THEORETICAL INVESTIGATION OF THE INITIAL RESPONSE OF A THIN RING TO A RADIAL SHOCK PULSE

by

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LIST OF SYMBOLS

A = Cross sectional area of ring

A_{mn} = Modal constants

A_n = Modal constants

a_n = Modal constants

a_n' = Modal constants

B_{mn} = Modal constants

B_n = Time functions

b_n = Modal constants

b_n' = Modal constants

c = Pressure pulse propagation velocity

C_{mn} = Modal constants

c_n = Modal constants

D_{mn} = Modal constants

D_n = Time functions

d_n = Modal constants

E = Young's Modulus

$_e$ = Subscript $_e$ denotes extensional components

e = r/ct_o Constant

$_f$ = Subscript $_f$ denotes flexural components

f_n = Separation constants

G = Moment stress resultant

g_n = Functions of θ

H = Load decay constant

$_h$ = Subscript $_h$ denotes homogeneous solution

h_n = Separation constants

I = Principal moment of inertia of ring cross section normal to plane of ring

I_n = Time functions

k = $EA/\rho r^2$ Constant

L = $2r/c$ Constant

ℓ_n = Separation constants

m = Integers

N = Shear stress resultant

n = Integers

P = Concentrated force magnitude

p = Distributed force magnitude (Force/Length)

q = Radial loading on ring

r = Radius to centerline of ring

s = Dummy time variable

s_1 = Dummy time variable

s_2 = Dummy time variable

T = Axial stress resultant

T_n = Time functions

t = Time

t_o = Pressure pulse duration

t_1 = $t_o + r/c$ Constant

$u(\theta, t)$ = Radial displacement of a point on ring centerline

$\dot{u}$ = Radial velocity

$v(\theta, t)$ = Tangential displacement of a point on ring centerline

$\dot{v}$ = Tangential velocity

w = Function of θ

x = Time function

Z_n = Time functions

Γ_n = Functions of θ

γ = Angular time coordinate

Δ = Envelopment Angle

δ_n = Phase angles

δ_n' = Phase angles

ϵ = Angular length

Θ_n = Functions of θ

θ = Angular position coordinates

Λ_n = Functions of θ

$\lambda(\theta)$ = Pressure pulse decay function

ρ = Mass per-unit-circumferential-length of ring

τ_n = Time functions

Φ_n = Functions of θ

ϕ = Angular time coordinate

ω_{en} = Extensional circular frequency for n^{th} mode

ω_{fn} = Flexural circular frequency for n^{th} mode

Ω = Angular time coordinate

I. INTRODUCTION

A. BACKGROUND

The problem of determining the response of a circular cylinder to a transient, transverse loading arises in connection with the study of blast and wind loads on missiles and with the study of underwater blast loads on submarine hulls. As an approximation to this physical problem, the subject of this paper will be the determination of the response of a thin circular ring to a moving, radial pressure pulse.

The dynamical behavior of elastic rings has been investigated by numerous authors. Among the first were Hoppe [1],* who derived the classical free flexural vibration frequency equation, and Rayleigh [2], who deduced the existence of primarily extensional and primarily flexural modes of vibration. Love [3] later contributed the classical sixth order partial differential equation governing the tangential displacement for free vibrations. He also discussed torsional and out-of-plane bending vibrations. Timoshenko [4] and Lamb [5] reviewed and summarized the theory of flexural vibrations of rings. Federhofer [6] studied the effect of damping on flexural vibrations of rings.

Waltking [7] and Federhofer [8], [9] investigated the influence of extension, shear, and rotary inertia on the frequency of flexural vibrations of various curved beam and circular arched type structures. Den Hartog [10], Nelson [11], and Raymond [12] considered the dynamics of incomplete rings.

Michell [13] and Brown [14] studied out-of-plane vibrations. Buckens [15] investigated the influence of the thickness of a ring on its

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*Numbers in brackets refer to References on page 69.

frequency. The dynamics of thick rings were treated by Erdelyi and Seidel [16]. Carrier [17] investigated the rotating circular ring, and Evensen [18] considered nonlinear effects in flexural vibrations of rings.

Carver [19] developed a sixth order equation analogous to Love's equation, but including extensional effects. A more general set of equations on the ring displacements was derived by Philipson [20]; this set included the effects of rotatory inertia and extension of the ring's neutral fiber. He solved the two coupled sixth order equations governing the motion for the case of steady state forced harmonic vibration.

Wenk [21] and Federhofer [22], working independently, developed the same equation for the forced flexural vibrations of a ring. The former solved the cases of a half sine pulse, concentrated load on a ring with elastic supports, and of a continuous steady state sinusoidal concentrated load on the same ring. The latter solved the problem of a steady state harmonic, distributed load. Reynolds [23] extended Wenk's work to include uniform external pressure and viscous damping. None of these three considered extensional effects.

Palmer [24] investigated the variation of flexural frequency due to a lumped mass on a ring. He also approximated the response of a ring to a shock loading [25].

The response of a ring to a distributed, transient loading was treated by Suzuki [26]. The analogous problem of a transverse pressure pulse acting on a cylindrical shell has been considered by several authors, including Baron and Bleich [27], Forrestal and Hermann [28], Reck [29], Johnson and Greif [30], Sheng [31], Cottis [32], and Humphreys and Winter [33], but only the first three of these considered moving loads.

Generally the topic of moving loads has not received too widespread attention in the literature; however, the subject of moving loads on beams has been studied by Timoshenko [34], Rogers [35], Florence [36] and Ayre, Jacobsen, and Hsu [37].

B. OBJECTIVE AND SCOPE

The purpose of this paper is to investigate the theoretical initial response of a thin circular ring to a transverse moving pressure pulse. The response is assumed to be linear, to consist only of small deflections with respect to the size of the radius of the ring, and to lie in the plane of the ring. The ring is assumed to be free in space.

The pressure pulse is assumed to have a short duration compared to the fundamental period of the ring; thus, it may be called a shock pulse.

In Section II analytical expressions for the tangential and radial components of displacement and velocity are determined as functions of time and position. Also presented in Section II is a solution using the method of double Fourier series and an approximate solution using this double series method.

Results for a specific numerical example are given in Section III; a discussion of these results and a summary are presented in Sections IV and V respectively. The discussion also contains some comments on analogous experimental results.

II. THEORETICAL ANALYSIS

A. STATEMENT OF THE PROBLEM

The plane circular ring under consideration is assumed to be of a homogeneous isotropic material and of uniform and symmetric cross section with one of the principal axes of the cross section lying in the plane of the ring. All motion is restricted to that plane. The dimensions of the cross section are small compared to the radius, so the ring satisfies the criterion of thin curved beam theory as postulated by Timoshenko [38]. Damping, shear effects, body forces other than inertia forces, and rotatory inertia are all neglected and the only forces experienced by the ring are due to the pressure pulse. Also, it is assumed that the motion of the ring does not affect the pressure pulse and that the pulse exerts only radial forces on the ring.

It is convenient in describing the load mathematically to introduce, in addition to the independent variable of time t, an angular time variable ϕ (t) given by

$$\phi(t) = \mathrm{Cos}^{-1}\left(1 - \frac{tc}{r}\right) \tag{1}$$

where r is the radius of the ring and c is the velocity of the moving pressure wave. This angular coordinate of time is valid only in the range

$$0 \leqq t \leqq \frac{2r}{c} = L .$$

The plane pressure pulse is assumed to strike the ring initially at time t = 0 at the location $\theta = 0$ (see Figure 1) and then to pass normal to the diameter at $\theta = 0$ across the external edge of the ring with constant velocity c. At any time t the position of the front of the wave

is given by $\gamma(t)$
where

$$\gamma(t) = \begin{cases} \phi(t) & \text{for} \quad \phi \leqq \frac{\pi}{2} \\ \frac{\pi}{2} & \text{for} \quad \phi > \frac{\pi}{2} . \end{cases} \tag{2}$$

The position of the termination of the wave is given by $\phi - \Delta$
where

$$\phi - \Delta = \begin{cases} 0 & \text{for} \quad t \leqq t_o \\ \cos^{-1}\left(1 - \frac{tc}{r} + \frac{t_o c}{r}\right) & \text{for} \quad t \geqq t_o . \end{cases} \tag{3}$$

This choice of $\gamma(t)$ specifies a wave front which moves with constant velocity until it attains the position $\theta = \frac{\pi}{2}$ where it stops.

It may be noticed that the angular envelopment of the pulse around the ring is given by Δ when $\phi \leqq \frac{\pi}{2}$ and by $\frac{\pi}{2} - (\phi - \Delta)$ when $\phi > \frac{\pi}{2}$ which gradually diminishes to zero as $\phi - \Delta$ approaches $\frac{\pi}{2}$.

For any position θ = constant on the ring the pressure wave is assumed to consist of an instantaneous rise followed by a parabolic decay back to zero, as shown by Figure 1. The load duration time for all points on the ring is the same, t_o.

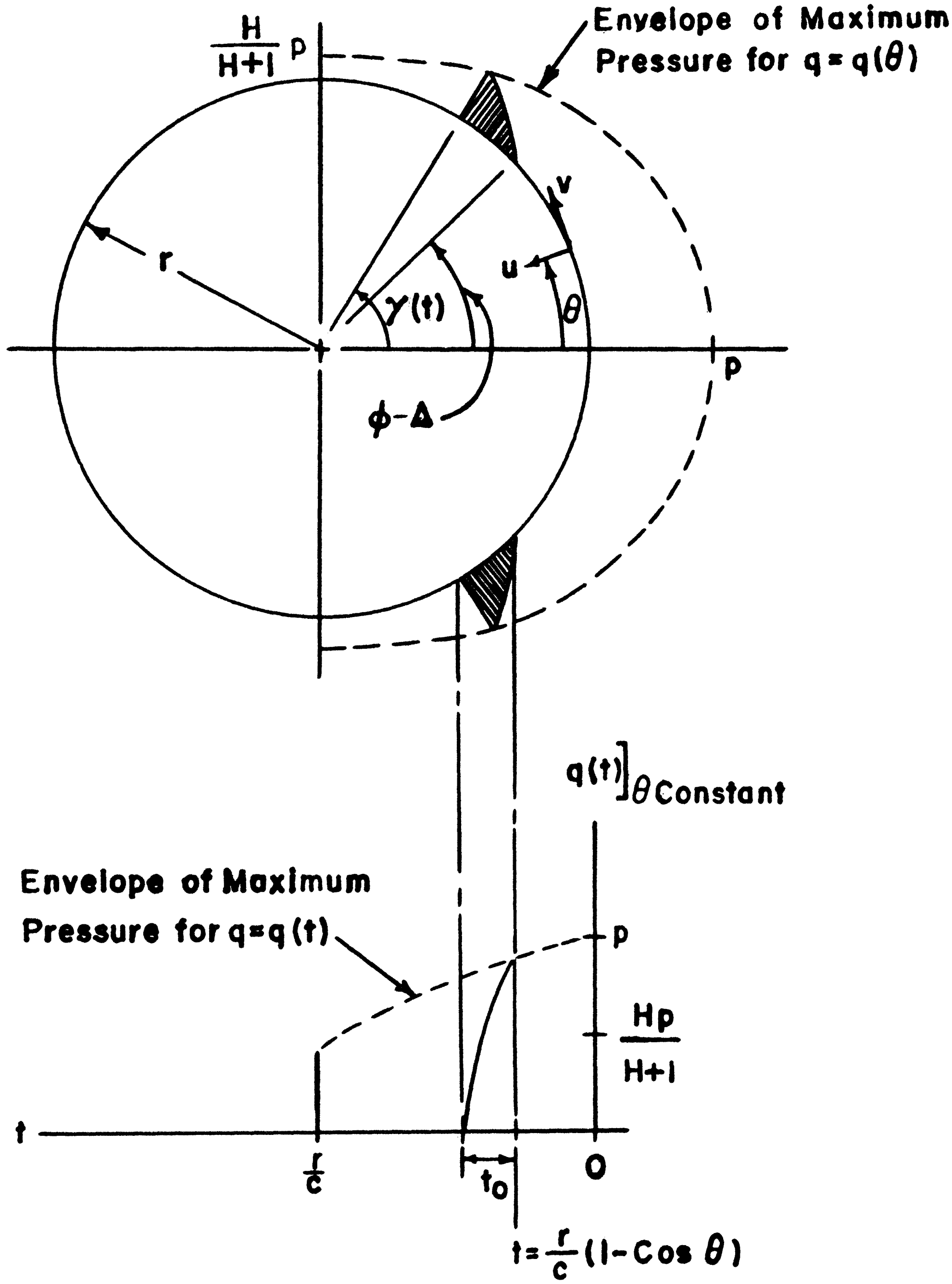

Figure 1 - Loading and Coordinates of Ring

It is further assumed that the maximum pressure in the wave at any time is governed by the following function of time

$$\lambda(\phi) = \frac{H + \cos\phi}{H + 1} \tag{4}$$

where H is a parameter that can be adjusted to vary the time rate of decay of maximum pressure. The envelope of maximum pressure is shown in Figure 1. It may be noted that the maximum pressure decays with time at a somewhat faster rate than given by (4) for times $\phi > \frac{\pi}{2}$ until $\phi - \Delta = \frac{\pi}{2}$ when the pressure has dropped back to zero everywhere.

The mathematical representation of the loading which has just been described is

$$q(\pm\theta, t) = \begin{cases} p\lambda(\phi)\left[e(\cos\phi - \cos\theta) + 1\right] & \text{for } \phi - \Delta \leqq |\theta| \leqq \gamma \\ 0 & \text{for } |\theta| > \gamma \text{ or } |\theta| < \phi - \Delta \end{cases} \tag{5}$$

where

$$e = \frac{r}{ct_0} \, .$$

B. SERIES SOLUTION

Love [3] has indicated that for a thin ring the two principal modes of vibration, extensional and flexural, can be treated separately. Also Baron and Bleich [27] have noted that when a ring's dimensions are such that

$$\frac{\text{radius of gyration of cross section}}{\text{radius of the ring}} \ll 1$$

then the flexural and extensional motions are essentially uncoupled for low modal numbers. Hence the following analysis will take advantage of the simplifications which arise by assuming the two types of motion to be uncoupled.

1. Forced Flexural Vibrations

By applying Newton's law to the arbitrary element of the ring shown in Figure 2, the equilibrium equations for rotation and for the radial and tangential directions are respectively

$$\frac{\partial G}{\partial \theta} + rN = 0 \tag{6}$$

$$qr + \frac{\partial N}{\partial \theta} + T = \rho r \frac{\partial^2 u}{\partial t^2} \tag{7}$$

$$\frac{\partial T}{\partial \theta} - N = \rho r \frac{\partial^2 v}{\partial t^2} . \tag{8}$$

The notation is defined on pages four through six.

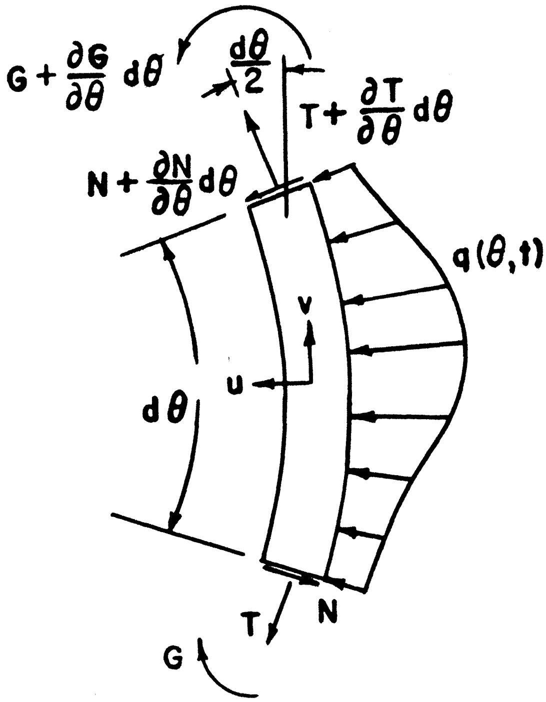

Figure 2 - Ring Element Undergoing Flexural and Extensional Vibrations

With the exception of the load term, these are the same equations which Love [3] derived using variational techniques. For the case of bending without extension of the ring's centerline, another equation can be derived by setting the circumferential strain equal to zero [3]. This gives

$$u_f = \frac{\partial v_f}{\partial \theta} \quad . \tag{9}$$

The moment-displacement relation is given by Love [3] as

$$G = \frac{EI}{r^2}\left(\frac{\partial^2 u_f}{\partial \theta^2} + u_f\right) . \tag{10}$$

Substitution of (10) into (6) will produce the shear force-displacement solution

$$N = -\frac{EI}{r^3}\left(\frac{\partial^3 u_f}{\partial \theta^3} + \frac{\partial u_f}{\partial \theta}\right). \tag{11}$$

Noting that the circumferential force T has not been assumed to be zero, this force is eliminated between Equations (7) and (8); the substitution of (11) for the shear force N in the previous results will give a second governing equation for the displacements

$$\frac{EI}{\rho r^4}\left(\frac{\partial^6 v_f}{\partial \theta^6} + 2\frac{\partial^4 v_f}{\partial \theta^4} + \frac{\partial^2 v_f}{\partial \theta^2}\right) + \frac{\partial^4 v_f}{\partial t^2 \partial \theta^2} - \frac{\partial^2 v_f}{\partial t^2} = \frac{1}{\rho}\frac{\partial q}{\partial \theta}. \tag{12}$$

The subscript f is now used to denote the displacements which are due only to flexural motion.

A solution may be determined for the two governing partial differential Equations (9) and (12) by the technique of separation of variables. This technique will produce a normal mode solution when the following series are assumed for the displacements and when the functions $\Theta_n(\theta)$ and $T_n(t)$ are evaluated to satisfy the governing equations and the load system:

$$\begin{aligned} v_f(\theta,t) &= \sum_{n=0}^{\infty} \Theta_n(\theta) \cdot T_n(t) \\ u_f(\theta,t) &= \sum_{n=0}^{\infty} \frac{\partial \Theta_n(\theta)}{\partial \theta} \cdot T_n(t) . \end{aligned} \tag{13}$$

Also the load must be expressed in the series

$$q(\theta,t) = \sum_{n=0}^{\infty} g_n(\theta) \cdot \tau_n(t) . \tag{14}$$

Since the load is known, if either g_n or τ_n can be determined, then the other becomes determinable.

Substitution of (13) and (14) into (12) yields

$$\sum_{n=0}^{\infty} \left\{ \tau_n \frac{EI}{\rho r^4} \left[\frac{d^6\Theta_n}{d\theta^6} + 2\frac{d^4\Theta_n}{d\theta^4} + \frac{d^2\Theta_n}{d\theta^2} \right] + \frac{d^2\tau_n}{dt^2} \left[\frac{d^2\Theta_n}{d\theta^2} - \Theta_n \right] - \frac{\tau_n}{\rho} \frac{dg_n}{d\theta} \right\}$$
$$= 0 .$$

This summation is certainly satisfied if it is required that the equation be satisfied for each individual value of n. The equation may then be rewritten for each n

$$\frac{\frac{EI}{\rho r^4} \left[\frac{d^6\Theta_n}{d\theta^6} + 2\frac{d^4\Theta_n}{d\theta^4} + \frac{d^2\Theta_n}{d\theta^2} \right]}{\frac{d^2\Theta_n}{d\theta^2} - \Theta_n} + \frac{\frac{d^2\tau_n}{dt^2}}{\tau_n} - \frac{\tau_n}{\rho \tau_n} \frac{\frac{dg_n}{d\theta}}{\frac{d^2\Theta_n}{d\theta^2} - \Theta_n} = 0 . \tag{15}$$

There are now two ways to effect the separation of variables in (15):

(a) select $\dfrac{\tau_n}{\tau_n} = 1$

(b) select $\dfrac{dg_n}{d\theta} = \dfrac{d^2\Theta_n}{d\theta^2} - \Theta_n .$

Although either seems to be an acceptable choice, it will be noticed that the first method will produce the time equation

$$\frac{d^2 T_n}{d t^2} + f_n T_n = 0 .$$

The only solution satisfying the conditions of a ring initially at rest in equilibrium is the trivial one $T_n = 0$. Thus the second choice must be used.

By using

$$\frac{d g_n}{d\theta} = \frac{d^2 \Theta_n}{d\theta^2} - \Theta_n \tag{16}$$

and f_n as the separation constant, the governing equation on Θ_n is obtained:

$$\frac{EI}{\rho r^4}\left[\frac{d^6\Theta_n}{d\theta^6} + 2\frac{d^4\Theta_n}{d\theta^4} + \frac{d^2\Theta_n}{d\theta^2}\right] - f_n\left[\frac{d^2\Theta_n}{d\theta^2} - \Theta_n\right] = 0 . \tag{17}$$

According to Love [3] the normal mode function which satisfies this equation is

$$\Theta_n = A_n \operatorname{Sin}(n\theta + \delta_n) .$$

However, for the case presently under consideration, which assumes symmetrical excitation with respect to θ, δ_n must equal zero in order to make v odd in θ. Since the factor A_n will later cancel in the final equations for v and u, it is chosen as 1 for convenience; thus Θ_n becomes

$$\Theta_n = \operatorname{Sin} n\theta . \tag{18}$$

Furthermore, it is seen from Love [3] that for a complete ring $v(\theta) = v(\theta + 2\pi)$; hence n must be an integer. Since $n = 0$ contributes nothing to u or v in (13), the $n = 0$ terms will be deleted from the series.

The separation constant is determined by substitution of (18) into (17) and it is the same as the expression for the square of the circular frequency of free flexural vibration of the ring [3], that is,

$$f_n = \omega_{fn}^2 = \frac{EI}{\rho r^4} \frac{n^2(n^2-1)^2}{n^2+1} . \tag{19}$$

From (15) the governing equation on T_n is

$$\frac{d^2 T_n}{dt^2} + \omega_{fn}^2 T_n = \frac{\mathcal{T}_n}{\rho} . \tag{20}$$

The homogeneous solution of (20) is of the form (21) except that it is identically zero for an initially quiescent ring:

$$T_n(t)\Big]_h = a_n' \cos \omega_{fn} t + b_n' \sin \omega_{fn} t . \tag{21}$$

A particular solution of (20) is given by Duhamel's integral:

$$T_n(t)\Big]_p = \int_0^t \frac{\mathcal{T}_n(s)}{\rho \omega_{fn}} \sin[\omega_{fn}(t-s)]\, ds . \tag{22}$$

The case for $n = 1$ must be treated separately since $\omega_{f1} = 0$ and the ω_{f1} would appear in the denominator of (22). By use of the relation

$$\lim_{\omega_{f1} \to 0} \frac{\sin[\omega_{fn}(t-s)]}{\omega_{f1}} = t-s$$

Equation (22) may be rewritten for n = 1 as

$$T_1(t) = \int_0^t \frac{\tau_1(s)}{\rho}(t-s)\,ds \,. \tag{23}$$

The functions τ_n must now be determined so that the integration in (22) and (23) can be performed; these functions are obtained as follows. First from (16) and (18) comes

$$g_n = \frac{n^2+1}{n} \cos n\theta \quad , \quad n \neq 0$$

which when substituted in (14) gives

$$q(\theta,t) = \sum_{n=1}^{\infty} \tau_n(t) \frac{n^2+1}{n} \cos n\theta \,.$$

Then using the orthogonality condition for the functions cos nθ, this may be rewritten again as

$$\tau_n(t) = \frac{2n}{\pi(1+n^2)} \int_0^{\pi} q(\theta,t) \cos n\theta \, d\theta \,.$$

Now the determination of the τ_n functions requires only the substitution of (5) into this and the integration of the results. Two special cases for the results arise for n = 1

$$\frac{\tau_1(t)}{p t_o^2} = \begin{cases} \dfrac{\lambda(\phi)}{\pi t_o^2}\left\{\left[e \operatorname{Cos}\phi - 1\right]\left[\operatorname{Sin}\gamma - \operatorname{Sin}(\phi - \Delta)\right] - \dfrac{e}{2}(\gamma - \phi + \Delta) \right. \\ \left. \qquad - \dfrac{e}{4}\left[\operatorname{Sin} 2\gamma - \operatorname{Sin}\left(2(\phi - \Delta)\right)\right]\right\} , \quad \phi - \Delta \leq \dfrac{\pi}{2} \\ 0 \, , \quad \phi - \Delta > \dfrac{\pi}{2} \end{cases} \quad (24)$$

and n > 1

$$\frac{\tau_n(t)}{p t_o^2} = \begin{cases} \dfrac{2\lambda(\phi)}{\pi(1+n^2)t_o^2}\left(e \operatorname{Cos}\phi + 1\right)\left\{\operatorname{Sin} n\gamma - \operatorname{Sin}\left[n(\phi - \Delta)\right]\right\} \\ \quad - \dfrac{n e \lambda(\phi)}{\pi(1+n^2)t_o^2}\left\{\dfrac{1}{n-1}\left[\operatorname{Sin}\left(\gamma(n-1)\right) - \operatorname{Sin}\left((n-1)(\phi - \Delta)\right)\right]\right. \\ \quad \left. - \dfrac{1}{n+1}\left[\operatorname{Sin}\left(\gamma(n+1)\right) - \operatorname{Sin}\left((n+1)(\phi - \Delta)\right)\right]\right\} , \quad \phi - \Delta \leq \dfrac{\pi}{2} \\ 0 \, , \quad \phi - \Delta > \dfrac{\pi}{2} \end{cases} \quad (25)$$

It will be noticed that the time variable t appears only implicitly in the τ_n functions.

Substitution of (24) and (25) into (22) and (23) respectively and integration of the results with respect to time will give the time functions $T_n(t)$ for the displacements u and v. However, because of the complexity of the functions involved, this integration cannot be performed in closed form.

By use of the trigonometric identity for the difference of two angles and with (13) and (18) the dimensionless displacements and velocities may be written

$$\left.\begin{aligned}
\frac{u_f(\theta,t)\cdot\rho}{pt_o^2} &= t\cos\theta\int_0^t \frac{T_1(s)}{pt_o^2}\,ds - \cos\theta\int_0^t s\,\frac{T_1(s)}{pt_o^2}\,ds \\
&\quad + \sum_{n=2}^{\infty}\frac{n\cos n\theta}{\omega_{fn}}\Big[\sin(\omega_{fn}t)\int_0^t \frac{T_n(s)}{pt_o^2}\cos(\omega_{fn}s)\,ds \\
&\quad - \cos(\omega_{fn}t)\int_0^t \frac{T_n(s)}{pt_o^2}\sin(\omega_{fn}s)\,ds\Big] \\
\frac{v_f(\theta,t)\cdot\rho}{pt_o^2} &= t\sin\theta\int_0^t \frac{T_1(s)}{pt_o^2}\,ds - \sin\theta\int_0^t s\,\frac{T_1(s)}{pt_o^2}\,ds \\
&\quad + \sum_{n=2}^{\infty}\frac{\sin n\theta}{\omega_{fn}}\Big[\sin(\omega_{fn}t)\int_0^t \frac{T_n(s)}{pt_o^2}\cos(\omega_{fn}s)\,ds \\
&\quad - \cos(\omega_{fn}t)\int_0^t \frac{T_n(s)}{pt_o^2}\sin(\omega_{fn}s)\,ds\Big]
\end{aligned}\right\} \quad (26)$$

$$\left.\begin{aligned}
\frac{\dot{u}_f(\theta,t)\cdot\rho}{pt_0} &= \cos\theta\int_0^t \frac{\tau_1(s)}{pt_0}\,ds \\
&\quad+\sum_{n=2}^{\infty} n\cos n\theta\left[\cos(\omega_{fn}t)\int_0^t \frac{\tau_n(s)}{pt_0}\cos(\omega_{fn}s)\,ds\right. \\
&\quad\left.+\sin(\omega_{fn}t)\int_0^t \frac{\tau_n(s)}{pt_0}\sin(\omega_{fn}s)\,ds\right] \\
\frac{\dot{v}_f(\theta,t)\cdot\rho}{pt_0} &= \sin\theta\int_0^t \frac{\tau_1(s)}{pt_0}\,ds \\
&\quad+\sum_{n=2}^{\infty} \sin n\theta\left[\cos(\omega_{fn}t)\int_0^t \frac{\tau_n(s)}{pt_0}\cos(\omega_{fn}s)\,ds\right. \\
&\quad\left.+\sin(\omega_{fn}t)\int_0^t \frac{\tau_n(s)}{pt_0}\sin(\omega_{fn}s)\,ds\right]
\end{aligned}\right\} \quad . \qquad (26)$$

Results from the numerical integration of (26) are given for a particular case in Section III. It should be remembered that (26) represents only the ring response to flexural motion without extension.

2. Forced Extensional Vibration

The contributions to the radial and tangential motions of the ring by the extensional modes of vibrations are determined from consideration of a set of modified governing equations. Equations (7) and (8) of the previous section on forced flexural vibration can be altered for the present case by letting the shear resultant N and the moment resultant G both be zero. Thus the equilibrium equations become

$$rq + T = \rho r \frac{\partial^2 u_e}{\partial t^2}$$
$$\frac{\partial T}{\partial \theta} = \rho r \frac{\partial^2 v_e}{\partial t^2} \; . \tag{27}$$

A third equation is found from the relation between the circumferential stress resultant T and the circumferential strain at the center of the ring's cross section [3].

$$T = \frac{EA}{r}\left(\frac{\partial v_e}{\partial \theta} - u_e\right). \tag{28}$$

Substituting (28) in (27) yields two coupled linear partial differential equations on the displacements u_e and v_e, that is,

$$rq + \frac{EA}{r}\left(\frac{\partial v_e}{\partial \theta} - u_e\right) = \rho r \frac{\partial^2 u_e}{\partial t^2}$$

and

$$\frac{EA}{r}\left(\frac{\partial^2 v_e}{\partial \theta^2} - \frac{\partial u_e}{\partial \theta}\right) = \rho r \frac{\partial^2 v_e}{\partial t^2} \; . \tag{29}$$

A solution to (29) can be obtained by use of separation of variables. Again the load and displacements are expressed in infinite series, but in this case the two displacements u_e and v_e are functions of different functions of time. Thus it will be noticed that this will not be a normal mode solution provided the two time functions do not turn out identical:

$$u_e(\theta,t) = \sum_{n=0}^{\infty} \Phi_n(\theta) \cdot B_n(t)$$

$$v_e(\theta,t) = \sum_{n=0}^{\infty} \Lambda_n(\theta) \cdot D_n(t) \qquad (30)$$

$$q(\theta,t) = \sum_{n=0}^{\infty} \Gamma_n(\theta) \cdot Z_n(t) \,.$$

Substitution of (30) into (29) produces

$$\sum_{n=0}^{\infty} \left[r\Gamma_n Z_n + \frac{EA}{r}\left(\frac{d\Lambda_n}{d\theta} D_n - \Phi_n B_n\right) - \rho r \Phi_n \frac{d^2 B_n}{dt^2} \right] = 0$$

and

$$\sum_{n=0}^{\infty} \left[\frac{EA}{r}\left(\frac{d^2\Lambda_n}{d\theta^2} D_n - \frac{d\Phi_n}{d\theta} B_n\right) - \rho r \Lambda_n \frac{d^2 D_n}{dt^2} \right] = 0 .$$

After requiring these equations to be satisfied for each value of n we may rewrite them as

$$\frac{\Gamma_n Z_n}{\rho \Phi_n D_n} + k \frac{d\Lambda_n}{d\theta} \frac{1}{\Phi_n} - k \frac{B_n}{D_n} = \frac{1}{D_n} \frac{d^2 B_n}{dt^2} \qquad (31)$$

$$k \frac{1}{\Lambda_n} \frac{d^2\Lambda_n}{d\theta^2} - k \frac{1}{\Lambda_n} \frac{d\Phi_n}{d\theta} \frac{B_n}{D_n} = \frac{1}{D_n} \frac{d^2 D_n}{dt^2} \tag{32}$$

where $k = \dfrac{EA}{\rho r^2}$ and $n = 1, 2, 3, \cdots$

There are two possible choices in (31) to make that equation separable. By previous arguments, however, only the choice of $\Gamma_n(\theta) = \Phi_n(\theta)$ will be considered. With this choice and letting $h_n(n)$ be the separation constant will allow (31) to be immediately written as

$$\frac{d^2 B_n}{dt^2} + k B_n = \frac{Z_n}{\rho} - h_n k D_n \tag{33}$$

and

$$\frac{d\Lambda_n}{d\theta} + h_n \Phi_n = 0. \tag{34}$$

Substitution of (34) into (32) will make that equation separate also, where $-\ell_n^2$ is the constant of separation:

$$\frac{d^2 D_n}{dt^2} + \ell_n^2 k D_n = -\frac{\ell_n^2 k}{h_n} B_n \tag{35}$$

and

$$\frac{d^2\Lambda_n}{d\theta^2} + \ell_n^2 \Lambda_n = 0. \tag{36}$$

The solution of (36) gives the space function

$$\Lambda_n = \operatorname{Sin}(\ell_n \theta + \delta_n').$$

Since v_e is odd in θ, this function becomes

$$\Lambda_n = \sin \ell_n \theta = \sin n\theta \tag{37}$$

and it will be noted that n must be an integer, hence $\ell_n = n$.

To determine the spatial function Φ_n, first differentiate (34) once with respect to θ and eliminate the term $d^2 \Lambda_n/d\theta^2$ between these results and (36) to get

$$n^2 \Lambda_n = f_n \frac{d\Phi_n}{d\theta} .$$

Now differentiation of this again with respect to θ and substitution of (34) into the results for the term $d\Lambda_n/d\theta$ will give

$$\frac{d^2\Phi_n}{d\theta^2} + n^2 \Phi_n = 0 .$$

Hence the solution may be written, assuming u_e even in θ, as

$$\Phi_n = \cos n\theta . \tag{38}$$

Substitution of (37) and (38) into (34) yields

$$f_n = -n .$$

Now the two simultaneous differential equations on the time functions B(t) and D(t) may be rewritten from (33) and (35) as

$$\left.\frac{d^2B_n}{dt^2} + kB_n = \frac{Z_n}{\rho} + nkD_n \right\} \tag{39}$$

and

$$\left.\frac{d^2 D_n}{dt^2} + kn^2 D_n = nk B_n \right\} . \quad (39)$$

It is now desirable to uncouple Equations (39); multiplying the first by n and adding both equations produces

$$n \frac{d^2 B_n}{dt^2} + \frac{d^2 D_n}{dt^2} = \frac{n}{\rho} Z_n$$

which can be integrated twice from time zero to time t to give

$$n B_n(t) - n B_n(0) + D_n(t) - D_n(0) + \left(n \frac{dB_n(0)}{dt} + \frac{dD_n(0)}{dt} \right) t =$$

$$\frac{n}{\rho} \int_0^t \left(\int_0^s Z_n(s_1) \, ds_1 \right) ds = \frac{n}{\rho} I_n(t) .$$

Using the conditions $B_n(0) = D_n(0) = \frac{dB_n(0)}{dt} = \frac{dD_n(0)}{dt} = 0$ for an initially quiescent ring will change this to

$$n B_n(t) + D_n(t) = \frac{n}{\rho} I_n(t) . \quad (40)$$

The uncoupling of Equations (39) is accomplished by the substitution of (40), that is,

$$\frac{d^2 B_n}{dt^2} + \omega_{en}^2 B_n = \frac{Z_n}{\rho} + \frac{n^2 k}{\rho} I_n \quad (41)$$

and

$$\frac{d^2 D_n}{dt^2} + \omega_{en}^2 D_n = \frac{nk}{\rho} I_n \tag{42}$$

where $\omega_{en} = k(n^2 + 1)$ is the extensional free vibration frequency [3].

Before (41) and (42) can be solved the Z_n functions must be determined. This is done by the same procedure that was previously used in finding the τ_n functions. By use of (5) and (30) the following are obtained

(a) for $n = 0$, $t \leqq t_o + \frac{r}{c}$

$$\frac{Z_0(t)}{p t_o^2} = \frac{\lambda(\phi)}{\pi t_o^2}\left\{\left(e \operatorname{Cos}\phi + 1\right)\left(\gamma - \phi + \Delta\right) - e\left[\operatorname{Sin}\gamma - \operatorname{Sin}(\phi - \Delta)\right]\right\}$$

(b) for $n = 1$, $t \leqq t_o + \frac{r}{c}$

$$\frac{Z_1(t)}{p t_o^2} = \frac{2\lambda(\phi)}{\pi t_o^2}\left(e \operatorname{Cos}\phi + 1\right)\left[\operatorname{Sin}\gamma - \operatorname{Sin}(\phi - \Delta)\right]$$

$$- \frac{e\lambda(\phi)}{\pi t_o^2}\left\{\gamma - \phi + \Delta + \frac{1}{2}\operatorname{Sin}(2\gamma) - \frac{1}{2}\operatorname{Sin}\left[2(\phi - \Delta)\right]\right\} \tag{43}$$

(c) for $n \geqq 2$, $t \leqq t_o + \frac{r}{c}$

$$\frac{Z_n(t)}{p t_o^2} = \frac{2\lambda(\phi)}{n\pi t_o^2}\left(e \operatorname{Cos} + 1\right)\left\{\operatorname{Sin}(n\gamma) - \operatorname{Sin}\left[n(\phi - \Delta)\right]\right\}$$

$$- \frac{e\lambda(\phi)}{\pi t_o^2}\left\{\frac{1}{n-1}\left[\operatorname{Sin}\left(\gamma(n-1)\right) - \operatorname{Sin}\left((n-1)(\phi - \Delta)\right)\right]\right.$$

$$\left. + \frac{1}{n+1}\left[\operatorname{Sin}\left(\gamma(n+1)\right) - \operatorname{Sin}\left((n+1)(\phi - \Delta)\right)\right]\right.$$

(d) for all n, $t \geqq t_o + \frac{r}{c}$

$$Z_n(t) = 0 \qquad (43)$$

By use of Duhamel's integral the particular solutions for $B_n(t)$ come from (41),

$$B_n(t) = \int_0^t \frac{Z_n(s) + n^2 k\, I_n(s)}{\rho\, \omega_{en}} \operatorname{Sin}\left[\omega_{en}(t-s)\right] ds$$

and from (40) come $D_n(t)$ once the $B_n(t)$ are known,

$$D_n(t) = n\left[\frac{I_n(t)}{\rho} \quad B_n(t)\right].$$

Again the homogeneous solution is identically zero.

Now that the coefficients $B_n(t)$ and $D_n(t)$ have been determined, the dimensionless displacements and velocities may be written as

$$\frac{u_e(\theta,t)\cdot\rho}{p\,t_o^2} = \sum_{n=0}^{\infty} \frac{\operatorname{Cos} n\theta}{\omega_{en}}\Big[\operatorname{Sin}(\omega_{en}t)\int_0^t\left(\frac{Z_n(s)}{p\,t_o^2} + n^2 k\,\frac{I_n(s)}{p\,t_o^2}\right)\operatorname{Cos}(\omega_{en}s)\,ds$$

$$- \operatorname{Cos}(\omega_{en}t)\int_0^t\left(\frac{Z_n(s)}{p\,t_o^2} + n^2 k\,\frac{I_n(s)}{p\,t_o^2}\right)\operatorname{Sin}(\omega_{en}s)\,ds\Big] \qquad (44)$$

$$\frac{v_e(\theta,t)\cdot\rho}{p\,t_o^2} = \sum_{n=0}^{\infty} n \operatorname{Sin} n\theta \Big[\operatorname{Cos}(\omega_{en}t)\int_0^t\left(\frac{Z_n(s)}{p\,t_o^2} + n^2 k\,\frac{I_n(s)}{p\,t_o^2}\right)\operatorname{Sin}(\omega_{en}s)\,ds$$

$$- \operatorname{Sin}(\omega_{en}t)\int_0^t\left(\frac{Z_n(s)}{p\,t_o^2} + n^2 k\,\frac{I_n(s)}{p\,t_o^2}\right)\operatorname{Cos}(\omega_{en}s)\,ds + \frac{I_n(t)}{p\,t_o^2}\Big]$$

$$\left.\begin{aligned}\frac{\dot{u}_e(\theta,t)\cdot\rho}{p\,t_o} &= \sum_{n=0}^{\infty}\cos n\theta\left[\sin(\omega_{en}t)\int_0^t\left(\frac{Z_n(s)}{p t_o}+n^2k\,\frac{I_n(s)}{p t_o}\right)\sin(\omega_{en}s)\,ds\right.\\ &\qquad\left.+\cos(\omega_{en}t)\int_0^t\left(\frac{Z_n(s)}{p t_o}+n^2k\,\frac{I_n(s)}{p t_o}\right)\cos(\omega_{en}s)\,ds\right]\\ \frac{\dot{v}_e(\theta,t)\cdot\rho}{p\,t_o} &= \sum_{n=0}^{\infty}-n\sin n\theta\left[\sin(\omega_{en}t)\int_0^t\left(\frac{Z_n(s)}{p t_o}+n^2k\,\frac{I_n(s)}{p t_o}\right)\sin(\omega_{en}s)\,ds\right.\\ &\qquad\left.+\cos(\omega_{en}t)\int_0^t\left(\frac{Z_n(s)}{p t_o}+n^2k\,\frac{I_n(s)}{p t_o}\right)\cos(\omega_{en}s)\,ds-\int_0^t\frac{Z_n(s)}{p t_o}\,ds\right]\end{aligned}\right\}. \quad (44)$$

The overall response is then given by

$$\left.\begin{aligned}u(\theta,t) &= u_e + u_f\\ v(\theta,t) &= v_e + v_f\\ \dot{u}(\theta,t) &= \dot{u}_e + \dot{u}_f\\ \dot{v}(\theta,t) &= \dot{v}_e + \dot{v}_f\end{aligned}\right\} \quad (45)$$

C. DOUBLE FOURIER SERIES SOLUTION

Another way of solving the problem stated in Part A of this section is by the use of the Fourier integral. In this method the displacements and the load are expressed in a Fourier integral of the time variable and a solution is pursued in a manner very similar to the previous solution where the displacements and load were expressed in an infinite series of trigonometric functions in θ. However, since only an early time response is desired, a simplification in the Fourier integral approach is possible. By restricting the solution to the interval $0 \leqq t \leqq L$ where $L = \frac{2r}{c}$ a series can be used in place of the integral thus eliminating the need of evaluating the Fourier integral with its infinite limits.

The solution thus derived is for an infinite succession of pressure waves, each starting at $\theta = 0$ at the times 0, L, 2L, $\cdots$ respectively and proceeding across the ring. In the time interval 0 to L, the solution represents the ring response to the first of these waves alone. It is to be remembered that the solution which is developed below is not the solution of the stated problem for $t > L$.

Expressing the time function in series form while the space function is also in such form then produces a double Fourier series.

Again the extensional and flexural parts of the response are treated separately for convenience.

1. Flexural Vibrations

The governing equations for flexural vibrations developed in the previous section will again be utilized. The displacements and load are taken to be

$$u_f(\theta,t) = u_{fh} + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} n A_{mn} \cos n\theta \cos\frac{m\pi t}{L}$$

$$v_f(\theta,t) = v_{fh} + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin n\theta \cos\frac{m\pi t}{L} \qquad (46)$$

and

$$q(\theta,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} B_{mn} \cos n\theta \cos\frac{m\pi t}{L} \qquad (47)$$

where u_{fh} and v_{fh} are the homogeneous solutions of the governing Equations (9) and (12); also $L = \frac{2r}{c}$. The substitution of (46) and (47) into these governing equations yields

$$\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \sin n\theta \cos\frac{m\pi t}{L}\left\{\left[\rho\left(\frac{m\pi}{L}\right)^2(n^2+1) - \frac{EI}{r^4}n^2(n^2-1)^2\right]A_{mn}\right.$$

$$\left. + B_{mn}\, n\right\} = 0 .$$

Requiring that this be satisfied for each n and rearranging the results will produce

$$A_{mn} = \frac{\frac{n}{1+n^2}\,\frac{B_{mn}}{\rho}}{\omega_{fn}^2 - \left(\frac{m\pi}{L}\right)^2} . \tag{48}$$

Obviously when $\omega_{fn} = \frac{m\pi}{L}$ the double Fourier series method is not applicable. This represents the resonance condition of one of the ring harmonic frequencies equaling one of the frequencies of the loading infinite series. In working any specific problem it is necessary to be sure that two such frequencies do not coincide.

The homogeneous solution to (12) is given by Love [3],

$$v_{fh} = \sum_{n=1}^{\infty} \frac{1}{n}\left[a_n \cos(\omega_{fn}t) + b_n \sin(\omega_{fn}t)\right] \sin n\theta .$$

Substitution of this v_{fh} into (46) and using the initial conditions $v(\theta, 0) = \dot{v}(\theta, 0) = 0$ yields

$$a_n = -\sum_{m=0}^{\infty} n A_{mn} \quad ; \qquad b_n = 0 \qquad ; \; n = 1, 2, \ldots$$

Thus from (46) the dimensionless flexural displacements and velocities may be

$$\left.\begin{aligned}
\frac{u_f(\theta,t)\rho}{p\, t_o^2} &= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \frac{\frac{n^2}{1+n^2}\,\frac{B_{mn}}{p t_o^2}}{\omega_{fn}^2 - \left(\frac{m\pi}{L}\right)^2}\left[\cos\frac{m\pi t}{L} - \cos(\omega_{fn} t)\right]\cos n\theta \\
\frac{v_f(\theta,t)\rho}{p\, t_o^2} &= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \frac{\frac{n}{1+n^2}\,\frac{B_{mn}}{p t_o^2}}{\omega_{fn}^2 - \left(\frac{m\pi}{L}\right)^2}\left[\cos\frac{m\pi t}{L} - \cos(\omega_{fn} t)\right]\sin n\theta \\
\frac{\dot{u}_f(\theta,t)\rho}{p\, t_o} &= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \frac{\frac{n^2}{1+n^2}\,\frac{B_{mn}}{p t_o}}{\omega_{fn}^2 - \left(\frac{m\pi}{L}\right)^2}\left[\omega_{fn}\sin(\omega_{fn} t) \right. \\
&\qquad \left. - \frac{m\pi}{L}\sin\frac{m\pi t}{L}\right]\cos n\theta \\
\frac{\dot{v}_f(\theta,t)\cdot\rho}{p\, t_o} &= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \frac{\frac{n}{1+n^2}\,\frac{B_{mn}}{p t_o}}{\omega_{fn}^2 - \left(\frac{m\pi}{L}\right)^2}\left[\omega_{fn}\sin(\omega_{fn} t) \right. \\
&\qquad \left. - \frac{m\pi}{L}\sin\frac{m\pi t}{L}\right]\sin n\theta
\end{aligned}\right\} \cdot (49)$$

The solution will be complete when the load coefficients B_{mn} are determined and substituted into (49). The determination of the B_{mn} is accomplished by evaluating the Fourier coefficient in (47),

$$B_{mn} = \frac{4}{L\pi}\int_0^t \left[\int_0^\pi q(\theta, t)\ \text{Cos}\ n\theta\, d\theta\right] \text{Cos}\ \frac{m\pi t}{L}\, dt .$$

Substitution of (5) into the above for $q(\theta, t)$ will give

$$B_{mn} = \frac{4p}{L\pi}\int_0^{t_1} \lambda(\phi)\Big[(e\ \text{Cos}\ \phi + 1)\int_{\phi-\Delta}^{\gamma} \text{Cos}\ n\theta\, d\theta$$

$$- e\int_{\phi-\Delta}^{\gamma} \text{Cos}\ \theta\ \text{Cos}\ n\theta\, d\theta\Big]\ \text{Cos}\ \frac{m\pi t}{L}\ dt$$

where $t_1 = t_o + \frac{r}{c}$.

Performing the integration with respect to θ yields for $n = 1$

$$\left.\begin{aligned} \frac{B_{m1}}{p t_o^2} = \frac{4}{L\pi t_o^2}\int_0^{t_1} \lambda(\phi)\Big\{&(e\ \text{Cos}\ \phi + 1)[\text{Sin}\ \gamma - \text{Sin}\ (\phi - \Delta)] \\ &- \frac{e}{2}(\gamma - \phi + \Delta) \\ &- \frac{e}{4}\left[\text{Sin}\ 2\gamma - \text{Sin}\big(2(\phi - \Delta)\big)\right]\Big\}\ \text{Cos}\ \frac{m\pi t}{L} dt \end{aligned}\right\} \quad (50)$$

and for $n \geqq 2$

$$\left.\begin{aligned}\frac{B_{mn}}{pt_o^2} = \frac{4}{L\pi t_o^2}\int_0^{t_1} \lambda(\phi)\Big\{(e\,\mathrm{Cos}\,\phi+1)\frac{1}{n}\big[\mathrm{Sin}\;n\gamma - \mathrm{Sin}\,(n(\phi-\Delta))\big]\\ -\frac{e}{2(n-1)}\big[\mathrm{Sin}\,(\gamma(n-1)) - \mathrm{Sin}\,((\phi-\Delta)(n-1))\big]\\ -\frac{e}{2(n+1)}\big[\mathrm{Sin}\,(\gamma(n+1)) - \mathrm{Sin}\,((\phi-\Delta)(n+1))\big]\Big\}\,\mathrm{Cos}\,\frac{m\pi t}{L}\,dt\end{aligned}\right\} . \quad (50)$$

Due to the complexity of the integrals with respect to time involved in (50), the B_{mn} cannot be readily obtained in closed form; for each different ring and shock wave combination these B_{mn} constants must be found by numerical integration.

2. Extensional Vibrations

The extensional part of the double Fourier series solution is developed from consideration of the governing Equations (29) from the previous section. The displacements are taken to be

$$\left.\begin{aligned}u_e(\theta,t) &= u_{eR} + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} C_{mn}\,\mathrm{Cos}\,n\theta\,\mathrm{Cos}\,\frac{m\pi t}{L}\\ v_e(\theta,t) &= v_{eR} + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} D_{mn}\,\mathrm{Sin}\,n\theta\,\mathrm{Cos}\,\frac{m\pi t}{L}\end{aligned}\right\} . \quad (51)$$

Substitution of (51) into the second of Equations (29) produces

$$D_{mn} = \frac{kn\,C_{mn}}{n^2k - \left(\frac{m\pi}{L}\right)^2} . \quad (52)$$

By use of (47), (51), (52) and the first of Equations (29), a second relation between the constants is obtained

$$n B_{mn} = -\rho \left(n C_{mn} + D_{mn}\right)\left(\frac{m\pi}{L}\right)^2 .$$

Using this equation with (52) we obtain

$$\left.\begin{aligned} C_{mn} &= \frac{B_{mn}\left[1 - k\left(\frac{nL}{m\pi}\right)^2\right]}{\rho\left[\omega_{en}^2 - \left(\frac{m\pi}{L}\right)^2\right]} \\ D_{mn} &= \frac{-nkB_{mn}}{\rho\left[\omega_{en}^2 - \left(\frac{m\pi}{L}\right)^2\right]\left(\frac{m\pi}{L}\right)^2} \end{aligned}\right\} . \quad (53)$$

Again the case of $\omega_{en} = \frac{m\pi}{L}$ is an exceptional one that cannot be handled by the double Fourier series method. The homogeneous, extensional solution is [3]

$$u_{eh} = \sum_{n=0}^{\infty} \left[c_n \operatorname{Sin}(\omega_{en} t) + d_n \operatorname{Cos}(\omega_{en} t)\right] \operatorname{Cos} n\theta$$

$$v_{eh} = -\sum_{n=0}^{\infty} n\left[c_n \operatorname{Sin}(\omega_{en} t) + d_n \operatorname{Cos}(\omega_{en} t)\right] \operatorname{Sin} n\theta .$$

Substitution of these and (53) into (51) and use of the initial conditions $u_e(\theta, 0) = \dot{u}_e(\theta, 0) = 0$ gives

$$c_n = 0 \quad ; \quad d_n = -\sum_{m=0}^{\infty} C_{mn} \quad ; \quad n = 1, 2, \cdots$$

Thus the extensional response may be written in dimensionless form as (54), and (55) gives the total response:

$$
\left.
\begin{aligned}
\frac{u_e(\theta,t)\cdot\rho}{p t_o^2} &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\frac{B_{mn}}{p t_o^2}\left[1-k\left(\frac{nL}{m\pi}\right)^2\right]}{\omega_{en}^2-\left(\frac{m\pi}{L}\right)^2}\left[\mathrm{Cos}\,\frac{m\pi t}{L}\right. \\
&\quad \left. -\mathrm{Cos}\,(\omega_{en}t)\right]\mathrm{Cos}\;n\theta \\
\frac{v_e(\theta,t)\cdot\rho}{p t_o^2} &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{n\,\frac{B_{mn}}{p t_o^2}}{\omega_{en}^2-\left(\frac{m\pi}{L}\right)^2}\left\{\left[1-k\left(\frac{nL}{m\pi}\right)^2\right]\mathrm{Cos}(\omega_{en}t)\right. \\
&\quad \left. -k\left(\frac{L}{m\pi}\right)^2\mathrm{Cos}\,\frac{m\pi t}{L}\right\}\mathrm{Sin}\;n\theta \\
\frac{\dot{u}_e(\theta,t)\cdot\rho}{p t_o} &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\frac{B_{mn}}{p t_o}\left[1-k\left(\frac{nL}{m\pi}\right)^2\right]}{\omega_{en}^2-\left(\frac{m\pi}{L}\right)^2}\left[\omega_{en}\,\mathrm{Sin}\,(\omega_{en}t)\right. \\
&\quad \left. -\frac{m\pi}{L}\,\mathrm{Sin}\,\frac{m\pi t}{L}\right]\mathrm{Cos}\;n\theta \\
\frac{\dot{v}_e(\theta,t)\cdot\rho}{p t_o} &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{n\,\frac{B_{mn}}{p t_o}}{\omega_{en}^2-\left(\frac{m\pi}{L}\right)^2}\left\{\frac{kL}{m\pi}\,\mathrm{Sin}\,\frac{m\pi t}{L}\right. \\
&\quad \left. -\left[1-k\left(\frac{nL}{m\pi}\right)^2\right]\omega_{en}\,\mathrm{Sin}(\omega_{en}t)\right\}\mathrm{Sin}\;n\theta
\end{aligned}
\right\} \quad (54)
$$

The overall response is then given by (55)

$$\left.\begin{aligned} u(\theta,t) &= u_e + u_f \\ v(\theta,t) &= v_e + v_f \\ \dot{u}(\theta,t) &= \dot{u}_e + \dot{u}_f \\ \dot{v}(\theta,t) &= \dot{v}_e + \dot{v}_f \end{aligned}\right\} . \quad (55)$$

D. APPROXIMATE SOLUTION

Since the exact solution of the problem of a distributed, time and position dependent, moving load on a ring cannot be obtained in closed form, this part of Section II presents the exact solution to an analogous problem whose solution is an approximate answer to the original problem. The problem to be considered is that of a concentrated radial load on the ring which moves around the ring with constant angular velocity and which diminishes as it moves.

The solution is obtained by means of the double Fourier series method which was presented in Part C of Section II. Since the major difficulty of this method was that the load coefficients could not be obtained in closed form for the loading of Equation (5), this method can be used with a simpler loading.

The loading to be considered is given in Figure 3; the load is distributed over the angular length ϵ, assumed small, and transverses the ring with the constant angular velocity $\dot{\Omega}$ while decaying at the linear rate in time of $1 - \frac{tc}{r}$. Using the conditions that

$$\Omega = \frac{\pi}{2} \quad \text{at} \quad t = \frac{r}{c}$$

$$\Omega = 0 \quad \text{at} \quad t = 0$$

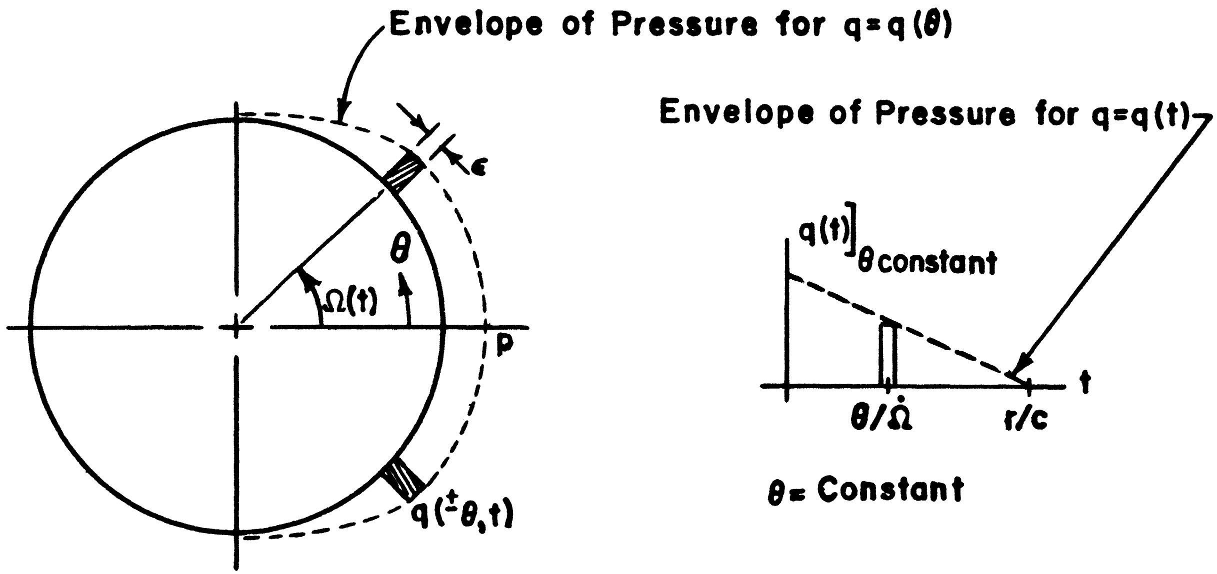

Figure 3 - Geometry of Approximate Loading

gives

$$\Omega = \frac{c\pi t}{2r} = \frac{\pi t}{L}$$

The load may then be expressed as

$$q(\theta,t) = \begin{cases} p\left(1 - \frac{tc}{r}\right) & \text{for } \Omega + \frac{\epsilon}{2} \geqq |\theta| \geqq \Omega - \frac{\epsilon}{2} \\ 0 & \text{for } |\theta| > \Omega + \frac{\epsilon}{2} \\ & \text{or } |\theta| < \Omega - \frac{\epsilon}{2} \\ & \text{or } |\theta| > \frac{\pi}{2} \end{cases} \quad . \tag{56}$$

Substitution of (56) into (47) will lead to a new set of B_{mn} coefficients:

$$B_{mn} = \frac{4p}{L\pi}\int_0^{\frac{L}{2}}\left(1 - \frac{tc}{r}\right)\int_{\Omega - \frac{\epsilon}{2}}^{\Omega + \frac{\epsilon}{2}} \cos n\theta \quad \cos\frac{m\pi t}{L}\, d\theta\, dt .$$

Performing the integration with respect to θ and using the condition that ϵ is small, that is, less than 5 degrees, we obtain

$$B_{mn} = \frac{4p\epsilon}{L\pi}\int_0^{\frac{L}{2}}\left(1 - \frac{tc}{r}\right)\cos\frac{n\pi t}{L} \quad \cos\frac{m\pi t}{L} \quad dt .$$

The B_{mn} may be determined in closed form by performing the indicated integration with respect to time. Also the limit is taken as $p \to \infty$ and $\epsilon \to 0$ so that the concentrated force P is defined by

$$P = \lim_{\substack{\epsilon \to 0 \\ p \to \infty}} p\epsilon r.$$

Thus

$$B_{on} = \begin{cases} \dfrac{8P}{rn^2\pi^3} & n \text{ odd} \\ \dfrac{16P}{rn^2\pi^3} & n \text{ even, } n = 2, 6, 10, \cdots \\ 0 & n \text{ even, } n = 4, 8, 12, \cdots \end{cases}$$

$$B_{oo} = \frac{P}{r\pi}$$

$$B_{mn} = \begin{cases} \dfrac{16\,P\,mn(-1)^{\frac{m+n}{2}}}{\pi^3(n-m)^2(n+m)^2\,r} & m, n \text{ odd and } m \neq n \\ \dfrac{-8P(m^2+n^2)(-1)^{\frac{m+n}{2}}}{\pi^3(n-m)^2(n+m)^2\,r} & m, n \text{ even and } m \neq n \\ \dfrac{P}{r}\left[\dfrac{1}{2\pi} + \dfrac{1-(-1)^n}{n^2\pi^3}\right] & m = n \\ 0 & m \text{ odd and } n \text{ even or } n \text{ odd and } m \text{ even} \end{cases} \qquad (57)$$

Substitution of (57) into (49) and (54) will produce the flexural and extensional responses respectively of the ring to this simplified loading scheme.

III. EXAMPLE PROBLEM

As an illustration of the application of the methods presented in Section II, a numerical example is presented in this section. Part A includes the pertinent data for the particular ring and shock pulse, and Part B gives the results of the various methods.

A. PROBLEM STATEMENT

The ring to be considered has a rectangular cross section and the following properties:

outside radius of ring = 9.0 (ft)

radius to neutral fiber = 8.65 (ft)

cross section area = 68.5 (sq in.)

Young's modulus = 30×10^6 (psi)

moment of inertia about neutral axis normal to the plane of the ring = 1435 (in^4)

mass density of ring = 15.2 (lb sec^2/ft^4).

The shock-pulse parameters used are:

duration time* = t_o = 0.50 (msec)

decay factor = H = 1.00

pulse velocity = c = 5.0 (ft/msec).

This particular choice was made so that the results could be compared to existing experimental data. Some comments on this comparison appear in Section IV.

The solution of this problem involved numerical integrations of several functions of time, which were carried out with the help of a digital computer. The integrations were performed with the trape-

*Except for one case where t_o = 0.25 was used.

zoidal rule technique; care was taken to ensure that enough subdivisions were used in the integration to give reasonably accurate results by plotting the integrand of several cases. The integrands were oscillatory curves, and about 1/2 percent accuracy, or possibly three significant place accuracy, was obtained by using at least 30 time subdivisions per period.

B. NUMERICAL RESULTS

The results of the integration of Equations (26) and (44) for the series solution using the previously stated values for the ring and pressure wave parameters are presented in Figures 4 through 12 for the location $\theta = 0$. The effect of using different numbers of modes is shown in Figures 4, 7, 8, 9 and 10.

For reasons to be discussed later no results are given from the second method of solution which employs the double Fourier series.

The results for the double series approximate problem are given in Figures 13 and 14 for the location $\theta = 0$, and again the effect of adding different numbers of terms of the series in n are shown. In all these double series results a sufficient number of terms in m were used with n held constant to ensure three place accuracy.

No results are presented from either method for the tangential response at $\theta = 0$ since this is always identically zero.

Table 1 gives a list of the natural frequencies of the ring under consideration.

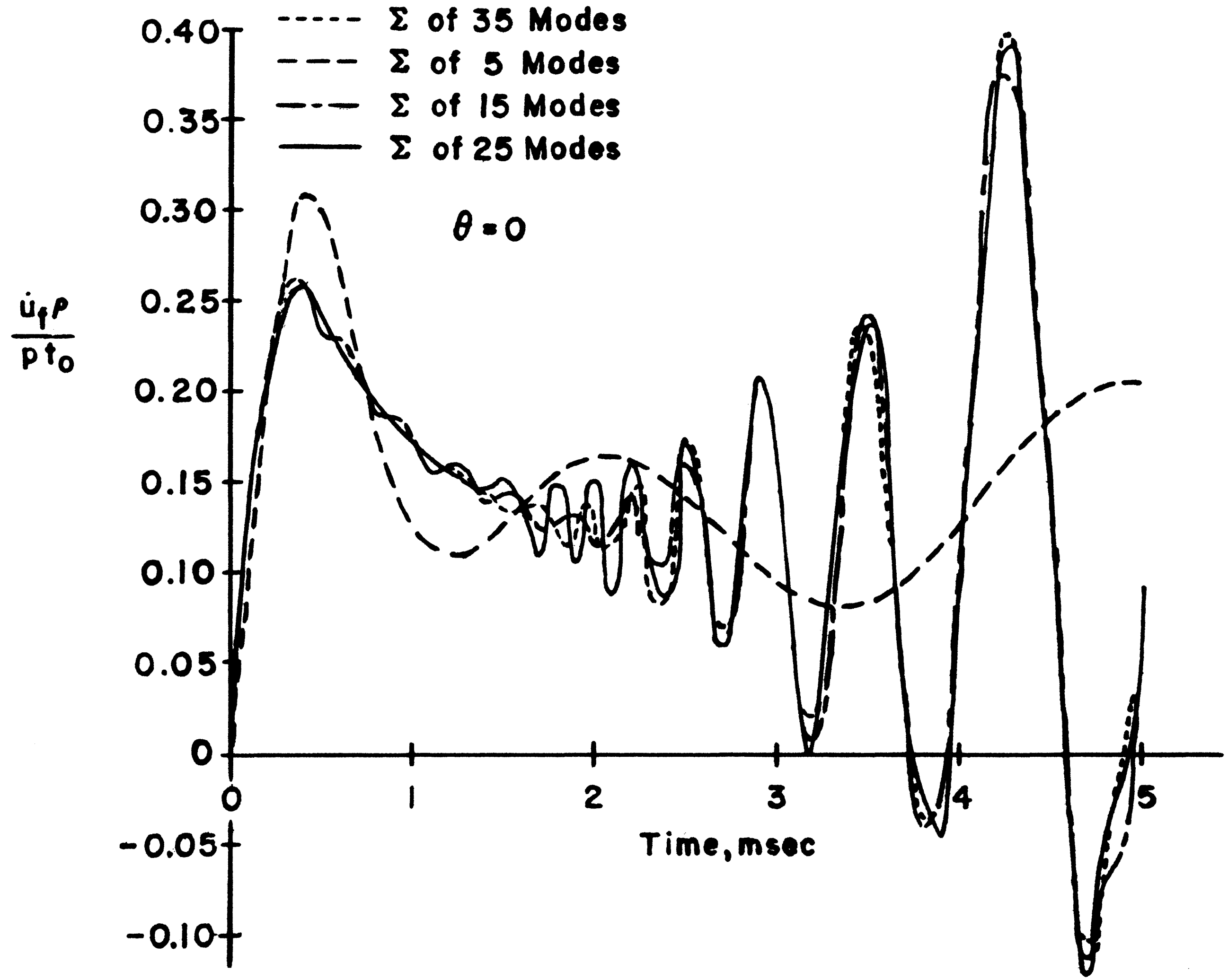

Figure 4 - Flexural Series Solution for Radial Velocity

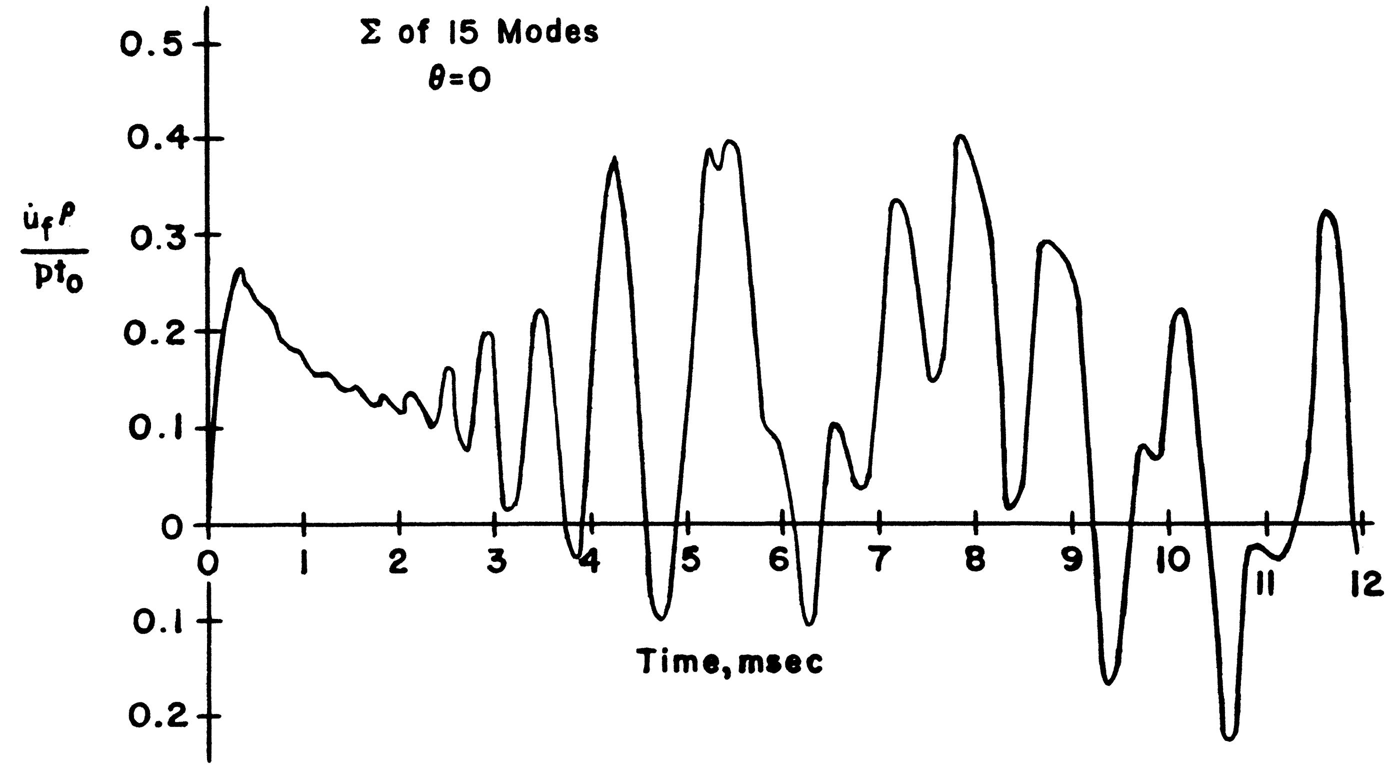

Figure 5 - Flexural Series Solution for Radial Velocity for Extended Time

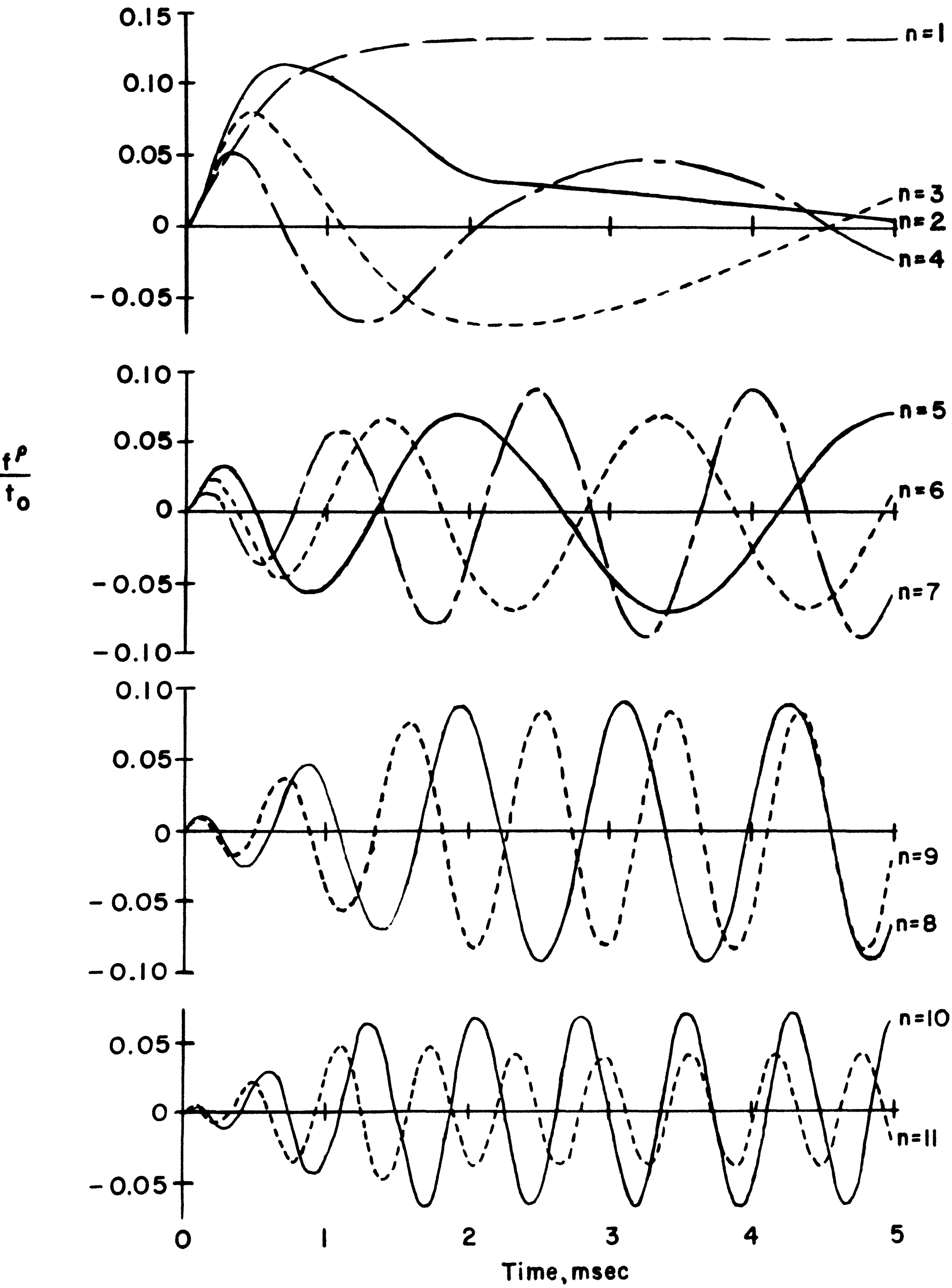

Figure 6 - Individual Flexural Modes for Radial Velocity from Series Solution, θ = 0

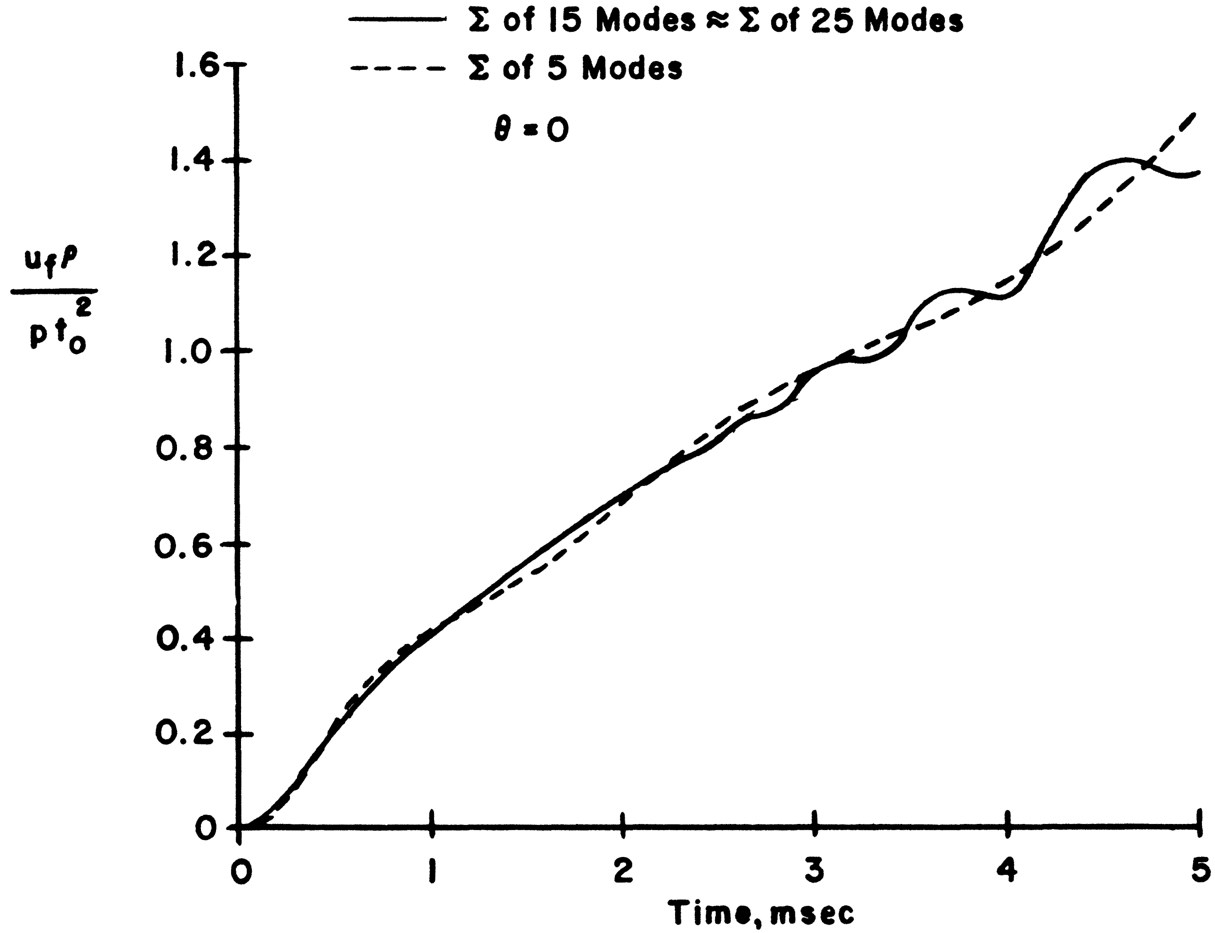

Figure 7 - Flexural Series Solution for Radial Displacement

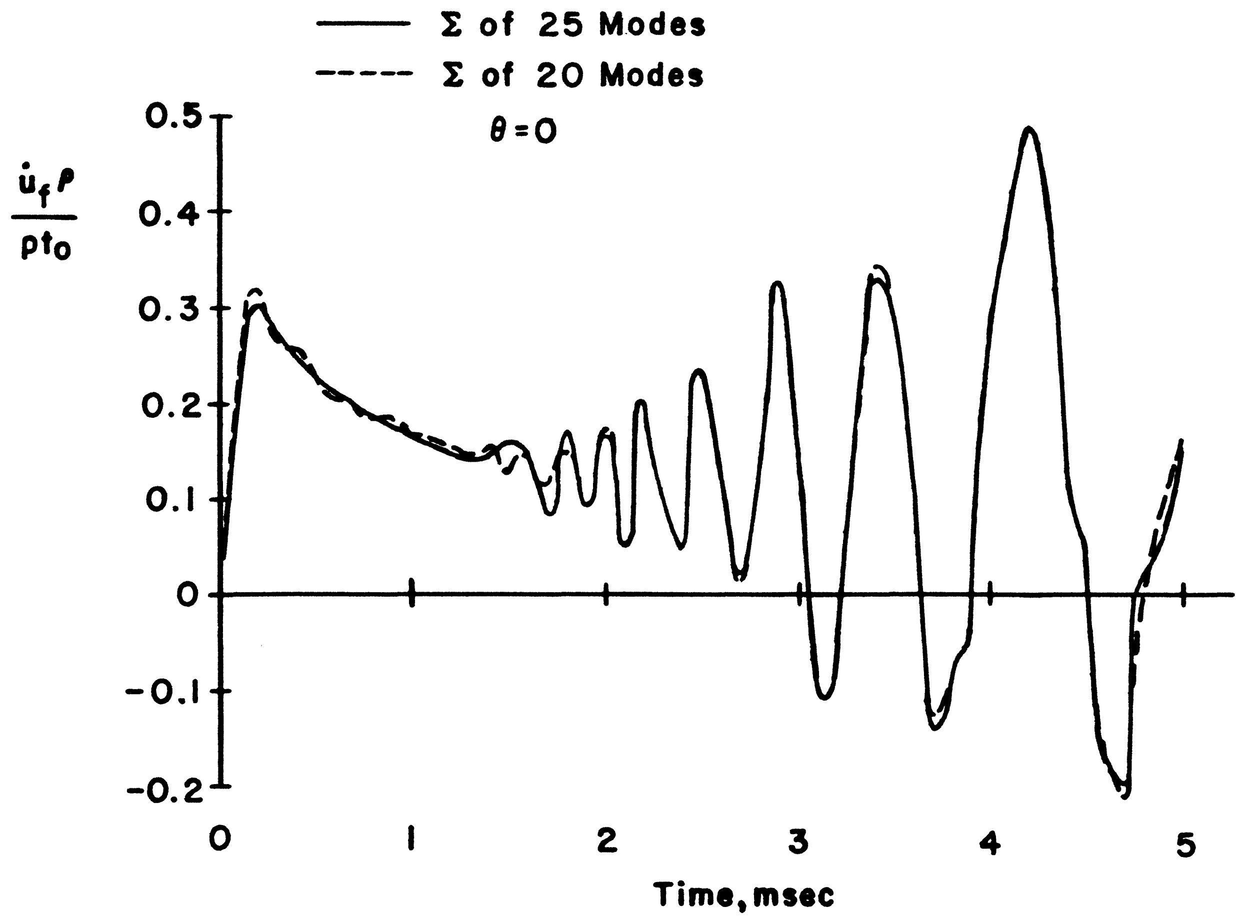

Figure 8 - Flexural Series Solution for Radial Velocity for t_o = 0.25 msec

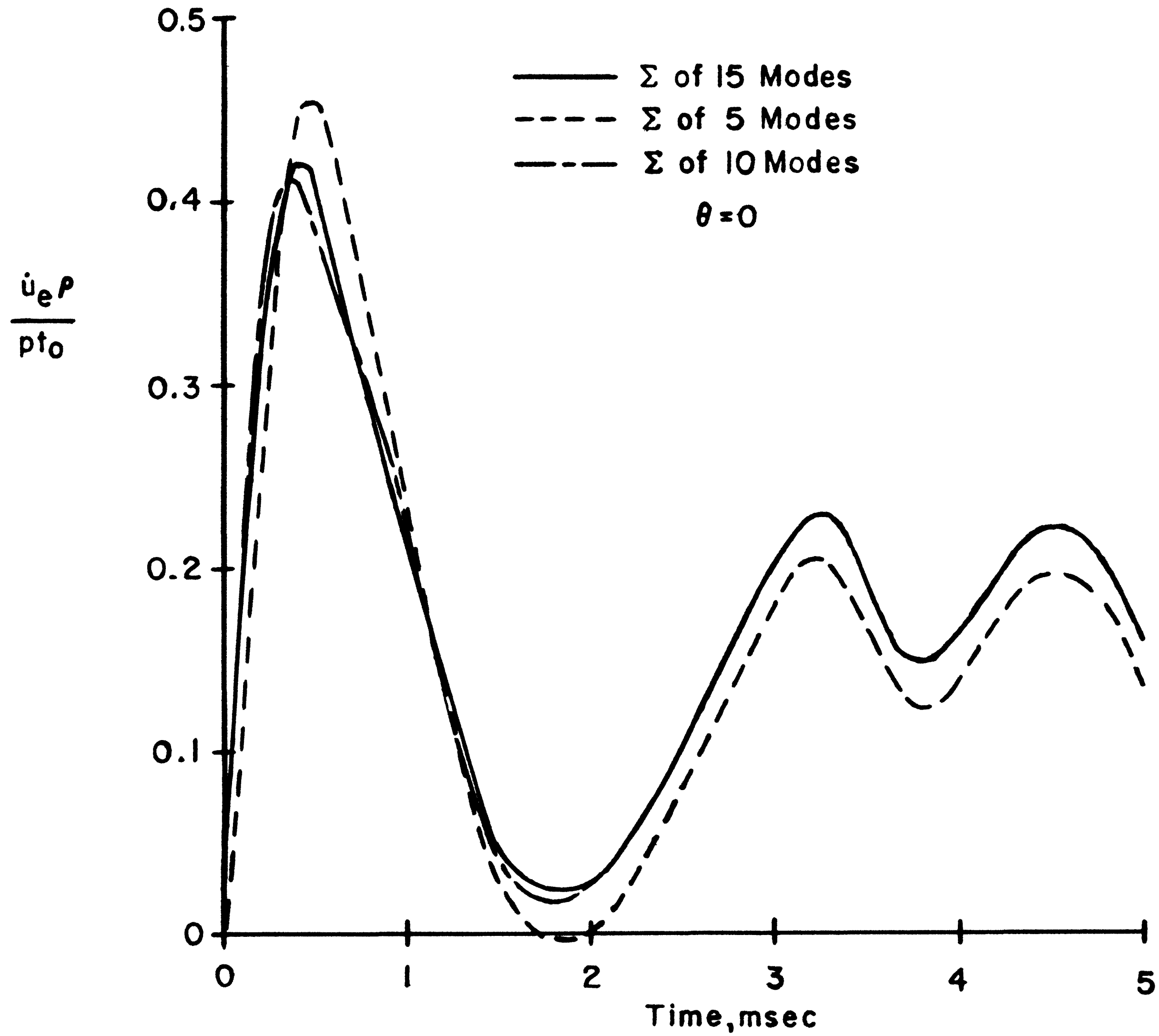

Figure 9 - Extensional Series Solution for Radial Velocity

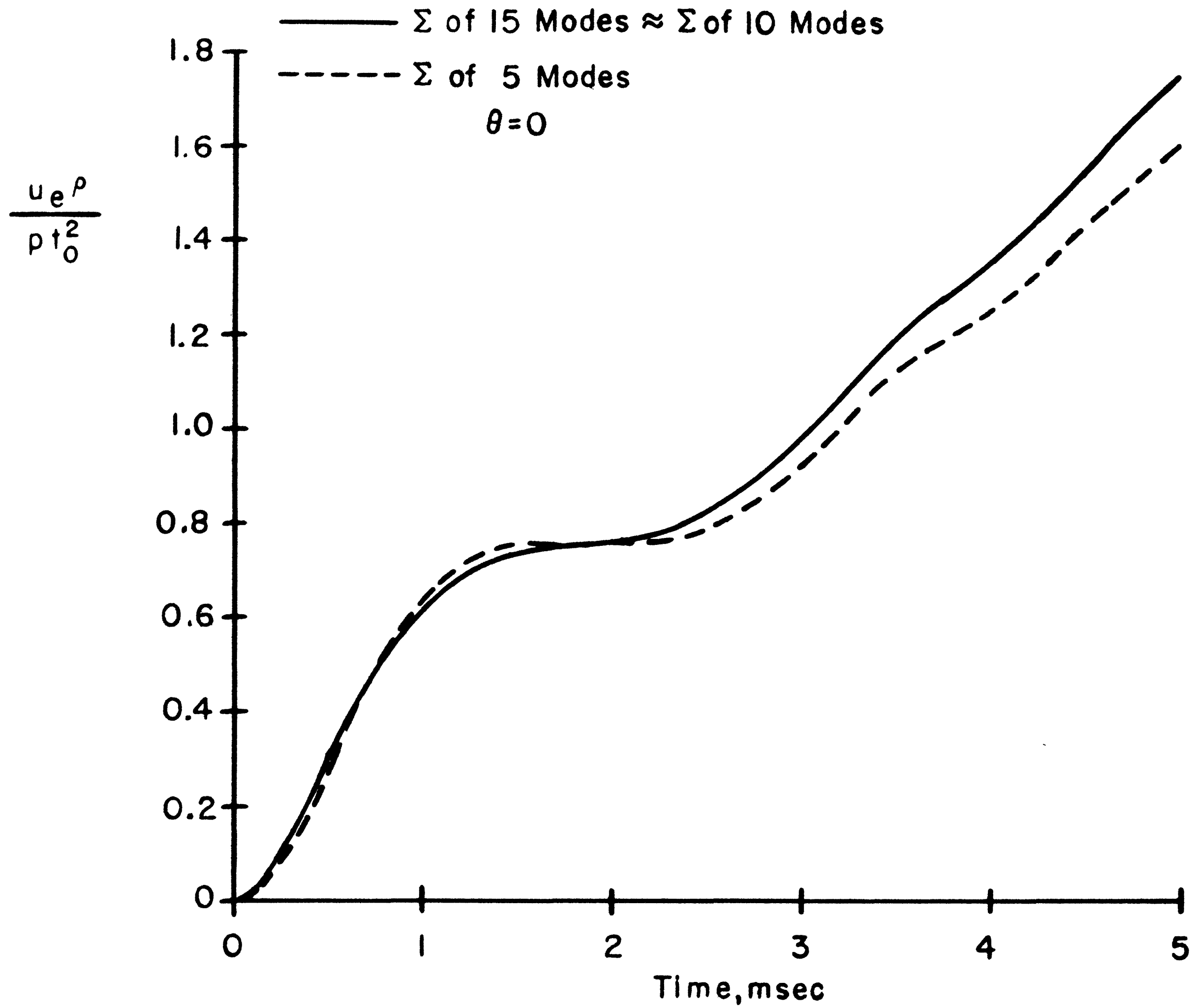

Figure 10 - Extensional Series Solution for Radial Displacement

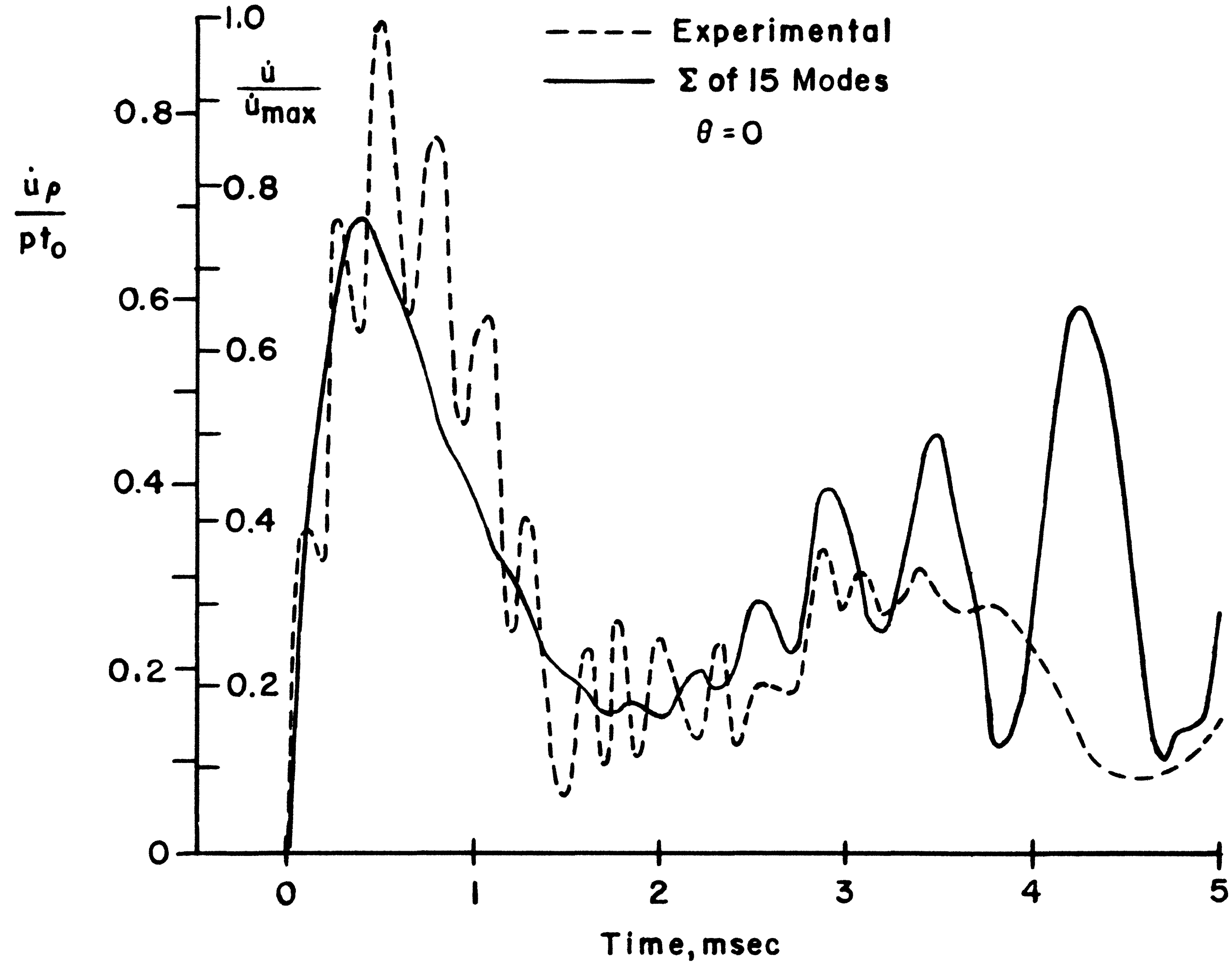

Figure 11 - Complete Series Solution for Radial Velocity

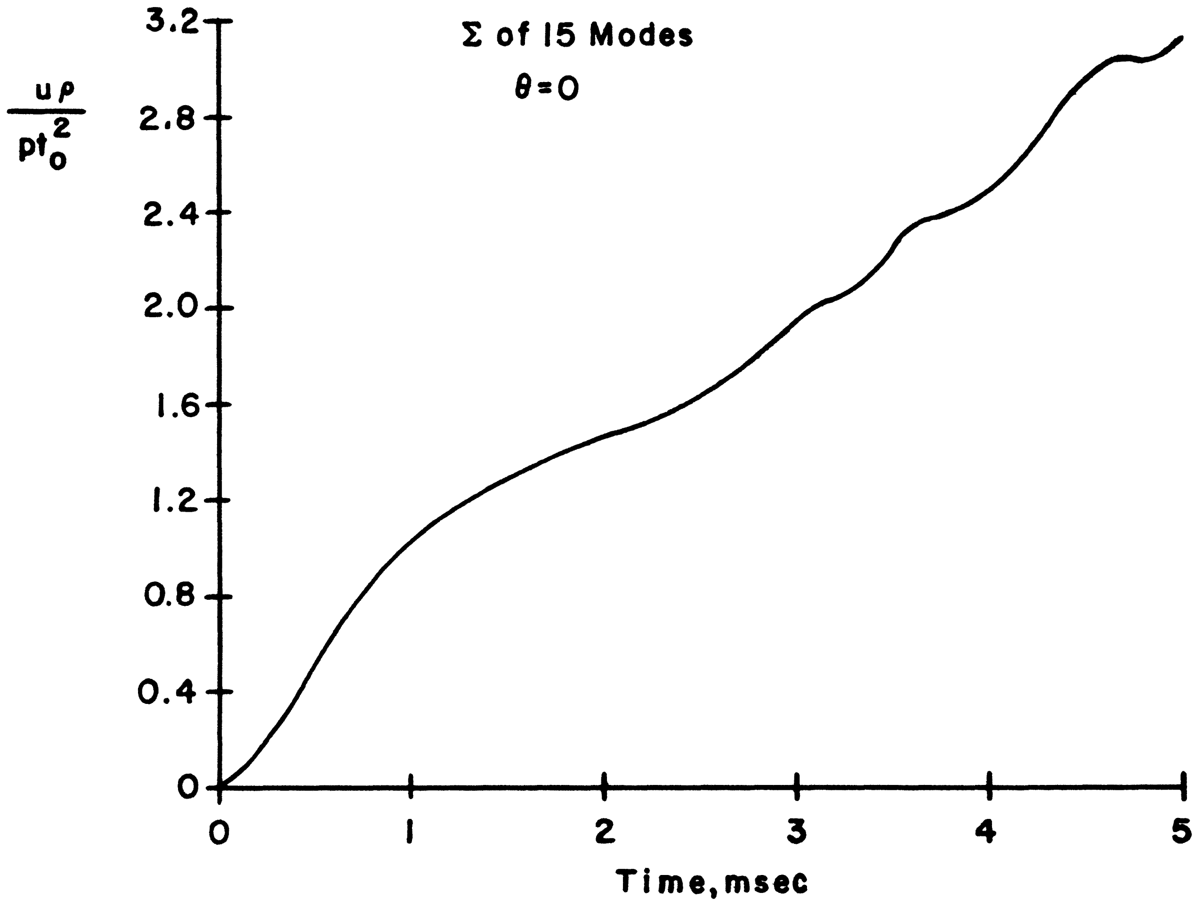

Figure 12 - Complete Series Solution for Radial Displacement

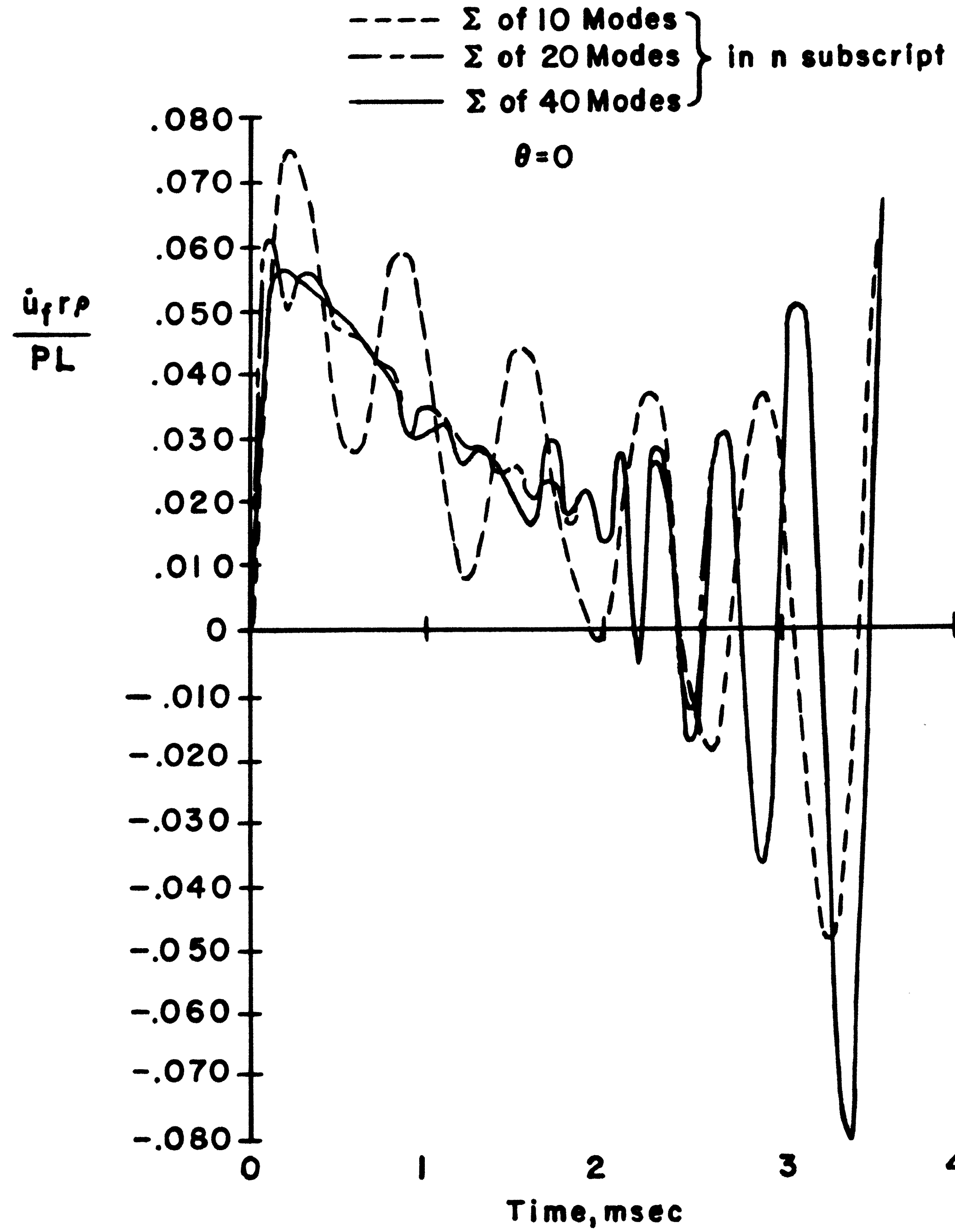

Figure 13 - Double Series Solution for Radial Velocity Due to Moving Concentrated Load on Ring

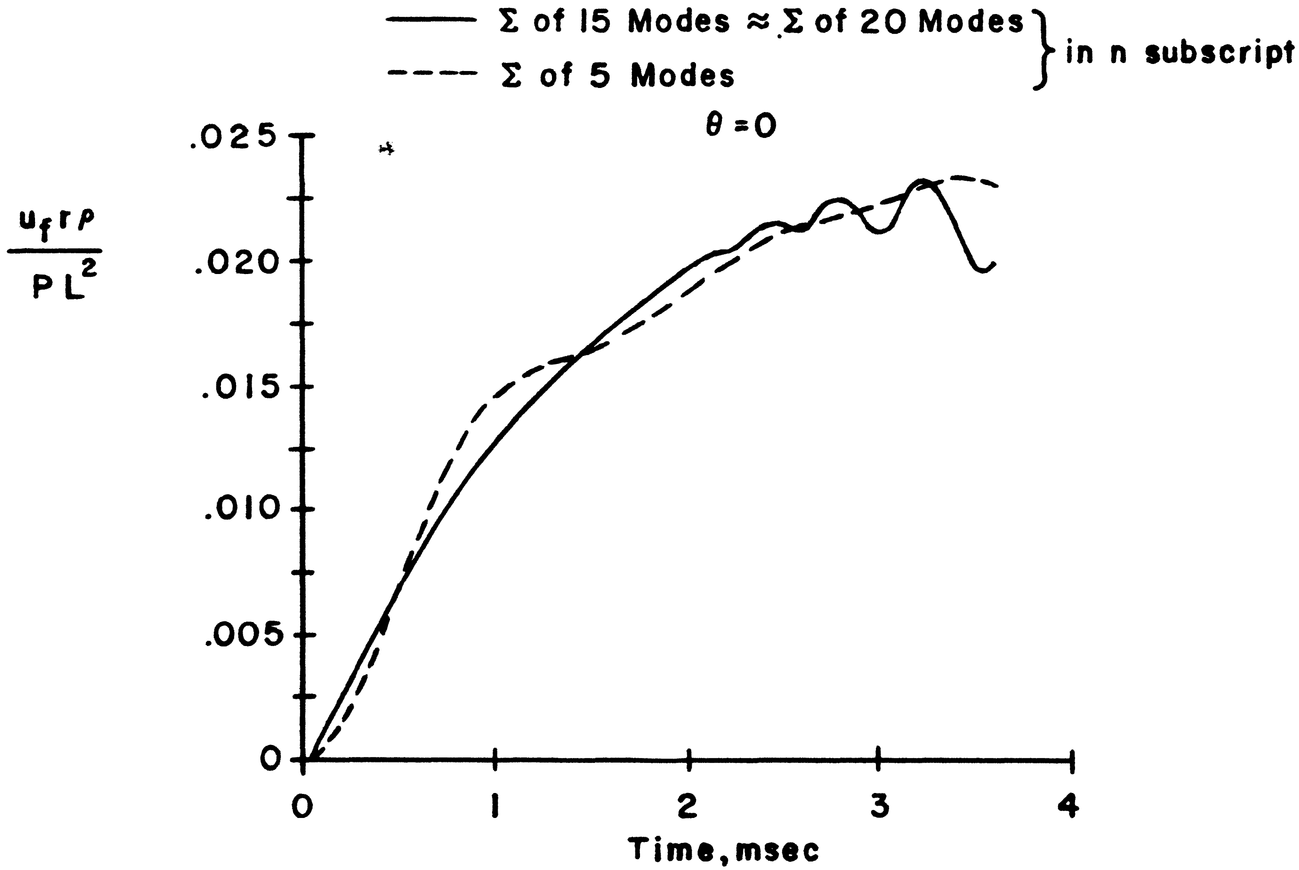

Figure 14 - Double Fourier Series Solution for Radial Displacement Due to Moving Concentrated Load on Ring

TABLE 1
Natural Frequencies

Mode Number	Flexural Frequency (radian/msec)	Extensional Frequency (radian/msec)
0	-	1.947
1	0	2.753
2	0.2302	4.354
3	0.6511	6.157
4	1.248	8.028
5	2.019	9.928
6	2.962	11.843
7	4.076	13.767
8	5.363	15.697
9	6.821	17.632
10	8.452	19.567
11	10.253	21.506
12	12.226	23.445
13	14.371	25.386
14	16.687	27.328
15	19.176	29.270
16	21.836	31.213
17	24.666	33.157
18	27.669	35.100
19	30.843	37.045
20	34.189	
21	37.707	
22	41.396	
23	45.257	
24	49.289	
25	53.493	

IV. DISCUSSION

A. SERIES SOLUTION

The predicted flexural velocity as shown in Figure 4 at first seems somewhat unrealistic because of the large velocities and accelerations which appear after the loading phase is completed. Some possible causes for this response are mentioned below.

1. Some arithmetic or algebraic error could exist in the numerical computation scheme used to determine the response; however, due care was exercised in the numerical phases of this problem and it is felt that the trouble lies elsewhere.

2. Some discrepancy could exist between the series solution as developed for displacements and as used to predict velocities. It is known that in some cases a function can be represented by a series but the derivative of the function is not represented by the derivative of series taken term by term since the differentiated series is divergent. However, the series in question is a function of the variable θ while the differentiation is with respect to the variable t, so the theorems governing differentiation of Fourier series [39], [40] are not applicable. Also the differentiated series is not obtainable in closed form, so standard ways of investigating the convergence, e.g., Cauchy's ratio test, are not applicable. By using the implied assumption that if the differentiated series does converge, then it does converge to the correct series for the velocity, the problem of determining the validity of the series for velocity becomes one of determining the convergence of that series.

3. The question of whether the series does converge, and if so, how many terms of the series must be included in a summation to give reasonably accurate results cannot be rigorously determined; however,

by observation of Figure 4, it is apparent that using 25 terms of the series will produce a response very similar to that of using 35 terms. Although this certainly does not prove convergence, it does at least lead one to suspect it.

4. Another cause for concern is the fact that in Equation (12) the term $\partial q/\partial \theta$ appears whereas Figure 1 shows that at the front and tail of the pressure distribution, i.e., at $\theta = \gamma$ and $\theta = \phi - \Delta$ respectively, the term $\partial q/\partial\theta$ is undefined. Physically the term $\partial q/\partial\theta$ is the rate of change of the load with respect to the angular position coordinate θ with time t held constant. The nonexistence of $\partial q/\partial\theta$ at these two points did not interfere with the development of the analytical solution of the flexural response; however, it does seem odd that a solution to Equation (12) is possible while some term of that equation is not completely defined. It seems possible that the nonexistence of $\partial q/\partial\theta$ at two points could cause some difficulties that are not readily evident but whose effects are seen in the predicted velocity response.

However, it must be pointed out that Wenk [21] solved the problem of a stationary, harmonic, concentrated load on a ring using Equations (9) and (12) but he did not encounter any such trouble involving the discontinuities in the loading scheme. Since the discontinuities of this concentrated load did not cause any difficulty in reaching a numerical solution, it seems reasonable to assume that the discontinuities of the distributed load are not causing the unexpected response.

5. A plausible explanation for the velocity response is that for the chosen loading system a resonance condition is possibly excited. Figure 6 seems to substantiate this theory since it shows that the flexural modes 7, 8, and 9 have amplitudes greater than some lower modes, whereas from modes 11 on up, the amplitudes are more

rapidly decreasing. From Table 1 it is seen that mode Number 9 has the period of 0.92 millisecond, closest of all modes to twice the load duration time which was 0.5 millisecond.

When it is remembered that all damping has been neglected and that in many transient vibration problems, where the loading time is small as compared to lowest natural period, the maximum response occurs a considerable time after the loading phase is past, then the predicted flexural response becomes more credible.

In regard to the extensional series solution, only one comment is noteworthy. The velocity equation contains the time function $I_n(t)$

$$I_n(t) = \int_0^t \int_0^s Z_n(s_1)\, ds_1\, ds$$

and after time $t_1 = t_o + \frac{r}{c}$ has passed, the Z_n becomes zero, thus $I_n(t)$ becomes

$$I_n(t)\Big]_{t>t_1} = (t - t_1)\int_0^{t_1} Z_n(s)\, ds + \int_0^{t_1}\int_0^s Z_n(s_1)\, ds_1\, ds$$

which gives rise to the secular term of t times a constant; this leads to an infinite velocity, which of course does not occur in nature. However, all of this trouble is bypassed because the velocity equation contains two such secular terms which are identical except for having opposite signs.

B. DOUBLE FOURIER SERIES SOLUTION

Essentially this solution is the same as the single series solution except that it has the time function expressed in a series. One advantage of this method is that it allows the use of theorems from the theory of Fourier series in investigating the convergence of the differentiated series; however, this single advantage is lost since the Fourier coefficients cannot be obtained in closed form and hence the theorems are not applicable.

One of the main disadvantages of this method is that all the Fourier coefficients must be determined by numerical integration and there are many more such coefficients in this method than in the single series solution. The solution of the approximate problem, which was obtained in closed form and which will be discussed later, gives some insight into the magnitude of the problem of finding enough of the coefficients to give a reasonably accurate solution.

The second major difficulty with this double series method lies in the additional concern it creates in regard to convergence. Both the series in m and the one in n must be convergent for the solution to be meaningful. Also if either series were only slowly convergent, the computation effort would be greatly increased.

Furthermore, it can be seen from (53) that for large n values

$$\lim_{n \to \infty} C_{mn} = \frac{B_{mn} L^2}{\rho m^2 \pi^2} \quad . \tag{58}$$

Thus the convergence or divergence of the displacement series of u_e with respect to n depends strictly on the behavior of the load coefficients B_{mn} with increasing n. Since the B_{mn} for the moving distributed load from (50) cannot be obtained in closed form, it is questionable whether these B_{mn} form a convergent set of coefficients.

The time function was expanded in a symmetric series for convenience; no loss of generality is involved since only the interval 0 to L is of concern and the interval -L to 0 is irrelevant.

A numerical solution for the original problem of a moving distributed load was not attempted by this double series method because of the uncertainty of the existence of the extensional part of the response and the large computation effort that would be involved in evaluating the B_{mn} coefficients.

C. APPROXIMATE SOLUTION

The solution of the approximate problem was determined in order to help in understanding the flexural part of the solution of the given problem; accordingly no numerical results were obtained for the extensional part of this approximate solution. It is significant to note that the flexural parts of both the series solution of the given problem and the double series solution of the approximate problem have the same characteristic shape. Thus any comments that can be made regarding the approximate solution might be applicable to the other solution also.

With regard to the analytical proof of convergence of the velocity series, no theorems from the theory of double Fourier Series as given by Hobson [39] were presented for the convergence of a differentiated series. However, there is a theorem [40] that if f(x) is expanded over the interval $-\pi \leq x \leq \pi$ and if $f(\pi)$ equals $f(-\pi)$, then the differentiated series does correspond to df/dx, whether it converges or not. In the case at hand, if the series on θ is disregarded or treated as a constant and the solution for displacement u is thought of as a function of time only, then the condition $u(L) = u(-L)$ is satisfied by Equation (49); thus $\dot{u}$ is the differentiation of the series for u, regardless of the aspect of convergence. Since the series for $\dot{u}$ is apparently convergent by observation, it is reasonable to say that the $\dot{u}$ of (49) is a valid velocity expression. Accordingly, this conclusion tends to indicate that the differentiation of the series performed in the series solution of

Section II, Part B is also a valid operation.

Some indication of the amount of error incurred by truncating this series is given by an investigation of the mode participation factor [41] for the double Fourier series solution to the concentrated, moving load problem. The factor for the n^{th} mode is

$$F_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} w(\theta)\, \text{Cos}\; n\theta\, d\theta$$

where f(θ) is described below. The load is written as

$$q(\theta,t) = p\; w(\theta)\, \chi(t)$$

and by analogy with Equation (56) the factors w and x become

$$w(\theta) = \begin{cases} 1 & |\theta| \leq \frac{\pi}{2} \\ 0 & |\theta| > \pi/2 \end{cases}$$

$$\chi(t) = \begin{cases} 1 - \frac{tc}{r} & \text{for } \Omega + \frac{\epsilon}{2} \geq |\theta| \geq \Omega - \frac{\epsilon}{2} \\ 0 & \text{for } \Omega + \frac{\epsilon}{2} < |\theta| \\ & \phantom{\text{for }} \Omega - \frac{\epsilon}{2} > |\theta| . \end{cases}$$

Thus F_n becomes

$$F_n = \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \text{Cos}\; n\theta\, d\theta = \begin{cases} \dfrac{(-1)^{\frac{n-1}{2}}}{n\pi} & , \text{ n odd} \\ 0 & , \text{ n even} \end{cases}$$

The fraction of total displacement response included by the summation of the first 15 terms in n (and using enough terms in m to get three place accuracy) is then

$$\frac{\sum_{n=1}^{15} F_n}{\sum_{n=1}^{\infty} F_n} = \frac{.246}{.250} = .984.$$

Thus at most, a 1.6 percent error exists in the displacement series solution due to truncating the series at 15 terms. If this 1.6 percent difference were used to smooth out the oscillations in the displacement curve of Figure (14) it is readily seen that the curve would not be altered enough to affect the oscillations appreciably; thus the slopes of this curve, i. e., the velocity history, will still exhibit the same characteristic behavior as before in Figure (13).

The suggestion from this is that the strange behavior of the flexural velocity response is not due to using an insufficient number of terms of the series in the numerical computations.

Using a constant angular velocity instead of a constant linear velocity for the motion of the load is an approximation that makes the closed form solution possible, although it introduces an additional difference between the given problem and the approximate problem.

This solution does give some idea of the number of terms necessary in a double series solution; namely, that for each single value of the n subscript used, 30 to 60 of the terms in m must be summed to get three place results. Equation (48) shows that no solution is possible when $\omega_{fn} = \frac{m\pi}{L}$; this condition did not occur in this solution and the closest these two quantities did get to one another was for n = m = 10 which produced the largest value of B_{mn} . Accordingly, the B_{mn} where $m \approx n$ were the largest, so in computing the responses shown in Figures (13) and (14) the values of m used were dictated by the value of n. That is, for n = 10, m ran from zero to 40 while n = 30 required m to go from 10 to 50 to achieve the desired accuracy.

D. GENERAL COMMENTS

Many problems involving the dynamic response of an elastic system to some transient loading can be solved rather efficiently by the use of Laplace transformations. However, the Laplace transform for the given load cannot be readily obtained because the transform involves integrals very similar to those involved in the Duhamel integral solution, neither of which can be found in closed form.

This brings up another matter worthy of recognition; the Laplace technique has been used successfully by several authors working forced, cylindrical shell problems, but they have not considered moving, distributed loads. The added realism of using such a loading brings with it complications which render the Laplace technique useless. Thus, there seems to be an area worth further investigation involving the question of whether this additional complication is worth the trouble it causes, or whether the other loading systems where the load comes on the shell everywhere at the same time will give a satisfactory solution for the case where a moving load is acting.

In regard to the number of modes that must be used for convergence, it may be said that, if a large number of modes, e.g., several hundred, must be considered in order to get convergence, then considerably more computation effort would be required in the form of a more sophisticated numerical integration scheme and in having to keep more significant places in the lower modes so that the effects of the higher ones would be preserved.

Also, the inclusion of many higher modes leads to a breakdown of the elementary methods used in this analysis of the ring response;

the effects of shear, which have been omitted from this analysis, become increasingly important in the higher modes. Thus, there is a built-in inaccuracy in the higher modes, and any solution which depends heavily on higher modes for convergence is necessarily in error.

There is an inconsistency in the method Love [3] used to separate the general ring vibration problem into the two distinct parts of flexural and extensional motions. While considering flexure without extension, the extensional strain is set equal to zero, but while considering extension without flexure, the bending stress and transverse shear stress are set equal to zero. It would seem more consistent to let either the strains or the stresses be zero in each case instead of mixing the two.

A set of governing equations on displacements which include both extensional and flexural motion together was derived by Baron and Bleich [27]. By use of their exact (neglecting shear effects and rotatory inertia) expressions for mode shapes and frequencies, it is seen that for the ring used as an example in Section III and for $n = 10$, an 18 percent error in mode shape and an 0.20 percent error in the frequencies result from the decision to consider the extensional and flexural motions separately.

For the sake of completeness, an experimental response is plotted with the series solution on Figure 11. The poor agreement is due to the vast differences in the two situations; the experimental results are from an actual explosion test of a submerged, axisymmetric, stiffened cylindrical shell with finite length and heavy end bulkheads. The effects of the interaction of the pressure wave and the elastic cylinder have not been accounted for analytically in this analysis; also the effects of damping in the cylinder were not considered.

In conclusion, a list of some recommendations and suggestions for researchers who desire to consider this area of investigation is presented.

1. Use the governing equations given by Philipson [20] which include shear effects to predict the ring response.

2. Non-dimensionize the governing equations eliminating as many parameters as possible so that a solution for one problem would be applicable to a ring of any material or geometric dimensions.

3. Develop a solution using the finite difference method or extend the static finite element analysis by Pletta, Liessner and Yeh [42] to include the dynamical aspects also.

4. If a normal mode analysis is used, investigate the possibility of using the Williams method described by Sheng [31] which will lead to a more rapidly converging solution. The references listed by Sheng should be consulted as it is not clear how this method can be applied in the case of a moving load.

5. Take a close look at some of the recent papers published in the aerospace field concerning transverse loadings on missiles, especially the work of Cottis [32] which gives a solution involving a Green's Function might prove interesting.

6. Compare some of these shell studies to see the relative importance of using a moving load as opposed to using the all-at-once loading scheme. A comparison of the loading used herein with that of Payton's work [43], which includes the effects of the motion of the elastic cylinder on the pressure pulse, would show which of the two types of loadings gives the most satisfactory results.

V. SUMMARY

An attempt is made to secure a solution for the tangential and radial displacement and velocity components resulting from a moving, radial, distributed load on the ring. It was found that a series solution could not be obtained in closed form, but a numerical example is presented for which the results are somewhat difficult to explain.

Approaching the problem with a double Fourier series type solution leads to too much computation effort to justify the attempt.

An approximate solution is developed but for various reasons it has only limited usefulness.

It appears that further investigation is warranted into the problem of the relative importance of a sweeping versus an all-at-once type loading.

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THEORETICAL INVESTIGATION OF THE INITIAL RESPONSE OF A THIN RING TO A RADIAL SHOCK PULSE

by

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ABSTRACT

In this analytical investigation of the initial response of a thin, circular, homogeneous, and isotropic ring to a transverse shock pulse, the radial and tangential components of displacement and velocity are found in series form by use of Duhamel's integral. A plane shock front is assumed to propagate normal to a diameter of the ring with constant linear velocity and to be followed by a parabolic decay. It is also assumed that the motion of the ring does not influence the pressure of the wave and that the wave exerts only radial forces on the ring. Classical, small-deflection linear theory, neglecting rotatory inertia and shear effects, is used in conjunction with the classical treatment of distinct extensional and flexural modes.

For the stated loading scheme, Duhamel's integral cannot be obtained in closed form; however, by use of numerical integration an example problem is solved and the resulting displacement and velocity histories are plotted. The flexural velocity showed unexpectedly large values at relatively late times.

An alternate analytical solution using a double Fourier series is also developed, but no numerical results were determined. The flexural response from the solution of an approximate problem consisting of a moving concentrated force on a ring was also investigated to help explain this unexpected response.

An area deserving further consideration is raised by the problem of the relative importance of using a sweeping type load as opposed to using the mathematically simpler all-at-once type loading.