A GENERAL SOLUTION FOR THE THERMAL STRESSES AND STRAINS IN AN INFINITE, HOLLOW, CASE-BONDED ROCKET GRAIN

by

George D. Iverson

Thesis submitted to the Graduate Faculty of the

Virginia Polytechnic Institute

in candidacy for the degree of

MASTER OF SCIENCE

in

AEROSPACE ENGINEERING

October 1962

Blacksburg, Virginia

TABLE OF CONTENTS

		Page
I.	INTRODUCTION	6
II.	SOLUTION OF THE HEAT CONDUCTION EQUATION FOR A COMPOSITE CYLINDER	8
III.	DEVELOPMENT OF GENERAL STRESS AND STRAIN EQUATIONS FOR AN ELASTIC CASE-BONDED ROCKET GRAIN	14
IV.	INTEGRATION OF THE EXPRESSION $\int_g^r (r) \Delta T \, dr$	23
V.	NUMERICAL PROCEDURE	26
VI.	DISCUSSION OF RESULTS	27
VII.	METHODS OF DETERMINING FAILURE AND MAXIMUM TEMPERATURE VARIATIONS	30
VIII.	CONCLUSIONS	32
IX.	ACKNOWLEDGEMENTS	33
X.	BIBLIOGRAPHY	34
XI.	VITA	36

LIST OF TABLES AND FIGURES

Tables

Page

TABLE 1 ROOTS OF THE EQUATION $Y_1(mx)\left[x\,J_1(x)-\beta\,J_o(x)\right]-J_1(mx)\left[x\,Y_1(x)-\beta\,Y_o(x)\right]$ 37

TABLE 2 CASE AND PROPELLANT PARAMETERS 38

TABLE 3 VALUES OF NON-DIMENSIONAL STRESSES AND STRAINS FOR DIFFERENT RADIAL POSITIONS 39

Figures

FIGURE 1 CROSS-SECTION OF CASE-BONDED ROCKET GRAIN 40

FIGURE 2 DIMENSIONLESS RADIAL STRESS VERSUS RADIAL POSITION 41

FIGURE 3 DIMENSIONLESS TANGENTIAL STRESS VERSUS RADIAL POSITION 42

FIGURE 4 RADIAL STRAIN VERSUS RADIAL POSITION 43

FIGURE 5 TANGENTIAL STRAIN VERSUS RADIAL POSITION 44

LIST OF SYMBOLS

T_g	Temperature of surrounding medium
T	Temperature
T_i	Initial temperature
a	Inner radius of propellant
b	Radius to case
c	Exterior radius of case
t	Time
λ	Thermal conductivity
ρ	Density of a substance
C	Specific heat capacity of a substance
K	Thermal diffusivity of a substance, $\lambda/\rho C$
h	Convection coefficient
θ	Dimensionless temperature difference
F	Fourier number, Kt/c^2
β	Biot number, hc/λ
P	Radial position, r/c
m	Diameter ratio, a/c
n	Diameter ratio, c/b
α	Coefficient of linear expansion
E	Young's modulus
ν	Poisson's ratio

K	Bulk modulus
σ	Stress
ϵ	Strain
s	Diameter ratio, b/a

Subscripts

p	Propellant
c	Case

I. INTRODUCTION

In recent years there has been an increased interest in solid propellant rockets. This is due, in part, to the fact that such propellant systems can be assembled in the rocket, along with the other systems, and stored as a single integral unit. Since the range of storage temperatures is generally different from the propellant cure temperatures, thermal stresses and strains will occur in the grain. These stresses and strains may cause cracks in the grain which will render it useless.

Before solving the thermal stress problem it is necessary to solve the heat conduction equation. The general solution of this equation, for an infinite hollow cylinder, is given by Carslaw and Jaeger(3) for a number of possible boundary conditions. Using this solution and the eigenvalues compiled by Lipow and Zwick(6), Geckler(4) calculated the stress distribution in an infinite hollow circular cylinder by using the thermal stress expressions developed by Timoshenko(7).

Zwick(9) and Au(1) developed expressions for thermal stresses in an infinite hollow case-bonded cylinder. However, no attempt was made by either of them to develop a solution for the heat conduction equation and incorporate it in the solution for the thermal stresses and strains.

In this thesis Carslaw and Jaeger's general solution to the heat conduction equation for a hollow cylinder was used to obtain a solution for a composite cylinder. In order to use this solution it was necessary to assume that the thermal conductivity and thermal diffusivity of the propellant and the case were the same. This temperature distribution was then used in the appropriate expressions (for thermal stresses and strains) to give the general solution for the thermal stresses and strains in a case-bonded rocket grain.

The numerical example presented indicates that the tangential stress and strain have their maxima at the inner radius of the propellant. Also, the stresses and strains were found to increase with "m", the ratio of the inner radius, a, to the outer radius, c. Application of failure criteria and the methods of computing maximum allowable storage temperature variation is discussed.

II. SOLUTION OF THE HEAT CONDUCTION EQUATION FOR A COMPOSITE CYLINDER

In order to solve the heat conduction equation for a composite cylinder two assumptions were made. First, it was assumed that the thermal diffusivity, K, and the thermal conductivity, λ, were the same for the propellant and the case. The letters a, c, and r are used to define the inner radius of the propellant, the exterior radius of the case, and an arbitrary radius between a and c, respectively. The differential equation describing the heat conduction for an infinite, hollow, composite circular cylinder, at time, t, greater than zero, is then

$$\frac{\partial T}{\partial t} = K \left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right) \quad . \qquad (1)$$

The boundary conditions imposed on this equation are

$$\lambda \frac{\partial T}{\partial r} = 0 \qquad \text{at} \quad r = a \quad t \geq 0 \qquad (2)$$

$$\lambda \frac{\partial T}{\partial r} = h(T_g - T) \qquad \text{at} \quad r = c \quad t \geq 0 \qquad (3)$$

where h represents the convection coefficient and T represents the temperature of the surrounding medium.

The initial condition is

$$T = T_i \quad \text{at } t = 0$$

Since it is convenient to work with dimensionless variables the following dimensionless quantities are defined:

$$F = Kt/c^2$$

$$\beta = hc/\lambda$$

$$P = r/c$$

$$m = a/c$$

and

$$\theta = \frac{(T_g - T)}{(T_g - T_i)} \quad .$$

Using the dimensionless variables the boundary conditions imposed on Eq. (1) can be rewritten as

$$\frac{\lambda}{c} \frac{\partial T}{\partial \left(\frac{r}{c} \right)} = 0 \qquad \text{at} \quad r = a \quad t \geq 0 \tag{4}$$

$$\frac{\partial T}{\partial \left(\frac{r}{c} \right)} = \frac{\beta}{c} (T_g - T) \qquad \text{at} \quad r = c \quad t \geq 0 \tag{5}$$

The analytical solution to Eq. (1) with a general set of boundary conditions has been given by Carslaw and Jaeger. To adopt the general solution to a specific situation, the general values of k that appear in Carslaw and Jaeger's solution are assigned specific values. These values are obtained from the specified boundary conditions.

The general solution, as given by Carslaw and Jaeger, is

$$T = \frac{-a\,k_3\left[k_1' - b\,k_2'\,\ln(r/c)\right] + b\,k_3'\left[k_1 + a\,k_2\,\ln(r/a)\right]}{a\,k_2\,k_1' + b\,k_1\,k_2' + ab\,k_2\,k_1'\,\ln(c/a)} -$$

$$\pi \sum_{n=1}^{\infty} \frac{e^{-K\,\alpha_n^{\,2} t}}{F(\alpha_n)}$$

$$\left\{k_1'\,\alpha_n\,J_1\,(b\,\alpha_n) - k_2'\,J_o\,(b\,\alpha_n)\right\}\;C_o\,(r,\alpha_n)$$

$$\left[k_3\left\{k_1'\,\alpha_n\,J_1\,(b\,\alpha_n) - k_2'\,J_o\,(b\,\alpha_n)\right\} -\right.$$

$$\left. k_3'\left\{k_1\,\alpha_n\,J_1\,(a\,\alpha_n) + k_2\,J_o\,(a\,\alpha_n)\right\}\right] \qquad (6)$$

where $F(\alpha_n)$ and $C_o(r,\alpha_n)$ are defined by the following equations,

$$F(\alpha_n) = (k_1^{\,2}\,\alpha_n^{\,2} + k_2'^{\,2})\left[k_1\,\alpha_n\,J_1\,(a\,\alpha_n) + k_2\,J_o\,(a\,\alpha_n)\right]^2 -$$

$$(k_1^{\,2}\,\alpha_n^{\,2} + k_2^{\,2})\left[k_1'\,\alpha_n\,J_1\,(b\,\alpha_n) - k_2'\,J_o\,(b\,\alpha_n)\right]^2 \qquad (7)$$

$$C_o(r,\alpha_n) = J_o\,(r\,\alpha_n)\left[k_1\,\alpha_n\,Y_1\,(a\,\alpha_n) + k_2\,Y_o\,(a\,\alpha_n)\right] -$$

$$Y_o(r\,\alpha_n)\left[k_1\,\alpha_n\,J_1\,(a\,\alpha_n) + k_2\,J_o\,(a\,\alpha_n)\right] \qquad (8)$$

with the values of α_n being the roots of the equation

$$\left[k_1 \alpha J_1 (a\alpha) + k_2 J_o (\alpha a)\right]\left[k_1' \alpha Y_1 (b\alpha) - k_2' Y_1 (b\alpha)\right] -$$

$$\left[k_1 \alpha Y_1 (a\alpha) + k_2 Y_o (a\alpha)\right]\left[k_1' \alpha J_1 (b\alpha) - k_2' J_o (b\alpha)\right] . \quad (9)$$

The values of k, to be used in Eqs. (6) through (9), are obtained from the boundary conditions (4) and (5). These values are

$$k_1 = \lambda/c \qquad k_2 = k_3 = 0$$

and

$$k_1' = 1 \qquad k_2' = \beta/c \qquad k_3' = \beta T_g/c$$

The values of k, as given above, were substituted into Eq. (6) to determine the expression which would give the eigenvalues for the specific case under consideration. That is,

$$\frac{\lambda}{c} \alpha J_1 (\alpha a) \left[\alpha Y_1 (\alpha c) - \frac{\beta}{c} Y_o (\alpha c)\right] -$$

$$\frac{\lambda}{c} \alpha Y_1 (\alpha a)\left[\alpha J_1 (c\alpha) - \frac{\beta}{c} J_o (c\alpha)\right] = 0$$

$$J_1 (\alpha a)\left[\alpha Y_1 (\alpha c) - \frac{\beta}{c} Y_o (\alpha c)\right] -$$

$$Y_1 (\alpha a)\left[\alpha J_1 (c\alpha) - \frac{\beta}{c} J_o (c\alpha)\right] = 0 \qquad (10)$$

Now define

$$x = \alpha c$$

and

$$\alpha = x/c \quad .$$

Equation (10) then reduces to

$$J_1(mx)\left[x\,Y_1(x) - \beta\,Y_o(x)\right] - Y_1(mx)\left[x\,J_1(x) - \beta\,J_o(x)\right] = 0 \tag{11}$$

This equation has been solved by Lipow and Zwick, and gives the necessary eigenvalues for the desired solution.

Using the above values of k the solution to Eq. (1), with boundary conditions ((4) and (5)), and the initial conditions, is

$$T = T_g - \frac{\pi \sum_{n=1}^{\infty} e^{-K\alpha_n^2 t}\left\{\alpha\,J_1(\alpha_n c) - \frac{\beta}{c}\,J_o(\alpha_n c)\right\}\left\{-\beta\,\frac{(T_g - T_i)}{c}\left[\frac{\lambda\alpha_n}{c}\,J_1(\alpha_n a)\right]\right\} C_o(r,\alpha_n)}{F(\alpha_n)} \tag{12}$$

where

$$F(\alpha_n) = \left(\alpha_n^2 + \frac{\beta^2}{c^2}\right)\left[\frac{\lambda\alpha_n}{c}\,J_1(\alpha_n a)\right]^2 - \left(\frac{\lambda^2}{c^2}\,\alpha_n^2\right)\left[\alpha_n\,J_1\,(\alpha_n) - \frac{\beta}{c}\,J_o\,(\alpha_n)\right]^2$$

and

$$C_o(r,\alpha_n) = J_o(r\,\alpha_n)\left[\frac{\lambda}{c}\,\alpha_n\,Y_1(a\alpha_n)\right] - Y_o(r\,\alpha_n)\left[\frac{\lambda\alpha_n}{c}\,J_1(a\alpha_n)\right]$$

Now, define

$$\delta_i = \alpha_n c$$

and

$$\alpha_n = \delta_i / c \quad .$$

Thus using the above definition of α_n, and recalling that $P = r/c$, Eq. (12) is rearranged to give

$$(T_g - T) = \Delta T = \pi \sum_{i=1}^{\infty} \frac{e^{(-F\,\delta_i^2)}\, \beta (T_g - T_i) \left\{ \delta_i\, J_1(\delta_i) - \beta\, J_o\,(\delta_i) \right\} J_1(m\delta_i)}{\left[\delta_i\, J_1\,(\delta_i) - \beta\, J_o\,(\delta_i) \right]^2 - (\delta_i{}^2 + \beta^2)\, J_1{}^2\,(m\delta_i)} \cdot$$

$$\left\{ J_o\,(P\delta_i)\, Y_1\,(m\delta_i) - Y_o(P\delta_i)\, J_1\,(m\delta_i) \right\} \qquad (13)$$

Letting

$$f_1(m, \beta, \delta_i) = \frac{\pi\, \beta\, J_1\,(m\delta_i) \left\{ \delta_i\, J_1\,(\delta_i) - \beta\, J_o\,(\delta_i) \right\}}{\left[\delta_i\, J_1(\delta_i) - \beta\, J_o(\delta_i) \right]^2 + \left[\delta_i{}^2 + \beta^2 \right] J_1{}^2(m\delta_i)}$$

and

$$f_2(P\delta_i,\, m\delta_i) = J_o\,(P\delta_i)\, Y_1\,(m\delta_i) - Y_o\,(P\delta_i)\, J_1(m\delta_i)$$

Eq. (13) reduces to

$$\theta = \sum_{i=1}^{\infty} \exp\,(-\,F\,\delta_i{}^2)\; f_1(m, \beta, \delta_i) \quad f_2(P\delta_i,\, m\delta_i) \quad . \qquad (14)$$

Equation (14) is the same solution to the heat conduction equation as was obtained by Geckler for the homogeneous solid cylinder.

III. DEVELOPMENT OF THE GENERAL STRESS AND STRAIN EQUATIONS FOR AN ELASTIC CASE-BONDED ROCKET GRAIN

The general expression for the stress in an elastic solid is given by

$$\sigma_{ij} = \lambda \, \epsilon_{kk} \, \delta_{ij} + 2\mu \, \epsilon_{ij} - 3K\alpha \, \Delta T \tag{15}$$

where λ and μ are Lame's moduli, α is the coefficient of linear expansion, and K is the bulk modulus.

The strain equation is written as

$$\epsilon_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right] \tag{16}$$

where u_k denotes deformation of the elastic solid.

The propellant and the case are considered to be a thick walled circular cylinder which satisfies the following conditions:

(a) Rotational symmetry exists, thus

$$\frac{\partial}{\partial \theta} = 0, \quad \sigma_{r\theta} = \sigma_{\theta z} = 0, \quad v = 0 \ .$$

(b) Plane sections remain plane, or

$$\epsilon_{zz} = \frac{\partial w}{\partial z} = \text{const.}$$

(c) Internal pressure and temperature are independent of z, hence

$$\frac{\partial}{\partial z} \sigma_{ij} = 0, \quad \frac{\partial \epsilon_{ij}}{\partial z} = 0 \quad \text{if } i \neq j \ .$$

The problem as set forth is one dimensional in r. Applying the above conditions to Eq. (15) and Eq. (16) the relationships to be used in this problem reduce to

$$\sigma_{rr} = \lambda(\epsilon_{rr} + \epsilon_{\theta\theta} + \epsilon_{zz}) + 2\mu\, \epsilon_{rr} - 3\alpha\, K\, \Delta T \qquad (17)$$

$$\sigma_{\theta\theta} = \lambda(\epsilon_{rr} + \epsilon_{\theta\theta} + \epsilon_{zz}) + 2\mu\, \epsilon_{\theta\theta} - 3\alpha\, K\, \Delta T \qquad (18)$$

$$\sigma_{zz} = \lambda(\epsilon_{rr} + \epsilon_{\theta\theta} + \epsilon_{zz}) + 2\mu\, \epsilon_{zz} - 3\alpha\, K\, \Delta T \qquad (19)$$

$$\sigma_{r\theta} = \sigma_{rz} = \sigma_{\theta z} = 0 \qquad (20)$$

$$\epsilon_{r\theta} = \epsilon_{rz} = \epsilon_{\theta z} = 0 \qquad (21)$$

$$\epsilon_{rr} = \frac{\partial u}{\partial r} \qquad (22)$$

$$\epsilon_{\theta\theta} = u/r \qquad (23)$$

$$\epsilon_{zz} = \text{const.} \qquad (24)$$

The first equilibrium equation

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\ \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} + \sigma_{\theta\theta}}{r} + R = 0 \qquad (25)$$

reduces to the following, considering the sum of the external forces, R, is equal to zero,

$$\frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr} + \sigma_{\theta\theta}}{r} = 0 \quad . \qquad (26)$$

Using Eqs. (17), (18), and (26) one finds that u is governed by the differential equation

$$(\lambda + 2\mu) \frac{d}{dr} \left[\frac{1}{r} \frac{d}{dr} (ru) \right] = 3\alpha K \frac{d}{dr} (\Delta T) \quad . \qquad (27)$$

The parameters λ, μ, and K can be defined in terms of Poisson's ratio, ν, and Young's modulus, E, as

$$\lambda = \frac{\nu E}{(1 + \nu)(1 - 2\nu)} , \quad \mu = \frac{E}{2(1 + \nu)} , \text{ and } \quad K = \frac{E}{3(1 - 2\nu)} .$$

Substituting the expression for λ, μ, and K into Eq. (27) gives

$$\frac{d}{dr} \left[\frac{1}{r} \frac{d}{dr} (ru) \right] = \alpha \left(\frac{1 + \nu}{1 - \nu} \right) \frac{d}{dr} (\Delta T) . \qquad (28)$$

The double integration of Eq. (28) yields

$$u = \frac{\alpha}{r} \left(\frac{1 + \nu}{1 - \nu} \right) \int_{g}^{r} (r) \, \Delta T \, dr + \frac{C_1 r}{2} + \frac{C_2}{r} \quad . \qquad (29)$$

In the above equation "g" is a radius which can be chosen arbitrarily. When working with the propellant it will be chosen as radius, a, and when working with the case it will be chosen as radius,c. Under the assumptions made earlier Eq. (29) holds for both the propellant and the case. The subscripts assigned to α, ν, C_1, and C_2 will designate whether Eq. (29) is being used for the propellant or the case.

From Eq. (29) the expressions for ϵ_{rr} and $\epsilon_{\theta\theta}$ are now obtained as

$$\epsilon_{rr} = \frac{\partial u}{\partial r} = \alpha \left(\frac{1 + \nu}{1 - \nu} \right) \left[\Delta T - \frac{1}{r^2} \int_g^r (r)\, \Delta T\, dr \right] + \frac{C_1}{2} - \frac{C_2}{r^2} \tag{30}$$

and

$$\epsilon_{\theta\theta} = \frac{u}{r} = \frac{\alpha}{r^2} \left(\frac{1 + \nu}{1 - \nu} \right) \int_g^r (r)\, \Delta T\, dr + \frac{C_1}{2} + \frac{C_2}{r^2} \ . \tag{31}$$

The substitution of Eq. (30) and (31) into (17), (18), and (19) gives the following expressions for the stresses,

$$\sigma_{rr} = \lambda \left[\alpha \left(\frac{1 + \nu}{1 - \nu} \right) \Delta T \right] + 2\mu \left[\alpha \left(\frac{1 + \nu}{1 - \nu} \right) \Delta T - \frac{\alpha}{r^2} \left(\frac{1 + \nu}{1 - \nu} \right) \int_g^r (r)\, \Delta T\, dr \right] +$$

$$\lambda \left[C_1 + \epsilon_{zz} \right] + 2\mu \left[\frac{C_1}{2} - \frac{C_2}{r^2} \right] - 3\alpha\, K\, \Delta T \tag{32}$$

$$\sigma_{\theta\theta} = \lambda \left[\alpha \left(\frac{1 + \nu}{1 - \nu} \right) \Delta T \right] + 2\mu \left[\frac{\alpha}{r^2} \left(\frac{1 + \nu}{1 - \nu} \right) \int_g^r (r)\, \Delta T\, dr \right] +$$

$$\lambda \left[C_1 + \epsilon_{zz} \right] + 2\mu \left[\frac{C_1}{2} + \frac{C_2}{r^2} \right] - 3\alpha\, K\, \Delta T \tag{33}$$

and

$$\sigma_{zz} = \lambda \left[\alpha \left(\frac{1 + \nu}{1 - \nu} \right) \Delta T + C_1 + \epsilon_{zz} \right] + 2\mu\, \epsilon_{zz} - 3\alpha\, K\, \Delta T \ . \tag{34}$$

In the equations for σ_{rr} and $\sigma_{\theta\theta}$ there are several terms which are independent of the radial position r. These terms are grouped together and defined in terms of two new constants A and B. Thus, using the definitions given for λ, μ, and K, the general expressions for the radial, tangential, and axial thermal stresses, in a case-bonded cylindrical rocket grain, can be written as

$$\sigma_{rr} = - \frac{E\alpha}{r^2 (1 - \nu)} \int_g^r (r) \Delta T \, dr + A - \frac{B}{r^2} \tag{35}$$

$$\sigma_{\theta\theta} = \frac{E\alpha}{r^2 (1 - \nu)} \int_g^r (r) \Delta T \, dr - \frac{E\alpha \Delta T}{(1 - \nu)} + A + \frac{B}{r^2} \tag{36}$$

and

$$\sigma_{zz} = 2\nu A - \frac{\alpha E \Delta T}{(1 - \nu)} + E \epsilon_{zz} \ . \tag{37}$$

The displacement now takes the form

$$u = \frac{\alpha}{r} \left(\frac{1 + \nu}{1 - \nu} \right) \int_g^r (r) \Delta T \, dr + r \left\{ \left(\frac{1 + \nu}{E} \right) \left[A(1 - 2\nu) + \frac{B}{r^2} \right] - \nu \epsilon_{zz} \right\} \tag{38}$$

From the following boundary conditions the five constants A_p and B_p for the propellant, A_c and B_c for the case, and ϵ_{zz} may be evaluated:

(1) $\sigma_{rr_p} = 0$ at $r = a$.

(2) At the bonded surface the stress must be continuous, or

$\sigma_{rr_p} = \sigma_{rr_c}$ at $r = b$.

(3) At the bonded surface the displacements must be continuous, thus

$$u_p = u_c \qquad \text{at} \qquad r = b \quad .$$

(4) $$\sigma_{rr_c} = 0 \qquad \text{at} \qquad r = c \quad .$$

(5) The net force in the axial direction must be zero, or

$$\int_a^c \sigma_{zz} \, (2\pi r) \, dr = 0 \quad .$$

The above boundary conditions, when applied to Eqs. (35) through (38), result in the following expressions for A_p, A_c, B_p, B_c, and ϵ_{zz}. These expressions can be shown to be

$$A_p = \frac{B_p}{a^2} \tag{39}$$

$$A_c = \frac{B_c}{c^2} \tag{40}$$

$$B_c = \left(\frac{n^2}{n^2 - 1} \right) \left\{ \frac{E_p \, \alpha_p}{(1 - \nu_p)} \int_a^b (r) \, \Delta T \, dr - \frac{E_c \, \alpha_c}{(1 - \nu_p)} \int_c^b (r) \, \Delta T \, dr - B_p \, (s^2 - 1) \right\} \tag{41}$$

$$\epsilon_{zz} = \frac{2E_p\alpha_p \left(\frac{1 - \nu_c}{1 - \nu_p} \right) \int_a^b (r) \, \Delta T \, dr - 2E_c\alpha_c \int_c^b (r) \, \Delta T \, dr - 2B_p (s^2 - 1)(\nu_p - \nu_c)}{E_p \, a^2 \, (s^2 - 1) + E_c \, b^2 \, (n^2 - 1)} \tag{42}$$

$$B_p = \frac{\left(\frac{E_p}{E_c}\right)\left\{\left(\frac{1+\nu_c}{1+\nu_p}\right) - \left(\frac{1+\nu_c}{1+\nu_p}\right)\left(\frac{1-2\nu_c+n^2}{n^2-1}\right) - \frac{2\left(\frac{\nu_p-\nu_c}{n^2-1}\right)\left(\frac{1-\nu_c}{1+\nu_p}\right)}{1+\left(\frac{E_p}{E_c}\right)\left(\frac{s^2-1}{n^2-1}\right)\left(\frac{1}{s^2}\right)}\right\}\frac{\alpha_c E_c}{(1-\nu_c)}\int_c^b (r)\,\Delta T\,dr}{\left\{\left[s^2(1-2\nu_p)+1\right] + \left(\frac{E_p}{E_c}\right)\left(\frac{1+\nu_c}{1+\nu_p}\right)\left(\frac{s^2-1}{n^2-1}\right)(1-2\nu_c+n^2) + \frac{2\left(\frac{E_p}{E_c}\right)\left(\frac{s^2-1}{n^2-1}\right)\frac{(\nu_p-\nu_c)^2}{1+\nu_p}}{1+\left(\frac{E_p}{E_c}\right)\left(\frac{s^2-1}{n^2-1}\right)\left(\frac{1}{s^2}\right)}\right\}} -$$

$$\frac{\left\{1 - \left(\frac{E_p}{E_c}\right)\left(\frac{1+\nu_c}{1+\nu_p}\right)\left(\frac{1-2\nu_c+n^2}{n^2-1}\right) - \frac{2(\nu_p-\nu_c)\left(\frac{1-\nu_c}{1+\nu_p}\right)\left(\frac{E_p}{E_c}\right)}{1+\left(\frac{E_p}{E_c}\right)\left(\frac{s^2-1}{n^2-1}\right)\left(\frac{1}{s^2}\right)}\right\}\frac{\alpha_p E_p}{(1-\nu_p)}\int_a^b (r)\,\Delta T\,dr}{\left\{\left[s^2(1-2\nu_p)+1\right] + \left(\frac{E_p}{E_c}\right)\left(\frac{1+\nu_c}{1+\nu_p}\right)\left(\frac{s^2-1}{n^2-1}\right)(1-2\nu_c+n^2) + \frac{2\left(\frac{E_p}{E_c}\right)\left(\frac{s^2-1}{n^2-1}\right)\frac{(\nu_p-\nu_c)^2}{1+\nu_p}}{1+\left(\frac{E_p}{E_c}\right)\left(\frac{s^2-1}{n^2-1}\right)\left(\frac{1}{s^2}\right)}\right\}} \qquad (43)$$

With the substitution of the above constants into Eqs. (35) through (37) the general expressions for the stresses and strains in the case and the propellant are obtained.

Since it is convenient to use dimensionless quantities in numerical calculations, the stress equations are made dimensionless on dividing through by the quantity $\frac{(1 - \nu)}{E\alpha(T_g - T_i)}$. If σ_{r_n}, σ_{θ_n}, and σ_{z_n} denote the dimensionless stresses, then

$$\sigma_{r_n} = \frac{\sigma_{rr}\ (1-\nu)}{E\alpha\ (T_g - T_i)} \tag{44}$$

$$\sigma_{\theta_n} = \frac{\sigma_{\theta\theta}\ (1-\nu)}{E\alpha\ (T_g - T_i)} \tag{45}$$

and

$$\sigma_{z_n} = \frac{\sigma_{zz}\ (1-\nu)}{E\alpha\ (T_g - T_i)} \tag{46}$$

The strains are, by definition, dimensionless and may be obtained from Eq. (38). There equations are written as

$$\epsilon_{rr} = \alpha \left(\frac{1 + \nu}{1 - \nu} \right) \Delta T - \frac{\alpha}{r^2} \int_g^r (r)\, \Delta T\, dr +$$

$$\frac{(1 + \nu)}{E} \left[A(1 - 2\nu) - \frac{B}{r^2} \right] - \nu\, \epsilon_{zz} \tag{47}$$

$$\epsilon_{\theta\theta} = \frac{\alpha}{r^2}\left(\frac{1+\nu}{1-\nu}\right)\int_g^r (r)\,\Delta T\,dr + \frac{(1+\nu)}{E}\left[A(1-2\nu) + \frac{B}{r^2}\right] - \nu\epsilon_{zz} \tag{48}$$

and

$\epsilon_{zz} =$

$$\frac{2\,E_p\alpha_p\left(\frac{1+\nu_c}{1+\nu_p}\right)\int_a^b (r)\,\Delta T\,dr - 2\,E_c\alpha_c\int_c^b (r)\,\Delta T\,dr - 2\,B_p(s^2-1)(\nu_p-\nu_c)}{a^2\left[E_p\,(s^2-1) + E_c\,s^2\,(n^2-1)\right]} . \tag{49}$$

IV. INTEGRATION OF THE EXPRESSION $\int_g^r (r)\,\Delta T\,dr$

From the general stress and strain equations in Section 3 it is seen that the integral $\int_g^r (r)\,\Delta T\,dr$ must be evaluated before a numerical solution is possible. The integration results in a series which converges rapidly. It is sufficient to use the first few terms to determine the numerical value of the integral. When the value for ΔT, from Section 2, is substituted into the integral the result is

$$\int_g^r (r)\,\Delta T\,dr = \frac{(T_g - T_i) \sum_{n=1}^{\infty} e^{(-F\,\delta_i^2)}\,\pi\,\beta\,J_1(m\delta_i)\left[\delta_i\,J_1(\delta_i) - \beta\,J_o(\delta_i)\right]}{\left[\delta_i\,J_1(\delta_i) - \beta\,J_o(\delta_i)\right]^2 - (\beta^2 + \delta_i^2)\,J_1^2(m\delta_i)} \cdot$$

$$\int_g^r (r)\left[J_o(P\delta_i)\,Y_1(m\delta_i) - Y_o(P\delta_i)\,J_1(m\delta_i)\right] dr \quad . \qquad (50)$$

Defining the quantity $\emptyset$ as

$$\frac{(T_g - T_i) \sum_{n=i}^{\infty} e^{(-F\,\delta_i^2)}\,\pi\,\beta\,J_1(m\delta_i)\left[\delta_i\,J_1(\delta_i) - \beta\,J_o(\delta_i)\right]}{\left[\delta_i\,J_1(\delta_i) - \beta\,J_o(\delta_i)\right]^2 - (\beta^2 + \delta_i^2)\,J_1^2(m\delta_i)}$$

then Eq. (50) can be rewritten as

$$\int_g^r (r)\ \Delta T\ dr = \emptyset \Big\{ Y_1\ (m\delta_i) \int_g^r (r)\ J_o\ (P\delta_i)\ dr\ -$$

$$J_1\ (m\delta_i) \int_g^r (r)\ Y_o\ (P\delta_i)\ dr \Big\} . \tag{51}$$

The terms in Eq. (51) which require integration are

$$\int_g^r (r)\ J_o\ (P\delta_i)\ dr \tag{52}$$

and

$$\int_g^r (r)\ Y_o\ (P\delta_i)\ dr\ . \tag{53}$$

Carrying out the integration of Eq. (52) one obtains

$$\int_g^r (r)\ J_o\ (\frac{\delta_i r}{c})\ dr = \frac{c^2}{\delta_i^2} \int_{g\delta_i/c}^{r\delta_i/c} (\frac{r\delta_i}{c})\ J_o\ (\frac{r\delta_i}{c})\ d\ (\frac{r\delta_i}{c})$$

$$= \frac{c^2}{\delta_i^2} \left[\frac{r\delta_i}{c}\ J_1\ (\frac{\delta_i r}{c}) - \frac{g\delta_i}{c}\ J_1\ (\frac{g\delta_i}{c}) \right] \tag{54}$$

In a similar manner, the result for Eq. (53) is

$$\int_g^r (r)\ Y_o\ (\frac{\delta_i r}{c})\ dr = \frac{c^2}{\delta_i^2} \left[\frac{r\delta_i}{c}\ Y_1\ (\frac{\delta_i r}{c}) - \frac{g\delta_i}{c}\ Y_1\ (\frac{g\delta_i}{c}) \right] \tag{55}$$

Eq. (50) is now rewritten as

$$\int_g^r (r)\, \Delta T\, dr = \frac{\rho(c^2)}{\delta_i}\left\{Y_1(m\delta_i)\left[P\,J_1\,(P\delta_i) - \frac{g}{c}\,J_1\,(\frac{g\delta_i}{c})\right] - \right.$$

$$\left. - J_1\,(m\delta_i)\left[P\,Y_1\,(P\delta_i) - \frac{g}{c}\,Y_1\,(\frac{g\delta_i}{c})\right]\right\} \qquad (56)$$

For g = a and c Eq. (56) takes the form

$$\int_a^r (r)\Delta T\, dr = \frac{\rho c^2}{\delta_i}\left\{Y_1\,(m\delta_i)\left[P\,J_1\,(P\delta_i) - m\,J_1\,(m\delta_i)\right] - \right.$$

$$\left. J_1(m\delta_i)\left[P\,Y_1\,(P\delta_i) - m\,Y_1(m\delta_i)\right]\right\} \qquad (57)$$

and

$$\int_r^c (r)\Delta T\, dr = \frac{\rho c^2}{\delta_i}\left\{Y_1\,(m\delta_i)\left[P\,J_1\,(P\delta_i) - J_1(\delta_i)\right] - \right.$$

$$\left. J_1(m\delta_i)\left[P\,Y_1(P\delta_i) - Y_1(\delta_i)\right]\right\} \qquad (58)$$

For g = a and c and r = b Eq. (56) results in

$$\int_a^b (r)\, \Delta T\, dr = \frac{\rho c^2}{\delta_1}\left\{Y_1\,(m\delta_i)\left[\frac{1}{n}\,J_1\,(\frac{\delta_i}{n}) - m\,J_1(m\delta_i)\right] - \right.$$

$$\left. J_1(m\delta_i)\left[\frac{1}{n}\,Y_1\,(\frac{\delta_i}{n}) - m\,Y_1\,(m\delta_i)\right]\right\} \qquad (59)$$

and

$$\int_c^b (r)\, \Delta T\, dr = \frac{\rho c^2}{\delta_i}\left\{Y_1\,(m\delta_i)\left[\frac{1}{n}\,J_1\,(\frac{\delta_i}{n}) - J_1\,(\delta_i)\right] - \right.$$

$$\left. J_1(m\delta_i)\left[\frac{1}{n}\,Y_1\,(\frac{\delta_i}{n}) - Y_1(\delta_i)\right]\right\} \qquad (60)$$

V. NUMERICAL PROCEDURE

In order to study the predictions of the general expressions a numerical example is now presented. The propellant chosen is a type of Polybutadiene Acrylic Acid, the case consists of wound glass fibers and resin. The thermal and mechanical properties for the propellant and the case are shown in Table 2[2]. These properties are substituted into the general equations and the dimensionless values of the stresses and strains are computed at several radial positions throughout the grain and the case. After the stresses and strains are computed at one ratio "m", the ratio is changed and the stresses and strains are recalculated. The recalculation will indicate whether the stresses and strains increase or decrease with an increase in "m". Graphs are presented to show the variations of stress and strain with radial position at a ratio of $m = 0.2$. The numerical values of the Bessel functions were taken from Watson[8], and the calculations were done on a desk calculator.

VI. DISCUSSION OF RESULTS

In the numerical example presented, the cure temperature was chosen to be $60^{o}F$. The propellant and case were assumed to be in an environment having a constant temperature of $100^{o}F$. The maximum temperature difference, ΔT, occurs initially. This is due to the negative exponential that appears in the numerator of the expression for ΔT. At any time greater than $t = 0$ this exponential will cause the value of ΔT to decrease. The stresses and strains were computed at $t = 0$ since, at this time, ΔT was at its largest value and therefore the stresses and strains were at a maximum for the conditions under consideration.

From the graph of the dimensionless radial stress versus radial position it is seen that the zero stress in the propellant occurred at its inner radius, and the zero stress in the case at the radius c. This is due to the boundary conditions imposed on the problem.

From the inner radius "a" the stresses build up to a maximum at the bond point, and then decrease through the case.

The dimensionless tangential stress is a maximum at the inner radius of the propellant. The stress decreases as the bond point is approached, and at this point the stress changes from a compression in the grain to a tension in the case. The maximum tangential stress in the case occurs at the bond point, and the stress decreases to a minimum at the case exterior.

The axial stress is found to be constant, and of a much smaller value than either the radial or tangential stress.

The strains are next taken into consideration. The radial strain is a maximum at the inner radius and decreases to a minimum at the exterior of the case.

The tangential strain is found to be negative and a maximum at the inner radius. The strain decreases as the radius increases, and reaches zero approximately three quarters of the way through the grain. At this point the strain becomes positive and increases to a maximum at the bond point. The strain in the case remains positive and decreases to a minimum at the exterior radius. This strain pattern is expected since the tangential stress has changed from a compression to a tension at the bond point. This stress change is reflected a small distance into the grain as is indicated by the zero strain point in the grain.

The axial strain, like the stress, is a small constant value.

The portions of the graphs which show the stress and strain in the case indicate only the trends of stress and strain. Due to the small ratio of fiberglass case thickness as compared to the propellant the values in this region are not obtained to within the same degree of accuracy as those in the propellant. They do indicate, however, the trend of the stress and strain distribution from the bond point to the exterior of the case.

Once it was determined that the maximum stress and strain encountered were at the inner radius, the ratio "m" was increased. The stresses and strains were computed again, and it was observed that the maximum stress and strain encountered were again at the inner radius, and that the values had increased with the increase in "m". When comparing two grains, under the same conditions, this indicates that the stresses and the strains will build up faster in the grain with the larger value of "m". Therefore the grain with the larger value of "m", all other conditions being equal, would fail first.

In applying the results it must be remembered that certain assumptions have been made which limit the accuracy of the values obtained. The properties in Table 2 have been considered to be independent of temperature. Solid propellants are plastic materials and their properties are not independent of temperature. To obtain more accurate values for the stress and strain the temperature dependency of the properties in Table 2 must be taken into consideration.

VII. METHODS OF DETERMINING FAILURE AND MAXIMUM TEMPERATURE VARIATION

Due to the complex stress and strain conditions in a solid, such as a case-bonded solid propellant, a true criterion for failure is somewhat in doubt. There has been, however, several theories suggested for determining failure conditions. For a one-dimensional problem, such as discussed in this thesis, only two theories are important. They are the "maximum stress theory" and the "maximum strain theory".

A given propellant will sustain a certain amount of stress and strain prior to failure, and this design stress and strain is a characteristic of the propellant. The two theories predict failure when the maximum stress or strain, in the grain, exceeds the design stress or strain. When working with composite grains the maximum strain theory is usually employed since failure phenomenon, in this type of grain, is more closely associated with strain than with stress.

The maximum allowable stresses and strains for the propellant used in this example were not available. In the absence of appropriate values it is impossible to show whether the grain will fail in stress or strain. The temperature conditions imposed on the example were in the region of the cure temperature. If the grain is designed properly it can be assumed that no failure will occur due to the small temperature difference imposed on the case and propellant.

The value chosen for T_g has a large effect on the problem. The greater the difference between T_g and T_i, the greater the stress and strain will be. It is indicated, in the numerical example presented, that the maximum stress occurs at the inner radius of the propellant. In this case T_g was chosen to be larger than T_i. The stress at the inner radius was found to be a compression. If T_g had been chosen less than T_i, the stress would have been a tension. For a given propellant the maximum allowable compressive and tensile stresses would be known. By varying T_g, and computing the resulting stress, an upper and lower limit on T_g would be obtained. This would give the maximum deviation from cure temperature allowed, before grain was damage/caused by excessive stresses. The same analysis can be applied for the strains. Finally the upper and lower limits on T_g can be established so as to prevent failure from either excessive stress or excessive strain.

VIII. CONCLUSIONS

In consideration of the results obtained, it is concluded that:

(1) The maximum stress and strain in a case-bonded rocket grain, when subjected to a temperature change, are developed at the inner radius of the grain. It is at this point that failure will first occur.

(2) The maximum stress and strain at the inner radius will increase as the ratio "m" increases. All conditions being equal the grain with the largest value for "m" will fail first.

(3) Grain failure and maximum allowable temperature variation can be determined by the theories of maximum stress and strain when the stress and strain characteristics of the grain are known.

IX. ACKNOWLEDGEMENTS

The author wishes to express his sincere appreciation to Dr. Hassan A. Hassan and Dr. James B. Eades, Jr. for their advice and valuable assistance during the course of this investigation.

X. BIBLIOGRAPHY

1. Au, N. N., A METHOD OF STRENGTH ANALYSIS OF SOLID PROPELLANT ROCKET GRAINS, SOLID PROPELLANT ROCKET RESEARCH, Vol. 1, pp. 101-120, 1960.

2. Boyd, A. B., GRAIN DESIGN AND DEVELOPMENT PROBLEMS FOR VERY LARGE ROCKET MOTORS, SOLID PROPELLANT ROCKET RESEARCH, Vol. 1, pp. 3-31, 1960.

3. Carslaw, H. S. and Jaeger, J. C., CONDUCTION OF HEAT IN SOLIDS, pp. 332-334, Oxford University Press, London, 1960.

4. Geckler, R., THERMAL STRESSES IN SOLID PROPELLANT GRAINS, JET PROPULSION, Vol. 26, pp. 93-97, 1956.

5. Geckler, R., TRANSIENT RADIAL HEAT CONDUCTION IN HOLLOW CIRCULAR CYLINDER, JET PROPULSION, Vol. 25, pp. 31-35, 1955.

6. Lipow, M. and Zwick, S. A., ON THE ROOTS OF THE EQUATION $J_1(mx)\left[x\, Y_1(x) - \beta\, Y_o(x)\right] - Y_1(mx)\left[x\, J_1(x) - \beta\, J_o(x)\right] = 0$, JOURNAL OF MATHEMATICS AND PHYSICS, Vol. 34, pp. 308-315, 1956.

7. Timoshenko, S. and Goodier, J., THEORY OF ELASTICITY, pp. 412, McGraw-Hill, New York, 1951.

8. Watson, G. N., A TREATISE ON THE THEORY OF BESSEL FUNCTIONS, pp. 606-697, University Press, New York, 1945.

9. Zwick, S. A., THERMAL STRESSES IN A INFINITE, HOLLOW CASE-BONDED CYLINDER, JET PROPULSION, Vol. 27, pp. 872-876, 1957.

The vita has been removed from the scanned document

$\theta = 10$

δ_i	m = 0.2	m = 0.4	m = 0.6
δ_1	02.3059	02.6958	03.5278
δ_2	05.5950	07.0317	09.9453
δ_3	09.1118	11.7525	17.0855
δ_4	12.7662	16.6760	24.5606

ROOTS OF THE EQUATION

$$Y_1(mx)\left[x\,J_1(x) - \beta\,J_o(x)\right] - J_1(mx)\left[x\,Y_1(x) - \beta\,Y_o(x)\right]$$

TABLE 1

Parameter	Propellant	Case
Thermal Diffusivity	0.010878 ft^2/hr	0.010878 ft^2/hr
Convection Coefficient	--	1.08 BTU ft^{-2} $hr^{-1}F^{o-1}$
Coefficient of Linear Expansion	5.0 x 10 F^{o-1}	5.0 x 10 F^{o-1}
Young's Modulus	300 psi	9 x 10 psi
Poisson's Ratio	0.5	0.22
Biot Number	--	10

CASE AND PROPELLANT PARAMETERS

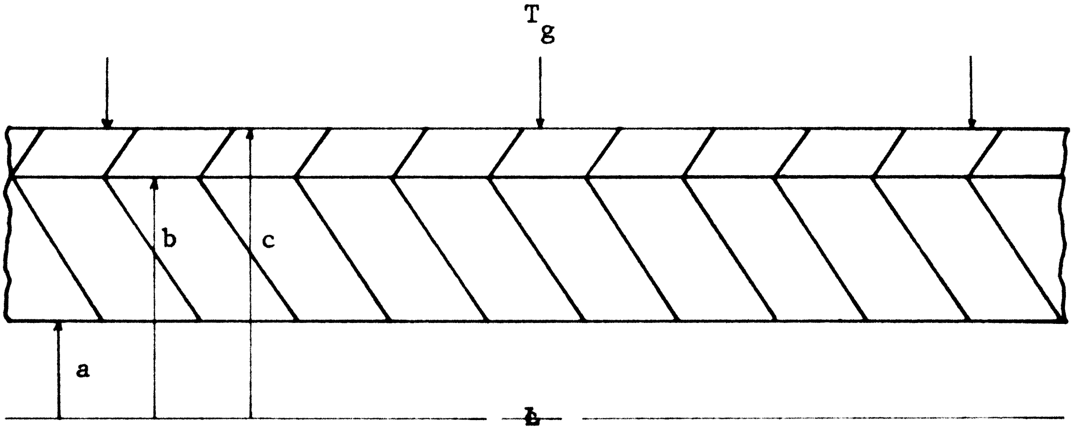
TABLE 2

Radial Position	σ_{r_n}	σ_{θ_n}	ϵ_{rr}	$\epsilon_{\theta\theta}$
08.65	0.00	-17.40	$137.70\text{x}10^{-5}$	$-123.20\text{x}10^{-5}$
15.00	-5.74	-11.67	$051.64\text{x}10^{-5}$	$-037.08\text{x}10^{-5}$
29.00	-7.91	-09.50	$019.64\text{x}10^{-5}$	$-004.64\text{x}10^{-5}$
36.00	-8.14	-09.16	$014.15\text{x}10^{-5}$	$+000.41\text{x}10^{-5}$
43.00	-8.32	-09.07	$013.03\text{x}10^{-5}$	$+001.53\text{x}10^{-5}$
43.25	0.00	+07.83	$000.23\text{x}10^{-5}$	$+000.82\text{x}10^{-5}$

VALUES OF NON-DIMENSIONAL STRESSES AND STRAINS

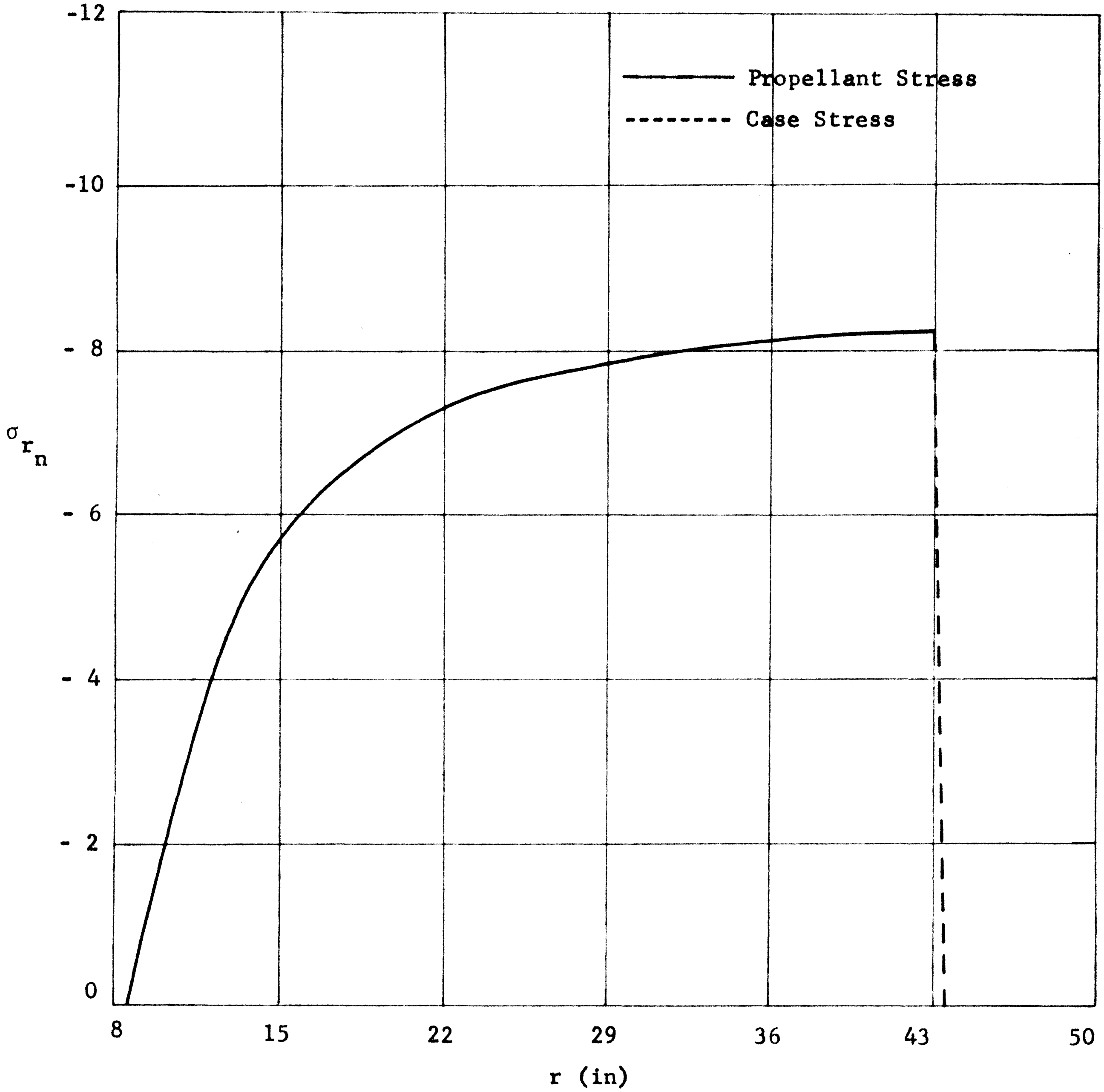
FOR DIFFERENT RADIAL POSITIONS

TABLE 3



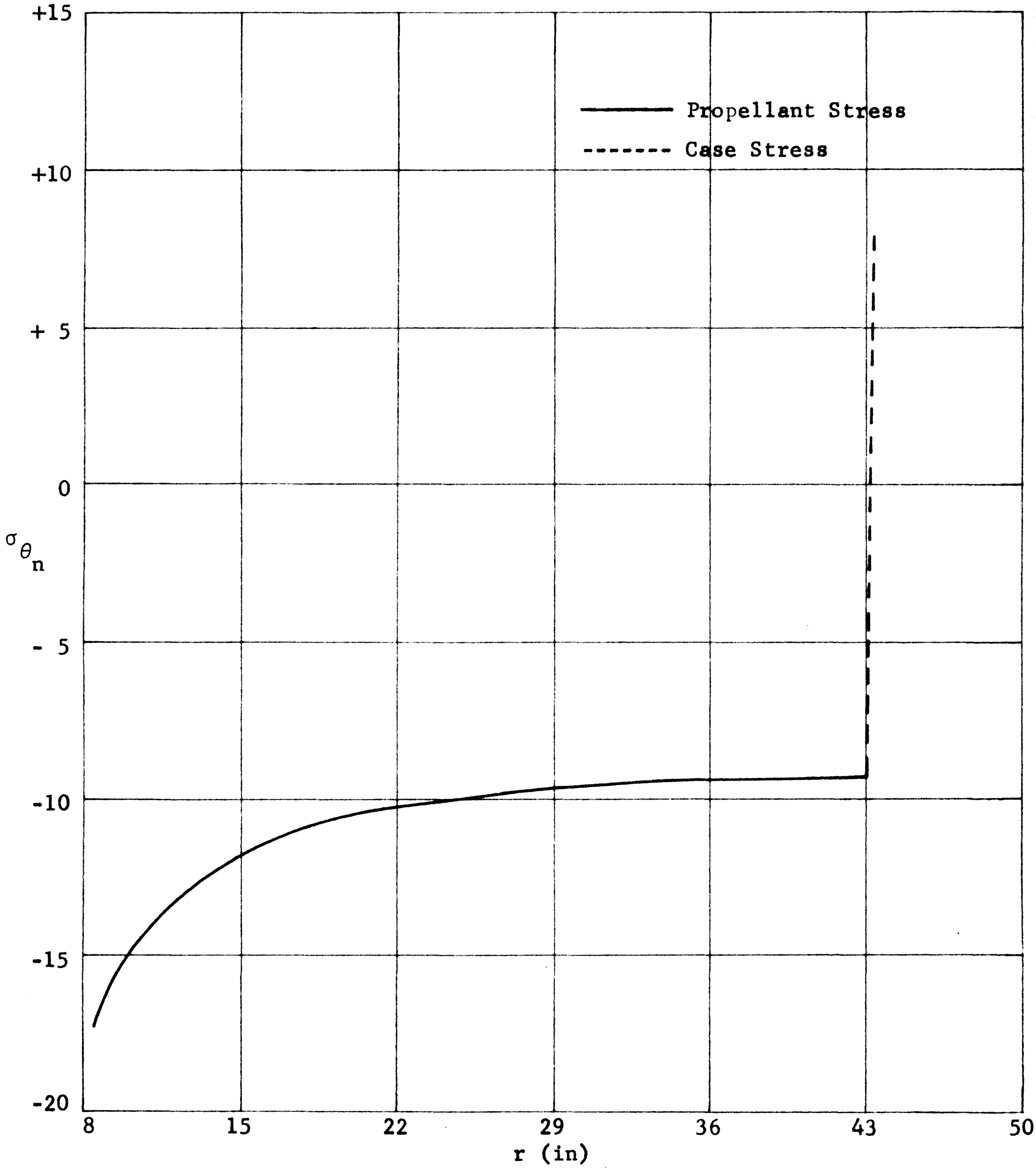
CROSS-SECTION VIEW OF A CASE-BONDED ROCKET GRAIN

FIGURE 1



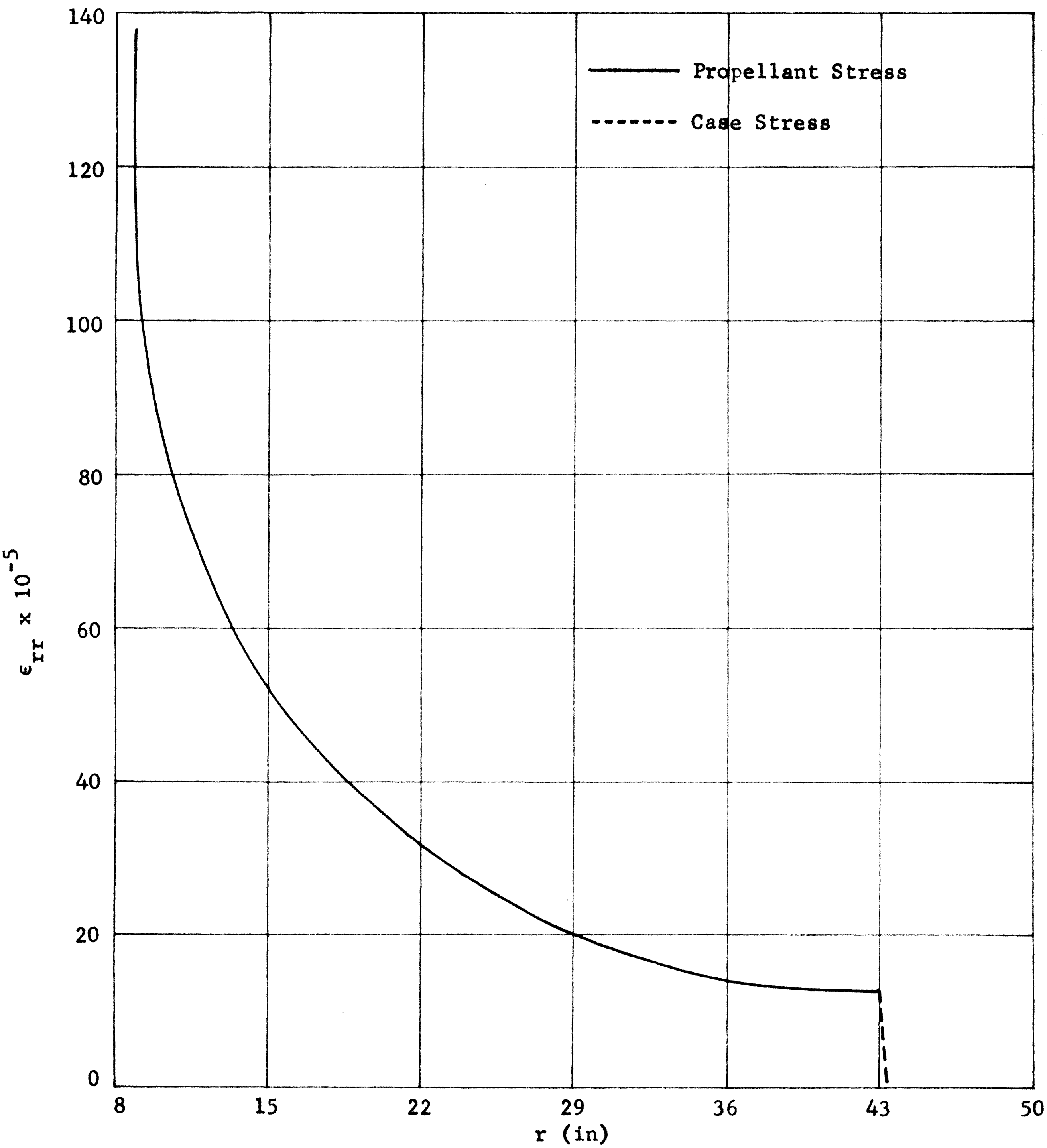
DIMENSIONLESS RADIAL STRESS VERSUS RADIAL POSITION

FIGURE 2



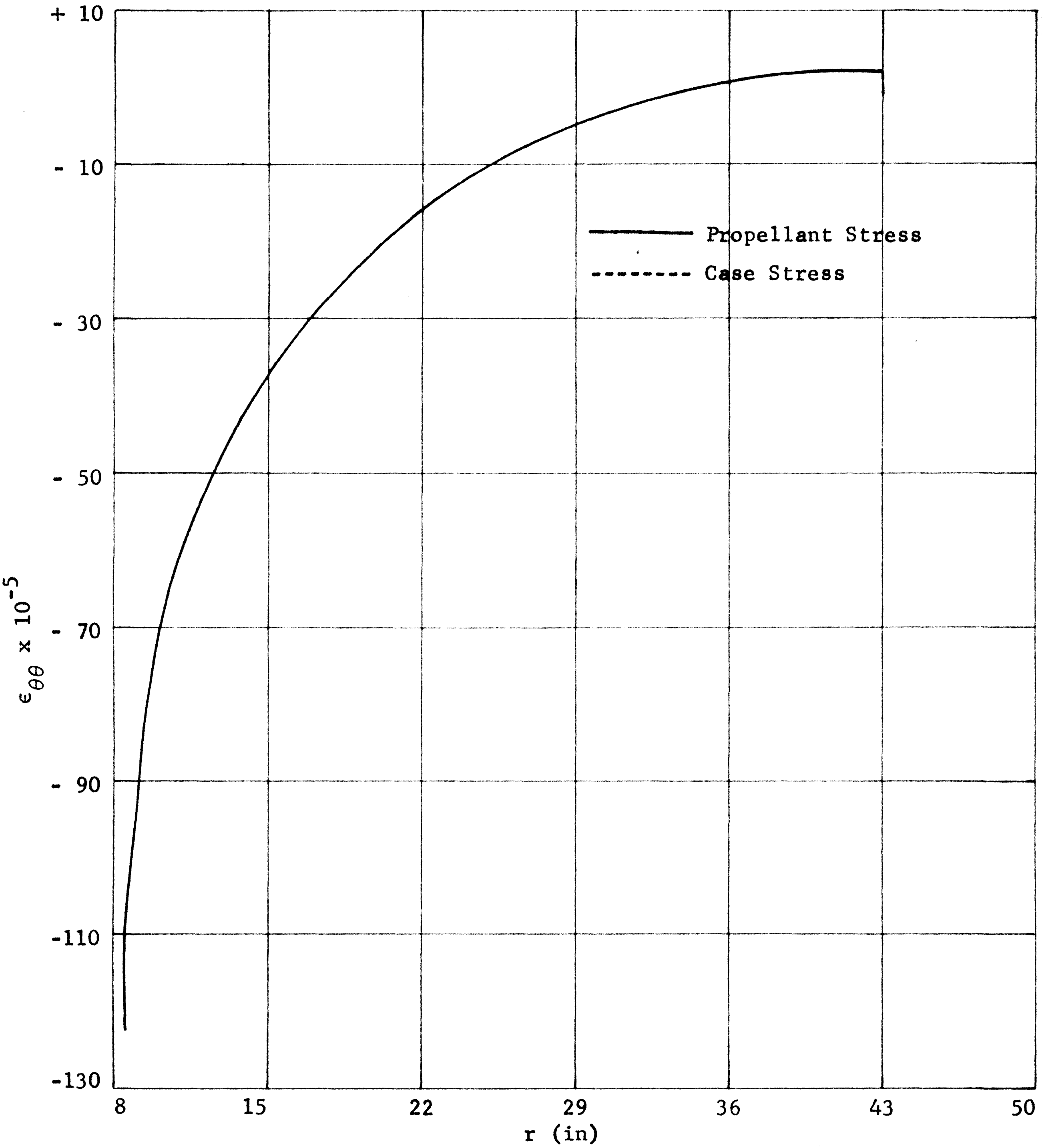
DIMENSIONLESS TANGENTIAL STRESS VERSUS RADIAL POSITION

FIGURE 3



RADIAL STRAIN VERSUS RADIAL POSITION

FIGURE 4



TANGENTIAL STRAIN VERSUS RADIAL POSITION

FIGURE 5

ABSTRACT

The object of this investigation was to develop a general solution for the thermal stresses and strains in a hollow cylindrical case-bonded solid propellant.

The heat conduction equation, as solved by Carslaw and Jaeger, was applied to a hollow composite cylinder. The temperature distribution from this equation was used in conjunction with the stress and strain for an elastic solid propellant. The boundary conditions were employed to solve for the constants and the general solution for the stresses and strains were obtained.

In order to study the predictions of the general expressions, a numerical example was presented. It was found that the maximum stress and strain appeared at the inner radius of the grain. It was also observed that the stress and strain increased with an increase in the radius ratio "m".

Failure criteria for the grain under consideration were discussed. A method for obtaining the maximum allowable temperature variation (from cure temperature) was investigated. Knowing the stress and strain characteristics of the grain the equations developed would indicate failure conditions and also allow calculations of the maximum allowable temperature variations prior to grain failure.