SENSOR ERROR MODEL FOR A UNIFORM LINEAR ARRAY

Aditya Gadre, Michael Roan, Daniel Stilwell

Virginia Center for Autonomous Systems
Virginia Polytechnic Institute & State University
Blacksburg, VA 24060
www.unmanned.vt.edu

May 5, 2008

Technical Report No. VaCAS-2008-01

Abstract

We derive a measurement error model for a uniform linear array whose output is the bearing to a single narrowband acoustic source. The measurement error depends on various array as well as environmental parameters, which include the number of hydrophones in the array, spacing between adjacent hydrophones, frequency of the acoustic signal, speed of sound and signal-to-noise ratio. Most importantly, we show that the measurement error is a function of the true bearing from the array to the acoustic source.

1 Introduction

We consider a uniform linear array (ULA) with N hydrophones. The output of the ULA is the bearing to a single acoustic source relative to the broadside of the array. The output of

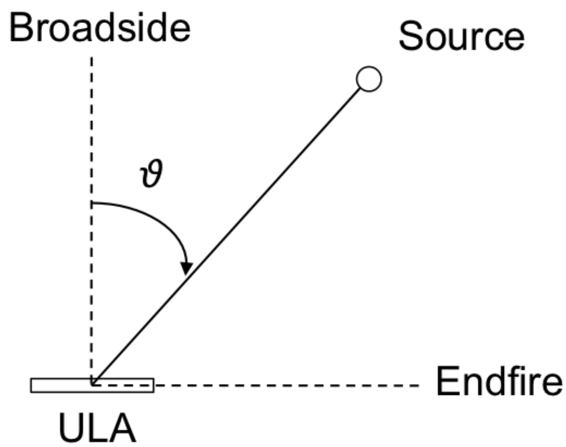

Figure 1: Bearing from uniform linear array to acoustic source

the array $\hat{\theta}$ can be written as

$$\hat{\theta} = \theta + \nu \tag{1}$$

where, θ is the true bearing to the acoustic source and ν is zero mean Gaussian noise with variance σ_ν^2; that is, $\nu \in \mathcal{N}(0, \sigma_\nu^2)$. The variance σ_ν^2 dependent on a variety of array and environmental parameters, including the number of hydrophones, separation between adjacent hydrophones, frequency of the input signal, speed of sound, signal-to-noise ratio (SNR) and bearing of the source relative to the array. Our goal is this technical report is to derive a sensor noise model for the ULA, which characterizes the measurement error variance σ_ν^2.

Our model is based on earlier works reported in [1], [2] and [3], in which an error model for the measurement of incremental phase shift of a ULA is derived by considering quadrature components of the input signal and noise. We restrict our analysis to real signals and derive a model to estimate uncertainty in the bearing angle measurement instead of the incremental phase shift.

2 Problem Statement

We consider a uniform linear array, with uniformly spaced N hydrophones, used to determine bearing angle θ to a single narrowband acoustic source. We assume that the array is sufficiently far from the acoustic source so that the incident waves are planar.

The ULA geometry is depicted in Figure 2. We designate the reference hydrophone as H_0, with the rest of the hydrophones being H_1, H_2,…,H_{N-1}. Adjacent hydrophones are

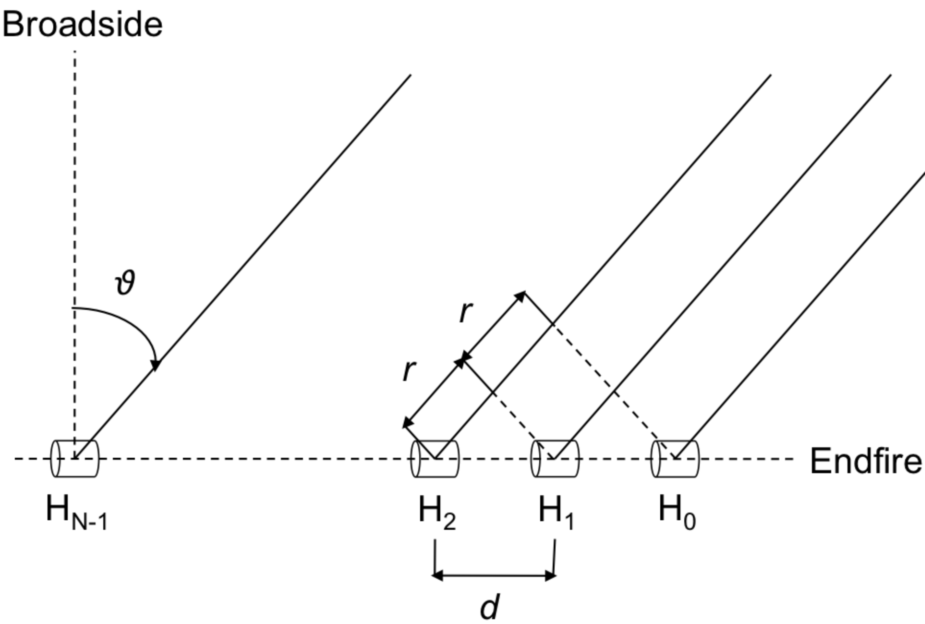

Figure 2: Geomety of a uniform linear array

separated by distance d. The acoustic planar waves are at an angle θ relative to the array broadside.

Let the signal received at the reference hydrophone H_0 be

$$x_0(t) = a \sin(\omega t + \phi) + \nu_0(t) \tag{2}$$

where, a is the amplitude of the signal, $\omega = 2\pi f$ is the angular frequency corresponding to the acoustic signal frequency f, ϕ is the phase of the signal and $\nu_0 \in \mathcal{N}(0, \sigma^2)$.

From the geometry of the ULA, we see that after reaching a hydrophone H_k, the incident waves need to travel an incremental distance r to reach hydrophone H_{k+1}. This incremental distance r can be expressed as

$$r = d \sin\theta \tag{3}$$

This incremental distance results in incremental time shift

$$\tau = \frac{r}{c} = \frac{d \sin\theta}{c} \tag{4}$$

where c is the speed of sound in the surrounding medium. Thus the signal received at hydrophone H_{k+1} is a time-shifted version of the signal received at hydrophone H_k. For an acoustic signal with frequency f, this time shift results in equivalent incremental phase shift

$$\Delta = 2\pi f \tau \tag{5}$$

which is equivalent to

$$\Delta = \frac{2\pi f}{c} d \sin\theta \tag{6}$$

We define

$$\alpha := \frac{2\pi f}{c} d \tag{7}$$

Thus the incremental phase shift in (6) can be expressed by

$$\Delta = \alpha \sin\theta \tag{8}$$

Using the expression for incremental phase shift, we can express the signal received at hydrophone k by

$$x_k(t) = a \sin\left(\omega t + \phi - k\alpha \sin\theta\right) + \nu_k(t), \quad k = 0, \ldots, N-1 \tag{9}$$

where $\nu_k \in \mathcal{N}(0, \sigma^2)$.

We assume that additive noise terms corresponding to distinct hydrophones are independent from each other and hence

$$\mathsf{E}\left[\left(x_k - a \sin\left(\omega t + \phi - k\alpha \sin\theta\right)\right)\left(x_l - a \sin\left(\omega t + \phi - l\alpha \sin\theta\right)\right)\right] = \delta_{kl}\sigma^2 \tag{10}$$

where

$$\delta_{kl} = \begin{cases} 0, & \text{if } k \neq l \\ 1, & \text{if } k = l \end{cases} \tag{11}$$

The signal-to-noise ratio at every hydrophone input is defined as

$$SNR = \frac{a^2}{\sigma^2} \tag{12}$$

In the next section, we derive the lower bound on the measurement error in $\hat{\theta}$.

3 Lower Bound on the error in $\hat{\theta}$

The accuracy with which bearing angle θ can be measured depends on the noise components present at individual hydrophones. To determine the lower bound on the error in the measurement $\hat{\theta}$, we consider a single snapshot of signals present at all hydrophones at a given instance t. Such a snapshot can be expressed by

$$S = \{x_0, x_1, \ldots, x_{N-1}\} \tag{13}$$

We consider two cases. In the first case, we assume that the phase of the acoustic signal is known whereas in the second case no such information about the phase is available.

3.1 Known signal phase ϕ

We assume that the exact phase ϕ of the input signal is known. Under this assumption, the likelihood function for $x_k(t)$ is conditioned only on the bearing angle θ and is given by

$$f(x_k|\theta) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{\left(x_k - a\sin\left(\omega t + \phi - k\alpha\sin\theta\right)\right)^2}{2\sigma^2}\right) \tag{14}$$

where dependence on t is dropped for notational convenience. The joint likelihood function for N samples from one snapshot S is

$$f(x_0,\ldots,x_{N-1}|\theta) = \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} \prod_{k=0}^{N-1} \exp\left(-\frac{\left(x_k - a\sin\left(\omega t + \phi - k\alpha\sin\theta\right)\right)^2}{2\sigma^2}\right) \tag{15}$$

The log-likelihood function is defined by

$$L(x_0,\ldots,x_{N-1}|\theta) := \ln f(x_0,\ldots,x_{N-1}|\theta) \tag{16}$$

Substituting (15) in (16) yields

$$L(x_0,\ldots,x_{N-1}|\theta) = \ln\frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} - \frac{1}{2\sigma^2}\sum_{k=0}^{N-1}\left(x_k - a\sin\left(\omega t + \phi - k\alpha\sin\theta\right)\right)^2 \tag{17}$$

From the Cramer-Rao lower bound, the variance of the estimation error satisfies [4]

$$\sigma_\nu^2 \geq \frac{1}{\mathsf{E}\left[\left(\frac{\partial}{\partial\theta}L(x_0,\ldots,x_{N-1}|\theta)\right)^2\right]} \tag{18}$$

Differentiating (17),

$$\frac{\partial}{\partial\theta}L(x_0,\ldots,x_{N-1}|\theta) = -\frac{a\alpha\cos\theta}{\sigma^2}\sum_{k=0}^{N-1} k\left(x_k - a\sin\left(\omega t + \phi - k\alpha\sin\theta\right)\right)\cos\left(\omega t + \phi - k\alpha\sin\theta\right) \tag{19}$$

Toward simplification of (18), we utilize the well known relationships [5]

$$\sum_{k=0}^{N-1} k = \frac{N(N-1)}{2}, \tag{20}$$

and

$$\sum_{k=0}^{N-1} k^2 = \frac{N(N-1)(2N-1)}{6} \tag{21}$$

Substituting (19) in (18) and simplifying using (20) and (21) yields

$$\sigma_\nu^2 \geq \frac{2\sigma^2}{a^2\alpha^2\cos^2\theta} \cdot \frac{6}{N(N-1)(2N-1)} \tag{22}$$

which can be expressed as

$$\sigma_\nu^2 \geq \frac{12}{\alpha^2\cos^2\theta\,(\mathrm{SNR})\,N(N-1)(2N-1)} \tag{23}$$

For large N, (23) can be approximated by

$$\sigma_\nu^2 \geq \frac{6}{\alpha^2\cos^2\theta\,(\mathrm{SNR})\,N^3} \tag{24}$$

This approximation gets better as the number of hydrophones N is increased.

From Equation (24) we see that the variance of the measurement error depends on the array parameters, which include the number of hydrophones N and the separation d between two adjacent hydrophones. If the number of hydrophones is increased, it results in effectively larger array aperture [6] and lower measurement error. Effects are similar if the separation d is increased. However, there is a limit on how much d can be increased. The upper limit on d is $\frac{\lambda}{2}$, beyond which aliasing results. From Equation (24) we also see that bearing measurements are more accurate for higher SNR values. Assuming spherical spreading, SNR is a function of the range of the array from the source and decreases as the range increases.

The most important aspect of the bearing measurement error is its dependence on the measurement itself. As the acoustic source moves from broadside ($\theta = 0$) to endfire ($\theta = \pm\pi/2$) of the array, the measurement error increases. When the source moves towards endfire, it reduces the effective aperture thus significantly increasing the measurement error [6].

3.2 Unknown signal phase ϕ

In deriving (24), we assumed that the exact phase of the acoustic signal was known. However, in practice we do not have access to such information. Although we do not need to estimate the phase of the input signal, it does affect estimation of the bearing angle. We rewrite the log-likelihood function (17) to reflect the fact that it is a function of ϕ as well.

$$L(x_0,\ldots,x_{N-1}|\theta,\phi) = \ln\frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} - \frac{1}{2\sigma^2}\sum_{k=0}^{N-1}\left[x_k - a\sin\left(\omega t + \phi - k\alpha\sin\theta\right)\right]^2 \tag{25}$$

Differentiating (25) with respect to ϕ and θ yields

$$\frac{\partial}{\partial\phi}L(x_0,\ldots,x_{N-1}|\theta,\phi) = \frac{a}{\sigma^2}\sum_{k=0}^{N-1}\left[x_k - a\sin\left(\omega t + \phi - k\alpha\sin\theta\right)\right]\cos\left(\omega t + \phi - k\alpha\sin\theta\right) \tag{26}$$

and

$$\frac{\partial}{\partial\theta}L(x_0,\ldots,x_{N-1}|\theta,\phi) = -\frac{a\alpha\cos\theta}{\sigma^2}\sum_{k=0}^{N-1} k\left[x_k - a\sin\left(\omega t+\phi-k\alpha\sin\theta\right)\right]\cos\left(\omega t+\phi-k\alpha\sin\theta\right) \tag{27}$$

The variance of error in the estimate of the bearing angle is given by [1]

$$\sigma_\nu^2 \geq \frac{\mathrm{E}\left[\left(\frac{\partial L}{\partial\phi}\right)^2\right]}{\mathrm{E}\left[\left(\frac{\partial L}{\partial\phi}\right)^2\right]\mathrm{E}\left[\left(\frac{\partial L}{\partial\theta}\right)^2\right] - \left[\mathrm{E}\left(\frac{\partial L}{\partial\phi}\frac{\partial L}{\partial\delta}\right)\right]^2} \tag{28}$$

From (26) and (27) we can write,

$$\mathrm{E}\left[\left(\frac{\partial L}{\partial\phi}\right)^2\right] = \frac{a^2 N}{2\sigma^2} \tag{29}$$

$$\mathrm{E}\left[\left(\frac{\partial L}{\partial\theta}\right)^2\right] = \frac{a^2\alpha^2\cos^2\theta}{2\sigma^2}\cdot\frac{N(N-1)(2N-1)}{6} \tag{30}$$

$$\left[\mathrm{E}\left(\frac{\partial L}{\partial\phi}\frac{\partial L}{\partial\delta}\right)\right] = \frac{-a^2\alpha\cos\theta}{2\sigma^2}\cdot\frac{N(N-1)}{2} \tag{31}$$

Substituting (29), (30) and (31) in (28) and simplifying yields

$$\sigma_\nu^2 \geq \frac{24}{\alpha^2\cos^2\theta\,(\mathrm{SNR})\,N(N^2-1)} \tag{32}$$

For large N, we can approximate (32) by

$$\sigma_\nu^2 \geq \frac{24}{\alpha^2\cos^2\theta\,(\mathrm{SNR})\,N^3} \tag{33}$$

This approximation gets better as N is increased.

The expression for measurement error in (33) is similar to that in (24). We see that not having any information about the phase of the input signal increases the measurement error.

It is important to note that the lower bound on the measurement error is obtained using only one snapshot of signals present at all hydrophones. In practice, a large number of successive snapshots are used to obtain the bearing measurement. A larger number of snapshots decreases the measurement error. As such, with just one snapshot, the lower bounds in (24) and (33) are very conservative.

References

[1] L. Brennan, "Angular accuracy of a phased array radar," *Antennas and Propagation, IEEE Transactions on [legacy, pre - 1988]*, vol. 9, no. 3, pp. 268–275, May 1961.

[2] D. Rife and R. Boorstyn, "Single tone parameter estimation from discrete-time observations," *Information Theory, IEEE Transactions on*, vol. 20, no. 5, pp. 591–598, Sep 1974.

[3] P. Stoica and N. Arye, "Music, maximum likelihood, and cramer-rao bound," *Acoustics, Speech, and Signal Processing [see also IEEE Transactions on Signal Processing], IEEE Transactions on*, vol. 37, no. 5, pp. 720–741, May 1989.

[4] A. Papoulis and S. U. Pillai, *Probability, Random Variables and Stochastic Processes*, 4th ed. McGraw-Hill, 2002.

[5] S. Libeskind, "Summation of finite series - a unified approach," *The Two-Year College Mathematics Journal*, vol. 12, pp. 41–50, 1981.

[6] H. L. V. Trees, *Optimum Array Processing*, ser. Detection, Estimation and Modulation Theory. A. John Wiley and Sons Inc, New York, 2002, vol. IV.