# Firm's Optimal Resource Portfolio under Consumer Choice, and Supply and Demand Risks

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(ABSTRACT)

We study the optimal resource portfolio for a price-setter firm under a consumer choice model with supply and demand risks. The firm sells two products that are vertically differentiated, and has the option to invest in both dedicated and flexible resources. Our objective is to understand the effectiveness of the two hedging mechanisms, resource flexibility and demand management through production differentiation, under demand and supply risks.

We show that the presence of consumer-driven substitution does not always reduce the need for the firm to offer differentiated products. In particular, when the firm faces demand risk and differential production costs, it might invest in the flexible resource and offer differentiated products for a wider range of parameters. Interestingly, more uncertainty (in the form of additional supply risk) does not always make the firm more eager to adopt a hedging mechanism. This depends on the relationship between resource risks, product attributes, and resource investment costs. On the other hand, when the firm invests in the flexible resource, this never completely replaces the dedicated resources, and always results in a "diverse" resource portfolio. While this happens in the supply risk setting mainly due to resource diversification advantage, it also happens in the demand risk setting due to the vertical differentiation between the products. Finally, in the absence of differential production costs, demand management by itself (without resource flexibility) becomes powerful enough to hedge against the demand risk, but not the supply risk, due to the additional resource diversification benefit of the flexible resource in the latter setting.

To my father Huiyi Chen, mother Zhihua Luo, and Sister Yaping Chen
献给我的父亲陈惠怡，母亲罗志华，和妹妹陈雅萍

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# Chapter 1

# Introduction and Motivation

A major decision faced by firms producing "differentiated (substitutable)" products (i.e., products that serve similar consumer needs) is their *resource investment decision.* This decision encompasses the firm's technology choice (i.e., "product-dedicated" resources versus a more expensive "flexible" resource[1] that can produce multiple products simultaneously) as well as the investment capacity for each type of resource. When making the resource investment decision, the firm faces risks from both demand and supply sides. A major cause of the demand side risk is the long lead times needed for resource acquisition, which force the resource investment decision to be made long before market conditions are observed (see Van Mieghem [25] for further discussion). The risk from the supply side may occur because the usable resource capacity at the production stage may differ from the target investment level. In this dissertation, we consider one form of this supply risk, that of resource disruption, which may arise due to accidents (e.g., fire), natural disasters (e.g., earthquake, hurricane), labor-related causes (e.g., labor strikes), or supply chain disruptions (see, for instance, Dada et. al. [9], Snyder and Shen [20], Tomlin and Wang [23]), rendering the resource unavailable for production.

A variety of hedging mechanisms can be used to mitigate the different supply and demand risks. In this dissertation, our focus is on research flexibility and demand management through product differentiation. Resource flexibility allows the firm to redistribute the capacity among the different products the resource can produce in response to how uncertainty is resolved ("firm-driven substitution"). On the demand side, an integration of pricing and product differentiation enables a powerful demand management, as it allows the firm to induce consumers to switch – through appropriate pricing mechanisms – to the product it wants to sell ("consumer-driven substitution"), based on how uncertainty is resolved. Although demand management with product differentiation does not require resource flexibility, its power can be enhanced by it, since then the firm can use a combined strategy of firm- and consumer-driven substitution. This is yet another advantage of the flexible resource –

[1] This is also referred to as "product-mix flexibility" in the literature.

in addition to the well-studied risk pooling and contribution margin benefits (Van Mieghem [24]). In the marketplace, firms that utilize product differentiation and resource flexibility disjointly or together co-exist. For example, "Ford managed to earn $7.2 billion (in 1999), more than any auto maker in history," in part due to a new pricing strategy that helped change the mix of vehicles it sells (Coy [8]). Consider Pontiac Aztek and Buick Rendezvous, both built in GM's Ramos, Mexico plant. "While the demand for Aztec was well less than forecasted, the Rendezvous became very popular. Product-mix flexibility was key to meeting demand and enhancing profitability in this situation. Nonetheless, price flexibility was also an important lever in balancing supply and demand, as significant discounts were offered to spark demand for Aztek"(Biller et al. [3]). During the Taiwan earthquake in 1999, Dell was able to steer customers to buy computer configurations that Dell could make from the available components, by giving them either a free or cheap upgrade (Tomlin [22]). *An important question then is, when is product differentiation (in conjunction with pricing), by itself (i.e., without resource flexibility), sufficient to hedge against supply and demand risks, and when it is not, so the firm has to acquire the more expensive flexible resource to empower or even replace it?*

Most research that studies the firm's optimal resource portfolio with dedicated and flexible resources considers that the flexible resource produces "independent" (i.e., neither substitutable nor complementary) products; thus consumer-driven substitution does not exist (e.g., Bish and Wang [4], Fine and Freund [10], Van Mieghem [24]). However, examples of flexible resources producing vertically or horizontally differentiated products are abundant. Sony can quickly shift from one model of camcorder to another (Nakamoto [18]). "Nissan's new Canton, Mississippi assembly plant can send a minivan, pickup truck, and sport-utility vehicle down the same assembly line, one after the other, without interruption." (Welch [26]). " Mazda's plant in Hiroshima builds the RX-7 (a rear-wheel-drive sports car in standard and convertible versions), the 929 (a rear-wheel-drive luxury car), the 121 (a front-wheel-drive mini car), and the 323 (a front-wheel-drive compact car) on the same assembly line" (Goyal et al. [11]). Ford's plant in Norfolk, VA, builds eight different truck models on two platforms (Mcmurray [14]). *Another important question then is how producing differentiated products (hence the existence of consumer-driven substitution) alters the firm's hedging mechanism under demand and supply risks?*

To answer these questions, we consider a firm that can offer two "vertically differentiated" products, and employs a responsive pricing strategy under which it delays its pricing decision until after market conditions are observed (see Chod and Rudi [6] for further discussion on responsive pricing). Vertical differentiation occurs when different products can be ranked according to some characteristic, such as quality. That is, if offered at the same price, all consumers will rank the vertically differentiated products in the same way (i.e., a higher quality product is always preferred to a lower quality product). We utilize a discrete choice model, which deals with a group of consumers making mutually exclusive decisions on a set of differentiated products, because this choice model is considered to be a realistic framework for modeling the demands for differentiated products. It can also capture consumer preference and product characteristics explicitly (see, for instance, Anderson et al [1]). The firm has the option to invest in two product-dedicated resources and/or one flexible resource that can produce both products. On the risk side, we consider both the demand risk, coming from the uncertainty in market size, and the supply risk, coming from imperfectly reliable resources. The effectiveness of product differentiation and resource flexibility strategies depends, among other things, on the interplay between firm-driven substitution and consumer-driven substitution, both of which we explicitly model. To isolate the effect of product differentiation, we first consider a "dedicated-only setting," in which the flexible resource is not available to the firm as a technology choice. We then study the "flexible setting," in which both dedicated and flexible resources are an investment option.

Most previous research that studies the firm's resource portfolio decision in the presence of dedicated and flexible resource options considers only the demand risk and assumes a perfectly reliable supply, and, as stated above, either completely ignores the consumer-driven substitution effect, or uses aggregate linear demand models that fail to accurately represent the complexities of the consumer choice process ( Lus and Muriel [13]). On the other hand, the economics and marketing literature that more realistically models the demand of differentiated products (using a discrete consumer choice process) typically assumes that capacity is infinite (so there is no supply risk) (e.g., Moorthy [15], [16], Mussa and Rosen [17]). Finally, research that studies the firm's resource portfolio decision under both supply and demand risks is very recent, and ignores the demand management aspect of the problem (i.e., product differentiation and pricing) (Tomlin and Wang [23]). In that sense, our research is one of the first to combine these different streams of research to study the firm's optimal resource portfolio, considering differentiated products, whose demands are derived from a consumer choice process, under both supply and demand risks. As far as we are aware, there is only one dissertation that studies the firm's investment decision under a consumer choice model (Kouvelis and Yu [12]), but considering only dedicated resources and demand risk.

Our contributions are as follows.

- Our consumer choice model gives rise to a *multiplicative* form of demand shock, as a result of which demand uncertainty in the dedicated-only system does *not* affect the resource mix in the optimal portfolio, but only their capacities. This makes the firm partially immune to demand forecast errors (as long as the demand shock is multiplicative; this result does not hold for additive demand shock). However, this immunity disappears when there is supply uncertainty, or when there is demand uncertainty and the flexible resource is an investment option.

- The presence of consumer-driven substitution does *not* always reduce the need for the firm to offer differentiated products. In particular, when the firm faces demand risk and differential production costs, it might invest in the flexible resource and offer differentiated products for a wider range of parameters.

- More uncertainty (in the form of additional supply risk) does *not* always make the firm more eager to adopt a hedging mechanism (i.e., resource flexibility or enhanced demand management through product differentiation), both in dedicated-only and flexible settings. This depends on the relationship between resource risks, product attributes, and resource investment costs.

- When the firm invests in the flexible resource, this never completely replaces the dedicated resources, and always results in a "diverse" resource portfolio. Thus, it is never optimal to acquire a portfolio with the flexible resource only. While this happens in the supply risk setting mainly due to the resource diversification advantage, it also happens in the demand risk setting due to the vertical differentiation between the products.

- In the absence of differential production costs, demand management by itself becomes powerful enough to hedge against the demand risk. Thus, a necessary condition for the flexible resource in this setting is differential production costs. This finding is different from the existing literature (e.g., Bish and Wang [4], Van Mieghem [24], both of which study independent products, with no consumer-driven substitution), and arises due to the presence of consumer-driven substitution in our model. This, however, no longer holds in the supply risk setting due to the additional resource diversification benefit of the flexible resource.

The remainder of this dissertation is organized as follows. We first introduce the notation and the discrete choice model in Chapter 2. In Chapter 3, we provide the mathematical formulation of the problem and discuss our solution methodology. In Chapter 4, we study the effectiveness of product differentiation considering dedicated-only system. We further investigate the effectiveness of an integrated product differentiation and resource diversification strategy in the flexible system in Chapter 5. We discuss the implications of relaxing two of our major assumptions in Chapter 6, and provide the conclusions and future research directions in Chapter 7. To improve the presentation, we delegate all proofs to the technical Appendix.

# Chapter 2

# The Model

We consider a monopolist selling two “vertically differentiated” products that only differ by a single *attribute*, such as quality, a higher level of which is preferred by all consumers. (Representing each product by a single attribute is a simplification, since in reality products will differ in several attributes. However, a single attribute representation is sufficient for our purpose of gaining insights.) Consequently, if both products are offered at equal prices, then there exists a natural ordering of products. However, the value of the attribute may be different for different consumers.

To represent the price-demand relationship, we utilize a consumer choice (discrete choice) model of vertical differentiation, commonly used in the economics and marketing literature (e.g., Moorthy [16], Mussa and Rosen [17], Tirole [21]) as well as the operations management literature (e.g., Choudhary et al. [7], Kouvelis and Yu [12], Rhee [19]). Specifically, let $s_i$ denote the attribute value of product $i$, $i = 1, 2$, and assume, without loss of generality (wlog), that $s_2 > s_1$. Thus, while products are vertically differentiated, they are not perfect substitutes. To present the consumer choice model, we follow the description in Moorthy [16] and Tirole [21]. We consider a population (market) of $N$ *heterogenous* consumers, and model the distribution of consumer type, $T$, by a continuous uniform distribution having support in $[0, b]$, as commonly done in the related literature (e.g., Choudhary [7], Moorthy [15] and [16], Rhee [19]). Let $f_T(.)$ and $F_T(.)$ respectively denote the probability density function and cumulative distribution function of $T$. A consumer of type $t$ has the following preferences:

$$u \equiv \begin{cases} ts - p, & \text{if the consumer buys a product with attribute value } s \text{ at price } p \\ 0, & \text{otherwise,} \end{cases}$$

where $u$ can be thought of as the “utility surplus” the consumer derives from consuming the particular product. Thus, the utility is separable in both attribute value and price. While all consumers prefer a higher value of the attribute for a given price, a consumer with a higher value of $t$ will be willing to pay more for the same attribute. We assume there are no income effects in the utility functions of these consumers (i.e., each consumer's expenditure on the product is only a small amount of her total expenditure so that her choice does not affect her marginal utility of income).

Each consumer buys *at most* one product from the monopolist's product line, based on the principle of maximization of her individual utility, that is, the consumer's choice set is given by $\{0, 1, 2\}$, where 0 corresponds to the "no purchase" option (from which, wlog, the consumer derives a utility of zero), and $i, i = 1, 2$, corresponds to purchasing product $i$. Let $p_i, i = 1, 2$, denote the price of product $i$, and let $d_i(p_1, p_2), i = 1, 2$, and $d_0(p_1, p_2)$ respectively denote the number of consumers who purchase product $i$, and who do not purchase any product. Then, $d_0(p_1, p_2) = N\ \Pr(Ts_i - p_i \leq 0, i = 1, 2)$ and $d_i(p_1, p_2) = N\ \Pr(Ts_i - p_i \geq \max\{Ts_{3-i} - p_{3-i}, 0\})$, for $i = 1, 2$. Thus, a direct implication of our consumer choice model is that consumers may be induced to *substitute* the products with each other by the firm's pricing strategy (consumer-driven subsitution).

We assume that the monopolist's product design (i.e., $0 \leq s_1 < s_2$) is exogenously determined. Then, faced with the above consumer choice process, the monopolist's problem is to determine the "resource investment portfolio," consisting of flexible and dedicated resources, ex-ante, under demand and supply uncertainty, and production and pricing strategy, $\overrightarrow{q} = (q_1, q_2)$ and $\overrightarrow{p} = (p_1, p_2)$, ex-post. This represents the fact that resource investment is a long-term decision, while production and pricing decisions can be made on a relatively shorter term. We denote the resource portfolio by $\overrightarrow{K} = (K_1, K_2, K_f)$, where $K_i, i = 1, 2$, denotes the acquired capacity for dedicated resource $i$, and $K_f$ denotes that for the flexible resource. As mentioned in Chapter 1, demand side uncertainty arises in the form of market size ($N$) uncertainty. We let $f_N(.)$ denote the probability density function of $N$, with support in $[0, \infty)$. We make no distributional assumptions on $N$. On the supply side, we consider uncertainty arising in the form of *resource disruptions* in which the failed resource becomes completely unusable (as in Snyder and Shen [20] and Tomlin and Wang [23]). The supply process may have a Bernoulli nature, due, for example, to labor strikes, natural calamities, accidents (e.g., fires), or supply chain disruptions. To model this phenomenon, we associate a Bernoulli random variable, $Y_i$, for resource type $i = 1, 2, f$, where $Y_i = 1$ with probability $\theta_i$, and $Y_i = 0$ with probability $1 - \theta_i$, where $0 < \theta_i < 1$. We assume $Y_1, Y_2, Y_f$ are independent and define $\overrightarrow{Y} = (Y_1, Y_2, Y_f)$. Note that the supply uncertainty considered here is different from *yield uncertainty*, which, in the context of our model, refers to the production capacity of a resource differing from its investment capacity by a random amount (see Yano and Lee [27] for a review of yield uncertainty).[1] In Chapter 6.2, we discuss the implications of supply disruptions versus yield uncertainty on our results.

On the financial side, the monopolist incurs a unit cost of $c_i, i = 1, 2, f$, for investing in resource type $i$, with $\max\{c_1, c_2\} < c_f < c_1 + c_2$ (this assumption is made to rule out trivial solutions), and a unit production cost of $\alpha s_i^2$ for product $i, i = 1, 2$, with $\alpha \geq 0$.

[1] Another form of supply uncertainty is lead time uncertainty (see Çakanyildirim and Bookbinder [5] for a review), which is not relevant in our single-period production setting.

(Both assumptions are common in the operations management and industrial organization literature, see, for example, Bish and Wang [4], Fine and Freund [10], and Van Mieghem [24] for the investment cost assumption; and Banker [2], Moorthy [16], and Rhee [19] for the production cost assumption). The production cost that is convex increasing in the attribute value can well capture the fact that it will be increasingly more expensive for the firm to improve the attribute value (e.g., quality) by a certain amount, due, in part, to costs of acquiring high quality raw material, operating high precision equipment, and additional organizational training to produce a higher valued product (Banker [2]). Here, $\alpha$ represent the production efficiency of the firm, where a low value of $\alpha$ means more efficient production technology (Rhee [19]). We further assume that $(s_1+s_2) < b/\alpha$, that is, observing the upper limit of consumer type and its production efficiency, the firm does not position its products at a too high attribute level such that it is never optimal for the firm to produce the higher attribute product even without any resource capacity constraints.

We model the monopolist's decision problem as a two-stage stochastic programming problem. In the first stage, the market size ($N$) and resource disruption vector ($\overrightarrow{Y}$) are random variables, and the firm determines its optimal resource portfolio ($\overrightarrow{K}$) so as to maximize its expected profit. Then, in the second stage, the firm observes the realization of the market size ($n$) and usable resource capacities ($y_iK_i, i = 1, 2, f$), and sets product prices ($\overrightarrow{p}$) and production quantities ($\overrightarrow{q}$) so as to maximize its profit subject to the resource capacity constraints imposed by its first stage decision. This two-stage framework leads to the following mathematical formulation:

<u>Problem **P**</u>:

Stage 1 Problem $\mathbf{P}_1$:

$$\max_{\overrightarrow{K}} V \equiv E_{N,\overrightarrow{Y}}[\Pi^*(\overrightarrow{K}, N, \overrightarrow{Y})] - \sum_{i=1,\ 2,\ f} c_i K_i \tag{2.1a}$$

$$\text{subject to } K_i \geq 0, \quad i = 1,\ 2,\ f. \tag{2.1b}$$

Stage 2 Problem $\mathbf{P_2}$:

$$\Pi^*(\overrightarrow{K}, n, \overrightarrow{y}) \equiv \max_{\overrightarrow{p}, \overrightarrow{q}} \Pi(\overrightarrow{K}, n, \overrightarrow{y}) = \max_{\overrightarrow{p}, \overrightarrow{q}} \sum_{i=1}^{2} q_i(p_i - \alpha s_i^2) \tag{2.2a}$$

subject to

$$q_i \leq y_i K_i + y_f K_f, \quad i = 1, 2 \tag{2.2b}$$

$$q_1 + q_2 \leq y_1 K_1 + y_2 K_2 + y_f K_f, \tag{2.2c}$$

$$q_i \leq d_i(\overrightarrow{p}), \quad i = 1, 2 \tag{2.2d}$$

$$p_i \geq 0, \quad i = 1, 2 \tag{2.2e}$$

$$q_i \geq 0, \quad i = 1, 2, \tag{2.2f}$$

$$d_i(\overrightarrow{p}) \geq 0, \quad i = 1, 2. \tag{2.2g}$$

In the above formulation, constraints (2.2b) and (2.2c) respectively ensure that the production quantity of each product, and the total production, do not exceed the respective capacities. Constraint (2.2d) ensures that the production of each product does not exceed its demand. Constraints (2.2e) - (2.2g) are the nonnegativity constraints for prices, production quantities, and demands, respectively.

To understand the effect of the different demand and supply risks, we study several variations of Problem **P**: with no uncertainty (the deterministic system), with demand uncertainty only (DU), with supply uncertainty only (SU), and with both demand and supply uncertainty ((S+D)U). When we say a property holds for the "general setting," this means it holds for variations DU, SU, and (S+D)U. In addition, we define the "dedicated-only system" as one in which the flexible resource is not available to the firm as an investment option, and the "flexible system" as one in which both dedicated and flexible resources are available for investment. We let $\overrightarrow{K}^D = (K_1^D, K_2^D)$ denote the optimal portfolio in the dedicated-only system and $\overrightarrow{K}^* = (K_1^*, K_2^*, K_f^*)$ the optimal portfolio in the flexible system.

# Chapter 3

# Preliminaries: Properties of the Optimal Resource Portfolio

In this Chapter, we first derive some properties of Problem **P** in the general setting, and then characterize its optimal solution for the demand uncertainty only (DU) setting (i.e., when $Y_i = 1$ with probability one, for $i = 1, 2, f$). The characterization gets quite complex for supply uncertainty only (SU) and supply and demand uncertainty ((S+D)U) systems, whose analysis we therefore limit to the $\alpha = 0$ case.

**Proposition 1.** For the general setting, given $n$, $\overrightarrow{y}$, and $\overrightarrow{K}$, the optimal prices to Problem $\mathbf{P_2}$, $(p_1^*, p_2^*)$, satisfy $0 \leq \frac{p_1^*}{s_1} \leq \frac{p_2^* - p_1^*}{s_2 - s_1} \leq b$, leading to

$$d_1(p_1^*, p_2^*) = n\ Pr(\frac{p_1^*}{s_1} \leq T \leq \frac{p_2^* - p_1^*}{s_2 - s_1}) = \frac{n}{b}(\frac{p_2^* - p_1^*}{s_2 - s_1} - \frac{p_1^*}{s_1}) \text{ and}$$
$$d_2(p_1^*, p_2^*) = n\ Pr(T \geq \frac{p_2^* - p_1^*}{s_2 - s_1}) = \frac{n}{b}(b - \frac{(p_2^* - p_1^*)}{s_2 - s_1}).$$

**Proof:** See Appendix A.1. □

**Proposition 2.** For the general setting, there exists an optimal solution to Problem $\mathbf{P_2}$ in which $q_i^* = d_i(\overrightarrow{p^*}), i = 1, 2$ (i.e., constraints (2.2d) are tight).

**Proof:** See Appendix A.2. □

Utilizing Propositions 1 and 2[1], the formulation for Problem $\mathbf{P_2}$ reduces to the following equivalent formulation in which prices are the only decision variables:

[1]We note that Propositions 1 and 2 in fact hold for any arbitrary, continuous distribution of $T$.

Stage 2 Problem $\mathbf{P_2}$:

$$\Pi^*(\overrightarrow{K}, n, \overrightarrow{y}) = \max_{\overrightarrow{p}} \frac{n}{b}[(\frac{p_2 - p_1}{s_2 - s_1} - \frac{p_1}{s_1})(p_1 - \alpha s_1^2) + (b - \frac{(p_2 - p_1)}{s_2 - s_1})(p_2 - \alpha s_2^2)] \quad (3.1a)$$

subject to

$$\frac{n}{b}(\frac{p_2 - p_1}{s_2 - s_1} - \frac{p_1}{s_1}) \leq y_1 K_1 + y_f K_f \quad (3.1b)$$

$$\frac{n}{b}(b - \frac{(p_2 - p_1)}{s_2 - s_1}) \leq y_2 K_2 + y_f K_f \quad (3.1c)$$

$$\frac{n}{b}(b - \frac{p_1}{s_1}) \leq y_1 K_1 + y_2 K_2 + y_f K_f \quad (3.1d)$$

$$p_i \geq 0, \quad i = 1, 2 \quad (3.1e)$$

$$\frac{p_1}{s_1} \leq \frac{p_2}{s_2} \quad (3.1f)$$

$$\frac{p_2 - p_1}{s_2 - s_1} \leq b. \quad (3.1g)$$

We can show that constraints (3.1e) - (3.1g) are redundant, and hence can be dropped from the formulation (see Appendix A.3). (Note that constraint (3.1f) comes from the relationship that $\frac{p_2 - p_1}{s_2 - s_1} \geq \frac{p_1}{s_1} \Leftrightarrow \frac{p_2}{s_2} \geq \frac{p_1}{s_1}$.)

We can show that the objective function of Problem $\mathbf{P_2}$ (given in (3.1a)) is strictly jointly concave in $p_1$ and $p_2$. Then, since constraints (3.1b) - (3.1d) are linear, the first-order Karush-Kuhn-Tucker (KKT) conditions are necessary and sufficient for optimality, leading to a characterization of the stage 2 optimal solution.

**Proposition 3.** For the general setting, given $n$, $\overrightarrow{y}$, and $\overrightarrow{K}$, the optimal solution to Problem $\mathbf{P_2}$ is as follows:

$$(p_1^*, p_2^*) = \begin{cases} (\frac{bs_1 + \alpha s_1^2}{2}, \ \frac{bs_2 + \alpha s_2^2}{2}), & \text{if } \Omega_1, \\ (bs_1 - \frac{bs_1(y_1K_1 + y_2K_2 + y_fK_f)}{n}, \ \frac{b(s_1 + s_2)}{2} - \frac{bs_1(y_1K_1 + y_2K_2 + y_fK_f)}{n} + \frac{\alpha(s_2^2 - s_1^2)}{2}), & \text{if } \Omega_2, \\ (\frac{bs_1 + \alpha s_1 s_2}{2} - \frac{b(y_1K_1 + y_fK_f)(s_2 - s_1)s_1}{ns_2}, \ \frac{(b + \alpha s_2)s_2}{2}), & \text{if } \Omega_3, \\ (bs_1 - \frac{bs_1(y_1K_1 + y_2K_2 + y_fK_f)}{n}, \ bs_2 - \frac{b(y_1K_1 + y_fK_f)s_1}{n} - \frac{by_2K_2s_2}{n}), & \text{if } \Omega_4, \\ (\frac{(b + \alpha s_1)s_1}{2}, \ b(s_2 - s_1) - \frac{b(y_2K_2 + y_fK_f)(s_2 - s_1)}{n} + \frac{(b + \alpha s_1)s_1}{2}), & \text{if } \Omega_5, \\ (bs_1 - \frac{bs_1(y_1K_1 + y_2K_2 + y_fK_f)}{n}, \ bs_2 - \frac{b(y_fK_2 + y_fK_f)s_2}{n} - \frac{by_1K_1s_1}{n}), & \text{if } \Omega_6, \end{cases}$$

where

$$\Omega_1 = \{n \le \frac{2b(y_1K_1 + y_fK_f)}{\alpha s_2},\ n \le \frac{2b(y_2K_2 + y_fK_f)}{b - \alpha(s_1+s_2)},\ n \le \frac{2b(y_1K_1 + y_2K_2 + y_2K_f)}{b - \alpha s_1}\},$$
$$\Omega_2 = \{n > \frac{2b(y_1K_1 + y_2K_2 + y_fK_f)}{b - \alpha s_1},\ n > \frac{2by_2K_2}{b - \alpha(s_1+s_2)},\ n \le \frac{2b(y_2K_2 + y_fK_f)}{b - \alpha(s_1+s_2)}\},$$
$$\Omega_3 = \{n > \frac{2b(y_1K_1 + y_fK_f)}{\alpha s_2},\ n \le \frac{2b[y_2K_2s_2 + (y_1K_1 + y_fK_f)s_1]}{(b - \alpha s_2)s_2}\},$$
$$\Omega_4 = \{n > \frac{2b[(y_1K_1 + y_fK_f)s_1 + y_2K_2s_2]}{(b - \alpha s_2)s_2},\ n < \frac{2by_2K_2}{b - \alpha(s_1+s_2)}\},$$
$$\Omega_5 = \{n > \frac{2b(y_2K_2 + y_fK_f)}{b - \alpha(s_1+s_2)},\ n < \frac{2b(y_1K_1 + y_2K_2 + y_fK_f)}{b - \alpha s_1}\},$$
$$\Omega_6 = \{n > \frac{2b(y_2K_2 + y_fK_f)}{b - \alpha(s_1+s_2)},\ n > \frac{2b(y_1K_1 + y_2K_2 + y_fK_f)}{b - \alpha s_1}\},$$

with $\Omega_i \cap \Omega_j = \emptyset, i, j = 1, ..., 6, i \ne j$, and $\cup_{i=1}^{6}\Omega_i = \mathbf{\Omega}$, where $\mathbf{\Omega}$ denotes the universal set.

**Proof:** See Appendix A.3. □

**Proposition 4.** Depending on the relationship between $y_1K_1 \ge 0, y_2K_2 \ge 0,$ and $y_fK_f \ge 0$, one of the following three regions is possible in stage 2 of the general setting:

Region I: If $y_1K_1 \ge \frac{\alpha s_2(y_2K_2+y_fK_f)}{b-\alpha(s_1+s_2)}$, then the sample space consists of events $\Omega_1, \Omega_5,$ and $\Omega_6$ only, see Figure 3.1.

Region II: If $\frac{\alpha s_2 y_2K_2}{b-\alpha(s_1+s_2)} - y_fK_f \le y_1K_1 \le \frac{\alpha s_2(y_2K_2+y_fK_f)}{b-\alpha(s_1+s_2)}$, then the sample space consists of events $\Omega_1, \Omega_2,$ and $\Omega_6$ only, see Figure 3.2.

Region III: If $\frac{\alpha s_2 y_2K_2}{b-\alpha(s_1+s_2)} - y_fK_f \ge y_1K_1$, then the sample space consists of events $\Omega_1$, $\Omega_3$, $\Omega_4$, $\Omega_2$, and $\Omega_6$ only, see Figure 3.3.

**Proof:** See Appendix A.4. □

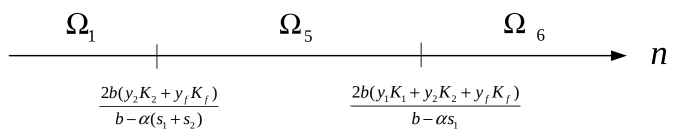



Figure 3.1: The Sample Space of the Stage 2 Problem for Region I.

While the objective function of the stage 1 problem ($V$) is continuous everywhere, Proposition 4 indicates that it takes on different forms in the three regions of the feasible $\overrightarrow{K}$-space. Furthermore, our numerical study shows that $V$ is not well-behaved everywhere in the feasible $\overrightarrow{K}$-space (i.e., it may have multiple local maxima). For this reason, in the remainder of this Chapter we focus on the DU setting (i.e., $\overrightarrow{Y} = (1, 1, 1)$ with certainty) for which we are able to reduce the feasible set to a Pareto efficient set over which $V$ is well-behaved everywhere.

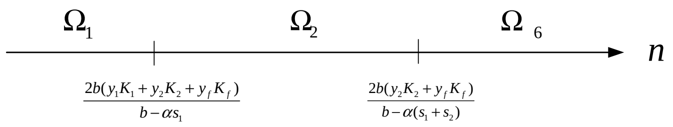



Figure 3.2: The Sample Space of the Stage 2 Problem for Region II.

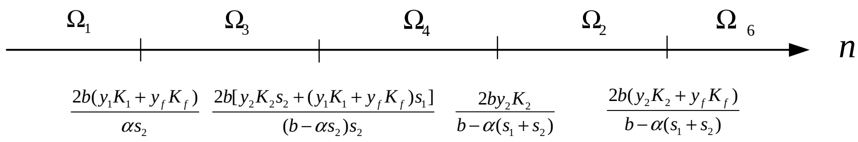


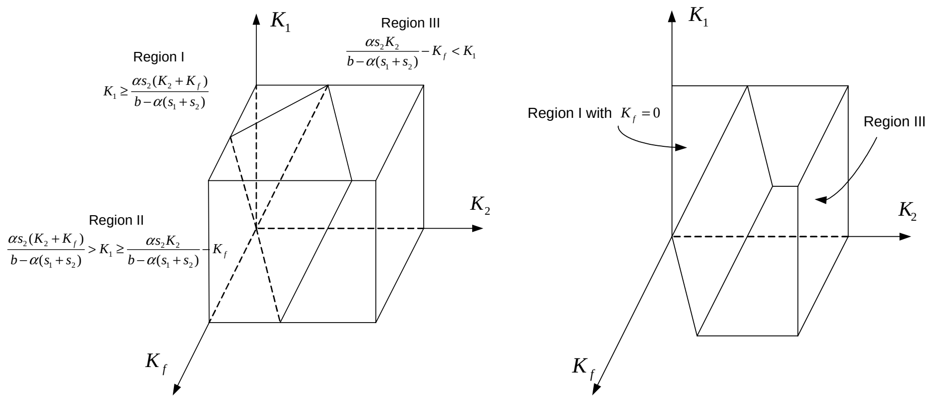

Figure 3.3: The Sample Space of the Stage 2 Problem for Region III.

**Theorem 1.** Consider Problem $\mathbf{P}_1$ in the DU setting.

(i) For any solution with $K_f > 0$ in Region I (i.e., $K_1 \geq \frac{\alpha s_2(K_2+K_f)}{b-\alpha(s_1+s_2)}$), there exists a *dominating* solution with $K_f = 0$. Thus, any solution with $K_f > 0$ in Region I cannot be in the *Pareto efficient* set.

(ii) For any solution with $\frac{\alpha s_2 K_2}{b-\alpha(s_1+s_2)} - K_f < K_1 \leq \frac{\alpha s_2(K_2+K_f)}{b-\alpha(s_1+s_2)}$ in Region II, there exists a *dominating* solution with $K_1 + K_f = \frac{\alpha s_2 K_2}{b-\alpha(s_1+s_2)}$. Thus, any solution with $K_1 + K_f > \frac{\alpha s_2 K_2}{b-\alpha(s_1+s_2)}$ in Region II cannot be in the *Pareto efficient* set.

**Proof:** See Appendix A.5.




(a) The Feasible Region. (b) The Pareto Efficient Set.

Figure 3.4: The Different Regions in Stage 1 Depicted in the Unit Decision Space for the DU Setting.

As Theorem 1 shows, the flexible resource is not beneficial in Region I, and any solution in Region II is dominated by solutions on the boundary of Regions II and III. Consequently, the Pareto efficient set consists only of Region I with $K_f = 0$ and Region III with $K_f \geq 0$ (see Figure 3.4 for the feasible region and the Pareto efficient set in a unit $\overrightarrow{K}$-space – limitation to unit space is for illustration purposes only). Therefore, we study Region I (with $K_f = 0$) and Region III (with $K_f \geq 0$) problems separately (see Appendix B.2) and characterize their optimal solutions. The best of these two optimal solutions then gives the global optimal solution, as characterized in the subsequent Chapters.

In order to understand the impact of *consumer-driven substitution* (coming from our consumer choice model of vertical differentiation), we also study a "no-consumer-driven-substitution" version of Problem $\mathbf{P}$ under each of demand and supply risks, which we refer to as Problems $\mathbf{P}-\mathbf{DU}(\mathbf{NC})$ and $\mathbf{P}-\mathbf{SU}(\mathbf{NC})$, respectively. Specifically, under no consumer-driven substitution, we consider that the two products are independent (i.e., neither substitutable nor complementary), each having a market size of $N_i, i = 1, 2$. Thus, a product $i, i = 1, 2$, consumer of type $t$ will purchase product $i$ only if $ts_i - p_i \geq 0$, and under no condition will she switch to the other product. Similar to our original model (Problem $\mathbf{P}$), we assume that the consumer type in each market follows a uniform distribution in $[0, b]$. All other assumptions in our original model apply. Then, in Problem $\mathbf{P}-\mathbf{DU}(\mathbf{NC})$, the only uncertainty in stage 1 comes from $N_i, i = 1, 2$, whereas in Problem $\mathbf{P}-\mathbf{SU}(\mathbf{NC})$, it comes from $Y_i, i = 1, 2$. Observe that in the dedicated-only system, both Problems $\mathbf{P}-\mathbf{DU}(\mathbf{NC})$ and $\mathbf{P}-\mathbf{SU}(\mathbf{NC})$ decompose by product.

Our main focus in this dissertation is on the effectiveness of the two strategies, resource flexibility and product differentiation, to hedge against supply and demand risks. Our agenda in the remainder of the dissertation is as follows. In Chapter 4, we first isolate the effect of product differentiation by studying dedicated-only systems in which resource flexibility is not an option (i.e., flexible resource is not a technology choice). Then, in Chapter 5, we study the additional effect of resource flexibility by considering flexible systems in which both dedicated and flexible resources are available to the firm for investment.

# Chapter 4

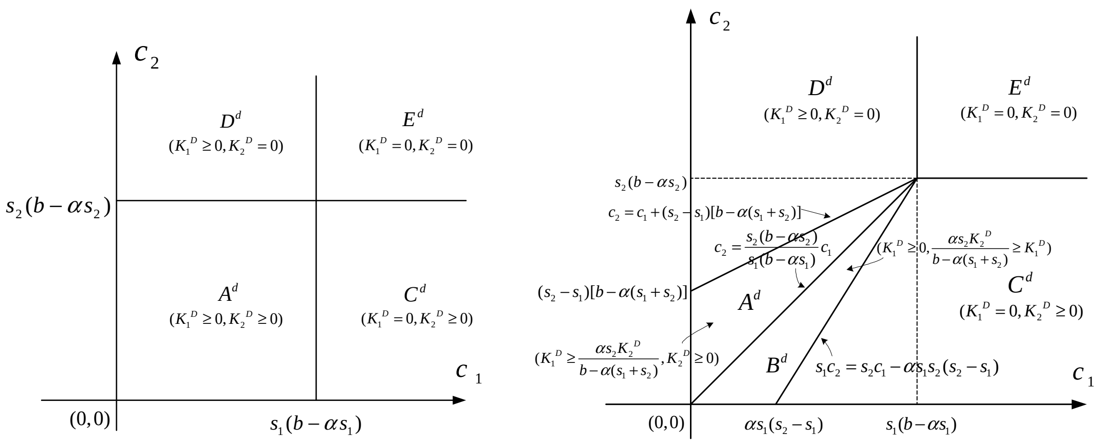
# The Effectiveness of Product Differentiation

In this Chapter, our objective is to understand when it is optimal for the firm to offer differentiated products versus a single product to hedge against uncertainty, and how the different demand and supply risks change this condition. Obviously, the firm's resource mix (i.e., the types of resources acquired) and product offering are interrelated, i.e., the resource mix dictates the product offering capability. We are also interested in understanding how the existence of consumer-driven substitution affects the firm's product differentiation strategy under the different risks. In order to isolate the impact of product differentiation, throughout this Chapter we assume that the flexible resource is *not* available as a technology choice, and consider the dedicated-only system under various settings with consumer-driven substitution (Problem $\mathbf{P}$) and without (Problem $\mathbf{P-NC}$). As stated in Chapter 3, while we analyze the DU setting for all $\alpha \geq 0$, we limit the analysis of SU and (S+D)U settings to $\alpha = 0$ only as the characterization gets quite complex otherwise.

Observe that in the absence of resource flexibility, the firm has to completely rely on demand management (through pricing and/or product differentiation) to hedge against uncertainty, so that it can take advantage of consumer-driven substitution and sell the consumers what it wants to sell. Furthermore, the firm's demand management can be enhanced if it has the capability to produce both of the differentiated products (i.e., it invests in both dedicated resources). This is because, in the case of demand risk, this allows the firm to change its product mix in response to market conditions; and in the case of supply risk where a resource might completely fail, it enables the firm to switch production to the other resource (hence to the other product), in a way similar to a dual sourcing strategy. In the following, we study each of these effects.

## 4.1 Characterization of the optimal portfolio in the dedicated-only system

We first characterize the optimal portfolio in the dedicated-only system both with and without consumer substitution.




(a) Without Consumer-Driven Substitution. (b) With Consumer-Driven Substitution.

Figure 4.1: Structure of the Optimal Portfolio for the Dedicated-only DU System ($\alpha \geq 0$).

**Proposition 5.** For $\alpha \geq 0$, the optimal resource vector, $(K_1^D, K_2^D)$, in the dedicated-only DU system is unique for both Problem **P** (with consumer-driven substitution) and Problem $\mathbf{P} - \mathbf{DU}(\mathbf{NC})$ (without consumer-driven substitution), and can be characterized as in Figures 4.1(a) and 4.1(b), respectively.

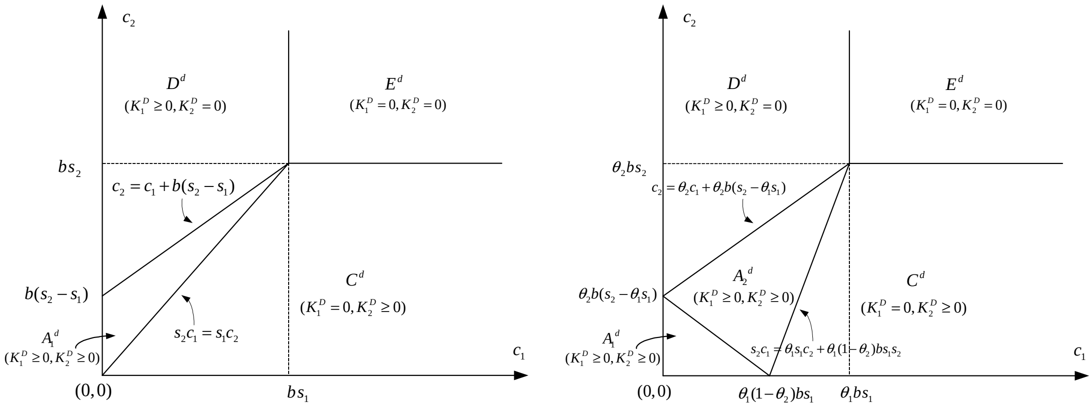
**Proof:** See Appendix B.3.

**Proposition 6.** Consider the dedicated-only system with $\alpha = 0$.

(i) If there is consumer-driven substitution, then the optimal resource vector $(K_1^D, K_2^D)$ is unique for each of the deterministic, DU, SU, and (S+D)U settings, and can be characterized as in Table 4.1 and Figures 4.2(a) and 4.2(b).

(ii) If there is no consumer-driven substitution, then the optimal resource vector $(K_1^D, K_2^D)$ is unique in the SU setting and can be characterized as in Figure 4.3.

**Proof:** See Appendix C.1 for part (i) and Appendix C.2 for part (ii). □




(a) Deterministic and DU Settings. (b) SU and (S+D)U Settings.

Figure 4.2: Structure of the Optimal Portfolio for the Dedicated-only System ($\alpha = 0$) with Consumer-driven Substitution

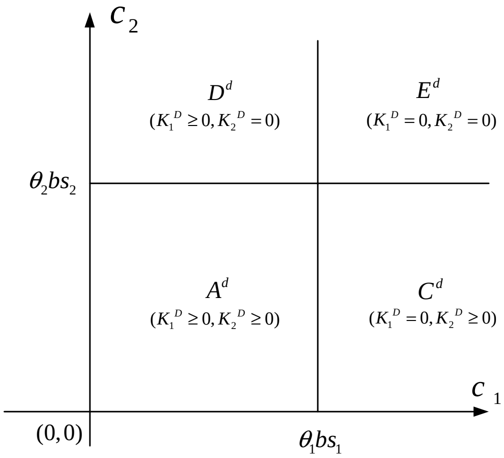



Figure 4.3: Structure of the Optimal Portfolio for the Dedicated-only SU System ($\alpha = 0$) without Consumer-driven Substitution.

## 4.2 The consumer-driven substitution effect

Not surprisingly, the existence of consumer-driven substitution reduces the need for product differentiation (hence, the need for two dedicated resources) under both demand and supply risks (compare region $A^d$+$B^d$ in Figure 4.1(b) with $A^d$ in Figure 4.1(a) for the DU setting, and $A_1^d + A_2^d$ in Figure 4.2(b) with $A^d$ in Figure 4.3), since in some cost regions it might be more beneficial for the firm to have the consumers switch (rather than investing in both dedicated resources). Furthermore, the impact of consumer-driven substitution on the optimal resource mix rises (further shrinking the product differentiation region) as either $s_2 - s_1$ decreases

Table 4.1: Characterization of the Optimal Portfolio in the Dedicated-only System under Various Settings ($\alpha = 0$).

| Region | $K_1^D$ | $K_2^D$ |
|---|---|---|
| | Deterministic Setting | |
| $A^d$ | $\frac{n(c_2-s_2)}{2b(s_2-s_1)}$ | $\frac{n}{2}[1-\frac{(c_2-c_1)}{b(s_2-s_1)}]$ |
| $C^d$ | 0 | $\frac{n}{2}(1-\frac{c_2}{bs_2})$ |
| $D^d$ | $\frac{n}{2}(1-\frac{c_1}{bs_1})$ | 0 |
| $E^d$ | 0 | 0 |
| | DU Setting | |
| Region | $K_1^D, K_2^D$ are solutions to: | |
| $A^d$ | $\int_{2(K_1+K_2)}^{\infty} bs_1[1-\frac{2(K_1+K_2)}{n}]f_N(n)dn = c_1$ <br> $\int_{2K_2}^{2(K_1+K_2)} b(s_2-s_1)(1-\frac{2K_2}{n})f_N(n)dn + \int_{2(K_1+K_2)}^{\infty}[bs_2 - 2b\frac{(K_2s_2+K_1s_1)}{n}]f_N(n)dn = c_2$ | |
| $C^d$ | $K_1^D = 0,\ \int_{2K_2}^{\infty} bs_2(1-\frac{2K_2}{n})f_N(n)dn = c_2$ | |
| $D^d$ | $\int_{2K_1}^{\infty} bs_1(1-\frac{2K_1}{n})f_N(n)dn = c_1,\ K_2^D = 0$ | |
| $E^d$ | $K_1^D = 0, K_2^D = 0$ | |
| | SU Setting | |
| Region | $K_1^D$ | $K_2^D$ |
| $A_1^d$ | $\frac{n}{2}[1-\frac{c_1}{\theta(1-\theta)bs_1}]$ | $\frac{n}{2}[1-\frac{c_2}{\theta b(s_2-\theta s_1)}]$ |
| $A_2^d$ | $\frac{ns_2}{2\theta s_1}[1-\frac{\theta b(s_2-\theta s_1)-c_2+\theta s_1}{\theta b(s_2-\theta^2 s_1)}] - \frac{nc_2}{2\theta^2 bs_1}$ | $\frac{n[\theta b(s_2-\theta s_1)-c_2+\theta c_1]}{2\theta b(s_2-\theta^2 s_1)}$ |
| $C^d$ | 0 | $\frac{n}{2}(1-\frac{c_2}{\theta bs_2})$ |
| $D^d$ | $\frac{n}{2}(1-\frac{c_1}{\theta bs_1})$ | 0 |
| $E^d$ | 0 | 0 |
| | (S+D)U Setting | |
| Region | $K_1^D, K_2^D$ are solutions to: | |
| $A_1^d$ or $A_2^d$ | $\theta^2\int_{2(K_1+K_2)}^{\infty} bs_1[1-\frac{2(K_1+K_2)}{n}]f_N(n)dn + \theta(1-\theta)\int_{2K_1}^{\infty} bs_1(1-\frac{2K_1}{n})f_N(n)dn = c_1$ <br> $\theta^2\int_{2K_2}^{2(K_1+K_2)} b(s_2-s_1)(1-\frac{2K_2}{n})f_N(n)dn + \theta^2\int_{2(K_1+K_2)}^{\infty}[bs_2 - 2b\frac{(K_2s_2+K_1s_1)}{n}]f_N(n)dn$ <br> $+(1-\theta)\theta\int_{2K_2}^{\infty} bs_2(1-\frac{2K_2}{n})f_N(n)dn = c_2$ | |
| $C^d$ | $K_1^D = 0,\ \theta\int_{2K_2}^{\infty} bs_2(1-\frac{2K_2}{n})f_N(n)dn = c_2$ | |
| $D^d$ | $\theta\int_{2K_1}^{\infty} bs_1(1-\frac{2K_1}{n})f_N(n)dn = c_1,\ K_2^D = 0$ | |
| $E^d$ | $K_1^D = 0, K_2^D = 0$ | |

(i.e., products become less differentiated), or $\alpha$ increases (i.e., production costs become more differentiated) because in the former case consumers will be more willing to substitute the products with one another, and in the latter case, the firm will be less willing to produce product 2 due to its much higher production cost, and attempt to save on production costs by inducing consumers to switch to the other product.

What is interesting is, in the DU setting with independent products (markets), the firm's resource mix does *not* depend on the magnitude of demand uncertainty in either market, but in the SU setting with independent products, it does. Similar results continue to hold for the settings with consumer-driven substitution, as we discuss in detail below.

Unless otherwise noted, all models we study in the remainder of the dissertation consider the consumer-driven substitution (i.e., Problem **P**).

## 4.3 The demand risk versus supply risk effect(under consumer-driven substitution)

**Proposition 7.** The existence of demand risk does *not* alter the resource mix in the optimal portfolio (hence the firm's product offering) in the dedicated-only system (for $\alpha \geq 0$). However, the existence of supply risk does (for $\alpha = 0$).

**Proof:** The first part follows because the optimal resource mix in the dedicated-only deterministic setting also follows Figure 4.1(b); see Appendix C.3. The second part is a direct consequence of Proposition 6 (ii). □

Thus, demand and supply risks impact the optimal portfolio differently (with the supply risk having a more significant effect). In particular, the demand risk affects only the resource capacities and not the types of resources in the portfolio, making the firm partially immune to forecast errors (i.e., the firm will always acquire the right resource mix) as long as the firm estimates correctly that the demand shock is multiplicative. However, this partial immunity disappears in the case of supply risk. This is because of our consumer choice model that leads to a *multiplicative* demand shock, under which the cost thresholds for investment decisions become independent of the distribution and parameters of the demand uncertainty, both with and without consumer-driven substitution. (Not surprisingly, this result no longer holds when the firm considers the flexible resource as an investment option, see Chapter 5.1.) In the additive demand shock case, a similar result holds *only if* production is always "profitable," independent of how demand uncertainty is resolved (e.g., the firm is a price-taker with $p_i \geq c_i, i = 1, 2$, see Van Mieghem [24]). On the other hand, if the demand shock is additive and the profitability of production depends on market conditions (e.g., the firm is a price-setter, that is, the optimal price charged for each product depends on the realization of the demand curve, see Bish and Wang [4]), then the resource mix will be a function of the demand distribution parameters.

On the other hand, supply risk affects *both* the resource mix and capacities in the dedicated-only system. Furthermore, we can show, for some special cases, that this remains true even when supply risk is in the form of yield uncertainty (i.e., $\overrightarrow{Y}$ follows a continuous distribution in $[0, 1]$; see Chapter 6.2 for discussion). Thus, whether uncertainty comes from the market or from the firm's own resources has a large effect on the firm's resource portfolio. This highlights an important advantage of offering differentiated products for a price-setter firm when resource flexibility is not an option: Offering differentiated products that serve a common population may make the firm partially immune to errors in demand forecasts when the market size is uncertain. The existence of supply risk, however, eliminates this advantage.

The following proposition shows how changes in the supply risk alters the firm's product offering.

**Proposition 8.** For $\alpha = 0$, the product differentiation region in both SU and (S+D)U settings ($A_1^d + A_2^d$ in Figure 4.2(b), corresponding to $(K_1^D \geq 0, K_2^D \geq 0)$) expands in $\theta_1(\theta_2)$ for $\theta_1 < \frac{s_2(2-\theta_2)}{2s_1}(\theta_2 < \frac{2s_2-\theta_1 s_1}{2s_2})$, and shrinks in $\theta_1(\theta_2)$ otherwise. When $\theta_1 = \theta_2 = \theta$, if $s_2 > 3s_1$, then the product differentiation region always expands in $\theta$; otherwise, it first expands, then shrinks in $\theta$, reaching its maximum area at $\theta = \frac{4s_2}{3(s_1+s_2)}$.

**Proof:** See Appendix C.4. □

As $\theta_i, i = 1$ or 2, increases, dedicated resource $i$ becomes more reliable. As a result, in some cost regions (with medium $c_i$) where the firm would originally invest only in the other resource $(3 - i)$, the firm can now afford to invest in both resources to take advantage of an enhanced demand management through product differentiation. On the other hand, in some cost regions (with medium $c_{3-i}$) where the firm would originally invest in both resources, the firm invests in resource $i$ only since the benefit from its higher reliability outweighs the investment cost of product differentiation. The rates of change of these regions dictate that the product differentiation region first expands, then shrinks.

When resources have the same risk level, $\theta$, we can see how the degree of product substitution plays a role in altering the resource mix. As $\theta$ increases, both dedicated resources become more reliable, encouraging the firm to invest in both resources for a wider range of $(c_1, c_2)$ due to an enhanced demand management and dual sourcing capability (due to resource diversification). As the resources get more reliable (hitting some reliability threshold of $\frac{4s_2}{3(s_1+s_2)}$), the benefits of a diversified portfolio reduce, and the firm becomes motivated to move towards a single resource (single product) investment, with the hope of managing demand through consumer-driven substitution only, which works well only when products are not much differentiated. As a result, for low product differentiation ($s_2 < 3s_1$), product differentiation region first expands, then shrinks in $\theta$. For high differentiation ($s_2 > 3s_1$) on the other hand, a higher value of $\theta$ cannot encourage the firm enough to move towards a single product regime (due to a weak consumer-driven substitution), and the product differentiation region expands for all values of $\theta$. This again emphasizes that demand management capability of product differentiation is valuable under both demand and supply risks.

We next study the effect of the supply risk in terms of how it forces the firm to switch from product differentiation ($PD$) to no product differentiation ($NPD$), i.e., single product offering, and vice versa. We first eliminate the effect of different resource risks and study the case where $\theta_1 = \theta_2 = \theta$ (i.e., both dedicated resources have the same risk level). We then consider different resource risks ($\theta_i < \theta_j$, i.e., resource $j$ is less risky than resource $i$, $i, j = 1, 2, i \neq j$).

**Corollary 1.** Consider the dedicated-only system. When both resources have the same risk level ($\theta_1 = \theta_2 = \theta, 0 < \theta < 1$) and $\alpha = 0$, the additional supply risk (in the SU versus deterministic settings, or in the (S+D)U versus DU settings) will force the firm to change its product offering as follows (see Figure 4.4[1]and Table 4.2):

(I) : **No Product Differentiation → Product Differentiation (NPD → PD)**

(a) $\mathbf{K_2} \rightarrow (\mathbf{K_1}, \mathbf{K_2})$ if $c_2 \le \frac{s_2}{s_1} c_1$ and $c_1 \le \frac{\theta s_1}{s_2} c_2 + \theta(1-\theta) b s_1$ (see regions $PD$ in Figure 4.4(a) and $PD1$ in 4.4(b)).

(b) $\mathbf{K_1} \rightarrow (\mathbf{K_1}, \mathbf{K_2})$ if $s_2 < (1+\theta)s_1,\ c_2 \le \theta c_1 + \theta b(s_2 - \theta s_1),$ and $c_2 \ge c_1 + b(s_2 - s_1)$ (see region $PD2$ in Figure 4.4(b)).

(II) : **Product Differentiation → No Product Differentiation (PD → NPD)**

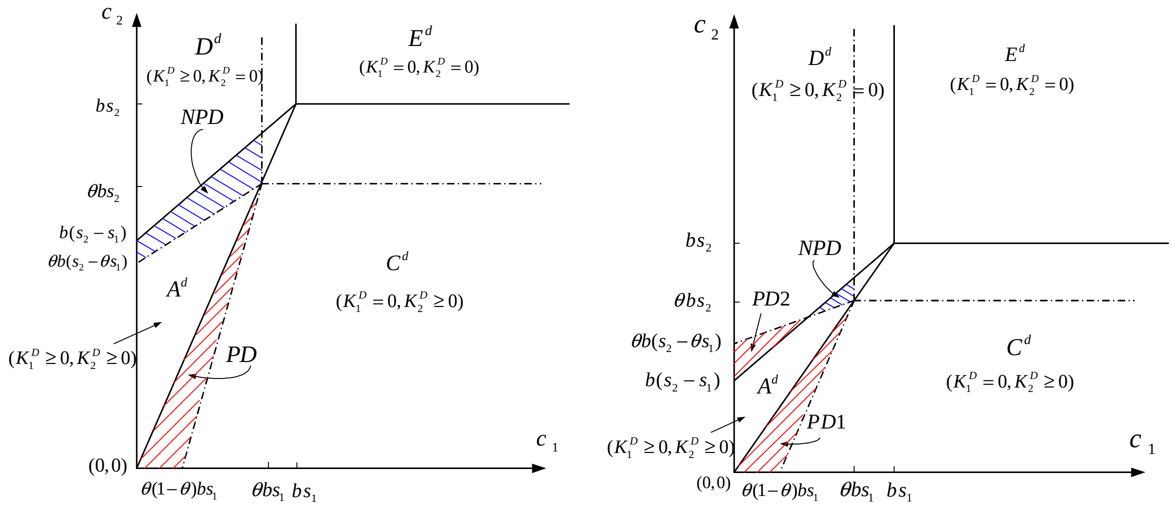
(a) $(\mathbf{K_1}, \mathbf{K_2}) \rightarrow \mathbf{K_1}$ if $c_2 \le c_1 + b(s_2 - s_1)$, $c_2 \ge \theta c_1 + \theta b(s_2 - \theta s_1)$, and $c_1 \le \theta b s_1$ (see regions $NPD$ in Figures 4.4(a) and 4.4(b)).

(III) : **Investment → No Investment**

(a) $(\mathbf{K_1}, \mathbf{K_2}) \rightarrow$ **No Investment** if $c_2 \le c_1 + b(s_2 - s_1)$, $c_2 \ge \frac{s_2}{s_1} c_1$, and $c_1 \ge \theta b s_1$.

(b) $\mathbf{K_2} \rightarrow$ **No Investment** if $c_2 \le \frac{s_2}{s_1} c_1$ and $b s_2 \ge c_2 \ge \theta b s_2$.

(c) $\mathbf{K_1} \rightarrow$ **No Investment** if $c_2 \ge c_1 + b(s_2 - s_1)$ and $b s_1 \ge c_1 \ge \theta b s_1$.




(a) High Product Differentiation ($s_2 \ge (1+\theta)s_1$). (b) Low Product Differentiation ($s_2 < (1+\theta)s_1$).

Figure 4.4: Optimal Resource Mix for the Dedicated-only System in Deterministic and DU Settings (solid line), and SU and (S+D)U Settings (dashed line) when $\theta_1 = \theta_2 = \theta$ and $\alpha = 0$.

One might think that in the absence of the resource flexibility option, more uncertainty, in terms of the additional supply risk (in the SU versus the deterministic settings, or (S+D)U

[1]Figures 4.4(a) and 4.4(b) are in the same scale, with the only difference being the value of parameter $s_2$.

Table 4.2: How Supply Risk Affects the Firm's Product Offering when $\theta_1 = \theta_2 = \theta$ ($\alpha = 0$).

| | High Product Differentiation ($s_2 \geq (1+\theta)s_1$) | Low Product Differentiation ($s_2 < (1+\theta)s_1$) |
|---|---|---|
| $NPD \to PD$ | $K_2 \to (K_1, K_2)$ low $\frac{c_2}{c_1}$ ratio, low to medium $c_1, c_2$ (see Figure 4.4(a) $PD$) | $K_2 \to (K_1, K_2)$ low $\frac{c_2}{c_1}$ ratio, low to medium $c_1, c_2$ (see Figure 4.4(b) $PD1$) |
| | | $K_1 \to (K_1, K_2)$ high $\frac{c_2}{c_1}$ ratio, low $c_1, c_2$ (see Figure 4.4(b) $PD2$) |
| $PD \to NPD$ | $(K_1, K_2) \to K_1$ high $\frac{c_2}{c_1}$ ratio, low to medium $c_1$, medium to high $c_2$ (see Figure 4.4(a) $NPD$) | $(K_1, K_2) \to K_1$ medium $\frac{c_2}{c_1}$ ratio, medium $c_1, c_2$ (see Figure 4.4(b) $NPD$) |

versus DU settings) will make the firm more eager to adopt product differentiation (equivalently, a diversified portfolio), either due to enhanced demand management capability or resource diversification advantage. However, Corollary 1 indicates that this is *not* always the case. This depends on the relationship between resource risks, product attributes, and resource investment costs. For the case of $\theta_1 = \theta_2 = \theta$, when $s_2 < (1+\theta)s_1$ (low product differentiation), the firm will move from $NPD$ to $PD$ when it faces a low $\frac{c_2}{c_1}$ ratio with low to medium $c_1$ and $c_2$, or high $\frac{c_2}{c_1}$ ratio with low $c_1$ and $c_2$. On the other hand, when the firm faces a medium $\frac{c_2}{c_1}$ ratio with medium $c_1$ and $c_2$, it will be better off by moving from $PD$ to $NPD$ (acquiring the capacity to produce product 1 only). (Similar results hold for high product differentiation, see Table 4.2.) The reason is that, with supply risk, the firm becomes more sensitive to the investment cost because there is a positive probability that a resource will be totally useless. As a result, for high enough investment costs, the firm chooses to rely solely on limited demand management for a single product, and forgoes to take advantage of product differentiation. Consequently, that the firm is a price-setter for each product makes it easier to switch from a product differentiation ($PD$) to a single product strategy ($NPD$) when resources are expensive, and a higher value of $s_2$ encourages this switch (to a single product strategy with the lower value product), as the firm can take less advantage of consumer-driven substitution when $s_2 - s_1$ is large (compare the $NPD$ region in Figures 4.4(a) and 4.4(b)).

Finally observe that only when product differentiation is low ($s_2 < (1+\theta)s_1$), will the additional supply risk force the firm to move from $K_1$ to $(K_1, K_2)$ (see cost region $PD2$ in Figure 4.4(b)); on the other hand, $K_2 \to (K_1, K_2)$ shift is possible under all levels of product differentiation. This, again, can be explained by the investment cost effect. With high product differentiation (i.e., $s_2 \geq (1+\theta)s_1$), product 2 is a more valuable product and the firm is willing to pay a higher investment cost for its dedicated resource. Thus, the

$(K_1^D \geq 0, K_2^D = 0)$ portfolio in the DU setting appears only when the investment cost, $c_2$, is very high, and at such a high value of $c_2$, the optimal resource mix cannot change with the additional supply risk (and is still comprised of dedicated resource 1 only). When $s_2$ is lower (i.e., $s_2 < (1+\theta)s_1$), so is the threshold for investing in dedicated resource 2 in DU, and additional supply risk may shift $K_1 \rightarrow (K_1, K_2)$ for low enough $c_2$ (i.e., in cost region $PD2$). Also observe that for high product differentiation (Figure 4.4(a)), cost region $PD2$ corresponds to product differentiation under both DU (dedicated) and SU ((S+D)U) settings.

When resources have different risk levels, similar results to Corollary 1 can be obtained, with the difference that now the firm may also change the types of resources acquired. More specifically, facing additional supply risk, when $\theta_2 > \theta_1$ ($\theta_2 < \theta_1$), the firm may switch from investing in dedicated resource 1(2) to dedicated resource 2(1) when it faces a medium $\frac{c_2}{c_1}$ ratio with medium $c_1$ and $c_2$. This is intuitive: When $\theta_2 > \theta_1(\theta_2 < \theta_1)$, dedicated resource 2(1) is more reliable, hence more preferable to the firm under a certain cost structure. See Tables C.3 - C.4 and Figures C.4 - C.3 in Appendix C.5 for details.

# Chapter 5

# The Effectiveness of an Integrated Product Differentiation and Resource Flexibility Strategy

When the firm invests in the flexible resource, it also acquires the product differentiation capability, whereas product differentiation is obviously possible without the flexible resource. Furthermore, while it may be optimal for the firm to adopt product differentiation even in a deterministic setting (see Proposition 6), a necessary condition for the flexible resource is the existence of uncertainty, either on the demand or supply side. Consequently, in this section, we are interested in the following questions: Under what conditions is product differentiation, by itself (i.e., through the dedicated resources), sufficient to hedge against demand and supply risks, and under what conditions it is not (so the firm has to rely on the more expensive strategy of acquiring the flexible resource)? For this purpose, we first study the DU setting with $\alpha \geq 0$. Then, we restrict ourselves to the $\alpha = 0$ case and study the effect of the additional supply risk in the (S+D)U setting. In the presence of supply risk, the value of the flexible resource highly depends on the risk level of the resources, with the value increasing in its own reliability and decreasing in the reliability of the dedicated resources. In order to eliminate the effect of different risk levels and focus solely on the impact of investment costs and the type of risk on the optimal portfolio, in the (S+D)U setting we assume that $\theta_1 = \theta_2 = \theta_f = \theta$ (i.e., each resource faces the same disruption risk).

## 5.1 The demand risk effect

We first characterize the optimal portfolio in the flexible DU system (for $\alpha \geq 0$).

**Proposition 9.** The optimal resource portfolio in the flexible DU system with $\alpha \geq 0$ can be characterized as in Figure 5.1 and Table 5.1, where $\varphi \equiv \frac{s_2(b-\alpha s_2)}{s_1(b-\alpha s_1)} + \frac{\alpha s_2}{b-\alpha(s_1+s_2)}$. (See Appendix

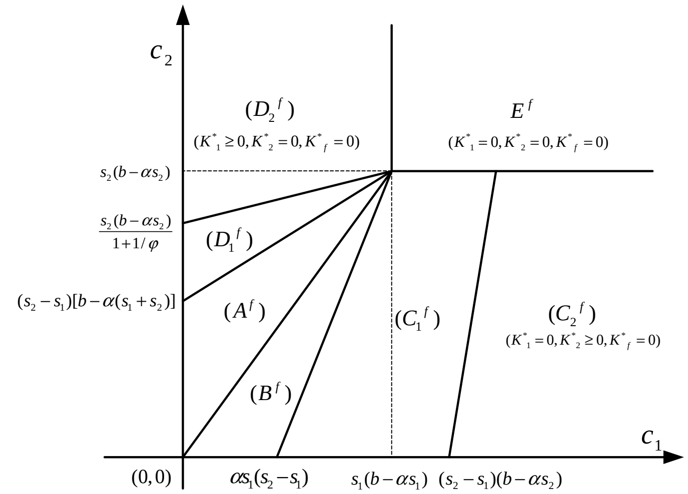



Figure 5.1: Structure of the Optimal Portfolio in the Flexible DU System ($\alpha \geq 0$).

D.1 for the optimality conditions.)

**Proof:** See Appendix D.1. □

Table 5.1: The Optimal Resource Portfolio for the Flexible DU System ($\alpha \geq 0$).

| Region | Is $K_f > 0$ Possible? | Condition | Optimal Solution |
|---|---|---|---|
| $A^f$ | **Yes** | $c_f < \underline{c}_f^3$ | $K_1^* = 0, K_2^* > 0, K_f^* > 0$ or $K_1^* > 0, K_2^* > 0, K_f^* > 0$ |
| | | $c_f \geq \underline{c}_f^3$ | $K_1^* \geq 0, K_2^* \geq 0, K_f^* = 0$ |
| $B^f$ | **Yes** | $c_f < \underline{c}_f^1$ | $K_1^* = 0, K_2^* > 0, K_f^* > 0$ or $K_1^* > 0, K_2^* > 0, K_f^* > 0$ |
| | | $c_f \geq \underline{c}_f^1$ | $K_1^* \geq 0, K_2^* \geq 0, K_f^* = 0$ |
| $C_1^f$ | **Yes** | $c_f < \underline{c}_f^2$ | $K_1^* = 0, K_2^* > 0, K_f^* > 0$ or $K_1^* > 0, K_2^* > 0, K_f^* > 0$ |
| | | $c_f \geq \underline{c}_f^2$ | $K_1^* = 0, K_2^* \geq 0, K_f^* = 0$ |
| $D_1^f$ | **Yes** | $c_f < \underline{c}_f^4$ | $K_1^* = 0, K_2^* > 0, K_f^* > 0$ or $K_1^* > 0, K_2^* > 0, K_f^* > 0$ |
| | | $c_f \geq \underline{c}_f^4$ | $K_1^* \geq 0, K_2^* = 0, K_f^* = 0$ |
| $C_2^f$ | No | NA | $K_1^* = 0, K_2^* \geq 0, K_f^* = 0$ |
| $D_2^f$ | No | NA | $K_1^* \geq 0, K_2^* = 0, K_f^* = 0$ |
| $E^f$ | No | NA | $K_1^* = 0, K_2^* = 0, K_f^* = 0$ |

It is never optimal to acquire the flexible resource in cost regions $C_2^f, D_2^f$, and $E^f$, whereas in regions $A^f, B^f, C_1^f$, and $D_1^f$, flexible resource is valuable only if its investment cost is below a threshold, which is a complex function of demand distribution parameters, dedicated resource investment costs, and attribute parameters. Thus, not surprisingly, when the flexible resource is an option, the existence of demand risk *does* alter the resource mix in the optimal portfolio, as opposed to the dedicated-only system (see Proposition 7). In cost regions $C_2^f, D_2^f$, and $E^f$, high investment costs prohibit the firm from using resource flexibility and product differentiation; even when the firm invests, it does so to produce a single product only (thus limiting its demand management capability). On the other hand, in cost regions $A^f, B^f, C_1^f$, and $D_1^f$, the firm always adopts product differentiation, which provides it with a more effective demand management capability. If, in addition, the flexible resource is cheap enough (i.e., $c_f < \underline{c}_f$) in these regions, then the firm enhances its demand management capability by acquiring the flexible resource, which enables it to alter the production quantities of the products in response to how uncertainty is revealed so as to pursue the maximum profit, that is, in these regions, the priority with which the firm allocates the flexible resource to the products changes with demand realization, making the flexible resource a valuable option (e.g., in region $B^f$, the firm acquires only a small capacity (if any) of dedicated resource 1; then, it allocates the flexible resource to product 1 when $n$ is small (due to small $K_1$), and to the more desirable product 2 when $n$ is large. The flexible resource can also be allocated to both products at the same time for certain demand realizations).

Equally important is the result that the flexible resource never completely replaces the dedicated resources. When $K_f^* > 0$, the optimal portfolio must be one of the following forms: (i) $(K_1^* = 0, K_2^* > 0, K_f^* > 0)$ or (ii) $(K_1^* > 0, K_2^* > 0, K_f^* > 0)$ (see Table 5.1). Thus, when the firm invests in the flexible resource, it always adopts resource diversification, even in the absence of supply risk. In this setting with only demand risk, the value of resource diversification comes from vertical differentiation. Product 2 is a more valuable product and the firm is always willing to offer product 2, however its production quantity depends on market conditions. Therefore, if the flexible resource is acquired, then it always accompanies dedicated resource 2. This last result extends Theorem 4.2 in Bish and Wang [4] and Proposition 2 in Van Mieghem [24] to the case where products are vertically differentiated and the demand is driven by a consumer choice model.

For the special case with $\alpha = 0$ (i.e., zero production costs), Proposition 9 leads to an interesting result.

**Corollary 2.** For $\alpha = 0$, it is never optimal for the firm to invest in the flexible resource in the flexible DU setting, i.e., $K_f^* = 0$.

Thus, when $\alpha = 0$, even when the firm has the option to acquire the flexible resource, it is never optimal to do so. This is because when production costs are negligible (or the same), demand management through product differentiation and responsive pricing is powerful enough to hedge against the demand risk, and more importantly, is costless (i.e., the firm does not lose, in terms of production costs, if it needs consumers to switch from product 1 to product 2. This is no longer the case for $\alpha > 0$, which implies a higher production cost for product 2). On the other hand, the flexible resource comes at an increased cost (over dedicated resources). Hence, any portfolio with the flexible resource is dominated by dedicated resources only, resulting in the firm's solely relying on demand management. The result that in the demand uncertainty case a necessary condition for the flexible resource is differential production costs for the products comes from the presence of consumer-driven substitution in our model. Both in Van Mieghem's model [24] (two *independent* products with exogenous, price-independent demands) and Bish and Wang's model [4] (two *independent* products with price-dependent demands and responsive pricing) flexible resource still remains valuable to hedge against the demand uncertainty even when production costs of the products are negligible or the same. This production cost effect exists because of consumer-driven substitution.

**Corollary 3.** Consider the DU setting. For $\alpha = 0$, the firm's product offering does not change with its technology choice, whereas for $\alpha > 0$ it does.

Thus, when $\alpha = 0$, neither the firm's resource portfolio nor product offering changes with its technology choice. However this result no longer holds in the case of differential production costs. In particular, for $\alpha > 0$, in regions $A^f$ and $B^f$, it invests in the capability to produce both products in both dedicated-only and flexible systems (see Propositions 5 and 9). However, in region $C_1^f$ ($D_1^f$), the firm invests only in the capability to produce product 2(1) in the dedicated-only system, but in the capability to produce both products in the flexible system. This, again, follows due to investment cost effect, and highlights another benefit of the flexible resource: a more effective demand management through the ability to offer differentiated products.

Finally, comparing the structure of the portfolio in this setting (with consumer-driven substitution) with that without consumer-driven substitution (Figures 5.1 and 4.1(a)), we observe that it is no longer true that consumer-driven substitution reduces the need for product differentiation (as was the case for the dedicated-only DU and SU settings, see Chapter 4.1). In particular, this occurs when there is a production cost differential ($\alpha > 0$), and follows because product differentiation is now possible with an investment in the flexible resource, which not only provides a more powerful demand management (as was the case for the dedicated-only system), but brings the other benefits of flexible resource (firm-driven substitution), making it more desirable.

## 5.2 The supply risk effect

In order to understand the effect of the additional supply risk, we next characterize the optimal portfolio in the (S+D)U setting (for the case of $\alpha = 0$).

**Proposition 10.** For $\alpha = 0$, the optimal resource portfolio in the flexible (S+D)U system is unique and can be characterized as in Figure 5.2 and Table 5.2.

**Proof:** See Appendix D.2. □

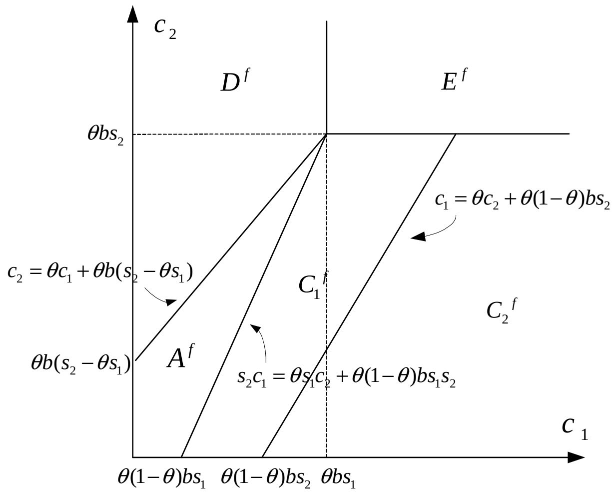



Figure 5.2: Structure of the Optimal Portfolio for the Flexible (S+D)U System ($\alpha = 0$).

Table 5.2: The Optimal Resource Portfolio in the Flexible (S+D)U System ($\alpha = 0$).

| Region | Is $K_f > 0$ Possible? | Condition | Optimal Solution |
|---|---|---|---|
| $A^f$ | **Yes** | $c_f < \underline{c}_f^1$ | $K_1^* = 0, K_2^* > 0, K_f^* > 0$ or $K_1^* > 0, K_2^* > 0, K_f^* > 0$ |
| | | $c_f \geq \underline{c}_f^1$ | $K_1^* \geq 0, K_2^* \geq 0, K_f^* = 0$ |
| $C_1^f$ | **Yes** | $c_f < \underline{c}_f^2$ | $K_1^* = 0, K_2^* > 0, K_f^* > 0$ [1] |
| | | $c_f \geq \underline{c}_f^2$ | $K_1^* = 0, K_2^* \geq 0, K_f^* = 0$ |
| $C_2^f$ | No | NA | $K_1^* = 0, K_2^* \geq 0, K_f^* = 0$ |
| $D^f$ | No | NA | $K_1^* \geq 0, K_2^* = 0, K_f^* = 0$ |
| $E^f$ | No | NA | $K_1^* = 0, K_2^* = 0, K_f^* = 0$ |

Note that when the firm acquires the flexible resource, the structure of its optimal portfolio will be of the form ($K_1^* = 0, K_2^* > 0, K_f^* > 0$) or ($K_1^* > 0, K_2^* > 0, K_f^* > 0$), similar to the DU setting but with $\alpha > 0$, see Proposition 5. However, for $\alpha = 0$, while the flexible resource has no value in the DU setting (see Corollary 2), it becomes valuable with the additional supply risk in the (S+D)U setting. Specifically, while in region $A^f$ the firm values both the dual sourcing and product differentiation capabilities of the flexible resource, in region $C_1^f$ the firm does *not* utilize its product differentiation capability, and always chooses to allocate it to product 2 in stage 2, independent of how uncertainty is resolved (see Table 5.2). However, the product differentiation capability of the flexible resource, which also exists in the DU setting, is not, by itself sufficient to make it valuable; it is the combination of this with its dual sourcing capability in the (S+D)U setting that now includes it in the optimal portfolio.

**Corollary 4.** In the SU setting with $\alpha = 0$, the firm's product offering does not change with its technology choice.

Observe that in terms of the product differentiation capability, the only difference between dedicated-only and flexible systems in the (S+D)U setting is region $C_1^f$. Although in this region the firm acquires the flexible resource, it does not utilize its product differentiation capability, as stated above. Thus, how the technology choice affects product offering is similar for demand and supply risks, see Corollary 3. This again follows due to the presence of consumer-driven substitution, which enables a powerful (and costless) demand management under both demand and supply risks.

We now study the effect of the additional supply risk on the firm's resource portfolio. In the following, $RF$-only $PD$-only, and $(PD+RF)$ respectively denote the resource flexibility strategy only, the product differentiation strategy only (through dedicated resources), and the integrated product differentiation and resource flexibility strategy.

**Corollary 5.** Consider the flexible system. When all resources have the same risk level ($\theta_1 = \theta_2 = \theta_f = \theta, 0 < \theta < 1$) and $\alpha = 0$, the supply risk (in the (S+D)U versus DU settings) will force the firm to change its product offering as follows (see Figure 5.3):

(I) : **No Product Differentiation → Product Differentiation with Resource Flexibility** (**NPD → (PD+RF)**)

(a) $\mathbf{K_2} \rightarrow (\mathbf{K_1}, \mathbf{K_2}, \mathbf{K_f})$ **or** $(\mathbf{K_2}, \mathbf{K_f})$ if $c_f \leq \underline{c}_f^1$, $c_2 \leq \frac{s_2}{s_1} c_1$, and $s_2 c_1 \leq \theta s_1 c_2 + \theta(1-\theta) b s_1 s_2$ (see region $(PD+RF)1$ in Figures 5.3(a) and 5.3(b));

(b) $\mathbf{K_1} \rightarrow (\mathbf{K_1}, \mathbf{K_2}, \mathbf{K_f})$ if $c_f \leq \underline{c}_f^1$, $s_2 < (1+\theta)s_1$, $c_2 \leq \theta c_1 + \theta b(s_2 - \theta s_1)$, and $c_2 \geq c_1 + b(s_2 - s_1)$ (see region $(PD+RF)2$ in Figure 5.3(b)).

(II) : **No Product Differentiation → Product Differentiation Only** (**NPD → PD-only**)

(a) $\mathbf{K_2} \rightarrow (\mathbf{K_1}, \mathbf{K_2})$ if $c_f > \underline{c}_f^1$, $c_2 \leq \frac{s_2}{s_1} c_1$, and $s_2 c_1 \leq \theta s_1 c_2 + \theta(1-\theta) b s_1 s_2$ (see region $(PD+RF)1$ in Figures 5.3(a) and 5.3(b));

(b) $\mathbf{K_1} \rightarrow (\mathbf{K_1}, \mathbf{K_2})$ if $c_f > \underline{c}_f^1$, $s_2 < (1+\theta)s_1$, $c_2 \leq \theta c_1 + \theta b(s_2 - \theta s_1)$, and $c_2 \geq c_1 + b(s_2 - s_1)$ (see region $(PD+RF)2$ in Figure 5.3(b)).

(III) **No Product Differentiation → Resource Flexibility Only** (**NPD → RF-only**)

(a) $\mathbf{K_2} \rightarrow (\mathbf{K_2}, \mathbf{K_f})$ if $c_f \leq \underline{c}_f^2$, $c_2 \leq \theta b s_2$, $s_2 c_1 \geq \theta s_1 c_2 + \theta(1-\theta) b s_1 s_2$, and $c_1 \leq \theta c_2 + \theta(1-\theta) b s_2$ (see region $RF$ in Figures 5.3(a) and 5.3(b)).

[1] While the flexible resource offers the firm product differentiation capability, the firm does not utilize this in region $C_1^f$, and always uses the flexible resource to produce product 2 in stage 2, no matter how uncertainty is resolved.

(IV) : **Product Differentiation → No Product Differentiation (PD → NPD)**

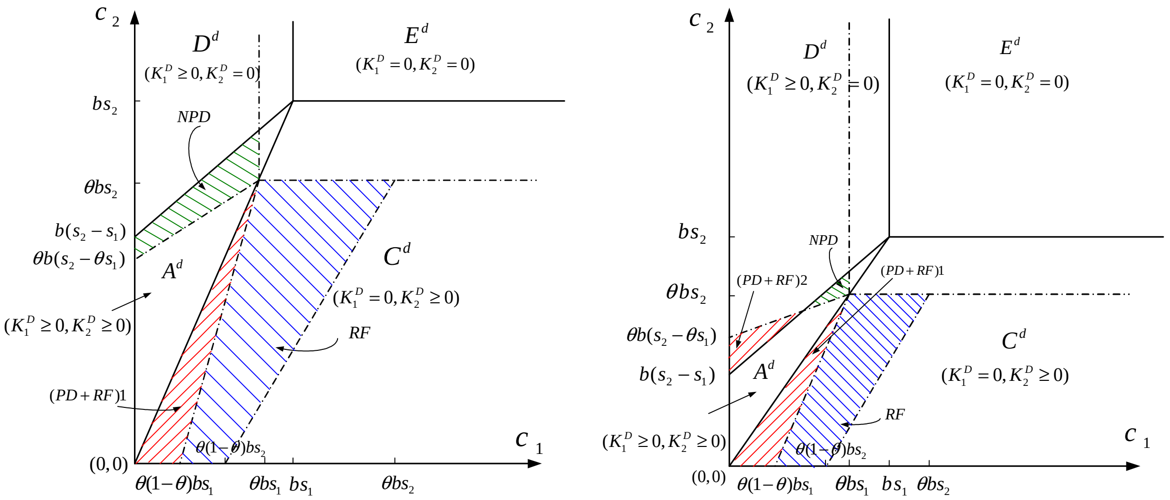
(a) $(\mathbf{K_1}, \mathbf{K_2}) \rightarrow \mathbf{K_1}$ if $c_2 \leq c_1 + b(s_2 - s_1)$, $c_2 \geq \theta c_1 + \theta b(s_2 - \theta s_1)$, and $c_1 \leq \theta b s_1$ (see regions $NPD$ in Figures 5.3(a) and 5.3(b)).

(V) : **Investment → No Investment**

(a) $(\mathbf{K_1}, \mathbf{K_2}) \rightarrow$ **No Investment** if $c_2 \leq c_1 + b(s_2 - s_1)$, $c_2 \geq \frac{s_2}{s_1} c_1$, and $c_1 \geq \theta b s_1$.

(b) $\mathbf{K_2} \rightarrow$ **No Investment** if $c_2 \leq \frac{s_2}{s_1} c_1$ and $b s_2 \geq c_2 \geq \theta b s_2$.

(c) $\mathbf{K_1} \rightarrow$ **No Investment** if $c_2 \geq c_1 + b(s_2 - s_1)$ and $b s_1 \geq c_1 \geq \theta b s_1$.




(a) High Product Differentiation ($s_2 \geq (1+\theta)s_1$). (b) Low Product Differentiation ($s_2 < (1+\theta)s_1$).

Figure 5.3: Optimal Resource Mix for the Flexible DU (solid line) and (S+D)U (dashed line) Systems when $\theta_1 = \theta_2 = \theta_f = \theta$ ($\alpha = 0$).

As discussed above, when $\alpha = 0$, the firm's product offering does not change with its technology choice under either the demand or supply risk. Consequently, Corollary 5 (in the flexible system) gives similar results to Corollary 1 (in the dedicated-only system), with the difference of region $RF$. As explained above, in region $RF$ the firm may acquire the flexible resource in the (S+D)U setting, accompanied with dedicated resource 2, but uses both resources to offer product 2 only, not changing its product offering over the DU setting.

**Proposition 11.** Consider the flexible system studied in Corollary 5. The $K_1 \rightarrow (K_1, K_2, K_f)$ region (see $(PD + RF)2$ in Figure 5.3) expands in $\theta$ for $\frac{s_2 - s_1}{s_1} < \theta < \frac{s_1 + s_2}{3s_1}$, and shrinks in $\theta$ for $\theta > \frac{s_1 + s_2}{3s_1}$.

Thus, even when the resource risk level increases, the firm may change from product differentiation and resource flexibility to investing in a single resource. Other than resource reliability, the firm's product differentiation and resource flexibility decision also depends on the cost structure and product and market characteristics.

# Chapter 6

# Discussion of Major Assumptions

In this chapter, we discuss the possible impact of the two assumptions that our analysis relies on: uniform distribution of consumer type T and Bernoulli distribution of supply risk. It is quite difficult to relax these assumptions analytically. Therefore, in what follows, we relax them for some special cases so as to gain insights.

## 6.1 Uniform distribution of consumer type $T$

Although the assumption that $T$, consumer type, is uniformly distributed has been commonly used in the marketing and operations management literature (e.g., Choudhary [7], Moorthy [15] and [16], Rhee [19]), it may not be realistic for some items. For instance, the willingness to pay for luxury products may be low for the majority of the population, and high for only a small fraction of the population. In order to understand the effect of this assumption on our findings, we now consider the DU setting with $\alpha = 0$.

**Proposition 12.** Consider that $T$ follows an arbitrary continuous distribution with support in $[0, b]$. Then, for all distribution functions $f_T(t)$ such that $f_T(t)$ is:

(i) increasing in $t, \forall t \in [0, b]$, or
(ii) concave decreasing in $t, \forall t \in [0, b]$, or
(iii) strictly unimodal such that it is first increasing and then concave decreasing in $t, \forall t \in [0, b]$,

$K_f^* = 0$ in an optimal solution to the flexible DU system with $\alpha = 0$.

**Proof:** See Appendix E.1. □
Thus, Proposition 12 states that Corollary 2 (which assumes that $T$ is uniformly distributed) in fact extends to a wide range of distributions of $T$. Furthermore, the following result provides several examples of distribution functions that do not satisfy the conditions in Proposition 12, but nevertheless yield the same result (i.e., $K_f^* = 0$). In fact, we have failed

to find a counter-example in which it is optimal for the firm to invest in the flexible resource in the flexible DU system with $\alpha = 0$. We, therefore, speculate that this result holds for a wider range of distributions than indicated by Propositions 12 and 13.

**Proposition 13.** Consider the flexible DU system with $\alpha = 0$. $K_f^* = 0$ in an optimal solution for the following family of distributions:

(i) $f_T(t) = a(\gamma + \beta t)^{-p}$, where $a > 0, \gamma > 0, \beta > 0, p \geq 2$.
(ii) $T = X|X < b$, where $X$ is Exponential with parameter $\lambda$, that is, $T$ follows a truncated exponential distribution.
(iii) $T = X|X < b$, where $X$ is Normal with parameters $\mu$ and $\sigma^2$, that is, $T$ follows a truncated normal distribution.

**Proof:** See Appendix E.2. □
It is quite difficult to relax the uniform $T$ assumption for the DU setting with $\alpha > 0$ as well as for the SU and (S+D)U settings. However, we speculate that most of our findings will continue to hold for other continuous distributions of $T$.

## 6.2 Bernoulli distribution assumption of supply uncertainty

As stated in Chapter 3, the forms of supply risk commonly used in the operations management literature include supply disruptions and yield uncertainty. In our analysis, we consider supply disruptions and assume a Bernoulli distribution for resource availability. In this section, we study the effect of modeling the supply risk as yield uncertainty.

For ease of analysis, we assume that the firm can invest in only the flexible resource, and consider that $Y_f$ follows an arbitrary continuous distribution with support in [0,1]. Thus, the flexible resource does not need to be completely "up" or "down;" it might be partially available.

**Proposition 14.** Consider the case when the firm can invest in only the flexible resource.

(i) Consider the SU setting of Problem P.
  (a) If $Y_f$ follows an arbitrary continuous distribution in [0,1], then the firm will invest in the flexible resource if and only if $c_f < s_2(b - \alpha s_2)E[Y_f]$.
  (b) If $Y_f$ follows a Bernoulli distribution where $Y_f = 1$ with probability $\theta$, and $Y_f = 0$ otherwise, then the firm will invest in the flexible resource if and only if $c_f < s_2(b - \alpha s_2)E[Y_f]$.
(ii) Consider the DU setting of Problem **P**. The firm will invest in the flexible resource if and only if $c_f < s_2(b - \alpha s_2)$.
(iii) Consider the deterministic setting of Problem **P**. The firm will invest in the flexible resource if and only if $c_f < s_2(b - \alpha s_2)$.

**Proof:** See Appendix F.1. □

Proposition 14 indicates that while the demand risk does *not* alter the investment decision (i.e., whether or not to invest), this decision depends on parameters of supply uncertainty (i.e., its first moment) under both supply disruptions (see (i)-a) and yield uncertainty (see (ii)-b), similar to the results we have obtained in Proposition 6. Moreover, the cost threshold is the same under both types of supply uncertainty. Furthermore, we can show that in the dedicated-only system, when $Y_1$ and $Y_2$ are continuous random variables in $[0, 1]$, investment thresholds continue to be functions of parameters of yield uncertainty (as was the case for supply disruptions, see Proposition 6). Therefore, the fact that supply uncertainty has a higher impact on the optimal portfolio than the demand uncertainty does not only hold for supply disruptions, but also for yield uncertainty. Although it is quite difficult to extend our analysis to the flexible system with all resources possible, we speculate that similar results will hold in the general system as well.

# Chapter 7

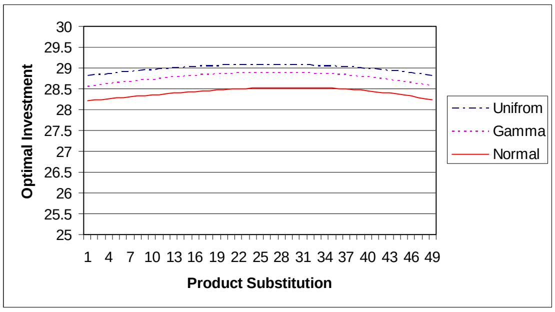
# Conclusions and Future Research Directions

We study the optimal resource portfolio for a price-setter firm under a consumer choice model of vertical product differentiation, and with supply and demand risks. Our analysis shows the effect of consumer-driven substitution, demand and supply risks, the firm's technology choice, as well as other environmental parameters (such as production and investment costs) on the optimal resource portfolio and the hedging mechanism (i.e., product differentiation versus resource flexibility).

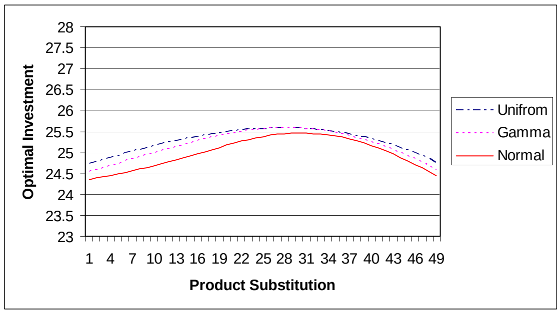
One extension of our study would be to perform a comparative statics analysis on product substitution and demand uncertainty. It is interesting to see how the firm's optimal investment level changes in terms of product substitution and demand uncertainty. For illustration purposes, we next give two numerical examples by studying a $K_f$-only system where the firm can only invest in a flexible resource, with the capability to produce both products, under the demand risk only. We model the degree of product substitution by $s_2 - s_1$, the difference between the attribute values of the two products. The magnitude of demand uncertainty is measured by the variance of the market size, $N$. Let $K_f^*$ be the optimal resource investment in the $K_f$-only system.

**Example 1.** Let $b = 10, s_2 = 5, c_f = 5$. Figure 7.1 shows how the firm's optimal investment level, $K_f^*$, changes when $s_1$ increases from 0.1 to 4.9, assuming $N$ respectively follows continuous Uniform (in $[50, 100]$), Gamma (with scale parameter 25 and shape parameter 3), and Normal (with mean 75 and variance 100) distributions, and $\alpha = 0.3, 0.5, 0.7,$ and 0.9.

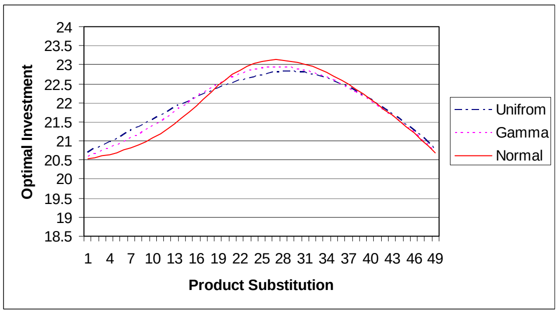
From Figure 7.1, we can see that when the degree of product substitution increases, $K_f^*$ increases first, then it decreases. This is because when $s_1$ is small, the firm will benefit from an expanding market share when $s_1$ increases, and therefore, $K_f^*$ increases. When $s_1$ is large (i.e., $s_1$ is close to $s_2$, hence product 1 and 2 are more substitutable), the production cost and cannibalization effect between these two products dominate the market share effect.



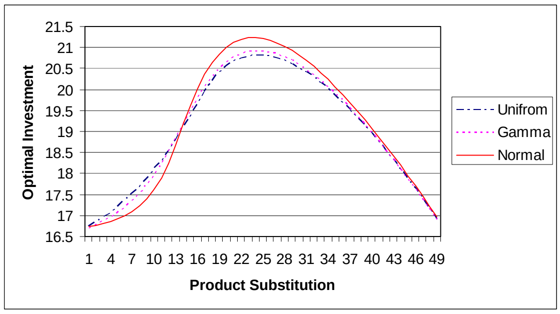

(a) $\alpha = 0.3$.




(b) $\alpha = 0.5$.




(c) $\alpha = 0.7$.




(d) $\alpha = 0.9$.

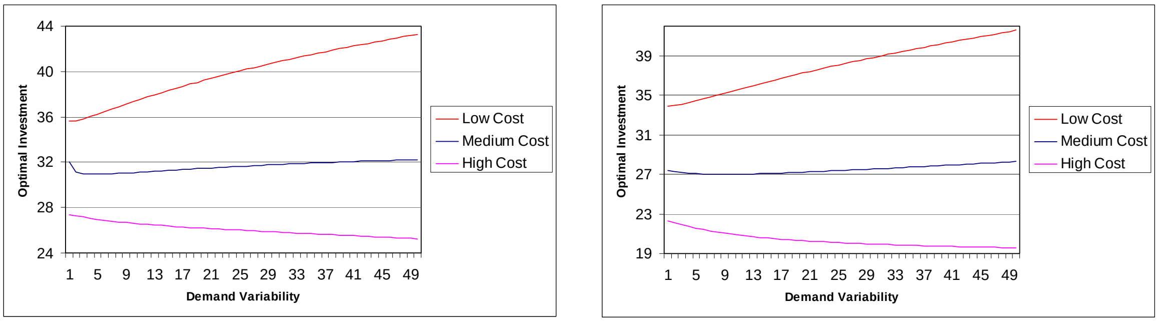
Figure 7.1: Optimal Resource Investment for the $K_f$-only System in the DU Setting under Various Product Substitution Levels.

Therefore, $K_f^*$ decreases.

**Example 2.** Let $b = 10, s_1 = 3, s_2 = 5, \alpha = 0.5$. Figure 7.2 shows how the firm's optimal investment level, $K_f^*$, changes under different investment cost levels as the variance of random variable $N$ increases from 100 to 5000, while maintaining a mean of 100. Gamma and Normal distributions are considered for $N$, and (Low Cost, Medium Cost, High Cost) are assumed to be (4, 7, 10) and (5, 10, 15), respectively, for these two distributions.

From Figure 7.2, we note that when demand variability increases, the optimal resource investment changes differently under different resource investment cost levels: $K_f^*$ increases when $c_f$ is low, decreases when $c_f$ is high, and decreases first and then increases when $c_f$ is in between. This is because if $c_f$ is high, the cost of unused resource (if the demand realization is small) outweighs the gain from a large demand realization. On the other hand, the firm can gain more from a large demand realization if $c_f$ is low. Similar results can be obtained by studying how $(K_1^D, K_2^D)$ (for the dedicated-only system) and $(K_1^*, K_2^*, K_f^*)$ (for the flexible system) are affected by the demand variability in the (S+D)U setting (see Example 3).

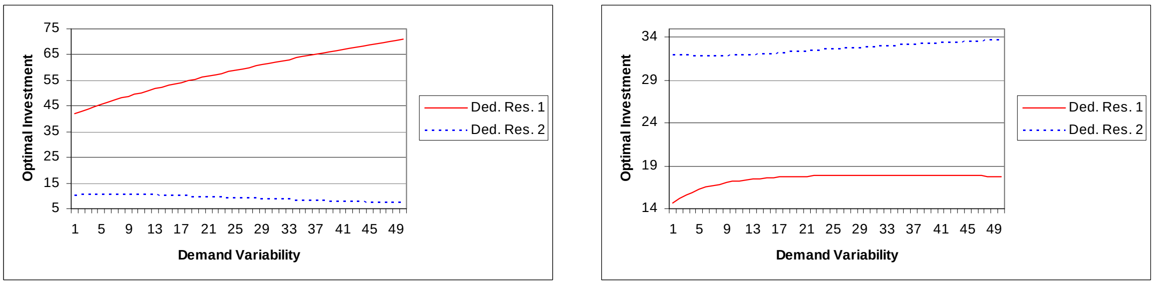
**Example 3.** Let $b = 10, s_1 = 4, s_2 = 5, \alpha = 0, \theta = 0.8$. Figures 7.3 and 7.4 respectively show how $(K_1^D, K_2^D)$ and $(K_1^*, K_2^*, K_f^*)$ change under different investment cost levels as the variance of Gamma random variable $N$ increases from 100 to 5000, while maintaining a




(a) Gamma Distribution. (b) Normal Distribution.

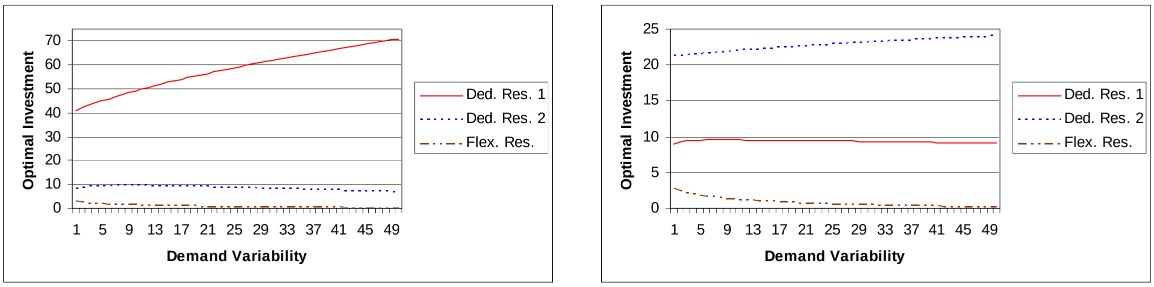
Figure 7.2: Optimal Resource Investment for the $K_f$-only System in the DU Setting under Various Demand Variability Levels.

mean 100. We assume $c_1 = 1.5, c_2 = 12, c_f = 12.75$ in the high $c_2/c_1$ ratio case, and $c_1 = 6.4, c_2 = 7, c_f = 7.5$ in the low $c_2/c_1$ ratio case.




(a) High $c_2/c_1$ Ratio. (b) Low $c_2/c_1$ Ratio.

Figure 7.3: Optimal Resource Investment for the (S+D)U Dedicated-only System under Various Demand Variability Levels.




(a) High $c_2/c_1$ Ratio. (b) Low $c_2/c_1$ Ratio.

Figure 7.4: Optimal Resource Investment for the (S+D)U Flexible System under Various Demand Variability Levels.

Similar analyses can be performed on different models that we discussed in the dissertation. Our numerical results provide different insights than those found in the extant literature that uses aggregate linear demand models. Thus, the next step would be to compare/contrast our results with the literature and explain why differences arise.

Furthermore, several other extensions of our models deserve further study. One obvious, albeit difficult, extension will be to relax some of our modeling assumptions, such as the uniform distribution of consumer type or the Bernoulli distribution of resource disruptions, for the more complex models that we study here, and understand their effect on our results. Although we are able to relax these assumptions in Chapter 6 for some of our simple models, this is still a valuable direction. In our current model, the firm can acquire at most one type of each resource (dedicated or flexible). As a result, if the firm wants a diverse resource portfolio without resource flexibility, then it needs an investment in each of the two dedicated resources, which also gives it the ability to produce differentiated products. Thus, another important extension is to separate the product differentiation and resource diversification strategies by allowing the firm to acquire multiple resources of the same type. This would, in turn, require more complex, perhaps location-based, investment cost structure. In a similar direction, the assumption that $Y_1, Y_2, Y_f$ are independent needs to be relaxed as disruptions of resources in the same location may be correlated. Our model is developed for vertically differentiated products. Another important extension would be to consider horizontally differentiated products, or a mix of vertically and horizontally differentiated line of products that the firm can select from. Finally, other interesting directions include introducing competition in our framework, and incorporating measures of risk-averse or risk-seeking behavior into the objective function (rather than the risk-neutral form of maximizing the expected profit that we utilize here).

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# Appendix A

Let $H(A)$ denote the hessian matrix of $A$. Let $\Pi_i \equiv \Pi^*(\overrightarrow{K}, n|n \in \Omega_i), i = 1, ..., 6$, that is, $\Pi_i$ is the conditional profit in stage 2 given that $n \in \Omega_i$, where $\Omega_i$ is defined in Chapter 3 of the main dissertation.

## 1. Proof of Proposition 1

Define $M_i$, the set of consumer types who purchase product $i, i = 1, 2$, as

$$M_i \equiv \{t \in [0, b] : ts_i - p_i \geq \max\{ts_{3-i} - p_{3-i}, 0\}\},$$

and the set of consumer types who do not purchase any product as

$$M_0 \equiv \{t \in [0, b] : ts_i - p_i \leq 0, \text{ for } i = 1, 2\}.$$

Observe that when $s_2 > s_1$ and $p_2 \geq p_1$, we have

$$\frac{p_2}{s_2} \geq \frac{p_1}{s_1} \Leftrightarrow \frac{p_2 - p_1}{s_2 - s_1} \geq \frac{p_2}{s_2}.$$

Hence, to determine an optimal solution to Problem $\mathbf{P_2}$, it is sufficient to consider the following four cases of relationship between $p_1$ and $p_2$:

| | | |
|---|---|---|
| Case 1. | $0 \leq \frac{p_1}{s_1} \leq \frac{p_2-p_1}{s_2-s_1} \leq b$ | $\Rightarrow M_0 = [0, \frac{p_1}{s_1}]$, $M_1 = [\frac{p_1}{s_1}, \frac{p_2-p_1}{s_2-s_1}]$, and $M_2 = [\frac{p_2-p_1}{s_2-s_1}, b]$. |
| Case 2. | $0 \leq \frac{p_1}{s_1} \leq b \leq \frac{p_2-p_1}{s_2-s_1}$ | $\Rightarrow M_0 = [0, \frac{p_1}{s_1}]$, $M_1 = [\frac{p_1}{s_1}, b]$, and $M_2 = \emptyset$. |
| Case 3. | $\frac{p_2-p_1}{s_2-s_1} \leq \frac{p_2}{s_2} \leq b$ | $\Rightarrow M_0 = [0, \frac{p_2}{s_2}]$, $M_1 = \emptyset$, and $M_2 = [\frac{p_2}{s_2}, b]$. |
| Case 4. | $\frac{p_1}{s_1} \geq b$ and $\frac{p_2}{s_2} \geq b$ | $\Rightarrow M_0 = \emptyset$, $M_1 = \emptyset$, and $M_2 = \emptyset$. |

Then, observing that each of Cases 2, 3, and 4 arises as a feasible solution to Case 1, which leads to

$$d_1(p_1^*, p_2^*) = n\ Pr(\frac{p_1^*}{s_1} \leq T \leq \frac{p_2^* - p_1^*}{s_2 - s_1}) = \frac{n}{b}(\frac{p_2^* - p_1^*}{s_2 - s_1} - \frac{p_1^*}{s_1}),$$

$$d_2(p_1^*, p_2^*) = n\ Pr(T \geq \frac{p_2^* - p_1^*}{s_2 - s_1}) = \frac{n}{b}(b - \frac{(p_2^* - p_1^*)}{s_2 - s_1}).$$

completes the proof. □

**2. Proof of Proposition** 2

Utilizing Proposition 1, we can write $d_1(\overrightarrow{p}) = \frac{n}{b}(\frac{p_2-p_1}{s_2-s_1} - \frac{p_1}{s_1})$ and $d_2(\overrightarrow{p}) = \frac{n}{b}(b - \frac{(p_2-p_1)}{s_2-s_1})$. Suppose that in an optimal solution to Problem $\mathbf{P_2}$, we have $d_1(\overrightarrow{p}^*) > q_1^* \geq 0$ and $d_2(\overrightarrow{p}^*) = q_2^* \geq 0$ (the cases where $d_2(\overrightarrow{p}^*) > q_2^*$ and $d_1(\overrightarrow{p}^*) = q_1^*$, and $d_i(\overrightarrow{p}^*) > q_i^*, i = 1, 2$, can be proven similarly). Then $\Pi^*(\overrightarrow{K}) = \sum_{i=1}^{2} q_i^*(p_i^* - \alpha s_i^2)$. In what follows, we construct an alternative feasible solution, $(\overrightarrow{p}', \overrightarrow{q}')$ with profit $\Pi'(\overrightarrow{K})$, such that $q_i' = d_i(\overrightarrow{p}'), i = 1, 2$, and $\Pi'(\overrightarrow{K}) > \Pi^*(\overrightarrow{K})$. Let $p_1' = p_1^* + \Delta_1$ and $p_2' = p_2^* + \Delta_2$, such that

$$q_1' = q_1^* = \frac{n}{b}(\frac{p_2^* + \Delta_2 - p_1^* - \Delta_1}{s_2 - s_1} - \frac{(p_1^* + \Delta_1)}{s_1}) = d_1(\overrightarrow{p}') < d_1^* = \frac{n}{b}(\frac{p_2^* - p_1^*}{s_2 - s_1} - \frac{p_1^*}{s_1}) \quad \text{(A.1)}$$

$$q_2' = q_2^* = \frac{n}{b}(b - \frac{(p_2^* + \Delta_2 - p_1^* - \Delta_1)}{s_2 - s_1}) = \frac{n}{b}(b - \frac{(p_2^* - p_1^*)}{s_2 - s_1}) = d_2(\overrightarrow{p}') = d_2^*. \quad \text{(A.2)}$$

From (A.1) and (A.2),

$$\Delta_1 = \Delta_2 = \frac{s_1(p_2^* - p_1^*)}{s_2 - s_1} - \frac{b}{n}q_1^* s_1 - p_1^*, \text{ and}$$
$$\frac{\Delta_1}{s_1} = \frac{b}{n}(d_1^* - q_1^*) > 0 \Rightarrow \Delta_1 = \Delta_2 > 0.$$

Observe that $(\overrightarrow{p}', \overrightarrow{q}')$ satisfies constraints (2.2b) - (2.2g), because $q_i' = q_i^* \leq y_i K_i + y_f K_f, i = 1, 2$, $q_1' + q_2' = q_1^* + q_2^* \leq y_1 K_1 + y_2 K_2 + y_f K_f$, $p_i' > p_i^*, i = 1, 2$, and $q_i' = d_i(\overrightarrow{p}') = q_i^* > 0, i = 1, 2$.

Furthermore, we can show that $(\overrightarrow{p}', \overrightarrow{q}')$ satisfies Proposition 1, that is

$$\frac{p_2'}{s_2} \geq \frac{p_1'}{s_1} \Leftrightarrow \frac{p_2^* + \Delta_2}{s_2} \geq \frac{p_1^* + \Delta_1}{s_1}$$
$$\Leftrightarrow \frac{s_1(p_2^* - p_1^*)}{s_2(s_2 - s_1)} - \frac{bs_1}{ns_2}q_1^* + \frac{p_2^* - p_1^*}{s_2} \geq \frac{p_2^* - p_1^*}{s_2 - s_1} - \frac{b}{n}q_1^*$$
$$\Leftrightarrow q_1^* \frac{b}{n}(1 - \frac{s_1}{s_2}) \geq (p_2^* - p_1^*)[\frac{1}{s_2 - s_1} - \frac{s_1}{s_2(s_2 - s_1)} - \frac{1}{s_2}] = 0$$
$$\Leftrightarrow q_1^* \geq 0, \text{ which holds since } \overrightarrow{q}^* \text{ is feasible to (2.2b) - (2.2g).}$$

In addition, we have $\frac{(p_2' - p_1')}{s_2 - s_1} = \frac{(p_2^* - p_1^*)}{s_2 - s_1} \leq b$. Therefore, $(\vec{p}', \vec{q}')$ is a feasible solution to Stage 2 Problem $\mathbf{P_2}$. In addition, observe that $\Pi'(\overrightarrow{K}) = \sum_{i=1}^{2} q_i'(p_i' - \alpha s_i^2) > \Pi^*(\overrightarrow{K}) = \sum_{i=1}^{2} q_i^*(p_i^* - \alpha s_i^2)$, since $q_i' = q_i^*$ and $p_i' > p_i^*, i = 1, 2$. Thus, there exists an optimal solution to Problem $\mathbf{P_2}$ in which demand equals the production quantity of each product. □

### 3. Proof of Proposition 3

We derive the Hessian matrix, $\mathbf{H}(\mathbf{\Pi}(\overrightarrow{p}))$, with respect to (w.r.t.) $p_1$ and $p_2$:

$$\mathbf{H}(\mathbf{\Pi}(\overrightarrow{p})) = \frac{2n}{b} \begin{bmatrix} \frac{-s_2}{(s_2-s_1)s_1} & \frac{1}{s_2-s_1} \\ \frac{1}{s_2-s_1} & \frac{-1}{s_2-s_1} \end{bmatrix}.$$

Since $s_2 > s_1$, we know $\frac{-s_2}{(s_2-s_1)s_1} < 0$, $\frac{-1}{s_2-s_1} < 0$, and

$$\frac{-s_2}{(s_2-s_1)s_1}(\frac{-1}{s_2-s_1}) - (\frac{1}{s_2-s_1})^2 = \frac{1}{(s_2-s_1)s_1} > 0.$$

Therefore, the objective function (3.1a) is strictly, jointly concave in $p_1$ and $p_2$. Since constraints (3.1b) - (3.1g) are all linear, the first-order Karush-Kuhn-Tucker (KKT) conditions are necessary and sufficient for optimality of the constrained problem. In what follows, we first consider constraints (3.1b), (3.1c), and (3.1d) only, and obtain the optimal solution using the first-order KKT conditions. Then, we show that these solutions always satisfy constraints (3.1e) - (3.1g). Hence, (3.1e) - (3.1g) are redundant.

Define $\lambda_1, \lambda_2$, and $\lambda_3$ as the KKT multipliers respectively corresponding to constraints (3.1b), (3.1c), and (3.1d). The first-order KKT conditions are given as:

$$\frac{n}{b}[\frac{2(p_2-p_1)}{s_2-s_1} - \frac{2p_1}{s_1} - \alpha s_2] + s_2\lambda_1 - \lambda_2 + \lambda_3 = 0 \quad \text{(A.3a)}$$

$$\frac{n}{b}[b - \frac{2(p_2-p_1)}{s_2-s_1} + \alpha(s_1+s_2)] - s_1\lambda_1 + \lambda_2 = 0 \quad \text{(A.3b)}$$

$$\lambda_1[p_1 s_2 - p_2 s_1 + \frac{b}{n}(y_1K_1 + y_fK_f)s_1(s_2-s_1)] = 0 \quad \text{(A.3c)}$$

$$\lambda_2[p_2 - p_1 - b(s_2-s_1) + \frac{b}{n}(y_2K_2 + y_fK_f)(s_2-s_1)] = 0 \quad \text{(A.3d)}$$

$$\lambda_3[p_1 - bs_1 + \frac{b}{n}(y_1K_1 + y_2K_2 + y_fK_f)s_1] = 0 \quad \text{(A.3e)}$$

$$\lambda_i \geq 0, \quad i = 1,2,3. \quad \text{(A.3f)}$$

Observe that in an optimal solution, either $\lambda_i > 0$ or $\lambda_i = 0, i = 1,2,3$, leading to eight potential solutions. Consider two of these solutions, given by $(\lambda_1 > 0, \lambda_2 > 0, \lambda_3 > 0)$, and $(\lambda_1 > 0, \lambda_2 > 0, \lambda_3 = 0)$. Observe that if $y_fK_f^* > 0$, then neither of these solutions is feasible (with respect to complementary slackness (A.3c) and (A.3d)), since constraints (3.1b) and (3.1c) cannot be binding at the same time due to (3.1d). On the other hand, if $y_fK_f^* = 0$, then one of (3.1b), (3.1c), and (3.1d) becomes redundant, and these solutions reduce to the solution where $(\lambda_i > 0, \lambda_{3-i} = 0, \lambda_3 > 0$ for $i = 1$ or $2)$, respectively analyzed in Cases

4 and 6. (When $y_f K_f^* = 0$, solutions to Cases 4 and 6 are identical.) Consequently, it is sufficient to consider only the remaining six possible $\lambda$ vector combinations. We characterize the solution for each of these six $\lambda$ vectors and determine the conditions under which each solution satisfies primal feasibility (i.e., (3.1b) - (3.1g)), dual feasibility (i.e., (A.3a), (A.3b), (A.3f)), and complementary slackness (i.e., (A.3c) - (A.3e)). See Table A.1 for the summary of the results. □

Table A.1: Optimal Solution for Stage 2 of the DU Dedicated-only System.

| | $(\lambda_1, \lambda_2, \lambda_3)$ | $\overrightarrow{p}^*$ | $\overrightarrow{q}^*$ |
|---|---|---|---|
| $\Omega_1$ | $(=0,=0,=0)$ | $p_1^* = \frac{bs_1+\alpha s_1^2}{2}$<br>$p_2^* = \frac{bs_2+\alpha s_2^2}{2}$ | $q_1^* = \frac{n\alpha s_2}{2b}$<br>$q_2^* = \frac{nb-\alpha n(s_1+s_2)}{2b}$ |
| $\Omega_2$ | $(=0,=0,>0)$ | $p_1^* = bs_1 - \frac{bs_1(y_1K_1+y_2K_2+y_fK_f)}{n}$<br>$p_2^* = \frac{b(s_1+s_2)}{2} - \frac{bs_1(y_1K_1+y_2K_2+y_fK_f)}{n}$<br>$+ \frac{\alpha(s_2^2-s_1^2)}{2}$ | $q_1^* = y_1K_1 + y_2K_2 + y_fK_f$<br>$- \frac{(nb-\alpha n(s_1+s_2))}{2b}$<br>$q_2^* = \frac{nb-\alpha n(s_1+s_2)}{2b}$ |
| $\Omega_3$ | $(>0,=0,=0)$ | $p_1^* = \frac{bs_1+\alpha s_1 s_2}{2} - \frac{b(y_1K_1+y_fK_f)(s_2-s_1)s_1}{ns_2}$<br>$p_2^* = \frac{(b+\alpha s_2)s_2}{2}$ | $q_1^* = y_1K_1 + y_fK_f$<br>$q_2^* = \frac{n(b-\alpha s_2)}{2b} - \frac{(y_1K_1+y_fK_f)s_1}{s_2}$ |
| $\Omega_4$ | $(>0,=0,>0)$ | $p_1^* = bs_1 - \frac{bs_1(y_1K_1+y_2K_2+y_fK_f)}{n}$<br>$p_2^* = bs_2 - \frac{b(y_1K_1+y_fK_f)s_1}{n} - \frac{bK_2s_2}{n}$ | $q_1^* = y_1K_1 + y_fK_f$<br>$q_2^* = y_2K_2$ |
| $\Omega_5$ | $(=0,>0,=0)$ | $p_1^* = \frac{(b+\alpha s_1)s_1}{2}$<br>$p_2^* = b(s_2-s_1) - \frac{b}{n}(y_2K_2+y_fK_f)(s_2-s_1)$<br>$+ \frac{(b+\alpha s_1)s_1}{2}$ | $q_1^* = \frac{n(b-\alpha s_1)}{2b} - (y_2K_2+y_fK_f)$<br>$q_2^* = y_2K_2 + y_fK_f$ |
| $\Omega_6$ | $(=0,>0,>0)$ | $p_1^* = bs_1 - \frac{bs_1(y_1K_1+y_2K_2+y_fK_f)}{n}$<br>$p_2^* = bs_2 - \frac{b(y_2K_2+y_fK_f)s_2}{n} - \frac{by_1K_1s_1}{n}$ | $q_1^* = y_1K_1$<br>$q_2^* = y_2K_2 + y_fK_f$ |

**4. Proof of Proposition** 4

**Case 1:** $y_1K_1 \geq \frac{\alpha s_2(y_2K_2+y_fK_f)}{b-\alpha(s_1+s_2)}$: Let us first consider $\Omega_5$. For it to be a nonempty set, we need

$$\begin{aligned}
&\frac{2b(y_2K_2+y_fK_f)}{b-\alpha(s_1+s_2)} < n \leq \frac{2b(y_1K_1+y_2K_2+y_fK_f)}{b-\alpha s_1} \\
\Rightarrow\quad &\frac{y_2K_2+y_fK_f}{b-\alpha(s_1+s_2)} \leq \frac{y_1K_1+y_2K_2+y_fK_f}{b-\alpha s_1} \\
\Rightarrow\quad &by_1K_1 \geq \alpha s_1 y_1 K_1 + \alpha s_2(y_1K_1+y_2K_2+y_fK_f). \qquad \text{(A.4)}
\end{aligned}$$

When (A.4) holds, we have

$$\frac{2b(y_1K_1 + y_fK_f)}{\alpha s_2} \geq \frac{2b(y_1K_1 + y_2K_2 + y_fK_f)}{b - \alpha s_1}$$
$$\Rightarrow \quad b(y_1K_1 + y_fK_f) - \alpha s_1(y_1K_1 + y_fK_f) \geq \alpha s_2(y_1K_1 + y_2K_2 + y_fK_f)$$
$$\Rightarrow \quad b(y_1K_1 + y_fK_f) > by_1K_1 + \alpha s_1 y_fK_f \geq \alpha s_1 y_1K_1 + \alpha s_2(y_1K_1 + y_2K_2 + y_fK_f) + \alpha s_1 y_fK_f.$$

Also from (A.4), it is easy to see that $\Omega_2$ is an empty set.

For $\Omega_3$ to be nonempty, we need

$$\frac{2b(y_1K_1 + y_fK_f)}{\alpha s_2} \leq \frac{y_2K_2s_2 + (y_1K_1 + y_fK_f)s_1}{(b - \alpha s_2)s_2}$$
$$\Rightarrow b(y_1K_1 + y_fK_f) \leq \alpha s_2(y_1K_1 + y_2K_2 + y_fK_f) + \alpha s_1(y_1K_1 + y_fK_f),$$

which is not true by (A.4).

For $\Omega_4$ to be nonempty, we need

$$\frac{2b[y_2K_2s_2 + (y_1K_1 + y_fK_f)s_1]}{(b - \alpha s_2)s_2} \leq \frac{2by_2K_2}{b - \alpha(s_1 + s_2)}$$
$$\Rightarrow by_2K_2s_2 + bs_1(y_1K_1 + y_fK_f) - \alpha y_2K_2s_2(s_1 + s_2) - \alpha(s_1 + s_2)s_1(y_1K_1 + y_fK_f)$$
$$\leq y_2K_2s_2(b - \alpha s_2)$$
$$\Rightarrow b(y_1K_1 + y_fK_f) \leq \alpha s_2(y_1K_1 + y_2K_2 + y_fK_f) + \alpha s_1(y_1K_1 + y_fK_f),$$

which is not true by (A.4).

To summarize, if $y_1K_1 \geq \frac{\alpha s_2(y_2K_2 + y_fK_f)}{b - \alpha(s_1 + s_2)}$, the sample space consists of sets $\Omega_1, \Omega_5,$ and $\Omega_6$ only, as shown in Figure 3.1.

**Case 2:** If $\frac{\alpha s_2(y_2K_2 + y_fK_f)}{b - \alpha(s_1 + s_2)} \geq y_1K_1 \geq \frac{\alpha s_2 y_2K_2}{b - \alpha(s_1 + s_2)} - y_fK_f$, we know that $\Omega_3,\ \Omega_4,$ and $\Omega_5$ are empty sets, and $\Omega_2$ is nonempty with

$$\frac{2by_2K_2}{b - \alpha(s_1 + s_2)} < \frac{2b(y_1K_1 + y_2K_2 + y_fK_f)}{b - \alpha s_1}.$$

We also know

$$\frac{2b(y_1K_1 + y_2K_2 + y_fK_f)}{b - \alpha s_1} < \frac{2b(y_1K_1 + y_fK_f)}{\alpha s_2}, \text{ and}$$
$$\frac{2b(y_2K_2 + y_fK_f)}{b - \alpha(s_1 + s_2)} > \frac{2b(y_1K_1 + y_2K_2 + y_fK_f)}{b - \alpha s_1}.$$

Hence, the sample space consists of sets $\Omega_1, \Omega_2,$ and $\Omega_6$ only, as shown in Figure 3.2.

**Case 3:** If $\frac{\alpha s_2 y_2 K_2}{b-\alpha(s_1+s_2)} - y_f K_f \geq y_1 K_1$, we know that $\Omega_2,\ \Omega_3,$ and $\Omega_4$ are nonempty, and $\Omega_5$ is empty. We further know

$$\frac{2b(y_1K_1 + y_fK_f)}{\alpha s_2} \leq \frac{2b(y_1K_1 + y_2K_2 + y_fK_f)}{b - \alpha s_1} \leq \frac{2by_2K_2}{b - \alpha(s_1 + s_2)}.$$

Hence, the sample space consists of $\Omega_1, \Omega_3, \Omega_4, \Omega_2,$ and $\Omega_6$ only, as shown in Figure 3.3. □

**5. Proof of Theorem 1**

(i) First note that for any solution with $K_f^{'} > 0$ in Region I, we must have $K_1^{'} > 0$. Consider such a solution, given by $\overrightarrow{K'} = (K_1^{'} > 0, K_2^{'} \geq 0, K_f^{'} > 0)$ such that $K_1^{'} \geq \frac{\alpha s_2(K_2^{'}+K_f^{'})}{b-\alpha(s_1+s_2)}$. We can always find an alternative solution in Region I, given by $\overrightarrow{K^a} = (K_1^a = K_1^{'} > 0, K_2^a = K_2^{'} + K_f^{'} > 0, K_f^a = 0)$, with $K_1^a \geq \frac{\alpha s_2(K_2^a+K_f^a)}{b-\alpha(s_1+s_2)}$. We can show that $q_j(\overrightarrow{K}') = q_j(\overrightarrow{K}^a), j = 1, 2,$ in each set, and hence, $\Pi_i(\overrightarrow{K}') = \Pi_i(\overrightarrow{K}^a)$, $i = 1, 5, 6$ (see Appendix B.1. for expressions for the conditional profit functions in each set). Furthermore, $\Omega_i(\overrightarrow{K}^a) = \Omega_i(\overrightarrow{K}')$, for $i = 1, 5, 6$, i.e., the boundaries of sets $\Omega_1, \Omega_5,$ and $\Omega_6$ do not change in $\overrightarrow{K}'$ and $\overrightarrow{K}^a$ (see Figure 3.1). Thus, we have $E_N[\Pi^*(\overrightarrow{K}')] = E_N[\Pi^*(\overrightarrow{K}^a)] \Rightarrow V(\overrightarrow{K^a}) > V(\overrightarrow{K'})$, since $c_2 < c_f$. This completes the proof.

(ii) For any solution $\overrightarrow{K'}$ satisfying $\frac{\alpha s_2(K_2^{'}+K_f^{'})}{b-\alpha(s_1+s_2)} \geq K_1^{'} > \frac{\alpha s_2 K_2^{'}}{b-\alpha(s_1+s_2)} - K_f^{'}$, we can always find an alternative solution $\overrightarrow{K^a}$ in Region II that satisfies $K_1^a = K_1^{'}$, $\frac{\alpha s_2 K_2^a}{b-\alpha(s_1+s_2)} - K_f^a = K_1^a$, and $K_2^a + K_f^a = K_2^{'} + K_f^{'}$. Note that

$$\begin{aligned}
K_1^a &= K_1^{'} \geq 0, \\
K_2^a &= \frac{(b - \alpha(s_1 + s_2))(K_1^{'} + K_2^{'} + K_f^{'})}{b - \alpha s_1} > 0, \\
K_f^a &= K_2^{'} + K_f^{'} - K_2^a = \frac{\alpha s_2(K_2^{'} + K_f^{'})}{b - \alpha s_1} - \frac{(b - \alpha(s_1 + s_2))K_1^{'}}{b - \alpha s_1} \\
&\geq \frac{(b - \alpha(s_1 + s_2))K_1^{'}}{b - \alpha s_1} - \frac{(b - \alpha(s_1 + s_2))K_1^{'}}{b - \alpha s_1} = 0.
\end{aligned}$$

Hence, $\overrightarrow{K^a}$ is a feasible solution. Similar to Part (i), we can show that $q_j(\overrightarrow{K}') = q_j(\overrightarrow{K}^a), j = 1, 2,$ in each set $\Omega_1$, $\Omega_2$, and $\Omega_6$, leading to $\Pi_i(\overrightarrow{K}') = \Pi_i(\overrightarrow{K}^a)$, $i = 1, 2, 6$. Since the boundaries of sets $\Omega_1, \Omega_2,$ and $\Omega_6$ do not change in $\overrightarrow{K}'$ and $\overrightarrow{K}^a$ (see Figure 3.2), we have $E_N[\Pi^*(\overrightarrow{K}')] = E_N[\Pi^*(\overrightarrow{K}^a)] \Rightarrow V(\overrightarrow{K^a}) > V(\overrightarrow{K'})$, since $c_2 < c_f$. □

# Appendix B

### 1. Conditional Profit Functions and Their Derivatives

We first derive the first- and second-order derivatives for the conditional profit functions in Stage 2 under DU setting with respect to $K_1$, $K_2$, and $K_f$ for all six disjoint sets. These derivations will be used in our subsequent proofs. We have

$$\begin{array}{ll}\Omega_1: & \Pi_1 = \frac{n\alpha(b-\alpha s_1)s_1 s_2}{4b} + \frac{n(b-\alpha s_2)[b-\alpha(s_1+s_2)]s_2}{4b} \\ \Rightarrow & \frac{\partial \Pi_1}{\partial K_i} = 0 \text{ for } i = 1, 2, f.\end{array}$$

$$\begin{array}{ll}\Omega_2: & \Pi_2 = [bs_1 - \frac{bs_1(K_1+K_2+K_f)}{n} - \alpha s_1^2][K_1 + K_2 + K_f - \frac{n(b-\alpha s_1-\alpha s_2)}{2b}] \\ & \quad + [\frac{b(s_2+s_1)}{2} - \frac{bs_1(K_1+K_2+K_f)}{n} + \frac{\alpha(s_2^2-s_1^2)}{2} - \alpha s_2^2][\frac{n(b-\alpha s_1-\alpha s_2)}{2b}] \\ \Rightarrow & \frac{\partial \Pi_2}{\partial K_1} = \frac{\partial \Pi_2}{\partial K_2} = \frac{\partial \Pi_2}{\partial K_f} = bs_1 - \alpha s_1^2 - \frac{2bs_1(K_1+K_2+K_f)}{n} \\ & \frac{\partial^2 \Pi_2}{\partial K_1^2} = \frac{\partial^2 \Pi_2}{\partial K_2^2} = \frac{\partial^2 \Pi_2}{\partial K_f^2} = \frac{\partial^2 \Pi_2}{\partial K_1 \partial K_2} = \frac{\partial^2 \Pi_2}{\partial K_1 \partial K_f} = \frac{\partial^2 \Pi_2}{\partial K_2 \partial K_f} = -\frac{2bs_1}{n}.\end{array}$$

$$\begin{array}{ll}\Omega_3: & \Pi_3 = [\frac{(b+\alpha s_2)s_1}{2} - \frac{b(K_1+K_f)(s_2-s_1)s_1}{ns_2} - \alpha s_1^2](K_1 + K_f) \\ & \quad + [\frac{(b+\alpha s_2)s_2}{2} - \alpha s_2^2][\frac{n(b-\alpha s_2)}{2b} - \frac{(K_1+K_f)s_1}{s_2}] \\ \Rightarrow & \frac{\partial \Pi_3}{\partial K_1} = \frac{\partial \Pi_3}{\partial K_f} = \alpha s_1 s_2 - \alpha s_1^2 - \frac{2b(K_1+K_f)(s_2-s_1)s_1}{ns_2} \\ & \frac{\partial^2 \Pi_3}{\partial K_1^2} = \frac{\partial^2 \Pi_3}{\partial K_f^2} = \frac{\partial^2 \Pi_3}{\partial K_1 \partial K_f} = -\frac{2b(s_2-s_1)s_1}{ns_2} \\ & \frac{\partial \Pi_3}{\partial K_2} = \frac{\partial^2 \Pi_3}{\partial K_1 \partial K_2} = \frac{\partial^2 \Pi_3}{\partial K_2 \partial K_f} = \frac{\partial^2 \Pi_3}{\partial K_2^2} = 0.\end{array}$$

$$\begin{array}{ll}\Omega_4: & \Pi_4 = [bs_1 - \frac{bs_1(K_1+K_2+K_f)}{n} - \alpha s_1^2](K_1 + K_f) + [bs_2 - \frac{bs_1(K_1+K_f)}{n} - \frac{bK_2 s_2}{n} - \alpha s_2^2]K_2 \\ \Rightarrow & \frac{\partial \Pi_4}{\partial K_1} = \frac{\partial \Pi_4}{\partial K_f} = bs_1 - \alpha s_1^2 - \frac{2bs_1(K_1+K_2+K_f)}{n} \\ & \frac{\partial \Pi_4}{\partial K_2} = bs_2 - \alpha s_2^2 - \frac{2bs_1(K_1+K_f)}{n} - \frac{2bK_2 s_2}{n} \\ & \frac{\partial^2 \Pi_4}{\partial K_1^2} = \frac{\partial^2 \Pi_4}{\partial K_f^2} = \frac{\partial^2 \Pi_4}{\partial K_1 \partial K_2} = \frac{\partial^2 \Pi_4}{\partial K_1 \partial K_f} = \frac{\partial^2 \Pi_4}{\partial K_2 \partial K_f} = -\frac{2bs_1}{n} \\ & \frac{\partial^2 \Pi_4}{\partial K_2^2} = -\frac{2bs_2}{n}.\end{array}$$

$$\begin{aligned}
\Omega_5: \quad & \Pi_5 = [\tfrac{(b+\alpha s_1)s_1}{2} - \alpha s_1^2][\tfrac{n(b-\alpha s_1)}{2b} - (K_2 + K_f)] \\
& \quad + [b(s_2 - s_1) - \tfrac{b(K_2+K_f)(s_2-s_1)}{n} + \tfrac{(b+\alpha s_1)s_1}{2} - \alpha s_2^2](K_2 + K_f) \\
\Rightarrow \quad & \tfrac{\partial \Pi_5}{\partial K_2} = \tfrac{\partial \Pi_5}{\partial K_f} = \alpha s_1^2 - \alpha s_2^2 + b(s_2 - s_1) - \tfrac{2b(K_2+K_f)(s_2-s_1)}{n} \\
& \tfrac{\partial \Pi_5}{\partial K_1} = \tfrac{\partial^2 \Pi_5}{\partial K_1^2} = \tfrac{\partial^2 \Pi_5}{\partial K_1 \partial K_2} = \tfrac{\partial^2 \Pi_5}{\partial K_1 \partial K_f} = 0 \\
& \tfrac{\partial^2 \Pi_5}{\partial K_2^2} = \tfrac{\partial^2 \Pi_5}{\partial K_f^2} = \tfrac{\partial^2 \Pi_5}{\partial K_2 \partial K_f} = -\tfrac{2b(s_2-s_1)}{n}.
\end{aligned}$$

$$\begin{aligned}
\Omega_6: \quad & \Pi_6 = [bs_1 - \tfrac{bs_1(K_1+K_2+K_f)}{n} - \alpha s_1^2]K_1 + [bs_2 - -\tfrac{bs_2(K_2+K_f)}{n} - \tfrac{bK_1 s_1}{n} - \alpha s_2^2](K_2 + K_f) \\
\Rightarrow \quad & \tfrac{\partial \Pi_6}{\partial K_1} = bs_1 - \alpha s_1^2 - \tfrac{2bs_1(K_1+K_2+K_f)}{n} \\
& \tfrac{\partial \Pi_6}{\partial K_2} = \tfrac{\partial \Pi_6}{\partial K_f} = bs_2 - \alpha s_2^2 - \tfrac{2bs_2(K_2+K_f)}{n} - \tfrac{2bK_1 s_1}{n} \\
& \tfrac{\partial^2 \Pi_6}{\partial K_1^2} = \tfrac{\partial^2 \Pi_6}{\partial K_1 \partial K_2} = \tfrac{\partial^2 \Pi_6}{\partial K_1 \partial K_f} = -\tfrac{2bs_1}{n} \\
& \tfrac{\partial^2 \Pi_6}{\partial K_2^2} = \tfrac{\partial^2 \Pi_6}{\partial K_f^2} = \tfrac{\partial^2 \Pi_6}{\partial K_2 \partial K_f} = -\tfrac{2bs_2}{n}.
\end{aligned}$$

## 2. Formulation and Optimality Conditions of Region I and Region III Problems

We first give the formulation of the Region I and Region III problems for the DU only setting. Then, Lemmas 1 and 2 establish the necessary and sufficient conditions for optimality of the Region I (with $K_f = 0$) and Region III (with $K_f \geq 0$) problems, respectively. These results will be used in our subsequent proofs.

The formulation for the **Region I problem**, $\mathbf{P_1^I}$, with $K_f = 0$ reduces to:

$$\mathbf{P_1^I}(K_f = 0): \quad V_I^* \equiv \max_{K_1, K_2} V = E_N[\Pi^*(K_1, K_2, N)] - \sum_{i=1,\ 2} c_i K_i \tag{B.1a}$$

subject to

$$K_1 \geq 0 \tag{B.1b}$$

$$K_2 \geq 0 \qquad \longleftarrow v_2 \tag{B.1c}$$

$$K_1 - \frac{\alpha s_2 K_2}{b - \alpha(s_1 + s_2)} \geq 0, \qquad \longleftarrow v_a \tag{B.1d}$$

where $v_2$ and $v_a$ denote the KKT multipliers respectively corresponding to constraints (B.1c) and (B.1d). Observe that (B.1c) and (B.1d) imply (B.1b); hence, (B.1b) is redundant.

**Lemma 1.** An investment vector $(K_1, K_2) \in R_+^2$ is the unique optimal solution to Problem $\mathbf{P_1^I}(K_f = 0)$ if and only if there exists a $\overrightarrow{v} = (v_2, v_a) \in R_+^2$ that satisfies the following conditions:

$$\int_{t_2}^{\infty} s_1(b-\alpha s_1)\frac{n-t_2}{n} f_N(n)dn = c_1 - v_a \tag{B.2a}$$

$$\int_{t_1}^{t_2} (s_2-s_1)[b-\alpha(s_1+s_2)]\frac{n-t_1}{n} f_N(n)dn + \int_{t_2}^{\infty}[bs_2 - \alpha s_2^2 - \frac{2b(K_2 s_2 + K_1 s_1)}{n}]f_N(n)dn = c_2 - v_2 + \frac{\alpha s_2 v_a}{b-\alpha(s_1+s_2)} \tag{B.2b}$$

$$v_2 K_2 = 0 \tag{B.2c}$$

$$v_a[K_1 - \frac{\alpha s_2 K_2}{b-\alpha(s_1+s_2)}] = 0 \tag{B.2d}$$

$$v_2, v_a \geq 0, \tag{B.2e}$$

where $t_1 \equiv \frac{2bK_2}{b-\alpha(s_1+s_2)}$ and $t_2 \equiv \frac{2b(K_1+K_2)}{b-\alpha s_1}$.

**Proof:** Consider the Region I problem $\mathbf{P_1^I}(K_f = 0)$, where $K_1 \geq \frac{\alpha s_2 K_2}{b-\alpha(s_1+s_2)} (\geq 0)$, and $K_f = 0$.

$$V(\overrightarrow{K}) = E_N[\Pi^*(\overrightarrow{K}, N)] - \sum_{i=1,2} c_i K_i = \sum_{i=1}^{6}\int_{n\in\ \Omega_i} \Pi_i(n) f_N(n)dn - \sum_{i=1,2} c_i K_i$$
$$= \int_0^{\frac{2bK_2}{b-\alpha(s_1+s_2)}} \Pi_1(n) f_N(n)dn + \int_{\frac{2bK_2}{b-\alpha(s_1+s_2)}}^{\frac{2b(K_1+K_2)}{b-\alpha s_1}} \Pi_5(n) f_N(n)dn + \int_{\frac{2b(K_1+K_2)}{b-\alpha s_1}}^{\infty} \Pi_6(n) f_N(n)dn - \sum_{i=1,2} c_i K_i$$

$$\Rightarrow \frac{\partial V(\overrightarrow{K})}{\partial K_1} = \int_{\frac{2b(K_1+K_2)}{b-\alpha s_1}}^{\infty} \frac{\partial \Pi_6(n)}{\partial K_1} f_N(n)dn - c_1$$

$$\frac{\partial V(\overrightarrow{K})}{\partial K_2} = \int_{\frac{2bK_2}{b-\alpha(s_1+s_2)}}^{\frac{2b(K_1+K_2)}{b-\alpha s_1}} \frac{\partial \Pi_5(n)}{\partial K_2} f_N(n)dn + \int_{\frac{2b(K_1+K_2)}{b-\alpha s_1}}^{\infty} \frac{\partial \Pi_6(n)}{\partial K_2} f_N(n)dn - c_2$$

$$\frac{\partial^2 V(\overrightarrow{K})}{\partial K_1^2} = \int_{\frac{2b(K_1+K_2)}{b-\alpha s_1}}^{\infty} \frac{\partial^2 \Pi_6(n)}{\partial K_1^2} f_N(n)dn - \frac{2b}{(b-\alpha s_1)} \frac{\partial \Pi_6(n)}{\partial K_1} f_N(n)|_{n=\frac{2b(K_1+K_2)}{b-\alpha s_1}}$$
$$= -2bs_1 \int_{\frac{2b(K_1+K_2)}{b-\alpha s_1}}^{\infty} \frac{1}{n} f_N(n)dn$$

$$\frac{\partial^2 V(\overrightarrow{K})}{\partial K_1 \partial K_2} = \int_{\frac{2b(K_1+K_2)}{b-\alpha s_1}}^{\infty} \frac{\partial^2 \Pi_6(n)}{\partial K_1 \partial K_2} f_N(n)dn - \frac{2b}{(b-\alpha s_1)} \frac{\partial \Pi_6(n)}{\partial K_1} f_N(n)|_{n=\frac{2b(K_1+K_2)}{b-\alpha s_1}}$$
$$= -2bs_1 \int_{\frac{2b(K_1+K_2)}{b-\alpha s_1}}^{\infty} \frac{1}{n} f_N(n)dn$$

$$\begin{aligned}\frac{\partial^2 V(\overrightarrow{K})}{\partial K_2^2} &= \int_{\frac{2bK_2}{b-\alpha(s_1+s_2)}}^{\frac{2b(K_1+K_2)}{b-\alpha s_1}} \frac{\partial^2 \Pi_5(n)}{\partial K_2^2} f_N(n)dn + \int_{\frac{2b(K_1+K_2)}{b-\alpha s_1}}^{\infty} \frac{\partial^2 \Pi_6(n)}{\partial K_2^2} f_N(n)dn \\ &+ \frac{2b}{b-\alpha s_1}\frac{\partial \Pi_5(n)}{\partial K_2} f_N(n)|_{n=\frac{2b(K_1+K_2)}{b-\alpha s_1}} - \frac{2b}{b-\alpha(s_1+s_2)}\frac{\partial \Pi_5(n)}{\partial K_2} f_N(n)|_{n=\frac{2bK_2}{b-\alpha(s_1+s_2)}} \\ &- \frac{2b}{(b-\alpha s_1)}\frac{\partial \Pi_6(n)}{\partial K_2} f_N(n)|_{n=\frac{2b(K_1+K_2)}{b-\alpha s_1}} \\ &= -2b(s_2-s_1)\int_{\frac{2bK_2}{b-\alpha(s_1+s_2)}}^{\frac{2b(K_1+K_2)}{(b-\alpha s_1)}} \frac{1}{n} f_N(n)dn - 2bs_2 \int_{\frac{2b(K_1+K_2)}{b-\alpha s_1}}^{\infty} \frac{1}{n} f_N(n)dn.\end{aligned}$$

Let $d \equiv \int_{\frac{2b(K_1+K_2)}{b-\alpha s_1}}^{\infty} \frac{1}{n} f_N(n)dn$ and $e \equiv \int_{\frac{2bK_2}{b-\alpha(s_1+s_2)}}^{\frac{2b(K_1+K_2)}{b-\alpha s_1}} \frac{1}{n} f_N(n)dn$. Then, the Hessian of $V(\overrightarrow{K})$ with respect to $K_1$ and $K_2$ is given by

$$H(V(\overrightarrow{K})) = \begin{bmatrix} -2bs_1 d & -2bs_1 d \\ -2bs_1 d & -2b(s_2-s_1)e - 2bs_2 d \end{bmatrix}.$$

We know $-2bs_1 d < 0$ and $-2b(s_2-s_1)e - 2bs_1 d < 0$. We also have

$$\begin{bmatrix} -2bs_1 d & -2bs_1 d \\ -2bs_1 d & -2b(s_2-s_1)e - 2bs_2 d \end{bmatrix} = \begin{bmatrix} -2bs_1 d & -2bs_1 d \\ 0 & -2b(s_2-s_1)(d+e) \end{bmatrix}.$$

Therefore, $H(V(\overrightarrow{K}))$ is negative definite, and hence, $V(\overrightarrow{K})$ is strictly jointly concave in $K_1$ and $K_2$. Since the constraints are linear in $K_1$ and $K_2$, the result follows by the first-order KKT conditions, which are necessary and sufficient for optimality of problem $\mathbf{P_1^I}(K_f = 0)$. □

Similarly, the formulation for the **Region III** (with $K_f \geq 0$) problem, $\mathbf{P_1^{III}}$, is given as:

$$\mathbf{P_1^{III}}: \quad V_{III}^* \equiv \max_{\overrightarrow{K}} V = E_N[\Pi^*(\overrightarrow{K}, N)] - \sum_{i=1,\ 2,\ f} c_i K_i \qquad \text{(B.3a)}$$

subject to

$$K_1 \geq 0 \qquad \longleftarrow v_1 \qquad \text{(B.3b)}$$

$$K_2 \geq 0 \qquad \text{(B.3c)}$$

$$K_f \geq 0 \qquad \longleftarrow v_f \qquad \text{(B.3d)}$$

$$\frac{\alpha s_2 K_2}{b-\alpha(s_1+s_2)} - K_1 - K_f \geq 0, \qquad \longleftarrow v_{a2} \qquad \text{(B.3e)}$$

where $v_1$, $v_f$, and $v_{a2}$ denote the KKT multipliers respectively corresponding to constraints (B.3b),(B.3d), and (B.3e). Observe that (B.3b),(B.3d), and (B.3e) imply (B.3c), which is redundant.

**Lemma 2.** An investment vector $\overrightarrow{K} = (K_1, K_2, K_f) \in R_+^3$ is the unique optimal solution to Problem $\mathbf{P_1^{III}}$ if and only if there exists a $\overrightarrow{v} = (v_1, v_f, v_{a2}) \in R_+^3$ that satisfies the following conditions:

$$\int_{t_1}^{t_2} \gamma f_N(n)dn + \int_{t_2}^{\infty} \beta f_N(n)dn = c_1 - v_1 + v_{a2} \tag{B.4a}$$

$$\int_{t_2}^{t_3} \omega_1 f_N(n)dn + \int_{t_3}^{t_4} \beta f_N(n)dn + \int_{t_4}^{\infty} \omega_2 f_N(n)dn = c_2 - \frac{\alpha s_2 v_{a2}}{b-\alpha(s_1+s_2)} \tag{B.4b}$$

$$\int_{t_1}^{t_2} \gamma f_N(n)dn + \int_{t_2}^{t_4} \beta f_N(n)dn + \int_{t_4}^{\infty} \omega_2 f_N(n)dn = c_f - v_f + v_{a2} \tag{B.4c}$$

$$v_1 K_1 = 0 \tag{B.4d}$$

$$v_f K_f = 0 \tag{B.4e}$$

$$v_{a2}[\frac{\alpha s_2 K_2}{b-\alpha(s_1+s_2)} - K_1 - K_f] = 0 \tag{B.4f}$$

$$v_1, v_f, v_{a2} \geq 0, \tag{B.4g}$$

where

$$\beta \equiv bs_1 - \alpha s_1^2 - \frac{2bs_1(K_1+K_2+K_f)}{n}, \quad \gamma \equiv \alpha s_1 s_2 - \alpha s_1^2 - \frac{2b(K_1+K_f)(s_2-s_1)s_1}{ns_2},$$
$$\omega_1 \equiv bs_2 - \alpha s_2^2 - \frac{2bs_1(K_1+K_f)}{n} - \frac{2bK_2 s_2}{n}, \quad \omega_2 \equiv bs_2 - \alpha s_2^2 - \frac{2bs_2(K_2+K_f)}{n} - \frac{2bK_1 s_1}{n},$$
$$t_1 \equiv \frac{2b(K_1+K_f)}{\alpha s_2}, \quad t_2 \equiv \frac{2b(K_2 s_2 + (K_1+K_f)s_1)}{(b-\alpha s_2)s_2},$$
$$t_3 \equiv \frac{2bK_2}{b-\alpha(s_1+s_2)}, \quad t_4 \equiv \frac{2b(K_2+K_f)}{b-\alpha(s_1+s_2)}.$$

**Proof:** Consider the Region III problem, $\mathbf{P_1^{III}}$, where $\frac{\alpha s_2 K_2}{b-\alpha(s_1+s_2)} - K_f \geq K_1$. Let $t_1 \equiv \frac{2b(K_1+K_f)}{\alpha s_2}$, $t_2 \equiv \frac{2b[K_2 s_2+(K_1+K_f)s_1]}{(b-\alpha s_2)s_2}$, $t_3 \equiv \frac{2bK_2}{b-\alpha(s_1+s_2)}$, and $t_4 \equiv \frac{2b(K_2+K_f)}{b-\alpha(s_1+s_2)}$. We have

$$V(\overrightarrow{K}) = E_N[\Pi^*(\overrightarrow{K}, N)] - \sum_{i=1,2,f} c_i K_i = \int_0^{t_1} \Pi_1(n) f_N(n)dn + \int_{t_1}^{t_2} \Pi_3(n) f_N(n)dn + \int_{t_2}^{t_3} \Pi_4(n) f_N(n)dn$$
$$+ \int_{t_3}^{t_4} \Pi_2(n) f_N(n)dn + \int_{t_4}^{\infty} \Pi_6(n) f_N(n)dn - \sum_{i=1,2,f} c_i K_i$$

$$\Rightarrow \frac{\partial V(\overrightarrow{K})}{\partial K_1} = \int_{t_1}^{t_2} \frac{\partial \Pi_3(n)}{\partial K_1} f_N(n)dn + \int_{t_2}^{t_3} \frac{\partial \Pi_4(n)}{\partial K_1} f_N(n)dn$$
$$+ \int_{t_3}^{t_4} \frac{\partial \Pi_2(n)}{\partial K_1} f_N(n)dn + \int_{t_4}^{\infty} \frac{\partial \Pi_6(n)}{\partial K_1} f_N(n)dn - c_1$$

$$\begin{aligned}\frac{\partial^2 V(\overrightarrow{K})}{\partial K_1^2} = &\int_{t_1}^{t_2} \frac{\partial^2 \Pi_3(n)}{\partial K_1^2} f_N(n)dn + \int_{t_2}^{t_3} \frac{\partial^2 \Pi_4(n)}{\partial K_1^2} f_N(n)dn + \int_{t_3}^{t_4} \frac{\partial^2 \Pi_2(n)}{\partial K_1^2} f_N(n)dn \\ &+ \int_{t_4}^{\infty} \frac{\partial^2 \Pi_6(n)}{\partial K_1^2} f_N(n)dn - \frac{2b}{\alpha s_2}\frac{\partial \Pi_3(n)}{\partial K_1} f_N(n)|_{n=t_1} + \frac{2bs_1}{(b-\alpha s_2)s_2}\frac{\partial \Pi_3(n)}{\partial K_1} f_N(n)|_{n=t_2} \\ &- \frac{2bs_1}{(b-\alpha s_2)s_2}\frac{\partial \Pi_4(n)}{\partial K_1} f_N(n)|_{n=t_2} \\ = &-\frac{2b(s_2-s_1)s_1}{s_2}\int_{t_1}^{t_2}\frac{f_N(n)}{n}dn - 2bs_1\int_{t_2}^{\infty}\frac{f_N(n)}{n}dn\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 V(\overrightarrow{K})}{\partial K_1 \partial K_2} &= -2bs_1\int_{t_2}^{t_3}\frac{f_N(n)}{n}dn - 2bs_1\int_{t_3}^{t_4}\frac{f_N(n)}{n}dn - 2bs_1\int_{t_4}^{\infty}\frac{f_N(n)}{n}dn = -2bs_1\int_{t_2}^{\infty}\frac{f_N(n)}{n}dn \\ \frac{\partial^2 V(\overrightarrow{K})}{\partial K_1 \partial K_f} &= \frac{\partial^2 V(\overrightarrow{K})}{\partial K_1^2} = -\frac{2b(s_2-s_1)s_1}{s_2}\int_{t_1}^{t_2}\frac{f_N(n)}{n}dn - 2bs_1\int_{t_2}^{\infty}\frac{f_N(n)}{n}dn\end{aligned}$$

$$\frac{\partial V(\overrightarrow{K})}{\partial K_2} = \int_{t_2}^{t_3}\frac{\partial \Pi_4(n)}{\partial K_2} f_N(n)dn + \int_{t_3}^{t_4}\frac{\partial \Pi_2(n)}{\partial K_2} f_N(n)dn + \int_{t_4}^{\infty}\frac{\partial \Pi_6(n)}{\partial K_2} f_N(n)dn - c_2$$

$$\begin{aligned}\frac{\partial^2 V(\overrightarrow{K})}{\partial K_2^2} &= -2bs_2\int_{t_2}^{t_3}\frac{f_N(n)}{n}dn - 2bs_1\int_{t_3}^{t_4}\frac{f_N(n)}{n}dn - 2bs_2\int_{t_4}^{\infty}\frac{f_N(n)}{n}dn \\ \frac{\partial^2 V(\overrightarrow{K})}{\partial K_2 \partial K_f} &= -2bs_1\int_{t_2}^{t_3}\frac{f_N(n)}{n}dn - 2bs_1\int_{t_3}^{t_4}\frac{f_N(n)}{n}dn - 2bs_2\int_{t_4}^{\infty}\frac{f_N(n)}{n}dn \\ \frac{\partial^2 V(\overrightarrow{K})}{\partial K_f^2} &= -\frac{2b(s_2-s_1)s_1}{s_2}\int_{t_1}^{t_2}\frac{f_N(n)}{n}dn - 2bs_1\int_{t_2}^{t_3}\frac{f_N(n)}{n}dn - 2bs_1\int_{t_3}^{t_4}\frac{f_N(n)}{n}dn - 2bs_2\int_{t_4}^{\infty}\frac{f_N(n)}{n}dn.\end{aligned}$$

Let $d \equiv \int_{t_1}^{t_2}\frac{f_N(n)}{n}dn$, $e \equiv \int_{t_2}^{t_3}\frac{f_N(n)}{n}dn$, $h \equiv \int_{t_3}^{t_4}\frac{f_N(n)}{n}dn$, $g \equiv \int_{t_4}^{\infty}\frac{f_N(n)}{n}dn$. The Hessian of $V(\overrightarrow{K})$ w.r.t. $K_1$, $K_2$, and $K_f$ is given by:

$$\begin{aligned}H(V(\overrightarrow{K})) = &-2b\begin{bmatrix} s_1(d+e+h+g)-\frac{s_1^2}{s_2}d & s_1(e+h+g) & s_1(d+e+h+g)-\frac{s_1^2}{s_2}d \\ s_1(e+h+g) & s_2e+s_1h+s_2g & s_1e+s_1h+s_2g \\ s_1(d+e+h+g)-\frac{s_1^2}{s_2}d & s_1e+s_1h+s_2g & s_1(d+e+h)+s_2g-\frac{s_1^2}{s_2}d \end{bmatrix} \\ = &-2b\begin{bmatrix} s_1(d+e+h+g)-\frac{s_1^2}{s_2}d & s_1(e+h+g) & s_1(d+e+h+g)-\frac{s_1^2}{s_2}d \\ 0 & s_2e+s_1h+s_2g-\frac{s_1^2(e+h+g)^2}{s_1(d+e+h+g)-\frac{s_1^2}{s_2}d} & (s_2-s_1)g \\ 0 & (s_2-s_1)g & (s_2-s_1)g \end{bmatrix}.\end{aligned}$$

Since $s_1 d - \frac{s_1^2}{s_2} d = \frac{s_1(s_2-s_1)}{s_2} d > 0$, we know $s_1(d+e+h+g) - \frac{s_1^2}{s_2} d > s_1(e+h+g)$. Hence,

$$\begin{aligned} s_2e + s_1h + s_2g - \frac{s_1^2(e+h+g)^2}{s_1(d+e+h+g) - \frac{s_1^2}{s_2}d} >& s_2e + s_1h + s_2g - s_1(e+h+g) \\ =& (s_2 - s_1)e + (s_2 - s_1)g \\ >& (s_2 - s_1)g > 0 \\ \Rightarrow [s_2e + s_1h + s_2g - \frac{s_1^2(e+h+g)^2}{s_1(d+e+h+g) - \frac{s_1^2}{s_2}d}]&\cdot(s_2 - s_1)g - (s_2 - s_1)^2g^2 > 0. \end{aligned}$$

We also know $s_1(d+e+h+g) - \frac{s_1^2}{s_2}d > 0$ and $(s_2 - s_1)g > 0$. Then, $V(\overrightarrow{K})$ is strictly jointly concave in $K_1$, $K_2$, and $K_f$. Since the constraints are linear in $K_1$, $K_2$, and $K_f$, the result follows by the first-order KKT conditions, which are necessary and sufficient for optimality of Problem $\mathbf{P_1^{III}}$.□

### 3. Proof of Proposition 5

(i) In the case of no consumer-driven substitution, the two-stage stochastic programming problem formulation for product $i$ becomes

Problem $\mathbf{P_d^i}$

Stage 1 Problem $\mathbf{P_{d1}^i}$:

$$\max_{\overrightarrow{K}} V_i \equiv E_N[\Pi^*(K_i, N)] - c_i K_i \tag{B.5a}$$

$$\text{subject to } K_i \geq 0. \tag{B.5b}$$

Stage 2 Problem $\mathbf{P_{d2}^i}$:

$$\Pi^*(K_i, n) = \max_{\overrightarrow{p}} \frac{n}{b}(b - \frac{p_i}{s_i})(p_i - \alpha s_i^2) \tag{B.6a}$$

subject to

$$\frac{n}{b}(b - \frac{p_i}{s_i}) \leq K_i \tag{B.6b}$$

$$b - \frac{p_i}{s_i} \geq 0. \tag{B.6c}$$

The solution to the stage 2 problem directly follows from Proposition 3. For the stage 1 problem, we know the objective function (B.5a) is strictly concave in $K_i$ and the first-order KKT conditions for the stage 1 problem are given by

$$\int_{2K_i}^{\infty} (bs_i - \alpha s_i^2)(1 - \frac{2K_i}{n}) f_{N_i}(n) dn = c_i - v_i \tag{B.7a}$$

$$K_i v_i = 0 \tag{B.7b}$$

$$v_i \geq 0. \tag{B.7c}$$

When $v_i = 0$, we need $c_i \leq s_i(b - \alpha s_i)$ for $K_i \geq 0$. Also, when $v_i > 0$, we need $c_i > s_i(b - \alpha s_i)$. Therefore, the structure of the optimal resource portfolio in this case can be characterized as follows (see Figure 4.1(a)):

($A^d$): If $c_1 \leq bs_1 - \alpha s_1^2$ and $c_2 \leq bs_2 - \alpha s_2^2$, then $K_1^D$ and $K_2^D$ are the unique solutions to:

$$\int_{2K_1}^{\infty}(bs_1 - \alpha s_1^2)(1 - \frac{2K_1}{n})f_{N_1}(n)dn = c_1, \qquad \int_{2K_2}^{\infty}(bs_2 - \alpha s_2^2)(1 - \frac{2K_2}{n})f_{N_2}(n)dn = c_2.$$

($C^d$): If $c_2 \leq bs_2 - \alpha s_2^2$ and $c_1 > bs_1 - \alpha s_1^2$, then $K_1^D = 0$ and $K_2^D$ is the unique solution to $\int_{2K_2}^{\infty}(bs_2 - \alpha s_2^2)(1 - \frac{2K_2}{n})f_{N_2}(n)dn = c_2$.

($D^d$): If $c_1 \leq bs_1$ and $c_2 > bs_2 - \alpha s_2^2$, then $K_2^D = 0$ and $K_1^D$ is the unique solution to $\int_{2K_1}^{\infty}(bs_1 - \alpha s_1^2)(1 - \frac{2K_1}{n})f_{N_1}(n)dn = c_1$.

($E^d$): If $c_1 \geq bs_1 - \alpha s_1^2$ and $c_2 \geq bs_2 - \alpha s_2^2$, then $K_1^D = 0$ and $K_2^D = 0$.

(ii) We study both Region I and III problems with $K_f = 0$, and compare the optimal solution to each to determine the global optimal solution. We denote $\overrightarrow{K}^I = (K_1^I, K_2^I)$ as the optimal solution to Problem $\mathbf{P_1^I}(K_f = 0)$, and $\overrightarrow{K}^{III} = (K_1^{III}, K_2^{III})$ as the optimal solution to Problem $\mathbf{P_1^{III}}(K_f = 0)$.

**Region I problem with $K_f = 0$ ($\mathbf{P_1^I}(K_f = 0)$):**

Recall that $v_2$ and $v_a$ denote the KKT multipliers respectively corresponding to constraints (B.1c) and (B.1d). We study the optimality conditions in Lemma 1 and summarizes the structure of the optimal solution for the Region I problem with $K_f = 0$ in Table B.1 (see also Figure B.1(a)).

**Region III problem with $K_f = 0$ ($\mathbf{P_1^{III}}(K_f = 0)$):**

Let $t_1 \equiv \frac{2bK_1}{\alpha s_2}$ and $t_2 \equiv \frac{2b(K_1 s_1 + K_2 s_2)}{(b - \alpha s_2)s_2}$. The following conditions arise as a special case of

Table B.1: Structure of the Optimal Solution for Problem $\mathbf{P_1^I}(K_f = 0)$.

| **Feasible Region :** $K_1 \geq \frac{\alpha s_2 K_2}{b-\alpha(s_1+s_2)} \geq 0$ | | |
|---|---|---|
| Cases | Solution Structure | Necessary and Sufficient Conditions |
| $(v_2 = 0, v_a = 0)$ | I(1) $K_1^I \geq 0$ and $K_2^I \geq 0$ | $c_2 \leq c_1 + (s_2 - s_1)[b - \alpha(s_1 + s_2)]$<br>$c_2 \geq \frac{s_2(b-\alpha s_2)}{s_1(b-\alpha s_1)} c_1$ |
| $(v_2 = 0, v_a > 0)$ | I(2) $K_1^I = \frac{\alpha s_2 K_2}{b-\alpha(s_1+s_2)}$ and $K_2^I \geq 0$ | $c_2 + \frac{\alpha s_2}{b-\alpha(s_1+s_2)} c_1 \leq \frac{\alpha s_1 s_2 (b-\alpha s_1)}{b-\alpha(s_1+s_2)} + s_2(b - \alpha s_2)$<br>$c_2 < \frac{s_2(b-\alpha s_2)}{s_1(b-\alpha s_1)} c_1$ |
| $(v_2 > 0, v_a = 0)$ | I(3) $K_1^I \geq 0$ and $K_2^I = 0$ | $c_2 > c_1 + (s_2 - s_1)[b - \alpha(s_1 + s_2)]$<br>$c_1 \leq s_1(b - \alpha s_1)$ |
| $(v_2 > 0, v_a > 0)$ | I(4) $K_1^I = 0$ and $K_2^I = 0$ | $c_2 + \frac{\alpha s_2}{b-\alpha(s_1+s_2)} c_1 > \frac{\alpha s_1 s_2 (b-\alpha s_1)}{b-\alpha(s_1+s_2)} + s_2(b - \alpha s_2)$<br>$c_1 > s_1(b - \alpha s_1)$ |

the optimality conditions in Lemma 2 (by setting $K_f = 0$).

$$\int_{t_1}^{t_2} \alpha s_1(s_2 - s_1)\frac{n - t_1}{n} f_N(n)dn + \int_{t_2}^{\infty} [bs_1 - \alpha s_1^2 - \frac{2b(K_2 + K_1)s_1}{n}] f_N(n)dn = c_1 - v_1 + v_{a2} \tag{B.8a}$$

$$\int_{t_2}^{\infty} [bs_2 - \alpha s_2^2 - \frac{2b(K_1 s_1 + K_2 s_2)}{n}] f_N(n)dn = c_2 - \frac{\alpha s_2 v_{a2}}{b - \alpha(s_1 + s_2)} \tag{B.8b}$$

$$v_1 K_1 = 0 \tag{B.8c}$$

$$v_{a2}[\frac{\alpha s_2 K_2}{b - \alpha(s_1 + s_2)} - K_1] = 0 \tag{B.8d}$$

$$v_1, v_{a2} \geq 0, \tag{B.8e}$$

where $v_1$ and $v_{a2}$ respectively denote the KKT multipliers corresponding to constraints (B.3b) and (B.3e). We analyze the Region III problem with $K_f = 0$ and summarizes the structure of the optimal solution in Table B.2 (see also Figure B.1(b)).

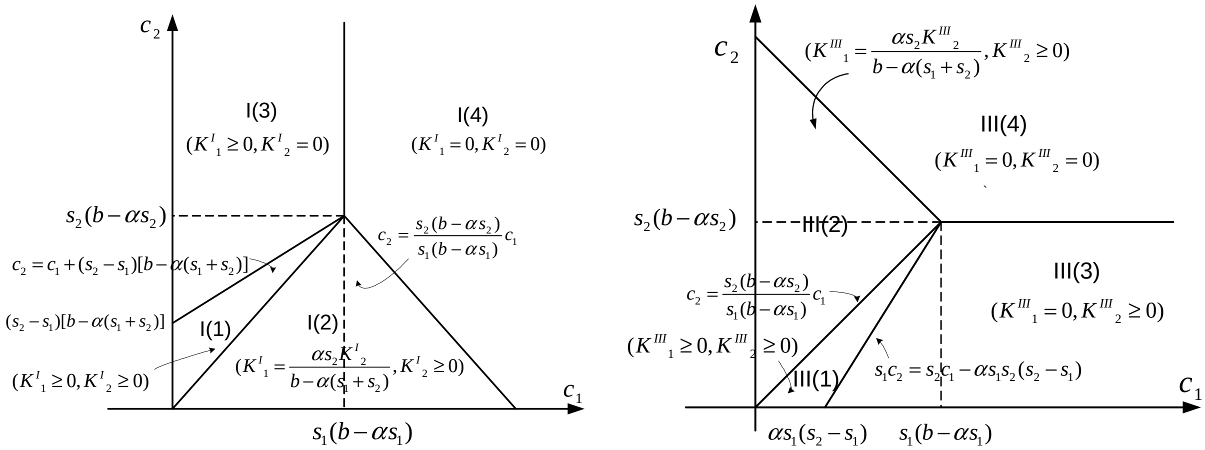
**The global optimal solution with $K_f = 0$:**

Comparing the optimal solutions in each region (see Tables 2 and 3), we obtain the global optimal solution as follows:

$(A^d)$: If $c_2 \leq c_1 + (s_2 - s_1)[b - \alpha(s_1 + s_2)]$ and $c_2 \geq \frac{s_2(b-\alpha s_2)}{s_1(b-\alpha s_1} c_1$, then the optimal solution to Problem $\mathbf{P_1^{III}}(K_f = 0)$ is given by Case III(2), which lies on the boundary of $K_1 = \frac{\alpha s_2 K_2}{b-\alpha(s_1+s_2)}$. Hence, in this region, the optimal solution to Problem $\mathbf{P_1^{III}}(K_f = 0)$ is feasible to Problem $\mathbf{P_1^I}(K_f = 0)$, but not optimal for the latter. Therefore, the optimal solution to Problem $\mathbf{P_1^{III}}(K_f = 0)$ is dominated by the optimal solution to

Table B.2: Structure of the Optimal Solution for Problem $\mathbf{P_1^{III}}(K_f = 0)$.

| **Feasible Region :** $0 \le K_1 \le \frac{\alpha s_2 K_2}{b-\alpha(s_1+s_2)}$ | | |
|---|---|---|
| Cases | Solution Structure | Necessary and Sufficient Conditions |
| $(v_1 = 0, v_{a2} = 0)$ | III(1) $K_1^{III} \ge 0$ and $K_2^{III} \ge 0$ | $s_1c_2 \ge s_2c_1 - \alpha s_1 s_2(s_2 - s_1)$ <br> $c_2 \le \frac{s_2(b-\alpha s_2)}{s_1(b-\alpha s_1)} c_1$ |
| $(v_1 = 0, v_{a2} > 0)$ | III(2) $K_1^{III} = \frac{\alpha s_2 K_2}{b-\alpha(s_1+s_2)}$ and $K_2^{III} \ge 0$ | $c_2 + \frac{\alpha s_2}{b-\alpha(s_1+s_2)} c_1 \le \frac{\alpha s_1 s_2(b-\alpha s_1)}{b-\alpha(s_1+s_2)} + s_2(b - \alpha s_2)$ <br> $c_2 > \frac{s_2(b-\alpha s_2)}{s_1(b-\alpha s_1)} c_1$ |
| $(v_1 > 0, v_{a2} = 0)$ | III(3) $K_1^{III} = 0$ and $K_2^{III} \ge 0$ | $s_1c_2 < s_2c_1 - \alpha s_1 s_2(s_2 - s_1)$ <br> $c_2 \le s_2(b - \alpha s_2)$ |
| $(v_1 > 0, v_{a2} > 0)$ | III(4) $K_1^{III} = 0$ and $K_2^{III} = 0$ | $c_2 + \frac{\alpha s_2}{b-\alpha(s_1+s_2)} c_1 > \frac{\alpha s_1 s_2(b-\alpha s_1)}{b-\alpha(s_1+s_2)} + s_2(b - \alpha s_2)$ <br> $c_2 > s_2(b - \alpha s_2)$ |




(a) Region I Problem with $K_f = 0$.

(b) Region III Problem with $K_f = 0$..

Figure B.1: The Structure of the Optimal Solution for the Region I and III Problems with $K_f = 0$.

Problem $\mathbf{P_1^I}(K_f = 0)$, which then is the optimal solution and $K_1^D$ and $K_2^D$ are the unique solutions to:

$$\int_{t_2}^{\infty} s_1(b - \alpha s_1)\frac{n - t_2}{n} f_N(n)dn = c_1$$

$$\int_{t_1}^{t_2} (s_2 - s_1)[b - \alpha(s_1 + s_2)]\frac{n - t_1}{n} f_N(n)dn + \int_{t_2}^{\infty} [bs_2 - \alpha s_2^2 - 2b\frac{(K_2 s_2 + K_1 s_1)}{n}]f_N(n)dn = c_2,$$

where $t_1 \equiv \frac{2bK_2}{b-\alpha(s_1+s_2)}$ and $t_2 \equiv \frac{2b(K_1+K_2)}{b-\alpha s_1}$.

$(B^d)$: If $c_2 \le \frac{s_2(b-\alpha s_2)}{s_1(b-\alpha s_1)} c_1$ and $s_1 c_2 \ge s_2 c_1 - \alpha s_1 s_2(s_2 - s_1)$, then the optimal solution to Problem $\mathbf{P_1^I}(K_f = 0)$ is given by Case I(2), which lies on the boundary of $K_1 = \frac{\alpha s_2 K_2}{b-\alpha(s_1+s_2)}$. Hence, in this region, the optimal solution to Problem $\mathbf{P_1^I}(K_f = 0)$ is feasible to

Problem $\mathbf{P_1^{III}}(K_f = 0)$, but not optimal for the latter. Therefore, the optimal solution to Problem $\mathbf{P_1^{I}}(K_f = 0)$ is dominated by the optimal solution to Problem $\mathbf{P_1^{III}}(K_f = 0)$, which then is the global optimal solution and $K_1^D$ and $K_2^D$ are the unique solutions to:

$$\int_{t_1}^{t_2} \alpha s_1(s_2 - s_1)\frac{n - t_1}{n} f_N(n)dn + \int_{t_2}^{\infty} [bs_1 - \alpha s_1^2 - \frac{2b(K_2 + K_1)s_1}{n}]f_N(n)dn = c_1$$
$$\int_{t_2}^{\infty} [bs_2 - \alpha s_2^2 - \frac{2b(K_1 s_1 + K_2 s_2)}{n}]f_N(n)dn = c_2,$$

where $t_1 \equiv \frac{2bK_1}{\alpha s_2}$ and $t_2 \equiv \frac{2b(K_1 s_1 + K_2 s_2)}{(b - \alpha s_2)s_2}$.

$(C^d)$: If $c_2 \le s_2(b - \alpha s_2)$ and $s_1 c_2 \le s_2 c_1 - \alpha s_1 s_2(s_2 - s_1)$, then we need to consider two subcases:

(i) If $s_1 c_2 \le s_2 c_1 - \alpha s_1 s_2(s_2 - s_1)$ and $c_2 + \frac{\alpha s_2}{b - \alpha(s_1 + s_2)} c_1 \le \frac{\alpha s_1 s_2(b - \alpha s_1)}{b - \alpha(s_1 + s_2)} + s_2(b - \alpha s_2)$, then similar to $(B^d)$, the optimal solution to Problem $\mathbf{P_1^{I}}(K_f = 0)$ is feasible for Problem $\mathbf{P_1^{III}}(K_f = 0)$, but not optimal for the latter. Hence, in this region, the optimal solution to Problem $\mathbf{P_1^{I}}(K_f = 0)$ is dominated by the optimal solution to Problem $\mathbf{P_1^{III}}(K_f = 0)$.

(ii) If $c_2 \le s_2(b - \alpha s_2)$ and $c_2 + \frac{\alpha s_2}{b - \alpha(s_1 + s_2)} c_1 \ge \frac{\alpha s_1 s_2(b - \alpha s_1)}{b - \alpha(s_1 + s_2)} + s_2(b - \alpha s_2)$, then the optimal solution to Problem $\mathbf{P_1^{I}}(K_f = 0)$ is $(K_1^I = 0, K_2^I = 0)$, which is dominated by the optimal solution to Problem $\mathbf{P_1^{III}}(K_f = 0)$.

Therefore, the global optimal solution is given by Case III(3). Then $K_1^D = 0$ and $K_2^D$ is the unique solution to $\int_{\frac{2bK_2}{b - \alpha s_2}}^{\infty} (bs_2 - \alpha s_2^2 - \frac{2bK_2 s_2}{n}) f_N(n)dn = c_2$.

$(D^d)$: If $c_2 \ge c_1 + (s_2 - s_1)[b - \alpha(s_1 + s_2)]$ and $c_1 \le s_1(b - \alpha s_1)$, then we need to consider two subcases:

(i) If $c_2 \ge c_1 + (s_2 - s_1)[b - \alpha(s_1 + s_2)]$ and $c_2 + \frac{\alpha s_2}{b - \alpha(s_1 + s_2)} c_1 \le \frac{\alpha s_1 s_2(b - \alpha s_1)}{b - \alpha(s_1 + s_2)} + s_2(b - \alpha s_2)$, then similar to $(A^d)$, the optimal solution to Problem $\mathbf{P_1^{III}}(K_f = 0)$ is feasible for Problem $\mathbf{P_1^{I}}(K_f = 0)$, but not optimal for the latter. Hence, in this region, the optimal solution to $\mathbf{P_1^{III}}(K_f = 0)$ is dominated by the optimal solution to $\mathbf{P_1^{I}}(K_f = 0)$.

(ii) If $c_1 \le s_1(b - \alpha s_1)$ and $c_2 + \frac{\alpha s_2}{b - \alpha(s_1 + s_2)} c_1 \ge \frac{\alpha s_1 s_2(b - \alpha s_1)}{b - \alpha(s_1 + s_2)} + s_2(b - \alpha s_2)$, then the optimal solution to Problem $\mathbf{P_1^{III}}(K_f = 0)$ is $(K_1^{III} = 0, K_2^{III} = 0)$, which is dominated by the optimal solution to Problem $\mathbf{P_1^{I}}(K_f = 0)$.

Therefore, the global optimal solution is given by Case I(3). Then $K_2^D = 0$ and $K_1^D$ is the unique solution to $\int_{\frac{2bK_1}{b - \alpha s_1}}^{\infty} (bs_1 - \alpha s_1^2 - \frac{2bK_1 s_1}{n}) f_N(n)dn = c_1$.

$(E^d)$: If $c_1 \ge s_1(b - \alpha s_1)$ and $c_2 \ge s_2(b - \alpha s_2)$, then the optimal solution in each region is $(K_1^I = K_1^{III} = 0, K_2^I = K_2^{III} = 0)$. Then $K_1^D = 0$ and $K_2^D = 0$. □

# Appendix C

Define $H_{ij}$ as the element on the $i^{th}$ row and $j^{th}$ column of matrix H. We also let $det(H)$ denote the determinant of matrix $H$. Define

$$\begin{aligned}
Pr_1 &\equiv Pr\{Y_1 = 1, Y_2 = 1, Y_f = 1\} = \theta_1\theta_2\theta_3, \\
Pr_2 &\equiv Pr\{Y_1 = 1, Y_2 = 1, Y_f = 0\} = \theta_1\theta_2(1-\theta_3), \\
Pr_3 &\equiv Pr\{Y_1 = 1, Y_2 = 0, Y_f = 1\} = \theta_1(1-\theta_2)\theta_3, \\
Pr_4 &\equiv Pr\{Y_1 = 1, Y_2 = 0, Y_f = 0\} = \theta_1(1-\theta_2)(1-\theta_3), \\
Pr_5 &\equiv Pr\{Y_1 = 0, Y_2 = 1, Y_f = 1\} = (1-\theta_1)\theta_2\theta_3, \\
Pr_6 &\equiv Pr\{Y_1 = 0, Y_2 = 1, Y_f = 0\} = (1-\theta_1)\theta_2(1-\theta_3), \\
Pr_7 &\equiv Pr\{Y_1 = 0, Y_2 = 0, Y_f = 1\} = (1-\theta_1)(1-\theta_2)\theta_3, \\
Pr_8 &\equiv Pr\{Y_1 = 0, Y_2 = 0, Y_f = 0\} = (1-\theta_1)(1-\theta_2)(1-\theta_3).
\end{aligned}$$

Also define $Pr_{i+j} \equiv Pr_i + Pr_j$, for $i \neq j, i, j = 1, 2, ..., 8$. Then, $Pr_{1+2} = \theta_1\theta_2, Pr_{3+4} = \theta_1(1-\theta_2), Pr_{5+6} = (1-\theta_1)\theta_2$, and $Pr_{7+8} = (1-\theta_1)(1-\theta_2)$.

The following results (Lemma 3 and Propositions 15 and 16) will be used in the subsequent proofs.

**Lemma 3.** An investment vector $(K_1, K_2, K_f) \in R^3_+$ is the unique optimal solution to Problem **P((S+D)U)** if and only if there exists a $\overrightarrow{v} = (v_1, v_2, v_f) \in R^3_+$ that satisfies the following conditions:

$$\begin{aligned}
&Pr_1 \int_{2(K_1+K_2+K_f)}^{\infty} bs_1(1 - \frac{2(K_1+K_2+K_f)}{n}) f_N(n)dn + Pr_2 \int_{2(K_1+K_2)}^{\infty} bs_1(1 - \frac{2(K_1+K_2)}{n}) f_N(n)dn \\
&+ Pr_3 \int_{2(K_1+K_f)}^{\infty} bs_1(1 - \frac{2(K_1+K_f)}{n}) f_N(n)dn + Pr_4 \int_{2K_1}^{\infty} bs_1(1 - \frac{2K_1}{n}) f_N(n)dn = c_1 - v_1 \qquad \text{(C.1a)}
\end{aligned}$$

$$Pr_1 \int_{2(K_2+K_f)}^{2(K_1+K_2+K_f)} b(s_2 - s_1)(1 - \frac{2(K_2+K_f)}{n}) f_N(n)dn$$
$$+ Pr_1 \int_{2(K_1+K_2+K_f)}^{\infty} [bs_2 - \frac{2b(K_2 s_2 + K_f s_2 + K_1 s_1)}{n}] f_N(n)dn$$
$$+ Pr_2 \int_{2K_2}^{2(K_1+K_2)} b(s_2 - s_1)(1 - \frac{2K_2}{n}) f_N(n)dn + Pr_2 \int_{2(K_1+K_2)}^{\infty} [bs_2 - \frac{2b(K_2 s_2 + K_1 s_1)}{n}] f_N(n)dn$$
$$+ Pr_5 \int_{2(K_2+K_f)}^{\infty} [bs_2 - \frac{2bs_2(K_2+K_f)}{n}] f_N(n)dn + Pr_6 \int_{2K_2}^{\infty} [bs_2 - \frac{2bs_2 K_2}{n}] f_N(n)dn = c_2 - v_2 \quad \text{(C.1b)}$$

$$Pr_1 \int_{2(K_2+K_f)}^{2(K_1+K_2+K_f)} b(s_2 - s_1)(1 - \frac{2(K_2+K_f)}{n}) f_N(n)dn$$
$$+ Pr_1 \int_{2(K_1+K_2+K_f)}^{\infty} [bs_2 - \frac{2b(K_2 s_2 + K_f s_2 + K_1 s_1)}{n}] f_N(n)dn$$
$$+ Pr_3 \int_{2K_f}^{2(K_1+K_f)} b(s_2 - s_1)(1 - \frac{2K_f}{n}) f_N(n)dn + Pr_3 \int_{2(K_1+K_f)}^{\infty} [bs_2 - \frac{2b(K_f s_2 + K_1 s_1)}{n}] f_N(n)dn$$
$$+ Pr_5 \int_{2(K_2+K_f)}^{\infty} [bs_2 - \frac{2bs_2(K_2+K_f)}{n}] f_N(n)dn + Pr_7 \int_{2K_f}^{\infty} [bs_2 - \frac{2bs_2 K_f}{n}] f_N(n)dn = c_f - v_f \quad \text{(C.1c)}$$

$$v_i K_i = 0, \quad i = 1, 2, f. \quad \text{(C.1d)}$$

**Proof:** We derive

$$\frac{\partial^2 V(\overrightarrow{K})}{\partial K_1^2} = -2bs_1 Pr_1 \int_{2(K_1+K_2+K_f)}^{\infty} \frac{1}{n} f_N(n)dn - 2bs_1 Pr_2 \int_{2(K_1+K_2)}^{\infty} \frac{1}{n} f_N(n)dn$$
$$- 2bs_1 Pr_3 \int_{2(K_1+K_f)}^{\infty} \frac{1}{n} f_N(n)dn - 2bs_1 Pr_4 \int_{2K_1}^{\infty} \frac{1}{n} f_N(n)dn$$
$$< 0,$$

$$\frac{\partial^2 V(\overrightarrow{K})}{\partial K_2^2} = -2b(s_2 - s_1) Pr_1 \int_{2(K_2+K_f)}^{2(K_1+K_2+K_f)} \frac{1}{n} f_N(n)dn - 2bs_2 Pr_1 \int_{2(K_1+K_2+K_f)}^{\infty} \frac{1}{n} f_N(n)dn$$
$$- 2b(s_2 - s_1) Pr_2 \int_{2K_2}^{2(K_1+K_2)} \frac{1}{n} f_N(n)dn - 2bs_2 Pr_2 \int_{2(K_1+K_2)}^{\infty} \frac{1}{n} f_N(n)dn$$
$$- 2bs_2 Pr_5 \int_{2(K_2+K_f)}^{\infty} \frac{1}{n} f_N(n)dn - 2bs_2 Pr_6 \int_{2K_2}^{\infty} \frac{1}{n} f_N(n)dn$$
$$< 0,$$

$$\frac{\partial^2 V(\overrightarrow{K})}{\partial K_f^2} = -2b(s_2-s_1)Pr_1\int_{2(K_2+K_f)}^{2(K_1+K_2+K_f)}\frac{1}{n}f_N(n)dn - 2bs_2Pr_1\int_{2(K_1+K_2+K_f)}^{\infty}\frac{1}{n}f_N(n)dn$$
$$- 2b(s_2-s_1)Pr_3\int_{2K_f}^{2(K_1+K_f)}\frac{1}{n}f_N(n)dn - 2bs_2Pr_3\int_{2(K_1+K_f)}^{\infty}\frac{1}{n}f_N(n)dn$$
$$- 2bs_2Pr_5\int_{2(K_2+K_f)}^{\infty}\frac{1}{n}f_N(n)dn - 2bs_2Pr_7\int_{2K_f}^{\infty}\frac{1}{n}f_N(n)dn$$
$$< 0,$$

$$\frac{\partial^2 V(\overrightarrow{K})}{\partial K_1\partial K_2} = \frac{\partial^2 V(\overrightarrow{K})}{\partial K_2\partial K_1} = -2bs_1Pr_1\int_{2(K_1+K_2+K_f)}^{\infty}\frac{1}{n}f_N(n)dn - 2bs_1Pr_2\int_{2(K_1+K_2)}^{\infty}\frac{1}{n}f_N(n)dn,$$
$$\frac{\partial^2 V(\overrightarrow{K})}{\partial K_1\partial K_f} = \frac{\partial^2 V(\overrightarrow{K})}{\partial K_f\partial K_1} = -2bs_1Pr_1\int_{2(K_1+K_2+K_f)}^{\infty}\frac{1}{n}f_N(n)dn - 2bs_1Pr_3\int_{2(K_1+K_f)}^{\infty}\frac{1}{n}f_N(n)dn,$$
$$\frac{\partial^2 V(\overrightarrow{K})}{\partial K_2\partial K_f} = \frac{\partial^2 V(\overrightarrow{K})}{\partial K_f\partial K_2} = -2b(s_2-s_1)Pr_1\int_{2(K_2+K_f)}^{2(K_1+K_2+K_f)}\frac{1}{n}f_N(n)dn$$
$$- 2bs_2Pr_1\int_{2(K_1+K_2+K_f)}^{\infty}\frac{1}{n}f_N(n)dn - 2bs_2Pr_5\int_{2(K_2+K_f)}^{\infty}\frac{1}{n}f_N(n)dn.$$

Let

$$d \equiv Pr_1\int_{2(K_1+K_2+K_f)}^{\infty}\frac{1}{n}f_N(n)dn, \qquad e \equiv Pr_2\int_{2(K_1+K_2)}^{\infty}\frac{1}{n}f_N(n)dn,$$
$$r \equiv Pr_3\int_{2(K_1+K_f)}^{\infty}\frac{1}{n}f_N(n)dn, \qquad g \equiv Pr_4\int_{2K_1}^{\infty}\frac{1}{n}f_N(n)dn$$
$$h \equiv Pr_1\int_{2(K_2+K_f)}^{2(K_1+K_2+K_f)}\frac{1}{n}f_N(n)dn, \qquad i \equiv Pr_2\int_{2K_2}^{2(K_1+K_2)}\frac{1}{n}f_N(n)dn,$$
$$j \equiv Pr_5\int_{2(K_2+K_f)}^{\infty}\frac{1}{n}f_N(n)dn, \qquad k \equiv Pr_6\int_{2K_2}^{\infty}\frac{1}{n}f_N(n)dn,$$
$$l \equiv Pr_3\int_{2K_f}^{2(K_1+K_f)}\frac{1}{n}f_N(n)dn, \qquad m \equiv Pr_7\int_{2K_f}^{\infty}\frac{1}{n}f_N(n)dn.$$

Note that $d, e, r, g, h, i, j, k, l, m \geq 0$. Then, the Hessian Matrix of the objective function with respect to $K_1, K_2$, and $K_f$ is given as

$$\mathbf{H}(\mathbf{V}(\overrightarrow{K})) = -2b\begin{bmatrix} s_1(d+e+r+g) & s_1(d+e) & s_1(d+r) \\ s_1(d+e) & \begin{matrix} s_2(d+e+h+i+j+k) \\ -s_1(h+i)\end{matrix} & s_2(h+d+j)-s_1h \\ s_1(d+r) & s_2(h+d+j)-s_1h & \begin{matrix} s_2(d+r+h+j+m+l) \\ -s_1(h+l)\end{matrix}\end{bmatrix}$$

$$= -2b \begin{bmatrix} s_1(d+e+r+g) & s_1(d+e) & s_1(d+r) \\ 0 & \begin{array}{c} s_2(d+e+h+i+j+k) \\ -s_1(h+i) - \frac{s_1(d+e)^2}{(d+e+r+g)} \end{array} & \begin{array}{c} s_2(h+d+j) - s_1 h \\ -\frac{s_1(d+e)(d+r)}{(d+e+r+g)} \end{array} \\ 0 & \begin{array}{c} s_2(h+d+j) - s_1 h \\ -\frac{s_1(d+e)(d+r)}{(d+e+r+g)} \end{array} & \begin{array}{c} s_2(d+r+h+j+m+l) \\ -s_1(h+l) - \frac{s_1(d+r)^2}{(d+e+r+g)} \end{array} \end{bmatrix}.$$

Obviously, $H_{11} = s_1(d+e+r+g) > 0$. We also know

$$\begin{aligned} H_{22} &= s_2(j+k) + (s_2 - s_1)(h+i) + \frac{(s_2 - s_1)(d+e)^2 + s_2(d+e)(r+g)}{(d+e+r+g)} \\ &> 0. \end{aligned}$$

Similarly, we can show that

$$s_2(d+r+h+l+j+m) - s_1(h+l) - \frac{s_1(h+r)^2}{(d+e+r+g)} > 0.$$

Notice that

$$\begin{aligned} H_{22} - H_{23} &= (s_2 - s_1)i + \frac{[s_2 k(d+e+r+g) + (s_2 - s_1)(d+e)e + s_2 e(r+g) + s_1 r(d+e)]}{(d+e+r+g)} \\ &> 0. \end{aligned}$$

Similarly,

$$\begin{aligned} H_{33} - H_{23} &= (s_2 - s_1)l + \frac{[s_2 m(d+e+r+g) + (s_2 - s_1)(d+e)r + s_2 r(r+g) + s_1 e(d+e)]}{(d+e+r+g)} \\ &> 0. \end{aligned}$$

Hence, $det(H) > 0$, and $\mathbf{H}(\mathbf{V}(\overrightarrow{K}))$ is negative definite. Therefore, $V(\overrightarrow{K})$ is strictly, jointly concave in $\overrightarrow{K}$, and the first-order KKT conditions are necessary and sufficient for optimality. The result follows. □

**Proposition 15.** The unique optimal investment vector $(K_1^D, K_2^D)$ in the dedicated-only (S+D)U system can be characterized as follows (see Figure 4.2(b)):

$(A^d)$: If $c_2 \le \theta_2 c_1 + \theta_2 b(s_2 - \theta_1 s_1)$ and $c_1 \le \frac{\theta_1 s_1}{s_2} c_2 + \theta_1(1-\theta_2) b s_1$, then $K_1^D$ and $K_2^D$ are the unique solutions to:

$$\theta_1\theta_2 \int_{2(K_1+K_2)}^{\infty} bs_1[1 - \frac{2(K_1+K_2)}{n}] f_N(n)dn + \theta_1(1-\theta_2)\int_{2K_1}^{\infty} bs_1[1-\frac{2K_1}{n}] f_N(n)dn = c_1$$

$$\theta_1\theta_2 \int_{2K_2}^{2(K_1+K_2)} b(s_2-s_1)(1-\frac{2K_2}{n}) f_N(n)dn + \theta_1\theta_2 \int_{2(K_1+K_2)}^{\infty} [bs_2 - 2b\frac{(K_2s_2+K_1s_1)}{n}] f_N(n)dn$$
$$+ (1-\theta_1)\theta_2 \int_{2K_2}^{\infty} bs_2(1-\frac{2K_2}{n}) f_N(n)dn = c_2.$$

$(B^d)$: If $c_2 \le \theta_2 b s_2$ and $c_1 > \frac{\theta_1 s_1}{s_2} c_2 + \theta_1(1-\theta_2) b s_1$, then $K_1^D = 0$ and $K_2^D$ is the unique solution to $\theta_2 \int_{2K_2}^{\infty} bs_2(1-\frac{2K_2}{n}) f_N(n)dn = c_2$.

$(C^d)$: If $c_1 \le \theta_1 b s_1$ and $c_2 > \theta_2 c_1 + \theta_2 b(s_2 - \theta_1 s_1)$, then $K_2^D = 0$ and $K_1^D$ is the unique solution to $\theta_1 \int_{2K_1}^{\infty} bs_1(1-\frac{2K_1}{n}) f_N(n)dn = c_1$.

$(D^d)$: If $c_1 \ge \theta_1 b s_1$ and $c_2 \ge \theta_2 b s_2$, then $K_1^D = 0$ and $K_2^D = 0$.

**Proof:** We have the following four cases.

**Case 1.** $v_1 = 0$ and $v_2 = 0$ ($\Rightarrow K_1 \ge 0$ and $K_2 \ge 0$):

In this case, (C.1a) and (C.1b) reduce to

$$\theta_1\theta_2 \int_{2(K_1+K_2)}^{\infty} bs_1[1 - \frac{2(K_1+K_2)}{n}] f_N(n)dn + \theta_1(1-\theta_2)\int_{2K_1}^{\infty} bs_1(1-\frac{2K_1}{n}) f_N(n)dn = c_1 \quad \text{(C.2)}$$

$$\theta_1\theta_2 \int_{2K_2}^{2(K_1+K_2)} b(s_2-s_1)(1-\frac{2K_2}{n}) f_N(n)dn + (1-\theta_1)\theta_2 \int_{2K_2}^{\infty} bs_2(1-\frac{2K_2}{n}) f_N(n)dn$$
$$+ \theta_1\theta_2 \int_{2(K_1+K_2)}^{\infty} [bs_2 - 2b\frac{(K_2s_2+K_1s_1)}{n}] f_N(n)dn = c_2. \quad \text{(C.3)}$$

The left-hand-side (LHS) of (C.2) is strictly decreasing in $K_1$. Then, from $K_1 \ge 0$, we have

$$c_1 \le \theta_1\theta_2 \int_{2K_2}^{\infty} bs_1(1-\frac{2K_2}{n}) f_N(n)dn + \theta_1(1-\theta_2) b s_1.$$

The LHS of (C.3) is strictly decreasing in $K_1$. Then, from $K_1 \ge 0$, we have

$$\theta_2 \int_{2K_2}^{\infty} bs_2(1-\frac{2K_2}{n}) f_N(n)dn \ge c_2.$$

Therefore,

$$c_1 \le \frac{\theta_1 s_1 c_2}{s_2} + \theta_1(1-\theta_2) b s_1.$$

The LHS of (C.2) is also strictly decreasing in $K_2$. Then, from (C.2), we have

$$\theta_1 \int_{2K_1}^{\infty} bs_1(1 - \frac{2K_1}{n}) f_N(n) dn \geq c_1.$$

From (C.3) - (C.2), we have

$$\theta_1\theta_2 \int_{2K_2}^{\infty} b(s_2 - s_1)(1 - \frac{2K_2}{n}) f_N(n) dn + (1-\theta_1)\theta_2 \int_{2K_2}^{\infty} bs_2(1 - \frac{2K_2}{n}) f_N(n) dn$$
$$- \theta_1(1-\theta_2) \int_{2K_1}^{\infty} bs_1(1 - \frac{2K_1}{n}) f_N(n) dn = c_2 - c_1. \tag{C.4}$$

Note that the left-hand-side of (C.4) is strictly decreasing in $K_2$. Then, from $K_2 \geq 0$, we have

$$c_2 - c_1 \leq \theta_1\theta_2 b(s_2 - s_1) + (1-\theta_1)\theta_2 bs_2 - \theta_1(1-\theta_2) \int_{2K_1}^{\infty} bs_1(1 - \frac{2K_1}{n}) f_N(n) dn.$$

Therefore,

$$c_2 - c_1 \leq \theta_1\theta_2 b(s_2 - s_1) + (1-\theta_1)\theta_2 bs_2 - (1-\theta_2)c_1$$
$$\Rightarrow c_2 \leq \theta_2 c_1 + \theta_2 b(s_2 - \theta_1 s_1).$$

**Case 2.** $v_1 > 0$ and $v_2 = 0$ ($\Rightarrow K_1 = 0$ and $K_2 \geq 0$):

In this case, (C.1a) and (C.1b) become

$$\theta_1\theta_2 \int_{2K_2}^{\infty} bs_1(1 - \frac{2K_2}{n}) f_N(n) dn + \theta_1(1-\theta_2) bs_1 + v_1 = c_1 \tag{C.5}$$
$$\theta_2 \int_{2K_2}^{\infty} bs_2(1 - \frac{2K_2}{n}) f_N(n) dn = c_2. \tag{C.6}$$

From (C.5) and $K_2 \geq 0$, we have $c_2 \leq \theta_2 bs_2$ and

$$\int_{2K_2}^{\infty} (1 - \frac{2K_2}{n}) f_N(n) dn = \frac{c_2}{\theta_2 bs_2}.$$

Therefore, we have

$$v_1 = c_1 - \frac{\theta_1 s_1 c_2}{s_2} - \theta_1(1-\theta_2) bs_1 > 0$$
$$\Rightarrow c_1 > \frac{\theta_1 s_1 c_2}{s_2} - \theta_1(1-\theta_2) bs_1.$$

**Case 3.** $v_1 = 0$ and $v_2 > 0$ ($\Rightarrow K_1 \geq 0$ and $K_2 = 0$):

In this case, (C.1a) and (C.1b) become

$$\theta_1 \int_{2K_1}^{\infty} bs_1(1 - \frac{2K_1}{n}) f_N(n) dn = c_1 \tag{C.7}$$

$$\theta_1\theta_2 \int_0^{2K_1} b(s_2 - s_1) f_N(n) dn + \theta_1\theta_2 \int_{2K_1}^{\infty} (bs_2 - \frac{2bs_1K_1}{n}) f_N(n) dn + (1-\theta_1)\theta_2 bs_2 + v_2 = c_2. \tag{C.8}$$

From (C.7) and $K_1 \geq 0$, we have $c_1 \leq \theta_1 bs_1$ and

$$\int_{2K_1}^{\infty} (1 - \frac{2K_1}{n}) f_N(n) dn = \frac{c_1}{\theta_1 bs_1}.$$

Therefore, from (C.7) and (C.8), we have

$$\begin{aligned} &v_2 = c_2 - \theta_2 c_1 + \theta_1\theta_2 bs_1 - \theta_2 bs_2 > 0 \\ &\Rightarrow c_2 > \theta_2 c_1 + \theta_2 b(s_2 - \theta_1 s_1). \end{aligned}$$

**Case 4.** $v_1 > 0$ and $v_2 > 0$ ($\Rightarrow K_1 = 0$ and $K_2 = 0$):

In this case, (C.1a) and (C.1b) become

$$\begin{aligned} \theta_1 bs_1 + v_1 &= c_1 \\ \theta_2 bs_2 + v_2 &= c_2. \end{aligned}$$

Therefore, from $v_1 > 0$ and $v_2 > 0$, we need $c_1 > \theta_1 bs_1$ and $c_2 > \theta_2 bs_2$. □

**Proposition 16.** The unique optimal investment vector $(K_1^D, K_2^D)$ in the dedicated-only SU system can be characterized as follows see Table 4.1 and Figure 4.2(b):

($A_1^d$): If $\frac{c_1}{\theta(1-\theta)bs_1} + \frac{c_2}{\theta b(s_2-\theta s_1)} < 1$, then $K_1^D = \frac{n}{2}[1 - \frac{c_1}{\theta(1-\theta)bs_1}]$ and $K_2^D = \frac{n}{2}[1 - \frac{c_2}{\theta b(s_2-\theta s_1)}]$.

($A_2^d$): If $c_2 \leq \theta c_1 + \theta b(s_2 - \theta s_1)$, $c_1 \leq \frac{\theta s_1}{s_2} c_2 + \theta(1-\theta)bs_1$, and $\frac{c_1}{\theta(1-\theta)bs_1} + \frac{c_2}{\theta b(s_2-\theta s_1)} \leq 1$, then $K_1^D = \frac{ns_2}{2\theta s_1}[1 - \frac{\theta b(s_2-\theta s_1)-c_2+\theta s_1}{\theta b(s_2-\theta^2 s_1)}] - \frac{nc_2}{2\theta^2 bs_1}$ and $K_2^D = \frac{n[\theta b(s_2-\theta s_1)-c_2+\theta s_1]}{2\theta b(s_2-\theta^2 s_1)}$.

($C^d$): If $c_2 \leq \theta bs_2$ and $c_1 > \frac{\theta s_1}{s_2} c_2 + \theta(1-\theta)bs_1$, then $K_1^D = 0$ and $K_2^D = \frac{n}{2}(1 - \frac{c_2}{\theta bs_2})$.

($D^d$): If $c_1 \leq \theta bs_1$ and $c_2 > \theta c_1 + \theta b(s_2 - \theta s_1)$, then $K_1^D = \frac{n}{2}(1 - \frac{c_1}{\theta bs_1})$ and $K_2^D = 0$.

($E^d$): If $c_1 > \theta bs_1$ and $c_2 > \theta bs_2$, then $K_1^D = 0$ and $K_2^D = 0$.

**Proof:** The solution of the Stage 2 problem of **P((S+D)U)** dedicated-only system (see Proposition 3) are giving as follows:

$$(p_1^*, p_2^*) = \begin{cases} (\frac{bs_1}{2},\ \frac{bs_2}{2}), & \text{if } \Omega_1, \\ (\frac{bs_1}{2},\ b(s_2 - s_1)(1 - \frac{y_2K_2}{n}) + \frac{bs_1}{2}), & \text{if } \Omega_5, \\ (bs_1 - \frac{bs_1(y_1K_1+y_2K_2)}{n},\ \ bs_2 - \frac{by_2K_2s_2}{n} - \frac{by_1K_1s_1}{n}), & \text{if } \Omega_6, \end{cases}$$

$$(q_1^*, q_2^*) = \begin{cases} (0,\ \frac{n}{2}), & \text{if } \Omega_1, \\ (\frac{n}{2} - y_2K_2,\ y_2K_2), & \text{if } \Omega_5, \\ (y_1K_1,\ \ y_2K_2), & \text{if } \Omega_6, \end{cases}$$

where

$$\begin{aligned} \Omega_1 &= \{n \le 2y_2K_2\}, \\ \Omega_5 &= \{2y_2K_2 \le n \le 2(y_1K_1 + y_2K_2)\}, \\ \Omega_6 &= \{n \ge 2(y_1K_1 + y_2K_2)\}. \end{aligned}$$

Therefore, with the realization of $\overrightarrow{Y}$, we need to consider the following four cases:

Case I: $(y_1 = 1, y_2 = 1)$

(a) When $K_2 \ge \frac{n}{2}$, $\Pi_1 = \frac{nbs_2}{4}$;

(b) When $K_2 < \frac{n}{2}$ and $K_1 + K_2 \ge \frac{n}{2}$, $\Pi_1 = \frac{nbs_1}{4} + b(s_2 - s_1)(1 - \frac{K_2}{n})K_2$;

(c) When $K_1 + K_2 < \frac{n}{2}$, $\Pi_1 = bs_1(1 - \frac{(K_1+K_2)}{n})K_1 + (bs_2 - \frac{bs_2K_2}{n} - \frac{bs_1K_1}{n})K_2$.

Case II: $(y_1 = 0, y_2 = 1)$

(a) When $K_2 \ge \frac{n}{2}$, $\Pi_2 = \frac{nbs_2}{4}$;

(b) When $K_2 < \frac{n}{2}$, $\Pi_2 = bs_2(1 - \frac{K_2}{n})K_2$.

Case III: $(y_1 = 1, y_2 = 0)$

(a) When $K_1 \ge \frac{n}{2}$, $\Pi_3 = \frac{nbs_1}{4}$;

(b) When $K_2 < \frac{n}{2}$, $\Pi_3 = bs_1(1 - \frac{K_1}{n})K_1$.

Case IV: $(y_1 = 0, y_2 = 0)$

(a) $\Pi_4 = 0$.

Now we start to solve the stage 1 problem. With different values of $K_1$ and $K_2$, the objective function has five different forms.

Case 1: $K_1 \geq \frac{n}{2}, K_2 \geq \frac{n}{2}$

We have $V(\overrightarrow{K}) = \theta\frac{nbs_2}{4} + \theta(1-\theta)\frac{nbs_1}{4} - c_1K_1 - c_2K_2$, $\frac{\partial V}{\partial K_1} = -c_1 < 0$, and $\frac{\partial V}{\partial K_2} = -c_2 < 0$. Hence, the optimal solution is given by $K_1 = K_2 = \frac{n}{2}$.

Case 2: $K_1 < \frac{n}{2}, K_2 \geq \frac{n}{2}$

We have

$$V(\overrightarrow{K}) = \theta\frac{nbs_2}{4} + \theta(1-\theta)bs_1(1-\frac{K_1}{n})K_1 - c_1K_1 - c_2K_2,$$
$$\frac{\partial V}{\partial K_1} = \theta(1-\theta)bs_1(1-\frac{2K_1}{n}) - c_1, \quad \frac{\partial^2 V}{\partial K_1^2} = -\frac{2\theta(1-\theta)bs_1}{n} < 0, \quad \frac{\partial V}{\partial K_2} = -c_2 < 0.$$

Then, the optimal solution is given by $K_1 = \frac{n}{2}[1 - \frac{c_1}{\theta(1-\theta)bs_1}], K_2 = \frac{n}{2}$.

Case 3: $K_1 \geq \frac{n}{2}, K_2 < \frac{n}{2}$

We have

$$V(\overrightarrow{K}) = \theta\frac{nbs_1}{4} + \theta b(s_2 - \theta s_1)(1-\frac{K_2}{n})K_2 - c_1K_1 - c_2K_2,$$
$$\frac{\partial V}{\partial K_1} = -c_1 < 0, \quad \frac{\partial V}{\partial K_2} = \theta b(1-\frac{2K_2}{n})(s_2 - \theta s_1) - c_2, \quad \frac{\partial^2 V}{\partial K_2^2} = -\frac{2\theta b(s_2 - \theta s_1)}{n} < 0.$$

Then, the optimal solution is given by $K_1 = \frac{n}{2}, K_2 = \frac{n}{2}[1 - \frac{c_2}{\theta b(s_2 - \theta s_1)}]$.

Case 4: $K_1 < \frac{n}{2}, K_2 < \frac{n}{2}, K_1 + K_2 \geq \frac{n}{2}$

We have

$$V(\overrightarrow{K}) = \theta^2\frac{nbs_1}{4} + \theta b(s_2 - \theta s_1)(1-\frac{K_2}{n})K_2 + \theta(1-\theta)bs_1(1-\frac{K_1}{n})K_1 - c_1K_1 - c_2K_2,$$
$$\frac{\partial V}{\partial K_1} = \theta(1-\theta)bs_1(1-\frac{2K_1}{n}) - c_1, \quad \frac{\partial^2 V}{\partial K_1^2} = -\frac{2\theta(1-\theta)bs_1}{n} < 0,$$
$$\frac{\partial V}{\partial K_2} = \theta b(1-\frac{2K_2}{n})(s_2 - \theta s_1) - c_2, \quad \frac{\partial^2 V}{\partial K_2^2} = -\frac{2\theta b(s_2 - \theta s_1)}{n} < 0.$$

Then, the optimal solution is given by $K_1 = \frac{n}{2}[1 - \frac{c_1}{\theta(1-\theta)bs_1}], K_2 = \frac{n}{2}[1 - \frac{c_2}{\theta b(s_2-\theta s_1)}]$.

<u>Case 5: $K_1 < \frac{n}{2}, K_2 < \frac{n}{2}, K_1 + K_2 < \frac{n}{2}$</u>

We have

$$\begin{aligned}
V(\overrightarrow{K}) &= \theta^2[bs_1 - \frac{bs_1(K_1+K_2)}{n}]K_1 + \theta^2[bs_2 - \frac{bs_2K_2}{n} - \frac{bs_1K_1}{n}]K_2 \\
&\quad + \theta(1-\theta)(bs_2 - \frac{bs_2K_2}{n})K2 + \theta(1-\theta)(bs_1 - \frac{bs_1K_1}{n})K_1 - c_1K_1 - c_2K_2, \\
\frac{\partial V}{\partial K_1} &= \theta bs_1(1 - \frac{2K_1}{n}) - \theta^2\frac{2bs_1K_2}{n} - c_1, \quad \frac{\partial^2 V}{\partial K_1^2} = -\frac{2\theta bs_1}{n}, \quad \frac{\partial^2 V}{\partial K_1 \partial K_2} = -\frac{2\theta^2 bs_1}{n}, \\
\frac{\partial V}{\partial K_2} &= \theta bs_2(1 - \frac{2K_2}{n}) - \theta^2\frac{2bs_1K_1}{n} - c_2, \quad \frac{\partial^2 V}{\partial K_2^2} = -\frac{2\theta bs_2}{n}.
\end{aligned}$$

We derive the Hessian matrix, $\mathbf{H}(\mathbf{V}(\overrightarrow{K}))$, with respect to (w.r.t.) $K_1$ and $K_2$:

$$\mathbf{H}(\mathbf{V}(\overrightarrow{K})) = \frac{2b\theta}{n}\begin{bmatrix} -s_1 & -\theta s_1 \\ -\theta s_1 & -s_2 \end{bmatrix}.$$

Since $s_2 > s_1$, we know $-s_1 < 0$, $-s_2 < 0$, and $s_1 s_2 - \theta^2 s_1^2 > 0$. Therefore, $V(\overrightarrow{K})$ is strictly, jointly concave in $K_1$ and $K_2$. The optimal solution is given by

$$K_1 = \frac{ns_2}{2\theta s_1}[1 - \frac{\theta b(s_2 - \theta s_1) - c_2 + \theta c_1}{\theta b(s_2 - \theta^2 s_1)}] - \frac{nc_2}{2\theta^2 bs_1}, \quad K_2 = \frac{n}{2}[\frac{\theta b(s_2 - \theta s_1) - c_2 + \theta c_1}{\theta b(s_2 - \theta^2 s_1)}].$$

Note that the optimal solutions of Cases 1, 2, and 3 lie on the boundary of Case 4. Therefore, we just need to consider Cases 4 and 5 for solving the stage 1 problem. Next, we will solve the stage 1 problem for Cases 4 and 5 separately, and then compare their solutions, the better solution of which will be the global optimal solution.

The problem formulation of Case 4 is given as follows:

$$V(\overrightarrow{K}) = \theta^2\frac{nbs_1}{4} + \theta b(s_2 - \theta s_1)(1 - \frac{K_2}{n})K_2 + \theta(1-\theta)bs_1(1 - \frac{K_1}{n})K_1 - c_1K_1 - c_2K_2 \quad \text{(C.9a)}$$

subject to

$$K_1 \le \frac{n}{2} \qquad \longleftarrow v_1 \qquad \text{(C.9b)}$$

$$K_2 \le \frac{n}{2} \qquad \longleftarrow v_2 \qquad \text{(C.9c)}$$

$$K_1 + K_2 \ge \frac{n}{2} \qquad \longleftarrow v_3 \qquad \text{(C.9d)}$$

where $v_1$, $v_2$, and $v_3$ denote the KKT multipliers respectively corresponding to constraints (C.9b),(C.9c), and (C.9d). The first-order KKT conditions are given as:

$$\theta(1-\theta)bs_1(1-\frac{2K_1}{n}) - v_1 + v_3 = c_1 \tag{C.10a}$$

$$b(s_2-\theta s_1)(1-\frac{2K_2}{n}) - v_2 + v_3 = c_2 \tag{C.10b}$$

$$v_1(\frac{n}{2} - K_1) = 0 \tag{C.10c}$$

$$v_2(\frac{n}{2} - K_2) = 0 \tag{C.10d}$$

$$v_3(K_1 + K_2 - \frac{n}{2}) = 0 \tag{C.10e}$$

$$v_i \geq 0, i = 1, 2, 3. \tag{C.10f}$$

We solve the Case 4 problem and characterize the optimal solution in Table C (see also Figures C.1(a) and C.2(a)).

Table C.1: Optimal Solution for Case 4 of the SU Dedicated-only System.

| | Necessary and Sufficient Conditions | Solution |
|---|---|---|
| $A_4$ | $c_1 \geq 0, \quad c_2 \geq 0$<br>$\frac{c_1}{\theta(1-\theta)bs_1} + \frac{c_2}{\theta b(s_2-\theta s_1)} \leq 1$ | $K_1 = \frac{n}{2}[1 - \frac{c_1}{\theta(1-\theta)bs_1}]$<br>$K_2 = \frac{n}{2}[1 - \frac{c_2}{\theta b(s_2-\theta s_1)}]$ |
| $B_4$ | $c_2 \geq c_1 - \theta(1-\theta)bs_1$<br>$c_2 \leq c_1 + \theta b(s_2 - \theta s_1)$<br>$\frac{c_1}{\theta(1-\theta)bs_1} + \frac{c_2}{\theta b(s_2-\theta s_1)} > 1$ | $K_1 = \frac{n}{2}[\frac{c_2-c_1+\theta(1-\theta)bs_1}{\theta b(s_2+s_1-2\theta s_1)}]$<br>$K_2 = \frac{n}{2}[1 - \frac{c_2-c_1+\theta(1-\theta)bs_1}{\theta b(s_2+s_1-2\theta s_1)}]$ |
| $C_4$ | $c_1 \geq 0, \quad c_2 < 0$<br>$c_1 \leq \theta(1-\theta)bs_1$ | $K_1 = \frac{n}{2}[1 - \frac{c_1}{\theta(1-\theta)bs_1}]$<br>$K_2 = \frac{n}{2}$ |
| $D_4$ | $c_1 > \theta(1-\theta)bs_1$<br>$c_2 < c_1 - \theta(1-ta)bs_1$ | $K_1 = 0$<br>$K_2 = \frac{n}{2}$ |
| $E_4$ | $c_1 < 0, \quad c_2 \geq 0$<br>$c_2 \leq \theta b(s_2 - \theta s_1)$ | $K_1 = \frac{n}{2}$<br>$K_2 = \frac{n}{2}[1 - \frac{c_2}{\theta b(s_2-\theta s_1)}]$ |
| $F_4$ | $c_2 > \theta b(s_2 - \theta s_1)$<br>$c_2 > c_1 + \theta b(s_2 - \theta s_1)$ | $K_1 = \frac{n}{2}$<br>$K_2 = 0$ |
| $G_4$ | $c_1 < 0, \quad c_2 < 0$ | $K_1 = \frac{n}{2}, \; K_2 = \frac{n}{2}$ |

Similarly, the problem formulation of Case 5 is given as follows:

$$V(\overrightarrow{K}) = \theta^2[bs_1 - \frac{bs_1(K_1+K_2)}{n}]K_1 + \theta^2[bs_2 - \frac{bs_2K_2}{n} - \frac{bs_1K_1}{n}]K_2$$
$$+\theta(1-\theta)(bs_2 - \frac{bs_2K_2}{n})K2 + \theta(1-\theta)(bs_1 - \frac{bs_1K_1}{n})K_1 - c_1K_1 - c_2K_2 \quad \text{(C.11a)}$$

subject to

$$K_1 \geq 0 \qquad \longleftarrow v_1 \qquad \text{(C.11b)}$$
$$K_2 \geq 0 \qquad \longleftarrow v_2 \qquad \text{(C.11c)}$$
$$\frac{n}{2} - (K_1+K_2) \geq 0 \qquad \longleftarrow v_3 \qquad \text{(C.11d)}$$

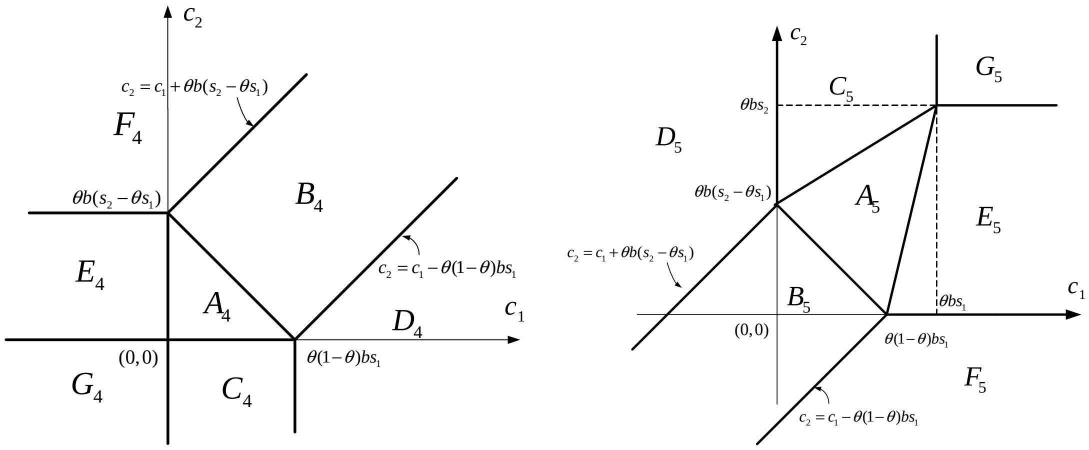
where $v_1$, $v_2$, and $v_3$ denote the KKT multipliers respectively corresponding to constraints (C.11b),(C.11c), and (C.11d). The first-order KKT conditions are given as:

$$\theta bs_1(1 - \frac{2K_1}{n}) - \theta^2 bs_1\frac{2K_2}{n} + v_1 - v_3 = c_1 \quad \text{(C.12a)}$$
$$\theta bs_2(1 - \frac{2K_2}{n}) - \theta^2 bs_1\frac{2K_1}{n} + v_2 - v_3 = c_2 \quad \text{(C.12b)}$$
$$v_1K_1 = 0 \quad \text{(C.12c)}$$
$$v_2K_2 = 0 \quad \text{(C.12d)}$$
$$v_3(\frac{n}{2} - (K_1+K_2)) = 0 \quad \text{(C.12e)}$$
$$v_i \geq 0, i = 1,2,3. \quad \text{(C.12f)}$$

Table C.2: Optimal Solution for Case 5 of the SU Dedicated-only System.

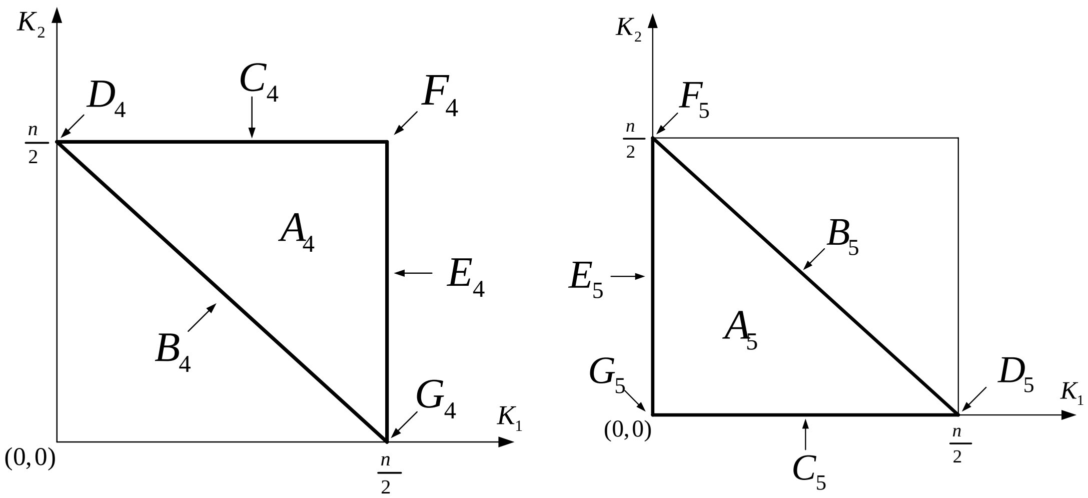
| | Necessary and Sufficient Conditions | Solution |
|---|---|---|
| $A_5$ | $\theta s_1c_2 \geq s_2c_1 - \theta s_1(1-\theta)bs_2$<br>$c_2 \leq \theta c_1 + \theta b(s_2 - \theta s_1)$<br>$\frac{c_1}{\theta(1-\theta)bs_1} + \frac{c_2}{\theta b(s_2-\theta s_1)} > 1$ | $K_1 = \frac{ns_2}{2\theta s_1}[1 - \frac{\theta b(s_2-\theta s_1)-c_2+\theta c_1}{\theta b(s_2-\theta^2 s_1)}] - \frac{nc_2}{2\theta^2 bs_1}$<br>$K_2 = \frac{n}{2}[\frac{\theta b(s_2-\theta s_1)-c_2+\theta c_1}{\theta b(s_2-\theta^2 s_1)}]$ |
| $B_5$ | $c_2 \geq c_1 - \theta(1-\theta)bs_1$<br>$c_2 \leq c_1 + \theta b(s_2 - \theta s_1)$<br>$\frac{c_1}{\theta(1-\theta)bs_1} + \frac{c_2}{\theta b(s_2-\theta s_1)} \leq 1$ | $K_1 = \frac{n}{2}[1 - \frac{(c_1-c_2+\theta bs_2-\theta^2 bs_1)}{\theta b(s_2+s_1-2\theta s_1)}]$<br>$K_2 = \frac{n}{2}[\frac{c_1-c_2+\theta bs_2-\theta^2 bs_1}{\theta b(s_2+s_1-2\theta s_1)}]$ |
| $C_5$ | $c_1 \geq 0, \quad c_1 \leq \theta bs_1$<br>$c_2 > \theta c_1 + \theta b(s_2 - \theta s_1)$ | $K_1 = \frac{n}{2}(1 - \frac{c_1}{\theta bs_1})$<br>$K_2 = 0$ |
| $D_5$ | $c_1 < 0, \quad c_2 > c_1 + \theta b(s_2 - \theta s_1)$ | $K_1 = \frac{n}{2},\ K_2 = 0$ |
| $E_5$ | $c_2 \geq 0, \quad c_2 \leq \theta bs_2$<br>$\theta s_1c_2 \leq s_2c_1 - \theta^2(1-\theta)bs_1s_2$ | $K_1 = 0$<br>$K_2 = \frac{n}{2}(1 - \frac{c_2}{\theta bs_2})$ |
| $F_5$ | $c_2 < 0,\ c_2 < c_1 - \theta(1-\theta)bs_1$ | $K_1 = 0,\ K_2 = \frac{n}{2}$ |
| $G_5$ | $c_1 > \theta bs_1, \quad c_2 > \theta bs_2$ | $K_1 = 0,\ K_2 = 0$ |

We solve the Case 5 problem and characterize the optimal solution in Table C.2 (see also Figures C.1(b) and C.2(b)).




(a) The Case 4 Problem. (b) The Case 5 Problem.

Figure C.1: Optimal Solution for the Dedicated-only SU System ($\alpha = 0$) in the Cost-Space.




(a) The Case 4 Problem. (b) The Case 5 Problem.

Figure C.2: Optimal Solution for the Dedicated-only SU System ($\alpha = 0$) in the $\vec{K}$-Space.

Compare Figures C.1 and C.2, we know $A_5$ dominates $B_4$, $A_4$ dominates $B_5$, $C_5$ dominates $B_4$ and $F_4$, $E_5$ dominates $B_4$ and $D_4$, and $G_5$ dominates $B_4$ in their regions, respectively. The result follows. □

**1. Proof of Proposition 6 Part (i)**

The results for the DU deterministic and dedicated-only systems with $\alpha = 0$ follow directly from Propositions 7. Similary, Propositions 15 and 16 characterize the optimal solutions for the (S+D)U and SU dedicated-only systems. □

## 2. Proof of Proposition 6 Part (ii)

The problem decomposes into two independent problems, one for each product. The formulation for product $i, i = 1, 2$, is given as

Problem $\mathbf{P_s^i}$

Stage 1 Problem $\mathbf{P_{s1}^i}$:

$$\max_{\overrightarrow{K}} V_i \equiv E_{Y_i}[\Pi^*(K_i, Y_i)] - c_i K_i \tag{C.13a}$$

$$\text{subject to } K_i \geq 0. \tag{C.13b}$$

Stage 2 Problem $\mathbf{P_{s2}^i}$:

$$\Pi^*(K_i, n_i) = \max_{\overrightarrow{p}} \frac{n_i}{b}(b - \frac{p_i}{s_i})(p_i - \alpha s_i^2) \tag{C.14a}$$

subject to

$$\frac{n_i}{b}(b - \frac{p_i}{s_i}) \leq y_i K_i \tag{C.14b}$$

$$b - \frac{p_i}{s_i} \geq 0. \tag{C.14c}$$

The solution to the stage 2 problem directly follows from Proposition 3. We next solve the stage 1 problem.

If $K_i \geq \frac{n_i}{2}$, we have $V_i = \frac{\theta n_i b s_i}{4} - c_i K_i$ and $\frac{\partial V_i}{\partial K_i} = -c_i < 0$. Then, the optimal solution is $K_i = \frac{n_i}{2}$.

If $K_i \leq \frac{n_i}{2}$, we have $V_i = \theta(bs_i - \frac{bs_i K_i}{n_i})K_i - c_i K_i$, $\frac{\partial V_i}{\partial K_i} = \theta b s_i(1 - \frac{2K_i}{n_i}) - c_i$, and $\frac{\partial^2 V_i}{\partial K_i^2} = -\frac{2bs_i}{n_i} < 0$. Then, the optimal solution is $K_i = \frac{n_i(\theta b s_i - c_i)}{2\theta b s_i}$.

Therefore, for $K_i \geq 0$, we need $c_i \leq \theta b s_i$. □

## 3. Proof of Proposition 7

In the deterministic case, the formulation of the problem becomes:

$$\Pi^*(\overrightarrow{K}) = \max_{\overrightarrow{p}} \frac{n}{b}[(\frac{p_2-p_1}{s_2-s_1} - \frac{p_1}{s_1})(p_1 - \alpha s_1^2 - c_1) + (b - \frac{(p_2-p_1)}{s_2-s_1})(p_2 - \alpha s_2^2 - c_2)] \quad \text{(C.15a)}$$

subject to

$$\frac{n}{b}(\frac{p_2-p_1}{s_2-s_1} - \frac{p_1}{s_1}) \geq 0 \quad \text{(C.15b)}$$

$$\frac{n}{b}(b - \frac{(p_2-p_1)}{s_2-s_1}) \geq 0 \quad \text{(C.15c)}$$

$$p_i \geq 0, \quad i = 1, 2. \quad \text{(C.15d)}$$

Similar to the Stage 2 problem in Proposition 3, the objective function (C.15a) is strictly jointly concave in $p_1$ and $p_2$. Therefore, the first-order conditions are necessary and sufficient for optimality and we can characterize the optimal solution as follows:

($A^d$ or $B^d$): If $c_2 s_1 \geq c_1 s_2 - \alpha s_1 s_2 (s_2 - s_1)$ and $c_2 \leq c_1 + [b - \alpha(s_1+s_2)](s_2 - s_1)$,

$$p_1^* = \frac{bs_1 + \alpha s_1^2 + c_1}{2}, \quad p_2^* = \frac{bs_2 + \alpha s_2^2 + c_2}{2}$$

$$K_1^D = \frac{n}{b}[\frac{\alpha s_2}{2} + \frac{c_2 - c_1}{2(s_2 - s_1)} - \frac{c_1}{2s_1}], \quad K_2^D = \frac{n}{b}[\frac{b - \alpha(s_1+s_2)}{2} - \frac{c_2-c_1}{2(s_2-s_1)}].$$

($C^d$): If $c_2 s_1 \leq c_1 s_2 - \alpha s_1 s_2 (s_2 - s_1)$ and $c_2 \leq bs_2 - \alpha s_2^2$,

$$p_1^* = \frac{(bs_2 + \alpha s_2^2 + c_2)s_1}{2s_2}, \quad p_2^* = \frac{bs_2 + \alpha s_2^2 + c_2}{2}$$

$$K_1^D = 0, \quad K_2^D = \frac{n}{b}(\frac{bs_2 - \alpha s_2^2 - c_2}{2s_2}).$$

($D^d$): If $c_2 \geq c_1 + [b - \alpha(s_1+s_2)](s_2 - s_1)$ and $c_1 \leq bs_1 - \alpha s_1^2$,

$$p_1^* = \frac{bs_1 + \alpha s_1^2 + c_1}{2}, \quad p_2^* = \frac{bs_1 + \alpha s_1^2 + c_1}{2} + b(s_2 - s_1)$$

$$K_1^D = \frac{n}{b}(\frac{bs_1 - \alpha s_1^2 - c_1}{2s_1}), \quad K_2^D = 0.$$

($E^d$): If $c_2 \geq bs_2 - \alpha s_2^2$ and $c_1 \geq bs_1 - \alpha s_1^2$,

$$p_1^* = bs_1, \quad p_2^* = bs_2, \quad K_1^D = 0, \quad K_2^D = 0.\Box$$

## 4. Proof of Proposition 8

The area of the ($K_1^D \geq 0, K_2^D \geq 0$) region is given by $R_D \equiv \frac{b^2(s_2-s_1)s_1}{2}$ in the deterministic and DU settings (see Figure 4.2(a)) and by $R_S \equiv \theta_1\theta_2 b^2 s_1(s_2 - \frac{1}{2}\theta_1 s_1 - \frac{1}{2}\theta_2 s_2)$ in the SU and (S+D)U settings (see Figure 4.2(b)). Observe that $R_S|_{\theta_1=\theta_2=\theta} = R_D$. Then, the proof follows because $\frac{\partial R_S}{\partial \theta_1} > 0$ for $\theta_1 < \frac{s_2(2-\theta_2)}{2s_1}$, and $\frac{\partial R_S}{\partial \theta_1} < 0$ otherwise. Similarly, $\frac{\partial R_S}{\partial \theta_2} > 0$ for $\theta_2 < \frac{2s_2-\theta_1 s_1}{2s_2}$, and $\frac{\partial R_S}{\partial \theta_2} < 0$ otherwise. Finally, when $\theta_1 = \theta_2 = \theta$, $\frac{\partial R_S}{\partial \theta} > 0$ for $\theta < \frac{4s_2}{3(s_1+s_2)}$, and $\frac{\partial R_S}{\partial \theta} < 0$ otherwise, with $\frac{4s_2}{3(s_1+s_2)} > 1$ when $s_2 > 3s_1$. $\square$

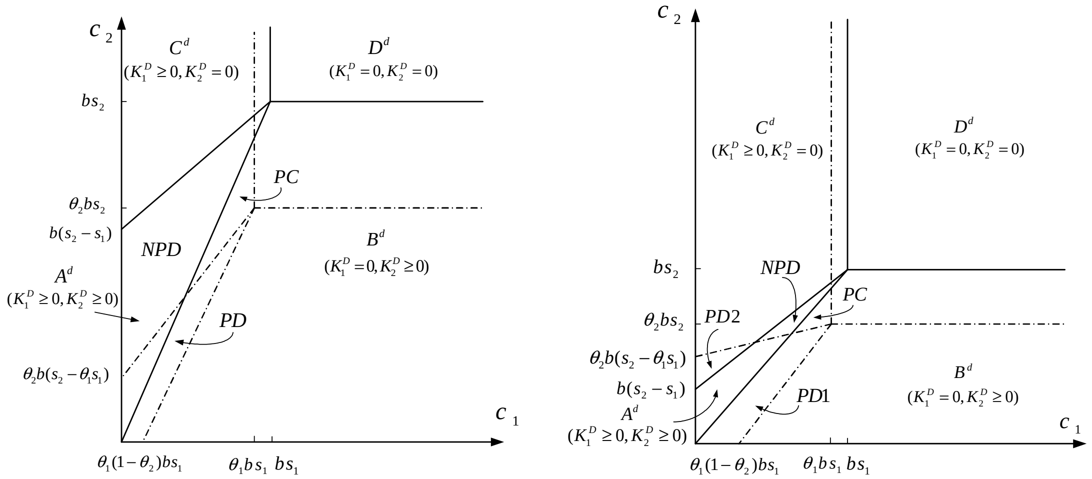
**5. Summary of strategies to hedge against supply risk when $\theta_2 > \theta_1$ or $\theta_2 < \theta_1$** (see Tables C.3 and C.4, and Figures C.4 and C.3).

Table C.3: Strategies to Hedge Against Supply Uncertainty when $\theta_2 > \theta_1$.

<table>
<tr><th></th><th>High Product Diff.<br>$s_2 \geq \frac{(1-\theta_1\theta_2)s_1}{1-\theta_2}$</th><th>Medium Product Diff.<br>$\frac{(1-\theta_1)s_1}{1-\theta_2} < s_2 < \frac{(1-\theta_1\theta_2)s_1}{1-\theta_2}$</th><th>Low Product Diff.<br>$s_2 \leq \frac{(1-\theta_1)s_1}{1-\theta_2}$</th></tr>
<tr><td rowspan="2">$NPD \rightarrow PD$</td><td rowspan="2">$K_2 \rightarrow (K_1, K_2)$<br>low $\frac{c_2}{c_1}$ ratio, low $c_1, c_2$<br>(see Figure C.4(a) $PD$)</td><td>$K_2 \rightarrow (K_1, K_2)$<br>low $\frac{c_2}{c_1}$ ratio, low $c_1, c_2$<br>(see Figure C.4(b) $PD1$)</td><td>$K_2 \rightarrow (K_1, K_2)$<br>low $\frac{c_2}{c_1}$ ratio, low $c_1, c_2$<br>(see Figure C.4(c) $PD1$)</td></tr>
<tr><td>$K_1 \rightarrow (K_1, K_2)$<br>high $\frac{c_2}{c_1}$ ratio, low $c_1$<br>medium $c_2$<br>(see Figure C.4(b) $PD2$)</td><td>$K_1 \rightarrow (K_1, K_2)$<br>high $\frac{c_2}{c_1}$ ratio<br>low to medium $c_1, c_2$<br>(see Figure C.4(c) $PD2$)</td></tr>
<tr><td rowspan="2">$PD \rightarrow NPD$</td><td>$(K_1, K_2) \rightarrow K_2$<br>medium $\frac{c_2}{c_1}$ ratio<br>medium $c_1, c_2$<br>(see Figure C.4(a) $NPD1$)</td><td>$(K_1, K_2) \rightarrow K_2$<br>low to medium $\frac{c_2}{c_1}$ ratio<br>medium $c_1, c_2$<br>(see Figure C.4(b) $NPD1$)</td><td rowspan="2">$(K_1, K_2) \rightarrow K_2$<br>low to medium $\frac{c_2}{c_1}$ ratio<br>medium to high $c_1$<br>low to medium $c_2$<br>(see Figure C.4(c) $NPD$)</td></tr>
<tr><td>$(K_1, K_2) \rightarrow K_1$<br>high $\frac{c_2}{c_1}$ ratio, low $c_1$<br>medium to high $c_2$<br>(see Figure C.4(a) $NPD2$)</td><td>$(K_1, K_2) \rightarrow K_1$<br>medium to high $\frac{c_2}{c_1}$ ratio<br>low $c_1$, medium $c_2$<br>(see Figure C.4(b) $NPD2$)</td></tr>
<tr><td>$PC$</td><td>$NA$</td><td>$NA$</td><td>$K_1 \rightarrow K_2$<br>medium $\frac{c_2}{c_1}$ ratio<br>medium $c_1, c_2$<br>(see Figure C.4(c) $PC$)</td></tr>
</table>

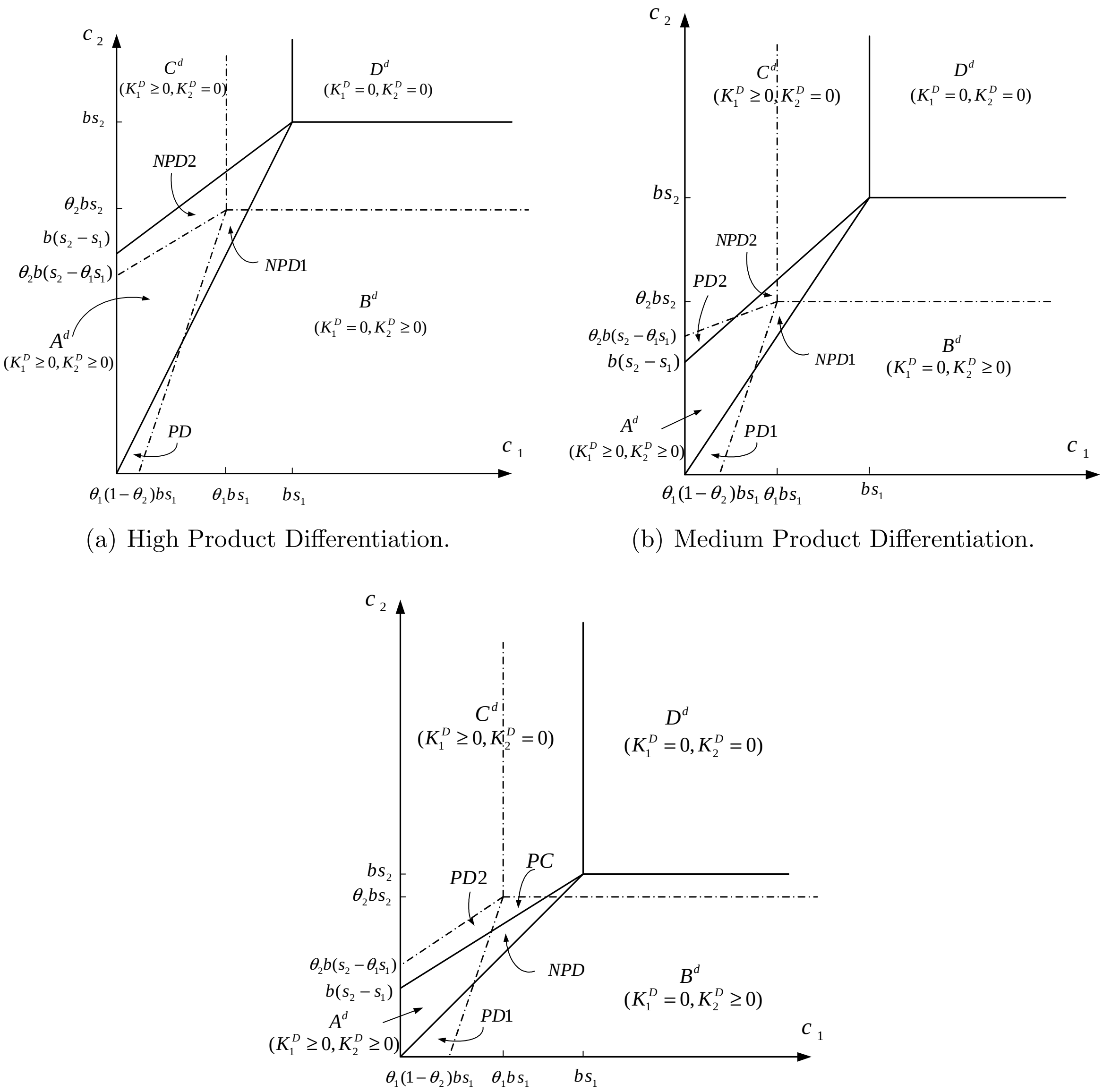
Table C.4: Strategies to Hedge Against Supply Uncertainty when $\theta_2 < \theta_1$.

| | High Product Differentiation $s_2 \geq (1+\theta)s_1$ | Low Product Differentiation $s_2 < (1+\theta)s_1$ |
|---|---|---|
| $NPD \rightarrow PD$ | $K_2 \rightarrow (K_1, K_2)$ low $\frac{c_2}{c_1}$ ratio, low to medium $c_1, c_2$ (see Figure C.3(a) $PD$) | $K_2 \rightarrow (K_1, K_2)$ low $\frac{c_2}{c_1}$ ratio, low to medium $c_1, c_2$ (see Figure C.3(b) $PD1$) |
| | | $K_1 \rightarrow (K_1, K_2)$ high $\frac{c_2}{c_1}$ ratio, low $c_1, c_2$ (see Figure C.3(b) $PD2$) |
| $PD \rightarrow NPD$ | $(K_1, K_2) \rightarrow K_1$ high $\frac{c_2}{c_1}$ ratio, low to medium $c_1$, medium to high $c_2$ (see Figure C.3(a) $NPD$) | $(K_1, K_2) \rightarrow K_1$ medium $\frac{c_2}{c_1}$ ratio, medium $c_1, c_2$ (see Figure C.3(b) $NPD$) |
| $PC$ | $K_2 \rightarrow K_1$ medium $\frac{c_2}{c_1}$ ratio, medium $c_1$, medium to high $c_2$ (see Figure C.3(a) $PC$) | $K_2 \rightarrow K_1$ medium $\frac{c_2}{c_1}$ ratio, medium $c_1, c_2$ (see Figure C.3(b) $PC$) |




(a) High Product Differentiation. (b) Low Product Differentiation.

Figure C.3: Optimal Resource Mix for the Dedicated-only System in Deterministic and DU Settings (solid line), and SU and (S+D)U Settings when $\theta_2 < \theta_1$.




(a) High Product Differentiation.

(b) Medium Product Differentiation.

(c) Low Product Differentiation.

Figure C.4: Optimal Resource Mix for the Dedicated-only System in Deterministic and DU Settings (solid line), and SU and (S+D)U Settings (dashed line) when $\theta_2 > \theta_1$.

# Appendix D

**1. Proof of Proposition 9**
The proof follows from the necessary and sufficient optimality conditions given in Lemmas 1 and 2. Recall that $\overrightarrow{K^D} = (K_1^D \geq 0, K_2^D \geq 0, K_f^D = 0)$ denotes the optimal solution to the dedicated-only system.

**Case 1.** $c_2 \leq \frac{s_2(b-\alpha s_2)}{s_1(b-\alpha s_1)} c_1$ and $s_1 c_2 \geq s_2 c_1 - \alpha s_1 s_2 (s_2 - s_1)$ (see $B^f$ in Figure 5.1):

The optimal dedicated-only solution $(K_1^D, K_2^D)$ is given by Case III(1) of Proposition 7, which satisfies $\frac{\alpha s_2 K_2^D}{b-\alpha(s_1+s_2)} - K_1^D \geq 0$, $K_f^D = 0$, and $v_{a2} = 0$. From (B.4c),

$$\int_{t_1}^{t_2} \gamma f_N(n)dn + \int_{t_2}^{t_4} \beta f_N(n)dn + \int_{t_4}^{\infty} \omega_2 f_N(n)dn = c_f - v_f = \underline{c}_f^1, \tag{D.1}$$

where $\beta, \gamma, \omega_2, t_1, t_2, t_3, t_4$ are as defined in Lemma 2 with $K_f = 0$. Observe that when $c_f < \underline{c}_f^1$, $\overrightarrow{K}_D$ cannot be optimal since (D.1) cannot hold with a nonnegative $v_f$. Thus, if $c_f < \underline{c}_f^1$, then we must have $K_f^* > 0$. Otherwise ($c_f \geq \underline{c}_f^1$), the optimal investment vector has the form $(K_1^* \geq 0, K_2^* \geq 0, K_f^* = 0)$.

**Case 2.** $c_2 \leq s_2(b - \alpha s_2)$ and $s_1 c_2 \leq s_2 c_1 - \alpha s_1 s_2 (s_2 - s_1)$:

The optimal dedicated-only solution $(K_1^D, K_2^D)$ is given by Case III(3) of Proposition 7, which satisfies $K_1^D = 0$, $K_2^D \geq 0$, $K_f^D = 0$, and $v_{a2} = 0$. From (B.4c), we have

$$\int_{t_1}^{t_2} \gamma f_N(n)dn + \int_{t_2}^{t_4} \beta f_N(n)dn + \int_{t_4}^{\infty} \omega_2 f_N(n)dn = c_f - v_f = \underline{c}_f^2, \tag{D.2}$$

where $\beta, \omega_2, t_1, t_2, t_3, t_4$ are as defined in Lemma 2 with $K_1 = K_f = 0$. From (B.4a), we know

$$\int_{t_1}^{t_2} \gamma f_N(n)dn + \int_{t_2}^{\infty} \beta f_N(n)dn = c_1 - v_1.$$

Hence, we have

$$
\begin{aligned}
& c_1 \geq \underline{c}_f^2 \\
& \Rightarrow \int_{t_1}^{t_2} \gamma f_N(n)dn + \int_{t_2}^{\infty} \beta f_N(n)dn + v_1 \\
& \quad \geq \int_{t_1}^{t_2} \gamma f_N(n)dn + \int_{t_2}^{t_4} \beta f_N(n)dn + \int_{t_4}^{\infty} \omega_2 f_N(n)dn \\
& \Rightarrow v_1 \geq \int_{t_4}^{\infty} (\omega_2 - \beta) f_N(n)dn = (s_2 - s_1)(b - \alpha(s_2 + s_1)) \int_{t_4}^{\infty} \frac{n - t_4}{n} f_N(n)dn \\
& \Rightarrow v_1 \geq (s_2 - s_1)(b - \alpha(s_1 + s_2)), \quad \text{since } t_4 \geq 0.
\end{aligned}
$$

In this case, we have $s_2 v_1 = s_2 c_1 - s_1 c_2 - \alpha s_1 s_2 (s_2 - s_1)$. Therefore, $\underline{c}_f^2 \leq c_1 \Rightarrow s_2 c_1 \geq s_1 c_2 + s_2(s_2 - s_1)(b - \alpha s_2)$. We need to consider two subcases:

**Case 2.1.** $c_2 \leq s_2(b - \alpha s_2)$, $s_1 c_2 \leq s_2 c_1 - \alpha s_1 s_2 (s_2 - s_1)$, and $s_2 c_1 \leq s_1 c_2 + s_2(s_2 - s_1)(b - \alpha s_2)$ (see $C_1^f$ in Figure 5.1): Observe that when $c_f < \underline{c}_f^2$, $\overrightarrow{K}_D$ cannot be optimal since (D.2) cannot hold with a nonnegative $v_f$. Thus, if $c_f < \underline{c}_f^2$, then we must have $K_f^* > 0$. Otherwise ($c_f \geq \underline{c}_f^2$), the optimal investment vector has the form ($K_1^* = 0, K_2^* \geq 0, K_f^* = 0$).

**Case 2.2.** $c_2 \leq s_2(b - \alpha s_2)$ and $s_2 c_1 \geq s_1 c_2 + s_2(s_2 - s_1)(b - \alpha s_2)$ (see $C_2^f$ in Figure 5.1): We have $\underline{c}_f^2 \leq c_1$, and hence, $c_f < \underline{c}_f^2$ cannot hold since we must have $c_f > c_1$ by assumption. Therefore, the optimal investment vector is of the form ($K_1^* = 0, K_2^* \geq 0, K_f^* = 0$).

**Case 3.** $c_2 \geq \frac{s_2(b-\alpha s_2)}{s_1(b-\alpha s_1)} c_1$ and $c_2 \leq c_1 + (s_2 - s_1)[b - \alpha(s_1 + s_2)]$ (see $A^f$ in Figure 5.1):

In this case, the optimal dedicated-only solution $\overrightarrow{K_D}$ is given by Case I(1) of Proposition 7. From Lemma 1, we know that a solution with $K_f > 0$ cannot be optimal in Region I. Hence, for the flexible system, we need to study the Region III problem with $K_f \geq 0$ in this case and compare its solution with the Region I problem. Note that $V_{III}^*$ is a nonincreasing function of $c_f$. Define $\underline{c}_f^3 \equiv \min\{\underline{c}_f \geq 0 : V_I^*(\overrightarrow{K}^D) = V_{III}^*(\overrightarrow{K}^*, \underline{c}_f)\}$. (Note that $\underline{c}_f^3$ may not exist.) Hence, if $\underline{c}_f^3$ exists, then for $c_f < \underline{c}_f^3$, $K_f^* > 0$. Otherwise ($c_f \geq \underline{c}_f^3$), the optimal investment vector has the form ($K_1^* \geq 0, K_2^* \geq 0, K_f^* = 0$).

**Case 4.** $c_1 \leq s_1(b - \alpha s_1)$ and $c_2 \geq c_1 + (s_2 - s_1)[b - \alpha(s_1 + s_2)]$:

In this case, the optimal dedicated-only solution $\overrightarrow{K_D}$ is given by Case I(3) of Proposition 7. Similar to Case 3, we need to study the Region III problem with $K_f \geq 0$ and compare its solution with the Region I problem. We consider the following subcases:
**Case 4.1.** $c_1 \leq s_1(b - \alpha s_1)$ and $c_2 + \frac{\alpha s_2}{b - \alpha(s_1 + s_2)} c_1 \geq \frac{\alpha s_1 s_2 (b - \alpha s_1)}{b - \alpha(s_1 + s_2)} + s_2(b - \alpha s_2)$ (see $D_2^f$ in Figure 5.1):

In this subcase, ($K_1^I = 0$, $K_2^I = 0$) is the optimal solution to Case III(4) in Proposition

7, which is dominated by the optimal solution to Case I(3) ($K_1^{III} \geq 0$, $K_2^{III} = 0$). Hence, it cannot be optimal to invest in the flexible capacity and the global optimal investment vector has the form $(K_1^* \geq 0, K_2^* = 0, K_f^* = 0)$.

**Case 4.2.** $c_2 \geq c_1 + (s_2 - s_1)[b - \alpha(s_1 + s_2)]$ and $c_2 + \frac{\alpha s_2}{b-\alpha(s_1+s_2)} c_1 \leq \frac{\alpha s_1 s_2 (b-\alpha s_1)}{b-\alpha(s_1+s_2)} + s_2(b - \alpha s_2)$:

In this subcase, any dedicated-only solution to the Region III problem is of the form $\frac{\alpha s_2 K_2^{III}}{b-\alpha(s_1+s_2)} = K_1^{III}, K_2^{III} \geq 0$. We first analyze the conditions under which the dedicated-only solution is optimal to Region III problem in the flexible system (i.e., with $K_f \geq 0$). Optimality conditions (B.4a) and (B.4b) reduce to

$$s_1(b - \alpha s_1) \int_{t_1}^{\infty} \frac{n - t_1}{n} f_N(n) dn = c_1 + v_{a2}$$
$$s_2(b - \alpha s_2) \int_{t_1}^{\infty} \frac{n - t_1}{n} f_N(n) dn = c_2 - \frac{\alpha s_2 v_{a2}}{b - \alpha(s_1 + s_2)}.$$

Solving the above equations for $v_{a2}$, we get

$$v_{a2} = [c_2 - \frac{s_2(b - \alpha s_2)}{s_1(b - \alpha s_1)} c_1]/[\frac{s_2(b - \alpha s_2)}{s_1(b - \alpha s_1)} + \frac{\alpha s_2}{b - \alpha(s_1 + s_2)}].$$

Let $\varphi \equiv \frac{s_2(b-\alpha s_2)}{s_1(b-\alpha s_1)} + \frac{\alpha s_2}{b-\alpha(s_1+s_2)}$. From (B.4c), we have

$$\int_{t_1}^{\infty} s_2(b - \alpha s_2) \frac{n - t_1}{n} f_N(n) dn - [c_2 - \frac{s_2(b - \alpha s_2)}{s_1(b - \alpha s_1)} c_1] \frac{1}{\varphi} = c_f - v_f = \underline{c}_f^{4'}. \tag{D.3}$$

When $c_f < \underline{c}_f^{4'}$, $\overrightarrow{K_D}$ cannot be optimal to Region III problem since (D.2) cannot hold with a nonnegative $v_f$. Thus, if $c_f < \underline{c}_f^{4'}$, then we will have $K_f > 0$ for the Region III problem. Further, observe that $\underline{c}_f^{4'} \leq s_2(b - \alpha s_2) - [c_2 - \frac{s_2(b-\alpha s_2)}{s_1(b-\alpha s_1)} c_1] \frac{1}{\varphi}$. Thus,

$$s_2(b - \alpha s_2) + \frac{s_2(b - \alpha s_2) c_1}{s_1(b - \alpha s_1)\varphi} \leq c_2(1 + \frac{1}{\varphi}) \Rightarrow \underline{c}_f^{4'} \leq c_2. \tag{D.4}$$

If (D.4) holds, then $c_f < \underline{c}_f^{4'}$ cannot hold, since $c_f > c_2$ by assumption. Then, we will have $K_f = 0$ in the optimal solution to the Region III problem. Define $l_1(c_2) \equiv [s_2(b - \alpha s_2) + \frac{s_2(b-\alpha s_2)c_1}{s_1(b-\alpha s_1)\varphi}]/(1+\frac{1}{\varphi})$ and $l_2(c_2) \equiv c_1 + (s_2 - s_1)[b - \alpha(s_1 + s_2)]$. We have $l_1(s_1(b - \alpha s_1)) = l_2(s_1(b - \alpha s_1)) = s_2(b - \alpha s_2)$, and $l_1(0) = s_2(b - \alpha s_2)/(1 + \frac{1}{\varphi})$ and $l_2(0) = (s_2 - s_1)[b - \alpha(s_1 + s_2)]$). We further know

$$\begin{aligned}
& s_2(b - \alpha s_2) - (s_2 - s_1)[b - \alpha(s_1 + s_2)](1 + \frac{1}{\varphi}) \\
&= \frac{s_1^2 (b - \alpha s_1)^3}{s_2[(b - \alpha s_2)^2 + \alpha s_1(s_2 - s_1)]} > 0 \\
&\Rightarrow s_2(b - \alpha s_2)/(1 + \frac{1}{\varphi}) > (s_2 - s_1)[b - \alpha(s_1 + s_2)]. \\
&\Rightarrow l_1(0) > l_2(0).
\end{aligned}$$

Therefore, we need to consider two subcases:

**Case 4.2.1.** $c_2 + \frac{\alpha s_2 c_1}{b-\alpha(s_1+s_2)} \leq \frac{\alpha s_1 s_2(b-\alpha s_1)}{b-\alpha(s_1+s_2)} + s_2(b-\alpha s_2)$ and $s_2(b-\alpha s_2) + \frac{s_2(b-\alpha s_2)c_1}{s_1(b-\alpha s_1)\varphi} \leq c_2(1+\frac{1}{\varphi})$ (see $D_2^f$ in Figure 5.1):

We have $K_f = 0$ in an optimal solution to the Region III problem. Hence, in this case, the optimal solution to Region III problem will always be dominated by the Region I problem (with $K_f = 0$) by Proposition 7. Then, the global optimal solution has the form $(K_1^* \geq 0, K_2^* = 0, K_f^* = 0)$.

**Case 4.2.2.** $s_2(b-\alpha s_2) + \frac{s_2(b-\alpha s_2)c_1}{s_1(b-\alpha s_1)\varphi} > c_2(1+\frac{1}{\varphi})$ and $c_2 \geq c_1 + (s_2 - s_1)[b - \alpha(s_1+s_2)]$ (see $D_1^f$ in Figure 5.1):

Define $\underline{c}_f^4 \equiv \min\{\underline{c}_f \geq 0 : V_I^*(\overrightarrow{K}^D) = V_{III}^*(\overrightarrow{K}^*, \underline{c}_f)\}$. (Note that $\underline{c}_f^4$ may not exist.) Then, similar to Case 3, we can argue that if $\underline{c}_f^4$ exists, then for $c_f < \underline{c}_f^4$, $K_f^* > 0$ in the global optimal solution. Otherwise $(c_f \geq \underline{c}_f^4)$, the optimal solution has the form $(K_1^* \geq 0, K_2^* = 0, K_f^* = 0)$.

**Case 5.** $c_1 \geq s_1(b-\alpha s_1)$ and $c_2 \geq s_2(b-\alpha s_2)$ (see $E^f$ in Figure 5.1):

At solution $\overrightarrow{K_D} = (K_1^D = 0, K_2^D = 0, K_f^D = 0)$, optimality conditions (B.4a) - (B.4c) reduce to:

$$
\begin{aligned}
bs_1 - \alpha s_1^2 &= c_1 - v_1 + v_{a2} \\
bs_2 - \alpha s_2^2 &= c_2 - \frac{\alpha s_2 v_{a2}}{b - \alpha(s_1+s_2)} \\
bs_2 - \alpha s_2^2 &= c_f - v_f + v_{a2}.
\end{aligned}
$$

Solving for $v_{a2}$, $v_f$, and $v_1$, we get

$$
\begin{aligned}
v_{a2} &= \frac{b-\alpha(s_1+s_2)}{\alpha s_2}[c_2 - s_2(b-\alpha s_2)] \geq 0 \\
v_1 &= c_1 - s_1(b-\alpha s_1) + v_{a2} \geq 0 \\
v_f &= c_f - s_2(b-\alpha s_2) + v_{a2} \geq 0 \Rightarrow \text{(B.4g) is satisfied.}
\end{aligned}
$$

In addition, (B.4d) - (B.4f) are satisfied. Therefore, $\overrightarrow{K_f^*} = (K_1^* = 0, K_2^* = 0, K_f^* = 0)$ is the global optimal solution.

The optimal conditions in each region are given as follows:

$(A^f)$ If $c_2 \geq \frac{s_2(b-\alpha s_2)}{s_1(b-\alpha s_1)}c_1$ and $c_2 \leq c_1 + (s_2 - s_1)[b-\alpha(s_1+s_2)]$, then $K_f^* > 0$ if and only if $c_f < \underline{c}_f^3$, where $\underline{c}_f^3 \equiv \min\{\underline{c}_f \geq 0 : V_I^*(\overrightarrow{K}^D) = V_{III}^*(\overrightarrow{K}^*, \underline{c}_f)\}$. (Note that $\underline{c}_f^3$ may not exist.)

Otherwise (if $c_f \geq \underline{c}_f^3$), $\overrightarrow{K}^* = (K_1^* \geq 0, K_2^* \geq 0, K_f^* = 0)$.

($B^f$) If $c_2 \leq \frac{s_2(b-\alpha s_2)}{s_1(b-\alpha s_1)} c_1$ and $s_1 c_2 \geq s_2 c_1 - \alpha s_1 s_2 (s_2 - s_1)$, then $K_f^* > 0$ if and only if $c_f < \underline{c}_f^1 \equiv \int_{t_1}^{t_2} \gamma f(n)dn + \int_{t_2}^{t_4} \beta f(n)dn + \int_{t_4}^{\infty} \omega_2 f(n)dn$, where $\beta, \gamma, \omega_2, t_1, t_2, t_3, t_4$ are as defined in Lemma 2 with $K_f = 0$.

Otherwise (if $c_f \geq \underline{c}_f^1$), $\overrightarrow{K}^* = (K_1^* \geq 0, K_2^* \geq 0, K_f^* = 0)$.

($C_1^f$) If $c_2 \leq s_2(b-\alpha s_2)$, $s_1 c_2 \leq s_2 c_1 - \alpha s_1 s_2(s_2 - s_1)$, and $s_2 c_1 \leq s_1 c_2 + s_2(s_2 - s_1)(b - \alpha s_2)$, then $K_f^* > 0$ if and only if $c_f < \underline{c}_f^2 \equiv \int_{t_1}^{t_2} \gamma f(n)dn + \int_{t_2}^{t_4} \beta f(n)dn + \int_{t_4}^{\infty} \omega_2 f(n)dn$, where $\beta, \gamma, \omega_2, t_1, t_2, t_3, t_4$ are as defined in Lemma 2 with $K_1 = 0$ and $K_f = 0$.

Otherwise (if $c_f \geq \underline{c}_f^2$), $\overrightarrow{K}^* = (K_1^* = 0, K_2^* \geq 0, K_f^* = 0)$.

($D_1^f$) If $s_2(b - \alpha s_2) + \frac{s_2(b-\alpha s_2)c_1}{s_1(b-\alpha s_1)\varphi} \geq c_2(1 + \frac{1}{\varphi})$ and $c_2 \geq c_1 + (s_2 - s_1)[b - \alpha(s_1 + s_2)]$, then $K_f^* > 0$ if and only if $c_f < \underline{c}_f^4$, where $\underline{c}_f^4 \equiv \min\{\underline{c}_f \geq 0 : V_I^*(\overrightarrow{K}^D) = V_{III}^*(\overrightarrow{K}^*, \underline{c}_f)\}$. (Note that $\underline{c}_f^4$ may not exist.)

Otherwise (if $c_f \geq \underline{c}_f^4$), then $\overrightarrow{K}^* = (K_1^* \geq 0, K_2^* = 0, K_f^* = 0)$.

($C_2^f$) If $c_2 \leq s_2(b - \alpha s_2)$ and $s_2 c_1 \geq s_1 c_2 + s_2(s_2 - s_1)(b - \alpha s_2)$, then $\overrightarrow{K}^* = (K_1^* = 0, K_2^* \geq 0, K_f^* = 0)$.

($D_2^f$) If $s_2(b - \alpha s_2) + \frac{s_2(b-\alpha s_2)c_1}{s_1(b-\alpha s_1)\varphi} \leq c_2(1 + \frac{1}{\varphi})$ and $c_1 \leq s_1(b - \alpha s_1)$, then $\overrightarrow{K}^* = (K_1^* \geq 0, K_2^* = 0, K_f^* = 0)$.

($E^f$) If $c_1 \geq s_1(b - \alpha s_1)$ and $c_2 \geq s_2(b - \alpha s_2)$, then $\overrightarrow{K}^* = (K_1^* = 0, K_2^* = 0, K_f^* = 0)$,

where $\varphi \equiv \frac{s_2(b-\alpha s_2)}{s_1(b-\alpha s_1)} + \frac{\alpha s_2}{b-\alpha(s_1+s_2)}$.

Note that for $K_f \geq 0$, we need $\frac{\alpha s_2 K_2}{b-\alpha(s_1+s_2)} \geq K_1 + K_f$ (see constraint (B.4f) of the **Region III** (with $K_f \geq 0$) problem, $\mathbf{P_1^{III}}$). Hence, when $K_2^* = 0$ in the optimal solution, we must have $K_1^* = K_2^* = 0$. Therefore, the optimal solution has the form of $K_1^* = 0, K_2^* > 0, K_f^* > 0$ or $K_1^* > 0, K_2^* > 0, K_f^* > 0$. □

**2. Proof of Proposition** 10.

The proof follows from the necessary and sufficient conditions given in Lemma 3. Recall that $\overrightarrow{K} = (K_1^D \geq 0, K_2^D \geq 0, K_f^D = 0)$ denotes the optimal solution to the dedicated-only system.

**Case 1.** $c_2 \leq \theta c_1 + \theta b(s_2 - \theta s_1)$ and $c_1 \leq \frac{\theta s_1}{s_2} c_2 + \theta(1-\theta) b s_1$ (see $A^f$ in Figure 5.2):

The optimal dedicated-only solution $(K_1^D, K_2^D)$ is given by Case 1 of Proposition 16. From (C.1c),

$$\theta^3 \int_{2K_2}^{2(K_1+K_2)} b(s_2 - s_1)(1 - \frac{2K_2}{n}) f_N(n)dn + \theta^3 \int_{2(K_1+K_2)}^{\infty} (bs_2 - \frac{2b(K_1 s_1 + K_2 s_2)}{n}) f_N(n)dn$$
$$+ \theta^2(1-\theta) \int_{2K_1}^{\infty} bs_1(1 - \frac{2K_1}{n}) f_N(n)dn + \theta^2(1-\theta) \int_{2K_2}^{\infty} bs_2(1 - \frac{2K_2}{n}) f_N(n)dn$$
$$+ \theta(1-\theta)b(s_2 - \theta s_1) = c_f - v_f = \underline{c}_f^1 \qquad \text{(D.5)}$$

Observe that when $c_f < \underline{c}_f^1$, $\overrightarrow{K}_D$ cannot be optimal since (D.5) cannot hold with a nonnegative $v_f$. Thus, if $c_f < \underline{c}_f^1$, we must have $K_f^* > 0$. Otherwise, the optimal investment vector has the form $(K_1^* \geq 0, K_2^* \geq 0, K_f^* = 0)$.

**Case 2.** $c_2 \leq \theta bs_2$ and $c_1 > \frac{\theta s_1}{s_2} c_2 + \theta(1-\theta)bs_1$:

The optimal dedicated-only solution $(K_1^D, K_2^D)$ is given by Case 2 of Proposition 16, which satisfies $K_1^D = 0, K_2^D \geq 0, K_f^D = 0$, and $\theta \int_{2K_2}^{\infty} bs_2(1 - \frac{2K_2}{n}) f_N(n)dn = c_2$. From (C.1c),

$$\theta^2 \int_{2K_2}^{\infty} bs_2(1 - \frac{2K_2}{n}) f_N(n)dn + \theta(1-\theta)bs_2 = c_f - v_f = \underline{c}_f^2 \qquad \text{(D.6)}$$
$$\Rightarrow \quad \underline{c}_f^2 = \theta c_2 + \theta(1-\theta)bs_2.$$

From $\underline{c}_f^2 > c_2$, we need

$$\theta c_2 + \theta(1-\theta)bs_2 > c_2 \quad \Rightarrow \quad c_2 < \theta bs_2,$$

which is satisfied in this case.

From $\underline{c}_f^2 > c_1$, we need $\theta c_2 + \theta(1-\theta)bs_2 > c_1$. Thus, we need to consider two subcases:

**Case 2.1.** $c_2 \leq \theta bs_2$, $c_1 > \frac{\theta s_1}{s_2} c_2 + \theta(1-\theta)bs_1$, and $\theta c_2 + \theta(1-\theta)bs_2 > c_1$ (see $C_1^f$ in Figure 5.2): Observe that when $c_f < \underline{c}_f^2$, $\overrightarrow{K}_D$ cannot be optimal since (D.6) cannot hold with a nonnegative $v_f$. Thus, if $c_f < \underline{c}_f^2$, we must have $K_f^* > 0$. Otherwise ($c_f \geq \underline{c}_f^2$), the optimal investment vector has the form $(K_1^* = 0, K_2^* \geq 0, K_f^* = 0)$.

**Case 2.2.** $c_2 \leq \theta bs_2$ and $c_1 \geq \theta c_2 + \theta(1-\theta)bs_2$ (see $C_2^f$ in Figure 5.2): It cannot be optimal to invest in the flexible capacity and the global optimal investment vector has the form $(K_1^* = 0, K_2^* \geq 0, K_f^* = 0)$.

**Case 3.** $c_1 \leq \theta bs_1$ and $c_2 > \theta c_1 + \theta b(s_2 - \theta s_1)$ (see $D^f$ in Figure 5.2):

The optimal dedicated-only solution $(K_1^D, K_2^D)$ is given by Case 3 of Proposition 16, which satisfies $K_1^D \geq 0, K_2^D = 0, K_f^D = 0$, and $\theta \int_{2K_1}^{\infty} bs_1(1 - \frac{2K_1}{n}) f_N(n)dn = c_1$. From

(C.1c),

$$\theta^2 \int_{2K_1}^{\infty} bs_1(1-\frac{2K_1}{n})f_N(n)dn + \theta^3 b(s_2-s_1) + \theta^2(1-\theta)bs_2 + \theta(1-\theta)b(s_2-\theta s_1) = c_f - v_f = \underline{c}_f^3 \tag{D.7}$$

$$\Rightarrow \underline{c}_f^3 = \theta c_1 + \theta b(s_2 - \theta s_1).$$

From $\underline{c}_f^3 > c_2$, we need $\theta c_1 + \theta b(s_2 - \theta s_1)$, which cannot hold in this region. Therefore, in this case, it cannot be optimal to invest in the flexible capacity and the global optimal investment vector has the form $(K_1^* \geq 0, K_2^* = 0, K_f^* = 0)$.

**Case 4.** $c_1 \geq \theta bs_1$ and $c_2 \geq \theta bs_2$ (see $E^f$ in Figure 5.2):

From (C.1c), we have $\theta bs_2 = c_f - v_f = \underline{c}_f^4$. Since $c_2 \geq \theta bs_2$ and $c_f > c_2$, $c_f < \underline{c}_f^4$ cannot hold in this region. Thus, the global optimal investment vector has the form $(K_1^* = 0, K_2^* = 0, K_f^* = 0)$.

To summarize, the optimal investment portfolio in the flexible (S+D)U system can be characterized as follows (see Figure 5.2):

$(A^f)$: If $c_2 \leq \theta c_1 + \theta b(s_2 - \theta s_1)$ and $c_1 \leq \frac{\theta s_1}{s_2} c_2 + \theta(1-\theta)bs_1$, then $K_f^* > 0$ if and only if

$$\begin{aligned} c_f < \underline{c}_f^1 \equiv \theta^3 &\int_{2K_2}^{2(K_1+K_2)} b(s_2-s_1)(1-\frac{2K_2}{n})f_N(n)dn \\ &+ \theta^3 \int_{2(K_1+K_2)}^{\infty} (bs_2 - \frac{2b(K_1s_1+K_2s_2)}{n})f_N(n)dn \\ &+ \theta^2(1-\theta)\int_{2K_1}^{\infty} bs_1(1-\frac{2K_1}{n})f_N(n)dn + \theta^2(1-\theta)\int_{2K_2}^{\infty} bs_2(1-\frac{2K_2}{n})f_N(n)dn \\ &+ \theta(1-\theta)b(s_2-\theta s_1). \end{aligned}$$

Otherwise (if $c_f \geq \underline{c}_f^1$), $\overrightarrow{K}^* = (K_1^* \geq 0, K_2^* \geq 0, K_f^* = 0)$.

$(C_1^f)$: If $c_2 \leq \theta bs_2$, $c_1 > \frac{\theta s_1}{s_2} c_2 + \theta(1-\theta)bs_1$, and $\theta c_2 + \theta(1-\theta)bs_2 > c_1$, then $K_f^* > 0$ if and only if $c_f < \underline{c}_f^2 \equiv \theta c_2 + \theta(1-\theta)bs_2$.
Otherwise (if $c_f \geq \underline{c}_f^2$), $\overrightarrow{K}^* = (K_1^* = 0, K_2^* \geq 0, K_f^* = 0)$.

$(C_2^f)$: If $c_1 \leq \theta bs_1$ and $c_2 > \theta c_1 + \theta b(s_2 - \theta s_1)$, then $\overrightarrow{K}^* = (K_1^* \geq 0, K_2^* = 0, K_f^* = 0)$.

$(D^f)$: If $c_2 \leq \theta bs_2$ and $c_1 \geq \theta c_2 + \theta(1-\theta)bs_2$, then $\overrightarrow{K}^* = (K_1^* = 0, K_2^* \geq 0, K_f^* = 0)$.

$(E^f)$: If $c_1 \geq \theta bs_1$ and $c_2 \geq \theta bs_2$, then $\overrightarrow{K}^* = (K_1^* = 0, K_2^* = 0, K_f^* = 0)$.

Next we study the optimal portfolio mix when it is optimal for the firm to invest in the flexible resource. When $K_f > 0$ in an optimal solution, the solution must be one of the following forms:

$$\begin{aligned}
\overrightarrow{K}^F &= (K_1 = 0, K_2 = 0, K_f > 0),\\
\overrightarrow{K}^{1F} &= (K_1 > 0, K_2 = 0, K_f > 0),\\
\overrightarrow{K}^{2F} &= (K_1 = 0, K_2 > 0, K_f > 0),\\
\overrightarrow{K}^A &= (K_1 > 0, K_2 > 0, K_f > 0).
\end{aligned}$$

If solution $\overrightarrow{K}^F$ is an optimal solution, there must exist a $\overrightarrow{v} = (v_1 > 0, v_2 > 0, v_f = 0)$ that satisfies the following conditions:

$$\theta^2 \int_{2K_f}^{\infty} bs_1(1 - \frac{2K_f}{n}) f_N(n)dn + \theta(1-\theta)bs_1 = c_1 - v_1 \tag{D.8}$$

$$\theta^2 \int_{2K_f}^{\infty} bs_2(1 - \frac{2K_f}{n}) f_N(n)dn + \theta(1-\theta)bs_2 = c_2 - v_2 \tag{D.9}$$

$$\theta^2 \int_{2K_f}^{\infty} bs_2(1 - \frac{2K_f}{n}) f_N(n)dn = c_f \tag{D.10}$$

From (D.9)-(D.10), we have

$$\begin{aligned}
&c_2 - v_2 - c_f\\
&= -\theta(1-\theta) \int_{2K_f}^{\infty} bs_2(1 - \frac{2K_f}{n}) f(n)dn + \theta(1-\theta)bs_2\\
&> -\theta(1-\theta)bs_2 + \theta(1-\theta)bs_2 = 0\\
&\Rightarrow c_2 - v_2 > c_f,
\end{aligned}$$

which contradicts with $c_2 < c_f$ and $v_2 > 0$. Therefore, $\overrightarrow{K}^F$ cannot be the optimal solution.

Similarly, if solution $\overrightarrow{K}^{1F}$ is an optimal solution, there must exist a $\overrightarrow{v} = (v_1 = 0, v_2 > 0, v_f = 0)$ that satisfies the following conditions:

$$\theta^2 \int_{2(K_1+K_f)}^{\infty} bs_1[1 - \frac{2(K_1+K_f)}{n}] f_N(n)dn + \theta(1-\theta) \int_{2K_1}^{\infty} bs_1(1 - \frac{2K_1}{n}) f_N(n)dn = c_1 \tag{D.11}$$

$$\begin{aligned}
&\theta^3 \int_{2K_f}^{2(K_1+K_f)} b(s_2 - s_1)(1 - \frac{2K_f}{n}) f_N(n)dn + \theta^3 \int_{2(K_1+K_f)}^{\infty} [bs_2 - \frac{2b(K_1 s_1 + K_f s_2)}{n}] f_N(n)dn\\
&+ \theta^2(1-\theta) \int_{2K_1}^{\infty} bs_1(1 - \frac{2K_1}{n}) f_N(n)dn + \theta^2(1-\theta) \int_{2K_f}^{\infty} bs_2(1 - \frac{2K_f}{n}) f_N(n)dn\\
&+ \theta^2(1-\theta)b(s_2 - s_1) + \theta(1-\theta)^2 = c_2 - v_2
\end{aligned} \tag{D.12}$$

$$\begin{aligned}
&\theta^2 \int_{2K_f}^{2(K_1+K_f)} b(s_2 - s_1)(1 - \frac{2K_f}{n}) f_N(n)dn + \theta^2 \int_{2(K_1+K_f)}^{\infty} [bs_2 - \frac{2b(K_1 s_1 + K_f s_2)}{n}] f_N(n)dn\\
&+ \theta(1-\theta) \int_{2K_f}^{\infty} bs_2(1 - \frac{2K_f}{n}) f_N(n)dn = c_f
\end{aligned} \tag{D.13}$$

From (D.12)-(D.13), we have

$$
\begin{aligned}
&c_2 - v_2 - c_f \\
&= -\theta^2(1-\theta)\int_{2K_f}^{\infty} b(s_2-s_1)(1-\frac{2K_f}{n})f_N(n)dn - \theta(1-\theta)^2\int_{2K_f}^{\infty} bs_2(1-\frac{2K_f}{n})f_N(n)dn \\
&\quad + \theta^2(1-\theta)\int_{2K_1}^{\infty} bs_1(1-\frac{2K_1}{n})f_N(n)dn - \theta^2(1-\theta)\int_{2(K_1+K_f)}^{\infty} bs_1(1-\frac{2(K_1+K_f)}{n})f_N(n)dn \\
&\quad + \theta^2(1-\theta)b(s_2-s_1) + \theta(1-\theta)^2 bs_2 \\
&> -\theta^2(1-\theta)b(s_2-s_1) - \theta(1-\theta)^2 bs_2 + \theta^2(1-\theta)b(s_2-s_1) + \theta(1-\theta)^2 bs_2 = 0 \\
&\Rightarrow c_2 - v_2 > c_f
\end{aligned}
$$

which contradicts with $c_2 < c_f$ and $v_2 > 0$. Hence, $\overrightarrow{K}^{1F}$ cannot be the optimal solution. Therefore, when the firm acquires the flexible resource, the structure of its optimal portfolio will be of the form $(K_1^* = 0, K_2^* > 0, K_f^* > 0)$ or $(K_1^* > 0, K_2^* > 0, K_f^* > 0)$.

Furthermore, for the solution of $(K_1 > 0, K_2 > 0, K_f > 0)$, (C.1a) and (C.1b) become

$$
\begin{aligned}
&\theta^3\int_{2(K_1+K_2+K_f)}^{\infty} bs_1(1-\frac{2(K_1+K_2+K_f)}{n})f_N(n)dn + \theta^2(1-\theta)\int_{2(K_1+K_2)}^{\infty} bs_1(1-\frac{2(K_1+K_2)}{n})f_N(n)dn \\
&+ \theta^2(1-\theta)\int_{2(K_1+K_f)}^{\infty} bs_1(1-\frac{2(K_1+K_f)}{n})f_N(n)dn + \theta(1-\theta)^2\int_{2K_1}^{\infty} bs_1(1-\frac{2K_1}{n})f_N(n)dn = c_1 \qquad \text{(D.14)}
\end{aligned}
$$

$$
\begin{aligned}
&\theta^3\int_{2(K_2+K_f)}^{2(K_1+K_2+K_f)} b(s_2-s_1)(1-\frac{2(K_2+K_f)}{n})f_N(n)dn \\
&+ \theta^3\int_{2(K_1+K_2+K_f)}^{\infty} [bs_2 - \frac{2b(K_2s_2+K_fs_2+K_1s_1)}{n}]f_N(n)dn \\
&+ \theta^2(1-\theta)\int_{2K_2}^{2(K_1+K_2)} b(s_2-s_1)(1-\frac{2K_2}{n})f_N(n)dn + \theta^2(1-\theta)\int_{2(K_1+K_2)}^{\infty} [bs_2 - \frac{2b(K_2s_2+K_1s_1)}{n}]f_N(n)dn \\
&+ \theta^2(1-\theta)\int_{2(K_2+K_f)}^{\infty} [bs_2 - \frac{2bs_2(K_2+K_f)}{n}]f_N(n)dn + \theta(1-\theta)^2\int_{2K_2}^{\infty} [bs_2 - \frac{2bs_2K_2}{n}]f_N(n)dn = c_2
\end{aligned}
\qquad \text{(D.15)}
$$

We have

$$
\begin{aligned}
& s_2c_1 - \theta s_1 c_2 \\
= \; & \overbrace{\theta^3 b s_1(\theta s_1 - s_2)\Big(\int_{2(K_2+K_f)}^{\infty}[1-\frac{2(K_2+K_f)}{n}]f_N(n)dn - \int_{2(K_1+K_2+K_f)}^{\infty}(1-\frac{2(K_1+K_2+K_f)}{n})f_N(n)dn\Big)}^{<\,0} \\
& \overbrace{+\theta^2(1-\theta)bs_1(\theta s_1 - s_2)\Big(\int_{2K_2}^{\infty}(1-\frac{2K_2}{n})f_N(n)dn - \int_{2(K_1+K_2)}^{\infty}(1-\frac{2(K_1+K_2)}{n})f_N(n)dn\Big)}^{<\,0} \\
& +\theta^2(1-\theta)bs_1s_2\int_{2(K_1+K_f)}^{\infty}(1-\frac{2(K_1+K_f)}{n})f_N(n)dn + \theta(1-\theta)^2 bs_1s_2\int_{2K_1}^{\infty}(1-\frac{2K_1}{n})f_N(n)dn \\
& <\theta^2(1-\theta)bs_1s_2\int_{2(K_1+K_f)}^{\infty}(1-\frac{2(K_1+K_f)}{n})f_N(n)dn + \theta(1-\theta)^2 bs_1s_2\int_{2K_1}^{\infty}(1-\frac{2K_1}{n})f_N(n)dn \\
& <\theta^2(1-\theta)bs_1s_2 + \theta(1-\theta)^2bs_1s_2 = \theta(1-\theta)bs_1s_2 \\
\Rightarrow \; & s_2c_1 - \theta s_1c_2 < \theta(1-\theta)bs_1s_2.
\end{aligned}
$$

Therefore, $(K_1 > 0, K_2 > 0, K_f > 0)$ can not be feasible in $C_1^f$ which has the constraint of $s_2c_1 - \theta s_1 c_2 \geq \theta(1-\theta)bs_1s_2$. On the other hand, when $c_f < \underline{c}_f^2$, the optimal solution in region $C_1^f$ must be in the form of $(K_1^* = 0, K_2^* > 0, K_f^* > 0)$.

For the solution of $(K_1 = 0, K_2 > 0, K_f > 0)$, (C.1a) - (C.1c) become

$$
\begin{aligned}
&\theta^3\int_{2(K_2+K_f)}^{\infty} bs_1(1-\frac{2(K_2+K_f)}{n})f_N(n)dn + \theta^2(1-\theta)\int_{2(K_2)}^{\infty} bs_1(1-\frac{2K_2}{n})f_N(n)dn \\
&+\theta^2(1-\theta)\int_{2K_f}^{\infty} bs_1(1-\frac{2K_f}{n})f_N(n)dn + \theta(1-\theta)^2bs_1 = c_1 - v_1 \qquad \text{(D.16)}
\end{aligned}
$$

$$
\theta^2\int_{2(K_2+K_f)}^{\infty} bs_2(1-\frac{2(K_2+K_f)}{n})f_N(n)dn + \theta(1-\theta)\int_{2K_2}^{\infty} bs_2(1-\frac{2K_2}{n})f_N(n)dn = c_2 \qquad \text{(D.17)}
$$

$$
\theta^2\int_{2(K_2+K_f)}^{\infty} bs_2(1-\frac{2(K_2+K_f)}{n})f_N(n)dn + \theta(1-\theta)\int_{2K_f}^{\infty} bs_2(1-\frac{2K_f}{n})f_N(n)dn = c_f \qquad \text{(D.18)}
$$

We have

$$
\begin{aligned}
& c_f - c_2 = \theta(1-\theta)\int_{2K_f}^{\infty} bs_2(1-\frac{2K_f}{n})f_N(n)dn - \int_{2K_2}^{\infty} bs_2(1-\frac{2K_2}{n})f_N(n)dn > 0 \\
& \Rightarrow \int_{2K_f}^{\infty}(1-\frac{2K_f}{n})f_N(n)dn > \int_{2K_2}^{\infty}(1-\frac{2K_2}{n})f_N(n)dn \\
& \Rightarrow K_f < K_2.
\end{aligned}
$$

We also have

$$
\begin{aligned}
&c_2 - (c_1 - v_1) \\
&= \theta^2 b(s_2 - \theta s_1) \int_{2(K_2+K_f)}^{\infty} (1 - \frac{2(K_2+K_f)}{n}) f_N(n) dn + \theta(1-\theta) b(s_2 - \theta s_1) \int_{2K_2}^{\infty} (1 - \frac{2K_2}{n}) f_N(n) dn \\
&\quad - \theta^2 (1-\theta) b s_1 \int_{2K_f}^{\infty} (1 - \frac{2K_f}{n}) f_N(n) dn - \theta(1-\theta)^2 b s_1 \\
&< \theta^2 b(s_2 - \theta s_1) \int_{2K_2}^{\infty} (1 - \frac{2K_2}{n}) f_N(n) dn + \theta(1-\theta) b(s_2 - \theta s_1) \int_{2K_2}^{\infty} (1 - \frac{2K_2}{n}) f_N(n) dn \\
&\quad - \theta^2 (1-\theta) b s_1 \int_{2K_2}^{\infty} (1 - \frac{2K_2}{n}) f_N(n) dn - \theta(1-\theta)^2 b s_1 \\
&= [\theta b(s_2 - \theta s_1) - \theta^2(1-\theta) b s_1] \int_{2K_2}^{\infty} (1 - \frac{2K_2}{n}) f_N(n) dn - \theta(1-\theta)^2 b s_1.
\end{aligned}
$$

Note that $\theta b(s_2 - \theta s_1) - \theta^2(1-\theta) b s_1 > 0$ since $\theta b(s_2 - \theta s_1) > \theta^2(1-\theta) b s_1 \Leftrightarrow \frac{s_2}{s_1} > 2\theta - \theta^2$, which holds because $2\theta - \theta^2 < 1 \Leftrightarrow (\theta - 1)^2 > 0$.

Then, $c_2 - (c_1 - v_1) < \theta b(s_2 - \theta s_1) - \theta^2(1-\theta) b s_1 - \theta(1-\theta)^2 b s_1 = \theta b(s_2 - s_1) < b(s_2 - s_1)$. Therefore, for the region that satisfies $c_2 \geq c_1 + b(s_2 - s_1)$ in $A^f$, there will not exist a positive $v_1$ such that the KKT conditions holds. Hence, the optimal solution can not have a form of $(K_1 = 0, K_2 > 0, K_f > 0)$ in this region. On the other hand, when $c_f < \underline{c}_f^1$, the optimal solution must be in the form of $(K_1^* > 0, K_2^* > 0, K_f^* > 0)$ in region $A^f$ with $c_2 \geq c_1 + b(s_2 - s_1)$. □

## 3. Proof of Proposition 11

The area of region $(PD + RF)2$ (see Figure 5.3(b)) is given by $R_{(PD+RF)2} = \frac{1}{2}(1 - \theta)[(1+\theta) b s_1 - b s_2]^2$. Note that we assume $s_2 < (1+\theta) s_1$ in this case. Then, $\frac{\partial R_{(PD+RF)2}}{\partial \theta} = \frac{1}{2}[(1+\theta) b s_1 - b s_2][(1 - 3\theta) b s_1 + b s_2] > 0$ when $\frac{s_2 - s_1}{s_1} < \theta < \frac{s_1 + s_2}{3 s_1}$, and $\frac{\partial R_{(PD+RF)2}}{\partial \theta} < 0$ when $\theta > \frac{s_1 + s_2}{3 s_1}$. The result follows. □

# Appendix E

In what follows, we will first consider the "uncapacitated stage 2 problem" (i.e., without constraints (3.1b) - (3.1d)), and based on this analysis we will draw conclusions on when the flexible capacity will be beneficial. Assume, wlog, that the total number of customers (market size) realized in Stage 2, $n$, is 1. We denote the uncapacitated Stage 2 Problem as $\mathbf{P_2^u}$, whose formulation is given by:

$$\text{Problem } \mathbf{P_2^u}: \ \Pi^u \equiv \max_{\overrightarrow{p}} \quad p_1 \int_{\frac{p_1}{s_1}}^{\frac{p_2-p_1}{s_2-s_1}} f_T(t)dt + p_2 \int_{\frac{p_2-p_1}{s_2-s_1}}^{b} f_T(t)dt \tag{E.1a}$$

$$\text{subject to} \quad b \geq \tfrac{p_2-p_1}{s_2-s_1} \geq \tfrac{p_1}{s_1} \geq 0. \tag{E.1b}$$

Let $t_1 \equiv \frac{p_1}{s_1}$ and $t_2 \equiv \frac{p_2-p_1}{s_2-s_1}$. Let $\overrightarrow{p}^u = (p_1^u, p_2^u)$, or alternatively, $\overrightarrow{t}^u = (t_1^u, t_2^u)$ denote the optimal solution to Problem $\mathbf{P_2^u}$. We also let $g_T(x) \equiv x f_T(x) + F_T(x), x \in [0, b]$.

**Proposition 17.** The optimal solution to Problem $\mathbf{P_2^u}$ can occur either in the interior of the feasible region, given by $b \geq \frac{p_2-p_1}{s_2-s_1} \geq \frac{p_1}{s_1} \geq 0$, or on the boundary line $p_1 s_2 = p_2 s_1$.

**Proof:** Consider the feasible region, $b \geq \frac{p_2-p_1}{s_2-s_1} \geq \frac{p_1}{s_1} \geq 0$, see Figure E.1, which corresponds to:

$$p_1 \geq 0, \ p_1 \leq \frac{p_2 s_1}{s_2}, \text{ and } p_2 - p_1 \leq b(s_2 - s_1).$$

(i) First consider all *boundary points* of the feasible region, given by $(p_1, p_2) = (0, 0), (0, b(s_2 - s_1))$, and $(bs_1, bs_2)$. Observe that each of these points yields a profit of zero (see (E.1a)). However, $\Pi^u > 0$, since one feasible solution is $(p_1 = \frac{bs_1}{2}, p_2 = \frac{bs_2}{2})$ , with a profit of $p_2 \int_{\frac{b}{2}}^{b} f_T(t)dt > 0$. Hence, these boundary points cannot be optimal.

(ii) Consider points on the *boundary line* $p_1 = 0$, which are of the form $(p_1 = 0, p_2 \in (0, b(s_2 - s_1)))$, with $\bar{t}_1 \equiv \frac{p_1}{s_1} = 0, \bar{t}_2 \equiv \frac{p_2-p_1}{s_2-s_1} > 0$, and profit $\bar{\Pi}$. Then, consider an alternative solution on the line $p_2 s_2 = p_2 s_1$ with $\hat{t}_1 (= \frac{\hat{p}_1}{s_1}) = \hat{t}_2 (= \frac{\hat{p}_2-\hat{p}_1}{s_2-s_1}) = \bar{t}_2$, where $\hat{p}_1 > \bar{p}_1 = 0$, which implies $\hat{p}_2 > \bar{p}_2$, since $\hat{p}_2 - \hat{p}_1 = \bar{p}_2 - \bar{p}_1$. Hence, we have a profit

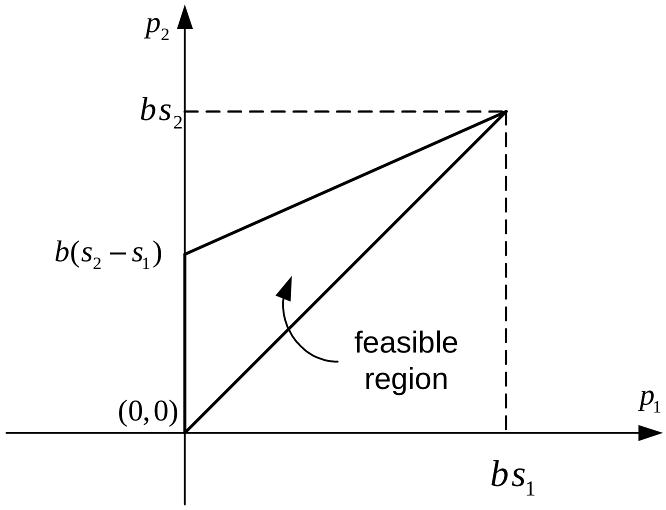



Figure E.1: Feasible Region of the Uncapacitated Problem $\mathbf{P_2^u}$.

of $\hat{\Pi} = \hat{p}_2 \int_{\hat{t}_2}^{b} f_T(t)dt > \bar{\Pi} = \bar{p}_2 \int_{\bar{t}_2}^{b} f_T(t)dt$. Thus, for any solution on the boundary line $p_1 = 0$, we can always find a dominating solution on the line $p_1 s_2 = p_2 s_1$. Therefore, a solution on this boundary line cannot be optimal.

(iii) Consider points on the *boundary line* $p_2 - p_1 = b(s_2 - s_1)$, which are of the form $(p_1 \in (0, bs_1), p_2 = p_1 + b(s_2 - s_1))$. We can show, following a similar reasoning to (ii), that there always exists a dominating solution on the line $p_1 s_2 = p_2 s_1$. Hence, the result follows. □

**Proposition 18.** The first order conditions (FOC) are always satisfied on the boundary line $s_2 p_1 = s_1 p_2$.

**Proof:** We first determine the first-order derivatives of $\Pi^u$ with respect to $p_1$ and $p_2$:

$$\frac{\partial \Pi^u}{\partial p_1} = \int_{\frac{p_1}{s_1}}^{\frac{p_2-p_1}{s_2-s_1}} f_T(t)dt + \frac{1}{s_2 - s_1} f_T(\frac{(p_2 - p_1)}{s_2 - s_1})(p_2 - p_1) - \frac{p_1}{s_1} f_T(\frac{p_1}{s_1})$$

$$\frac{\partial \Pi^u}{\partial p_2} = \int_{\frac{p_2-p_1}{s_2-s_1}}^{b} f_T(t)dt - \frac{(p_2 - p_1)}{s_2 - s_1} f_T(\frac{p_2 - p_1}{s_2 - s_1}).$$

Observe that the first-order conditions, $\frac{\partial \Pi^u}{\partial p_1} = 0$, $\frac{\partial \Pi^u}{\partial p_2} = 0$, reduce to:

$$\int_{t_1}^{t_2} f_T(t)dt = t_1 f_T(t_1) - t_2 f_T(t_2) \quad \text{and} \quad \int_{t_2}^{b} f_T(t)dt = t_2 f_T(t_2)$$

$$\Rightarrow \int_{t_1}^{b} f_T(t)dt = t_1 f_T(t_1) \quad \text{and} \quad \int_{t_2}^{b} f_T(t)dt = t_2 f_T(t_2),$$

or equivalently, to any $x \in [0, b]$ satisfying

$$\begin{aligned} &\int_x^b f_T(t)dt = x f_T(x) \\ \Rightarrow\quad & 1 - F_T(x) = x f_T(x) \\ \Rightarrow\quad & g_T(x) = x f_T(x) + F_T(x) = 1. \end{aligned} \tag{E.2}$$

Observe that one FOC point is given by $t_1 = t_2 = x^*$, where $x^*$ is a solution to (E.2), which implies

$$\frac{p_1}{s_1} = \frac{p_2 - p_1}{s_2 - s_1} = x^* \;\Rightarrow\; p_1 = \frac{p_2 s_1}{s_2}.$$

Hence, the result follows. □

Note that the objective function (E.1a) is continuous and differentiable with respect to $p_1$ and $p_2$ everywhere in the feasible region. Therefore, from Propositions 17 and 18, a global maxima occurs either at an interior point, or on the line $p_1 s_2 = p_2 s_1$, and satisfies the FOC. Thus, it is sufficient to consider only the FOC points.

**Lemma 4.** Let $m$ be the number of solutions to $g_T(x) \equiv x f_T(x) + F_T(x) = 1$. Then, there exist $\frac{m(m+1)}{2}$ FOC points to the objective function (E.1a) in the feasible region $b \geq \frac{p_2 - p_1}{s_2 - s_1} \geq \frac{p_1}{s_1} \geq 0$.

**Proof:** The FOC points are of the form $0 \leq t_1 \leq t_2 \leq b$ satisfying (E.2). Let $m$ be the number of solutions in $[0, b]$ satisfying (E.2). Thus, if $m = 1$, then $t_1 = t_2 = x^*$, where $x^*$ is the unique solution to (E.2), is the unique FOC point to the objective function (E.1a), and it lies in the feasible region (on the boundary line $p_1 = \frac{p_2 s_1}{s_2}$). If there exist $m \geq 2$ solutions, $x_i^*, i = 1, 2, ..., m$, to (E.1a), then there are $m$ FOC points with $t_1 = t_2 = x_i^*, i = 1, 2, ..., m$, and $\binom{m}{2}$ FOC points with $t_1 = x_i^*, t_2 = x_j^*$ for $i, j \in 1, 2, ..., m$, and $x_i^* < x_j^*$. Furthermore, all FOC points are in the feasible region. Then, the total number of FOC points in the feasible region is given by $m + \binom{m}{2} = \frac{m(m+1)}{2}$. □

**Proposition 19.** Consider that $T$ follows an arbitrary continuous distribution with support in $[0, b]$. Then, if

(i) $g_T(t)$ is nondecreasing in $t$ over $[0, b]$ and strictly increasing at $g_T^{-1}(1)$, or

(ii) $g_T(t)$ is strictly unimodal in $t$ over $[0, b]$,

$K_f^* = 0$ in an optimal solution to the flexible system.

**Proof:** (**i**) We first determine the second-order derivatives of $\Pi^u$ with respect to $p_1$ and $p_2$:

$$\frac{\partial^2\Pi^u}{\partial p_1^2} = -\frac{2}{s_2-s_1}f_T(\frac{p_2-p_1}{s_2-s_1}) - \frac{2}{s_1}f_T(\frac{p_1}{s_2}) - \frac{(p_2-p_1)}{(s_2-s_1)^2}f_T^{(1)}(\frac{p_2-p_1}{s_2-s_1}) - \frac{p_1}{s_1^2}f_T^{(1)}(\frac{p_1}{s_1})$$

$$\frac{\partial^2\Pi^u}{\partial p_2^2} = -\frac{2}{(s_2-s_1)}f_T(\frac{p_2-p_1}{s_2-s_1}) - \frac{(p_2-p_1)}{(s_2-s_1)^2}f_T^{(1)}(\frac{p_2-p_1}{s_2-s_1})$$

$$\frac{\partial^2\Pi^u}{\partial p_1\partial p_2} = \frac{\partial^2\Pi^u}{\partial p_2\partial p_1} = \frac{2}{s_2-s_1}f_T(\frac{p_2-p_1}{s_2-s_1}) + \frac{p_2-p_1}{(s_2-s_1)^2}f_T^{(1)}(\frac{p_2-p_1}{s_2-s_1}).$$

We also know $g_T^{(1)}(x) = 2f_T(x) + xf_T^{(1)}(x)$. Hence, we have

$$\frac{\partial^2\Pi^u}{\partial p_1^2} = \frac{-1}{s_2-s_1}g_T^{(1)}(t_2) - \frac{1}{s_1}g_T^{(1)}(t_1), \quad \frac{\partial^2\Pi^u}{\partial p_2^2} = \frac{-1}{s_2-s_1}g_T^{(1)}(t_2),$$

and

$$det(H) = \frac{1}{(s_2-s_1)s_1}g_T^{(1)}(t_1)g_T^{(1)}(t_2).$$

Notice that $g_T(0) = 0$ and $g_T(b) = bf_T(b) + 1 \geq 1$. Since $g_T(x)$ is nondecreasing in $x$ over $[0, b]$ and strictly increasing at $g_T^{-1}(1)$, there exists a unique point, $x^* \in (0, b]$, satisfying $g_T(x^*) = 1$ such that $g_T^{(1)}(x)|_{x=x^*} > 0$. Consequently, by Lemma 4, there exists only one possible FOC point, $\overrightarrow{t} = (t_1 = x^*, t_2 = x^*)$, which must be the global maxima. If $f_T(b) > 0$, then $x^* \in (0, b)$ (i.e., an internal point), leading to $d_0 > 0, d_1 = 0$, and $d_2 > 0$. If $f_T(b) = 0$, then $x^* = b$, leading to $d_0 = 1$ and $d_1 = d_2 = 0$.

(**ii**) We consider two cases:

(a) If $f_T(b) > 0$, then $g_T(b) = bf_T(b) + 1 > 1$. Since $g_T(x)$ is continuous and strictly unimodal in $x$ over $[0, b]$, there must exist a unique point, $x^* \in (0, b)$, satisfying $g_T(x^*) = 1$ and $g_T^{(1)}(x^*) > 0$. Then, by Lemma 4, $\overrightarrow{t} = (t_1 = x^*, t_2 = x^*)$ gives the unique local maximum solution.

(b) If $f_T(b) = 0$, then $g_T(b) = bf_T(b)+1 = 1$. Hence, there exist exactly two points, $x^*$ and $b$, with $g_T(x^*) = g_T(b) = 1$. Then, from Lemma (4), there must exist three FOC points to the objective function in the feasible region: $(t_1, t_2) = (x^*, x^*), (x^*, b),$ and $(b, b)$.

(1) For $\overrightarrow{t} = (x^*, x^*)$, we know $g_T^{(1)}(x^*) > 0$. Then

$$\frac{\partial^2\Pi}{\partial p_1^2}|_{\overrightarrow{t}=(x^*,x^*)} = -(\frac{1}{s_2-s_1} + \frac{1}{s_1})g_T^{(1)}(x^*) < 0,$$

$$\frac{\partial^2\Pi}{\partial p_2^2}|_{\overrightarrow{t}=(x^*,x^*)} = \frac{-1}{s_2-s_1}g_T^{(1)}(x^*) < 0,$$

$$det(H)|_{\overrightarrow{t}=(x^*,x^*)} = \frac{1}{(s_2-s_1)s_1}[g_T^{(1)}(x^*)]^2 > 0.$$

Thus, $\overrightarrow{t} = (t_1 = x^*, t_2 = x^*)$ is a local maxima.

(2) For $\overrightarrow{t} = (x^*, b)$, we have $g_T^{(1)}(x^*) > 0$ and $g_T^{(1)}(b) < 0$. Then,

$$det(H) = \frac{1}{(s_2 - s_1)s_1} g_T^{(1)}(x^*) g_T^{(1)}(b) < 0.$$

Hence, $\overrightarrow{t} = (x^*, b)$ is a saddle point, and is neither a local maxima nor a local minima.

(3) For $\overrightarrow{t} = (b, b)$, since $g_T^{(1)}(b) < 0$, we have

$$\begin{aligned}
\frac{\partial^2 \Pi}{\partial p_1^2}|_{\overrightarrow{t}=(b,b)} &= -1(\frac{1}{s_2 - s_1} g_T^{(1)} + \frac{1}{s_1}) g_T^{(1)}(b) > 0, \\
\frac{\partial^2 \Pi}{\partial p_2^2}|_{\overrightarrow{t}=(b,b)} &= \frac{-1}{s_2 - s_1} g_T^{(1)}(b) > 0, \\
\text{and } det(H)|_{\overrightarrow{t}=(b,b)} &= \frac{1}{(s_2 - s_1)s_1} [g_T^{(1)}(b)]^2 > 0,
\end{aligned}$$

and $\overrightarrow{t} = (b, b)$ is a local minima.

Therefore, $\overrightarrow{t} = (x^*, x^*)$ is the global maximum solution in both case (i) and (ii) (with $x^* \in (0, b)$), leading to $d_0 > 0, d_1 = 0, d_2 > 0$. □

## 1. Proof of Corollary 12

(i) The proof follows from Proposition 19($i$) by noting that when $f_T^{(1)}(x) > 0$, $g_T^{(1)}(x) = 2f_T(x) + x f_T^{(1)}(x) > 0$ for $x \in (0, b]$, $g_T(x)$ becomes increasing in $x$ over $(0, b]$.

(ii) Note that $g_T(0) = 0$ and $g_T(b) = b f_T(b) + 1 \geq 1$, with $g_T^{(1)}(x)|_{x=0} \geq 0$ and

$$\frac{\partial^2 g_T(x)}{\partial x^2} = 3 f_T^{(1)}(x) + x f_T^{(2)}(x) < 0, \quad \forall x \in [0, b].$$

Let $\hat{x} \equiv \arg\min_x \frac{\partial g_T(x)}{\partial x} = 0$. Note that $\hat{x}$ may not exist in $(0, b)$. There are two cases:

(a) $\hat{x} \in (0, b)$. Then, $\frac{\partial^2 g_T(x)}{\partial x^2}|_{\partial g_T(x)/\partial x = 0} < 0 \Rightarrow g_T(x)$ is strictly unimodal in $x$ and the result follows from Proposition 19($ii$).

(b) Otherwise (i.e., $\hat{x}$ does not exist in $(0, b)$), $g_T(x)$ is strictly increasing in $x$ and the result follows from Proposition 19($i$).

(iii) Let $\bar{x}$ denote the mode of $f_T(x)$. We have $g_T^{(1)}(x) = 2f_T(x) + x f_T^{(1)}(x) > 0$, for $x \in [0, \bar{x}]$. There are two cases. If $g_T^{(1)}(x) > 0$, for $x \in [\bar{x}, b]$, then the result follows from Proposition 19($i$). If $g_T^{(1)}(\bar{x}) = 0$ for some $\hat{x} \in (\bar{x}, b]$, since we have $f_T^{(1)}(\hat{x}) < 0, f_T^{(2)}(\hat{x}) < 0$, and

$$\frac{\partial^2 g_T(x)}{\partial x^2}|_{x=\hat{x}} = 3f_T^{(1)}(\hat{x}) + \hat{x} f_T^{(2)}(\hat{x}) < 0,$$

then the result follows from Proposition 19($ii$). □

## 2. Proof of Corollary 13

(i) We know $g_T^{(1)}(x) = 2f_T(x) + xf_T^{(1)}(x) = a(\gamma + \beta x)^{-p-1}[2\gamma + \beta x(2-p)]$.

  (a) For $p = 2$, $g_T^{(1)}(x) > 0$.

  (b) For $p > 2$, $g_T(x)$ is increasing in $x$ for $x < \frac{2\gamma}{\beta(p-2)}$, and decreasing for $x > \frac{2\gamma}{\beta(p-2)}$. If $b \leq \frac{2\gamma}{\beta(p-2)}$, then $g_T(x)$ is increasing in $x$ over $[0, b]$. If $b > \frac{2\gamma}{\beta(p-2)}$, then $g_T(x)$ is increasing in $x$ in $[0, \frac{2\gamma}{\beta(p-2)})$ and decreasing in $(\frac{2\gamma}{\beta(p-2)}, b]$. Hence, $g_T(x)$ is strictly unimodal over $[0, b]$.

  Hence in both cases it follows, from Proposition (19), that there exists a unique FOC point $t_1 = t_2 = x^*$ with $x^* = g_T^{-1}(1)$.

(ii) The pdf of $T$ is given as

$$f_T(t) = f_{X|X<b}(t) = \begin{cases} \frac{\lambda e^{-\lambda t}}{1-e^{-\lambda b}}, & \text{if } t \leq b; \\ 0, & \text{otherwise.} \end{cases}$$

We know $g_T^{(1)}(x) = \frac{\lambda e^{-\lambda x}(2-\lambda x)}{1-e^{-\lambda b}}$, for $x \in [0, b]$. Hence, $g_T(x)$ is increasing in x for $x < \frac{2}{\lambda}$ and decreasing for $x > \frac{2}{\lambda}$. Consequently, for $b \leq \frac{2}{\lambda}$, $g_T(x)$ is an increasing function, and for $b > \frac{2}{\lambda}$, $g(x)$ is strictly unimodal. Then, from Proposition (19), there exists a unique FOC point $t_1 = t_2 = x^*$ with $g_T(x^*) = 1$ for $x^* \in (0, b)$.

(iii) The pdf of $T$ is given as

$$f_T(x) = f_{X|X<b}(t) = \begin{cases} \frac{1}{a\sqrt{2\pi\sigma^2}} e^{-\frac{(t-\mu)^2}{2\sigma^2}}, & \text{if } t \in [0, b]; \\ 0, & \text{otherwise,} \end{cases}$$

We know $g_T^{(1)}(x) = f_T(x)(\frac{-x^2+\mu x+2\sigma^2}{\sigma^2})$, for $x \in [0, b]$. Note that for $\frac{\mu-\sqrt{\mu^2+8\sigma^2}}{2} \leq x \leq \frac{\mu+\sqrt{\mu^2+8\sigma^2}}{2}$, we have $-x^2 + \mu x + 2\sigma^2 \geq 0$, and for $x > \frac{\mu+\sqrt{\mu^2+8\sigma^2}}{2}$, we have $-x^2 + \mu x + 2\sigma^2 < 0$. Noting that $\frac{\mu-\sqrt{\mu^2+8\sigma^2}}{2} \leq 0 \leq \frac{\mu+\sqrt{\mu^2+8\sigma^2}}{2}$, we have

$$\begin{cases} \text{if } b \leq \frac{\mu+\sqrt{\mu^2+8\sigma^2}}{2}, & g^{(1)}(x) > 0; \\ \text{if } b < \frac{\mu+\sqrt{\mu^2+8\sigma^2}}{2}, & g(x) \text{ is strictly unimodal.} \end{cases}$$

Hence, from Lemma (4), there exists a unique FOC point $t_1 = t_2 = x^*$ with $g_T(x^*) = 1$ for $x^* \in (0, b)$. $\square$

# Appendix F

**1. Proof of Proposition** 14.

(i) The problem formulation in this case is given as follows:

Problem $\mathbf{P^c(SU)}$:

Stage 1 Problem $\mathbf{P_1^c(SU)}$:

$$\max_{K_f} V \equiv E_Y[\Pi^*(K_f, Y)] - c_f K_f \tag{F.1a}$$

$$\text{subject to } K_f \geq 0. \tag{F.1b}$$

Stage 2 Problem $\mathbf{P_2^c(SU)}$:

$$\Pi^*(\overrightarrow{K}, n) \equiv \max_{\overrightarrow{p}} \frac{n}{b}[(\frac{p_2 - p_1}{s_2 - s_1} - \frac{p_1}{s_1})(p_1 - \alpha s_1^2) + (b - \frac{(p_2 - p_1)}{s_2 - s_1})(p_2 - \alpha s_2^2)] \tag{F.2a}$$

subject to

$$yK_f - \frac{n}{b}(b - \frac{p_1}{s_1}) \geq 0 \tag{F.2b}$$

$$\frac{n}{b}(\frac{p_2 - p_1}{s_2 - s_1} - \frac{p_1}{s_1}) \geq 0 \tag{F.2c}$$

$$\frac{n}{b}(b - \frac{(p_2 - p_1)}{s_2 - s_1}) \geq 0 \tag{F.2d}$$

Solving the second stage problem, we obtain the optimal solution:

$$(p_1^*, p_2^*) = \begin{cases} (\frac{(b+\alpha s_1)s_1}{2}, \ \frac{(b+\alpha s_2)s_2}{2}), & \text{if } \ \Omega_1, \\ (bs_1(1 - \frac{yK_f}{n}), \ \frac{b(s_1+s_2)}{2} - \frac{bs_1 yK_f}{n} + \frac{\alpha(s_2^2 - s_1^2)}{2}), & \text{if } \ \Omega_2, \\ (bs_1(1 - \frac{yK_f}{n}), \ \ bs_2(1 - \frac{yK_f}{n})), & \text{if } \ \Omega_6, \end{cases}$$

$$(q_1^*, q_2^*) = \begin{cases} (\frac{n\alpha s_2}{2b}, \ \frac{n(b-\alpha(s_1+s_2))}{2b}), \ \text{if} \ \Omega_1, \\ (yK_f - \frac{n[b-\alpha(s_1+s_2)]}{2b}, \ \frac{n[b-\alpha(s_1+s_2)]}{2b}), \ \text{if} \ \Omega_2, \\ (0, \ yK_f), \ \text{if} \ \Omega_6, \end{cases}$$

where $\Omega_1 = \{1 \geq y \geq \frac{n(b-\alpha s_1)}{2bK_f}\}$, $\Omega_2 = \{\frac{n(b-\alpha s_1)}{2bK_f} \geq y \geq \frac{n(b-\alpha(s_1+s_2))}{2bK_f}\}$, $\Omega_6 = \{\frac{n(b-\alpha(s_1+s_2))}{2bK_f} \geq y \geq 0\}$.

Then, the objective function of stage 1 problem and its first- and second-order derivatives are given as

$$V = \int_0^{\frac{[b-\alpha(s_1+s_2)]n}{2bK_f}} \Pi_6 f_Y(y)dy + \int_{\frac{[b-\alpha(s_1+s_2)]n}{2bK_f}}^{\frac{(b-\alpha s_1)n}{2bK_f}} \Pi_2 f_Y(y)dy + \int_{\frac{(b-\alpha s_1)n}{2bK_f}}^{1} \Pi_1 f_Y(y)dy - c_f K_f$$

$$\begin{aligned} \frac{\partial V}{\partial K_f} &= \int_0^{\frac{[b-\alpha(s_1+s_2)]n}{2bK_f}} \frac{\partial \Pi_6}{\partial K_f} f_Y(y)dy + \int_{\frac{[b-\alpha(s_1+s_2)]n}{2bK_f}}^{\frac{(b-\alpha s_1)n}{2bK_f}} \frac{\partial \Pi_2}{\partial K_f} f_Y(y)dy - c_f \\ &= \int_0^{\frac{[b-\alpha(s_1+s_2)]n}{2bK_f}} (bs_2 - \alpha s_2^2 - \frac{2bs_2yK_f}{n}) y f_Y(y)dy \\ &\quad + \int_{\frac{[b-\alpha(s_1+s_2)]n}{2bK_f}}^{\frac{(b-\alpha s_1)n}{2bK_f}} (bs_1 - \alpha s_1^2 - \frac{2bs_1yK_f}{n}) y f_Y(y)dy - c_f \end{aligned}$$

$$\frac{\partial^2 V}{\partial K_f^2} = \int_0^{\frac{[b-\alpha(s_1+s_2)]n}{2bK_f}} -\frac{2bs_2y^2}{n} f_Y(y)dy + \int_{\frac{[b-\alpha(s_1+s_2)]n}{2bK_f}}^{\frac{(b-\alpha s_1)n}{2bK_f}} -\frac{2bs_1y^2}{n} f_Y(y)dy < 0.$$

Hence, the optimal solution is given by

$$\begin{aligned} \frac{\partial V}{\partial K_f} = 0 \ \Rightarrow \ & \int_0^{\frac{[b-\alpha(s_1+s_2)]n}{2bK_f}} (bs_2 - \alpha s_2^2 - \frac{2bs_2yK_f}{n}) y f_Y(y)dy \\ & + \int_{\frac{[b-\alpha(s_1+s_2)]n}{2bK_f}}^{\frac{(b-\alpha s_1)n}{2bK_f}} (bs_1 - \alpha s_1^2 - \frac{2bs_1yK_f}{n}) y f_Y(y)dy = c_f \end{aligned} \tag{F.3}$$

Note that the LHS of (F.3) is strictly decreasing in $K_f$. Then, for $K_f \geq 0$, we need $c_f \leq \int_0^1 (bs_2 - \alpha s_2^2) y f_Y(y)dy = (b - \alpha s_2)s_2 E[Y]$.

(ii) The problem formulation in this case is similar to the one in (i), while $Y$ follows a Bernoulli distribution with $\Pr(Y = 1) = \theta$. Then, when $y = 1$, we need to consider three cases for the Stage 1 problem:

Case 1: $K_f \geq \frac{n(b-\alpha s_1)}{2b}$

We have $V = \frac{n\alpha(b+\alpha s_1)s_1 s_2}{4b} + \frac{n(b-\alpha(s_1+s_2))(b+\alpha s_2)s_2}{4b} - c_f K_f$ and $\frac{\partial V}{\partial K_f} = -c_f < 0$. Then, the optimal solution is given by $K_f = \frac{n(b-\alpha s_1)}{2b}$.

Case 2: $\frac{n(b-\alpha s_1)}{2b} \geq K_f \geq \frac{n(b-\alpha(s_1+s_2))}{2b}$

We have

$$\begin{aligned}
V &= \theta b s_1(1-\frac{K_f}{n})(K_f - \frac{n[b-\alpha(s_1+s_2)]}{2b}) \\
&\quad + \theta[\frac{b(s_1+s_2)}{2} - \frac{bs_1K_f}{n} + \frac{\alpha(s_2^2-s_1^2)}{2}][\frac{n[b-\alpha(s_1+s_2)]}{2b}] - c_fK_f \\
\frac{\partial V}{\partial K_f} &= \theta(bs_1 - \alpha s_1^2 - \frac{2bs_1K_f}{n}) - c_f \\
\frac{\partial^2 V}{\partial K_f^2} &= -\frac{2\theta b s_1 K_f}{n} < 0.
\end{aligned}$$

The optimal solution is given by $\frac{\partial V}{\partial K_f} = 0 \;\Rightarrow\; K_f = \frac{(\theta b s_1 - \theta\alpha s_1^2 - c_f)n}{2\theta b s_1}$.

Case 3: $K_f \leq \frac{n(b-\alpha(s_1+s_2))}{2b}$

We have $V = bs_2(1-\frac{K_f}{n})K_f - c_fK_f$, $\frac{\partial V}{\partial K_f} = \theta(bs_2 - \alpha s_2^2 - \frac{2bs_2K_f}{n}) - c_f$, and $\frac{\partial^2 V}{\partial K_f^2} = -\frac{2\theta b s_2 K_f}{n} < 0$. Then, the optimal solution is given by $\frac{\partial V}{\partial K_f} = 0 \;\Rightarrow\; K_f = \frac{(\theta b s_2 - \theta\alpha s_2^2 - c_f)n}{2\theta b s_2}$.

Comparing the above three cases, we obtain the optimal solution to the Stage 1 problem:

(1) If $\theta\alpha s_1 s_2 \leq c_f \leq \theta(bs_2 - \alpha s_2^2)$, $K_f^* = \frac{(\theta b s_2 - \theta\alpha s_2^2 - c_f)n}{2\theta b s_2}$;

(2) If $c_f \leq \theta\alpha s_1 s_2$, $K_f^* = \frac{(\theta b s_1 - \theta\alpha s_1^2 - c_f)n}{2\theta b s_1}$.

Therefore, when $c_f \leq \theta(bs_2 - \alpha s_2^2)$, we have $K_f^* \geq 0$.

(iii) The problem formulation in the case is given as follows:

Problem $\mathbf{P^c}(\mathbf{DU})$:

Stage 1 Problem $\mathbf{P_1^c}(\mathbf{DU})$:

$$\max_{K_f} V \equiv E_N[\Pi^*(K_f, N)] - c_f K_f \tag{F.4a}$$

$$\text{subject to } K_f \geq 0. \tag{F.4b}$$

Stage 2 Problem $\mathbf{P_2^c(DU)}$:

$$\Pi^*(\overrightarrow{K}, n) \equiv \max_{\overrightarrow{p}} \frac{n}{b}[(\frac{p_2 - p_1}{s_2 - s_1} - \frac{p_1}{s_1})(p_1 - \alpha s_1^2) + (b - \frac{(p_2 - p_1)}{s_2 - s_1})(p_2 - \alpha s_2^2)] \tag{F.5a}$$

subject to

$$K_f - \frac{n}{b}(b - \frac{p_1}{s_1}) \geq 0 \tag{F.5b}$$

$$\frac{n}{b}(\frac{p_2 - p_1}{s_2 - s_1} - \frac{p_1}{s_1}) \geq 0 \tag{F.5c}$$

$$\frac{n}{b}(b - \frac{(p_2 - p_1)}{s_2 - s_1}) \geq 0 \tag{F.5d}$$

Solving the second stage problem, we obtain the optimal solution:

$$(p_1^*, p_2^*) = \begin{cases} (\frac{(b+\alpha s_1)s_1}{2}, \ \frac{(b+\alpha s_2)s_2}{2}), & \text{if } \ \Omega_1, \\ (bs_1(1 - \frac{K_f}{n}), \ \frac{b(s_1+s_2)}{2} - \frac{bs_1K_f}{n} + \frac{\alpha(s_2^2 - s_1^2)}{2}), & \text{if } \ \Omega_2, \\ (bs_1(1 - \frac{K_f}{n}), \ bs_2(1 - \frac{K_f}{n})), & \text{if } \ \Omega_6, \end{cases}$$

$$(q_1^*, q_2^*) = \begin{cases} (\frac{n\alpha s_2}{2b}, \ \frac{n(b-\alpha(s_1+s_2))}{2b}), & \text{if } \ \Omega_1, \\ (K_f - \frac{n[b-\alpha(s_1+s_2)]}{2b}, \ \frac{n[b-\alpha(s_1+s_2)]}{2b}), & \text{if } \ \Omega_2, \\ (0, \ K_f), & \text{if } \ \Omega_6, \end{cases}$$

where $\Omega_1 = \{0 \leq n \leq \frac{2bK_f}{b-\alpha s_1}\}$, $\Omega_2 = \{\frac{2bK_f}{b-\alpha s_1} \leq n \leq \frac{2bK_f}{b-\alpha(s_1+s_2)}\}$, $\Omega_6 = \{\frac{2bK_f}{b-\alpha(s_1+s_2)} \leq n\}$.

Then, the objective function of stage 1 problem and its first- and second-order derivatives are given as

$$
\begin{aligned}
V &= \int_0^{\frac{2bK_f}{b-\alpha s_1}} \Pi_1 f_N(n)dn + \int_{\frac{2bK_f}{b-\alpha s_1}}^{\frac{2bK_f}{b-\alpha(s_1+s_2)}} \Pi_2 f_N(n)dn + \int_{\frac{2bK_f}{b-\alpha(s_1+s_2)}}^{\infty} \Pi_6 f_N(n)dn - c_f K_f \\
\frac{\partial V}{\partial K_f} &= \int_{\frac{2bK_f}{b-\alpha s_1}}^{\frac{2bK_f}{b-\alpha(s_1+s_2)}} \frac{\partial \Pi_2}{\partial K_f} f_N(n)dn + \int_{\frac{2bK_f}{b-\alpha(s_1+s_2)}}^{\infty} \frac{\partial \Pi_6}{\partial K_f} f_N(n)dn - c_f \\
&= \int_{\frac{2bK_f}{b-\alpha s_1}}^{\frac{2bK_f}{b-\alpha(s_1+s_2)}} (bs_1 - \alpha s_1^2 - \frac{2bs_1K_f}{n}) f_N(n)dn \\
&\quad + \int_{\frac{2bK_f}{b-\alpha(s_1+s_2)}}^{\infty} (bs_2 - \alpha s_2^2 - \frac{2bs_2K_f}{n}) f_N(n)dn - c_f \\
\frac{\partial^2 V}{\partial K_f^2} &= \int_{\frac{2bK_f}{b-\alpha s_1}}^{\frac{2bK_f}{b-\alpha(s_1+s_2)}} -\frac{2bs_1}{n} f_N(n)dn + \int_{\frac{2bK_f}{b-\alpha(s_1+s_2)}}^{\infty} -\frac{2bs_2}{n} f_N(n)dn < 0.
\end{aligned}
$$

Hence, the optimal solution is given by

$$
\begin{aligned}
\frac{\partial V}{\partial K_f} = 0 \quad \Rightarrow \quad & \int_{\frac{2bK_f}{b-\alpha s_1}}^{\frac{2bK_f}{b-\alpha(s_1+s_2)}} (bs_1 - \alpha s_1^2 - \frac{2bs_1K_f}{n}) f_N(n)dn \\
& + \int_{\frac{2bK_f}{b-\alpha(s_1+s_2)}}^{\infty} (bs_2 - \alpha s_2^2 - \frac{2bs_2K_f}{n}) f_N(n)dn = c_f.
\end{aligned} \tag{F.6}
$$

Note that the LHS of (F.6) is strictly decreasing in $K_f$. Then, for $K_f \geq 0$, we need $c_f \leq \int_0^{\infty} (bs_2 - \alpha s_2^2) f_N(n)dn = (b - \alpha s_2)s_2$. □

# Vita

Weiping Chen was born in November, 1981 at Hunan, P.R.China. He completed his Ph.D. studies from the Grado Department of Industrial and Systems Engineering at Virginia Tech in Summer 2007. His research focuses on the development of optimal resource investment and pricing strategies for quality-differentiated products under supply and demand uncertainty. In July 1997, he obtained his B.S. degree in Mechanical Engineering from Beijing University of Aeronautics and Astronautics, Beijing, China, and in May 2005, he obtained her M.S. degree in Industrial and Systems Engineering from Virginia Tech.

While working towards his Ph.D. degree, Weiping was awarded Dover fellowship in 2004 and Pratt fellowship in 2006 by the ISE Department. In 2004, he also received the First Place award of Material Handling Design Competition from Material Handling Industry of America. He is a finalist in the 2007 Doctoral Dissertation Proposal Award in Supply Chain Optimization sponsored by the University of Florida Supply Chain And Logistic Engineering Center. He has served as vice president of SME Virginia Tech Chapter in 2005, and president of INFORMS Virginia Tech Chapter in 2006.