 \ddot{u}_o \delta u_o + b_1 \ddot{v}_o \delta v_o + b_1 \ddot{w}_o + (b_4 + b_{14})\ddot{\theta}_x \delta\theta_x$$

$$+(b_5 + b_{15})\ddot{\theta}_y \delta\theta_y - (b_{10} + b_{18})\ddot{\varphi}''\delta\varphi + (b_4 + b_5)\ddot{\varphi}\,\delta\varphi\Big) dz$$

$$-(b_{10} + b_{18})\ddot{\varphi}'\,\delta\varphi\,\Big|_0^L\ \Bigg] dt \tag{2.6.3}$$

$$\delta V = -\int_0^L \Big[T_z'\delta w_o + (M_y' - O_x)\delta\theta_y + (M_x' - Q_y)\delta\theta_x$$

$$+(B_w'' + M_z')\delta\varphi + Q_x'\delta u_o + Q_y'\delta v_o\Big] dz + \tag{2.6.4}$$

$$\Big[T_z \delta w_o + M_y \delta\theta_y + M_x \delta\theta_x - B_w \delta\varphi' + (B_w' + M_z)\delta\varphi + Q_x \delta u_o + Q_y \delta v_o\Big]\Big|_0^L$$

Substitution of Eqs. (2.6.2)-(2.6.4) into Eq. (2.6.1) yields the boundary-value problem for the most general case of anisotropy. In this sense we have:

a) the equations of motion expressed in terms of 1-D stress-resultants and stress-couple measures as [45]:

$$\begin{aligned}
&\delta u_o: \quad \underline{M_y''} + \underline{\underline{Q_x'}} - I_1 + p_x + \underline{m_y'} = 0 \\
&\delta v_o: \quad \underline{M_x''} + \underline{\underline{Q_y'}} - I_2 + p_y + \underline{m_x'} = 0 \\
&\delta w_o: \quad T_z' - I_3 + p_z = 0 \\
&\delta\varphi: \quad B_w'' + M_z' - (I_4 - I_9') + m_z + b_w' = 0 \\
&\underline{\underline{\delta\theta_x: \quad M_x' - Q_y - I_5 + m_x = 0}} \\
&\underline{\underline{\delta\theta_y: \quad M_y' - Q_x - I_7 + m_y = 0}}
\end{aligned} \tag{2.6.5 a-f}$$

b) the boundary conditions at the ends $(z = 0, L)$ for a cantilever beam:

$$
\begin{array}{lll}
 & \textit{The BCs at } z=0 & \textit{The BCs at } z=L \\
\delta u_o: & u_o=\bar{u}_o & \underline{M_y'}+\underline{\underline{Q_x}}=\bar{Q}_x \\
\delta v_o: & v_o=\bar{v}_o & \underline{M_x'}+\underline{\underline{Q_y}}=\bar{Q}_y \\
\delta w_o: & w_o=\bar{w}_o & T_z=\bar{T}_z \\
\delta\theta_y \text{ or } \delta u_o' & \underline{\underline{\theta_y=\bar{\theta}_y}} \text{ or } \underline{u_o'=\bar{u}_o'} & M_y=\bar{M}_y \\
\delta\theta_x \text{ or } \delta v_o' & \underline{\underline{\theta_x=\bar{\theta}_x}} \text{ or } \underline{v_o'=\bar{v}_o'} & M_x=\bar{M}_x \\
\delta\varphi': & \varphi=\bar{\varphi}' & \underline{B_w=\bar{B}_w} \\
\delta\varphi: & \varphi=\bar{\varphi} & \underline{B_w'}+M_z=\bar{M}_z-I_9
\end{array}
\qquad (2.6.6\ \text{a-g})
$$

In Eqs. (2.6.5) and (2.6.6), the single and double solid lines identify the terms pertaining to the infinitely rigid and flexible in transverse shear theories, respectively, while in Eq. (2.6.6) the overline sign affects the prescribed quantities. When the unshearable beam theory is considered, the terms indentified by the double solid lines have to be discarded. Conversely, when the shearable theory is adopted, the terms affected by a single solid line have to be dropped. The same convensions apply to the boundary conditions (BCs) as well. Moreover, the free warping counterpart of Eqs. (2.6.5) and (2.6.6) can be obtained by discarding the quantities underlined by the dotted line. In addition, $I_i\,(i=\overline{1,9})$ are the inertia terms whose expressions are displayed in the Appendix A.

2.7 The Governing System of Equations

The most convenient formulation of the governing system and of the associated BCs consists of their representation in terms of the kinematic quantities $u_o, v_o, w_o, \theta_x, \theta_y$ and φ. As in the theory of plates and shells, (see e.g. Librescu, 1975), this representation can be obtained by expressing the stress resultants and stress couples entering the equations of equilibrium motion in terms of the unknown kinematic variables [46]. From Eq. (2.6.5), the governing system of thin-walled beams which incorporates also transverse shear effect is expressed as

$$\delta w_o:\quad a_{11}w_o''+a_{12}\theta_y''+a_{13}\theta_x''+a_{14}(u_o''+\theta_y')+a_{15}(v_o''+\theta_x')+a_{16}\varphi'''+a_{17}\varphi''+p_z-T_z^{a\prime}=I_3$$

$$\begin{aligned}\delta\theta_y:\quad & a_{21}w_o''+a_{22}\theta_y''+a_{23}\theta_x''+a_{24}(u_o''+\theta_y')+a_{25}(v_o''+\theta_x')+a_{26}\varphi'''+a_{27}\varphi''-a_{41}w_o' \\ & -a_{42}\theta_y'-a_{43}\theta_x'-a_{44}(u_o'+\theta_y)-a_{45}(v_o'+\theta_x)-a_{46}\varphi''-a_{47}\varphi'+m_y-M_y^{a\prime}=I_7\end{aligned}$$

$$\begin{aligned}\delta\theta_x:\quad & a_{31}w_o''+a_{32}\theta_y''+a_{33}\theta_x''+a_{34}(u_o''+\theta_y')+a_{35}(v_o''+\theta_x')+a_{36}\varphi'''+a_{37}\varphi''-a_{51}w_o' \\ & -a_{52}\theta_y'-a_{53}\theta_x'-a_{54}(u_o'+\theta_y)-a_{55}(v_o'+\theta_x)-a_{56}\varphi''-a_{57}\varphi'+m_z-M_x^{a\prime}=I_5\end{aligned}$$

$$\delta u_o:\quad a_{41}w_o''+a_{42}\theta_y''+a_{43}\theta_x''+a_{44}(u_o''+\theta_y')+a_{45}(v_o''+\theta_x')+a_{46}\varphi'''+a_{47}\varphi''+p_x-Q_x^{a\prime}=I_1$$

$$\delta v_o:\quad a_{51}w_o''+a_{52}\theta_y''+a_{53}\theta_x''+a_{54}(u_o''+\theta_y')+a_{55}(v_o''+\theta_x')+a_{56}\varphi'''+a_{57}\varphi''+p_y-Q_y^{a\prime}=I_2$$

$$\begin{aligned}\delta\varphi:\ & -a_{61}w_o'''-a_{62}\theta_y'''-a_{63}\theta_x'''-a_{64}(u_o'''+\theta_y'')-a_{65}(v_o'''+\theta_x'')-a_{66}\varphi''''-a_{67}\varphi'''+a_{71}w_o'' \\ & +a_{72}\theta_y''+a_{73}\theta_x''+a_{74}(u_o''+\theta_y')+a_{75}(v_o''+\theta_x')+a_{76}\varphi'''+a_{77}\varphi''+m_z-B_w^{a\prime\prime}=I_4-I_9'\end{aligned}$$

(2.7.1 a-f)

For cantilevered TWBs, the BCs expressed in terms of kinematic variables:

at the root of the beam are:

$$w_o = \theta_y = \theta_x = u_o = v_o = \varphi' = \varphi = 0 \qquad (2.7.2\text{ a-g})$$

and at the tip are:

$$\delta w_o : a_{11} w_o' + a_{12}\theta_y' + a_{13}\theta_x' + a_{14}(u_o' + \theta_y) + a_{15}(v_o' + \theta_x) + a_{16}\varphi'' + a_{17}\varphi' = 0$$

$$\delta\theta_y : a_{21} w_o' + a_{22}\theta_y' + a_{23}\theta_x' + a_{24}(u_o' + \theta_y) + a_{25}(v_o' + \theta_x) + a_{26}\varphi'' + a_{27}\varphi' = 0$$

$$\delta\theta_x : a_{31} w_o' + a_{32}\theta_y' + a_{33}\theta_x' + a_{34}(u_o' + \theta_y) + a_{35}(v_o' + \theta_x) + a_{36}\varphi'' + a_{37}\varphi' = 0$$

$$\delta u_o : a_{41} w_o' + a_{42}\theta_y' + a_{43}\theta_x' + a_{44}(u_o' + \theta_y) + a_{45}(v_o' + \theta_x) + a_{46}\varphi'' + a_{47}\varphi' = 0$$

$$\delta v_o : a_{51} w_o' + a_{52}\theta_y' + a_{53}\theta_x' + a_{54}(u_o' + \theta_y) + a_{55}(v_o' + \theta_x) + a_{56}\varphi'' + a_{57}\varphi' = 0$$

$$\delta\varphi' : a_{61} w_o' + a_{62}\theta_y' + a_{63}\theta_x' + a_{64}(u_o' + \theta_y) + a_{65}(v_o' + \theta_x) + a_{66}\varphi'' + a_{67}\varphi' = -I_9$$

$$\begin{aligned}\delta\varphi : & -a_{61} w_o'' - a_{62}\theta_y'' - a_{63}\theta_x'' - a_{64}(\theta_y' + u_o'') - a_{65}(\theta_x' + v_o'') - a_{66}\varphi''' - a_{67}\varphi'' + \\ & a_{71} w_o' + a_{72}\theta_y' + a_{73}\theta_x' + a_{74}(\theta_y + u_o') + a_{75}(\theta_x + v_o') + a_{76}\varphi'' + a_{77}\varphi' = 0\end{aligned}$$

(2.7.3 a-g)

The meaning of the quantities intervening in these equations has been already defined.

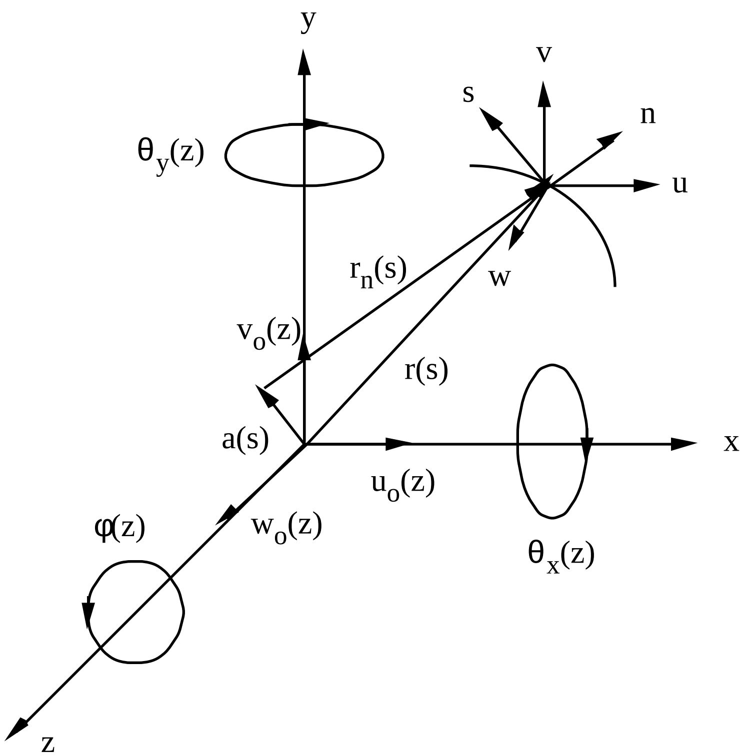

Fig. 2.1 Displacement field for the thin-walled beam

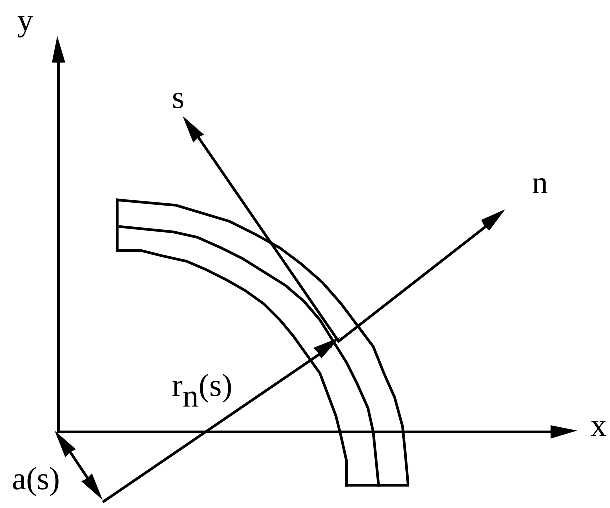

Fig. 2.2 Configuration of a cross-section of a thin-walled beam

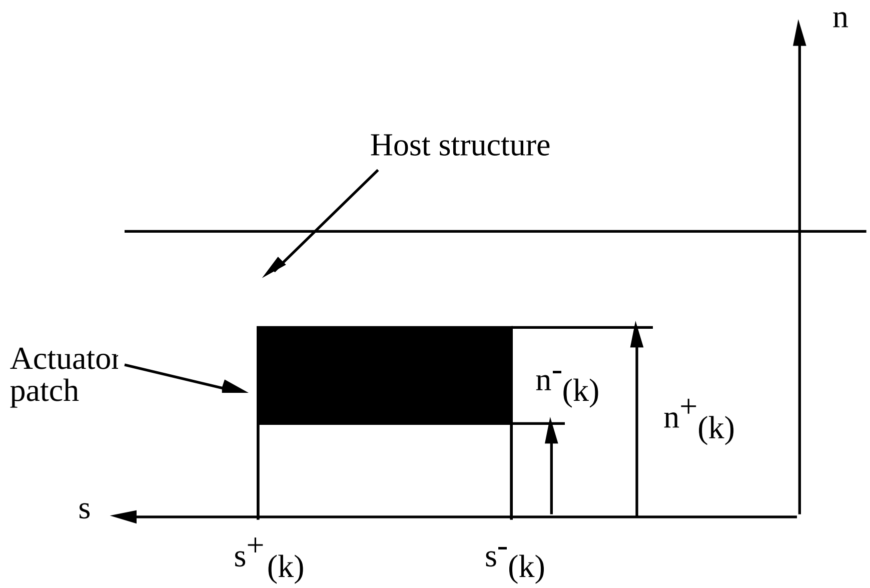

Fig. 2.3 Piezoactuator patch distribution

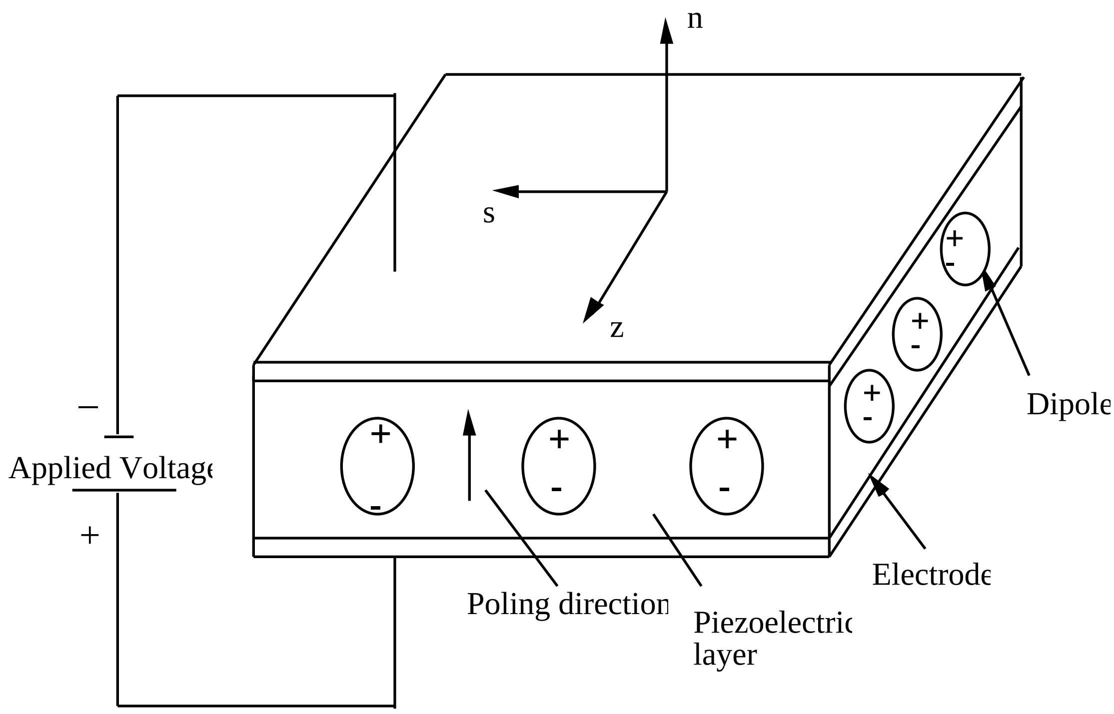

Fig. 2.4 Piezoelectric layer Nomenclature