AIP | Journal of Mathematical Physics

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Citation: Journal of Mathematical Physics **19**, 359 (1978); doi: 10.1063/1.523679
View online: http://dx.doi.org/10.1063/1.523679
View Table of Contents: http://scitation.aip.org/content/aip/journal/jmp/19/2?ver=pdfcov
Published by the AIP Publishing

A kinetic formulation of the three-dimensional quantum mechanical harmonic oscillator under a random perturbation

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(Received 18 July 1977)

The behavior of a three-dimensional, nonrelativistic, quantum mechanical harmonic oscillator is investigated under the influence of three distinct types of randomly fluctuating potential fields. Specifically, kinetic (or transport) equations are derived for the corresponding stochastic Wigner equation (the exact equation of evolution of the phase-space Wigner distribution density function) and the stochastic Liouville equation (correspondence limit approximation) using two closely related statistical techniques, the first-order smoothing and the long-time Markovian approximations. Several physically important averaged observables are calculated in special cases. In the absence of a deterministic inhomogeneous potential field (randomly perturbed, freely propagating particle), the results reduce to those reported previously by Besieris and Tappert.

1. INTRODUCTION

In a previous paper,[1] referred to in the sequel as Paper I, kinetic equations were derived for the stochastic Wigner equation (the exact equation of evolution of the phase-space Wigner distribution density function) and the stochastic Liouville equation (correspondence limit approximation) associated with the quantized nonrelativistic motion of a particle described by a stochastic Schrödinger equation having a deterministic background potential field independent of the space and time coordinates. It is our purpose in this paper to lift the latter restriction and investigate specifically the behavior of a three-dimensional quantum mechanical harmonic oscillator experiencing a random perturbation.

Consider the stochastic Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{x},t;\alpha)=H_{\rm op}\left(\mathbf{x},-i\hbar\frac{\partial}{\partial\mathbf{x}},t;\alpha\right)\psi(\mathbf{x},t;\alpha),\qquad t>t_0,\quad \mathbf{x}\in R^3, \tag{1.1a}$$

$$H_{\rm op}\left(\mathbf{x},-i\hbar\frac{\partial}{\partial\mathbf{x}},t;\alpha\right)=-\frac{\hbar^2}{2m}\nabla^2+V(\mathbf{x},t;\alpha), \tag{1.1b}$$

$$\psi(\mathbf{x},t_0;\alpha)=\psi_0(\mathbf{x}). \tag{1.1c}$$

Here, the Hamiltonian $H_{\rm op}$ is a self-adjoint, stochastic operator depending on a parameter $\alpha\in A$; (A,F,P) being an underlying probability measure space. In addition, $\psi(\mathbf{x},t;\alpha)$, the complex random wavefunction, is an element of an infinitely dimensional vector space H, and $V(\mathbf{x},t;\alpha)$ is the potential field which is assumed to be a real, space- and time-dependent random function.

In the course of this work we shall deal explicitly with the following three distinct categories of the potential field:

(i) $V(\mathbf{x},t;\alpha)=\frac{1}{2}kx^2+\delta V(\mathbf{x},t;\alpha)$, (1.2a)

(ii) $V(\mathbf{x},t;\alpha)=\frac{1}{2}kx^2[1+\delta G(t;\alpha)]$, (1.2b)

(iii) $V(\mathbf{x},t;\alpha)=\frac{1}{2}k[\mathbf{x}-\mathbf{a}\delta H(t;\alpha)]^2$, (1.2c)

where $x=|\mathbf{x}|$, k is a positive real constant number, and $\mathbf{a}$ is a fixed vector quantity. The first category corresponds to a linear harmonic oscillator immersed in a zero-mean, space- and time-dependent, random potential field $\delta V(\mathbf{x},t;\alpha)$; the second one is the case of a harmonic oscillator whose frequency is modulated by the zero-mean, time-dependent, random field $\delta G(t;\alpha)$; finally, the third type of potential is associated with a harmonic oscillator whose equilibrium position is perturbed via the zero-mean, time-dependent, random function $\delta H(t;\alpha)$. (This is also closely linked to the Brownian motion arising from a randomly forced harmonic oscillator.)

The random quantum mechanical harmonic oscillator problem corresponding to potential fields of types (ii) and (iii) has already been investigated extensively by several workers under specific restrictive assumptions regarding the random processes $\delta G(t;\alpha)$ and $\delta H(t;\alpha)$. We cite here the early treatment of the Brownian motion of a quantum oscillator by Schwinger,[2] and the quantum theory of a randomly modulated harmonic oscillator by Crosignani *et al.*[3] and Mollow.[4] A more complete account of the statistical analysis of the quantum mechanical oscillator, with applications to quantum optics, can be found in the recent review article by Agarwal.[5]

Besides its generic significance in quantum mechanics, the random harmonic oscillator is of fundamental importance in other physical areas since it provides a dynamic model incorporating salient features common to all of them. For example, Schrödinger-like equations of the form (1.1) and (1.2) play a significant role in plane and beam electromagnetic and acoustic wave propagation. They are usually derived from a scalar Helmholtz equation within the framework of the parabolic (or small-angle) approximation. Statistical analyses of optical wave propagation in randomly perturbed lenslike media have been undertaken by Vorob'ev,[6] Papanicolaou *et al.*,[7] McLaughlin,[8] Beran and Whitman,[9] and Chow.[10] Along the same vein, starting from a space—time parabolic approximation to the full wave equation, Besieris and Kohler[11] have recently considered the problem of underwater sound wave propagation in the presence of a randomly perturbed parabolic sound speed profile.

 0022-2488/78/1902-0359$1.00

It is our intent in this paper to present a unified stochastic kinetic analysis of the random harmonic oscillator, which is equally applicable to the three types of potential field in (1.2), without imposing physically unjustifiable restrictions on the random processes δV, δG, and δH. Special emphasis will be placed on the additional effects contained in our formulation as compared with previously reported results. Finally, it should be pointed out that although the discussion in this paper is restricted to the quantum mechanical random harmonic oscillator, the main results are also applicable to other physical problems by virtue of the statements made in the previous paragraph.

2. THE STOCHASTIC WIGNER DISTRIBUTION FUNCTION

The phase-space analog of the equal-time, two-point density function for a pure state,

$$\rho(\mathbf{x}_2, \mathbf{x}_1, t; \alpha) = \psi^*(\mathbf{x}_2, t; \alpha)\psi(\mathbf{x}_1, t; \alpha), \tag{2.1}$$

is provided by the Wigner distribution function which is defined as follows[12]:

$$f(\mathbf{x}, \mathbf{p}, t; \alpha) = (2\pi\hbar)^{-3} \int_{R^3} d\mathbf{y} \exp(i\mathbf{p}\cdot\mathbf{y}/\hbar) \times\rho(\mathbf{x}+\tfrac{1}{2}\mathbf{y}, \mathbf{x}-\tfrac{1}{2}\mathbf{y}, t; \alpha). \tag{2.2}$$

This quantity is real, but not necessarily positive everywhere. It can be shown (cf. Appendix A; also Ref. 13), in general, that $|f(\mathbf{x}, \mathbf{p}, t; \alpha)| \leq (\hbar\pi)^{-3}$ for any realization $\alpha \in A$. Provided that $f(\mathbf{x}, \mathbf{p}, t; \alpha)$ is normalized (to unity), this means that the Wigner distribution function is different from zero in a region of which the volume in phase space is at least equal to $(\hbar\pi)^3$. Hence, $f(\mathbf{x}, \mathbf{p}, t; \alpha)$ can never be sharply localized in $\mathbf{x}$ and $\mathbf{p}$. This situation is a reflection of the uncertainty principle.[14]

The total wave energy and wave action are given in terms of the Wigner distribution function as follows:

$$E = \int_{R^3} d\mathbf{x} \int_{R^3} d\mathbf{p} H(\mathbf{x}, \mathbf{p}, t; \alpha) f(\mathbf{x}, \mathbf{p}, t; \alpha), \tag{2.3a}$$

$$A = \int_{R^3} d\mathbf{x} \int_{R^3} d\mathbf{p} f(\mathbf{x}, \mathbf{p}, t; \alpha). \tag{2.3b}$$

Here, $H(\mathbf{x}, \mathbf{p}, t; \alpha)$ is the Weyl transform of the operator H_{op} and is given explicitly as

$$H(\mathbf{x}, \mathbf{p}, t; \alpha) = \frac{1}{2m} p^2 + V(\mathbf{x}, t; \alpha), \quad p \equiv |\mathbf{p}|. \tag{2.4}$$

The total wave energy is not conserved since the potential field is assumed to be time dependent. On the other hand, the total wave action is conserved because of the self-adjointness of the Hamiltonian operator, a property satisfied by the three types of potential fields in (1.2).

The time evolution of the Wigner distribution function is governed by the equation

$$\frac{\partial}{\partial t} f(\mathbf{x}, \mathbf{p}, t; \alpha) = Lf(\mathbf{x}, \mathbf{p}, t; \alpha), \tag{2.5a}$$

$$Lf(\mathbf{x}, \mathbf{p}, t; \alpha) = -\frac{1}{m}\mathbf{p}\cdot\frac{\partial}{\partial \mathbf{x}} f(\mathbf{x}, \mathbf{p}, t; \alpha) + \theta f(\mathbf{x}, \mathbf{p}, t; \alpha). \tag{2.5b}$$

The potential-dependent term on the right-hand side of (2.5b) can be cast into the following three useful representations:

(i) $$\theta f(\mathbf{x}, \mathbf{p}, t; \alpha) = \int_{R^3} d\mathbf{p}' K(\mathbf{x}, \mathbf{p}-\mathbf{p}', t; \alpha) f(\mathbf{x}, \mathbf{p}', t; \alpha),$$
$$K(\mathbf{x}, \mathbf{p}, t; \alpha) = (i\hbar)^{-1}(2\pi\hbar)^{-3} \int_{R^3} d\mathbf{y} \exp(i\mathbf{p}\cdot\mathbf{y}/\hbar) \times[V(\mathbf{x}-\tfrac{1}{2}\mathbf{y}, t; \alpha) - V(\mathbf{x}+\tfrac{1}{2}\mathbf{y}, t; \alpha)]; \tag{2.6a}$$

(ii) $$\theta f(\mathbf{x}, \mathbf{p}, t; \alpha) = (i\hbar)^{-1}(2\pi\hbar)^{-3} \int_{R^3} d\mathbf{y} \exp(i\mathbf{p}\cdot\mathbf{y}/\hbar) \times\rho(\mathbf{x}+\tfrac{1}{2}\mathbf{y}, \mathbf{x}-\tfrac{1}{2}\mathbf{y}, t; \alpha) \times[V(\mathbf{x}-\tfrac{1}{2}\mathbf{y}, t; \alpha) - V(\mathbf{x}+\tfrac{1}{2}\mathbf{y}, t; \alpha)]; \tag{2.6b}$$

(iii) $$\theta f(\mathbf{x}, \mathbf{p}, t; \alpha) = V(\mathbf{x}, t; \alpha)\frac{2}{\hbar} \times\sin\left[\frac{\hbar}{2}\left(\frac{\overleftarrow{\partial}}{\partial \mathbf{x}}\cdot\frac{\overrightarrow{\partial}}{\partial \mathbf{p}}\right)\right] f(\mathbf{x}, \mathbf{p}, t; \alpha). \tag{2.6c}$$

We shall refer to the exact equation of evolution of $f(\mathbf{x}, \mathbf{p}, t; \alpha)$ as the *stochastic Wigner equation*.

It is seen from (2.6c) that in the correspondence limit ($\hbar \to 0$),

$$\theta f(\mathbf{x}, \mathbf{p}, t; \alpha) = \frac{\partial}{\partial \mathbf{x}} V(\mathbf{x}, t; \alpha)\cdot\frac{\partial}{\partial \mathbf{p}} f(\mathbf{x}, \mathbf{p}, t; \alpha) + O(\hbar^2). \tag{2.7}$$

Within the limits of this approximation, we shall refer to (2.5) as the *stochastic Liouville equation*.

We shall next list the specific realization of $\theta f(\mathbf{x}, \mathbf{p}, t; \alpha)$ corresponding to the three choices of the potential field $V(\mathbf{x}, t; \alpha)$ in (1.2):

(i) $$\theta f(\mathbf{x}, \mathbf{p}, t; \alpha) = \left(k\mathbf{x}\cdot\frac{\partial}{\partial \mathbf{p}} + \frac{\partial}{\partial \mathbf{x}}\delta V(\mathbf{x}, t; \alpha)\right) \times f(\mathbf{x}, \mathbf{p}, t; \alpha) + O(\hbar^2); \tag{2.8a}$$

(ii) $$\theta f(\mathbf{x}, \mathbf{p}, t; \alpha) = \left(k\mathbf{x}\cdot\frac{\partial}{\partial \mathbf{p}} + kG(t; \alpha)\mathbf{x}\cdot\frac{\partial}{\partial \mathbf{p}}\right) \times f(\mathbf{x}, \mathbf{p}, t; \alpha); \tag{2.8b}$$

(iii) $$\theta f(\mathbf{x}, \mathbf{p}, t; \alpha) = \left(k\mathbf{x}\cdot\frac{\partial}{\partial \mathbf{p}} - k\delta H(t; \alpha)\mathbf{a}\cdot\frac{\partial}{\partial \mathbf{p}}\right) \times f(\mathbf{x}, \mathbf{p}, t; \alpha). \tag{2.8c}$$

It should be noted that the last two expressions for $\theta f(\mathbf{x}, \mathbf{p}, t; \alpha)$ are exact. This is due to the special forms of the representations for $V(\mathbf{x}, t; \alpha)$ in (1.2b) and (1.2c).

3. GENERAL EQUATIONS FOR THE MEAN WIGNER DISTRIBUTION FUNCTION

The stochastic Wigner distribution function f and the operator L [cf. Eq. (2.5a)] are next separated into mean and fluctuating parts:

$$f(\mathbf{x}, \mathbf{p}, t; \alpha) = E\{f(\mathbf{x}, \mathbf{p}, t; \alpha)\} + \delta f(\mathbf{x}, \mathbf{p}, t; \alpha), \tag{3.1a}$$

$$L = E\{L\} + \delta L. \tag{3.1b}$$

On the basis of the first-order smoothing approximation,[17–19] one obtains the following general kinetic equation for the ensemble average of f:

$$\left(\frac{\partial}{\partial t} - E\{L\}\right)E\{f(\mathbf{x},\mathbf{p},t;\alpha)\} = \int_0^t d\tau\, E\{\delta L(t)\exp[\tau E\{L\}]\delta L(t-\tau)\}E\{f(\mathbf{x},\mathbf{p},t-\tau;\alpha)\}. \tag{3.2}$$

In deriving (3.2) it has been assumed that $\delta f(\mathbf{x},\mathbf{p},0;\alpha) = 0$ and that $E\{L\}$ is independent of the time variable. [The latter condition is satisfied for the three types of potential fields prescribed in (1.2)]. This kinetic equation is uniformly valid in time. The right-hand side of (3.2) contains generalized operators (nonlocal, with memory) in phase space.

Various levels of simplification can be obtained by introducing additional constraints. For example, the long-time Markovian results in the simpler kinetic equation

$$\left(\frac{\partial}{\partial t} - E\{L\}\right)E\{f(\mathbf{x},\mathbf{p},t;\alpha)\} = \int_0^\infty d\tau\, E\{\delta L(t)\exp[\tau E\{L\}]\delta L(t-\tau)\} \times \exp[-\tau E\{L\}]E\{f(\mathbf{x},\mathbf{p},t;\alpha)\}. \tag{3.3}$$

This particular functional form is due to Van Kampen.[20] It should be pointed out, however, that this expression is identical to Eq. (3.3) of Paper I. A detailed discussion of the long-time Markovian approximation can be found in Refs. 20 and 21. Here, we mention simply that in addition to the usual assumptions entering into the first-order smoothing approximation (cf. Refs. 17–19), the derivation of (3.3) presupposes that $E\{f\}$ vary slowly on the scale of the correlation time of δL.

Having established an expression for the mean Wigner distribution function by solving either of the above kinetic equations, physical observables, such as the average probability density, the average probability current density, the centroid of a wavepacket, the spread of a wavepacket, etc., can be found by taking appropriate phase-space moments (cf. Paper I).

If the mean Wigner distribution function is normalized to unity, i.e.,

$$\int_{R^3} d\mathbf{x} \int_{R^3} d\mathbf{p}\, E\{f(\mathbf{x},\mathbf{p},t;\alpha)\} = 1, \tag{3.4}$$

the following general relationship holds:

$$D^2(t) \equiv (2\pi\hbar)^3 \int_{R^3} d\mathbf{x} \int_{R^3} d\mathbf{p}\, [E\{f(\mathbf{x},\mathbf{p},t;\alpha)\}]^2 \leq 1. \tag{3.5}$$

[A proof of (3.5) is outlined in Appendix B.] Equality holds if and only if $E\{f\}$ is a "pure" state. Otherwise, $E\{f\}$ is said to represent a "mixed" state, and D (which we shall call the *degree of coherence*) is less than unity.

4. KINETIC THEORY FOR THE STOCHASTIC WIGNER EQUATION

The results of the previous section are specialized here to the stochastic Wigner equation (2.5) corresponding to the potential field given in (1.2a), viz., $V(\mathbf{x},t;\alpha) = \frac{1}{2}kx^2 + \delta V(\mathbf{x},t;\alpha)$. It is convenient to use for this purpose the representation (2.6a) for $\theta f(\mathbf{x},\mathbf{p},t;\alpha)$.

A. The first-order smoothing approximation

The mean and fluctuating parts of the operator L in (2.5) are given explicitly as follows:

$$E\{L\} = -\frac{1}{m}\mathbf{p}\cdot\frac{\partial}{\partial\mathbf{x}} + k\mathbf{x}\cdot\frac{\partial}{\partial\mathbf{p}}, \tag{4.1}$$

$$\delta L = \int_{R^3} d\mathbf{p}'\, \delta K(\mathbf{x},\mathbf{p}-\mathbf{p}',t;\alpha)(\cdot), \tag{4.2a}$$

$$\delta K(\mathbf{x},\mathbf{p},t;\alpha) = (i\hbar)^{-1}(2\pi\hbar)^{-3}\int_{R^3} d\mathbf{y}\exp(i\mathbf{p}\cdot\mathbf{y}/\hbar) \times[\delta V(\mathbf{x}-\tfrac{1}{2}\mathbf{y},t;\alpha) - \delta V(\mathbf{x}+\tfrac{1}{2}\mathbf{y},t;\alpha)]. \tag{4.2b}$$

Introducing (4.1) and (4.2a) in (3.2), we determine the following equation for the ensemble average of the Wigner distribution function within the framework of the first-order smoothing approximation:

$$\left(\frac{\partial}{\partial t} + \frac{1}{m}\mathbf{p}\cdot\frac{\partial}{\partial\mathbf{x}} - k\mathbf{x}\cdot\frac{\partial}{\partial\mathbf{p}}\right)E\{f(\mathbf{x},\mathbf{p},t;\alpha)\} = \Theta E\{f(\mathbf{x},\mathbf{p},t;\alpha)\}, \tag{4.3a}$$

$$\begin{aligned}\Theta E\{f(\mathbf{x},\mathbf{p},t;\alpha)\} &= \int_0^t d\tau \int_{R^3} d\mathbf{p}' \int_{R^3} d\mathbf{p}''\, E\{\delta K(\mathbf{x},\mathbf{p}-\mathbf{p}',t;\alpha)\\ &\times \delta K[\mathbf{x}\cos\omega_0\tau - (\mathbf{p}'/m\omega_0)\sin\omega_0\tau, \mathbf{x}m\omega_0\sin\omega_0\tau\\ &+ \mathbf{p}''\cos\omega_0\tau - \mathbf{p}'', t-\tau;\alpha]\}E\{f[\mathbf{x}\cos\omega_0\tau\\ &- (\mathbf{p}'/m\omega_0)\sin\omega_0\tau, \mathbf{p}'', t-\tau]\},\end{aligned} \tag{4.3b}$$

where $\omega_0 = (k/m)^{1/2}$. In deriving this equation we have made use of the well-known propagator property

$$\exp\left[\tau\left(-\frac{1}{m}\mathbf{p}\cdot\frac{\partial}{\partial\mathbf{x}} + k\mathbf{x}\cdot\frac{\partial}{\partial\mathbf{p}}\right)\right]g(\mathbf{x},\mathbf{p}) = g[\mathbf{x}\cos\omega_0\tau - (\mathbf{p}/m\omega_0)\sin\omega_0\tau, \mathbf{x}m\omega_0\sin\omega_0\tau + \mathbf{p}\cos\omega_0\tau]. \tag{4.4}$$

For the sake of simplicity, we shall assume that $\delta V(\mathbf{x},t;\alpha)$ [which enters into (4.3b) via the defining equation (4.2b)] is a spatially homogeneous, wide-sense stationary random process, viz.,

$$\Gamma(\mathbf{y},\tau) = E\{\delta V(\mathbf{x},t;\alpha)\ \delta V(\mathbf{x}-\mathbf{y},t-\tau;\alpha)\}. \tag{4.5}$$

The correlation function is even in both $\mathbf{y}$ and τ. In our subsequent work we shall require the spectrum [i.e., the space–time Fourier transform of $\Gamma(\mathbf{y},\tau)$], viz., $\hat{\Gamma}(\mathbf{p},u) = F_4\{\Gamma(\mathbf{y},\tau)\}$. It is related to the space–time Fourier transform of $\delta V(\mathbf{x},t;\alpha)$, viz., $\delta\hat{V}(\mathbf{p},u) = F_4\{\delta V(\mathbf{x},t;\alpha)\}$ in the following manner:

$$E\{\delta\hat{V}(\mathbf{p},u)\ \delta\hat{V}(\mathbf{p}',u')\} = \delta(\mathbf{p}+\mathbf{p}')\delta(u+u')\hat{\Gamma}(\mathbf{p},u). \tag{4.6}$$

It should be noted that $\hat{\Gamma}(\mathbf{p},u)$ is real, nonnegative, and even in both $\mathbf{p}$ and u.

The operator Θ on the right-hand side of (4.3a) can now be evaluated explicitly. The resulting kinetic equation for the mean Wigner distribution function assumes the following form:

$$\begin{aligned}&\left(\frac{\partial}{\partial t} + \frac{1}{m}\mathbf{p}\cdot\frac{\partial}{\partial\mathbf{x}} - k\mathbf{x}\cdot\frac{\partial}{\partial\mathbf{p}}\right)E\{f(\mathbf{x},\mathbf{p},t;\alpha)\}\\ &= \frac{2}{\hbar^2}\int_{R^3} d\mathbf{p}' \int_0^t d\tau\, Q(\mathbf{x},\mathbf{p},\mathbf{p}',\tau)\Bigg(E\Bigg\{f\Bigg[\mathbf{x}\cos\omega_0\tau - \tfrac{1}{2}(\mathbf{p}+\mathbf{p}')\\ &\times\frac{1}{m\omega_0}\sin\omega_0\tau, -\tfrac{1}{2}(\mathbf{p}-\mathbf{p}') + \mathbf{x}m\omega_0\sin\omega_0\tau + \tfrac{1}{2}(\mathbf{p}+\mathbf{p}')\\ &\times\cos\omega_0\tau, t-\tau;\alpha\Bigg]\Bigg\} - E\Bigg\{f\Bigg[\mathbf{x}\cos\omega_0\tau - \tfrac{1}{2}(\mathbf{p}+\mathbf{p}')\end{aligned}$$

$$\times \frac{1}{m\omega_0}\sin\omega_0\tau, + \tfrac{1}{2}(\mathbf{p}-\mathbf{p}') + \mathbf{x}m\omega_0\sin\omega_0\tau + \tfrac{1}{2}(\mathbf{p}+\mathbf{p}')$$

$$\times\cos\omega_0\tau, t-\tau, \alpha\Big]\Big\}\Big), \quad (4.7a)$$

$$Q(\mathbf{x},\mathbf{p},\mathbf{p}',\tau) = (2\pi\hbar)^{-3}\int_{R^3} d\mathbf{y}\,\Gamma(\mathbf{y},\tau) \times \cos\Big[\mathbf{y}\cdot(\mathbf{p}-\mathbf{p}')/\hbar + (\mathbf{p}-\mathbf{p}') \times\Big(\mathbf{x}\cos\omega_0\tau - \tfrac{1}{2}(\mathbf{p}+\mathbf{p}')\frac{1}{m\omega_0}\sin\omega_0\tau - \mathbf{x}\Big)\Big/\hbar\Big]. \quad (4.7b)$$

This rather formidable integrodifferential equation constitutes a uniform approximation, valid for any value of time, from which short and long time limiting cases can be considered. (The latter will be dealt with in detail in the following subsection.) The right-hand side of (4.7) contains a generalized operator (nonlocal, with memory) in phase space due to the presence of random fluctuations in the potential field, as well as to the interaction of these random inhomogeneities with the deterministic profile of the potential field. No special assumptions concerning the scale lengths of the potential fluctuations have been made in deriving (4.7). The only condition (which is implicit in the first-order smoothing approximation) is that the potential fluctuations be sufficiently small. Finally, it should be noted that in the limit $\omega_0 \to 0$ (absence of deterministic inhomogeneities), (4.7) coincides with Eq. (4.5) of Paper I.

B. The long-time Markovian approximation

By imposing additional restrictions, the kinetic equation (4.7) can be simplified considerably. The long-time Markovian approximation [cf. Eq. (3.3)] yields the following expression:

$$\Big(\frac{\partial}{\partial t} + \frac{1}{m}\mathbf{p}\cdot\frac{\partial}{\partial\mathbf{x}} - k\mathbf{x}\cdot\frac{\partial}{\partial\mathbf{p}}\Big)E\{f(\mathbf{x},\mathbf{p},t;\alpha)\} = \int_{R^3} d\mathbf{p}'\,W(\mathbf{x},\mathbf{p},\mathbf{p}')[E\{f(\mathbf{x},\mathbf{p}',t;\alpha)\} - E\{f(\mathbf{x},\mathbf{p},t;\alpha)\}], \quad (4.8a)$$

$$W(\mathbf{x},\mathbf{p},\mathbf{p}') = \frac{2}{\hbar^2}\int_0^\infty d\tau\,\tilde{\Gamma}(\mathbf{p}-\mathbf{p}',\tau)\cos\Big[(\mathbf{p}-\mathbf{p}')\cdot\Big(\mathbf{x}\cos\omega_0\tau - \tfrac{1}{2}(\mathbf{p}+\mathbf{p}')\frac{1}{m\omega_0}\sin\omega_0\tau - \mathbf{x}\Big)\Big/\hbar\Big], \quad (4.8b)$$

where $\tilde{\Gamma}(\mathbf{p},\tau)$ is the spatial Fourier transform of the correlation function $\Gamma(\mathbf{y},\tau)$.

Equation (4.8) has the form of a radiation transport equation, or a Boltzmann equation for waves (quasiparticles in phase space). The expression for the transition probability [cf. Eq. (4.8b)] is space-dependent (in contradistinction to the case of a potential field having a constant deterministic part), and obeys the principle of detailed balance, viz., $W(\mathbf{x},\mathbf{p},\mathbf{p}') = W(\mathbf{x},\mathbf{p}',\mathbf{p})$. The latter implies conservation of probability (total mean action).

The integration over τ on the right-hand side of (4.8b) can be carried out explicitly resulting in the following more revealing form for the transition probability:

$$W(\mathbf{x},\mathbf{p},\mathbf{p}') = \sum_{n=-\infty}^{\infty} W_n(\mathbf{x},\mathbf{p},\mathbf{p}'), \quad (4.9a)$$

$$W_n(\mathbf{x},\mathbf{p},\mathbf{p}') = \frac{2\pi}{\hbar}J_n\Big(\frac{a}{\hbar}\Big)\cos\Big(\frac{b}{\hbar} - n\frac{\pi}{2}\Big)\Big\{\cos\Big[n\Big(\delta+\frac{\pi}{2}\Big)\Big] \times\hat{\Gamma}(\mathbf{p}-\mathbf{p}', n\hbar\omega_0) - \sin\Big[n\Big(\delta+\frac{\pi}{2}\Big)\Big] \hat{\Gamma}_H(\mathbf{p}-\mathbf{p}', n\hbar\omega_0)\Big\}, \quad (4.9b)$$

$$a = \{[\mathbf{x}\cdot(\mathbf{p}-\mathbf{p}')]^2 + [(p^2-p'^2)/(2m\omega_0)]^2\}^{1/2}, \quad (4.9c)$$

$$b = \mathbf{x}\cdot(\mathbf{p}-\mathbf{p}'), \quad (4.9d)$$

$$\delta = \tan^{-1}[-2m\omega_0 b/(p^2-p'^2)]. \quad (4.9e)$$

J_n in (4.9b) denotes an ordinary Bessel function of the nth order, and $\hat{\Gamma}(\mathbf{p}-\mathbf{p}', n\hbar\omega_0)$ is the Hilbert transform of the spectrum $\hat{\Gamma}(\mathbf{p}-\mathbf{p}', n\hbar\omega_0)$ with respect to the second argument, viz.,

$$\hat{\Gamma}_H(\cdot, n\hbar\omega_0) = \frac{1}{\pi}P\int_{-\infty}^{\infty} d\omega\,\frac{\hat{\Gamma}(\cdot,\omega)}{\omega - n\hbar\omega_0}. \quad (4.10)$$

The representation of the transition probability W in (4.9a) as an infinite sum is a manifestation of the discrete nature of the quantum mechanical stochastic harmonic oscillator. The term W_n, for example, can be interpreted as the transition probability of the scattering event that changes the energy of the particle by an amount equal to $n\hbar\omega_0$.

If the correlation function $\Gamma(\mathbf{y},\tau)$ decreases rapidly in τ, so does the spectrum $\hat{\Gamma}(\mathbf{p},u)$ in u, and its Hilbert transform $\hat{\Gamma}_H(\mathbf{p},u)$ with respect to its second argument. Under these conditions, since the Bessel functions and the sinusoidal terms in (4.9b) are bounded, it is possible to approximate the transition probability W in (4.9a) by a sum of the first few terms, i.e.,

$$W = \sum_{n=-N}^{N} W_n, \quad (4.11)$$

where the integer N can be estimated from our knowledge of the correlation time of the random process $\delta V(\mathbf{x},t;\alpha)$.

It is clear from (4.8b) that in the limiting case $\omega_0 \to 0$ (stochastically perturbed free particle),

$$W(\mathbf{p},\mathbf{p}') = \frac{2}{\hbar^2}\int_0^\infty d\tau\,\tilde{\Gamma}(\mathbf{p}-\mathbf{p}',\tau)\cos\Big[\tau\Big(\frac{p^2}{2m} - \frac{p'^2}{2m}\Big)\Big/\hbar\Big], \quad (4.12)$$

which, upon integration, yields the following expression for the transition probability,

$$W(\mathbf{p},\mathbf{p}') = \frac{2\pi}{\hbar}\hat{\Gamma}\Big(\mathbf{p}-\mathbf{p}', \frac{p^2}{2m} - \frac{p'^2}{2m}\Big) \quad (4.13)$$

[cf. Eq. (4.7), Paper I]. The same result can be also obtained from (4.9) provided that the operations $\lim_{\omega_0\to 0}$ and infinite summation are not interchanged.

We shall close this subsection with the following remark: If the lower limit in the integral on the right-hand side of (4.8b) were replaced by $-\infty$ (this corresponds to the specification of initial data at $t=-\infty$ instead of $t=0$), the expression for W_n in (4.9b) would be modified as follows:

$$W_n(\mathbf{x},\mathbf{p},\mathbf{p}') = \frac{4\pi}{\hbar} J_n\left(\frac{a}{\hbar}\right)\cos\left(\frac{b}{\hbar} - n\frac{\pi}{2}\right)\cos\left[n\left(\delta + \frac{\pi}{2}\right)\right] \times \hat{\Gamma}(\mathbf{p}-\mathbf{p}', n\hbar\omega_0). \quad (4.14)$$

The terms in (4.9b) proportional to the Hilbert transform $\hat{\Gamma}_H(\mathbf{p}-\mathbf{p}', n\hbar\omega_0)$, which are absent in (4.14), can be interpreted as representing the effect of "switching on" the interaction between the random fluctuations of the potential field and the inhomogeneous deterministic background at the finite time $t=0$. In the special case of a potential field with a constant deterministic part, one has the relationship

$$W^{\text{LTMA}}_{t_0=0} = \tfrac{1}{2} W^{\text{LTMA}}_{t_0=-\infty} \quad (4.15)$$

for the transition probabilities corresponding to initial data prescribed at $t_0=0$ and $t_0=-\infty$, respectively. (LTMA is an abbreviation for the term long-time Markovian approximation.)

C. Kinetic equations in special cases

We shall derive here the explicit form of the kinetic equation in the long-time Markovian approximation limit for several special types of the random function $\delta V(\mathbf{x}, t; \alpha)$.

Case (i): $\delta V(\mathbf{x}, t; \alpha)$ has δ-function correlations in time.

Let $\Gamma(\mathbf{y},\tau) = \gamma(\mathbf{y})\delta(\tau)$. It follows, then, that $\hat{\Gamma}(\mathbf{p}, u) = \hat{\gamma}(\mathbf{p})$, where $\hat{\gamma}(\mathbf{p})$ is the Fourier transform of $\gamma(\mathbf{y})$. The transport equation (4.8) specializes in this case to

$$\left(\frac{\partial}{\partial t} + \frac{1}{m}\mathbf{p}\cdot\frac{\partial}{\partial \mathbf{x}} - k\mathbf{x}\cdot\frac{\partial}{\partial \mathbf{p}}\right) E\{f(\mathbf{x},\mathbf{p},t;\alpha)\} = \int_{R^3} d\mathbf{p}'\, W(\mathbf{p},\mathbf{p}')[E\{f(\mathbf{x},\mathbf{p}',t;\alpha)\} - E\{f(\mathbf{x},\mathbf{p},t;\alpha)\}], \quad (4.16a)$$

$$W(\mathbf{p},\mathbf{p}') = \frac{1}{\hbar^2}\hat{\gamma}(\mathbf{p}-\mathbf{p}'). \quad (4.16b)$$

The right-hand side of (4.16a), with W given in (4.16b), is identical to Eq. (5.1) of Paper I. It is, therefore, due solely to the random fluctuations of the potential field. The terms in the more general kinetic equation (4.8) arising from the interaction of the deterministic profile and the random fluctuations of the potential field are completely eliminated in this special case.

The spectrum $\hat{\gamma}(\mathbf{p})$ is real, nonnegative, and even. As a consequence, the transition probability $W(\mathbf{p},\mathbf{p}')$ [cf. Eq. (4.16b)] is real, nonnegative, and obeys the (detailed balance) property $W(\mathbf{p},\mathbf{p}') = W(\mathbf{p}',\mathbf{p})$. The latter implies conservation of probability (total mean action). On the strength of the principle of detailed balance, together with the nonnegativity of the transition probability, it follows, also, that the degree of coherence introduced in Sec. 3 is a monotonically decreasing function of time, viz., $(d/dt)D(t) \le 0$.[22]

The *scattering rate* (also called the *extinction coefficient* or *collision frenquency*) is defined in general as

$$\nu(\mathbf{p}) = \int_{R^3} d\mathbf{p}'\, W(\mathbf{p},\mathbf{p}'). \quad (4.17)$$

In the case under consideration here, the scattering rate is independent of $\mathbf{p}$ and is given by

$$\nu = \frac{1}{\hbar^2}\gamma(0). \quad (4.18)$$

Using this result, (4.16) can be rewritten in the following form:

$$\left(\frac{\partial}{\partial t} + \frac{1}{m}\mathbf{p}\cdot\frac{\partial}{\partial \mathbf{x}} - k\mathbf{x}\cdot\frac{\partial}{\partial \mathbf{p}} + \frac{1}{\hbar^2}\gamma(0)\right) E\{f(\mathbf{x},\mathbf{p},t;\alpha)\} = \frac{1}{\hbar^2}\int_{R^3} d\mathbf{p}'\hat{\gamma}(\mathbf{p}-\mathbf{p}') E\{f(\mathbf{x},\mathbf{p}',t;\alpha)\}. \quad (4.19)$$

Starting from the convolution-type integro-differential equation (4.19), with the prescribed initial condition $E\{f(\mathbf{x},\mathbf{p},0;\alpha)\} = f_0(\mathbf{x},\mathbf{p})$, it is possible to determine a Green's function $G(\mathbf{x},\mathbf{x}',\mathbf{p},\mathbf{p}',t)$ such that[24]

$$E\{f(\mathbf{x},\mathbf{p},t;\alpha)\} = \int_{R^3} d\mathbf{x}' \int_{R^3} d\mathbf{p}'\, G(\mathbf{x},\mathbf{x}',\mathbf{p},\mathbf{p}',t) f_0(\mathbf{x}',\mathbf{p}'). \quad (4.20)$$

This is a useful expression because, for specific statistics $\gamma(\mathbf{y})$ [or, equivalently, $\hat{\gamma}(\mathbf{p})$] and initial data $f_0(\mathbf{x},\mathbf{p})$, physically important averaged observables can be found directly from (4.20) by taking phase-space moments, without having to solve first explicitly for the mean Wigner distribution function. (This procedure is illustrated in Appendix C.)

It can be shown by means of the Donsker—Furutsu—Novikov[26-28] functional method that for a potential field fluctuation $\delta V(\mathbf{x}, t; \alpha)$ which constitutes a δ-correlated (in time), homogeneous, wide-sense stationary, Gaussian random process, the kinetic equation (4.16) for the mean Wigner distribution function is the *exact* statistical equation. (The proof will not be presented here since it is similar to that given in the Appendix of Paper I.)

Case (ii): $\delta V(\mathbf{x}, t; \alpha)$ has no time dependence.

Assuming that $\Gamma(\mathbf{y},\tau) = \gamma(\mathbf{y})$, we have $\hat{\Gamma}(\mathbf{p},u) = \hat{\gamma}(\mathbf{p})\delta(u)$. The transition probability W becomes

$$W(\mathbf{x},\mathbf{p},\mathbf{p}') = \frac{2}{\hbar^2}\hat{\gamma}(\mathbf{p}-\mathbf{p}')\int_0^\infty d\tau \cos\left(\frac{a\sin(\omega_0\tau+\delta) - b}{\hbar}\right), \quad (4.21)$$

where a, b, and δ are defined in Eqs. (4.9c)–(4.9e). It must be pointed out that the condition for the applicability of the long-time Markovian approximation [i.e., $E\{f\}$ should vary slowly on the scale of the correlation time of $\delta V(\mathbf{x}, t; \alpha)$] is clearly violated in this case. In this sense, (4.21) should be considered only as a formal result. Finally, in the limit as $\omega_0 \to 0$, (4.21) reduces to Eq. (5.4) of Paper I, viz.,

$$W(\mathbf{p},\mathbf{p}') = \frac{2\pi}{\hbar}\hat{\gamma}(\mathbf{p}-\mathbf{p}')\delta\left(\frac{p^2}{2m} - \frac{p'^2}{2m}\right). \quad (4.22)$$

Case (iii): $\delta V(\mathbf{x}, t; \alpha)$ has δ-function correlations in space.

Let $\Gamma(\mathbf{y},\tau) = (2\pi\hbar)^3\delta(\mathbf{y})\gamma(\tau)$. It follows, then, that $\hat{\Gamma}(\mathbf{p},u) = \hat{\gamma}(u)$, where $\hat{\gamma}(u)$ denotes the time Fourier transform of $\gamma(\tau)$. The mean Wigner distribution function evolves in time according to (4.8a), with the transition probability given by

$$W(\mathbf{x},\mathbf{p},\mathbf{p}')=\sum_{n=-\infty}^{\infty} W_n(\mathbf{x},\mathbf{p},\mathbf{p}'), \tag{4.23a}$$

$$W_n(\mathbf{x},\mathbf{p},\mathbf{p}')=\frac{2\pi}{\hbar}J_n\left(\frac{a}{\hbar}\right)\cos\left(\frac{b}{\hbar}-n\frac{\pi}{2}\right)\left\{\cos\left[n\left(\delta+\frac{\pi}{2}\right)\right]\times\hat{\gamma}(n\hbar\omega_0)-\sin\left[n\left(\delta+\frac{\pi}{2}\right)\right]\hat{\gamma}_H(n\hbar\omega_0)\right\}. \tag{4.23b}$$

$\hat{\gamma}_H(n\hbar\omega_0)$ stands for the Hilbert transform of the temporal spectrum $\hat{\gamma}(n\hbar\omega_0)$ [cf., also, Eq. (4.10)].

5. KINETIC THEORY FOR THE STOCHASTIC LIOUVILLE EQUATION

The results of Sec. 3 will now be specialized to the stochastic Liouville equation, i.e., Eq. (2.5), with the specific realizations of $\theta f(\mathbf{x},\mathbf{p},t;\alpha)$ given in (2.8a)—(2.8c).

A. $V(\mathbf{x},t;\alpha)=1/2kx^2+\delta V(\mathbf{x},t;\alpha)$

The mean part of the operator L in (2.5) is given in (4.1). On the other hand, the fluctuating part of L assumes the following form,

$$\delta L=\frac{\partial}{\partial\mathbf{x}}\delta V(\mathbf{x},t;\alpha)\cdot\frac{\partial}{\partial\mathbf{p}}+O(\hbar^2). \tag{5.1}$$

On the basis of the first-order smoothing approximation only [cf. Eq. (3.2)], one has the kinetic equation

$$\left(\frac{\partial}{\partial t}+\frac{1}{m}\mathbf{p}\cdot\frac{\partial}{\partial\mathbf{x}}-k\mathbf{x}\cdot\frac{\partial}{\partial\mathbf{p}}\right)E\{f(\mathbf{x},\mathbf{p},t;\alpha)\}=\frac{\partial}{\partial\mathbf{p}}\cdot\left[\int_0^t d\tau\, E\left\{\frac{\partial}{\partial\mathbf{x}}\delta V(\mathbf{x},t;\alpha)\frac{\partial}{\partial\mathbf{x}'}\delta V(\mathbf{x}',t-\tau;\alpha)\right\}\times\frac{\partial}{\partial\mathbf{p}'}E\{f(\mathbf{x}',\mathbf{p}',t-\tau;\alpha)\}\right], \tag{5.2}$$

where

$$\mathbf{x}'=\mathbf{x}\cos\omega_0\tau-(1/m\omega_0)\mathbf{p}\sin\omega_0\tau, \tag{5.3a}$$

$$\mathbf{p}'=\mathbf{p}\cos\omega_0\tau+m\omega_0\mathbf{x}\sin\omega_0\tau. \tag{5.3b}$$

By virtue of the homogeneity and stationarity of the random function $\delta V(\mathbf{x},t;\alpha)$ (cf. Sec. 4 A),

$$E\left\{\frac{\partial}{\partial\mathbf{x}}\delta V(\mathbf{x},t;\alpha)\frac{\partial}{\partial\mathbf{x}'}\delta V(\mathbf{x}',t-\tau;\alpha)\right\}=-\frac{\partial^2}{\partial\mathbf{y}\,\partial\mathbf{y}}\Gamma(\mathbf{y},\tau), \tag{5.4}$$

where

$$\mathbf{y}=\mathbf{x}-\mathbf{x}'=\mathbf{x}(1-\cos\omega_0\tau)+(1/m\omega_0)\mathbf{p}\sin\omega_0\tau. \tag{5.5}$$

Finally, Eq. (5.2) can be written as follows:

$$\left(\frac{\partial}{\partial t}+\frac{1}{m}\mathbf{p}\cdot\frac{\partial}{\partial\mathbf{x}}-k\mathbf{x}\cdot\frac{\partial}{\partial\mathbf{p}}\right)E\{f(\mathbf{x},\mathbf{p},t;\alpha)\}=\frac{\partial}{\partial\mathbf{p}}\cdot\left[\int_0^\infty d\tau\int_{R^3}d\mathbf{y}\,\delta\left(\mathbf{y}-\mathbf{x}(1-\cos\omega_0\tau)+\frac{1}{m\omega_0}\mathbf{p}\sin\omega_0\tau\right)\times\left(-\frac{\partial}{\partial\mathbf{y}\,\partial\mathbf{y}}\Gamma(\mathbf{y},\tau)\right)\cdot\left(\frac{\sin\omega_0\tau}{m\omega_0}\frac{\partial}{\partial\mathbf{x}}\cos\omega_0\tau\frac{\partial}{\partial\mathbf{p}}\right)\times E\left\{f\left(\mathbf{x}\cos\omega_0\tau-\frac{1}{m\omega}\mathbf{p}\sin\omega_0\tau,\mathbf{p}\cos\omega_0\tau+m\omega_0\mathbf{x}\sin\omega_0\tau,t-\tau;\alpha\right)\right\}\right]. \tag{5.6}$$

For random fluctuations which are statistically homogeneous, wide-sense stationary, and δ correlated in time $[\Gamma(\mathbf{y},\tau)=\gamma(\mathbf{y})\delta(\tau)]$, the time integration on the right-hand side of (5.6) can be carried out explicitly, with the result

$$\left(\frac{\partial}{\partial t}+\frac{1}{m}\mathbf{p}\cdot\frac{\partial}{\partial\mathbf{x}}-k\mathbf{x}\cdot\frac{\partial}{\partial\mathbf{p}}\right)E\{f(\mathbf{x},\mathbf{p},t;\alpha)\}=\frac{\partial}{\partial\mathbf{p}}\cdot\left[\mathbf{D}\cdot\frac{\partial}{\partial\mathbf{p}}E\{f(\mathbf{x},\mathbf{p},t;\alpha)\}\right], \tag{5.7a}$$

$$\mathbf{D}=-\tfrac{1}{2}\lim_{\mathbf{y}\to 0}\left(\frac{\partial^2}{\partial\mathbf{y}\,\partial\mathbf{y}}\gamma(\mathbf{y})\right). \tag{5.7b}$$

The right-hand side of this transport equation is identical to that in Eq. (6.2) of Paper I, which was obtained under the assumption that $\omega_0=0$. This shows that there is no interaction between the deterministic potential field profile and the random variations under the presently specified statistical properties. It should also be noted that if, in addition to the prescribed properties, $\delta V(\mathbf{x},t;\alpha)$ is a Gaussian process, the kinetic equation (5.7) is the *exact* statistical equation for the mean Wigner distribution function within the stochastic Liouville approximation. (The proof of an analogous statement can be found in the second part of the Appendix in Paper I.)

Equation (5.7) is a variant of the *equation of Kramers*.[29] A fundamental solution for it can be found by a method introduced by Wang and Uhlenbeck.[30] Equation (5.7) can be also obtained from (4.19) or, equivalently, from the three-dimensional analog of (C1a) (cf. Appendix C). If, in the latter, the term $\gamma(\hbar u)$ is expanded to order $\hbar^2$, and an inverse Fourier transform is performed with respect to variables $\mathbf{u}$ and $\mathbf{q}$ [cf. Eq. (C2)], the ensuing transport equation is identical to (5.7). As a result, the expressions for the first- and second-order averaged observables listed in Appendix C remain unchanged. However, third- and higher-order observables calculated on the basis of (C1) will contain terms of at least first order in $\hbar$, which will be absent in the corresponding Liouville approximation.

In the long-time Markovian approximation, (5.2) simplifies to

$$\left(\frac{\partial}{\partial t}+\frac{1}{m}\mathbf{p}\cdot\frac{\partial}{\partial\mathbf{x}}-k\mathbf{x}\cdot\frac{\partial}{\partial\mathbf{p}}\right)E\{f(\mathbf{x},\mathbf{p},t;\alpha)\}=\frac{\partial}{\partial\mathbf{p}}\cdot\left(\mathbf{D}^{(1)}(\mathbf{x},\mathbf{p})\cdot\frac{\partial}{\partial\mathbf{p}}+\mathbf{D}^{(2)}(\mathbf{x},\mathbf{p})\cdot\frac{\partial}{\partial\mathbf{x}}\right)E\{f(\mathbf{x},\mathbf{p},t;\alpha)\}. \tag{5.8}$$

This is a Fokker—Planck equation in phase space. The space- and momentum-dependent dyadic diffusion coefficients are given by

$$\mathbf{D}^{(1)}(\mathbf{x},\mathbf{p})=\int_0^\infty d\tau\int_{R^3}d\mathbf{y}\,\delta\left(\mathbf{y}-\mathbf{x}(1-\cos\omega_0\tau)-\frac{1}{m\omega_0}\mathbf{p}\sin\omega_0\tau\right)\times\left(-\frac{\partial^2}{\partial\mathbf{y}\,\partial\mathbf{y}}\Gamma(\mathbf{y},\tau)\right)\cos\omega_0\tau, \tag{5.9a}$$

$\mathbf{D}^{(2)}(\mathbf{x},\mathbf{p})=$

$$\int_0^\infty d\tau \int_{R^3} d\mathbf{y}\,\delta\Big(\mathbf{y}-\mathbf{x}(1-\cos\omega_0\tau)-\frac{1}{m\omega_0}\mathbf{p}\sin\omega_0\tau\Big) \times\Big(-\frac{\partial^2}{\partial\mathbf{y}\,\partial\mathbf{y}}\Gamma(\mathbf{y},\tau)\Big)\frac{\sin\omega_0\tau}{m\omega_0}. \tag{5.9b}$$

Equation (5.8) can be derived by applying the long-time Markovian approximation directly to the stochastic Liouville equation. Alternatively, it can be derived from the transport equation corresponding to the long-time Markovian approximation of the stochastic Wigner equation [cf. Eq. (4.8)] under the restriction that $\delta V(\mathbf{x},t;\alpha)$ varies slowly in space. This can be done by following the method used by Landau to derive the Fokker–Planck equation for a plasma from a Boltzmann equation (cf. Paper I and Ref. 31).

B. $V(\mathbf{x},t;\alpha)=1/2\,kx^2\,[1+\delta G(t;\alpha)]$

The exact stochastic Wigner equation assumes in this case the form

$$\Big(\frac{\partial}{\partial t}+\frac{1}{m}\mathbf{p}\cdot\frac{\partial}{\partial\mathbf{x}}-k\mathbf{x}\cdot\frac{\partial}{\partial\mathbf{p}}\Big)f(\mathbf{x},\mathbf{p},t;\alpha) =k\delta G(t;\alpha)\mathbf{x}\cdot\frac{\partial}{\partial\mathbf{p}}f(\mathbf{x},\mathbf{p},t;\alpha). \tag{5.10}$$

One has, then, in the first-order smoothing approximation,

$$\Big(\frac{\partial}{\partial t}+\frac{1}{m}\mathbf{p}\cdot\frac{\partial}{\partial\mathbf{x}}-k\mathbf{x}\cdot\frac{\partial}{\partial\mathbf{p}}\Big)E\{f(\mathbf{x},\mathbf{p},t;\alpha)\} =k^2\frac{\partial}{\partial\mathbf{p}}\cdot\Big[\int_0^t d\tau\,\Gamma(\tau)\mathbf{x}\mathbf{x}'\cdot\frac{\partial}{\partial\mathbf{p}'}E\{f(\mathbf{x}',\mathbf{p}',t-\tau;\alpha)\}\Big], \tag{5.11}$$

where $\Gamma(\tau)=E\{\delta G(t;\alpha)\delta G(t-\tau;\alpha)\}$, and $\mathbf{x}',\mathbf{p}'$ are given in (5.3). The kinetic equation (5.11) can be rewritten as follows:

$$\Big(\frac{\partial}{\partial t}+\frac{1}{m}\mathbf{p}\cdot\frac{\partial}{\partial\mathbf{x}}-k\mathbf{x}\cdot\frac{\partial}{\partial\mathbf{p}}\Big)E\{f(\mathbf{x},\mathbf{p},t;\alpha)\} =k^2\frac{\partial}{\partial\mathbf{p}}\cdot\int_0^t d\tau\,\Gamma(\tau)\mathbf{x}\Big[\mathbf{x}\cos\omega_0\tau-\frac{1}{m\omega_0}\mathbf{p}\sin\omega_0\tau\Big] \cdot\Big(\frac{\sin\omega_0\tau}{m\omega_0}\frac{\partial}{\partial\mathbf{x}}+\cos\omega_0\tau\frac{\partial}{\partial\mathbf{p}}\Big)E\Big\{f\Big(\mathbf{x}\cos\omega_0\tau -\frac{1}{m\omega_0}\mathbf{p}\sin\omega_0\tau,\,\mathbf{p}\cos\omega_0\tau+m\omega_0\mathbf{x}\sin\omega_0\tau,\,t-\tau;\alpha\Big)\Big\}. \tag{5.12}$$

For a harmonic oscillator whose frequency is modulated by a wide-sense stationary, δ-correlated random process, viz., $\Gamma(\tau)=D\delta(\tau)$, where D is a constant, the integration on the right-hand side of (5.11) can be performed explicitly, yielding[32]

$$\Big(\frac{\partial}{\partial t}+\frac{1}{m}\mathbf{p}\cdot\frac{\partial}{\partial\mathbf{x}}-k\mathbf{x}\cdot\frac{\partial}{\partial\mathbf{p}}\Big)E\{f(\mathbf{x},\mathbf{p},t;\alpha)\} =k^2D\Big(\mathbf{x}\cdot\frac{\partial}{\partial\mathbf{p}}\Big)^2E\{f(\mathbf{x},\mathbf{p},t;\alpha)\}. \tag{5.13}$$

The one-dimensional version of this equation was derived previously by Mollow (cf. Ref. 4).

Equation (5.13) corresponds to a Fokker–Planck equation in phase space, with a quadratic diffusion coefficient. The latter is due entirely to the presence of random fluctuations. No exact fundamental solution for (5.13) seems to be possible to the general case. However, closed systems of equations for moments of any order can be obtained. For example, since

$$E\{\mathbf{x}(t;\alpha)\}=\int_{R^3}d\mathbf{x}\int_{R^3}d\mathbf{p}\,\mathbf{x}E\{f(\mathbf{x},\mathbf{p},t;\alpha)\}, \tag{5.14a}$$

$$E\{\mathbf{p}(t;\alpha)\}=\int_{R^3}d\mathbf{x}\int_{R^3}d\mathbf{p}\,\mathbf{p}E\{f(\mathbf{x},\mathbf{p},t;\alpha)\}, \tag{5.14b}$$

one derives from (5.13) the following equations of motion:

$$\frac{d}{dt}E\{\mathbf{x}(t;\alpha)\}=\frac{1}{m}E\{\mathbf{p}(t;\alpha)\}, \tag{5.15a}$$

$$\frac{d}{dt}E\{\mathbf{p}(t;\alpha)\}=-kE\{\mathbf{x}(t;\alpha)\}. \tag{5.15b}$$

The initial conditions required for their solution are obtainable from (5.14), viz.,

$$E\{\mathbf{x}(0;\alpha)\}=\int_{R^3}d\mathbf{x}\int_{R^3}d\mathbf{p}\,\mathbf{x}E\{f(\mathbf{x},\mathbf{p},0;\alpha)\}\equiv\mathbf{x}_0, \tag{5.16a}$$

$$E\{\mathbf{p}(0;\alpha)\}=\int_{R^3}d\mathbf{x}\int_{R^3}d\mathbf{p}\,\mathbf{p}E\{f(\mathbf{x},\mathbf{p},0;\alpha)\}\equiv\mathbf{p}_0. \tag{5.16b}$$

It then readily follows that

$$E\{\mathbf{x}(t;\alpha)\}=\mathbf{x}_0\cos\omega_0t+(k/m)^{-1/2}\mathbf{p}_0\sin\omega_0t, \tag{5.17a}$$

$$E\{\mathbf{p}(t;\alpha)\}=\mathbf{p}_0\cos\omega_0t-(k/m)^{1/2}\mathbf{x}_0\sin\omega_0t, \tag{5.17b}$$

where $\omega_0=(k/m)^{1/2}$. We next note the following: (1) The random perturbation $\delta G(t;\alpha)$ in this case has no effect whatsoever at the level of the first two moments. (This, of course, is not the case for higher moments); (2) Equation (5.17) gives the expressions for the position and momentum of a classical harmonic oscillator characterized by a frequency ω_0. This is due to the fact that the stochastic Liouville equation (5.10) is identical to the equation governing the classical distribution function $f_c(\mathbf{x},\mathbf{p},t;\alpha)=\delta[\mathbf{x}-\mathbf{x}(t;\alpha)]\delta[\mathbf{p}-\mathbf{p}(t;\alpha)]$, $f_c(\mathbf{x},\mathbf{p},0;\alpha)=\delta(\mathbf{x}-\mathbf{x}_0)\delta(\mathbf{p}-\mathbf{p}_0)$, where $(d/dt)\mathbf{x}(t;\alpha)=(1/m)\mathbf{p}(t;\alpha)$, $(d/dt)\mathbf{p}(t;\alpha)=-k[1+\delta G(t;\alpha)]\mathbf{x}(t;\alpha)$, and $\mathbf{x}(0;\alpha)=\mathbf{x}_0$, $\mathbf{p}(0;\alpha)=\mathbf{p}_0$.

In the long-time Markovian approximation, (5.11) simplifies to the Fokker–Planck equation

$$\Big(\frac{\partial}{\partial t}+\frac{1}{m}\mathbf{p}\cdot\frac{\partial}{\partial\mathbf{x}}-k\mathbf{x}\cdot\frac{\partial}{\partial\mathbf{p}}\Big)E\{f(\mathbf{x},\mathbf{p},t;\alpha)\} =\Big(\frac{\partial}{\partial\mathbf{p}}\cdot\mathbf{D}^{(1)}(\mathbf{x},\mathbf{p})\cdot\frac{\partial}{\partial\mathbf{p}}+\frac{\partial}{\partial\mathbf{p}}\cdot\mathbf{D}^{(2)}(\mathbf{x},\mathbf{p})\cdot\frac{\partial}{\partial\mathbf{x}}\Big) \times E\{f(\mathbf{x},\mathbf{p},t;\alpha)\}. \tag{5.18}$$

The two dyadic diffusion coefficients are given as follows:

$$\mathbf{D}^{(1)}(\mathbf{x},\mathbf{p})=\Big[k^2\int_0^\infty d\tau\,\Gamma(\tau)\cos^2\omega_0\tau\Big]\mathbf{x}\mathbf{x} -\Big(\frac{k^2}{2m\omega_0}\int_0^\infty d\tau\,\Gamma(\tau)\sin2\omega_0\tau\Big)\mathbf{x}\mathbf{p}, \tag{5.19a}$$

$$\mathbf{D}^{(2)}(\mathbf{x},\mathbf{p})=-\Big[\Big(\frac{k}{m\omega_0}\Big)^2\int_0^\infty d\tau\,\Gamma(\tau)\sin^2\omega_0\tau\Big]\mathbf{x}\mathbf{p} +\Big(\frac{k^2}{2m\omega_0}\int_0^\infty d\tau\,\Gamma(\tau)\sin^2\omega_0\tau\Big)\mathbf{x}\mathbf{x}. \tag{5.19b}$$

In general, no exact solution to (5.18) seems to be possible. Nevertheless, closed systems of equations for moments of any order can be obtained by taking appropriate phase-space moments. For example, using the definitions of the average position and momentum [cf. Eq. (5.14)], the following equations of motion can be readily derived from (5.18):

$$\frac{d}{dt}E\{\mathbf{x}(t;\alpha)\}=\frac{1}{m}E\{\mathbf{p}(t;\alpha)\}, \tag{5.20a}$$

$$\frac{d}{dt}E\{\mathbf{p}(t;\alpha)\}=-kE\{\mathbf{x}(t;\alpha)\}+c_1E\{\mathbf{x}(t;\alpha)\}-c_2E\{\mathbf{p}(t;\alpha)\}, \tag{5.20b}$$

with the constant coefficients c_1 and c_2 given by

$$c_1=\frac{k^2}{2m\omega_0}\int_0^\infty d\tau\,\Gamma(\tau)\sin 2\omega_0\tau, \tag{5.21}$$

$$c_2=\left(\frac{k}{m\omega_0}\right)^2\int_0^\infty d\tau\,\Gamma(\tau)\sin^2\omega_0\tau. \tag{5.22}$$

The latter one may be expressed in terms of the spectrum $\hat{\Gamma}(u)$ as follows,

$$c_2=\left(\frac{k}{m\omega_0}\right)^2\frac{\pi}{2}[\hat{\Gamma}(0)-\hat{\Gamma}(2\omega_0)]. \tag{5.23}$$

On the other hand, the former one may be written as

$$c_1=\frac{k^2}{2m\omega_0}\,\frac{\pi}{2}\,\hat{\Gamma}_H(2\omega_0), \tag{5.24}$$

where

$$\hat{\Gamma}_H(2\omega_0)=\frac{2}{\pi}\int_0^\infty d\tau\,\Gamma(\tau)\sin 2\omega_0\tau=\frac{1}{\pi}P\int_{-\infty}^\infty\frac{\hat{\Gamma}(u)}{u-2\omega_0}\,du \tag{5.25}$$

is the Hilbert transform of $\hat{\Gamma}(u)$.

Eliminating $E\{\mathbf{p}(t;\alpha)\}$ between (5.20a) and (5.20b), we obtain the second-order equation

$$\frac{d^2}{dt^2}E\{\mathbf{x}(t;\alpha)\}+c_2\frac{d}{dt}E\{\mathbf{x}(t;\alpha)\}+\omega_0^2\left(1-\frac{c_1}{\omega_0^2}\right)E\{\mathbf{x}(t;\alpha)\}=0 \tag{5.26}$$

for the mean position vector. It is clear from this expression that the presence of random fluctuations has a significant effect, even at the level of the first statistical moment. The average position is damped by an amount proportional to c_2. According to (5.23), this damping may be negative when the fluctuations are particularly strong at twice the unperturbed frequency. Furthermore, a shift in the oscillator frequency arises, which is determined by the Hilbert transform of the spectrum of the correlation function $\Gamma(\tau)$. Identical results have been reported recently by Van Kampen (cf. Ref. 20) who applied the long-time Markovian approximation directly to the equations of motion of one-dimensional classical harmonic oscillator. The coincidence of his results with ours is not surprising at all since the mean trajectory of the quantum mechanical oscillator is exactly the same with the path traversed by a classical harmonic oscillator. [More generally, this statement is valid whenever the potential field $V(\mathbf{x},t;\alpha)$ in (1.1) is such that the exact Wigner equation is of the form of a Liouville equation.]

C. $V(\mathbf{x},t;\alpha)=1/2\,k\,[\mathbf{x}-\mathbf{a}\delta H(t;\alpha)]^2$

The Wigner distribution function is governed in this case exactly by the stochastic Liouville equation

$$\left(\frac{\partial}{\partial t}+\frac{1}{m}\mathbf{p}\cdot\frac{\partial}{\partial\mathbf{x}}-k\mathbf{x}\cdot\frac{\partial}{\partial\mathbf{p}}\right)f(\mathbf{x},\mathbf{p},t;\alpha)=k\delta H(t;\alpha)\mathbf{a}\cdot\frac{\partial}{\partial\mathbf{p}}f(\mathbf{x},\mathbf{p},t;\alpha). \tag{5.27}$$

The corresponding kinetic equation for the mean Wigner distribution function in the first-order smoothing approximation has the form

$$\begin{aligned}\left(\frac{\partial}{\partial t}+\frac{1}{m}\mathbf{p}\cdot\frac{\partial}{\partial\mathbf{x}}-k\mathbf{x}\cdot\frac{\partial}{\partial\mathbf{p}}\right)E\{f(\mathbf{x},\mathbf{p},t;\alpha)\}&=k^2\frac{\partial}{\partial\mathbf{p}}\cdot\Bigg[\int_0^t d\tau\,\Gamma(\tau)\mathbf{a}\mathbf{a}\cdot\left(\frac{\sin\omega_0\tau}{m\omega_0}\frac{\partial}{\partial\mathbf{x}}+\cos\omega_0\tau\frac{\partial}{\partial\mathbf{p}}\right)\\&\times E\Big\{f\Big(\mathbf{x}\cos\omega_0\tau-\frac{1}{m\omega_0}\mathbf{p}\sin\omega_0\tau,\mathbf{p}\cos\omega_0\tau\\&+m\omega_0\mathbf{x}\sin\omega_0\tau,t-\tau;\alpha\Big)\Big\}\Bigg],\end{aligned} \tag{5.28}$$

where $\Gamma(\tau)=E\{\delta H(t;\alpha)\delta H(t-\tau;\alpha)\}$.

For a random process $\delta H(t;\boldsymbol{\alpha})$ which is wide-sense stationary and δ correlated in time, viz., $\Gamma(\tau)=D\delta(\tau)$, where D is a constant, the time integration in (5.28) can be carried out explicitly. The resulting transport equation is

$$\left(\frac{\partial}{\partial t}+\frac{1}{m}\mathbf{p}\cdot\frac{\partial}{\partial\mathbf{x}}-k\mathbf{x}\cdot\frac{\partial}{\partial\mathbf{p}}\right)E\{f(\mathbf{x},\mathbf{p},t;\alpha)\}=k^2D\left(\mathbf{a}\cdot\frac{\partial}{\partial\mathbf{p}}\right)^2E\{f(\mathbf{x},\mathbf{p},t;\alpha)\}. \tag{5.29}$$

If, in addition to the above assumptions $\delta H(t;\alpha)$ is a Gaussian random process, (5.29) is the *exact* statistical equation for $E\{f(\mathbf{x},\mathbf{p},t;\alpha)\}$.

The stochastic Liouville equation (5.27) is identical to the equation governing the classical distribution function $f_c(\mathbf{x},\mathbf{p},t;\alpha)=\delta[\mathbf{x}-\mathbf{x}(t;\alpha)]\delta[\mathbf{p}-\mathbf{p}(t;\alpha)]$, $f_c(\mathbf{x},\mathbf{p},0;\alpha)=\delta(\mathbf{x}-\mathbf{x}_0)\delta(\mathbf{p}-\mathbf{p}_0)$ associated with the Brownian motion of a simple, classical, harmonic oscillator, viz., $(d/dt)\mathbf{x}(t;\alpha)=(1/m)\mathbf{p}(t;\alpha)$, $(d/dt)\mathbf{p}(t;\alpha)=-k\mathbf{x}(t;\alpha)+\mathbf{a}\delta H(t;\boldsymbol{\alpha})$, with $\mathbf{x}(0;\alpha)=\mathbf{x}_0$, $\mathbf{p}(0;\alpha)=\mathbf{p}_0$. Equation (5.29) has an exact fundamental solution since, except for the initial condition, it is identical to the equation satisfied by $E\{f_c(\mathbf{x},\mathbf{p},t;\alpha)\}$, and the latter has been studied extensively (cf. Ref. 30).

In the long-time Markovian approximation, (5.28) reduces to the simpler transport equation

$$\left(\frac{\partial}{\partial t}+\frac{1}{m}\mathbf{p}\cdot\frac{\partial}{\partial\mathbf{x}}-k\mathbf{x}\cdot\frac{\partial}{\partial\mathbf{p}}\right)E\{f(\mathbf{x},\mathbf{p},t;\alpha)\}=\left(\frac{\partial}{\partial\mathbf{p}}\cdot\mathbf{D}^{(1)}\cdot\frac{\partial}{\partial\mathbf{p}}+\frac{\partial}{\partial\mathbf{p}}\cdot\mathbf{D}^{(2)}\cdot\frac{\partial}{\partial\mathbf{x}}\right)E\{f(\mathbf{x},\mathbf{p},t;\alpha)\}. \tag{5.30}$$

The dyadic diffusion coefficients are given by the expressions

$$\mathbf{D}^{(1)} = (k^2 \int_0^\infty d\tau\, \Gamma(\tau) \cos\omega_0\tau)\mathbf{a}\,\mathbf{a}, \tag{5.31a}$$

$$\mathbf{D}^{(2)} = \left(\frac{k^2}{m\omega_0} \int_0^\infty d\tau\, \Gamma(\tau) \sin\omega_0\tau\right)\mathbf{a}\,\mathbf{a}. \tag{5.31b}$$

They can be easily written in terms of the spectrum $\hat{\Gamma}(u)$ and its Hilbert transform $\hat{\Gamma}_H(u)$ as follows:

$$\mathbf{D}^{(1)} = \pi k^2 \mathbf{a}\,\mathbf{a}\hat{\Gamma}(\omega_0)/2, \tag{5.32a}$$

$$\mathbf{D}^{(2)} = \pi k^2 \mathbf{a}\,\mathbf{a}\hat{\Gamma}_H(\omega_0)/(2m\omega_0). \tag{5.32b}$$

Since both $\mathbf{D}^{(1)}$ and $\mathbf{D}^{(2)}$ are constant, it is possible to determine a general fundamental solution for the Fokker–Planck equation (5.30).

APPENDIX A: THE UNCERTAINTY PRINCIPLE IN PHASE SPACE

On the basis of the Schwartz inequality,

$$|f(\mathbf{x},\mathbf{p},t;\alpha)|^2 \leq (2\pi\hbar)^{-6}\left[\int_{R^3} d\mathbf{y}\, |\psi^*(\mathbf{x}+\tfrac{1}{2}\mathbf{y},t;\alpha)|^2\right] \times \left[\int_{R^3} d\mathbf{y}\, |\psi(\mathbf{x}-\tfrac{1}{2}\mathbf{y},t;\alpha)|^2\right]. \tag{A1}$$

Consider the integral

$$I_+ \equiv \int_{R^3} d\mathbf{y}\, |\psi^*(\mathbf{x}+\tfrac{1}{2}\mathbf{y},t;\alpha)|^2 = 6\int_{R^3} d\mathbf{x}\, \psi\,|(\mathbf{x},t;\alpha)|^2. \tag{A2}$$

The total action, however, is conserved for every realization $\alpha \in A$, and is assumed to be normalized to unity (cf. Sec. 2). Therefore, $I_+ = 6$. Similarly,

$$I_- \equiv \int_{R^3} d\mathbf{y}\, |\psi(\mathbf{x}-\tfrac{1}{2}\mathbf{y},t;\alpha)|^2 = 6. \tag{A3}$$

Using these results in (A1) we obtain, finally,

$$|f(\mathbf{x},\mathbf{p},t;\alpha)| \leq (\hbar\pi)^{-3}, \quad \forall\alpha \in A. \tag{A4}$$

APPENDIX B: DEGREE OF COHERENCE

Given a wavefunction $\psi(\mathbf{x},t;\alpha)$, the *degree of coherence*, $D(t)$, is defined as follows:

$$D^2(t) = (2\pi\hbar)^3 \int_{R^3} d\mathbf{x} \int_{R^3} d\mathbf{p}[E\{f(\mathbf{x},\mathbf{p},t;\alpha)\}]^2 = \int_{R^3} d\mathbf{x}_2 \int_{R^3} d\mathbf{x}_1\, |E\{\psi^*(\mathbf{x}_2,t;\alpha)\psi(\mathbf{x}_1,t;\alpha)\}|^2. \tag{B1}$$

This quantity is intimately linked with the irreversible loss of information (coherence) due to the statistical fluctuations.

The degree of coherence is characterized by the property

$$D^2(t) \leq 1, \tag{B2}$$

the equality holding for the case of a purely coherent state. To show this we note that in the absence of random fluctuations (B1) reduces to

$$D^2(t) = \int_{R^3} d\mathbf{x}_2 \int_{R^3} d\mathbf{x}_1\, |\psi^*(\mathbf{x}_2,t)\psi(\mathbf{x}_1,t)|^2 = \left[\int_{R^3} d\mathbf{x}_2\, |\psi(\mathbf{x}_2,t)|^2\right]\left[\int_{R^3} d\mathbf{x}_1\, |\psi(\mathbf{x}_1,t)|^2\right] = 1, \tag{B3}$$

the final equality following because of the conservation of the total action.

To prove the inequality $D^2(t) < 1$, which holds for a partially coherent (mixed) state, we use the Cauchy–Schwartz inequality,[33] viz.,

$$|E\{\psi^*(\mathbf{x},t;\alpha)\psi(\mathbf{x}_1,t;\alpha)\}|^2 \leq E\{|\psi(\mathbf{x}_2,t;\alpha)|^2\}E\{|\psi(\mathbf{x}_1,t;\alpha)|^2\}, \tag{B4}$$

in conjunction with (B1). We then have

$$D^2(t) \leq \left[\int_{R^3} d\mathbf{x}_2\, E\{|\psi(\mathbf{x}_2,t;\alpha)|^2\}\right]\left[\int_{R^3} d\mathbf{x}_1\, E\{|\psi(\mathbf{x}_1,t;\alpha)|^2\}\right] = 1, \tag{B5}$$

the last equality following from the fact that the total mean action is conserved and is normalized to unity.

APPENDIX C: INTEGRATION OF THE KINETIC EQUATION (4.19)

We shall integrate here the transport equation (4.19) and use the result to determine several averaged observables. For simplicity, we shall restrict the discussion to the one-dimensional case.

Taking a double Fourier transform of (4.19), we obtain the initial value problem

$$\left\{\frac{\partial}{\partial t} - \frac{1}{m}q\frac{\partial}{\partial u} - ku\frac{\partial}{\partial q} + \frac{1}{\hbar^2}[\gamma(0) - \gamma(\hbar u)]\right\}E\{\hat{f}(q,u,t;\alpha)\} = 0, \tag{C1a}$$

$$E\{\hat{f}(q,u,0;\alpha)\} = \hat{f}_0(q,u), \tag{C1b}$$

where

$$E\{\hat{f}(q,u,t;\alpha)\} = (2\pi)^{-2}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp\, \exp[-i(qx+up)]E\{f(x,p,t;\alpha)\}. \tag{C2}$$

We next introduce a new function

$$g(q,u,t) = \exp(\nu t)E\{\hat{f}(q,u,t;\alpha)\}, \tag{C3}$$

together with a new set of variables (r,ϕ), defined by the relations

$$u = (mk)^{-1/2} r\cos\phi, \tag{C4a}$$

$$q = r\sin\phi. \tag{C4b}$$

The equation for the time evolution of $\tilde{g}(\phi,t) = g[r\sin\phi, (mk)^{-1/2}r\cos\phi, t]$ now takes the following form,

$$\left(\frac{\partial}{\partial t} + \omega_0\frac{\partial}{\partial\phi} - \frac{1}{\hbar^2}\tilde{\gamma}(\phi)\right)\tilde{g}(\phi,t) = 0, \tag{C5a}$$

$$\tilde{g}(\phi,0) = \tilde{g}_0(\phi), \tag{C5b}$$

where $\tilde{\gamma}(\phi) = \gamma[(mk)^{-1/2}\hbar r\cos\phi]$.

The solution of (C5) can be found by the method of characteristics. It is given by

$$\tilde{g}(\phi,t) = \left\{\exp\frac{1}{\hbar^2}\int_0^t d\tau\, \tilde{\gamma}(\phi - \omega_0\tau)\right\}\tilde{g}_0(\phi - \omega_0 t). \tag{C6}$$

Returning to the original variables, we finally have

$$E\{\hat{f}(q,u,t;\alpha)\} = \exp\left[-\nu t + \frac{1}{\hbar^2}\int_0^t d\tau\, \gamma\left(\hbar u\cos\omega_0\tau + \frac{\hbar q}{m\omega_0}\sin\omega_0\tau\right)\right] \times \hat{f}_0\left(q\cos\omega_0 t - m\omega_0 u\sin\omega_0 t,\, u\cos\omega_0 t + \frac{q}{m\omega_0}\sin\omega_0 t\right). \tag{C7}$$

Many important averaged physical observables can be found directly from (C7), making use of the fact that the moments of $E\{f(x,p,t;\alpha)\}$ can be expressed in terms of derivatives of $E\{\hat{f}(q,u,t;\alpha)\}$. For example, the averaged total energy of the system is given by the formula

$$E\{E(t)\} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp \left(\frac{p^2}{2m} + \tfrac{1}{2}kx^2\right) E\{f(x,p,t;\alpha)\}$$

$$= -(2\pi)^2 \left(\frac{1}{2m}\frac{\partial^2}{\partial u^2} E\{\hat{f}(q,u,t;\alpha)\} + \tfrac{1}{2}k \frac{\partial^2}{\partial q^2} E\{\hat{f}(q,u,t;\alpha)\}\right)_{q=u=0}. \quad \text{(C8)}$$

Substituting (C7) into the above expression, we obtain

$$E\{E(t)\} = E\{E(0)\} - \gamma''(0)t/2m. \quad \text{(C9)}$$

Since $\gamma''(0) < 0$, we can see immediately that this model predicts amplification of the energy of the particle due the stochastic variations of the potential field. The formula (C9) is also valid for the case of free propagation ($\omega_0 \to 0$). For the three-dimensional case, (C9) is replaced by

$$E\{E(t)\} = E\{E(0)\} - 3\gamma''(0)t/2m. \quad \text{(C10)}$$

Expressions for other physical averaged observables are listed below:

(i) *Mean centroid of a wavepacket*:

$$E\{x_c(t)\} = E\{x_c(0)\}\cos\omega_0 t + \frac{1}{m\omega_0} E\{p_c(0)\}\sin\omega_0 t; \quad \text{(C11a)}$$

(ii) *Mean momentum*:

$$E\{p_c(t)\} = E\{p_c(0)\}\cos\omega_0 t - m\omega_0 E\{x_c(0)\}\sin\omega_0 t; \quad \text{(C11b)}$$

(iii) *Spatial spread of a wavepacket*:

$$E\{\sigma_x^2(t)\} = E\{\sigma_x^2(0)\}\cos^2\omega_0 t + (m\omega_0)^{-2}E\{\sigma^2(0)\}\sin^2\omega_0 t \times \frac{1}{m\omega_0} E\{\sigma_{xp}^2(0)\}\sin 2\omega_0 t - \frac{\gamma''(0)}{(m\omega_0)^2} \times \left(\frac{t}{2} - \frac{\sin 2\omega_0 t}{4\omega_0}\right); \quad \text{(C11c)}$$

(iv) *Momentum spread of a wavepacket*:

$$E\{\sigma_p^2(t)\} = E\{\sigma_p^2(0)\}\cos^2\omega_0 t + (m\omega_0)^2 E\{\sigma_x^2(0)\} \times \sin^2\omega_0 t - m\omega_0 E\{\sigma_{xp}^2(0)\}\sin 2\omega_0 t - \gamma''(0)\left(\frac{t}{2} + \frac{\sin 2\omega_0 t}{4\omega_0}\right). \quad \text{(C11d)}$$

In the limit as $\omega_0 \to 0$ (free propagation), these results simplify as follows:

(i) $$E\{x_c(t)\} = E\{x_c(0)\} + \frac{1}{m}E\{p\ (0)\}t; \quad \text{(C12a)}$$

(ii) $$E\{p_c(t)\} = E\{p_c(0)\}; \quad \text{(C12b)}$$

(iii) $$E\{\sigma_x^2(t)\} = E\{\sigma_x^2(0)\} + \frac{1}{m^2}E\{\sigma_p^2(0)\}t^2 - \frac{2}{m}E\{\sigma_{xp}^2(0)\}t - \frac{\gamma''(0)}{3m^2}t^3; \quad \text{(C12c)}$$

(iv) $$E\{\sigma_p^2(t)\} = E\{\sigma_p^2(0)\} - \gamma''(0)t. \quad \text{(C12d)}$$

It is interesting to note that the average spread of a wavepacket grows with time due to the presence of stochastic fluctuations. The growth is proportional to the first power of time for a particle in the field of an elastic force, and to the third power of time for a freely propagating particle. On the other hand, the spread of momentum grows linearly with time in both cases.

ACKNOWLEDGMENTS

The research reported in this paper was completed while two of the authors (I. M. B. and F. D. T.) participated in the Applied Mathematics Summer Institute, 1976 at Dartmouth College. The Institute was supported by the Office of Naval Research under Contract No. N00014-75-C-0921 with the Applied Instiute of Mathematics, Inc. In the course of this research, W. B. S. was sponsored by the Air Force Office of Scientific Research, Air Force Systems Command, USAF, under Grant No. AFOSR-74-2651C.

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