

**Optimal Experimental Design for
Poisson Impaired Reproduction Studies**

by

Jennifer Wade Huffman

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APPROVED

Raymond H. Myers, Chairman
Clint W. Coakley
Robert V. Foutz
Marvin Lentner
Keying Ye

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(ABSTRACT)

Impaired reproduction studies with Poisson responses are among a growing class of toxicity studies in the biological and medical realm. In recent years, little effort has been focused on the development of efficient experimental designs for impaired reproduction studies. This research concentrates on two areas: 1) the use of Bayesian techniques to make single regressor designs robust to parameter misspecification and 2) the extension of design optimality methods to the k -regressor model. The standard Poisson model with log link is used. Bayesian designs with priors on the parameters are explored using both the D and F-optimality criteria for the single regressor Poisson exponential model. Since these designs are found via numeric optimization techniques, Bayesian equivalence theory functions are derived to verify the optimality of these designs. Efficient Bayesian designs which provide for lack-of-fit testing are discussed. Characterizations of D, D_s , and interaction optimal designs which are factorial in nature are demonstrated for models involving interaction through k factors. The optimality of these designs is verified using equivalence theory. In addition, augmentations of these designs that result in desirable lack of fit properties are discussed. Also, a structure for fractional factorials is given in which specific points are added one at a time to the main effect design in order to gain estimability of the desired interactions. Robustness properties are addressed as well. Finally, this entire line of research is extended to industrial exponential models where different regressors work to increase and/or decrease a count data response produced by a process.

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