The Complete Pick Property and Reproducing Kernel Hilbert Spaces

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(ABSTRACT)

We present two approaches towards a characterization of the complete Pick property. We first discuss the lurking isometry method used in a paper by J.A. Ball, T.T. Trent, and V. Vinnikov. They show that a nondegenerate, positive kernel has the complete Pick property if $1/k$ has one positive square. We also look at the one-point extension approach developed by P. Quiggin which leads to a sufficient and necessary condition for a positive kernel to have the complete Pick property. We conclude by connecting the two characterizations of the complete Pick property.

Contents

Chapter 1

Introduction

1.1 The Classical Nevanlinna-Pick Problem

The classical Nevanlinna-Pick problem is an interpolation problem. We are given a finite set of points $z_1, z_2, \ldots, z_n$ in the unit disk $\mathbb{D}$ and a set of complex numbers $w_1, w_2, \ldots, w_n$. With this data, we must determine conditions for the existence of a holomorphic function $f(z)$ which interpolates the data

$$f(z_i) = w_i \text{ for } i = 1, 2, \ldots, n$$

and maintains norm control

$$\|f\|_\infty = \sup_{z \in \mathcal{D}} |f(z)| \leq 1.$$

The classical Nevanlinna-Pick interpolation problem was solved independently by G. Pick in 1916 [9] and R. Nevanlinna in 1919 [6]. The following solution is due to Pick.

Theorem 1.1. *There exists an interpolating function $f(z)$ which solves the Nevanlinna-Pick problem if and only if the Pick matrix*

$$P_n = \left(\frac{1 - w_i \overline{w_j}}{1 - z_i \overline{z_j}} \right)^n_{i,j=1}$$

is positive (here and in the following positive will be synonymous with positive semi-definite).

We omit the proof of this result here as we solve the problem in more generality later. In addition to investigating conditions for the existence of an interpolant, Nevanlinna successfully characterized the solution set [7] using the Schur algorithm [13]. An account of this approach to the Nevanlinna-Pick problem can be found in [2].

1.2 The Pick Problem

In 1967, D. Sarason [11] put the classical Nevanlinna-Pick problem in the context of operator theory. He formulated and solved a commutant lifting problem and found that the classical Nevanlinna-Pick problem followed. Later, B. Sz.-Nagy and C. Foias generalized his results into the Commutant Lifting Theorem.

From the perspective of the interpolation problem, Sarason's result is important because it is suited for generalization in ways that the classical approaches are not. Also, it has opened the door for others to apply operator theory techniques to interpolation problems.

Today, the Pick problem refers to various generalizations of the original interpolation problem [1]. In the following, we look at these generalized interpolation problems from the perspective of reproducing kernel Hilbert spaces.

1.3 Chapter Summaries

Chapter 2 - Scalar-Valued Kernels. We discuss the connection between scalar-valued kernels and there associated reproducing kernel Hilbert spaces. This chapter is an introduction to the tools and concepts which will be treated in more depth in later chapters. Many results are presented without proof.

Chapter 3 - Operator-Valued Kernels. We generalize the core results from the previous chapter with full details and proofs. We discuss multipliers on vector-valued reproducing kernel Hilbert spaces, and we introduce the interpolation problem which will be the subject of chapters 4 and 5.

Chapter 4 - Lurking Isometry. We use the lurking isometry method [4] to prove a sufficiency condition for nondegenerate, positive kernels to have the complete Pick property.

Chapter 5 - One-step Extension. We prove the result of Chapter 4 with a different method. We find a seemingly different condition (this time necessary and sufficient) and show that the conditions are in fact the same.

Chapter 2

Scalar Case

2.1 Definitions and Preliminary Results

We will begin by laying out some basic results and terminology to be used in the following discussion. Let X be a set of points. A *positive kernel* k is a $\mathbb{C}$-valued function on $X \times X$ such that

$$\sum_{i,j=1}^{n} \overline{c_i} c_j k(z_i, z_j) \geq 0 \tag{2.1}$$

for any finite $c_1, c_2, \ldots, c_n \in \mathbb{C}$ and $z_1, z_2, \ldots, z_n \in X$.

We may derive a Hilbert space from a positive kernel in the following way. Let H_0 be the vector space of finite linear combinations of the functions $k_z := k(\cdot, z)$. Define a sesquilinear form on $\mathcal{H}_0$ by

$$\left\langle \sum a_i k_{z_i}, \sum b_j k_{w_j} \right\rangle_{\mathcal{H}_0} := \sum \overline{b_j} a_i k(w_j, z_i).$$

By hypothesis, $\langle \cdot, \cdot \rangle_{\mathcal{H}_0}$ is positive semidefinite. Suppose $f = \sum a_i k_{z_i}$ satisfies $\langle f, f \rangle = 0$, then

$$|f(w)| = \left| \langle f, k_w \rangle_{\mathcal{H}_0} \right| \leq \langle f, f \rangle_{\mathcal{H}_0}^{1/2} k(w, w)^{1/2} = 0$$

for all $w \in X$ from which we see f is the zero function. We conclude that $\mathcal{H}_0$ is a pre-Hilbert space. Take the completion to create a Hilbert space $\mathcal{H}(k)$ where we identify $f \in \mathcal{H}(k)$ as a function via

$$f(z) = \langle f, k_z \rangle.$$

3

A *reproducing kernel Hilbert space* is a Hilbert space $\mathcal{H}$ whose elements are functions on X such that the point evaluations of the functions are bounded linear functionals (i.e. for any $f(\cdot) \in \mathcal{H}$ and $z \in X$, the map $ev_z : f(\cdot) \to f(z)$ is a bounded linear functional on $\mathcal{H}$). The preceding construction enables us to associate a reproducing kernel Hilbert space with any positive kernel.

Before, we saw that we could derive a Hilbert space from a kernel. Now, we will construct a kernel from a reproducing kernel Hilbert space $\mathcal{H}$. Fix $z \in X$. By the Riesz representation theorem, we may write the linear functional ev_z described above in terms of an inner product with a fixed vector in $\mathcal{H}$

$$f(z) = ev_z(f(\cdot)) = \langle f, k_z \rangle_{\mathcal{H}} . \tag{2.2}$$

The vector $k_z \in \mathcal{H}$ is called the *reproducing kernel* at z. Define $k : X \times X \to \mathbb{C}$ by

$$k(w, z) := \langle k_z, k_w \rangle .$$

The following calculation shows that the function k satisfies the positivity condition (2.1)

$$
\begin{aligned}
\sum_{i,j=1}^{n} \overline{c_i} c_j k(z_i, z_j) &= \sum_{i,j=1}^{n} \overline{c_i} c_j \left\langle k_{z_j}, k_{z_i} \right\rangle \\
&= \sum_{i,j=1}^{n} \left\langle c_j k_{z_j}, c_i k_{z_i} \right\rangle \\
&= \left\langle \sum_{j=1}^{n} c_j k_{z_j}, \sum_{i=1}^{n} c_i k_{z_i} \right\rangle \\
&= \left\| \sum_{i=1}^{n} c_i k_{z_i} \right\|^2 \\
&\geq 0,
\end{aligned}
$$

and so k is a positive kernel.

Thus, we can construct a kernel from a reproducing kernel Hilbert space, and we can construct a Hilbert space from a kernel. Is there a correspondence between these objects? The answer is the next theorem.

Theorem 2.1 (E. H. Moore). *There is a bijective correspondence between reproducing kernel Hilbert spaces on X and positive kernels on X.*

In the examples that follow, we will utilize this next proposition to derive kernels from reproducing kernel Hilbert spaces.

Proposition 2.1. *Let $\mathcal{H}$ be a reproducing kernel Hilbert space on X, and let $\{e_i\}_{i \in I}$ be any orthonormal basis for $\mathcal{H}$. Then*

$$k(z, w) = \sum_{i \in I} \overline{e_i(w)} e_i(z). \tag{2.3}$$

2.2 Examples

To bring these concepts to light, we will look at a few examples of reproducing kernel Hilbert spaces and kernels.

2.2.1 Hardy space

The Hardy space H^2 is composed of those holomorphic functions on $\mathbb{D}$ with square-summable Taylor coefficients. The inner product of H^2 is given by

$$\langle f, g \rangle = \sum_{n=0}^{\infty} \hat{f}(n) \overline{\hat{g}(n)}$$

where $\{\hat{g}(n)\}_{n=0}^{\infty}$ and $\{\hat{f}(n)\}_{n=0}^{\infty}$ are the Taylor coefficients of f and g respectively

$$f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n \text{ and } g(z) = \sum_{n=0}^{\infty} \hat{g}(n) z^n.$$

Consequently, an orthonormal basis on H^2 is given by $\{z^n\}_{n=0}^{\infty}$. With (2.3), we compute the kernel

$$k(z, w) = \sum_{n=0}^{\infty} e_n(z) \overline{e_n(w)}$$

$$= \sum_{n=0}^{\infty} (z\overline{w})^n \tag{2.4}$$

$$= \frac{1}{1 - z\overline{w}}. \tag{2.5}$$

2.2.2   Bergman space

The Bergman space $L_a^2(\mathbb{D})$ consists of those holomorphic functions which satisfy

$$\frac{1}{\pi} \int_{\mathbb{D}} |f(z)|^2 \, dm(z) < \infty$$

where m is Lebesgue area measure. Using the notation of the previous example, the inner product is

$$\langle f, g \rangle = \sum_{n=0}^{\infty} \frac{\hat{f}(n)\overline{\hat{g}(n)}}{(n+1)},$$

and an orthonormal basis is $\{(\sqrt{n+1})z^n\}_{n=0}^{\infty}$. Again with (2.3), we compute the kernel

$$k(z,w) = \sum_{n=0}^{\infty} e_n(z)\overline{e_n(w)}$$
$$= \sum_{n=0}^{\infty} (n+1)(z\overline{w})^n$$
$$= \frac{1}{(1-z\overline{w})^2}.$$

Note that the last line of the calculation follows from differentiating the last two lines of the previous kernel calculation (2.4) (2.5).

2.2.3   Drury-Arveson space

In the previous examples, a description of a reproducing kernel Hilbert space is given from a function space perspective and the kernel is derived. The Drury-Arveson space H_d^2 is more appropriately introduced in terms of its kernel

$$k(z,w) = \frac{1}{1 - \langle z, w \rangle_d} \tag{2.6}$$

for $z, w \in \mathbb{D}^d$ (where the dimension d may be countably infinite).

The Drury-Arveson kernel is also called the universal Pick kernel because any kernel with the complete Pick property, after normalization, can be expressed as the Drury-Arveson kernel composed with a change of variable $\Phi : X \to \mathbb{B}^d$ [1].

Chapter 3

Vector Case

In this chapter, we generalize and add to the results found in the previous chapter. The first result relates a kernel with its reproducing kernel Hilbert space and Kolmogorov decomposition in the operator-valued case. Next, we introduce multipliers in the context of vector-valued reproducing Hilbert spaces, and in the final section, we discuss the interpolation problem whose solution is the subject of the remaining chapters.

3.1   Organizational Lemma

The result below can be found in [3].

Lemma 3.1. *For an operator-valued kernel function given by $K(z, w) : X \times X \mapsto B(\mathcal{Y})$, the following are equivalent.*

    i. K is a positive kernel on X. For any $w_1, w_2, \ldots, w_n \in X$ and $y_1, y_2, \ldots, y_n \in \mathcal{H}$, we have the condition

$$\sum_{i,j=1}^{n} \langle K(w_i, w_j) y_j, y_i \rangle_{\mathcal{H}} \geq 0.$$

    ii. There exists a Hilbert space $\mathcal{H}(K)$ whose elements are $\mathcal{Y}$ valued functions on X such that

        a. For each $w \in X$ and $y \in \mathcal{Y}$ fixed, the function $f(x) = K(x, w)(y)$ is in $\mathcal{H}(K)$.

        b. For all $f \in \mathcal{H}(K)$,

$$\langle f(\cdot), K(\cdot, w)(y) \rangle_{\mathcal{H}(K)} = \langle f(w), y \rangle_{\mathcal{Y}}. \tag{3.1}$$

iii. *(Kolmogorov decomposition) There exists $H : X \mapsto B(\mathcal{U}, \mathcal{Y})$ ($\mathcal{U}$ auxiliary Hilbert Space)
such that*

$$K(w, x) = H(w)H(x)^*. \tag{3.2}$$

.

Proof. First, we show (i) implies (ii). Consider the vector space H_0 given by $H_0 = \text{span}\{\{K(\cdot, x)(y) : x \in X, y \in \mathcal{Y}\}\}$. Define a sesquilinear form on H_0 by extending the map $\phi(K(\cdot, x)y, K(\cdot, w)y') = \langle K(w, x)y, y' \rangle_{\mathcal{Y}}$ by sesquilinearity.

We check that ϕ defines an inner product. For $f \in H_0$, the fact that $\phi(f, f) \geq 0$ follows directly from premise (i). Hence, the sesquilinear form ϕ is semi-definite, and we may apply the Cauchy-Schwarz inequality. If $\phi(f, f) = 0$ and $g \in H_0$, then

$$|\phi(f, g)|^2 \leq \phi(f, f)\phi(g, g) = 0.$$

If we let $g = k(\cdot, w)y$, then

$$\begin{aligned}
0 &= \phi(f, g) \\
&= \phi(f, k(\cdot, w)y) \\
&= \langle f(w), y \rangle_{\mathcal{Y}}
\end{aligned}$$

which shows that for all $w \in X$ the vector $f(w)$ is perpendicular to y. Thus, f is the zero vector, and the function ϕ defines an inner product (i.e. positive definite, sesquilinear form) which we will denote $\phi(f, g) = \langle f, g \rangle_{H_0}$.

Let $H(K)$ be the Hilbert space completion of H_0, and (ii) follows from our construction.

Next, we show that (ii) implies (iii). Let $\mathcal{U} = \mathcal{H}(K)$. Define the operator $H(w) \in B(\mathcal{H}(K), \mathcal{Y})$ by

$$H(w)(f) : f \mapsto f(w).$$

We find the adjoint of $H(w)$ by direct calculation

$$\langle f, H(w)^* y \rangle_{\mathcal{Y}} = \langle f(w), y \rangle_{\mathcal{Y}} = \langle f, K(\cdot, w)(y) \rangle_{\mathcal{H}(K)},$$

thus $H(w)^*(y) : y \mapsto K(\cdot, w)(y)$ where we note that $K(\cdot, w)(y) \in \mathcal{H}(K)$.

The result follows as

$$H(w)H(x)^*(y) = H(w)K(\cdot, x)(y) = K(w, x)(y).$$

The last step is to show that (iii) implies (i). We check that the operator-valued kernel K is positive. For any $w_1, w_2, \ldots, w_n \in X$ and $y_1, y_2, \ldots, y_n \in \mathcal{H}$,

$$
\begin{aligned}
\sum_{i,j=1}^{n} \langle K(w_i, w_j) y_j, y_i \rangle_{\mathcal{H}} &= \sum_{i,j=1}^{n} \langle H(w_i) H^*(w_j) y_j, y_i \rangle_{\mathcal{H}} \\
&= \sum_{i,j=1}^{n} \langle H^*(w_j) y_j, H^*(w_i) y_i \rangle_{\mathcal{U}} \\
&= \left\langle \sum_{j=1}^{n} H^*(w_j) y_j, \sum_{i=1}^{n} H^*(w_i) y_i \right\rangle_{\mathcal{U}} \\
&\geq 0.
\end{aligned}
$$

$\square$

3.2 Multipliers

Consider a positive kernel k and its associated reproducing kernel Hilbert space $\mathcal{H}(k)$. In Chapter 2, we saw that the elements of $\mathcal{H}(k)$ are those functions on X such that the point evaluations of the functions are bounded linear functionals. Here we generalize to the vector-valued case.

Given a Hilbert space $\mathcal{E}$, we may construct a vector-valued reproducing kernel Hilbert space with the tensor product $\mathcal{H}(k) \otimes \mathcal{E}$. In Chapter 2, we discussed the scalar case where we identified $f \in \mathcal{H}(k)$ as a $\mathbb{C}$-valued function on X by

$$
f(z) = \langle f, k_z \rangle .
$$

In the vector case, we identify $f \in \mathcal{H}(k) \otimes \mathcal{E}$ as an $\mathcal{E}$-valued function on X by

$$
\langle f(z), e \rangle_{\mathcal{E}} = \langle f, k_z \otimes e \rangle_{\mathcal{H}(k) \otimes \mathcal{E}} .
$$

For the Hilbert spaces $\mathcal{E}$ and $\mathcal{E}_*$, we will denote their associated space of multipliers by

$$
\mathcal{M}_k(\mathcal{E}, \mathcal{E}_*) = \{ W : X \to B(\mathcal{E}, \mathcal{E}_*) : W(z) f(z) \in \mathcal{H}(k) \otimes \mathcal{E}_* \text{ for all } f(z) \in \mathcal{H}(k) \otimes \mathcal{E} \}. \quad (3.3)
$$

We will use M_W to denote the multiplication operator given by $W \in \mathcal{M}_k(\mathcal{E}, \mathcal{E}_*)$.

Lemma 3.2. *If $W \in \mathcal{M}_k(\mathcal{E}, \mathcal{E}_*)$, then $M_W^*(k_z \otimes e_*) = k_z \otimes W^*(z) e_*$.*

Proof. For any $f \in \mathcal{H}(k)$ and $e \in \mathcal{E}$, we have

$$\langle f \otimes e, M_W^*(k_z \otimes e_*)\rangle_{\mathcal{H}(k)\otimes\mathcal{E}} = \langle M_W(f \otimes e), (k_z \otimes e_*)\rangle_{\mathcal{H}(k)\otimes\mathcal{E}_*}$$
$$= f(z)\langle W(z)e, e_*\rangle_{\mathcal{E}_*}$$
$$= f(z)\langle e, W(z)^*e_*\rangle_{\mathcal{E}}$$
$$= \langle f \otimes e, k_z \otimes W(z)^*e_*\rangle_{\mathcal{H}(k)\otimes\mathcal{E}}.$$

The result follows. $\square$

3.3 The Interpolation Problem

We are given a positive kernel k on X and separable Hilbert spaces $\mathcal{E}$ and $\mathcal{E}_*$. The interpolation problem is to show that given $z_1, z_2, \ldots, z_n \in X$ and $W_1, W_2, \ldots, W_n \in B(\mathcal{E}, \mathcal{E}_*)$ there exists $W \in \mathcal{M}_k(\mathcal{E}, \mathcal{E}_*)$ such that $W(z_n) = W_n$ and $\|M_W\| \leq 1$.

Associated with our interpolation data is the Pick matrix

$$P_n = \left((I - W_i W_j^*)k(z_i, z_j)\right)_{i,j=1}^n. \tag{3.4}$$

We say that k has the complete Pick property if a positive Pick matrix is necessary and sufficient to solve the interpolation problem. The necessity side of the argument is simple and always holds. The sufficiency side of the is more challenging. In general, a positive Pick matrix does not always imply a solution to the interpolation problem.

In [4], J. Ball, T. Trent, and V. Vinnikov show that if a nondegenerate, positive kernel's inverse $\frac{1}{k(z,w)}$ has one positive square, then the kernel has the complete Pick property. This result is presented in Chapter 4.

In [10], P. Quiggin shows that a nondegenerate, positive kernel has the complete Pick property if the matrix

$$F_n = \left(1 - \frac{k(z_i, z_{n+1})k(z_{n+1}, z_j)}{k(z_i, z_j)k(z_{n+1}, z_{n+1})}\right)_{i,j=1}^n \tag{3.5}$$

is positive for any finite set of points in X. In [5], S. McCullough shows that the condition is also necessary. We present these results in Chapter 5.

Chapter 4

Lurking Isometry

In this section, we follow [4] to obtain sufficiency conditions for a positive kernel k to have the complete Pick property. First, we need to introduce some new terms. A positive kernel k is *nondegenerate* if $k(z, z) > 0$ for all $z \in X$. We say that the inverse of a positive kernel has one positive square if

$$\frac{1}{k(z, w)} = a(z)\overline{a(w)} - b(z)b(w)^* \tag{4.1}$$

where $a(z)$ is a scalar-valued function and $b(z)$ is a $B(\mathbb{C}^d, \mathbb{C})$-valued function on X (we associate $\mathbb{C}^d$ with ℓ^2 in the case where $d = \infty$).

For a nondegenerate, positive kernel, we may assume the following normalization

$$\frac{1}{k(z, w)} = 1 - b(z)b(w)^*. \tag{4.2}$$

4.1 Principle Result

Theorem 4.1. *(Ball-Trent-Vinnikov) Let k be a nondegenerate, positive kernel such that $\frac{1}{k(z,w)}$ has one positive-square, and let W be the operator valued function on X given by $W(z) : X \mapsto B(\mathcal{E}, \mathcal{E}_*)$. The following are equivalent:*

> *i. $W \in \mathcal{M}_k(\mathcal{E}, \mathcal{E}_*)$ with $\|M_W\|_{op} \leq 1$.*

> *ii. The kernel $K_W(z, w) := k(z, w)(I_{\mathcal{E}_*} - W(z)W(w)^*)$ is a positive $B(\mathcal{E}_*)$-valued kernel on X.*

iii. There exists a unitary V such that

$$V = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{E} \end{bmatrix} \to \begin{bmatrix} \oplus_1^d \mathcal{H} \\ \mathcal{E}_* \end{bmatrix}$$

such that $W(z) = D + C(I - b(z)A)^{-1}b(z)B$.

Remark 4.1. *We do not use the nondegenerate character of the kernel k to show (i) $\Leftrightarrow$ (ii). The fact that (i) and (ii) are equivalent holds for all positive kernels.*

Proof. To show that (i) implies (ii), we use Lemma 3.2.

$$\begin{aligned}
0 &\leq \sum_{i,j=1}^n \left\langle (I - M_W M_W^*)k_{z_j} \otimes e_j, k_{z_i} \otimes e_i \right\rangle \\
&= \sum_{i,j=1}^n \left\langle k_{z_j} \otimes e_j, k_{z_i} \otimes e_i \right\rangle - \left\langle (M_W M_W^*)k_{z_j} \otimes e_j, k_{z_i} \otimes e_i \right\rangle \\
&= \sum_{i,j=1}^n \left\langle k_{z_j} \otimes e_j, k_{z_i} \otimes e_i \right\rangle - \left\langle k_{z_j} \otimes W(z_i)W(z_j)^* e_j, k_{z_i} \otimes e_i \right\rangle \\
&= \sum_{i,j=1}^n \left\langle k_{z_j} \otimes e_j - W(z_i)W(z_j)^* e_j, k_{z_i} \otimes e_i \right\rangle \\
&= \sum_{i,j=1}^n k(z_i, z_j) \left\langle (I - W(z_i)W(z_j)^*)e_j, e_i \right\rangle
\end{aligned}$$

We show (ii) implies (i) by working the previous calculation backwards.

The next result is (iii) implies (ii) . By (3.2), we may write K_W in terms of its Kolmogorov decomposition

$$K_W(z,w) := k(z,w)(I_{\mathcal{E}_*} - W(z)W(w)^*) = H(z)H(w)^* \tag{4.3}$$

where $H(z)$ is a $B(\mathcal{H}_0, \mathcal{E}_*)$-valued function on X and $\mathcal{H}_0$ is an auxiliary Hilbert space. Divide by $k(z,w)$ and reorganize

$$I_{\mathcal{E}_*} - W(z)W(w)^* = H(z)(1 - b(z)b(w)^* \otimes I_{\mathcal{E}_*})H(w)^* \tag{4.4}$$
$$H(z)H(w)^* + W(z)W(w)^* = I_{\mathcal{E}_*} + (b(z) \otimes H(z))(b(w)^* \otimes H(w)^*). \tag{4.5}$$

The map $b(w)^* \otimes H(w)^*$ is in $\mathbb{C}^d \otimes \mathcal{H}_0$ which we may interpret as the direct sum $\oplus_1^d \mathcal{H}_0$ and rewrite

$$b(w)^* \otimes H(w)^* = \begin{bmatrix} b_1(w) \\ \vdots \\ b_d(w) \end{bmatrix} H(w)* : \mathcal{E}_* \to \oplus_1^d \mathcal{H}_0 \tag{4.6}$$

where $b(z) = [b_1(z) \cdots b_2(z)] : \mathbb{C}^d \to \mathbb{C}$. To be clear, the above is the operator-valued function on X give by $b(\cdot)^* \otimes H(\cdot)^* : X \to B(\mathcal{E}_*, \oplus_1^d \mathcal{H}_0)$.

Define the map V_0^* by

$$V_0^* : \begin{bmatrix} b(w)^* H(w)^* e_* \\ e_* \end{bmatrix} \to \begin{bmatrix} H(w)^* e_* \\ W(w)^* e_* \end{bmatrix} \tag{4.7}$$

then by (4.5)

$$
\begin{aligned}
\left\langle V_0^* \begin{bmatrix} b(w)^* H(w)^* e_* \\ e_* \end{bmatrix}, V_0^* \begin{bmatrix} b(z)^* H(z)^* e'_* \\ e'_* \end{bmatrix} \right\rangle &= \left\langle \begin{bmatrix} H(w)^* e_* \\ W(w)^* e_* \end{bmatrix}, \begin{bmatrix} H(z)^* e'_* \\ W(z)^* e'_* \end{bmatrix} \right\rangle \\
&= \langle H(w)^* e_*, H(z)^* e'_* \rangle + \langle W(w)^* e_*, W(z)^* e'_* \rangle \\
&= \langle (H(z) H(w)^* + W(z) W(w)^*) e_*, e'_* \rangle \\
&= \langle [(b(z) \otimes H(z))(b(w)^* \otimes H(w)^*) + I_{\mathcal{E}_*}] e_*, e'_* \rangle \\
&= \langle (b(w)^* \otimes H(w)^*) e_*, (b(z)^* \otimes H(z)^*) e'_* \rangle + \langle e_*, e'_* \rangle \\
&= \left\langle \begin{bmatrix} b(w)^* H(w)^* e_* \\ e_* \end{bmatrix}, \begin{bmatrix} b(z)^* H(z)^* e'_* \\ e'_* \end{bmatrix} \right\rangle.
\end{aligned}
$$

From this identity, we see that

$$\left\| V_0^* \left(\sum_{k=1}^n \begin{bmatrix} b(w_k)^* H(w_k)^* e_{*k} \\ e_{*k} \end{bmatrix} \right) \right\|^2 = \left\| \sum_{k=1}^n \begin{bmatrix} H(w_k)^* e_{*k} \\ W(w_k)^* e_{*k} \end{bmatrix} \right\|^2.$$

By linearity and continuity, we may extend V_0^* to an isometry (i.e. the "lurking isometry") from

$$\mathcal{D}_{V_0^*} = \overline{\mathrm{span}} \left\{ \begin{bmatrix} b(w)^* H(w)^* e_* \\ e_* \end{bmatrix} : w \in X \text{ and } e_* \in \mathcal{E}_* \right\}$$

onto

$$\mathcal{R}_{V_0^*} = \overline{\mathrm{span}} \left\{ \begin{bmatrix} H(w)^* e_* \\ W(w)^* e_* \end{bmatrix} : w \in X \text{ and } e_* \in \mathcal{E}_* \right\}.$$

If $\dim(\mathcal{D}_{V_0^*})^\perp = \dim(\mathcal{R}_{V_0^*})^\perp$, we can extend the isometry $V_0^* : \mathcal{D}_{V_0^*} \to \mathcal{R}_{V_0^*}$ to an isometry

$$V^* = \begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix} : \begin{bmatrix} \oplus_1^d \mathcal{H}_0 \\ \mathcal{E}_* \end{bmatrix} \to \begin{bmatrix} \mathcal{H}_0 \\ \mathcal{E} \end{bmatrix} \tag{4.8}$$

by letting $V_0^*|_{(\mathcal{D}_{V_0^*})^\perp}$ equal any isometry from $(\mathcal{D}_{V_0^*})^\perp$ onto $(\mathcal{R}_{V_0^*})^\perp$.

If it is not the case that $\dim(\mathcal{D}_{V_0^*})^\perp = \dim(\mathcal{R}_{V_0^*})^\perp$, extend to $\begin{bmatrix} \oplus_1^d \mathcal{H} \\ \mathcal{E}_* \end{bmatrix} \to \begin{bmatrix} \mathcal{H} \\ \mathcal{E} \end{bmatrix}$ where $\mathcal{H} \supset \mathcal{H}_0$
and $\dim(\mathcal{H} \ominus \mathcal{H}_0) = \infty$ so that $\dim\left(\begin{bmatrix} \oplus_1^d \mathcal{H} \\ \mathcal{E}_* \end{bmatrix} \ominus \mathcal{D}_{V_0^*} \right) = \left(\dim \begin{bmatrix} \mathcal{H} \\ \mathcal{E} \end{bmatrix} \ominus \mathcal{R}_{V_0^*} \right) = \infty.$

From (4.8), we arrive at the following system of equations

$$A^*b(w)^*H(w)^*e_* + C^*e_* = H(w)^*e_* \tag{4.9}$$
$$B^*b(w)^*H(w)^*e_* + D^*e_* = W(w)^*e_* \tag{4.10}$$

for $w \in X$ and $e_* \in \mathcal{E}_*$.

The next step is to use the system of equations above to to solve for $W(w)$ as follows. We solve for $H(w)^*e_*$ in (4.9)

$$H(w)^*e_* = (I - A^*b(w)^*)^{-1}C^*e_* \tag{4.11}$$

and plug the result into (4.10)

$$W(w)^*(e_*) = B^*b(w)^*(I - A^*b(w)^*)^{-1}C^*e_* + D^*e_*.$$

Take adjoints, and we have $W(\cdot) : X \to B(\mathcal{E}, \mathcal{E}_*)$ given by

$$W(\cdot) = D + C(I - b(\cdot)A)^{-1}b(\cdot)B$$

We must justify the step (4.11) where we have taken the inverse of $I - A^*b(w)^*$. That is, we must show that this inverse is well defined which is equivalent to showing that $\|A^*b(w)^*\| < 1$. Note that A^* is a block of a unitary operator matrix so $\|A^*\| \le 1$. The kernel k is assumed to be nondegenerate over X, and so

$$0 < \frac{1}{k(w,w)} = 1 - b(w)b(w)^*.$$

Thus, $b(w)^*$ is a strict contraction and $\|A^*b(w)^*\| \le \|A^*\|\,\|b(w)^*\| < 1$.

We find that (iii) implies (ii) from the following calculation.

$$
\begin{aligned}
I - W(z)W(w)^* &= I - \Big(D + C(I - b(z)A)^{-1}b(z)B\Big)\Big(D + C(I - b(w)A)^{-1}b(w)B\Big)^* \\
&= I - \Big(D + C(I - b(z)A)^{-1}b(z)B\Big)\Big(D^* + B^*b(w)^*(I - A^*b(w)^*)^{-1}C^*\Big) \\
&= I - DD^* - DB^*b(w)^*(I - A^*b(w)^*)^{-1}C^* - C(I - b(z)A)^{-1}b(z)BD^* \\
&\quad - C(I - b(z)A)^{-1}b(z)BB^*b(w)^*(I - A^*b(w)^*)^{-1}C^* ...
\end{aligned}
$$

By hypothesis, V^* is unitary and given by

$$
V^* = \begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix} : \begin{bmatrix} \oplus_1^d \mathcal{H} \\ \mathcal{E}_* \end{bmatrix} \to \begin{bmatrix} \mathcal{H} \\ \mathcal{E} \end{bmatrix}.
$$

From this representation, we deduce that

$$
\begin{aligned}
I - DD^* &= CC^* \\
I - AA^* &= BB^* \\
-BD^* &= AC^*.
\end{aligned}
$$

Continue by plugging in

$$
\begin{aligned}
... &= CC^* + CA^*b(w)^*(I - A^*b(w)^*)^{-1}C^* + C(I - b(z)A)^{-1}b(z)AC^* \\
&\quad - C(I - b(z)A)^{-1}b(z)(I - AA^*)b(w)^*(I - A^*b(w)^*)^{-1}C^* \\
&= C(I - b(z)A)^{-1}\Big((I - b(z)A)(I - A^*b(w)^*) + (I - b(z)A)A^*b(w)^* \\
&\quad + b(z)A(I - A^*b(w)^*) - b(z)(I - AA^*)b(w)^*\Big)(I - A^*b(w)^*)^{-1}C^* \\
&= C(I - b(z)A)^{-1}\Big(I - b(z)b(w)^*\Big)(I - A^*b(w)^*)^{-1}C^*.
\end{aligned}
$$

From (4.2), we arrive at the equation

$$
k(z,w)(I - W(z)W(w)^*) = H(z)H(w)^*
$$

where $H(z)$ is the $B(\mathcal{H}, \mathcal{E}_*)$-valued function over X given by $H(z) := C(I - b(z)A)^{-1}$. $\square$

4.2   Solution to the Interpolation Problem

Theorem 4.2. *A nondegenerate, positive kernel $k(z,w)$ whose inverse has one positive-square has the complete Pick property (see Section 3.3).*

Proof. We assume (4.2), and we must show that the positivity of the Pick matrix is both necessary and sufficient for the existence of a contractive interpolant.

We have already shown the necessity side of the argument. Notice that Theorem 4.1 (ii) implies the positivity of the Pick matrix. Thus, if we assume a contractive interpolant (i.e. Theorem 4.1 (i) holds), then the Pick matrix (3.4) must be positive.

The difficult part of the argument is to prove sufficiency, but most of the heavy lifting was performed in the proof of Theorem 4.1.

To prove sufficiency, we are given that the Pick matrix associated with some interpolation data is positive, and we must show the existence of a contractive interpolant.

The data points $z_1, z_2, \ldots, z_n$ are a finite set contained in X. If we restrict X to these points by letting $X_n = z_1, z_2, \ldots, z_n$, then we may apply Theorem 4.1 to this finite point set. By (iii) of the theorem, we find that there exists a unitary

$$V = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{E} \end{bmatrix} \to \begin{bmatrix} \oplus_1^d \mathcal{H} \\ \mathcal{E}_* \end{bmatrix}$$

such that the interpolant is given by $W(z_k) = D + C(I - b(z_k)A)^{-1}b(z_k)B$ for $k = 1, 2, \ldots, n$.

The final step is to define $W : X \to B(\mathcal{E}, \mathcal{E}_*)$ by $W(z) = D + C(I - b(z)A)^{-1}b(z)B$. We observe that $b(z)$ is well defined on all of X, and in turn, the interpolating function $W(z)$ is defined on X. Thus we have shown that Theorem 4.1 (iii) holds on X which implies (i) (i.e. $\|M_W\| \leq 1$). We conclude that W is an interpolating contractive multiplier.

$\square$

Chapter 5

One-step Extension

We follow [1] to obtain another solution to the interpolation problem presented in Section 3.3. In the following discussion, we consider positive kernels which are *irreducible.* An irreducible, positive kernel has kernel functions k_z that are linearly independent and $k(z, w) > 0$.

5.1 Another Solution to the Interpolation Problem

Theorem 5.1. *(McCullough, Quiggin) Let k be an irreducible, positive kernel on X, and let $\mathcal{E}$ and $\mathcal{E}_*$ be separable Hilbert spaces. A necessary and sufficient condition for k to have the complete Pick property is that the matrix*

$$F_n = \left(1 - \frac{k(z_i, z_{n+1})k(z_{n+1}, z_j)}{k(z_i, z_j)k(z_{n+1}, z_{n+1})} \right)_{i,j=1}^n \tag{5.1}$$

is positive for any finite collection of points from X.

Proof. First, we address sufficiency. That is, we assume that (5.1) is positive and show that a positive Pick matrix implies the existence of a contractive interpolant. Let

$$\mathcal{D}_n := \operatorname{span}\{k_{z_i} \otimes e_* : 1 \leq i \leq n, e_* \in \mathcal{E}_*\} \tag{5.2}$$
$$\mathcal{R}_n := \operatorname{span}\{k_{z_i} \otimes e : 1 \leq i \leq n, e \in \mathcal{E}\}.$$

Define an operator $(M_W^{(n)})^* : \mathcal{D}_n \to \mathcal{R}_n$ by $(M_w^{(n)})^* : k_{z_i} \otimes e_* \to k_{z_i} \otimes W(z_i)^* e_*$ for $1 \leq i \leq n$ and extend to all of $\mathcal{D}_n$ by linearity. By Lemma 3.2, if $W \in M_k(\mathcal{E}, \mathcal{E}_*)$ is an interpolant (i.e. $W(z_i) = W_i$ for $i = 1, 2, \ldots, n$), then $(M_W)^*|_{\mathcal{D}^n} = (M_W^{(n)})^* : \mathcal{D}^n \to \mathcal{R}^n$.

If the Pick matrix

$$P_n = \left((I - W_i W_j^*)k(z_i, z_j) \right)_{i,j=1}^n . \tag{5.3}$$

is positive, then by Theorem 4.1, the multiplication operator $(M_W^{(n)})^*$ is a contraction over $\mathcal{D}_n$ (i.e. consider the result of the theorem if $X = z_1, z_2, \ldots, z_n$). We will show that given an additional point $z_{n+1} \in X$ there exists an element W_{n+1} of $B(\mathcal{E}, \mathcal{E}_*)$ such that if $W(z_{n+1}) = W_{n+1}$, the multiplication operator $(M_W^{(n+1)})^*$ is contractive over $\mathcal{D}_{n+1}$. It can be shown by a Zorn's lemma argument for inductive systems due to Kurosh (see [1]) that this process of one-point extensions is sufficient to prove the existence of a multiplier W on all of X such that M_W is contractive.

Again, we know that $(M_W^{(n)})^*$ is a contractive multiplier over $\mathcal{D}_n$. To extend, we must find a $W_{n+1} \in B(\mathcal{E}, \mathcal{E}_*)$ such that $W(z_{n+1}) = W_{n+1}$ and $(M_W^{(n+1)})^*$ remains a contraction on $\mathcal{D}_{n+1}$.

Let W_{n+1} be arbitrary and presume $W(z_{n+1}) = W_{n+1}$. The operator M_W^* is defined with domain $\mathcal{D}_{n+1}$ and range $\mathcal{R}_{n+1}$. Decompose the domain and range as direct sums

$$\mathcal{D}_{n+1} = \mathcal{D}_n \oplus \mathcal{D}_n^\perp$$
$$\mathcal{R}_{n+1} = \mathcal{P} \oplus \mathcal{P}^\perp$$

where $\mathcal{P} = \text{span}\{k_{z_{n+1}} \otimes e : e \in \mathcal{E}\}$, and then decompose the operator

$$(M_W)^* = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{D}_n \\ \mathcal{D}_n^\perp \end{bmatrix} \to \begin{bmatrix} \mathcal{P}^\perp \\ \mathcal{P} \end{bmatrix} \tag{5.4}$$

The next step is to show that with the appropriate choice of W_{n+1}, the multiplication operator M_W is contractive over $\mathcal{D}_{n+1}$. To achieve this, we use Parrott's Lemma [8]. The lemma tells us that if the D entry of (5.4) is arbitrary then

$$\inf_{W_{n+1} \in B(\mathcal{E}, \mathcal{E}_*)} \|(M_W)^*\| = \max\left\{ \left\| \begin{bmatrix} A \\ C \end{bmatrix} \right\|, \left\| \begin{bmatrix} A & B \end{bmatrix} \right\| \right\} \tag{5.5}$$

Thus, we proceed by demonstrating that we can make D arbitrary by our choice of W_{n+1} and that the operators A, B, and C do not depend on W_{n+1}.

Consider the operator $D : \mathcal{D}_n^\perp \to \mathcal{P}$. As the vectors $\{k_{z_i}\}_1^{n+1}$ are linearly independent, we may construct a dual basis $\{v_j\}_1^{n+1}$ (i.e. $\langle k_{z_i}, v_j \rangle = \delta_{ij}$ for $n = 1, \ldots, n+1$). By our construction, the space $\mathcal{D}_n^\perp = \text{span}\{v_{n+1} \otimes \mathcal{E}_*\}$. Let $v_{n+1} = \sum_1^{n+1} c_i k_{z_i}$. Observe that $c_{n+1} \neq 0$; otherwise $\|v_{n+1}\| = 0$ which implies $v_{n+1} = 0$, a contradiction. Observe next that

$$D(v_{n+1} \otimes e_*) = P_{\mathcal{P}}\left(\sum_1^{n+1} c_i k_{z_i} \otimes W_i^* e_*\right)$$

$$D(v_{n+1} \otimes e_*) = \sum_1^n c_i \frac{\langle k_{z_i}, k_{z_{n+1}}\rangle}{\langle k_{z_{n+1}}, k_{z_{n+1}}\rangle} k_{z_{n+1}} \otimes W_i^* e_* + c_{n+1} k_{z_{n+1}} \otimes W_{n+1}^* e_*$$

$$D(v_{n+1} \otimes e_*) = k_{z_{n+1}} \otimes \left(\sum_1^n c_i \frac{\langle k_{z_i}, k_{z_{n+1}}\rangle}{\langle k_{z_{n+1}}, k_{z_{n+1}}\rangle} W_i^* + c_{n+1} W_{n+1}^*\right) e_*. \tag{5.6}$$

We have not picked W_{n+1}^* and $c_{n+1} \neq 0$, thus the term in parenthesis in (5.6) is arbitrary. Consequently, the operator D is arbitrary, and we may use the result of Parrott's Lemma (5.5) so long as the operators $\begin{bmatrix} A \\ C \end{bmatrix}$ and $\begin{bmatrix} A & B \end{bmatrix}$ do not depend on W_{n+1}^*.

The operator $\begin{bmatrix} A \\ C \end{bmatrix}$ is the multiplication operator $(M_W^{(n)})^*$ over the initial domain $\mathcal{D}_n$ and does not depend on W_{n+1}. Likewise, the action of the operator $\begin{bmatrix} A & B \end{bmatrix}$ does not depend on W_{n+1} since the operator matrix $\begin{bmatrix} A & B \end{bmatrix} = P_{\mathcal{P}^\perp}(M_W)^*$ where the projection eliminates terms containing W_{n+1}.

We conclude that Parrott's Lemma applies

$$\inf_{W_{n+1} \in B(\mathcal{E}, \mathcal{E}_*)} \|M_W^*\| = \max\{\left\|\begin{bmatrix} A \\ C \end{bmatrix}\right\|, \left\|\begin{bmatrix} A & B \end{bmatrix}\right\|\}. \tag{5.7}$$

As mentioned before, the operator $\left\|\begin{bmatrix} A \\ C \end{bmatrix}\right\|$ is the same as M_W acting on $\mathcal{D}_n$ which is a contraction by hypothesis (i.e. $P_n \geq 0$). We need only show that $\left\|\begin{bmatrix} A & B \end{bmatrix}\right\|$ is a contraction.

Let $\mathcal{Q} = \text{span}\{k_{z_{n+1}} \otimes e_* : e_* \in \mathcal{E}_*\}$ (i.e. the same as $\mathcal{P}$ except in the domain $\mathcal{D}_{n+1}$). From our assorted domain and range decompositions, we have

$$\begin{bmatrix} A & B \end{bmatrix} = P_{\mathcal{P}^\perp}(M_W)^* = P_{\mathcal{P}^\perp}(M_W)^* P_{\mathcal{Q}} + P_{\mathcal{P}^\perp}(M_W)^* P_{\mathcal{Q}^\perp}$$

where $P_{\mathcal{P}^\perp}(M_W)^* P_{\mathcal{Q}} = 0$ since $(M_W)^* k_{z_{n+1}} \otimes e_* = k_{z_{n+1}} \otimes W_{z_{n+1}}^* e_* \in \mathcal{P}$. Thus it suffices to show that $\|P_{\mathcal{P}^\perp}(M_W)^* P_{\mathcal{Q}^\perp}\| \leq 1$, or equivalently, the operator $P_{\mathcal{Q}^\perp} - P_{\mathcal{Q}^\perp} M_W P_{\mathcal{P}^\perp} P_{\mathcal{P}^\perp}(M_W)^* P_{\mathcal{Q}^\perp}$ is positive

$$\left\langle (P_{\mathcal{Q}^\perp} - P_{\mathcal{Q}^\perp} M_W P_{\mathcal{P}^\perp} P_{\mathcal{P}^\perp} (M_W)^* P_{\mathcal{Q}^\perp}) k_{z_j} \otimes e_{*j}, k_{z_i} \otimes e_{*i} \right\rangle =$$

$$\left\langle (I - P_{\mathcal{Q}}) k_{z_j} \otimes e_{*j}, k_{z_i} \otimes e_{*i} \right\rangle - \left\langle (I - P_{\mathcal{Q}}) M_W (I - P_{\mathcal{P}}) M_W^* (I - P_{\mathcal{Q}}) k_{z_j} \otimes e_{*j}, k_{z_i} \otimes e_{*i} \right\rangle =$$

$$\left\langle \left(k_{z_j} - \frac{\langle k_{z_j}, k_{z_{n+1}} \rangle}{\langle k_{z_{n+1}}, k_{z_{n+1}} \rangle} k_{z_{n+1}} \right) \otimes e_{*j}, k_{z_i} \otimes e_{*i} \right\rangle - \left\langle \left(k_{z_j} - \frac{\langle k_{z_j}, k_{z_{n+1}} \rangle}{\langle k_{z_{n+1}}, k_{z_{n+1}} \rangle} k_{z_{n+1}} \right) \otimes W_j^* e_{*j}, k_{z_i} \otimes W_i^* e_{*i} \right\rangle$$

$$\left\langle \left(k_{z_j} - \frac{\langle k_{z_j}, k_{z_{n+1}} \rangle}{\langle k_{z_{n+1}}, k_{z_{n+1}} \rangle} k_{z_{n+1}} \right) \otimes (I - W_i W_j^*) e_{*j}, k_{z_i} \otimes e_{*i} \right\rangle =$$

$$\left[\langle k_{z_j}, k_{z_i} \rangle - \frac{\langle k_{z_j}, k_{z_{n+1}} \rangle \langle k_{z_{n+1}}, k_{z_i} \rangle}{\langle k_{z_{n+1}}, k_{z_{n+1}} \rangle} \right] \left\langle (I - W_i W_j^*) e_{*j}, e_{*i} \right\rangle =$$

$$\left[1 - \frac{\langle k_{z_j}, k_{z_{n+1}} \rangle \langle k_{z_{n+1}}, k_{z_i} \rangle}{\langle k_{z_{n+1}}, k_{z_{n+1}} \rangle \langle k_{z_j}, k_{z_i} \rangle} \right] \langle k_{z_j}, k_{z_i} \rangle \left\langle (I - W_i W_j^*) e_{*j}, e_{*i} \right\rangle =$$

$$\left[1 - \frac{k(z_{n+1}, z_j) k(z_i, z_{n+1})}{k(z_{n+1}, z_{n+1}) k(z_i, z_j)} \right] k(z_i, z_j) \left\langle (I - W_i W_j^*) e_{*j}, e_{*i} \right\rangle.$$

We conclude by the calculation above that the operator $P_{\mathcal{Q}} - P_{\mathcal{Q}} M_W P_{\mathcal{P}} P_{\mathcal{P}} (M_W)^* P_{\mathcal{Q}}$ is positive when the n-by-n operator-valued matrix $F_n \cdot P_n$ is positive where F_n is given by (5.1), P_n is given by (5.3), and the dot denotes the Schur product. The Schur product theorem [12] states that the Schur product (entry-wise) of two positive matrices is a positive matrix. Therefore, the operator $P_{\mathcal{Q}} M_W P_{\mathcal{P}} P_{\mathcal{P}} (M_W)^* P_{\mathcal{Q}}$ is contractive, and this fact plus Parrott's Lemma gives that the multiplier $(M_W)^*$ is a contraction over $\mathcal{D}_{n+1}$.

For necessity, we assume that the positive kernel k has the complete Pick property and show that the matrix F_n (5.1) is positive. We will proceed by induction. First, consider the base case $n = 1$ which gives

$$F_1 = 1 - \frac{k(z_1, z_2) k(z_1, z_2)}{k(z_1, z_1) k(z_2, z_2)}.$$

This value is positive by the Cauchy-Schwarz inequality

$$\langle k_{z_2}, k_{z_1} \rangle \langle k_{z_1}, k_{z_2} \rangle \le \langle k_{z_1}, k_{z_1} \rangle \langle k_{z_2}, k_{z_2} \rangle$$

$$0 \le 1 - \frac{\langle k_{z_2}, k_{z_1} \rangle \langle k_{z_1}, k_{z_2} \rangle}{\langle k_{z_1}, k_{z_1} \rangle \langle k_{z_2}, k_{z_2} \rangle}$$

$$0 \le 1 - \frac{k(z_1, z_2) k(z_1, z_2)}{k(z_1, z_1) k(z_2, z_2)}.$$

Next, we assume the matrix F_n is positive and show that F_{n+1} is positive using the hypothesis that k has the complete Pick property.

Define the $(n+1)$-by-$(n+1)$ matrix H by

$$H_{ij} = \frac{k(z_i, z_{n+1})k(z_{n+1}, z_j)}{k(z_{n+1}, z_{n+1})}$$

$$= \frac{k(z_i, z_{n+1})\overline{k(z_j, z_{n+1})}}{k(z_{n+1}, z_{n+1})}.$$

Note that we may write

$$H = cc^*$$

where c is the vector in $\mathbb{C}^{n+1}$ given by $c_i = \frac{k(z_i, z_{n+1})}{\|k_{z_{n+1}}\|}$. We conclude that H is a rank-one, positive matrix.

Let G be the $(n+1)$-by-$(n+1)$ matrix given by

$$G_{ij} = 1 - \frac{H_{ij}}{k(z_i, z_j)}.$$

The matrix G is simply F_n with an added 0-column and 0-row

$$G = \begin{bmatrix} & & & 0 \\ & F_n & & 0 \\ & & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

By hypothesis, the matrix F_n is positive, so the matrix G is positive as well. Consequently, we may write G as a Gram matrix (i.e. $G_{ij} = \langle v_j, v_i \rangle = v_i^* v_j$ for some collection of vectors $v_i \in \mathbb{C}^{n+1}$.

Note that $H_{ij} = k(z_i, z_j)(1 - G_{ij})$ so the matrix $K \cdot (J - G)$ is positive. We now identify $k(z_i, z_j)(1 - G_{ij})$ with the entries of the Pick matrix $k(z_i, z_j)(1 - W_i W_j^*)$ by letting $W_i = v_i^*$ which we interpret as an operator in $B(\mathbb{C}, \mathbb{C}^n)$. Thus, the Pick matrix P_{n+1} is positive. From the sufficiency argument above, we found that a consequence of the complete Pick property is that $P_{n+1} \cdot F_{n+1}$ is positive when P_{n+1} is positive so

$$0 \le P_{n+1} \cdot F_{n+1} = H \cdot F_{n+1}.$$

and

$$0 \le F_{n+1} \cdot H \cdot \left(\frac{1}{H}\right) = F_{n+1}$$

where $\left(\frac{1}{H}\right) = \hat{c}\hat{c}^*$, $\hat{c}_i = c_i^{-1}$. $\qquad\square$

The next theorem relates the characterizations of the complete Pick property that we have discussed.

Theorem 5.2. *Let k be a positive, irreducible kernel. Then the matrix F_n is positive for any $z_1, z_2, \ldots, z_{n+1} \in X$ if and only if the inverse of k has one positive square.*

Proof. First, suppose that F_n is positive for any $z_1, z_2, \ldots, z_{n+1} \in X$. Then for any $z_0 \in X$

$$k_F(z, w) = \left(1 - \frac{k(z, z_0)k(z_0, w)}{k(z, w)k(z_0, z_0)} \right) \tag{5.8}$$

is a positive kernel which has the Kolmogorov decomposition $k_F(z, w) = b(z)b(w)^*$ where $b : X \to B(\mathbb{C}, \mathcal{H})$ for some auxiliary Hilbert space $\mathcal{H}$ (3.2). We use (5.8) to write $\frac{1}{k(z,w)}$ in terms of k_F

$$\begin{aligned}
\frac{1}{k(z, w)} &= (1 - k_F(z, w)) \frac{k(z_0, z_0)}{k(z, z_0)k(z_0, w)} \\
&= (1 - b(z)b(w)^*) \frac{k(z_0, z_0)}{k(z, z_0)k(z_0, w)}.
\end{aligned}$$

We rescale k in such a way to make the fraction disappear i.e. let $\hat{k}(z, w) = k(z, w)\frac{k(z_0,z_0)}{k(z,z_0)k(z_0,w)}$ (see [1] for more information on the rescaling of positive kernels). The result follows.

Next, suppose that the inverse of k has one-positive square

$$\frac{1}{k(z, w)} = 1 - b(z)b(w)^*.$$

As a result, the matrix

$$L_{n+1} = \left(\frac{1}{k(z_i, z_j)} \right)_{i,j=1}^{n+1}$$

has one positive eigenvalue for any set of points $z_1, z_2, \ldots, z_{n+1} \in X$. We proceed by showing that L_{n+1} has one positive eigenvalue if and only if F_n is positive.

Let

$$L_{n+1} = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$$

where $A = L_n$, B is a column, and D is a scalar. The matrix L_{n+1} is congruent to

$$\begin{bmatrix} A - BC^{-1}B^* & 0 \\ 0 & C \end{bmatrix} = \begin{bmatrix} F_n \cdot \frac{-k(z_{n+1}, z_{n+1})}{k(z_i, z_{n+1})k(z_{n+1}, z_j)} & 0 \\ 0 & \frac{1}{k(z_{n+1}, z_{n+1})} \end{bmatrix}, \tag{5.9}$$

so the matrix L_{n+1} will have the same number of positive, negative, and zero eigenvalues as (5.9). As the bottom right entry of (5.9) is positive, L_{n+1} has one positive eigenvalue if and only if the matrix

$$F_n \cdot \frac{-k(z_{n+1}, z_{n+1})}{k(z_i, z_{n+1})k(z_{n+1}, z_j)} \tag{5.10}$$

is negative (i.e. negative semi-definite). As the rank one matrix $\frac{-k(z_{n+1},z_{n+1})}{k(z_i,z_{n+1})k(z_{n+1},z_j)}$ is negative, the matrix (5.10) is negative if and only if F_n is positive.

$\square$

Bibliography

[1] J. Agler and J.E. McCarthy. *Pick interpolation and Hilbert function spaces*, volume 44 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2002.

[2] J.A. Ball. Nevanlinna-Pick interpolation: Generalizations and applications. *Recent Results in Operator Theory*, 1:51–94, 1988.

[3] J.A. Ball. Linear systems, operator model theory and scattering: multivariable generalizations. In *Operator theory and its applications (Winnipeg, MB, 1998)*, volume 25 of *Fields Inst. Commun.*, pages 151–178. Amer. Math. Soc., Providence, RI, 2000.

[4] J.A. Ball, T.T. Trent, and V. Vinnikov. Interpolation and commutant lifting for multipliers on reproducing kernel Hilbert spaces. In *Operator theory and analysis (Amsterdam, 1997)*, volume 122 of *Oper. Theory Adv. Appl.*, pages 89–138. Birkhäuser, Basel, 2001.

[5] S. McCullough. The local de Branges-Rovnyak construction and complete Nevanlinna-Pick kernels. In *Algebraic methods in operator theory*, pages 15–24. Springer, 1994.

[6] R. Nevanlinna. Über beschränkte Funktionen die in gegebene Punkten vorgeschriebene Werte annehmen. *Ann. Acad. Sci. Fenn. Ser. A*, 13(1), 1919.

[7] R. Nevanlinna. *Über beschränkte analytische Funktionen*. Suomalaisen Tiedeakatemian Kustantama, 1929.

[8] S. Parrott. On a quotient norm and the Sz.-Nagy - Foiaş lifting theorem. *J. Funct. Anal.*, 30(3):311–328, 1978.

[9] G. Pick. Über die beschränkungen analytischer Funktionen, welche durch vorgegebene Funktionswerte bewirkt werden. *Math. Ann.*, 77:7–23, 1916.

[10] P. Quiggin. For which reproducing kernel Hilbert spaces is Pick's theorem true? *Integral Equations and Operator Theory*, 16(2):244–266, 1993.

[11] D. Sarason. Generalized interpolation in H^∞. *Trans. Amer. Math. Soc.*, 127:179–203, 1967.

[12] I. Schur. Bemerkungen zur Theorie der beschränkten Bilinearformen mit unendlich vielen Veränderlichen. *Journal für die reine und Angewandte Mathematik*, 140:1–28, 1911.

[13] I. Schur. Über Potenzreihen, die im Innern des Einheitskreises beschränkt sind. *Journal für die reine und angewandte Mathematik*, 148:122–145, 1918.