Dimensionally Compatible System of Equations for Tree and Stand Volume, Basal Area, and Growth

By

Mahadev Sharma

Dissertation submitted to the Faculty of the

Virginia Polytechnic Institute and State University

in partial fulfillment of the requirement for the degree of

Doctor of Philosophy

in

Forestry

APPROVED:

Richard G. Oderwald, Chairman

Marion R. Reynolds, Jr.

Jay Sullivan

Wallace M. Aust

Ralph Amateis

December, 1999

Blacksburg, Virginia

Keywords: Nonlinear least squares regression, mathematically/algebraically consistent, Quadratic Mean Diameter, Lorey Mean Height, inflection points, tree profile

Dimensionally Compatible System of Equations for Tree and Stand Volume, Basal Area, and Growth

Mahadev Sharma

(ABSTRACT)

A dimensionally compatible system of equations for stand basal area, volume, and basal area and volume growth was derived using dimensional analysis. These equations are analytically and numerically consistent with dimensionally compatible individual tree volume and taper equations and share parameters with them. Parameters for the system can be estimated by fitting individual tree taper and volume equations or by fitting stand level basal area and volume equations. In either case the parameters are nearly identical. Therefore, parameters for the system can be estimated at the tree or stand level without changing the results.

Data from a thinning study in loblolly pine (*Pinus taeda L.*) plantations established on cutover site-prepared lands were used to estimate the parameters. However, the developed system of equations is general and can be applied to other tree species in other locales.

This work is dedicated to my late mother,

HARIKALA DEVI SHARMA

And my father,

DEVI PRASAD SHARMA

ACKNOWLEDGEMENTS

I wish to sincerely thank my advisor, Dr. Richard G. Oderwald for his guidance, support throughout my graduate education in the Department of Forestry at Virginia Tech. I thank him for providing me with such worthwhile opportunities. I also wish to thank the members of my advisory and examination committees, namely Dr. Timothy G. Gregoire, Dr. Marion R. Reynolds, Dr. Jay Sullivan, and Dr. Wallace M. Aust for their time and assistance to this endeavor. Special thanks also go to Ralph Amateis for his thoughtful suggestions and for serving in my committee after Dr. Gregoire left Virginia Tech.

I am grateful to Dr. A. L. Hammett who helped me a lot by providing material for class projects and other assistance during my stay in Blacksburg. I am also grateful to Dr. H. E. Burkhart and Dr. Gregory J. Buhyoff who were always concerned about my study and progress.

I am eternally grateful to the faculty, staff, and students in the Department of Forestry, the College of Forestry and Wildlife Resources, and the Department of Statistics for providing me the best environment and knowledge to pursue a graduate program successfully.

Finally, I wish to acknowledge the Loblolly Pine Growth and Research Cooperative at Virginia Tech for providing the data.

CONTENTS

LIST OF TABLES

LIST OF FIGURES

Chapter 1

INTRODUCTION

Efficient and effective management of timber resources depends on accurate determination and projection of forest stands. The goal of timber management is to provide the mix of timber quality, quantity, and size that will maximize owner satisfaction while meeting the imposed constraints (Newberry 1984). Consequently, accurate and flexible methods are required to estimate stand growth and yield for evaluating the numerous management and optimum utilization alternatives for timber resources.

Researchers have developed numerous models for predicting growth and yield accounting for morphological and intra-specific differences species caused by varying physiographic, climatic, and other environmental effects. These models have been steadily increasing in their complexity and detail from the earliest empirical yield tables and yield equations to diameter-distribution models, individual-tree models (distance dependent and distance independent), and physiological process models due to many factors including more refined statistical estimation techniques and the increasing availability of computers. Forest biometricians have continually made efforts at developing and refining these models for the enhancement and flexibility of their uses.

Since the introduction of regression methods into forestry, it has been common practice to select models for growth and yield based on diagnostic and model selection criteria, and to estimate equation coefficients for a specific data set. Mathematical properties such as derivatives, integrals, inflection points, and other mathematically derivable implications have been applied for the improvement of model specification and for algebraic and numerical compatibility of growth and yield systems. Since these models have not been based on theoretical information, there have been problems with compatibility in the regression method of coefficient estimation. This is because of the least squares principle of regression optimizes coefficients only for the dependent variable under consideration.

For example, if coefficients are estimated for a volume growth equation and a volume yield equation based on the growth model, coefficient values in the two equations will differ because of the difference in the sum of squares being minimized (Hans 1986). Therefore, while the two equations are algebraically compatible, they are not numerically compatible. The coefficients of two equations will not be numerically equivalent unless both of them have the same theoretical basis.

Models developed in this fashion are most often validated using statistical procedures that compare predictions from the model with data. The validity of the models is established only for the situations corresponding to the validation data. In some circumstances model validation may require additional data and data for model development and validation may not be available across the entire range of situations for which application is desired (Oderwald and Hans 1993).

In physics, dimensionality is considered to be the primary checking point for the validity of an equation. As a result, the dimension of left-hand side of an equation must be equal to the dimension of right hand side of that equation. In forestry, however, very few researchers have been found to have considered this fact in their model development. Sometimes the volume of a stand measured in cubic-foot or cubic-meter is expressed in terms of time (stand age) which is very unrealistic in the context of dimensionality.

Stands are made up of individual trees. The volume of an individual tree consists of the volumes of sections (logs) starting from the stump to the tip of the tree. The sum of tree volumes and basal areas should equal stand volume and stand basal area respectively. Further, tree volume is the integral of tree taper. Therefore, it is natural to assume a theoretical relationship between individual tree taper, individual tree volume, and stand volume.

A system of equations is called algebraically/mathematically compatible if one equation in the system can be expressed in terms of other equation(s). For example, the stand volume can be expressed mathematically in terms of individual tree volume obtained from the taper equation, and the basal area of the stand can be expressed in terms of individual tree basal areas. Therefore, the system of equations consisting of the taper equation, the individual tree volume equation, and the stand volume equation is algebraically/mathematically compatible. Furthermore, the system is consistent if it is numerically compatible. For example, if the estimates of the same parameter obtained independently from the taper equation, individual tree volume equation, and the stand volume equation are not significantly different from one another the system is consistent.

A consistent and mathematically and dimensionally compatible system of equations consisting of taper, individual tree volume, individual tree basal area, stand volume, and stand basal area equations for commercial tree species would be a very useful and flexible tool for both forestry practice and forest research. These equations could provide estimates of stem diameter at any height and estimates of total stem volume as well as merchantable volume at any stem diameter along the bole. In addition, they provide the volume, basal area, and growth of the stand from one point to another.

A system of dimensionally compatible volume equation for a tree and a taper equation algebraically and numerically consistent with the dimensionally compatible volume equation was developed in the previous study (MS Thesis) by this author (Sharma, 1995). Therefore, the objectives of this study are to:

(i) Develop a dimensionally compatible stand volume equation that is algebraically and numerically consistent with the individual tree equation;

(ii) Derive a stand basal area growth equation; and

(iii) Derive volume growth and yield equations consistent with the stand volume and basal area equations.

The organization of this dissertation is as follows. Chapter 2 contains a review of present state of the stand volume and basal area growth and yield modeling. Chapter 3 provides a brief theory of dimensional analysis and derives stand basal area, volume, and growth equations based on dimensional analysis. A description of the data used herein and the methods of parameter estimation and evaluation appear in Chapter 4. Chapter 5 summarizes the findings of this study. Finally, Chapter 6 brings the dissertation to the end with conclusions.

Chapter 2

LITERATURE REVIEW

2.1 Stand Modeling

The development of quantitative growth and yield models dates back to the pioneering work of MacKinney and Chaiken (1939) whose multiple regression methods were applied to the problem of variable density yield estimation. Since that time a variety of approaches have been taken to growth and yield estimation, each arising from the need for specific types of information. Available yield models reflect different silvicultural practices, modeling philosophies, and levels of mathematical complexity (Clutter et al. 1984).

In these models, the growth of a stand is normally driven by certain measurable variables, dbh, height, basal area, site index, age etc. Based on their structural complexity and output detail, these models are further divided into three broad categories: (i) whole stand models, (ii) size class distribution models, and (iii) individual tree models (Daniels and Burkhart 1988).

2.1.1 Whole Stand Models

Since Mackinney and Chaiken (1939), a number of investigators have used multiple regression techniques to construct stand aggregate growth and/or yield expressions (e.g., Bennett 1970, Bennett et al. 1959, Brender and Clutter 1970, Burkhart et al. 1972a, 1972b, Clutter 1963, Coile and Schumacher 1964, Goebel and Shipman 1964, Schumacher and Coile 1960, Sullivan and Clutter 1972). Whole stand growth and/or yield estimates can be calculated from these models in terms of stand level attributes such as age, density, and site index.

Clutter (1963) was the first researcher introducing the notion of compatibility in growth and yield equations. He defined compatible growth and yield models as those in which yield was obtained through mathematical integration of growth. Noncompatible growth and yield models, on the other hand, produce different values for yield and cumulative growth. He also noted that growth and yield observations should not be considered as independent phenomena but be analyzed as mathematically dependent quantities. He started with the following two equations;

Volume and yield equation

$$Ln(V) = b_0 + b_1 S + b_2 Ln(B) + b_3 A_1^{-1} \text{ and} \qquad (2.1.1)$$

Basal area growth rate equation

$$\frac{\partial B}{\partial A} = -B\,Ln(B)\,A^{-1} + c_0\,B\,A^{-1} + c_1\,B\,S\,A^{-1} \qquad (2.1.2)$$

where,

B = Basal area per acre,

S = Site index,

A = Age of the stand,

V = Total cubic-foot volume, and

b_0, b_1, b_2, b_3, c_0, and c_1 are coefficients which were estimated using linear regression methods. For the growth rate equation of volume, he differentiated volume yield equation and substituted $\frac{\partial B}{\partial A}$ from the second equation. His final volume growth equation was;

$$\frac{\partial V}{\partial A} = -b_2\,V\,Ln(B)\,A^{-1} + b_2\,c_0\,V\,A^{-1} + b_2\,c_1\,V\,S\,A^{-1} - b_3\,V\,A^{-2} \qquad (2.1.3)$$

This equation was also fitted using linear regression methods. Integration of both basal area and volume growth rate equations from initial to predicted age and from initial basal area/volume to predicted basal area/volume and solving for basal area/volume resulted in basal area and volume prediction equations.

But the estimates of the coefficients obtained by fitting regressions for different equations independently at different stages gave different estimates of the coefficients. This resulted in a system that was not only numerically inconsistent, but was algebraically inconsistent as well.

Sullivan and Clutter (1972) later refined the compatible models to provide numerically consistent growth and yield models by noting that the coefficients for Clutter's set of equations were not independent between the equations. Their refined equations for volume and basal area were;

$$Ln(V_1) = b_0 + b_1 S + b_2 A_1^{-1} + b_3 Ln(B_1) \tag{2.1.4}$$

$$Ln(V_2) = b_0 + b_1 S + b_2 A_2^{-1} + b_3 Ln(B_2) \tag{2.1.5}$$

$$Ln(B_2) = \left(\frac{A_1}{A_2}\right) Ln(B_1) + a_1 \left(1 - \frac{A_1}{A_2}\right) + a_2 \left(1 - \frac{A_1}{A_2}\right) S \tag{2.1.6}$$

where,

A_1 = Initial age,

A_2 = Future age,

B_1 = Basal area at age A_1

B_2 = Basal area at age A_2,

V_1 = Volume at age A_1, and

V_2 = Volume at age A_2.

Substituting $Ln(V_2)$ from equation (2.1.6) into equation (2.1.5), a projected cubic-foot volume equation can be written as;

$$Ln(V_2) = b_0 + b_1 S + b_2 A_2^{-1} + b_3 \left(\frac{A_1}{A_2}\right) Ln(B_1) + b_4 \left(1 - \frac{A_1}{A_2}\right) + b_5 \left(1 - \frac{A_1}{A_2}\right) S \tag{2.1.7}$$

It should be noted here that when $A_1 = A_2$, the equation collapses to the yield equation. This implies, therefore, that the equation is simultaneously a yield model and projection model. For the algebraical and numerical consistency, we must have;

$$b_4 = b_3\, a_1 \text{ and } b_5 = b_3\, a_2 \tag{2.1.8}$$

and the estimates of α_1 and α_2 should be obtained using these relationships. Although, these estimates for α_1 and α_2 are required for the system to be numerically consistent, these are not the most statistically efficient estimators. These estimates depend on the volume units and merchantability limits chosen for the dependent variables. Bailey and Ware (1983) modified the basal area growth equation and used it for estimating the basal area growth for thinned and unthinned stands.

Furnival and Wilson (1971) also noted that growth and yield models can be considered as systems of related equations and suggested using multistage regression techniques for fitting such systems. Many investigators have presented applications and theory of such parameter estimation techniques in various growth and yield modeling situations (Murphy and Sternitzke 1979, Murphy and Beltz 1981, Amateis et al. 1984, Borders and Bailey 1986, Gregoire 1987, Reed 1987, Borders 1989, Cao and Durand 1991).

Burkhart and Sprinz (1984) suggested another way of estimating the coefficients. In fitting the volume growth rate and basal area equations (2.1.6) and (2.1.7) they gave equal weights for volume and basal area. In another words, they minimized;

$$F = \frac{\sum_{i=1}^{n}(Y_i - \hat{Y}_i)^2}{\hat{s}_y^2} + \frac{\sum_{i=1}^{n}(B_i - \hat{B}_i)^2}{\hat{s}_B^2} \tag{2.1.9}$$

where, Y_i and B_i are the observed values for volume and basal area and $\hat{Y}_i$ and $\hat{B}_i$ are the predicted values. $\hat{s}_y^2$ and $\hat{s}_B^2$ were estimated using the mean square error from the least square fits of the volume and basal area equations, respectively. By imposing the relationships (2.1.8), they found that the coefficients for the basal area equations were almost identical regardless of merchantability definition used in the yield equation and equations that were numerically consistent.

Curtis (1967) also used the notion of compatibility that the growth rate expressed as a differential equation and the yield equation equaled to the integral of the growth rate. His growth rate equation was of the form;

$$\frac{\partial Y}{\partial A} = f(A)\, f(S)\, f(RD) \tag{2.1.10}$$

where,

A = age,

S = site index,

RD = relative density, and

Y = gross volume or basal area.

A gross volume equation was obtained by integrating the growth rate equation. After comparing growth rate and yield from permanent and temporary plots, he found that the satisfactory estimates could be obtained from the temporary plots.

The growth rate equation proposed by Curtis (1967) involves stand age. This equation, therefore, can not be applied to uneven-aged stands. Moser and Hall (1969) proposed a theoretical growth rate equation independent of stand age as;

$$\frac{\partial Y}{\partial T} = f(y) \tag{2.1.11}$$

where,

Y = cumulative growth and T = elapsed time.

Volume (V) was then expressed as a function of basal area (B) as;

$$V = b_0 B^{b_1} \tag{2.1.12}$$

where,

b_0 and b_1 are regression coefficients.

Differentiating this equation with respect to time results in the growth rate equation as;

$$\frac{\partial V}{\partial T} = b_1 V B^{-1} \frac{\partial B}{\partial T} \tag{2.1.13}$$

They used von Bertalanffy's generalized growth equation;

$$\frac{\partial B}{\partial T} = n B^m - k B \tag{2.1.14}$$

to obtain the final growth rate equation as;

$$\frac{\partial V}{\partial T} = b_1 V \left(n B^{m-1} - k \right) \tag{2.1.15}$$

These equations were then fitted using regression analysis techniques. The volume and basal area yield functions were then found by solving the respective growth rate equations for volume and basal area. This article provided a methodology for deriving time dependent yield functions from growth rate equations that do not have time or age as an independent variable.

Moser and Hall (1969) showed that solution to differential equations expressed as a function of dependent variables provided time-dependent yield functions for uneven aged stands. Later, Moser (1972) alone described the components of net growth for number of trees and basal area as a system of first-order ordinary differential equations for an uneven-aged forest stand. His equations were;

$$\frac{\partial Y_1}{\partial t} = -\frac{\partial Y_3}{\partial t} + \frac{\partial Y_4}{\partial t} \quad \text{and} \tag{2.1.16}$$

$$\frac{\partial Y_2}{\partial t} = -\frac{\partial Y_5}{\partial t} + \frac{\partial Y_6}{\partial t} + \frac{\partial Y_7}{\partial t} \tag{2.1.17}$$

where,

Y_1 = number of trees $\geq$ 7.0 inches present at a given instant,

Y_2 = basal area of the Y_1 trees,

Y_3 = number of mortality trees $\geq$ 7.0 inches in the interval $[t_0, t_n]$,

Y_4 = number of ingrowth trees in the interval $[t_0, t_n]$,

Y_5 = basal area of the Y_3 mortality trees,

Y_6 = basal area of the Y_4 ingrowth trees,

Y_7 = cumulative basal area growth of the Y_1 trees in the interval $[t_0, t_n]$.

He developed other models for each differential term in equations (2.1.16) and (2.1.17) and estimated the coefficients of each model by linear and nonlinear regression methods. His models were different from the even-aged stand models in that (i) the growth components were individually identified by governing equations, (ii) the future stand conditions were predicted by a simultaneous solution of the system components, and (iii) the right hand sides of the governing equations were expressed solely as a function of the dependent variables.

Pienaar and Turnbull (1973) also considered the theoretical growth-rate approach to express the growth and yield of a forest stand. They used Chapman-Richards generalization of von Bertalanffy's growth model, so that the parameters in the model can be physiologically or biologically interpreted. This model was formulated by von Bertalanffy (1957) by the reasoning that the growth of an organism is "given by the difference between the processes of building up and breaking down". The Chapman-Richards model is a deterministic model in which an effect, or response, is assumed to be absolutely determined by the formulated cause. The expression for Chapman-Richards model was;

$$\frac{\partial W}{\partial t} = h\, W^{m} - g\, W \qquad (2.1.18)$$

where,

$\frac{\partial W}{\partial t}$ = change in organism weight (W) over time (t),

h = constant of anabolism,

g = constant of catabolism, and

m = allometric constant.

Pienaar and Turnbull (1973) demonstrated the fixed allometric relationship between dbh, height, and volume of spruce trees and applied this growth model for basal area/acre. They reported that the Richards model worked well when fitted to basal area, height, and volume data.

Smith (1983) proposed a set of compatible basal area growth and yield functions consistent with general forest growth theory. He assumed the following relationships of stand basal area (B) and stand height (H) to stand volume (V) and basal area of an individual tree (b) to its height (h) as;

$$V = G + F\,B\,H, \text{ and} \tag{2.1.19}$$

$$b = k\,h \tag{2.1.20}$$

where,

G is an intercept, F is the stand form factor, and k is a proportionality constant. He further assumed;

$$B = N\,b \tag{2.1.21}$$

and hence from equation (2.1.20),

$$B = N\,k\,H \tag{2.1.22}$$

where,

N is the number of trees per unit area.

Basal area growth model obtained by differentiating equation (2.1.19) with respect to age (A) is given as;

$$\frac{\partial B}{\partial A} = \frac{1}{F\,H}\,\frac{dV}{dA} - \frac{dH}{dA}\,\frac{B}{H} \tag{2.1.23}$$

He substituted B and H from (2.1.21) and (2.1.22) into equation (2.1.23) and integrated with respect to age (*A*), to get a basal area yield model compatible with the growth model as;

$$B_f = \sqrt{\left(2\frac{N\,k}{F}\left(V_f - V_n\right) - \left(N\,k\right)^2\left(H_f^2 - H_n^2\right) + B_n^2\right)} \qquad (2.1.24)$$

where,

f and *n* denote the future and present values of the variables.

He found this compatible yield model superior to other empirical models.

Sadiq (1982) used logistic curves to forest stands for estimating basal area growth and yield. By fitting data from red pine into the reverse logistic function, he concluded that the function not only fitted data well, but also had high predictive power.

Amateis and McDill (1989) took dimensional analysis approach to develop dimensionally compatible growth and yield models. They assumed the following yield function;

$$Y = F\left(H_d, N, A\right) \qquad (2.1.25)$$

where,

Y = total volume per unit area,

H_d = height of the dominant stand,

N = number of trees per unit area, and

A = stand age.

Their analysis showed that the age (*A*) did not have any role in the final model. The final model they reached was;

$$Y = H_d\, h\left(H_d^2\, N\right) \qquad (2.1.26)$$

where,

h is some function.

They also derived a basal area growth model as a function of dominant height (H_d), the stand age (A), the ratio of the tree's dbh to the average stand dbh (D_r), and crown ratio (C_r). Their model for basal area growth (B_g) was;

$$B_g = H_d^2 \, A^{-1} \, h\left(D_r, C_r\right) \qquad (2.1.27)$$

Apart from these compatible and/or theoretically based growth and yield models, a number of empirical models have been developed for estimating growth and yield for even-aged and uneven-aged stands. These can be found in the work of Knoebel et al. (1986), Murphy and Farrar (1988), O'Hara and Oliver (1988), Matney and Farrar (1992), Pienaar and Rheney (1993, 1995), Zhang et al. (1993), Graney and Murphy (1994), and others.

2.1.2 Size Class Models

Diameter and other size class distributions described by probability functions have been used for predicting growth and yield in forest stand models. In most cases diameter class has been used. In this approach, the number of trees per unit area in each diameter class is estimated through the use of probability density function that provides the relative frequency of trees by diameters. For trees of given diameters in stands with specific characteristics such as specified age, site index, and stand density, mean total tree heights are predicted. Volume per diameter class is calculated by substituting mean tree heights and the diameter class midpoints into tree volume or taper equations. Finally, total stand yield is estimated by summing the volume of all diameter classes of the trees. Although only stand level attributes, such as age, site index, number of trees per unit area are used as input to these models, detailed stand distributional information can be obtained as output (Avery and Burkhart 1994).

A variety of probability density functions can be used to describe diameter distribution in the stands. In past, mostly two probability density functions, beta and Weibull, have been used by researchers. Bennett and Clutter (1968), Burkhart and Strub (1974), Lenhart (1972), and Lanhart and Clutter (1971) used the beta density function to describe diameter distributions. On the other hand, Dell et al. (1979), Feduccia et al. (1979), Lohrey and Bailey (1977), Smalley and Bailey (1974a and 1974b), and Nepal and Somers (1992) have adopted the Weibull density function. However, Hafley and Schreuder (1977) have shown Johnson's S_B distribution to perform the best for fitting diameter and height data in even-aged stands of Southern pine species. In addition, they used Johnson's bivariate S_{BB} distribution to describe joint diameter-height distributions.

The above studies have been found satisfactory for describing the relative frequencies by dbh class in unthinned even-aged stands. Hyink (1980) presented a generalized method for the projection of diameter distributions applied to uneven-aged stands. Yield was estimated by numerical integration of five simultaneous differential equations to arrive at the average annual changes in (i) the parameter of the Weibull distribution characterizing the diameter distribution, and (ii) the number of trees per acre. Stiff (1980) also showed the utility of Weibull density function for describing diameter distributions in mixed uneven-aged stands.

2.1.3 Individual Tree Models

Stand modeling approaches to predicting stand growth and yield that use individual tree as the basic unit, are classified as individual-tree models. The components of tree growing factors are generally linked together through a computer simulation program. These models provide detailed information about stand dynamics and structure, including the distribution of stand volume by size classes (Avery and Burkhart 1994).

Munro (1974) has further divided these models into two classes, distance independent and distance dependent. In the former, the spatial coordinates of the trees are not required, and the measures of competition included does not utilize inter-tree distances. Under this approach trees can be grown individually or in size classes to provide both detailed tree records and cumulative stand summaries. Due to the lack of actual individual stem locations, the effects of competition are not established directly. Instead, growth is usually governed by some function of aggregate stand characteristics and current tree size.

Examples of distance-independent models can be found in the work of Stage (1975), Dale (1975), Alder (1979), Harrison et al. (1986) Patterson and Stiff (1988), Yoshimoto et al. (1990), Borders and Patterson (1990), Wykoff (1990), Maguire et al. (1991), Short and Burkhart (1992), Avila and Burkhart (1992).

On the other hand, distance-dependent models use individual tree as the primary unit and requires the measurement of inter-tree distances. Most of the models following this approach either accept user input or internally generate individual tree records and/or spatial patterns. The growth of each tree is simulated as a function of the tree's size, site quality, and a measure of competition with the stand. The component that distinguishes distance-dependent from independent is the competition index that is used to assess the interdependence of individual trees.

The competition index may differ from model to model, but it is usually a function of both the subject tree's size and location in relation to the size and location of its competitors (Daniels and Burkhart 1988). Mortality may be controlled either probabilistically or deterministically as a function of competition and/or other individual tree attributes.

Distance-dependent models generally provide more detailed information about individual tree growth than distance-independent models. In addition, they offer the user the flexibility of thinning, pruning, site preparation, and the silvicultural treatments for the potential effects on both the growth of individual trees and the development of the stand. However, distance-dependent models are also more expensive to develop and employ than distance-independent models.

Examples of distance-dependent models may be found in Newnham and Smith (1964), Bella (1970), Arney (1972), Ek and Monserud (1974), Mitchell (1975), Daniels and Burkhart (1975), Daniels et al. (1979). Similarly, these models are also found in Larocque and Marshall (1988), Short and Burkhart (1992), Biging and Dobbertin (1992), and Liu and Burkhart (1994).

Besides these empirical models, ecosystem process models have also been developed to explicitly incorporate the physiological responses of the trees in evaluating projected changes in their environment. These models have not been mentioned here.

As mentioned in Chapter 1 that the stands are made up of individual trees and the sum of tree volumes and basal areas should equal stand volume and stand basal area, respectively. All of the growth and yield models mentioned above were developed for a stand and there was no relationship established between tree and stand level models. These models contain a varying degree of empiricism from one model to another. Moreover, these models are not dimensionally compatible.

This study will derive stand volume, basal area, and growth equations by incorporating theoretical information. This study will also establish a relationship between individual tree and stand volume and basal area equations. The equations derived in this study will be dimensionally compatible.

Chapter 3
MODEL DEVELOPMENT

The first task of the model development is to identify variables that affect the dependent variable and to determine, if possible, the functional relationship between dependent and independent variables. A model can then be established in the form of an equation either empirically from experimental data or by rigorous algebraic analysis based on the application of the fundamental laws and principles (Douglas 1969). Empirically developed models would provide numbers meaningful only in the range of the data applied and are applicable to the situations of similar geographic and environmental conditions from which the data were obtained. On the other hand, models based on the application of the fundamental laws and principles can yield consistent results, even applied beyond the range of the data obtained.

In order to obtain significant conclusions, the actual scientific information, however general it may be, should be investigated rather than estimating the empirical numbers (Khil'mi 1959). However, an appropriate theoretical approach is needed to establish the functional relationships in the form of an equation between quantities involved in any particular physical problem and obtain the scientific information. The mathematical relationships established should then be tested by comparison with numerical data obtained by experimentation or with data obtained through observation.

Dimensional analysis methods have been proved efficient in solving mechanical and physical problems and are independent of the specific properties of these problems (Khil'mi 1959). These methods are powerful tools for model identification and have been extensively used in the field of physics (mainly in mechanics, heat, and electromagnetism), engineering and other related disciplines. These methods can also be used to check models developed using other procedures. Hence, the dimensional analysis approach seems a justified ethodological technique to identify the variables and determine their functional relationships.

3.1 Dimensional Analysis

Dimensional analysis is a method by which information about a phenomenon is deduced from a single premise and the phenomenon can be described by a dimensionally correct equation among certain variables (Langhaar 1951). It is concerned with the nature of the relationship between the various quantities which enter the physical problem. Before dimensional analysis can be applied it must be known that one and only one relationship exists between a certain number of physical quantities, that no pertinent quantities have been omitted, and that no extraneous quantities have been included (Taylor 1974).

Dimensional analysis is a step toward the goal of describing a physical entity or phenomenon in terms of relationships between numbers. The ultimate goal can never be reached by dimensional analysis alone and it is, therefore, not a substitute for complete analysis or experiment. It is nevertheless a very useful tool, especially for improving physical insight. It can sometimes give useful results that are virtually impossible to obtain by any other methods (Taylor 1974).

3.2 Dimensions and Units

In order to describe either a single object or a complex system both qualitatively and quantitatively, it is necessary to decide what particular properties the object or the system possesses. Then methods can be devised by which these properties can be measured. These and other similar properties are dimensions and can be employed to satisfy a physical situation (Douglas 1969). The dimensions of a body, or of a system, are the measurable properties that can be used to describe its physical state.

For example, for a body the dimension of length is not usually sufficient. We also need the magnitude of this length to express the body numerically. For this purpose units of measurement are used. There are several different systems of units in common use. However, the system of units chosen to measure a particular dimension of a body or system does not of course affect the real size of the actual dimension. Thus, dimensions are properties that can be measured, and units are the standard elements in terms of which these dimensions may be measured numerically to describe a given situation quantitatively (Douglas 1969).

3.3 Fundamental and Derived Units and Dimensions

In mechanics, there are mainly three fundamental units. These are the units of length, units of mass, and units of time. These are called fundamental units, because one can not be expressed in terms of the others. For example, mass can not be expressed in terms of length and/or time, length can not be expressed in terms of mass and/or time, and the time can not be expressed in terms of length and/or mass. In the MKS system of units, these three units are expressed in meters, kilograms, and seconds respectively. These are expressed in feet, pounds, and seconds respectively in FPS system.

All other measurement units are expressed in terms of these three fundamental units by means of the physical laws that exist between them and the fundamental quantities. These are called derived units. For example, the unit of volume (cubic feet) needs only one fundamental unit (length) to be expressed. But the unit of force, defined as the product of mass and acceleration, is derived by a combination of all these fundamental units and is expressed as mass x length/time^2 (kg-meter/sec^2 or lb-ft/sec^2).

Dimensional analysis is concerned with the nature of properties involved in the situation and not with their numerical values. The dimension then, of any quantity 'X', is designated as $[X]$ and is expressed in terms of these fundamental units, where the square brackets denote "dimension of. " The fundamental units are symbolized by the letters, L for length, M for mass, and T for time. i.e.,

$$[L] = L, \tag{3.3.1}$$

$$[M] = M, \text{ and} \tag{3.3.2}$$

$$[T] = T \tag{3.3.3}$$

and dimension of other quantities (units) are expressed as;

$$[X] = M^{a} L^{b} T^{c} \quad (3.3.4)$$

where,

X is a quantity or unit and a, b, and c are rational numbers that may be positive, negative, or zero. For example,

Dimension of basal area [A]	=	L^2	=	$M^0 L^2 T^0$	(3.3.5)
Dimension of volume [V]	=	L^3	=	$M^0 L^3 T^0$	(3.3.6)
Dimension of volume growth [V/T]	=	L^3/T	=	$M^0 L^3 T^{-1}$	(3.3.7)
Dimension of basal area growth [B/T]	=	L^2/T	=	$M^0 L^2 T^{-1}$	(3.3.8)
Dimension of gravitational acceleration [G]	=	L/T^2	=	$M^0 L T^{-2}$	(3.3.9)
Dimension of weight [$W = M G$]	=	ML/T^2	=	$M L T^{-2}$	(3.3.10)
etc.					

3.4 Dimensional Homogeneity

An equation is said to be dimensionally homogeneous if the form of equation does not depend on the fundamental units of measurement. For example, the volume equation of the form;

$$V = k D^2 H \quad (3.4.1)$$

is valid whether volume is measured in cubic feet or in cubic meters. Therefore, by definition, the equation is dimensionally homogeneous. However, if k = 0.005454 is substituted in the equation, the equation is correct only in FPS system. The equation is no longer dimensionally homogeneous since the factor 0.005454 only applies if D is measured in inches and H is measured in feet. It should be remembered here that dimensions are not assigned to numbers. Otherwise, any equation can be regarded as dimensionally homogeneous. In general, an equation;

$$x = a + b + c + \ldots . \tag{3.4.2}$$

is dimensionally homogeneous if, and only if, the variables *x*, *a*, *b*, *c*, ... all have the same dimension. This principle is applied to differential equations and integral equations, as well as algebraic equations but may not be applied to empirical equations (models).

Thus, the first step of dimensional analysis for a problem is to decide which variables should enter the problem. If variables are omitted that logically may influence the phenomenon, an incomplete or erroneous result will be obtained. Even though some variables are essentially constants, they may be essential because they interact with other active variables to form dimensionless products (Langhaar 1951).

The application of dimensional analysis to a practical problem is based on the hypothesis that the solution of the problem is expressible by means of a dimensionally homogeneous equation in terms of specified variables. The equation should contain all the variables that would appear in an analytical derivation of the equation.

3.5 Mathematical Definition of Dimensional Homogeneity

Let y be a function of n variables expressed as;

$$y = f(x_1, x_2, x_3, \ldots\ldots, x_n) \tag{3.5.1}$$

The symbol *f* can be regarded as an operator that is applied to the independent variables $x_1, x_2, x_3, \ldots\ldots, x_n$ to yield the proper value of the dependent variable *y*. If the basic units of measurement are subjected to changes, the variables take new values, $x_1', x_2', x_3', \ldots, x_n', y'$. The relation between *x* and x' is given as;

$$x' = x\, A^a\, B^b\, C^c \tag{3.5.2}$$

where,

A, *B*, and *C* are such that

1 original mass unit = A new mass unit,

1 original length unit = B new length unit, and

1 original time unit = C new time unit.

By definition, the equation is dimensionally homogeneous if, and only if,

$$y' = f(x_1', x_2', x_3', \ldots, x_n') \tag{3.5.3}$$

in which f is the same operator as before.

Thus,

$$y' = y\, A^a\, B^b\, C^c = y\, K \tag{3.5.4}$$

$$x_1' = x_1\, A^{a_1}\, B^{b_1}\, C^{c_1} = x_1\, K_1 \tag{3.5.5}$$

$$x_2' = x_2\, A^{a_2}\, B^{b_2}\, C^{c_2} = x_2\, K_2 \tag{3.5.6}$$

$$\begin{matrix} . & . & . \\ . & . & . \end{matrix}$$

$$x_n' = x_n\, A^{a_n}\, B^{b_n}\, C^{c_n} = x_n\, K_n \tag{3.5.7}$$

where,

$$K = A^a\, B^b\, C^c \text{ and} \tag{3.5.8}$$

$$K_i = A^{a_i}\, B^{b_i}\, C^{c_i}\,. \tag{3.5.9}$$

Substituting these values of $x_1', x_2', x_3', \ldots, x_n',$ and y', equation (3.5.3) yields;

$$K\, f(x_1, x_2, x_3, \ldots\ldots, x_n) = f(K_1 x_1, K_2 x_2, K_3 x_3, \ldots\ldots, K_n x_n) \tag{3.5.10}$$

If the relationship between y and x's is given as;

$$y = x_1^{k_1}\, x_2^{k_2} \ldots\ldots x_n^{k_n}\,, \tag{3.5.11}$$

Such expressions are called "products." The product y is dimensionally homogeneous if, and only if, the exponents $(k_1, k_2, k_3, \ldots, k_n)$ are a solution of the linear equations,

$$a_1 k_1 + a_2 k_2 + \ldots\ldots a_n k_n = a \tag{3.5.12}$$

$$b_1 k_1 + b_2 k_2 + \ldots\ldots b_n k_n = b \tag{3.5.13}$$

$$c_1 k_1 + c_2 k_2 + \ldots\ldots c_n k_n = c \tag{3.5.14}$$

where, a, b, and c are dimensional exponents of left-hand side of equation (3.5.11) and a_1, a_2, a_3, ... a_n are constants.

Similarly, expressions like,

$$y_1^{k_1} y_2^{k_2} y_3^{k_3} \ldots\ldots y_n^{k_n} \quad = \text{ constant} \tag{3.5.15}$$

are called dimensionless products, where the exponents $(k_1, k_2, k_3, \ldots\ldots k_n)$ are the solution of the linear equations;

$$a_1 k_1 + a_2 k_2 + \ldots\ldots a_n k_n = 0 \tag{3.5.16}$$

$$b_1 k_1 + b_2 k_2 + \ldots\ldots b_n k_n = 0 \tag{3.5.17}$$

$$c_1 k_1 + c_2 k_2 + \ldots\ldots c_n k_n = 0 \tag{3.5.18}$$

3.6 Dimensional Analysis by the Group Method

If we denote the dimensionless product by p, equation (3.5.15) can be written in more general form as,

$$y\ (p_1, p_2, p_3, \ldots\ldots, p_n) = 0 \tag{3.6.1}$$

where, y is a function and $p_1, p_2, p_3, \ldots\ldots, p_n$ are dimensionless products.

The number of these products depends on the number of quantities (variables), *n*, to be related and the number of independent indicial equations, *m*. The total number of dimensionless products is equal to (*m* - *n*). Normally, there will be one indicial equation for each fundamental dimension and hence the minimum number of products equal to (*n* - *k*), where *k* is the number of fundamental dimensions. This leads to Buckingham's Pi theorem that is stated in the form (Buckingham 1914),

"A complete dimensionally homogeneous equation relating *n* physical quantities which are expressible in terms of *k* fundamental quantities can be reduced to a functional relationship between (*n* - *k*) dimensionless products."

Thus, if the quantities q_1, q_2, q_3, ... , q_n are related by the complete dimensionally homogeneous equation,

$$f(q_1, q_2, q_3, \ldots\ldots, q_n) = 0 \tag{3.6.2}$$

then according to Buckingham's Pi theorem, this relationship can be reduced to the form,

$$y\ (p_1, p_2, p_3, \ldots\ldots, p_{n-k}) = 0 \tag{3.6.3}$$

where, $p_1, p_2, p_3, \ldots\ldots, p_{n-k}$ are dimensionless products formed by the combinations of any *k* quantities with the remaining (*n* - *k*) quantities in turn.

3.7 Volume and Taper Equations

A dimensionally compatible volume equation for a tree and a taper equation compatible with the dimensionally compatible volume equation have been developed previously[1]. These are;

---

[1] Sharma, M., 1995. Volume and taper equations for loblolly pine trees using dimensional analysis. MS Thesis. Va. Polytech. Inst. State University, Blacksburg, VA.

Volume equation

$$V_t = a\ D^b\ H^{3-b} \tag{3.7.1}$$

where,

V_t = total volume of a tree,

D = dbh,

H = total height of the tree, and

a and b are parameters.

And the taper equation

$$d^2 = D^b\ g \left(1 - \frac{h}{H}\right) h^{2-b} \tag{3.7.2}$$

where,

d = diameter at height h of the tree.

From equation (3.6.2) when $d = D$ (dbh), $h = k$.

where,

k = 4.5 feet in FPS system and

= 1.37 meters in MKS system.

Then,

$$D^2 = D^b\ g \left(1 - \frac{k}{H}\right) k^{2-b} \text{ and hence} \tag{3.7.3}$$

$$g = \frac{\left(\frac{D}{k}\right)^{2-b}}{\left(1 - \frac{k}{H}\right)} \tag{3.7.4}$$

Substituting this value of γ into equation (3.6.2) and rearranging the terms we obtain the final taper equation as;

$$d^2 = D^2 \left(\frac{h}{k}\right)^{2-b} \left(\frac{H-h}{H-k}\right) \qquad (3.7.5)$$

It is obvious in the above equation (3.6.5) that if $h = k$, $d = D$ and if $h = H$, $d = 0$. These are very important characteristics of a taper equation.

3.8 Equations for Basal Area (BA) and its Growth

The basal area (*BA*) of a tree is written as;

$$BA = \frac{p}{4} D^2 \qquad (3.8.1)$$

Alternately, the basal area (*BA*) can be obtained by substituting the value of D^2 from equation (3.6.3) into the equation (3.7.1). The basal area obtained in this fashion is expressed as;

$$BA = \frac{p}{4} D^b\, g \left(1 - \frac{k}{H}\right) k^{2-b} \qquad (3.8.2)$$

To calculate the change in Basal area (*BA* growth) over time *T* (between times t_1 an t_2) we can write equation (3.7.2) as;

$$BA_1 = \frac{p}{4} D_1{}^b\, g_1 \left(1 - \frac{k}{H_1}\right) k^{2-b} \quad \text{and} \qquad (3.8.3)$$

$$BA_2 = \frac{p}{4} D_2{}^{b}\, g_2 \left(1 - \frac{k}{H_2}\right) k^{2-b} \tag{3.8.4}$$

where, the subscripts *1* and *2* are for times t_1 and t_2 respectively.

Then, the change in Basal Area between times t_1 and t_2 is expressed as;

$$\Delta BA = \frac{p}{4}\, k^{2-b} \left(D_2{}^{b}\, g_2 \left(1 - \frac{k}{H_2}\right) - D_1{}^{b}\, g_1 \left(1 - \frac{k}{H_1}\right)\right) \tag{3.8.5}$$

Now, the total basal area of the stand can be written in terms of individual tree basal area as;

$$BA_s = \sum_{i=1}^{N} (BA)_i \tag{3.8.6}$$

where,

BA_s = stand basal area,

$(BA)_i$ = basal area of the i^{th} tree, and

N = total number of trees in the stand.

Therefore, the change in total basal area of a stand between times t_1 and t_2 is;

$$\Delta BA_s = \frac{p}{4}\, k^{2-b} \left(\sum_{i=1}^{N_2} D_{2i}{}^{b}\, g_{2i} \left(1 - \frac{k}{H_{2i}}\right) - \sum_{j=1}^{N_1} D_{1j}{}^{b}\, g_{1j} \left(1 - \frac{k}{H_{1j}}\right)\right)$$

Equivalently,

$$\Delta BA_s = \frac{p}{4}\, k^{2-b} \left(N_2\, \overline{D}_2{}^{b}\, \bar{g}_2 \left(1 - \frac{k}{\overline{H}_2}\right) - N_1\, \overline{D}_1{}^{b}\, \bar{g}_1 \left(1 - \frac{k}{\overline{H}_1}\right)\right) \tag{3.8.7}$$

where,

N_1 = Number of trees on the stand at time t_1,

N_2 = Number of trees on the stand at time t_2,

$\overline{D}$ = Quadratic mean diameter,

$\bar{g}$ = value of γ calculated from average DBH and height,

$\overline{H}$ = average or Lorey mean height to be determined later, and

the subscripts 1 and 2 are for times t_1 and t_2 as mentioned earlier.

Lorey Mean Height (LMH) (Loetsch et al. 1973) is defined as;

$$LMH = \frac{\sum_i g_i \, h_i}{\sum_i g_i} \tag{3.8.8}$$

where, g_i is the basal area and h_i is the height of the i^{th} tree in the stand.

One of the characteristics of the models based on theoretical information is that the growth model should be compatible with time. Compatible with time is in the sense that the sum of basal area growths in the time interval t_1 and t_2 and in the time interval t_2 and t_3 should not depend on the intermediate basal area at time t_2. To show this analytically we can express the basal area at time t_3 as;

$$BA_3 = \frac{p}{4} D_3^{\,b} \, g_3 \left(1 - \frac{k}{H_3}\right) k^{2-b} \tag{3.8.9}$$

Then, the basal area growth between times t_1 and t_2 is given as;

$$BA_2 - BA_1 = \frac{p}{4} \, k^{2-b} \left(D_2^{\,b} \, g_2 \left(1 - \frac{k}{H_2}\right) - \; D_1^{\,b} \, g_1 \left(1 - \frac{k}{H_1}\right)\right) \tag{3.8.10}$$

And the basal area growth between times t_2 and t_3 is expressed as;

$$BA_3 - BA_2 = \frac{p}{4}\, k^{2-b} \left(D_3{}^{b}\, g_3 \left(1 - \frac{k}{H_3}\right) - D_2{}^{b}\, g_2 \left(1 - \frac{k}{H_2}\right)\right) \qquad (3.8.11)$$

Now, the basal area growth between time t_1 and t_3 is;

$$(BA_3 - BA_2) + (BA_2 - BA_1) = BA_3 - BA_1$$

i.e. $$(\Delta BA)_{t_3 - t_1} = \frac{p}{4}\, k^{2-b} \left(D_3{}^{b}\, g_3 \left(1 - \frac{k}{H_3}\right) - D_1{}^{b}\, g_1 \left(1 - \frac{k}{H_1}\right)\right) \qquad (3.8.12)$$

It is obvious that the basal area growth in the interval t_1 and t_3 is completely independent of the basal area at time t_2.

3.9 Equations for Stand Volume and its Growth

In forestry literature, stand volumes have been generally found to be expressed in terms of basal area, site quality, and age of the stand. These variables have been considered as potential variables in developing stand volume and growth equations in this study. Since volume is also a function of height, some form of stand height (average height, Lorey mean height, or some other function of the stand height) has also been considered as the potential variable determining the stand volume.

Assume that the stand volume (V_s) is expressed as,

$$V_s = d\,(BA_s)^{q}\; p^{y}\; q^{h}\; A^{f} \qquad (3.9.1)$$

where,

BA_s = stand basal area,

p = some function of stand height to be determined,

q = site quality,

A = stand age, and

d, θ, ψ, h, and f are parameters (constants) to be determined.

In expression (3.8.1), q represents some measure of site quality. In forestry, site quality is expressed in terms of average height of dominant and codominant trees at a predetermined age, and is called site index. Thus, being the site index as a function of age, it is only a conceptual measure of the site quality. It might be expressed in terms of stand height (site index) and/or stand biomass. It might also be expressed in terms of other site characteristics representing the site productivity. In this case, let us assume that the site quality is a function of height, biomass, and the age of the stand.

Since the function of height has already been incorporated by introducing the variable p, the dimensions of the site quality can be written as;

$$[q] = M^a L^0 T^b \qquad (3.9.2)$$

where,

a and b are the constants to be determined.

Then,

Dimension of stand volume (V_s) = $M^0 L^3 T^0$

Dimension of basal area (BA_s) = $M^0 L^2 T^0$

Dimension of height (p) = $M^0 L T^0$

Dimension of site quality (q) = $M^a L^0 T^b$

Dimension of stand age (A) = $M^0 L^0 T$

As mentioned earlier, the constant d does not have any dimensions. Hence,

dimensions of L. H. S. of equation (3.8.1) = $M^0 L^3 T^0$

and the dimensions of R. H. S. of this equation = $M^{a\eta} L^{2q+y} T^{bh+f}$

Therefore, the dimensional form of the volume equation (3.8.1) is written as;

$$M^0 L^3 T^0 = M^{a h} L^{2q+y} T^{bh+f} \qquad (3.9.3)$$

This equation is valid only when $ah = 0$, $bh + f = 0$, and $2q + y = 3$. This implies $h = 0$, $f = 0$, and $y = 3 - 2q$. This shows that if there is a contribution of site quality (q) to the stand volume, it will be reflected by the function of height p. However, the age (A) of the stand does not have any contribution to the stand volume (V_s). It should be noted that dimensional analysis, being qualitative, can not give the value to the constant d. The numerical value of d can be determined experimentally.

Thus, the final dimensional form stand volume equation is given as;

$$V_s = d\,(BA_s)^q\ p^y \tag{3.9.4}$$

where,

$2q + y = 3.$

We can express the dimensional form of stand volume (V_s) in terms of some function of DBH and stand height. The following steps will lead to the result.

$$\begin{aligned} V_s &= d\,(BA_s)^q\ p^{3-2q} \\ &= \sum_{i=1}^{N} d\,(BA_i)^q\ p^{3-2q} \\ &= N d\ (\overline{BA})^q\ p^{3-2q} \end{aligned}$$

$$\therefore \qquad V_s = N d \left(\frac{p}{4}\right)^q \left(\overline{D}^{\,2}\right)^q\ p^{3-2q} \tag{3.9.5}$$

The equation (3.8.5) to be compatible with individual tree volume equation (3.6.1) we must have;

$$d\left(\frac{p}{4}\right)^{q} = a \text{ and } 2q = b. \quad (3.9.6)$$

Therefore, the mathematical form of the stand volume equation in terms of diameter and height is expressed as;

$$V_s = N\,a\,\overline{D}^{\,b}\,p^{3-b} \quad (3.9.7)$$

where,

N = Total number of trees on the stand,

$\overline{D}$ = Quadratic mean diameter i. e. DBH of the tree with basal area equal to $\overline{BA}$, and

p = function of height e.g. average height, Lorey mean height or some other function of height to be determined from the data.

To derive the stand volume growth equation, we express the stand volume (V_s) at time t_1 as;

$$(V_s)_1 = N_1\,a\,\overline{D}_1^{\,b}\,p_1^{\,3-b} \quad (3.9.8)$$

Similarly, the stand volume (V_s) at time t_2 is written as;

$$(V_s)_2 = N_2\,a\,\overline{D}_2^{\,b}\,p_2^{\,3-b} \quad (3.9.9)$$

where, the subscripts *1* and *2* are for times t_1 and t_2 respectively and other terms are as defined earlier.

Then, the volume growth in the time interval t_1 and t_2 is calculated as;

$$\Delta V = (V_s)_2 - (V_s)_1 = N_2\,a\,\overline{D}_2^{\,b}\,p_2^{\,3-b} - N_1\,a\,\overline{D}_1^{\,b}\,p_1^{\,3-b}$$

i.e. $$(\Delta V)_{t_2-t_1} = a\left(N_2\,\overline{D}_2^{\,b}\,p_2^{\,3-b} - N_1\,\overline{D}_1^{\,b}\,p_1^{\,3-b}\right) \quad (3.9.10)$$

As we showed in the case of basal area equation, we can also show the compatibility of stand volume growth equation with time. For this, we can express stand volume at time t_3 as;

$$(V_s)_3 = N_3\, a\, \overline{D}_3^{\,b}\, p_3^{\,3-b} \tag{3.9.11}$$

Then, the stand volume growth between times t_1 and t_2 is given as;

$$(V_s)_2 - (V_s)_1 = N_2\, a\, \overline{D}_2^{\,b}\, p_2^{\,3-b} - N_1\, a\, \overline{D}_1^{\,b}\, p_1^{\,3-b} \tag{3.9.12}$$

And the stand volume growth between times t_2 and t_3 is expressed as;

$$(V_s)_3 - (V_s)_2 = N_3\, a\, \overline{D}_3^{\,b}\, p_3^{\,3-b} - N_2\, a\, \overline{D}_2^{\,b}\, p_2^{\,3-b} \tag{3.9.13}$$

Now, the stand volume growth between time t_1 and t_3 is;

$$\left((V_s)_3 - (V_s)_2\right) + \left((V_s)_2 - (V_s)_1\right) = \left((V_s)_3 - (V_s)_1\right)$$

i.e. $$(\Delta V)_{t_3 - t_1} = a\left(N_3\, \overline{D}_3^{\,b}\, p_3^{\,3-b} - N_1\, \overline{D}_1^{\,b}\, p_1^{\,3-b}\right) \tag{3.9.14}$$

This shows that the stand volume growth in the interval t_1 and t_3 is completely independent of the stand volume at time t_2. In other words, the sum of volume growths in the time interval t_1 and t_2 and the time interval t_2 and t_3 is independent of the intermediate volume at time t_2.

Chapter 4
MATERIALS AND METHODS

4.1 Data

In order to estimate the parameters of individual tree volume and taper equations, stem analysis data are required. These data are obtained by measuring diameter and height pairs at every 4 ft along the bole of each individual tree in a sample from a stand. The parameters associated with the individual tree volume and taper equations are the same as the parameters attached to the stand basal area and volume equations. Therefore, the estimates of the parameters obtained from individual tree and volume equations can be adopted into the stand basal area and volume equations. For the evaluation of stand basal area and volume equations, however, DBH and total height of each individual tree in the stand are needed. Therefore, the data obtained from Loblolly Pine Growth and Yield Research Cooperative, Department of Forestry at Virginia Polytechnic Institute and State University were used in this study.

These data were collected during a thinning study in loblolly pine Plantations that were established on cutover site-prepared lands (Burkhart et al. 1985). The thinning study established in 1980-1982 consisted of two thinning treatments and a control. At the time of plot establishment two levels of thinning treatments were applied. In the heavily thinned plots, approximately 50% of the total basal area was removed where as in lightly thinned plots, approximately 30% of the basal area was removed. A second thinning was applied in roughly half of the thinned plots during the 1992-1994 dormant seasons, 12 years after plot establishment and first thinning.

At the time of plot establishment, dominant, codominant, or intermediate crown class trees were felled for the stem analysis. The rest of the trees were measured for DBH and height and left in the stands to grow. All plots have been remeasured for DBH and height every three years since the establishment. So far five remeasurements have been conducted.

During the second thinning, two trees were randomly selected from each plot (control, light thin, and heavy thin) for stem analysis and the balance of the number of the trees prescribed for removal were marked using regular operational procedure. In light thinned and heavy thinned plots, the actual trees randomly selected for removal were felled. In the control plots, however, trees similar in DBH, total height, stem form, etc. to those randomly selected were felled from trees in the buffer zone bordering the plots.

The trees selected for stem analysis were sectioned at about 4-ft intervals beginning at the stump to a small top diameter of approximately 2 in. Two measures of diameter outside bark (dob) and two measures of diameter inside bark (dib) perpendicular to each other were collected and averaged to obtain dob and dib measurements up the stem. Volume of each section from stump to the small top diameter was calculated using Smalian's formula. The volume of a cone was calculated for the portion from the last cut to the tip of the tree. These section volumes were then accumulated for each tree to obtain total tree volume (both inside and outside bark).

In total, there were 10,114 observations of diameter, height pairs on 781 trees felled for stem analysis. Among these 781 trees, 470 trees were sampled at the time of plot establishment and the remaining 311 trees with 4544 observations were obtained during the second thinning. Out of these 311 trees, 104 trees were from unthinned plots, 104 trees were from light thinned plots, and the rest 103 trees were from the heavy thinned plots.

Since the treatments (light and heavy thinning) were applied for the first time during 1980-1982, at the time of plot establishment, the stem analysis data obtained after the plot establishment were used to estimate the parameters. Had we included all the trees measured at the time of plot establishment to estimate the parameters, the large number of small trees measured at the time of plot establishment would have made the data unbalanced and the results from thinned and unthinned stands could not have been compared. Summary statistics for these 311 trees are presented in Table 1.

Table 1. Summary statistics for the stem analysis data used in this study.

Variable	Number of trees	Mean	Standard deviation	Minimum	Maximum
Unthinned trees					
DBH (in)	104	7.54	2.08	3.80	13.50
Height (ft)	104	57.09	9.65	33.80	80.50
Volume (ob, cuft)	104	8.926	5.850	1.245	29.456
Volume(ib, cuft)	104	7.254	4.888	0.867	25.220
Light Thinned trees					
DBH (in)	104	8.58	1.84	4.90	13.90
Height (ft)	104	59.16	9.09	41.40	85.00
Volume (ob, cuft)	104	11.439	6.415	1.672	31.688
Volume(ib,cuft)	104	9.394	5.436	1.325	26.085
Heavy Thinned					
DBH (in)	103	9.21	1.78	4.60	13.80
Height (ft)	103	59.96	8.27	41.30	87.90
Volume (ob, cuft)	103	13.139	6.231	1.961	32.196
Volume(ib, cuft)	103	10.757	5.286	1.584	28.005

4.2 Estimation of Parameters

The diameters and heights of all the sampled and non-sampled trees in the stands were measured in English system of units. Moreover, the same values of the parameter estimates are obtained by fitting a dimensionally compatible system of equations no matter which system of units is used. Therefore, data in English units were used to estimate the parameters and evaluate the results.

The volume of each felled tree was calculated by summing the section volumes. These section volumes were based on section length and cross-sectional area. The cross-sectional area was calculated using the average value of diameters measured perpendicular to each other across the cut stem.

On the other hand, DBH of a standing tree is measured using a diameter tape along the circumference of the cross-sectional area at height 4.5 ft (breast height) from the ground. DBH measured in this fashion tends to be larger than the diameter measured between heights 4-4.5 ft in stem analysis. The data show that in most of the cases, DBH measured by the diameter tape is greater than the DBH obtained from stem analysis.

The volume equation has two parameters, α and β, and the taper equation has only one parameter, β, that is the same as in the volume equation. Moreover, the volume equation involves only two independent variables, DBH and total height, and the taper equation contains all the diameters and corresponding heights along the bole including both DBH and total height.

Therefore, if we estimate α and β in the volume equation utilizing only DBH and total height and adopt the estimated value of β to the taper equation, the taper function would not accurately estimate diameters along the bole of the tree. On the other hand, if we fit only the taper equation, the parameter α in the volume equation can not be estimated.

Thus, it would be appropriate to fit both the volume and the taper equation simultaneously. In this way, more information from all the trees (the larger the tree the more the information) is contributed in fitting the equations and consistent estimates of the parameters would be obtained.

A preliminary analysis was undertaken to determine whether the value of β is different for the thinned and unthinned plantations, using the taper equation with dummy variables. The equation was fitted twice, once using unthinned vs thinned data and again using light thinned vs heavily thinned data. Ordinary nonlinear regression procedures in Statistical Analysis System (SAS) were applied to estimate the difference in the value of β. The difference was estimated between unthinned and thinned and between light thinned and heavy thinned trees. These estimates along with their corresponding asymptotic fits are presented in Table 2.

Table 2. Estimates of the difference in the value of β and their asymptotic fits for thinned vs unthinned and light thinned vs heavy thinned plot trees.

Difference in β	Estimate	Std. error	95% Confidence interval
Unthinned vs Thinned	0.00472	0.00215	0.00051 - 0.00893
Light thinned vs Heavy thinned	0.00536	0.00216	0.00112 - 0.00961

Neither of the confidence intervals in Table 2 includes zero. This implies that the difference in the value of β for unthinned vs thinned and light thinned vs heavy thinned trees are significantly different from zero at the probability level 0.05. Partial F-tests was also performed to confirm the result. In both cases, p-values for partial F-test were less than 0.01. Therefore, parameters of the individual tree volume and the taper equations were estimated for unthinned, light thinned and heavy thinned trees independently.

The SYSNLIN procedure in SAS was used to estimate the parameters. This procedure combines iterative minimization methods for nonlinear regression with specialized estimation techniques for simultaneous equation systems to estimate parameters in a simultaneous system of nonlinear equations. Under this procedure there are 5 methods available; nonlinear ordinary least squares (OLS) method was used in this study.

The taper equation assumes that the diameter at height zero is very large (infinity) compared to the diameter above the stump height. Therefore, in fitting the data, the stump height was changed to 0.001 ft wherever it was equal to zero in the data set. The stump height greater than zero in the data set was left unchanged. The estimates of α and β for inside and outside bark volumes (vob and vib) along with their approximate standard errors have been presented in Table 3.

Table 3. Estimates of α and β for vob and vib and their corresponding approximate Standard Errors.

Stands	Parameters	Estimates	Approx. Std. Error
Outside Bark			
Unthinned	α	0.5009	0.0042
	β	2.0887	0.0017
Light thinned	α	0.5075	0.0035
	β	2.0874	0.0015
Heavy Thinned	α	0.4945	0.0035
	β	2.0819	0.0015
Inside Bark			
Unthinned	α	0.3666	0.0052
	β	2.0625	0.0031
Light thinned	α	0.3848	0.0045
	β	2.0667	0.0026
Heavy Thinned	α	0.3741	0.0042
	β	2.0610	0.0025

Both of the parameters (α and β) used in stand volume, basal area, and their growth equations are also used in the individual tree volume and taper equations. There are no other parameters or constraints imposed that would require estimations by least squares and/or other techniques.

4.3 Evaluation of Stand Volume Equation

Evaluation of a model is usually performed by predicting response values that are independent of the data that estimated the parameters (Myers, 1990). Therefore, in order to evaluate the system of equations (individual tree volume, tree taper, stand volume, basal area, and growth equations) DBH and height data obtained from the trees left on the stands were used. These trees were left on the stands by felling the sampled trees for stem analysis.

Amateis and Burkhart (1987) noted that the combined variable volume equation gave the better fit to the data from loblolly pine trees in cutover site-prepared plantations than other models they tested. A dimensionally compatible volume equation for a tree was evaluated in previous work by this author (Sharma, 1995) and was found superior to the combined variable volume equation.

The taper equation compatible with the volume equation was compared to other taper equations describing the paracone shape of the tree. The taper equation compatible with the volume equation was also found superior to other taper equations. Therefore, no further evaluations of individual tree volume and taper equations were made in this study. However, other characteristics of the taper equation will be explored with the help of graphics in the sections to come.

As mentioned earlier, the DBH and the height of all the trees left on the stands were measured at the time of plot establishment and remeasurements were made in every three years. These remeasurements were accomplished during the 1983-85, 1986-88, 1989-91, 1992-94 and 1995-96 dormant seasons. In addition, thinning treatments were applied during 1992-94. Summary statistics of DBH and height of these trees from unthinned, light thinned and heavy thinned stands have been displayed in the tables 4, 5 and 6 respectively.

Total volumes of these trees were calculated using the individual tree volume equation (3.6.1) with different values of parameters α and β for differently managed stands. Average DBH, average tree height, quadratic mean diameter, average tree volume and average tree basal area were calculated at each remeasurement for each differently managed stand. These data were applied to evaluate the BA, stand volume and their growth equations. To determine the value of p in the stand volume equation, Lorey Mean Height was also calculated for each remeasurement of each stand. Average height and Lorey Mean Height (functions of height) were substituted in the stand volume equation with average DBH and Quadratic Mean Diameter (functions of DBH) in turn.

Table 4. Summary statistics for DBH and Height from unthinned stands:

Variable	Number of trees	Mean	Standard deviation	Minimum	Maximum
At first remeasurement					
DBH (in)	4795	6.10	1.84	1.10	14.00
Height (ft)	4795	42.47	10.88	10.00	84.00
At second remeasurement					
DBH (in)	4391	6.62	1.89	1.30	15.10
Height (ft)	4391	47.83	10.64	15.00	87.00
At third remeasurement					
DBH (in)	3807	7.02	1.96	1.70	14.80
Height (ft)	3807	51.80	10.35	18.00	90.00
At fourth remeasurement					
DBH (in)	2861	7.62	2.01	2.00	15.80
Height (ft)	2861	56.52	10.31	20.00	93.00
At fifth remeasurement					
DBH (in)	1996	8.17	2.10	2.00	16.50
Height (ft)	1996	61.59	10.48	16.00	97.00

Table 5. Summary statistics for DBH and Height from light thinned stands:

Variable	Number of trees	Mean	Standard deviation	Minimum	Maximum
At first remeasurement					
DBH (in)	7113	6.82	1.67	1.40	14.40
Height (ft)	7113	44.12	10.93	14.00	84.00
At second remeasurement					
DBH (in)	6774	7.44	1.74	1.40	15.70
Height (ft)	6774	49.74	10.41	15.00	90.00
At third remeasurement					
DBH (in)	5958	7.94	1.84	1.50	16.00
Height (ft)	5958	53.93	9.93	15.00	91.00
At fourth remeasurement					
DBH (in)	4877	8.52	1.95	1.60	16.70
Height (ft)	4877	58.56	9.80	15.00	91.00
At fifth remeasurement					
DBH (in)	3063	9.29	1.94	1.60	17.40
Height (ft)	3063	64.05	9.69	15.00	101.00

Table 6. Summary statistics for DBH and Height from heavy thinned stands:

Variable	Number of trees	Mean	Standard deviation	Minimum	Maximum
At first remeasurement					
DBH (in)	5663	7.08	1.64	2.00	14.80
Height (ft)	5663	44.23	11.04	14.00	86.00
At second remeasurement					
DBH (in)	5468	7.82	1.66	2.60	14.90
Height (ft)	5468	49.89	10.37	17.00	85.00
At third remeasurement					
DBH (in)	4915	8.45	1.77	2.80	16.20
Height (ft)	4915	54.16	9.71	18.00	94.00
At fourth remeasurement					
DBH (in)	3974	9.08	1.86	2.40	17.10
Height (ft)	3974	59.06	9.71	28.00	91.00
At fifth remeasurement					
DBH (in)	2438	9.95	1.95	4.00	17.20
Height (ft)	2438	64.83	10.01	31.00	101.00

In this way, there were four possible combinations of functions of height and functions of diameter going into the volume equations for each stand managed with different tree intensities. These are Lorey Mean Height (LMH) and Quadratic Mean Diameter (QMD), LMH and average DBH, average height and QMD, and average height and average DBH. Average tree volumes were estimated using these four combinations of heights and diameters for each stand. These volumes along with their corresponding observed average tree volumes at remeasurements 1-5 for unthinned, light thinned and heavy thinned stands are presented in the tables 7, 8 and 9 respectively. Average DBH, average tree heights, quadratic mean diameters and Lorey mean heights are also displayed in the tables.

Table 7. Average DBH, average Height, Quadratic Mean Diameter, Lorey Mean Height, observed volume and average tree volumes calculated using four combinations of these diameters and heights for unthinned stands.

Remeasurement	1	2	3	4	5
Average DBH (in)	6.10	6.62	7.02	7.62	8.17
Average height (ft)	42.47	47.83	51.8	56.52	61.59
Quadratic Mean Diameter (QMD) (in)	6.38	6.88	7.29	7.88	8.43
Lorey Mean Height (LMH) (ft)	47.38	52.20	55.94	60.19	65.12
Observed Volume (cuft)	4.53	5.79	6.96	8.75	10.82
Volume Obtained by LMH and QMD (cuft)	4.51	5.78	6.94	8.74	10.81
Volume Obtained by LMH and average DBH (cuft)	4.11	5.33	6.42	8.14	10.12
Volume Obtained by average height and QMD (cuft)	4.08	5.34	6.47	8.25	10.27
Volume Obtained by average height and average DBH (cuft)	3.72	4.92	5.98	7.69	9.62

Table 8. Average DBH, average Height, Quadratic Mean Diameter, Lorey Mean Height, observed volume and average tree volumes calculated using four combinations of these diameters and heights for light thinned stands.

Remeasurement	1	2	3	4	5
Average DBH (in)	6.82	7.44	7.94	8.52	9.29
Average height (ft)	44.12	49.74	53.93	58.56	64.05
Quadratic Mean Diameter (QMD) (in)	7.02	7.64	8.16	8.74	9.49
Lorey Mean Height (LMH) (ft)	48.18	53.13	57.08	61.46	66.46
Observed Volume (cuft)	5.73	7.47	9.13	11.29	14.38
Volume Obtained by LMH and QMD (cuft)	5.70	7.44	9.10	11.25	14.34
Volume Obtained by LMH and average DBH (cuft)	5.37	7.04	8.60	10.66	13.72
Volume Obtained by average height and QMD (cuft)	5.26	7.00	8.64	10.76	13.87
Volume Obtained by average height and average DBH (cuft)	4.95	6.63	8.17	10.20	13.27

Table 9. Average DBH, average Height, Quadratic Mean Diameter, Lorey Mean Height, observed volume and average tree volumes calculated using four combinations of these diameters and heights for heavy thinned stands.

Remeasurement	1	2	3	4	5
Average DBH (in)	7.08	7.82	8.45	9.08	9.95
Average height (ft)	44.23	49.89	54.16	59.06	64.83
Quadratic Mean Diameter (QMD) (in)	7.27	8.00	8.64	9.27	10.14
Lorey Mean Height (LMH) (ft)	48.16	52.97	56.88	61.50	67.09
Observed Volume (cuft)	6.13	8.15	10.22	12.71	16.59
Volume Obtained by LMH and QMD (cuft)	6.10	8.12	10.18	12.67	16.54
Volume Obtained by LMH and average DBH (cuft)	5.78	7.76	9.73	12.14	15.91
Volume Obtained by average height and QMD (cuft)	5.64	7.69	9.74	12.21	16.03
Volume Obtained by average height and average DBH (cuft)	5.34	7.34	9.30	11.70	15.42

Among these four combinations, volume calculated by using the combination of Lorey Mean Height (LMH) and Quadratic Mean Diameter (QMD) is closer to the observed average tree volume for all stands. Volume obtained by utilizing average height and average DBH is the farthest apart (in magnitude) from the observed one in all three cases. Volumes from the other two combinations of heights and diameters have intermediate values.

Formal comparisons among the estimates were obtained by examining bias, precision, and mean square error. Bias was calculated as,

$$\overline{D} = \frac{\sum_{i=1}^{5} D_i}{5} \qquad (4.3.1)$$

where, $D_i = Y_i - \hat{Y}_i$, and Y_i and $\hat{Y}_i$ are measured and predicted volumes respectively, and 5 is the number of remeasurements. Precision was calculated using,

$$S_D = \sqrt{\frac{\sum_{i=1}^{5} D_i^2 - \frac{\left(\sum_{i=1}^{5} D_i\right)^2}{5}}{4}} \tag{4.3.2}$$

Measure of bias and precision were combined into mean square error as given below;

$$MS = \overline{D}^2 + S_{\overline{D}}^{\ 2} \tag{4.3.3}$$

where,

$$S_{\overline{D}}^{\ 2} = \frac{S_D^{\ 2}}{5} \tag{4.3.4}$$

Then $\sqrt{MS}$ was calculated. These results for unthinned, light thinned and heavy thinned stands have been presented and discussed in the next chapter. These statistics for the total volume of a stand can be calculated by multiplying the statistics of average tree volume and the corresponding number of trees of the stand.

4.4 Evaluation of Basal Area Equation

Observed average basal area per tree was compared to predicted average basal area per tree using parameters from QMD and LMH and from average DBH and average height in equation 3.7.2. Only these two combinations were used since they represented the extremes of bias and precision in the case of volume. These average tree basal areas calculated at remeasurements 1-5 for each stand are presented in Table 10.

Table 10. Average tree basal area (BA) observed and calculated by the combinations of different functions of diameter and height.

Remeasurement	1	2	3	4	5
Unthinned Stand					
Observed BA (sqft)	0.2216	0.2581	0.2898	0.3387	0.3876
BA obtained by QMD and LMH (sqft)	0.2218	0.2581	0.2898	0.3387	0.3876
BA obtained by average DBH and average height (sqft)	0.2029	0.2390	0.2688	0.3167	0.3580
Light thinned stand					
Observed BA (sqft)	0.2687	0.3184	0.3627	0.4166	0.4912
BA obtained by QMD and LMH (sqft)	0.2689	0.3184	0.3627	0.4166	0.4912
BA obtained by average DBH and average height (sqft)	0.2546	0.3029	0.3450	0.3972	0.4710
Heavy thinned stand					
Observed BA (sqft)	0.2880	0.3485	0.4067	0.4682	0.5604
BA obtained by QMD and LMH (sqft)	0.2881	0.3486	0.4069	0.4684	0.5607
BA obtained by average DBH and average height (sqft)	0.2734	0.3335	0.3894	0.4497	0.5400

Average basal area calculated using Quadratic Mean Diameter (QMD) and Lorey Mean Height (LMH) is nearly identical to observed average basal area for all three stands. The basal area obtained from average DBH and average height is smaller than the observed basal area in all 5 remeasurements for all differently managed stands. These results along with the ones obtained in the case of stand volume equation reveal that LMH should be used in conjunction with QMD in calculating average basal area and average volume of the trees in the stand. Since the basal area obtained from average DBH and average height is not comparable to the one obtained from QMD and LMH, bias, precision, and mean square error, in this case, were not calculated.

4.4 Evaluation of Volume and Basal Area Growth

It was shown in sections 3.7 and 3.8 that the basal area and volume growth of a stand between time interval t_1 and t_2 could be obtained by subtracting the basal area and volume at time t_1 from the basal area and volume at time t_2 respectively. It was also shown analytically that the basal area and volume growths were compatible across the time no matter what time interval was chosen.

To demonstrate these results using the data, volume and basal area growths between remeasurements *i* and *j* were calculated by subtracting the volume and basal area at remeasurement *i* from the volume and basal area at remeasurement *j* respectively. Average values of volume and basal area growth for successive time periods for these stands have been presented in the tables 11, 12, and 13. The predicted values of volumes and basal areas were obtained using QMD and LMH.

Table 11. Observed and predicted per tree volume and basal area growths at different periods for unthinned stand.

Growth period	Observed volume growth (cuft)	Predicted volume growth (cuft)	Observed BA growth (sqft)	Predicted BA growth (sqft)
Remeasurement 1-2	1.2660	1.2641	0.0365	0.0364
Remeasurement 2-3	1.1671	1.1659	0.0316	0.0316
Remeasurement 3-4	1.7901	1.7929	0.0489	0.0489
Remeasurement 4-5	2.0689	2.0688	0.0489	0.0489
Total growth	6.2921	6.2916	0.1659	0.1658
Remeasurement 1-3	2.4331	2.4299	0.0681	0.0680
Remeasurement 1-4	4.2232	4.2228	0.1171	0.1170
Remeasurement 1-5	6.2921	6.2916	0.1659	0.1658
Remeasurement 2-5	5.0261	5.0275	0.1294	0.1294
Remeasurement 3-5	3.8590	3.8616	0.0978	0.0978

Table 12. Observed and predicted per tree volume and basal area growths at different periods for light thinned stand.

Growth period	Observed volume growth (cuft)	Predicted volume growth (cuft)	Observed BA growth (sqft)	Predicted BA growth (sqft)
Remeasurement 1-2	1.7406	1.7354	0.0497	0.0495
Remeasurement 2-3	1.6669	1.6613	0.0443	0.0443
Remeasurement 3-4	2.1545	2.1498	0.0539	0.0539
Remeasurement 4-5	3.0922	3.0950	0.0745	0.0745
Total growth	8.6542	8.6416	0.2224	0.2223

Table 13. Observed and predicted per tree volume and basal area growths at different periods for heavy thinned stand.

Growth period	Observed volume growth (cuft)	Predicted volume growth (cuft)	Observed BA growth (sqft)	Predicted BA growth (sqft)
Remeasurement 1-2	2.0206	2.0191	0.0605	0.0605
Remeasurement 2-3	2.0679	2.0616	0.0582	0.0582
Remeasurement 3-4	2.4883	2.4839	0.0615	0.0615
Remeasurement 4-5	3.8839	3.8778	0.0922	0.0923
Total growth	10.4607	10.4423	0.2724	0.2726

It is obvious from these tables that the estimates of volume and basal area growth are accurate and compatible across the time for all unthinned, light thinned and heavy thinned stands. In addition, the sum of volume and basal area growth calculated for each consecutive period of remeasurements is equal to the total volume and basal area growth estimated between remeasurements 1 and 5.

Chapter 5

RESULTS AND DISCUSSIONS

In Chapter 4, parameters (α and β) associated with the system of equations were estimated using stem analysis data. Evaluations of stand volume, basal area, and growth equations were performed using the DBH and height pair data from the trees that were not sampled for stem analysis. Results from those evaluations for individual tree volume, tree taper, stand volume, basal area, and growth have been discussed in the following sections.

5.1 Tree Volume Equation

The dimensionally compatible volume equation did not contain the intercept term. Data to be used in estimating the parameters of an equation are always measured in some system of units. As a result, if we include the intercept term in the volume equation and estimate it using the data, the volume equation will no longer be dimensionally compatible. Therefore, the volume equation with intercept fitted with the data measured in MKS system of units can not be used in English system of units and vice versa.

Most often, researchers add an intercept term to a model containing dependent and independent variables. The intercept is especially useful in an empirical model. The intercept is estimated by regressing the dependent variable against the independent variables in the model. It is interpreted as the mean value of the response variable when all the regressor variables are zero. In the case of a volume equation of a tree, the intercept is justified as the volume of the tree when the DBH is equal to zero.

The necessity of the intercept term in the case of dimensionally compatible volume equation obtained by incorporating theoretical information was examined. This was accomplished by regressing the outside bark volume of the trees, *Vob*, against D^bH^{3-b}, where D is the DBH and H is the total height of the tree. The value of β obtained from the simultaneous fit of individual tree volume and taper equation was used to calculate $D^b H^{3-b}$. Thus, the form of the examined equation was;

$$Vob = \delta + \alpha D^b H^{3-b} \tag{5.1.1}$$

Regressions were performed for unthinned, lightly thinned, and heavily thinned stands separately, since the value of β was different for differently managed stands. Estimates for intercept (δ), α and their p-values are presented in Table 14.

Table 14. Estimates for intercept, α, and their p-values obtained from simple linear regression for unthinned, lightly thinned, and heavily thinned stands.

Stand	Parameters	Estimates	p-value
Unthinned (β = 2.0887)	δ	0.713343	0.0050
	α	0.473171	0.0001
Light thinned (β = 2.0874)	δ	0.043338	0.8697
	α	0.506019	0.0001
Heavy thinned (β = 2.0820)	δ	0.356783	0.3760
	α	0.483617	0.0001

It is obvious from Table 14 that the intercept is highly insignificant in the case of light and heavy thinned stands. In the case of unthinned stand, however, the intercept is significant at probability level 0.05. This implies that the intercept is inappropriate in the case of light and heavy thinned stands, but could be appropriate in the case of unthinned stands. To carry out further analysis in the case of unthinned stand, the prediction abilities of the volume equation with and without intercept were examined. For this purpose, the same stem analysis data (104 tree-data) from unthinned plots were randomly split into two equal parts. One part was used to estimate the parameters and the other part was used for evaluation.

For the equation with intercept, β was first estimated by fitting the taper equation using nonlinear regression procedure in SAS. The estimated value of β was then substituted in the volume equation to calculate $D^b H^{3-b}$. Afterwards, the intercept and α of equation (5.1.1) were estimated by linear regression procedure in SAS. The same 52 tree-data were used to estimate the parameters, α and β in the case of dimensionally compatible volume equation also. These parameters were estimated by the simultaneous fit of tree volume and taper equations using the SYSNLIN procedure in SAS. The estimates for these parameters are presented in Table 15.

Table 15. Estimates of intercept, α, and β for volume equations with and without intercept.

Volume Equation	Parameters	Estimates
With intercept	intercept	0.3675
	α	0.5007
	β	2.0883
Without intercept	α	0.5159
	β	2.0882

The magnitude of the measure of bias, $\overline{D}$, in the case of dimensionally compatible volume equation (volume equation without intercept) is smaller than the corresponding value obtained from the volume equation with intercept. However, the measure of precision, S_D, in the case of volume equation with intercept is smaller than the one obtained from dimensionally compatible volume equation. The combined measure of bias and precision, $\sqrt{MS}$, in the case of dimensionally compatible volume equation is superior to the corresponding $\sqrt{MS}$ for the volume equation with intercept.

Outside bark volumes were calculated for the trees separated for evaluation using these parameters in the volume equation with and without intercept. Measure of bias, $\overline{D}$, measure of precision, S_D, and combined measure of bias and precision, $\sqrt{MS}$, were calculated for both equations. These statistics are displayed in Table 16.

Table 16. Summary statistics of measure of bias, measure of precision, and combined measure of bias and precision for volume equations with and without intercept.

Volume Equation	$\overline{D}$	S_D	$\sqrt{MS}$
Without intercept	-0.20702	1.65216	0.30878
With intercept	-0.28256	1.55862	0.35575

These results showed that the dimensionally compatible volume equation had better predictive ability than the volume equation with intercept. Thus, if an equation is derived by incorporating theoretical information, adding the intercept does not necessarily make the equation superior.

The tree volume equation is compatible with the taper equation even in the presence of the intercept. This means the total volume of a tree can be obtained by integrating the taper equation from height equal to zero to the total height of the tree. This has been proved analytically in the following steps.

The total volume of a tree, V can be obtained by integrating the taper equation from height zero to the total height, H of the tree as;

$$V = \int_0^H k\, d^2 dh + c \qquad (5.1.2)$$

Where, d is the diameter of the tree along the bole at height h, c is integration constant, and k is another constant to be determined. In the case of the dimensionally compatible volume equation, c is equal to zero. Otherwise, c is equal to the intercept.

Substituting the value of d^2 from taper equation (3.6.2) into equation (5.1.2) we obtain;

$$V = \int_0^H kg\, D^b \left(1 - \frac{h}{H}\right) h^{3-b}\, dh + c \qquad (5.1.3)$$

Finally,

$$V = a\, D^b H^{3-b} + c \qquad (5.1.4)$$

The value of k in equation (5.1.2) and the value of α in equation (5.1.4) turn out to be;

$$k = \frac{b(4-b)(3-b)}{g} \qquad (5.1.5)$$

and

$$a = \frac{kg}{(3-b)(4-b)} \qquad (5.1.6)$$

We see that both volume equations (with and without intercept) can be derived from the same taper equation. In other words, both the volume equations are compatible with the taper equation. The volume equation with intercept, however, will not be a dimensionally compatible volume equation.

In addition, the volume of any section of a tree along the bole can be obtained by integrating the taper equation from the height at the lower end to the height at the upper end of the section. In mathematical form, the volume of a section, V_s of a tree from height h_1 to h_2 from the ground is expressed as;

$$V_s = \int_{h_1}^{h_2} k\,d^2 dh + c \qquad (5.1.7)$$

It is clear that if $h_1 = 0$ and $h_2 = H$ (total height), then $V_s = V$ (total volume of the tree).

5.2 Taper Equation

As mentioned in previous chapters, the taper equation (3.6.2) yields consistent estimates and has the following characteristics essential for a taper equation;

(1) Diameter at height $h = H$ is always equal to zero,

(2) Diameter at height $h = 4.5$ ft is always equal to DBH, and

(3) Accounts for the butt swell part of the tree almost perfectly.

To demonstrate these properties of the taper equation, a tree profile has been generated from taper equation by choosing a typical value of $b = 2.085$ (at the middle in the range of b for unthinned, light thinned, and heavy thinned trees). Average values of DBH and height (8 in and 60 ft respectively ignoring the digits after decimal) were used to generate the profile. The profile is presented graphically in figure 1.

Detail study of the taper equation was accomplished by estimating and plotting the diameters at every 0.5 ft along the bole of the tree. There were at least 4 inflection points on the curve;

(1) at the bottom of the tree (between 0.5 to 1 ft from the ground),

(2) at about 6 ft (10% of the total height) from the ground,

(3) at about 42 to 45 ft (70 to 75% of the total height) from the ground, and

(4) at about 54 ft (90% of the total height) from the ground.

In mathematical terminology, a point in a curve is called an inflection point if the curve has a tangent line at this point and the sense of concavity changes from the point (Leithold, 1986). The curve crosses the tangent line at this point. These inflection points can be obtained analytically by taking the first and the second derivatives of the taper function with respect to the bole height, setting the second derivative equal to zero and solving for the height. We could calculate the second derivative of the taper function (3.6.2) analytically. The numerical solution of the resulting equation, however, is complicated because the exponent of the height is a fractional number.

The size of the butt diameter of a tree mainly depends on the value of b and DBH. The greater the value of b and/or DBH, the larger is the diameter of the butt. However, the influence of DBH in butt swell is less sensitive than the influence of the value of b. In order to examine the effect of b, DBH, and the height of a tree on its profile, diameters along the bole were plotted against the height by varying one and holding the other two constants. These plots have been presented in figures 2 - 5.

Figure 2 displays the profiles of four trees with the same value of b (2.085), the same DBH (8.0 in) and different values of height (40, 50, 60, and 70 ft). It is obvious from this figure that the profile up to the breast height is identical for all the trees. Therefore, if the DBH and the value of b are the same for two different trees, we expect the same butt diameters of these trees irrespective of their total heights. After the breast height the shorter the tree the more it tapers.

Figure 3 shows the profiles of four trees with the same value of b (2.085), the same height (60 ft) and different values of DBH (6.0, 8.0, 10.0, and 12.0 in). The profile from the bottom to the top is similar for all the trees. A tree with smaller DBH has less taper and smaller diameter at the butt compared to the taper and butt diameters of trees with larger DBH. Therefore, if the total height and the value of b are the same for two different trees, butt diameter of these trees varies slightly with DBH. The larger the DBH the greater the size of the butt and the more the tree tapers. However, the ratio of the butt diameter to the DBH remains constant.

In figure 4, tree profiles with the same value of the DBH (8.0 in) and the same value of height (60 ft), but different values of b (2.00, 2.05, 2.10, and 2.15) have been displayed. There is a great deal of difference among the sizes of the butt obtained from the different values of b. For the value of b equal 2.0 the swelling at the butt is nil. As the value of b increases from 2.0 the butt size increases dramatically. The ratio of the diameter at the butt to its DBH increases with the value of b. Above the breast height the greater the value of b the more the tree tapers.

Figure 5 demonstrates the case where b is less than 2. Tree profiles with the same value of the DBH (8.0 in) and the same value of total height (60 ft), but different values of b (1.9, 1.8, 1.7, and 1.6) have been generated. The difference in shapes for different values of b is obvious. Since the sum of b (exponent of DBH) and the exponent of the height equals 3, the smaller the value of b the larger the exponent of the height. Therefore, the volume of a tree increases by decreasing the values of b given that the total height and DBH are constant. This is because the contribution of the total height in the volume equation for smaller b, becomes greater than its contribution for larger value of b. As the value of b decreases from 2.0 the butt size decreases dramatically. The ratio of the diameter at the butt to its DBH decreases with the value of b. Above the breast height the smaller the value of b the smaller the tree tapers.

A tree profile has been generated from taper equation by choosing an arbitrary value of $b = 1.9$. Average values of DBH and height (8 in and 60 ft respectively) were used to generate the profile. It is clear from the figure that the diameter decreases slowly as we go down from 4.5 ft to 0.0 ft along the bole of the tree.

An important fact to be mentioned at this point is that the total volume of the tree remains the same irrespective of the value of b, if the DBH and total height of the trees are the same. Since the DBH is very small compared to the total height of a tree, the value of D^bH^{3-b} for large b would be smaller than the value of D^bH^{3-b} for small b. In order to calculate the total volume of a tree accurately, the value of the parameter a would vary with the value of the parameter b; that is, the greater the value of b the smaller the value of a. In addition, the distribution of the total volume of the tree along the bole would be different for different values of b. This means that if b is chosen to be equal to 2.0, the total volume of the tree is estimated accurately. This is the reason why the combined variable volume equation is superior to other empirical models for estimating the volume of the trees. In combined variable volume equation, exponents of DBH and total height are 2 and 1 ($b = 2$ and $3\text{-}b = 1$) respectively. Since the exponents of DBH and total height sum to 3, this equation is a dimensionally valid volume equation.

Even though the estimate of the total volume of a tree obtained using the value of b equal to 2.0 is accurate, this value of b makes the shape of the tree completely parabolic from the beginning to the end of the tree. As a result, the volume estimated at the butt of the tree is smaller than the butt volume the tree actually holds. The difference between the actual and the estimated volume below the breast height is distributed evenly along the bole of the tree above the breast height.

Thus, the diameter estimated below the breast height tends to be smaller than the actual diameter. The diameter estimated above the breast height tends to be larger than the observed diameter at that point. Consequently, if we estimate the section volume between any two points (other than very end points) along the bole of a tree by integrating the taper equation the estimate would be biased. If b is less than 2.0 the diameters below breast height will be smaller than DBH. The results imply that the value of b should be greater than 2.0. The bigger the butt of the tree the greater is the value of b. Theoretically, the value of b can not exceed 3.0, but in practice it should be less than 2.2.

As we mentioned in chapter 3, the DBH measured by the diameter tape is greater than the diameter observed at 4.5 ft in stem analysis for most of the trees. Therefore, if we estimate the parameter b only from the individual tree volume equation using these data, according to the discussion above, the estimate should be less than 2. To examine whether the estimate for b is less than 2 using these data, outside bark volume was regressed against DBH and height in individual tree volume equation. Ordinary nonlinear least square methods in SAS were applied to perform this task. Estimates for b along with its standard error and 95% confidence intervals for unthinned, light thinned, and heavy thinned stands are displayed in Table 17.

Table 17. Estimates for β, their approximate standard errors, and corresponding asymptotic 95% confidence intervals for unthinned, light thinned, and heavy thinned stands obtained by fitting the individual tree volume equation.

Stand	Parameter	Estimate	Approx. Std. Error	Asymptotic 95% confidence interval
Unthinned	β	1.7122	0.0622	1.5887 - 1.8357
Light thinned	β	2.0695	0.0629	1.9446 - 2.1943
Heavy thinned	β	1.9638	0.0778	1.8095 - 2.1182

As suspected, in the case of unthinned and heavy thinned stands, the estimate for β is less than 2. In the case of light thinned stand, however, the estimate is greater than 2, but the 95% confidence interval for β shows that this estimate is not significantly different from any value between 1.9446 and 2.1943 at probability level 0.05. Therefore, even if the diameters smaller than DBH below 4.5 ft of the trees have not been used directly in regression, the results obtained are very consistent with the theoretical ones. Had we had the DBH measured in the same way as other diameters are measured along the bole of a tree in stem analysis, estimates for β from the regression of individual tree volume equation alone would have been greater than 2.

We discussed the characteristics and the mathematical properties of the taper equation. The taper equation was further evaluated by comparing predicted diameters to measured taper for two trees; one in the range of 5-9 inches and the other one in the range of 9-13 inches of DBH from each stand (unthinned, light thinned, and heavy thinned). Those trees selected for evaluation for which the DBH obtained from the tape and stem analysis were the same to avoid the problems caused by the difference. To make comparisons, the predicted and the observed diameters along the bole of the trees were plotted in the same graph. These plots for unthinned, light thinned, and heavy thinned stand trees are presented in figures 6, 7, and 8, respectively.

It is obvious from these plots that the stump diameters have been estimated very accurately for all stands. Since there was only one measurement for diameter below the DBH for each tree, diameters at other heights between stump and breast height could not be compared. Above the breast height, there was no fixed pattern. There were crests and dips between the DBH and the tip of the trees. These crests and dips might be due to the measuring point falling at the origin of branches.

On the other hand, the profile generated from the taper function follows a fixed pattern from the breast height to the tip of the trees. It obviously does not account for the irregularities caused by branches. Nevertheless, the taper equation seemed to be working very well. At several points along the bole, the estimated diameters coincided with the observed ones. Even at irregular points, the difference between observed and predicted diameters was, on average, less than one inch.

5.3 Basal Area Equation

We presented an alternate basal area equation (3.7.2) in chapter 3 in terms of DBH, total height, and parameters, β and γ. In chapter 4, we showed that the average basal area of the stand was estimated accurately using the Quadratic Mean Diameter (QMD) and Lorey Mean Height (LMH) instead of average DBH and average height of the stand. We also demonstrated that other combinations of these functions of DBH and height would lead to the biased estimates.

To check whether the same value of estimate for β can be obtained from the basal area equation, average basal area per unit tree of the stand was regressed against QMD and LMH. For this purpose, all remeasurement data (except stem analysis) from plot establishment to the 5^{th} remeasurement of the trees were used. Data obtained at the time of plot establishment were included here for two reasons; (1) to increase the number of observations for regression and (2) to look at how these equations work for trees that existed before plot establishment.

Basal area of each tree was calculated using the equation $BA = \frac{p}{4}(DBH)^2$, and average basal area per unit tree was obtained for each stand at each remeasurement. Value of γ was calculated using QMD and LMH. In this way, we had 6 observations (one observation at each remeasurement) for each stand to perform regression analysis. Had we not included the data from the plot establishment, we would have only five observations for each stand.

Nonlinear Least Square methods in SAS were applied to get an estimate for β for each differently managed stand using basal area equation (3.7.2). To make comparisons among the values of β estimated from different equations and methods, stem analysis data were also fit to the taper equation alone for each stand. This was also performed in SAS using Nonlinear Least Square methods. Estimates of β obtained in this fashion, along with their corresponding estimates obtained from stem analysis data using taper and volume equations for these stands are displayed in Table 18.

We see that the estimates of β from the fit of taper equation alone and from the simultaneous fit of volume and taper equation using the stem analysis data for light and heavy thinned stands are the same up to the 4^{th} decimal places. For unthinned stand these estimates are the same up to 3^{rd} decimal places. In the case of estimates from basal area equation, these values agree up to the 3rd decimal places with their corresponding values for all stands. After the 4th place from decimal (not shown here), however, estimates of β obtained from all three methods were different for all stands.

Table 18. Estimates of β obtained from nonlinear linear regression of basal area and taper equations separately and simultaneous fit of volume and taper equation for unthinned, light thinned and heavy thinned stands.

Equation	Estimate of β for Unthinned Stand	Estimate of β for Light thinned stand	Estimate of β for Heavy thinned stand
Basal area equation	2.0888	2.0871	2.0825
Taper equation	2.0888	2.0874	2.0820
Taper and volume equation	2.0887	2.0874	2.0820

These differences in β observed from three different methods mentioned above do not make any significant difference in calculated volume of a tree or stand from the volume equation and/or diameter of a tree along its bole from taper equation. Therefore, β can be estimated from either of these methods accurately. Estimating β from the basal area equation, however, needs γ which can be estimated only if the value of β is known.

Rather than the methods of parameter estimation, one purpose of the demonstration and discussion above was to examine how the variables QMD and LMH, and parameters β and γ are related. Another purpose of the analysis was to check whether these estimates were numerically consistent/compatible with algebraically consistent/compatible equations (volume, taper and basal area equations).

The results presented in chapter 4 and in this section above showed that instead of average DBH and average heights, QMD and LMH were the variables required for the basal area equation of the stand. Estimates of β obtained from the individual tree volume and the taper equation with variables QMD and LMH furnished the numerically consistent estimates for basal area of the stand. Using β estimated from the volume and the taper equation with other functions of DBH and height resulted to the biased estimates for the basal area of the stand. Moreover, regressing the basal area of the stand against QMD and LMH gave the numerically consistent estimate of β with the one obtained from individual tree volume and taper equations.

5.4 Stand Volume Equation

In section 5.1 we discussed the properties of the individual tree volume equation. The equation was compatible with the taper equation i.e. integrating the taper equation from height zero to the tip of the tree gave the total volume of the tree. We could also calculate the volume of a section of the tree by integrating the taper equation from the height where the section starts to the height where the section ends.

In order to obtain a stand volume equation we expressed the dimensional form of stand volume equation in terms of Quadratic Mean Diameter (QMD), some function of stand height (p), and total number of trees of the stand in section (3.8). In that section, we also noted that the function of height could be any function of stand height (e.g. average height, Lorey Mean Height (LMH) or some other function of height), but the dimension of p should be the dimension of length, $[L]$. This ruled out the possibility of p being a function such as square root, square, or some other higher order power functions of the height. Otherwise, the dimension of p could not be $[L]$.

Therefore, we substituted different combinations of QMD and two functions of height, average height and LMH that have the dimension of $[L]$ into the stand volume equation in section (4.3) to estimate the stand volumes. For the sake of completeness, we also estimated stand volumes by substituting different combinations of these heights and the average DBH into the stand volume equation. We found in that section that the estimated volumes for all stands by the combination of QMD and LMH were closer to the observed ones than the stand volumes estimated using other combinations of these heights and DBH.

To make formal comparisons, measure of bias, $\overline{D}$, measure of precision, S_D, and combined measure of bias and precision, $\sqrt{MS}$, were calculated for all combinations of these functions of heights and DBH. These statistics for unthinned, light thinned and heavy thinned stands are presented in tables 19, 20, and 21 respectively.

Table 19. Summary statistics of measure of bias, measure of precision, and combined measure of bias and precision for unthinned Stand.

Volume Equation	$\overline{D}$	S_D	$\sqrt{MS}$
Volume Obtained by LMH and QMD (cuft)	0.0185	0.0013	0.0185
Volume Obtained by LMH and average DBH (cuft)	0.5476	0.1136	0.5499
Volume Obtained by average height and QMD (cuft)	0.4912	0.0416	0.4916
Volume Obtained by average height and average DBH (cuft)	0.9854	0.1564	0.9879

Table 20. Summary statistics of measure of bias, measure of precision, and combined measure of bias and precision for light thinned Stand.

Volume Equation	$\overline{D}$	S_D	$\sqrt{MS}$
Volume Obtained by LMH and QMD (cuft)	0.0319	0.0062	0.0320
Volume Obtained by LMH and average DBH (cuft)	0.5191	0.1261	0.5221
Volume Obtained by average height and QMD (cuft)	0.4910	0.0281	0.4912
Volume Obtained by average height and average DBH (cuft)	0.9542	0.1481	0.9565

Table 21. Summary statistics of measure of bias, measure of precision, and combined measure of bias and precision for heavy thinned Stand.

Volume Equation	$\overline{D}$	S_D	$\sqrt{MS}$
Volume Obtained by LMH and QMD (cuft)	0.0373	0.0076	0.0374
Volume Obtained by LMH and average DBH (cuft)	0.4987	0.1328	0.5022
Volume Obtained by average height and QMD (cuft)	0.5002	0.0359	0.5005
Volume Obtained by average height and average DBH (cuft)	0.9411	0.1587	0.9438

These results show that in the case of the volume equation with QMD and LMH as the independent variables the measure of bias, the measure of precision, and combined measure of bias and precision are much smaller than their corresponding values obtained from other combinations of the functions of DBH and height. Thus, the stand volume equation with QMD and LMH as its predictor variables has a better fit and predictive ability than the volume equations with other combinations of DBH and height as their independent variables.

Consequently, as was the case for stand basal area equation, QMD and LMH are the variables required for the stand volume equation. Volume equations obtained from other combinations of the functions of DBH and height will lead to misspecified models and the estimates for the stand volumes from these models will not be accurate.

As we mentioned in previous sections the parameter β was estimated from simultaneous fit of individual tree volume and taper equations for each of the three differently managed stands. The same value of β was obtained by fitting the taper equation alone as was estimated from the simultaneous fit. In section (5.3) we fit mean basal area of a stand against QMD and LMH and obtained the same estimate for β for each stand showing the numerical consistency/compatibility among the system of equations.

To check whether this consistency/compatibility holds for the stand volume equation, mean volume per tree of a stand was regressed against QMD and LMH of the stand. The same remeasurement data obtained from plot establishment to the 5^{th} remeasurement of the trees and used in the case of basal area equation were used to perform the task. Volume of a tree for a particular stand was calculated using the dimensionally compatible volume equation with estimates of parameters for that stand. In this way, 6 observations (one from each remeasurement) were obtained for each stand to perform regression analysis.

Nonlinear Least Square methods in SAS were applied to get estimates for α and β in the stand volume equation (3.8.8) replacing $\overline{D}$ and p by QMD and LMH respectively. These estimates for α and β were obtained only for outside bark volumes for each differently managed stand. The estimates for α and β along with their corresponding approximate standard errors for differently managed stands are displayed in Table 22.

To make the comparisons convenient, estimates for α and β obtained from the simultaneous fit of individual tree volume and taper equations using stem analysis data are also displayed in the table. If the estimates obtained by fitting the stand volume equation are not significantly different from the ones obtained by the simultaneous fit of individual tree volume and taper equations then the system is numerically consistent. Otherwise, the stand volume equation is not numerically consistent with the individual tree volume and taper equations.

Table 22. Estimates for α, β, and their approximate standard errors for outside bark volumes obtained from the simultaneous fit of individual volume and taper equations and from the stand volume equation for unthinned, light thinned and heavy thinned stands.

Method	Parameters	Estimates	Approx. Std. Error	Asymptotic 95% confidence interval
Unthinned stand				
Simultaneous fit of volume and taper equations	α	0.5009	0.0042	
	β	2.0887	0.0017	
Fit of Stand volume equation only	α	0.6395	0.0719	0.4398 - 0.8392
	β	2.1415	0.0249	2.0724 - 2.2106
Light thinned Stand				
Simultaneous fit of volume and taper equations	α	0.5075	0.0035	
	β	2.0874	0.0015	
Fit of Stand volume equation only	α	0.4120	0.0786	0.1938 - 0.6301
	β	2.0393	0.0430	1.9198 - 2.1587
Heavy thinned Stand				
Simultaneous fit of volume and taper equations	α	0.4945	0.0035	
	β	2.0819	0.0015	
Fit of Stand volume equation only	α	0.4094	0.0236	0.3439 - 0.4749
	β	2.0382	0.0131	2.0016 - 2.0747

We can see in the table that the estimates for α and β obtained from the fit of stand volume equation in the case of unthinned stand are slightly higher than the corresponding estimates from the simultaneous fit of volume and taper equations. On the other hand, both of these estimates from the fit of stand volume equation in the case of light and heavy thinned stands are smaller than the corresponding estimates from the simultaneous fit.

The trend, however, remained consistent, i.e. the estimate for β in the case of unthinned stand is the largest and the estimate in the case of heavy thinned stand is the smallest as was obtained from the simultaneous fits. As suspected the approximate standard errors of estimates in the case of the stand volume equation are higher than the corresponding estimates in the case of simultaneous fit of volume and taper equation for all stands. As a result, the asymptotic 95% confidence intervals for α and β in the case of stand volume equation are wider than the ones (not shown) from the simultaneous fit of volume and taper equation. The main purpose of displaying 95% confidence intervals obtained from the regression of stand volume equation is to check whether these intervals contain the estimates from the simultaneous fit of individual tree volume and taper equations.

In the case of unthinned and light thinned stands, confidence intervals for both α and β obtained from the stand volume equation included the estimates obtained from the simultaneous fit. This implies that the estimates for α and β obtained from stand volume equation and from simultaneous fits of individual tree volume and taper equations are not significantly different at probability level 0.05 for these stands. In the case of light thinned stand, however, the 95% confidence interval for β includes the numbers less than or equal to 2. Therefore, it did not rule out the possibility of β being less than or equal to 2 for light thinned stands. In the case of heavy thinned stand, 95% confidence intervals did not contain the estimates from the simultaneous fit. This means the estimates for α and β obtained from the regression of the stand volume equation are significantly different from the ones obtained from the simultaneous fit at probability level 0.05 for heavy thinned stand.

These results raised the question that if the system of equations is algebraically compatible and the numerical compatibility was observed from individual tree volume and taper equations to the stand basal area equation, why does the numerical compatibility not hold for the stand volume equation? Moreover, even if numerical compatibility holds for unthinned and light thinned stands, why are the estimates for β not exactly the same as they were in the case of basal area equation up to the third decimal places?

There are three factors mainly involved in driving these incompatible results for these stands.

1. Volumes of the stands or trees calculated were only estimates, not the exact values calculated directly.
2. The range of the data (DBH and height) obtained from the remeasurements of the stands was different from the range of the data obtained from stem analysis.
3. The number of observations used in the case of regression of the stand volume equation were very small compared to the number of observations in the case of simultaneous fit of the individual tree volume and the taper equations.

These factors are discussed in detail in the rest of this section below.

In the case of volume of an individual tree, there could be many trees with the same DBH but different heights and butt diameters. Even if the DBH and the total height are the same for two different trees, it can not be guaranteed that the butt diameter of these trees would be the same. As a result, an estimate (not the exact value) of the volume of a tree can be obtained from a volume equation or a model based on the DBH and the total height.

On the other hand, basal area does not depend on the height or the stump diameter. Once the DBH of a tree is measured, the exact basal area of the tree can be calculated by applying the familiar formula irrespective of the species, butt diameter, total height, and/or site of the tree are. As opposed to basal area, calculation of exact content of the volume of a tree using only its DBH and height is almost impossible.

In the case of factor 2, we know that predictions outside the range of the data used in fitting the regression may not be consistent. Since statistical fitting procedures cannot take into account unmeasured data, the consistent results are obtained from a model if predictions are made only on the range of the data on which the model is based (Oderwald and Hans (1993). In the stem analysis data, the minimum and maximum values for DBH were 3.8 in (unthinned stand) and 13.9 in (light thinned stand), respectively and for height were 33.8 ft (unthinned) and 87.9 ft (heavy thinned), respectively.

In chapter 4, summary statistics for DBH and height were presented only for the data obtained at and after the first remeasurement. Since the data measured at the time of the plot establishment were included in the regression of stand volume equation, summary statistics for the DBH and height measured at the time of plot establishment for these differently managed stands are presented in Table 23.

Table 23. Summary statistics for DBH and height measured at the time of plot establishment on unthinned, light thinned, and heavy thinned stands.

Variable	Number of trees	Mean	Standard deviation	Minimum	Maximum
Unthinned trees					
DBH (in)	5554	5.44	1.79	0.50	14.10
Height (ft)	5554	36.75	11.64	6.00	79.00
Light Thinned trees					
DBH (in)	11867	5.51	1.80	0.20	13.40
Height (ft)	11867	37.09	11.84	7.00	80.00
Heavy Thinned trees					
DBH (in)	12206	5.46	1.80	0.10	14.00
Height (ft)	12206	36.91	11.85	5.00	81.00

The minimum and maximum values of DBH measured at the time of plot establishment are 0.1 (heavy thinned stand) and 17.4 (light thinned stand) in, respectively and for height are 6 ft (unthinned stand) and 101 ft (light and heavy thinned stand), respectively. These minimum values for both DBH and the height are far smaller than the minimum values of these variables measured in stem analysis.

Considering factor 3, the greater the number of observations the more precise is the estimator, the smaller is its standard error, and hence the narrower is the confidence interval. In the case of simultaneous fit of individual tree volume and taper equations, the number of observations used in fitting the regressions were 1465, 1537, and 1542 for unthinned, light thinned and heavy thinned stands, respectively. In the case of stand volume equation there were only 6 observations from each stand and two parameters for each stand were estimated. This is obviously the reason for wider confidence interval in the case of stand volume equation.

We can not control factors 1 and 3, i.e. we can not measure the volumes of all the trees in the stands directly, nor can we increase the number of observations in the regression. To do so, one need to wait 300 years to get 100 more observations, since the stands are remeasured in every three years. Moreover, if all the trees are felled down to get direct measurements of tree volumes no trees will be left for the next remeasurement.

However, we can check how the second factor has influenced our results. Remeasured trees with DBH and height in the range of stem analysis data were selected. Volumes of these trees were calculated using the method described above. Mean volumes per tree, QMD, and LMH were then calculated at each remeasurement for each stand using these trees. Mean volume per tree of each stand was regressed against its QMD and LMH using ordinary nonlinear least square methods in SAS. The estimates for α and β along with their corresponding asymptotic 95% confidence intervals are presented in Table 24. To make the comparisons convenient the estimates from the simultaneous fit of individual tree and taper equations are also presented in the table.

Results in the table show that the estimates for α and β obtained from the regression of stand volume equation are much closer to the corresponding estimates obtained from simultaneous fit of individual tree volume and taper equations than the ones obtained by using all tree data. In addition, approximate standard errors are smaller. As a result, confidence intervals obtained in this case are much narrower than the ones obtained from the regression of stand volume equation using all tree data. Most importantly, asymptotic 95% confidence intervals for α and β obtained in this case contain the estimates obtained from simultaneous fit of the individual tree volume and the taper equations for all the stands.

Table 24. Estimates for α, β, and their approximate standard errors for outside bark volumes obtained from the simultaneous fit of individual volume and taper equations and from the stand volume equation for unthinned, light thinned and heavy thinned stands using the trees with DBH and height in the range of stem analysis data.

Method	Parameters	Estimates	Approx. Std. Error	Asymptotic 95% confidence interval
Unthinned stand				
Simultaneous fit of volume and taper equations	α	0.5009	0.0042	
	β	2.0887	0.0017	
Fit of Stand volume equation only	α	0.5129	0.01256	0.4780 - 0.5478
	β	2.0930	0.00544	2.0779 - 2.1081
Light thinned Stand				
Simultaneous fit of volume and taper equations	α	0.5075	0.0035	
	β	2.0874	0.0015	
Fit of Stand volume equation only	α	0.4983	0.03258	0.4079 - 0.5888
	β	2.0824	0.01478	2.0413 - 2.1234
Heavy thinned Stand				
Simultaneous fit of volume and taper equations	α	0.4945	0.0035	
	β	2.0819	0.0015	
Fit of Stand volume equation only	α	0.4698	0.01170	0.4373 - 0.5023
	β	2.0696	0.00569	2.0538 - 2.0854

Based on these results, it can be concluded that the estimates for α and β obtained from the regression of stand volume equation are not significantly different from the estimates obtained from the simultaneous fit of volume and taper equations at the 0.05 probability level for all stands. As we noticed earlier, these estimates were significantly different for heavy thinned stand using all tree data. In that case, the minimum DBH and height were 0.1 in and 5 ft respectively and these values were far away from the minimum values of DBH and height in the case of stem analysis data. As we suspected, these out of range data pushed the estimates to be incompatible and made the confidence intervals wider than the ones to be resulted from the smaller number of observations.

Even though we did not have volumes of trees measured directly as pointed out in factor 1 above, estimates for β obtained from the regression of stand volume equation are consistent with the estimates obtained from the simultaneous fit. Estimates are consistent in the sense that the estimate for β is the largest in the case of unthinned stand and the smallest in the case of heavy thinned stand. The magnitude of the estimate for α is not a concern of great importance here. The concern of great importance is the magnitude of β that decides the shape of the trees.

In spite of small number of observations, the confidence intervals are narrower and these confidence intervals contain the estimates obtained from the simultaneous fit of individual tree volume and the taper equations for all stands. As we mentioned earlier that the number of observations in the case of regression of stand volume equation were only 6 as opposed to about 1500 observations used in fitting the individual tree volume and the taper equations simultaneously for each stand. Moreover, these 95% confidence intervals for β do not contain the numbers less than or equal to 2 for all stands. This means these estimates for β are significantly greater than 2 at probability level 0.05 for all the stands.

5.5 Stand Volume and Basal Area Growth

In chapter 3, we showed that the basal area and volume growth of a stand between time interval t_1 and t_2 were obtained by subtracting the basal area and volume at time t_1 from the basal area and volume at time t_2 respectively. We also presented the analytical proof of the compatibility of basal area and volume growth equations across time. In chapter 4, we demonstrated these results using the data. At that time, we saw that the volume and basal area growth estimated between remeasurements *i* and *j* were independent of their intermediate estimates for any *i* and *j*, showing the numerical compatibility across the time.

To summarize the results discussed in this chapter above, algebraic and numerical properties of the compatible system of equations are briefly mentioned here. In chapter 3, we derived the compatible system of equations starting from individual tree volume and taper equations to the stand basal area and volume equations. As we mentioned earlier, the stand basal area and volume growths were compatible across the time numerically and analytically. In addition, these equations are dimensionally compatible.

In the whole compatible system of equations, there were only two parameters α and β. These parameters were estimated by the simultaneous fit of individual tree volume and taper equations using the data obtained from the stem analysis for differently managed stands. To examine the numerical compatibility of this algebraically compatible system of equations, these parameters were estimated again from stand basal area and volume equations using stand remeasurement data.

Volumes for the trees used in stem analysis and the basal areas for the trees observed at the time of plot remeasurements were obtained directly (without estimation of parameters). Therefore, the same estimates for β were obtained from the simultaneous fit of individual tree volume and taper equations and the fit of basal area equation for a stand. The same estimates for β were also obtained by fitting the stand volume equation using the trees remeasured in every three years with DBH and height in the range of stem analysis data. Since the volumes of these remeasured trees were calculated using the estimated values of α and β in the volume equation, applying the data outside the range of stem analysis resulted in incompatible estimates for heavy thinned stands.

At the time of derivation of the system of equations, the function of height to be used in the stand volume and basal area equations was unknown. Lorey mean height (LMH) along with quadratic mean diameter (QMD) in the stand basal area and volume equation predicted the basal area and volumes of the stands precisely and accurately. Therefore, LMH was chosen as the appropriate function of height for the stand volume and basal area equations.

Finally, as we derived all the equations (individual tree, taper, stand basal area, and stand volume equations) as an algebraically compatible system of equations, these were also found to be numerically compatible. Numerical compatibility holds if the range of the data in estimating the parameters in one equation is nearly the same as the range of data in another equation and the predicted values from one equation are used in another equation.

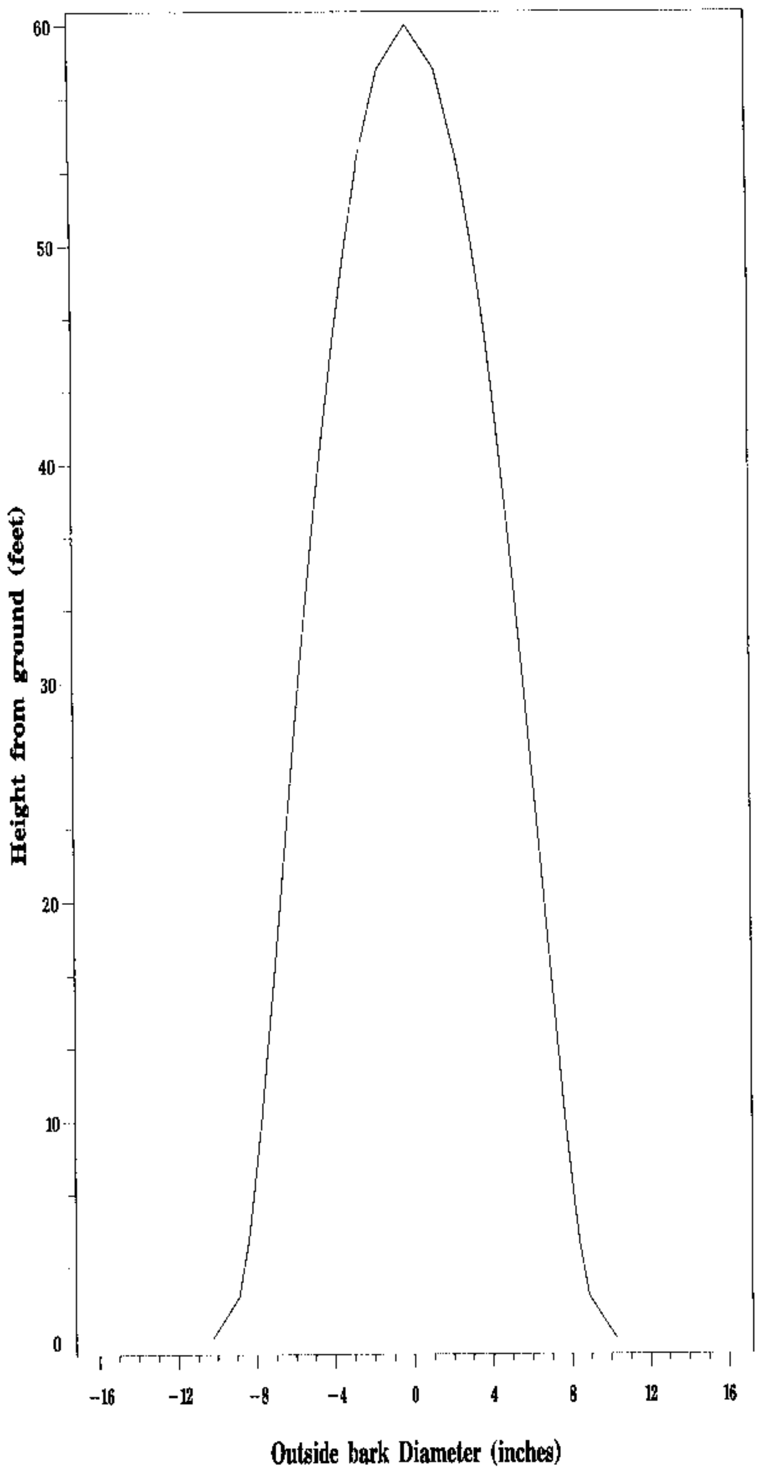


Figure 1. Tree profile generated from taper equation 3.6.2 with Beta = 2.085, DBH = 8 in, Total height = 60 ft,

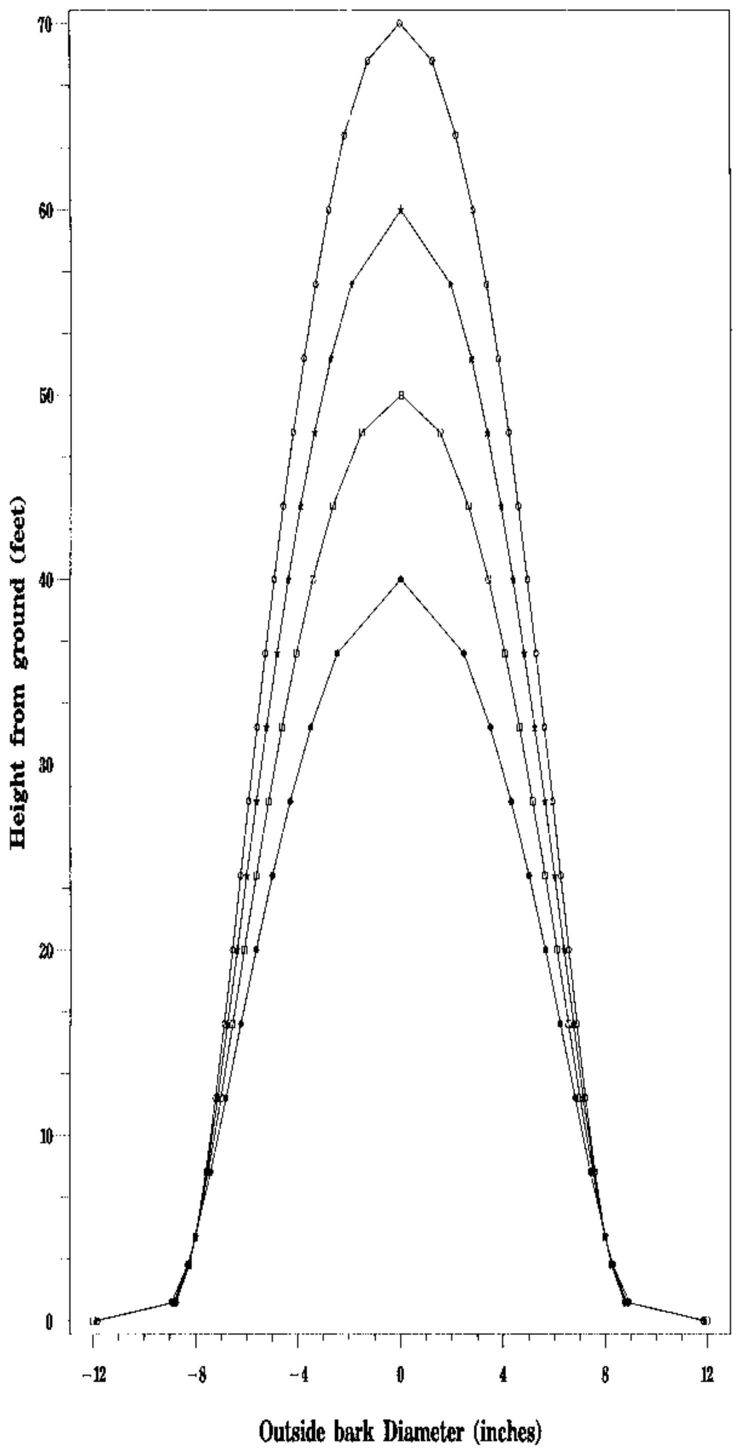


Figure 2. Tree profile generated by using different values of height
DBH = 8 in, beta = 2.085, Hights; dot=40 ft, square=50 ft, star=60 ft and circle=70 ft

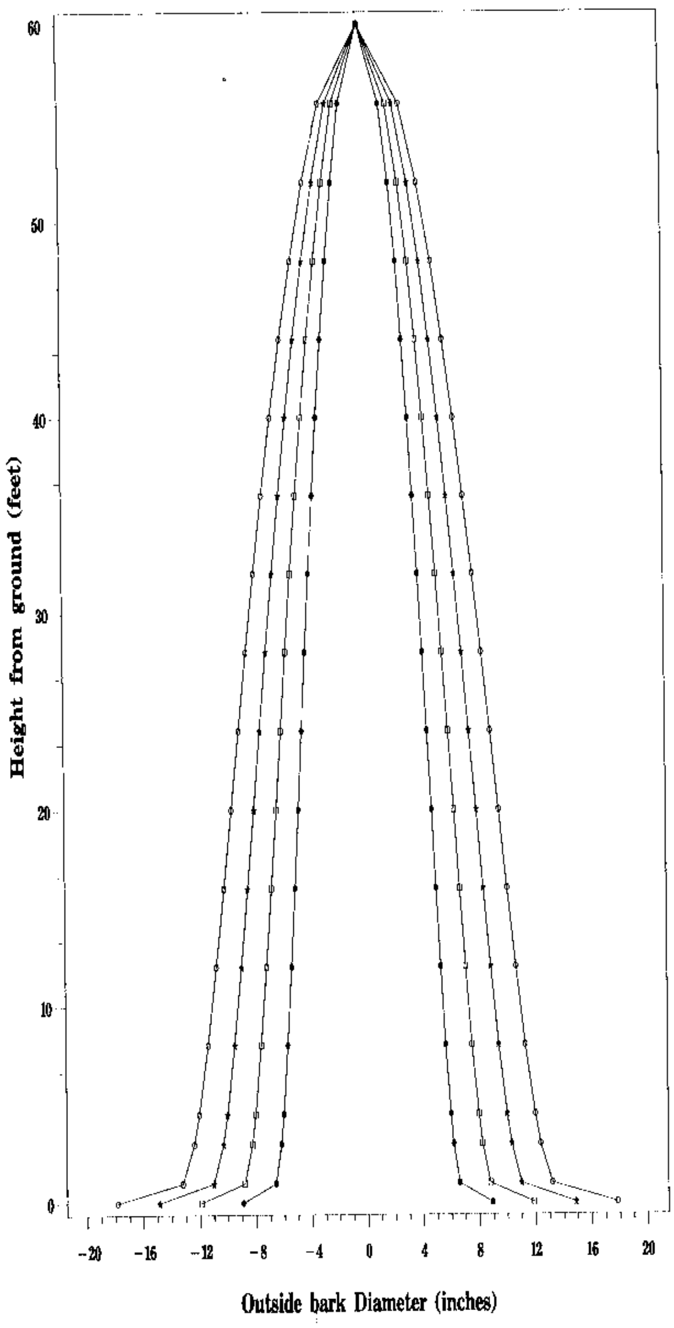


Figure 3. Tree profile generated by using different values of DBH
Height = 60 ft, beta = 2.085, DBH; dot = 6 in, square = 8 in, star = 10 in and circle = 12 in

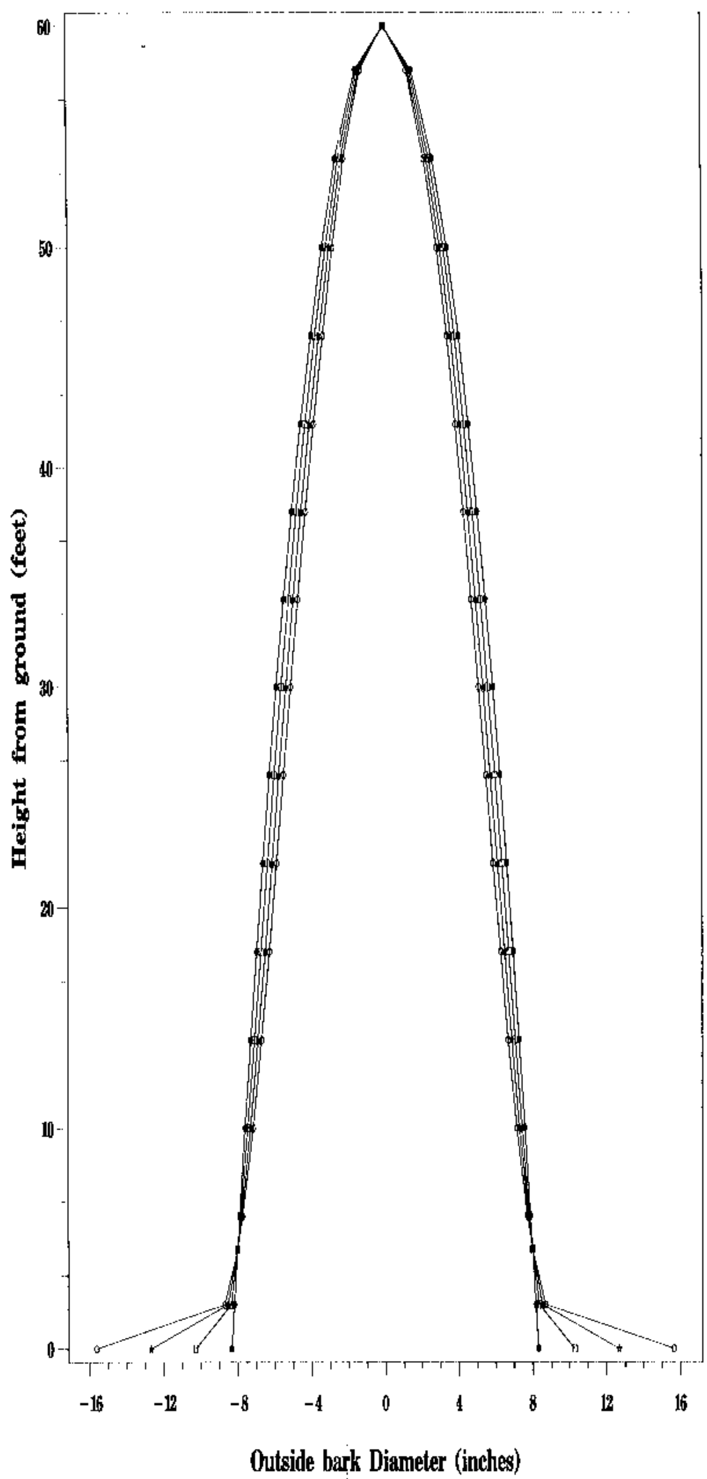


Figure 4. Tree profile generated by using different values of beta
DBH = 8 in, Height = 60 ft, Beta; dot = 2.0, square = 2.05, star = 2.1 and circle = 2.15

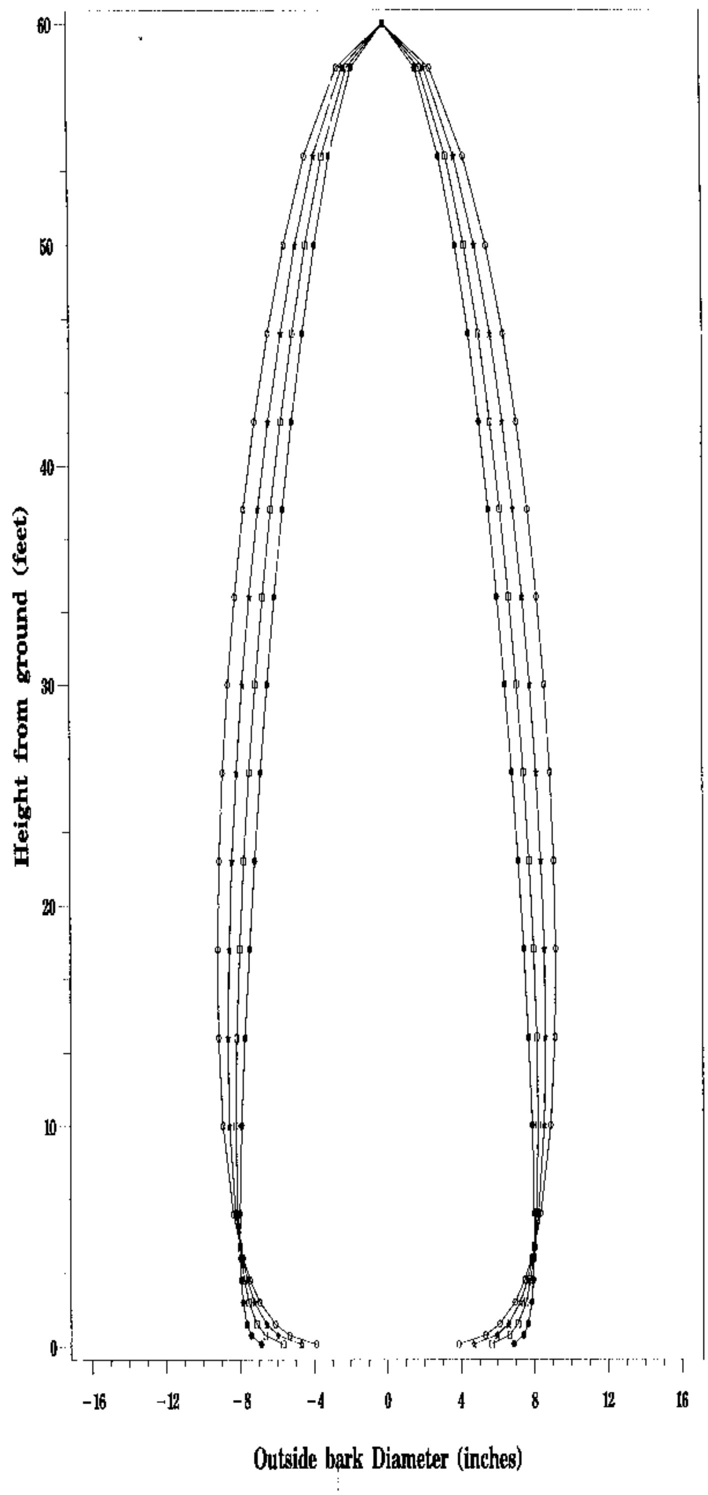
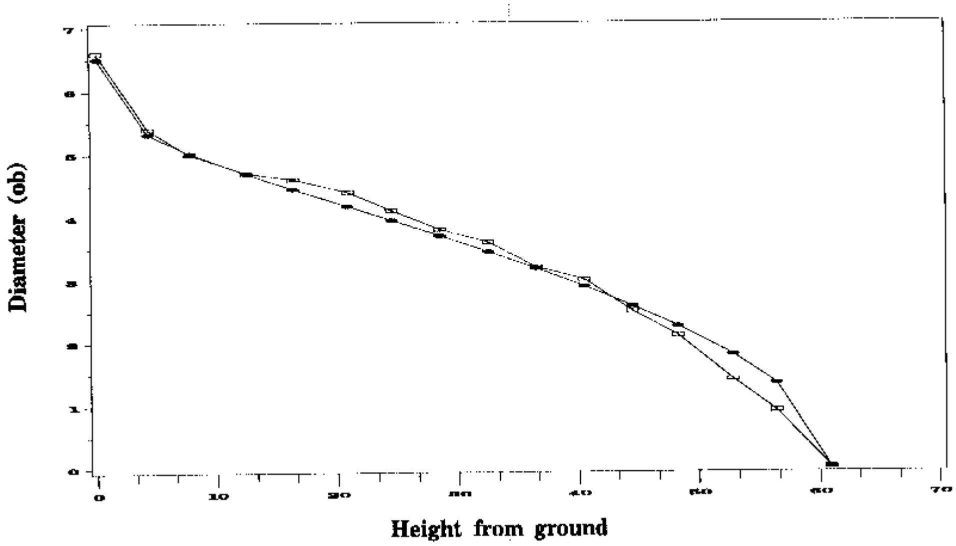
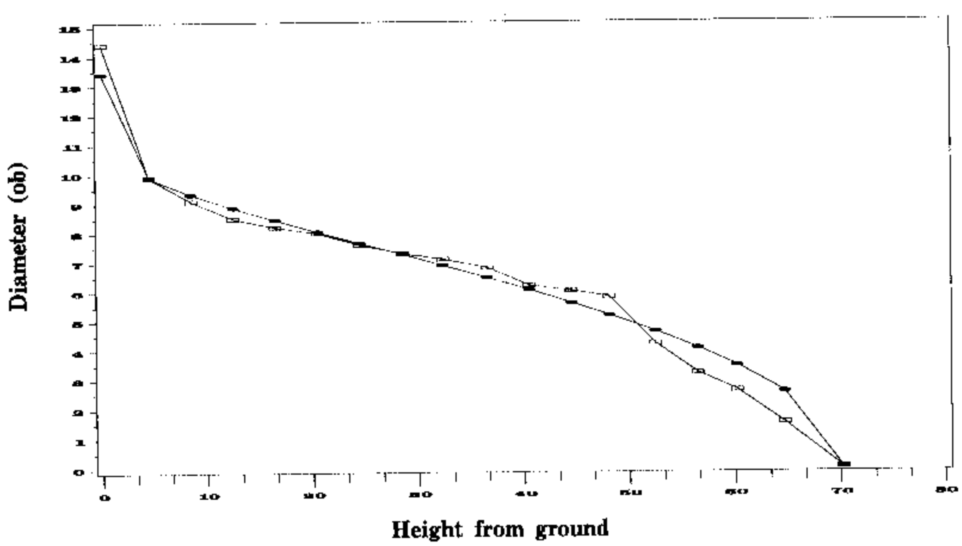


Figure 5. Tree profile generated by using different values of beta
DBH = 8 in, Height = 60 ft, Beta; dot = 1.9, square = 1.8, star = 1.7 and circle = 1.6

DBH = 5.3 in, Hight = 60.9 ft



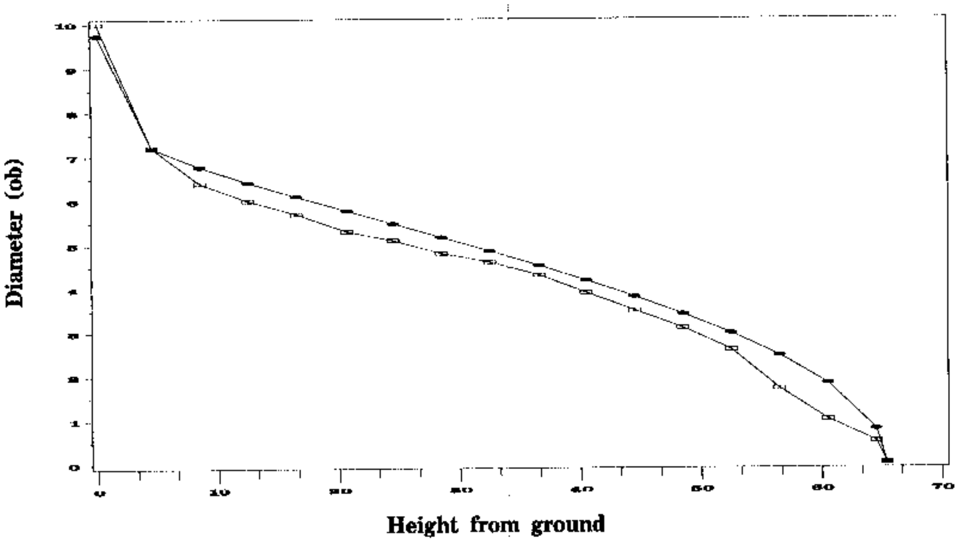
DBH = 9.9 in, Hight = 70.2 ft



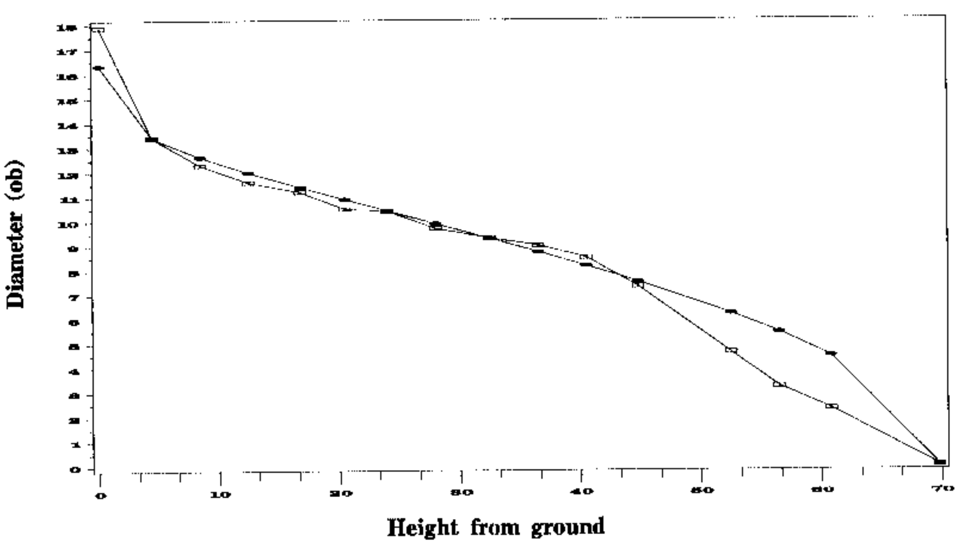
Symbols; dot=generated from taper equation, square=observed

Figure 6. Tree profile observed and generated from taper equation for unthinned stands

DBH = 7.2 in, Hight = 65.4 ft

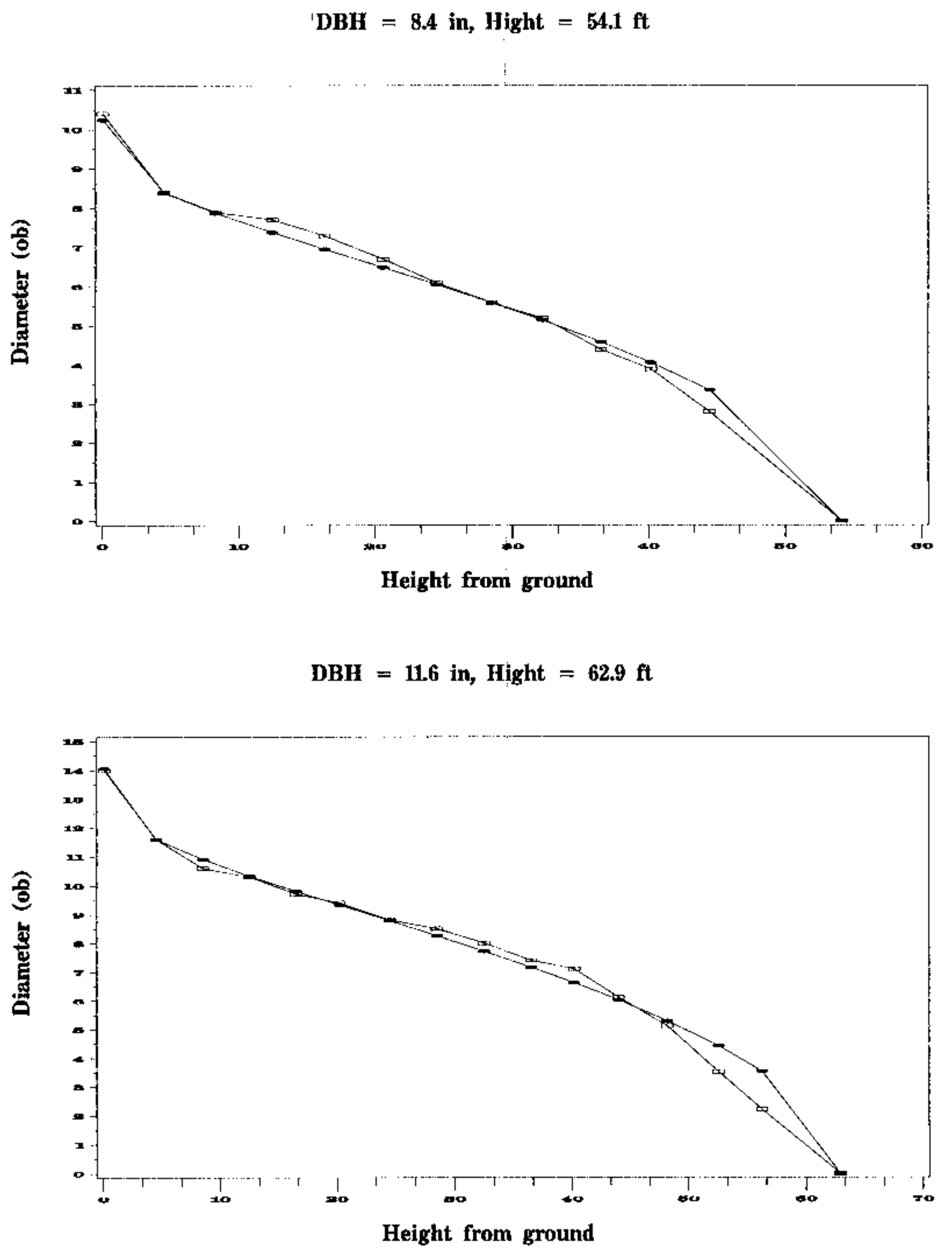


DBH = 13.4 in, Hight = 69.8 ft



Symbols; dot = generated from taper equation, square = observed

Figure 7. Tree profile observed and generated from taper equation for lightly thinned stands



Symbols; dot = generated from taper equation, square = observed

Figure 8. Tree profile observed and generated from taper equation for heavily thinned stands

Chapter 6

CONCLUSIONS

The concept of compatibility of the system of equations estimating tree and stand characteristics is probably as old as the idea of modeling these properties. Researchers have spent a great deal of energy in developing compatible taper and volume equations for a tree. The compatible system may be taper-based or volume-based depending on which function is derived first. There have also been tremendous efforts made at deriving compatible volume and basal area growth models of a stand. The main goal of these efforts was to make the system algebraically and numerically compatible.

Algebraic compatibility and consistency were achieved to some extent in other studies by applying mathematical properties such as derivatives, integrals, inflection points, and other mathematically derivable implications. Numerical compatibility, on the other hand, was rarely achieved because of the varying degrees of empiricism among the models. Models were based more on biological information than mathematical one in deriving the compatible system of equations.

For example, Clutter (1963) expressed the stand volume in terms of basal area, age, and the site productivity of the stand and took the derivative of this equation with respect to time in order to calculate the volume and basal area growths. Biologically, it is true that the volume of a stand is somehow related to its age and site productivity. But, in a mathematical context, the volume of a stand is directly related to DBH, height and taper of the trees and hence the age and the site productivity become irrelevant in calculating the volume and the basal area of the stand. Moreover, site index (height of dominant and co-dominant trees), mostly used as a measure of site quality, is only a conceptual measure of the site productivity. As has been realized there are numerous factors influencing the site productivity and these factors are barely represented by the site index.

Algebraically compatible system of models can be derived using these conceptual variables. But, the numerical compatibility among these models with the parameters estimated by statistical estimation procedures can never be achieved. In the past, the problems caused by empiricism in numerical compatibility hindered the process of the development of a compatible system of equations starting from individual tree taper to the stand volume equation. Efforts were limited to the compatibility within the tree or stand level and the compatibility between these two levels became an unattainable destination.

Algebraic and numerical compatibility among the system of equations starting from individual tree taper to the stand volume equation were achieved in this study by incorporating the theoretical information while deriving these equations. Individual tree volume equation was developed first. Other equations: taper, individual tree basal area, stand basal area, and stand volume equations were subsequently derived from this equation. Parameters (a and b) associated with the individual tree volume equation were the sole parameters throughout the system of equations in this study.

The system of equations is appropriate for all trees; variations are captured through parameter changes. If the goal is only the estimation of the total volume of a tree or a stand, the individual tree volume equation can be fitted to estimate the parameters. As mentioned earlier, the total volume of the tree can be estimated accurately irrespective of the value of b (exponent of DBH). Different values of b will make the distribution of the volume of the tree along the bole different, but the total volume of the tree will be the same for all the values of b.

As DBH measured by diameter tape is greater than the diameter at *4.5* ft observed in stem analysis for most of the trees. Therefore, the estimate for b obtained by fitting the individual tree volume equation alone will be different from the one obtained by fitting the tree volume and taper equations simultaneously or the taper equation alone. Since several observations along the bole of a tree are utilized in taper equation as opposed to one (DBH) in volume equation, estimate for b obtained from the simultaneous fit will not be significantly different from the one obtained from the fit of taper equation alone.

Therefore, b can be estimated from taper equation alone. The only reason of fitting the volume and taper equations simultaneously is to estimate the parameter a that is solely involved in the volume equation. Dimensional analysis requires that the sum of the exponents of DBH and height in the volume equation be 3. Theoretically, b can take any number between 0 and 3, but this study indicates that value of b should be between 2 and 2.2 depending on the diameter of the butt of the tree. In general, it was found that the bigger the butt-size the greater the value of b. If there is any tree for which the butt diameter is less then the DBH, the value of b can be less than 2.

A tree does not have a fixed shape across the bole that can be described by a single mathematical equation. In general, the shape is so complex that researchers have developed various forms of taper models in an attempt to describe a complete tree profile. A few of these models in increasing complexity are (i) segmented polynomial (Max and Burkhart, 1976), (ii) variable exponent (Kozak, 1988), and (iii) variable form (Newnham, 1992) models. But the shape of a tree, in general, can be approximated with (i) the lower part as a truncated neiloids, (ii) the middle part as a truncated paraboloid, and (iii) the upper part as a truncated conoid.

The taper equation developed in this study has captured at least these three shapes in a tree. As observed in the tree profile generated from the taper equation, the curve had at least 4 inflection points. The shape up to the second inflection point is similar to the truncated neiloid, the shape after the last inflection point is the truncated conoid, and the shape between these two sections is nearly a truncated paraboloid.

Even though the researchers were successful in describing the shape above the DBH by the means of complex models, the major drawback of most of these models was the prediction of butt diameter. The taper equation developed in this paper is shown to efficiently and accurately predict tree diameters from the butt to the tip of the tree. At height zero, this equation assumes the diameter to be very large (virtually infinity) compared to other diameters along the bole. This can be explained biologically. Even if the height of a tree is measured from the ground, the actual tree height starts from the point where the roots are spread in all directions. As a result, the diameter at this point is virtually infinity compared to other diameters along the bole. Therefore, if the diameter at height zero is desired it should be calculated for very small height (e.g. $h = 0.01$ft, 0.001ft etc.).

The most remarkable characteristics of the proposed taper equation are that it has theoretical basis and it is the simplest form of currently available taper equations. Since it does not contain any extraneous parameters and complexity in form, the diameters to any point along the bole can be calculated with only the use of a hand calculator. This taper equation has only one minor drawback in that height to any desired diameter can not be calculated directly, since the exponent of the height is a fractional number. Therefore, if desired the height to any diameter limit should be calculated iteratively.

The basal area equation expressed in term of DBH and total height of the tree also contains the same parameter b. The constraint g is calculated using the estimate of b. In order to calculate the average basal area per tree of a stand the DBH and the total height in the individual tree basal area equation should be replaced by quadratic mean diameter and Lorey mean height, respectively. Total basal area of the stand is then the product of this average basal area per tree and the total number of trees in the stand. Basal area growth between time t_1 and t_2 is just the difference between total basal areas at times t_1 and t_2.

The stand volume equation was derived from the stand basal area equation with the same parameters (a and b) as they were in the individual tree volume equation. Average volume per tree was obtained by substituting the DBH and total height with some functions of DBH and height respectively in the individual tree volume equation. Also in this case, the functions of DBH and height were found to be the quadratic mean diameter and Lorey mean height respectively. The total volume of the stand was obtained by multiplying this average volume per tree and the total number of the trees in the stand.

It should be noted that the average volume per tree is not the same as the average tree volume of the stand. Average volume per tree is the volume of a tree with quadratic mean diameter as its DBH and the Lorey mean height as its total height. On the other hand, average tree volume is the volume of a tree with average DBH and average height as its DBH and total height respectively. In order to estimate the stand volume growth for a certain period of time, total stand volumes were calculated at the beginning and at the end of that period and the stand volume at the beginning was subtracted from the stand volume at the end. Both of the stand basal area and the stand volume were algebraically and numerically compatible across the time.

The dimensionally compatible system of equations presented here is very general. There are only two parameters (a and b) in the whole system of equations. It is almost impossible to determine the actual values of these parameters that can be applied to any species grown in different sites with different intensities. Parameters estimated by statistical procedures have limited applications and the consistent/compatible results are obtained if predictions are made only on the range of the data on which the parameters were based. In this study, parameters were estimated using the data from loblolly pine trees grown in cutover, site-prepared plantations. The system of equations can be adapted for loblolly pine trees grown in different sites and for other species by fitting the developed volume and taper equation or stand volume equation to data specific from those sites in the United States and other countries.

This study represents the beginning of an effort to determine the proper theoretical relationships between dependent and independent variables. This study proves that if models are developed by incorporating proper mathematical information, numerical compatibility can be achieved among the algebraically compatible models. Any equation (model) from the algebraically compatible system of equations (models) can be fitted to estimate the parameters of the system. The same estimated values of parameters are obtained no matter which equation is fitted.

In summary, the dimensionally compatible system of equations presented here can be applied in any system of units to estimate tree and stand characteristics for loblolly pine trees grown in cutover, site-prepared plantations within the geographic area that the data were collected. Models based only on statistical assumptions can yield accurate results only when applied within the range of observed data. Equations developed in this paper yield consistent results even when applied to ranges beyond the observed data. However, numerical compatibility between any two equations holds if the range of the data in estimating the parameters in one equation is nearly the same as the range of the data in another equation and the predicted values from one equation are used in another equation.

LITERATURE CITED

Alder, D., 1979. A distance-independent tree model for exotic conifer plantations in East Africa. For. Sci., 25:59-71.

Amateis, R. L. and Burkhart, H. E., 1987. Cubic-foot volume equations for loblolly pine trees in cutover, site-prepared plantations. South. J. Appl. For. 11(4):190-192.

Amateis, R. L., Burkhart, H. E., Knoebel, B. R., and Sprinz, P. T., 1984. Yield and size class distributions for unthinned loblolly pine plantations on cutover site-prepared lands. Sch. For. Wildl. Resour. Va. Polytech. Inst. State Univ. Publ. FWS-2-84, 60 pp.

Amateis, R. L. and McDill, M. E., 1989. Developing growth and yield models using dimensional analysis. For. Sci., 35:329-337.

Arney, J. D., 1974. An individual tree model for stand simulation in Douglas-fir. In: J.Fries (Editor), Growth models for trees and stand simulation. Royal College of Forestry, Stockholm, 38-46 pp.

Avery, T. E. and Burkhart, H. E., 1994. Forest Measurements (4th Edition). McGraw-Hill, New York, 408 pp.

Avila, O. B. and Burkhart, H. E., 1992. Modeling survival of loblolly pine trees in thinned and unthinned plantations. Can. J. For. Res., 22:1878-1882.

Bailey, R. L., 1980. Individual tree growth derived from diameter distribution models. For. Sci., 26:626-632.

Bailey, R. L. and Ware, K. D., 1983. Compatible basal area growth and yield model for thinned and unthinned stands. Can. J. For. Res. 13:563-571.

Bella, I. E., 1971. A new competition model for individual trees. For. Sci., 17:364-372.

Bennett, F. A., 1970. Variable-density yield tables for managed stands of natural slash pine. UADA For. Serv. Res. Note SE-141, 7 pp.

Bennett, F. A. and Clutter, J. L., 1968. Multiple-product yield estimates for unthinned slash pine plantations - pulpwood, sawtimber, gum. USDA For. Serv. Res. Pap. SE-35, 21 pp.

Bennett, F. A., McGee, C. E. and Clutter, J. L., 1959. Yield of old-field slash pine plantations. USDA For. Serv. Stn. Pap. 107, 19 pp.

Biging, G. S. and Dobbertin, M., 1995. Evaluation of Competition indices in individual tree growth models. For. Sci., 41:360-377.

Borders, B. E. 1989. Systems of equations in forest stand modeling. For. Sci., 35:548-556.

Borders, B. E. and Bailey, R.. L., 1986. A compatible system of growth and yield equations for slash pine fitted with restricted three stage least squares. For. Sci., 32:185-201.

Borders, B. E. and Patterson, W. D., 1990. Projecting stand tables: a comparison of the Weibull diameter distribution method, a percentile-based projection method, and a basal area growth projection method. For Sci., 36:413-424.

Brender, E. V. and Clutter, J. L., 1970. Yield of even-aged natural stands of loblolly pine. Ga. For. Res. Counc. Rep. 23, 7 pp.

Buckingham, E., 1914. On physically similar systems: Illustrations of the use of dimensional equations. Phys. Rev. 4(4):345-376.

Burkhart, H. E., Cloeren, D. C., and Amateis, R. L., 1985. Yield relationships in unthinned loblolly pine plantations on cutover, site-prepared lands. South. J. Appl. For. 9:84-91.

Burkhart, H. E. and Sprinz, P. T., 1984. Compatible cubic volume and basal area projection equations for thinned old-field loblolly pine plantations. For. Sci., 30:86-93.

Burkhart, H. E. and Strub, M. R., 1974. A model for simulation of planted loblolly pine stands. In: J. Fries (Editor), Growth models for tree and stand simulation. Royal College of Forestry, Stockholm, 128-135 pp.

Burkhart, H. E., Parker, R. C., and Oderwald, R. G., 1972a. Yields for natural stands of loblolly pine. Div. For. Wildl. Resour. Va. Polytch. Inst. State. Univ. Publ. FWS-2-72, 63 pp.

Burkhart, H. E., Parker, R. C., Strub, M. R., and Oderwald, R. G., 1972b. Yields for natural stands of loblolly pine. Div. For. Wildl. Resour. Va. Polytch. Inst. State. Univ. Publ. FWS -3-72, 51 pp.

Cao, Q. V. and Durand, K. M., 1991. A growth and yield model for improved eastern cottonwood plantations in the lower Mississippi Delta. Suoth. J. Appl. For. 15:213-216.

Clutter, J. L., 1963. Compatible growth and yield models for loblolly pine. For. Sci., 9:354-371.

Clutter, J. L., Harms, W. R., Brister, G. H., and Rhenny, J. W., 1984. Stand structure and yields of site-prepared loblolly pine plantations in the lower coastal plain of the Carolinas, Georgia, and North Florida. USDA For. Serv. Gen. Tech. Rep. SE-27, 173 pp.

Coile, T. S. and Schumacher, F. X., 1964. Soil-site relations, stand structure, and yields of slash and loblolly pine plantations in the Southern United States. T. S. Coile, Inc., Durham, NC, 296 pp.

Curtis, R. O., 1967. A method of estimation of gross yield of Douglas-fir. For. Sci., monograph 13, 24 pp.

Dale, M. E., 1975. Individual tree growth and simulation of stand development of an 80-year-old white oak stand. In: A. R. Ek, J. W. Balsiger and L. C. Promnitz (Editors), Forest Modeling and Inventory. Department of Forestry, University of Wisconsin, 49-63 pp.

Daniels, R. F. and Burkhart, H. E., 1975. Simulation of individual tree growth and stand development in managed loblolly pine plantations. Div. For. Wildl. Resour. Va. Poly. Inst. State Univ. Publ. FWS-75, 69 pp.

Daniels, R. F. and Burkhart, H. E., 1988. An integrated system of forest stand models. For. Ecol. Manage. 23:159-177.

Daniels, R. F., Burkhart, H. E., and Strub, M. R. 1979. Yield estimates for loblolly pine plantations. J. For., 79:581-583, 586.

Dell, T. R., Feduccia, D. P., Campbell, T. E., Mann, W. E., Jr., and Polmer, B. H., 1979. Yields of unthinned slash pine plantations on cutover sites in the West Gulf region. UADA For. Serv. Res. Pap. SO-147, 84 pp.

Douglas, J. F., 1969. An introduction to dimensional analysis for engineers. Sir Isaac Pitman and Sons, London. 134 pp.

Ek, A. R. and Monserud, R. A., 1974. Forest. A computer model for simulating the growth and reproduction of mixed species forests stands. Res. Rep. R2635, School of Natural Resources, University of Wisconsin, 13 pp.

Feduccia, D. P., Dell, T. R., Mann, W. F., Jr., and Polmer, B. H., 1979. Yields of unthinned loblolly pine plantations on cutover sites in the West Gulf region. UADA For. Serv. Res.Pap. SO -148, 88 pp.

Furnival, G. M., and Wilson, R. W. Jr., 1971. Systems of equations for predicting forest growth and yield. Statist. Ecol. 3:43-55.

Goebel, N. B. and Shipman, R. D., 1964. Volume and yields of loblolly pine plantations for a variety of sites in the South Carolina Piedmont. Clemson Univ. Forest Res. Series No.13, 23 pp.

Graney, D. L., and Murphy, P. A., 1994. Growth and yield of thinned upland oak stands in the Boston Mountains of Arkansas. South. J. Appl. For. 18:10-14.

Gregoire, T. G., 1987. Generalized error structure for forestry yield models. For. Sci., 33:423-444.

Hafley, W. L. and Schreuder, H. T., 1977. Statistical distributions for fitting diameter and height data in even-aged stands. Can. J. For. Res., 7:481-487.

Hans, R. P., 1986. Estimating the coefficients in a system of compatible growth and yield equations for loblolly pine. M.S. Thesis, Va. Polytech. Inst. State Univ. Blacksburg VA. 46pp.

Harrison, W. C., Burk, T. E. and Beck, D. E., 1986. Individual tree basal area increment and total height equations for Appalachian mixed hardwoods after thinning. Sou. J. App. For., 10:99-104.

Hynk, D. M., 1980. Diameter distribution approaches to growth and yield modeling. Proc. Workshop forecasting forest stand dynamics, Lake head University, Thunder Bay, Ont., 138-163 pp.

Khil'mi, G. E., 1957. Theoretical forest biophysics. Academy of Sciences of the USSR (translated from Russian in 1962), 150 pp.

Knoebel, B. R., Burkhart, H. E., and Beck, D. E., 1986. A growth and yield model for thinned stands of Yellow-Poplar. For. Sci. Monograph. 27:39 pp.

Kovats, M., 1993. A comparison of British, Swedish and British Colombian growth and yield predictions for lodgepole pine. For. Chron. 69:450-457.

Kozac, A., 1988. A variable-exponent taper equation. Can. J. For. Research. 18(11): 1363-1368.

Langhaar, H. L., 1951. Dimensional analysis and theory of models. Weley, New York, 155 pp.

Larocque, G. and Marshall, P. L., 1988. Improving single-tree distance-dependent growth models. USDA For. Serv. Gen. Tech. Rep. NC, North. Cent. For. Exp. Stn. 120:94-101.

Leithold, L., 1986. The calculus of a single variable with analytic geometry (Fifth edition). Harper & Row, Publishers, New York. 942 pp.

Lenhart, J. D., 1972. Cubic volume yields for unthinned old-field loblolly pine plantations in the interior West Gulf Coastal Plain. Stephen F. Austin State Univ. Tex. For. Pap.14, 46 pp.

Lenhart, J. D. and Clutter J. L., 1971. Cubic-foot yield tables for old-field loblolly pine plantations in Georgia Piedmont. Ga. For. Res. Counc. Rep. 22 - Ser. 3, 12 pp.

Liu, J. and Burkhart, H. E., 1994. Spatial autocorrelation of diameter and height increment predictions from two stand simulators for loblolly pine. For. Sci., 40:349-356.

Loetsch F., Zohrer F., and Haller K. E., 1973. Forest Inventory (Volume II). BLV Verlagsgesellschaft Munchen Bern Wien, Germany, 469 pp.

Lohrey, R. E. and Bailey, R. L., 1977. Yield tables and stand structure for unthinned longleaf pine plantations in Louisiana and Texas. USDA For. Serv. Res. Pap. SO-133, 53 pp.

MacKinney, A. L. and Chaiken, L. E., 1939. Volume, Yield, and growth of loblolly pine in the Mid-Atlantic Coastal Region. USDA For. Serv. Tech. Note No.33. 30 pp.

Maguire, D. A., Kershaw, J. A., Jr., and Hann, D. W., 1991. Predicting the effects of silvicultural regime on branch size and crown wood core in Douglas-fir. For. Sci., 37:1409-1428.

Matney, T. G. and Farrar, R. M., Jr., 1992. A thinned/unthinned loblolly pine growth and yield simulator for planted cutover site-prepared land in the Mid-Gulf South. South. J. Appl. For. 16:70-75.

Max, T. A. and Burkhart, H. E. 1976. Segmented polynomial regression applied to taper equations. For. Science. 22(3):283-289.

Mitchell, K. J., 1975. Dynamics and simulated yields of Douglas-fir. For. Sci. Monograph 17, 39 pp.

Moser, J. W., 1972. Dynamics of an uneven-aged forest stand. For. Sci., 18:184-191.

Moser, J. W. and Hall, O. F., 1969. Deriving growth and yield functions for uneven-aged forest stands. For., Sci. 15:183-188.

Munro, D. D., 1974. Forest growth models - a prognosis. In: J. Fries (Editor), Growth Models for Tree and Stand Simulation. Royal College of Forestry, Stockholm, 7-21 pp.

Murphy, P. A. and Beltz, R. C., 1981. Growth and yield of Shortlief pine in the West Gulf region., USDA For. Serv. Res. Pap. SO-169. 15 pp.

Murphy, P. A. and Farrar, R. M., Jr., 1988. A framework for stand structure projection of uneven-aged loblolly-shortleaf pine stands. For. Sci., 34:321-332.

Murphy, P. A. and Sternitzke, H. S., 1979. Growth and yield estimation for loblolly pine in the West Gulf. USDA For. Serv. Res. Pap. SO-154. 8pp.

Myers, R. H., 1990. Classical and modern regression with applications (2nd Edition). Duxbury Press, Belmont, California, 488 pp.

Nepal, S. K. and Somers, G. L., 1992. A generalized approach to stand table projection. For. Sci., 38:120-133.

Newberry, J. D., 1984. Methods for modeling whole stem diameter growth and taper. Ph.D. Dissertation. Va. Polytech. Inst. and State Univ., Blacksburg, VA.

Newnham, R. M. 1992. Variable-form taper functions for four Alberta tree species. Can. J. For. Research. 22(2):210-223.

Oderwald, R. G. and Hans, R. P., 1993. Corroborating models with model properties. For. Ecol. Manage., 62:271-283.

O'Hara, K. L. and Oliver, C. D., 1988. Three-dimensional representation of Douglas-fir volume growth: comparison of growth and yield models with stand data. For. Sci., 34:724-743.

Patterson, D. A. and Stiff, C. T., 1988. An evaluation and comparison of two distance-independent forest projection models in the Inland Northwest. USDA For. Serv. Gen. Tech. Rep. NC, North. Cent. For. Exp. Stn. 120:804-809.

Piennar, L. V. and Rheney, J. W., 1993. Yield prediction for mechanically site-prepared slash pine plantations in the southern Coastal Plain. South. J. Appl. For. 17:163-173.

Piennar, L. V. and Rheney, J. W., 1995. Modeling stand level growth and yield response to silvicultural treatments. For. Sci., 41:629-638.

Pienaar, L. V. and Turnbull, K. J., 1973. The Chapman-Richards generalization of von Bertalanffy's growth model for basal area growth and yield in even-aged stands. For. Sci., 29:2-22.

Reed, D. D., 1987. Estimation procedures for analytically related growth and yield models. In: Proc. 1986, Soc. Am. For. Nat. Conv. Soc. Am. For., Bethesda, MD 352 pp.

Sadiq, R. A., 1983. Estimation of stand basal area growth and yield with a reverse logistic function. Can. J. For. Res. 13:289-297.

Schumacher, F. X. and Coile, T. S., 1960. Growth and yield of natural stands of the Shouthern pines. T. S. Coile, Inc., Durham, NC, 115 pp.

Sharma, M., 1995. Volume and taper equations for loblolly pine trees using dimensional analysis. MS Thesis. Va. Polytech. Inst. State Univ., Blacksburg, VA.

Short, E. A. and Burkhart, H. E., 1992. Predicting crown-height increment for thinned and unthinned loblolly pine plantations. For. Sci., 38:594-610.

Smalley, G. W. and Bailey, R. L., 1974a. Yield tables and stand structure for loblolly pine plantation in Tennessee, Alabama, and Georgia highlands. USDA For. Service Res. Paper SO-96, 81 pp.

Smalley, G. W. and Bailey, R. L., 1974b. Yield tables and stand structure for shortleaf pine plantation in Tennessee, Alabama, and Georgia highlands. USDA For. Service Res. Paper SO-97, 57 pp.

Smith, V. G., 1983. Compatible basal area growth and yield models consistent with forest growth theory. For. Sci., 29:279-288.

Stage, A. R., 1973. Prognosis model for stand development. USDA For. Serv. Res. Pap. INT-137, 32 pp.

Stiff, C. T., 1980. Modeling the growth dynamics of natural mixed-species Appalachian hardwoods. Ph.D. Dissertation, Va. Poly. Inst and state Univ. Blacksburg, VA.

Sullivan, A. D. and Clutter, J. L., 1972. A simultaneous growth and yield model for loblolly pine. For. Sci., 18:76-86.

Taylor, E. S., 1974. Dimensional analysis for engineers. Clarendon, Oxford. 162 pp.

Von Bertalanffy, L., 1957. Quantitative laws in metabolism and growth. The Quarterly Review of Biology. 32:217-231.

Wykoff, W. R., 1990. A basal area increment model for individual conifers in the northern Rockey Mountains. For. Sci., 36:1077-1104.

Yoshimoto, A., Haight, R. G., and Brodie, J. D., 1990. A comparison of the pattern search algorithm and the modified path algorithm for optimizing an individual tree model. For. Science. 36:394-412.

Zhang, L, J., Moore, J. A. and Newberry, J. D., 1993. A whole-stand growth and yield model for interior Douglas-fir. West. J. Appl. For. 8:120-125.

Vita

The author was born in Baglung, Nepal on September 22, 1959. He received a high school degree from U. P. Board, India as a private student in 1977. He then joined Gurukul Inter Science College, India and received an I. Sc. Degree in 1979. Afterwards, he attended Allahabed University, India and received a B. Sc. degree in 1982. While pursuing the B. Sc. degree, he successfully completed another Bachelors degree in Sanskrit (Shastri) from Sampurnanand Sanskrit University, India.

From 1982 to 1985, he taught Mathematics and Science in high schools at number of locations in Nepal. He then went to pursue a Masters degree in Physics at Tribhuvan University, Nepal and graduated in 1988. After graduation, he was employed as a Lecturer at the Institute of Forestry, Nepal until he joined Virginia Tech in January 1994. In December 1995, he graduated with an MS degree in Forest Biometrics from Virginia Tech. He started working on his Ph.D. degree in Forest Biometrics at Virginia Tech in August 1996. While working on his Ph.D. degree, he successfully completed another Masters degree in Statistics from Virginia Tech in May 1998.

He was married to Yamuna Sharma in February 1981. He is also the father of two children, Chetna and Rudra.

Mahadev Sharma