Chapter I

Introduction

1.1 Problem Description

In a variety of practical situations, a decision maker faces a problem of how to optimally serve a set of existing facilities that have fixed and known locations. The generalization of this problem has been first formally posed by Cooper (1964) as: “Given the location of each destination, its associated demand and a set of shipping costs for the region of interest, determine the number, location, and capacities of sources so as to minimize the total location and shipping cost”. However, as different practical situations induce the elements of the above generalization to diversify, several classes of problems arise. When the allocation variables are fixed, the above problem reduces to the class of pure location problems; otherwise, it is known as the location-allocation problem. The particular location-allocation problem discussed in this research is a class of mathematical programs that seeks a least-cost solution to the task of siting a set of n centers or facilities so as to best serve a set of m demand points, and simultaneously, finding the corresponding optimal allocation of service from the new facilities to the demand points, where the cost is directly proportional to the quantity shipped and the Euclidean distance separating the source and the destination (Cooper, 1963 and Ostresh, 1975). A common application of this class of problems involves the location of central warehouses that must receive products from manufacturing facilities and distribute them to various branch warehouses.

As noted, in this research, we will focus on solving a specific type of location and location-allocation problems that restrict using the Euclidean norm for representing the distance measured between the facilities. These problems are the **Euclidean distance multifacility location problem** and the **Euclidean distance capacitated location-allocation problem.** The following assumptions are considered for both problems.

(A1) A plane is considered to be a valid approximation of the region defining the problems.

(A2) The data defining the problems are deterministic (or deterministic equivalent data are employed).

(A3) The cost function is represented by the sum of weighted distances between every pair of interacting facilities.

(A4) Any point in the plane is permissible as a location for any source (there are no forbidden areas).

(A5) Any facility location is a point in the plane (has no area).

(A6) The number of new facilities is fixed, and subsequently the fixed cost associated with the installation of a source is ignored.

(A7) The Euclidean metric is used to measure distances between the facilities.

1.1.a The Euclidean Multifacility Location Problem.

The Euclidean Multifacility Location Problem (EMFLP) seeks to locate n new facilities at some points (x_i, y_i), $i = 1,.., n$ in R^2, in the presence of some m existing facilities located at specified points (a_j, b_j), $j = 1,.., m$, given certain interaction weights $w_{ij} > 0$ between designated pairs (i, j) of new and existing facilities in some index-pair set A_{NE}, as well as certain interaction weights $v_{kl} > 0$ between designated pairs (k, l), $k < l$, of new facilities themselves in some index-pair set A_{NN}. The cost for each pair of interacting facilities is assumed to be directly proportional to the Euclidean distance that separates these facilities. Each new facility is assumed to have unlimited production capacity. This problem may be mathematically stated as follows: (Note that although we consider this problem on a plane, a direct extension of our analysis to a $d > 2$-dimensional space is readily evident.)

EMFLP: Minimize

$$f(x, y) \equiv \sum\sum_{(i,j)\in A_{NE}} w_{ij}\{(x_i - a_j)^2 + (y_i - b_j)^2\}^{1/2} + \sum\sum_{(k,l)\in A_{NN}} v_{kl}\{(x_k - x_l)^2 + (y_k - y_l)^2\}^{1/2}. \tag{1.1}$$

The objective function of EMFLP is convex since it is the sum of norms that are convex functions. However, Francis and Cabot (1972) have proven that a necessary and sufficient condition for the objective function to be strictly convex is that for each new facility i, the set $S_i = \{ j : w_{ij} > 0\}$ is nonempty and that the location of the points in S_i are non-collinear. Note that this convexity property asserts that any local optimum is a global optimum. As is well known (see Francis *et al*. 1983), the optimal solution of EMFLP problem exists and lies in the convex hull of the existing facilities, and therefore, this optimal solution can be expressed as the convex combination of the existing facilities. These foregoing properties are very essential in designing algorithmic approaches for solving EMFLP.

In order to present our motivation for studying Problem EMFLP, we first need to consider one important property of this problem regarding the derivative of the objective function of (1.1) which can written as follows (when it exists):

$$\frac{\partial f}{\partial x_i} = \sum_{j=1}^{m} w_{ij}\,(x_i - a_j) / [(x_i - a_j)^2 + (y_i - b_j)^2]^{1/2} + \sum_{\substack{k=1\\k\neq i}}^{n} v_{ik}\,(x_i - x_k)/[(x_i - x_k)^2 + (y_i - y_k)^2]^{1/2} \quad \forall\, i = 1,\ldots, n \qquad (1.2)$$

$$\frac{\partial f}{\partial y_i} = \sum_{j=1}^{m} w_{ij}\,(y_i - a_j) / [(y_i - a_j)^2 + (y_i - b_j)^2]^{1/2} + \sum_{\substack{k=1\\k\neq i}}^{n} v_{ik}(y_i - y_k) / [(x_i - x_k)^2 + (y_i - y_k)^2]^{1/2} \quad \forall\, i = 1,\ldots, n. \qquad (1.3)$$

As we can see from expressions (1.2) and (1.3), the objective function has undefined first partial derivatives whenever any new facility coincides with either an existing or another new facility with which it interacts, and because of this feature, standard differentiable optimization techniques cannot be directly applied. The most famous approach that has been developed to mask this nondifferentiability is the hyperboloid approximation procedure (HAP) of Eyster *et al*.(1973) in which a smooth hyperboloid is employed to approximate the objective function in order to induce differentiability, and an iterative fixed-point descent scheme is developed to generate a sequence of solutions that converge to an optimal solution for EMFLP. Unfortunately, it is well known that the HAP approach suffers from ill-conditioning effects if the point of convergence is nondifferentiable (Charalambous , 1985).

Motivated by the efforts to overcome this nondifferentiability issue, we develop two equivalent differentiable, convex, constrained reformulations of EMFLP. The first of these is constructed directly in the primal space and is amenable to any differentiable nonlinear optimization technique. We establish the relationship between the Karush-Khun-Tucker (KKT) conditions for this problem and the necessary and sufficient optimality conditions for EMFLP, in order to explore the possible performance of standard differentiable nonlinear programming algorithms.

The second equivalent differentiable formulation is derived via a Lagrangian dual approach based on the optimum of a linear function over a unit ball, and recovers Francis and Cabot's (1972) dual formulation. The good feature of this formulation is that, as we show, primal locational decisions can be recovered via this dual problem. This settles an issue remained open in the literature since 1972 regarding the recovery of primal locations via this dual problem. Some encouraging computational results are also presented.

In another effort to avoid the nondifferentiablitity difficulty associated with Problems EMFLP, we develop an algorithmic approach that uses conjugate or deflected subgradient based methods along with suitable line-search strategies. The subgradient deflection method considered is the Average Direction Strategy (ADS) of Sherali and Ulular (1989) imbedded within the Variable Target Value Method (VTVM) of Sherali *et al.*(1995). We selected the VTVM algorithmic in our approach for solving EMFLP since it is highly adaptable in permitting the special design of both a strategy for computing step-lengths, as well as a strategy for employing a subgradient deflection scheme, while preserving convergence properties. However, we modify the restarting and the termination criteria prescribed by this procedure in order to improve its convergence behavior for the class of problems EMFLP. Within this modified framework, we investigate the generation of two types of subgradients to be employed in conjunction with the ADS deflection strategy. The first subgradient generation method , assigns zero values to contributions from the nondifferentiable terms in the objective function. The second method derives a low-norm subgradient based on using valid contributions other than zeros for the nondifferentiable terms in order to attain improved search directions. We also experimented with the use of an alternative step-size strategy known as the block-halving scheme, due to Sherali and Ulular (1989), within the VTVM, and we denote this by VTVM + BH. In addition, a Newton-based inexact line-search is developed and tested in conjunction with both step-size strategies in order to observe its impact on the convergence. The results show that the modified VTVM along with a certain combination of the two subgradient generation strategies and the use of a suitable inexact line-search technique provides an effective solution procedure in comparison with alternative solution techniques. Furthermore, the VTVM+BH in conjunction with the proposed line-search method yields a competitive performance.

1.1b The Capacitated Location-Allocation Problem

Given the fixed location of m existing facilities or customers on a continuous plane and their associated demands, the Euclidean location-allocation problem seeks to determine the location of n new facilities or sources with known capacities and the allocation of their supply in order to satisfy the demand requirements of the customer at a minimum total cost. The decisions are where to locate the n sources and how much shipment to send from each source to each customer. For convenience, we assume that the total supply is equal to the total demand. As before, the Euclidean metric is used to measure distances. This problem can be mathematically stated as follows.

EDLAP: Minimize $f(x,w) \equiv \sum_{i=1}^{n}\sum_{j=1}^{m} c_{ij}\, w_{ij}\, \{(x_i - a_j)^2 + (y_i - b_j)^2\}^{1/2}$

$$\text{subject to} \quad \sum_{j=1}^{m} w_{ij} = s_i \qquad i = 1,\ldots,n$$

$$\sum_{i=1}^{n} w_{ij} = d_j \qquad j = 1,\ldots,m$$

$$w_{ij} \geq 0 \qquad i = 1,\ldots,n; \;\; j = 1,\ldots,m$$

where m = the number of customers,

n = the number of supply centers,

X_i = $(x_i, y_i) \equiv$ location of supply center i,

P_j = $(a_j, b_j) \equiv$ location of the customer j,

s_i = (annual) capacity of supply center *i*,

d_j = (annual) demand of customer *j*,

c_{ij} = cost of unit flow per unit distance from supply center *i* to customer *j*.

and where the decision variables are:

(x_i, y_i) : location of source *i*,

w_{ij}: amount shipped from supply center *i* to customer *j*.

For convenience, let us also denote $W = \{w \equiv w_{ij} : w$ satisfies the (transportation) constraints of problem EDLAP$\}$.

The above problem is the **Capacitated Location-Allocation Problem** (EDLAP). Depending on which type of decision variable is fixed, two classes of problems result. For a fixed set of allocations $w \equiv (w_{ij},\ i = 1,..., n\ ;\ j = 1\ ,...,m)$, the problem EDLAP reduces to a pure location problem in which the sources are not interacting, whereas for a fixed set of locations (x_i,y_i), $i = 1,..., n$, the problem reduces to the ordinary transportation/ allocation problem.

As shown in Sherali and Nordai (1988), the problem EDLAP is NP hard even if all demand points are located on a straight line. Its objective function is nonconvex which results in a multiple local minima, and moreover, optimal flow solution occurs at an extreme point of the set of the feasible solutions *W* (Cooper, 1972), while the location of the sources lies in the convex hull of the destinations' location (Wendell and Hurter, 1973).

To demonstrate another difficulty associated with the solving the problem EDLAP, we consider the following expressions for the first partial derivatives (when they exists) of its objective function with respect to w_{ij}, x_i, and y_i, respectively:

$$\frac{\partial f}{\partial w_{ij}} = c_{ij}\ [(x_i - a_j)^2 + (y_i - b_j)^2]^{1/2} \qquad i = 1, ..., n,\ \ j = 1,.., m, \qquad (1.4)$$

$$\frac{\partial f}{\partial x_i} = \sum_{j=1}^{m} c_{ij}\ w_{ij}(x_i - a_j) / [(x_i - a_j)^2 + (y_i - b_j)^2]^{1/2} \qquad \forall\ i = 1, ..., n \qquad (1.5)$$

$$\frac{\partial f}{\partial y_i} = \sum_{j=1}^{m} c_{ij}\, w_{ij}(y_i - a_j) / \left[(y_i - a_j)^2 + (y_i - b_j)^2\right]^{1/2} \qquad \forall\, i = 1, ..., n. \qquad (1.6)$$

We observe from expression (1.5) and (1.6) that the objective function f has undefined partial derivatives $\frac{\partial f}{\partial x_i}$ and $\frac{\partial f}{\partial y_i}$ when the location of the supply source matches the location of any of the demand points. Therefore, this nondifferentiability issue precludes researchers from using gradient based optimization methods for solving the problem.

We propose an approach to solve the above capacitated location-allocation problem using a branch-and-bound procedure that uses the Reformulation-Linearization Technique (RLT) developed by Sherali and Tuncbilek (1992) for computing bounds. This method is comprised of two phases, namely, the Reformulation Phase and the Linearization Phase. In the Reformulation Phase, variable factors (of the form [variable - its lower bound] and [upper bound - the corresponding variable]) are used to multiply constraints as well as constraints are used to multiply constraints resulting in the generation of new implied constraints. In the Linearization Phase, an appropriate variable substitution strategy is used in linearizing the problem. However, certain (differentiable) convex constraints are retained in this problem and are handled via a specialized Lagrangian relaxation/dual approach. This process transforms the representation of the nonconvex EDLAP from the original defining space into a higher dimensional space associated with a lower bounding convex (but largely linear) program, henceforth referred to as RLT(Ω) based on hyper rectangle Ω that bounds the flows. This problem RLT(Ω) approximates the closure convex hull of feasible solutions to the nonconvex EDLAP, when its objective function is accommodated into the constraints. The lower bounding facility via problem RLT(Ω) is then imbedded in a branch-and-bound algorithm, where the partitioning is performed by altering the bounds on the allocation variables. Since the original allocation constraints and bounds (which have been possibly further restricted) of problem EDLAP have been included, a feasible solution that yields an upper bound on the original objective function value is available via the solution to any relaxation RLT(Ω), by solving the accompanying pure Euclidean distance location problem. An alternative lower bounding scheme based on a projection of the objective function onto the location space is also developed. In a hybrid approach, the maximum of the two lower bounds thus developed is used. The branch-and-bound process is continued until it converges to an optimal solution.

1.2 Organization of Dissertation.

In the next chapter, we review the literature on location and location-allocation problems, as well as present a brief review on the Reformulation-Linearization Technique. Chapter 3 derives two equivalent differentiable reformulations of the Euclidean distance location problem, and presents some computational experience on solving these reformulations using standard differentiable convex optimization software. Chapter 4 develops a conjugate subgradient approach for solving the Euclidean location problem and presents some related computational experience. Chapter 5 deals with the global optimization of the Euclidean distance capacitated location-allocation problem and designs a convex lower bounding relaxation for this problem. This is embedded within a branch-and-bound algorithm along with a suitable partitioning scheme that includes finite configuration to an optimal solution for this problem. Some computational experience is also presented for this procedure. Finally, Chapter 6 summarizes our contributions and suggests recommendations for further research.