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Parallel-plate waveguide with sinusoidally perturbed boundaries

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The method of multiple scales is used to obtain a uniformly valid asymptotic expansion for the propagation of TM modes on a parallel-plate waveguide with perfectly conducting boundary surfaces that are sinusoidally perturbed in the direction of propagation. The analysis shows that resonance occurs whenever the wave number of the wall distortion function is equal to the difference between the wave numbers of two propagating modes. It is further shown that the generated mode is the same order of magnitude as the excited mode due to resonance and that energy is continuously exchanged between the two modes as they propagate down the guide.

I. INTRODUCTION

In this paper we investigate the generation of a propagating mode as a result of the interaction of an excited mode with the geometrical perturbations of the walls of a parallel-plate waveguide. The walls are assumed to be perfectly conducting and have wall distortion functions of the form

$$\bar{x} = a\sin(\bar{k}_w \bar{z}) \quad \text{lower wall}, \tag{1a}$$

$$\bar{x} = d + a\sin(\bar{k}_w \bar{z} + \theta) \quad \text{upper wall}, \tag{1b}$$

where a is the amplitude of the wall's sinusoidal distortions, d is the undistorted separation of the plates, k_w is the wave number of the distortions, and θ is the phase difference between the distortions of the two walls. We treat the case of weak perturbations so that $a \ll d$.

The problem of a dielectric slab waveguide having wall distortion functions as those of Eqs. (1a) and (1b) was treated by Marcuse.[1] He considered the case of a slab with periodically varying thickness ($\theta = \pi$), and used an *ad hoc* approximation in order to derive equations for the amplitudes of two interacting modes satisfying the resonance condition

$$\bar{k}_m = \bar{k}_n - \bar{k}_w, \tag{2}$$

where $\bar{k}_m$ ($\bar{k}_n$) are the wave numbers of the generated (excited) mode. It is the purpose of the present analysis to treat the problem by systematic perturbation techniques taking into account the case of near resonance; that is

$$\bar{k}_m \approx \bar{k}_n - \bar{k}_w, \tag{3}$$

and also for any value of θ.

Following Nayfeh's treatment[2] of acoustic waves in two-dimensional ducts with sinusoidal walls, we use the method of multiple scales[3] to determine an approximate solution for the cases of TM and TE modes. To this end, we make length and time dimensionless using the average separation of the plates d so that Eqs. (1a) and (1b) take the form

$$x = \epsilon\sin(k_w z) \quad \text{lower wall}, \tag{4a}$$

$$x = 1 + \epsilon\sin(k_w z + \theta) \quad \text{upper wall}, \tag{4b}$$

where x and z are dimensionless coordinates and $\epsilon = a/d$ is a parameter much smaller than unity. We consider the case of TM modes as an illustration of the technique. The fields for a TM mode are derivable from a vector potential which for $H_z = 0$ is z directed. For a harmonic time variation $\exp(-i\omega t)$, the z-directed wave function ψ is governed by the Helmholtz equation

$$\nabla^2\psi + k^2\psi = 0, \tag{5}$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial z^2$ and k is the free-space dimensionless wave number. The boundary conditions on ψ for the case of infinite conductivity are the vanishing of the tangential component of the electric field at the boundaries. This gives

$$\left(\frac{\partial^2}{\partial z^2} + k^2\right)\psi = -\epsilon k_w \cos(k_w z)\frac{\partial^2\psi}{\partial x\,\partial z} \quad \text{at } x = \epsilon\sin(k_w z), \tag{6}$$

$$\left(\frac{\partial^2}{\partial z^2} + k^2\right)\psi = -\epsilon k_w \cos(k_w z + \theta)\frac{\partial^2\psi}{\partial x\,\partial z} \quad \text{at } x = 1 + \epsilon\sin(k_w z + \theta). \tag{7}$$

We first determine the resonant frequencies by obtaining a first-order straightforward perturbation expansion in Sec. II.

II. A STRAIGHTFORWARD EXPANSION

We seek an asymptotic expansion for ψ of the form

$$\psi(x,z) = \psi_0(x,z) + \epsilon\psi_1(x,z) + \cdots. \tag{8}$$

Substituting Eq. (8) into Eqs. (5)–(7), transferring the boundary conditions to $x=0$ and $x=1$ by developing ψ and its derivatives in Taylor series around $x=0$ and $x=1$, and equating coefficients of equal powers of ϵ, we obtain

$O(\epsilon^0)$

$$\nabla^2\psi_0 + k^2\psi_0 = 0, \tag{9}$$

$$\psi_0 = 0 \quad \text{at } x = 0, \tag{10}$$

$$\psi_0 = 0 \quad \text{at } x = 1, \tag{11}$$

$O(\epsilon)$

$$\nabla^2\psi_1 + k^2\psi_1 = 0, \tag{12}$$

$$\left(\frac{\partial^2}{\partial z^2} + k^2\right)\psi_1 = -\sin(k_w z)\left(\frac{\partial^2}{\partial z^2} + k^2\right)\frac{\partial\psi_0}{\partial x} - k_w\cos(k_w z)\frac{\partial^2\psi_0}{\partial x\,\partial z} \quad \text{at } x = 0, \tag{13}$$

$$\left(\frac{\partial^2}{\partial z^2}+k^2\right)\psi_1=-\sin(k_w z+\theta)\left(\frac{\partial^2}{\partial z^2}+k^2\right)\frac{\partial\psi_0}{\partial x} - k_w\cos(k_w z+\theta)\frac{\partial^2\psi_0}{\partial x\,\partial z} \quad \text{at } x=1. \tag{14}$$

The solution of Eq. (9) that satisfies the boundary conditions (10) and (11) is

$$\psi_0=A_n\sin(n\pi x)\exp(ik_n z), \tag{15}$$

where $n=1, 2, 3, \ldots$, and for a propagating mode

$$k_n^2=k^2-(n\pi)^2>0. \tag{16}$$

Substituting Eq. (15) into the boundary conditions (13) and (14), we obtain

$$\left(\frac{\partial^2}{\partial z^2}+k^2\right)\psi_1=\tfrac{1}{2}in\pi A_n\{(n^2\pi^2-k_nk_w)\exp[i(k_n+k_w)z]-(n^2\pi^2+k_nk_w)\exp[i(k_n-k_w)z]\} \quad \text{at } x=0, \tag{17}$$

$$\left(\frac{\partial^2}{\partial z^2}+k^2\right)\psi_1=\tfrac{1}{2}in\pi(-1)^n A_n\{(n^2\pi^2-k_nk_w)\exp[i(k_n+k_w)z+i\theta]-(n^2\pi^2+k_nk_w)\exp[i(k_n-k_w)z-i\theta]\} \quad \text{at } x=1. \tag{18}$$

Equations (17) and (18) suggest that ψ_1 has a particular solution of the form

$$\psi_1(x,z)=\tfrac{1}{2}in\pi A_n\{(n^2\pi^2-k_nk_w)\Phi_1(x)\exp[i(k_n+k_w)z] - (n^2\pi^2+k_nk_w)\Phi_2(x)\exp[i(k_n-k_w)z]\}. \tag{19}$$

Substituting Eq. (19) into Eqs. (12)—(14), we obtain

$$\frac{d^2\Phi_1}{dx^2}+\alpha_1^2\Phi_1=0, \quad \alpha_1^2=k^2-(k_n+k_w)^2, \tag{20}$$

$$\Phi_1(0)=\alpha_1^{-2}, \tag{21}$$

$$\Phi_1(1)=(-1)^n\alpha_1^{-2}\exp(i\theta), \tag{22}$$

$$\frac{d^2\Phi_2}{dx^2}+\alpha_2^2\Phi_2=0, \quad \alpha_2^2=k^2-(k_n-k_w)^2, \tag{23}$$

$$\Phi_2(0)=\alpha_2^{-2}, \tag{24}$$

$$\Phi_2(1)=(-1)^n\alpha_2^{-2}\exp(-i\theta). \tag{25}$$

The solutions of Eqs. (20)—(25) are

$$\Phi_j=[\alpha_j^2\sin(\alpha_j)]^{-1}\{[(-1)^n\exp[-i\theta\cos(\pi j)]-\cos(\alpha_j) \times\sin(\alpha_j x)+\sin(\alpha_j)\cos(\alpha_j x)\}, \tag{26}$$

where $j=1,2$.

We note that $\Phi_j\to\infty$ as $\alpha_j\to m\pi$ for integer m. In other words, the straightforward expansion breaks down near the resonant wave numbers

$$k_r^2=(k_n\pm k_w)^2+m^2\pi^2. \tag{27}$$

Uniformly valid expansions in the neighborhood of resonance will be obtained in Sec. III by using the method of multiple scales.[3] Since $k_r^2-m^2\pi^2=k_m^2$, the resonance condition can be written

$$k_m=k_n\pm k_w. \tag{28}$$

III. EXPANSIONS VALID NEAR RESONANCE

We seek an asymptotic expansion of the form

$$\psi(x,z)=\psi_0(x,z_0,z_1)+\epsilon\psi_1(x,z_0,z_1)+\cdots, \tag{29}$$

where $z_0=z$ is a length scale of the order of a wavelength and $z_1=\epsilon z$ is a long length scale characterizing the amplitude and phase modulations. Substituting Eq. (29) into Eqs. (5)—(7) and equating coefficients of equal powers of ϵ, we obtain

$O(\epsilon^0)$

$$\frac{\partial^2\psi_0}{\partial x^2}+\frac{\partial^2\psi_0}{\partial z_0^2}+k^2\psi_0=0, \tag{30}$$

$$\psi_0=0 \quad \text{at } x=0, \tag{31}$$

$$\psi_0=0 \quad \text{at } x=1, \tag{32}$$

$O(\epsilon)$

$$\frac{\partial^2\psi_1}{\partial x^2}+\frac{\partial^2\psi_1}{\partial z_0^2}+k^2\psi_1=-2\frac{\partial^2\psi_0}{\partial z_0\,\partial z_1}, \tag{33}$$

$$\left(\frac{\partial^2}{\partial z_0^2}+k^2\right)\psi_1=-\sin(k_w z_0)\left(\frac{\partial^2}{\partial z_0^2}+k^2\right)\frac{\partial\psi_0}{\partial x} - k_w\cos(k_w z_0)\frac{\partial^2\psi_0}{\partial x\,\partial z_0}-2\frac{\partial^2\psi_0}{\partial z_0\,\partial z_1} \quad \text{at } x=0, \tag{34}$$

$$\left(\frac{\partial^2}{\partial z_0^2}+k^2\right)\psi_1=-\sin(k_w z_0+\theta)\left(\frac{\partial^2}{\partial z_0^2}+k^2\right)\frac{\partial\psi_0}{\partial x} - k_w\cos(k_w z_0+\theta)\frac{\partial^2\psi_0}{\partial x\,\partial z_0}-2\frac{\partial^2\psi_0}{\partial z_0\,\partial z_1} \quad \text{at } x=1. \tag{35}$$

The solution of Eq. (30) for the case of resonance must contain both of the interacting modes, i.e.,

$$\psi_0=A_n(z_1)\sin(n\pi x)\exp(ik_n z_0)+A_m(z_1)\sin(m\pi x)\exp(ik_m z_0), \tag{36}$$

where $A_n(z_1)$ and $A_m(z_1)$ are to be determined at the next level of approximation. Substituting for ψ_0 into Eqs. (33)—(35), we obtain

$$\frac{\partial^2\psi_1}{\partial x^2}+\frac{\partial^2\psi_1}{\partial z_0^2}+k^2\psi_1=-2ik_nA_n'\sin(n\pi x)\exp(ik_n z_0) - 2ik_mA_m'\sin(m\pi x)\exp(ik_m z_0), \tag{37}$$

$$\left(\frac{\partial^2}{\partial z_0^2}+k^2\right)\psi_1=\tfrac{1}{2}i\pi\left(\sum_{j=n,m} j(j^2\pi^2-k_jk_w)A_j\exp[i(k_j+k_w)z_0] - \sum_{j=n,m} j(j^2\pi^2+k_jk_w)A_j\exp[i(k_j-k_w)z_0]\right) \quad \text{at } x=0, \tag{38}$$

$$\left(\frac{\partial^2}{\partial z_0^2}+k^2\right)\psi_1=\tfrac{1}{2}i\pi\left(\sum_{j=n,m}(-1)^j j(j^2\pi^2-k_jk_w)A_j\exp[i(k_j+k_w)z_0+i\theta] - \sum_{j=n,m}(-1)^j j(j^2\pi^2+k_jk_w)A_j \times\exp[i(k_j-k_w)z_0-i\theta]\right) \quad \text{at } x=1, \tag{39}$$

where primes denote differentiation with respect to z_1. Since the homogeneous equations (37)—(39) have a nontrivial solution, the inhomogeneous equations (37)—(39) have a solution if, and only if, a solvability condition is satisfied. To determine this solvability condition, we make use of the resonance condition given by Eq. (28). We note that Eq. (28) indicates that two resonant cases with different values of m are possible. The terms on the right-hand side of Eqs. (38) and (39) that lead to resonance are the same for both values of m except that when $k_m=k_n+k_w$, the roles of k_m and k_n are interchanged. Thus resonance occurs whenever k_w is nearly equal to the difference between the wave numbers of two propagating modes. We consider the case of near resonance $k_m \approx k_n - k_w$ and introduce a detuning parameter δ defined by

$$k_m = k_n - k_w + \epsilon\delta, \quad \delta = O(1), \tag{40}$$

and express $\exp[i(k_n - k_w)z_0]$ and $\exp[i(k_m + k_w)z_0]$ as

$$\exp[i(k_n - k_w)z_0] = \exp[ik_m z_0 - i\delta z_1], \tag{41a}$$

$$\exp[i(k_m + k_w)z_0] = \exp[ik_n z_0 + i\delta z_1]. \tag{41b}$$

To determine the solvability condition for Eqs. (37)—(39) we seek a particular solution of the form

$$\psi_1 = i\Phi_n(x, z_1)\exp(ik_n z_0) + i\Phi_m(x, z_1)\exp(ik_m z_0). \tag{42}$$

Substituting Eqs. (42) into Eqs. (37)—(39), using Eq. (41), and equating the coefficients of $\exp(ik_n z_0)$ and $\exp(ik_m z_0)$ on both sides, we obtain

$$\frac{\partial^2\Phi_n}{\partial x^2} + n^2\pi^2\Phi_n = -2k_n A_n' \sin(n\pi x), \tag{43a}$$

$$n^2\pi^2\Phi_n = \tfrac{1}{2}m\pi A_m(m^2\pi^2 - k_m k_w)\exp(i\delta z_1) \quad \text{at } x = 0, \tag{43b}$$

$$n^2\pi^2\Phi_n = \tfrac{1}{2}m\pi(-1)^m A_m(m^2\pi^2 - k_m k_w)\exp[i(\delta z_1 + \theta)] \quad \text{at } x = 1, \tag{43c}$$

$$\frac{\partial^2\Phi_m}{\partial x^2} + m^2\pi^2\Phi_m = -2k_m A_m' \sin(m\pi x), \tag{44a}$$

$$m^2\pi^2\Phi_m = -\tfrac{1}{2}n\pi(n^2\pi^2 + k_n k_w)A_n\exp(-i\delta z_1) \quad \text{at } x = 0, \tag{44b}$$

$$m^2\pi^2\Phi_m = \tfrac{1}{2}n\pi(-1)^{n+1}(n^2\pi^2 + k_n k_w)A_n\exp[-i(\delta z_1 + \theta)] \quad \text{at } x = 1. \tag{44c}$$

The general solution of Eq. (43a) is

$$\Phi_n(x, z_1) = c_1\cos(n\pi x) + c_2\sin(n\pi x) + (k_n/n\pi)xA_n'\cos(n\pi x). \tag{45}$$

Substituting Eq. (45) into Eqs. (43b) and (43c) yields

$$n^2\pi^2 c_1 = \tfrac{1}{2}m\pi A_m(m^2\pi^2 - k_m k_w)\exp(i\delta z_1), \tag{46a}$$

$$(-1)^n n^2\pi^2[c_1 + (k_n/n\pi)A_n'] = \tfrac{1}{2}m\pi(-1)^m A_m(m^2\pi^2 - k_m k_w) \times\exp[i(\delta z_1 + \theta)]. \tag{46b}$$

Eliminating c_1 from Eqs. (46a) and 46b), we obtain the following solvability condition for Eqs. (43a)—(43c):

$$A_n' = \tfrac{1}{2}(m/nk_n)A_m(k_m k_w - m^2\pi^2)[1 - (-1)^{m+n}\exp(i\theta)]\exp(i\delta z_1). \tag{47}$$

Similarly, the solvability condition for Eqs. (44a)—(44c) is

$$A_m' = \tfrac{1}{2}(n/mk_m)A_n(n^2\pi^2 + k_n k_w)[1 - (-1)^{m+n}\exp(-i\theta)] \times\exp(-i\delta z_1). \tag{48}$$

We seek a solution to Eqs. (47) and (48) of the form

$$A_m = a_m\exp(sz_1), \quad A_n = a_n\exp[(s + i\delta)z_1], \tag{49}$$

where a_j and s are constants. Substituting Eq. (49) into Eqs. (47) and (48) and eliminating the a's, we obtain

$$s(s + i\delta) = \Omega, \tag{50a}$$

where

$$\Omega = \tfrac{1}{2}(k_m k_n)^{-1}(k_m k_w - m^2\pi^2)(k_n k_w + n^2\pi^2)[1 - (-1)^{m+n}\cos\theta]. \tag{50b}$$

The solution of Eq. (50a) is

$$s_{1,2} = \tfrac{1}{2}i[-\delta \pm (\delta^2 - 4\Omega)^{1/2}]. \tag{51}$$

The coupling coefficient Ω is negative because

$$k_m k_w - m^2\pi^2 = k_n k_m - k^2 + \epsilon\delta k_m < 0,$$

since $k_n < k$ and $k_m < k$. Hence, s is purely imaginary and A_m and A_n are consequently bounded.

From Eqs. (49) and (50) we note that for two modes to interact the coupling coefficient Ω must be different from zero. For the special case of a guide of periodically varying direction ($\theta = 0$), $m + n$ must be an odd integer; i.e., odd modes generate even modes, or vice versa. For the case of a guide with periodically varying thickness ($\theta = \pi$), $m + n$ must be an even integer; i.e., odd (even) modes generate odd (even) modes. For values of θ different from 0 and π, m takes on all integer values.

The exact forms of $A_n(z_1)$ and $A_m(z_1)$ depend on the excitation. Let the excitation at $z = 0$ be

$$A_n(0) = 1, \quad A_m(0) = 0. \tag{52}$$

According to Eq. (49), $A_n(z_1)$ and $A_m(z_1)$ are of the form

$$A_n = a_1\exp[(s_1 + i\delta)z_1] + a_2\exp[(s_2 + i\delta)z_1], \tag{53a}$$

$$A_m = a_3\exp(s_1 z_1) + a_4\exp(s_2 z_1), \tag{53b}$$

where the a_i are complex constants. We substitute Eqs. (53a) and (53b) into Eqs. (52), (47), and (48), evaluate the latter at $z_1 = 0$, and obtain

$$a_1 + a_2 = 1, \tag{54a}$$

$$a_3 + a_4 = 0, \tag{54b}$$

$$(s_1 + i\delta)a_1 + (s_2 + i\delta)a_2 = 0, \tag{54c}$$

$$s_1 a_3 + s_2 a_4 = \gamma(n/mk_m)(n^2\pi^2 + k_n k_w), \tag{54d}$$

where $\gamma = \tfrac{1}{2}[1 - (-1)^{m+n}\exp(-i\theta)]$.

Solving for the a's and substituting the result into Eqs. (53a) and (53b), we obtain

$$A_n = (s_1 - s_2)^{-1}\{-(s_2 + i\delta)\exp[(s_1 + i\delta)z_1] + (s_1 + i\delta)\exp[(s_2 + i\delta)z_1]\}, \tag{55a}$$

$$A_m = (s_1 - s_2)^{-1}\gamma(n/mk_m)(n^2\pi^2 + k_n k_w)[\exp(s_1 z_1) - \exp(s_2 z_1)]. \tag{55b}$$

We can express A_n and A_m in the form

$$A_j = |A_j|\exp(i\beta_j), \tag{56a}$$

where

$$|A_j| = [(\mathrm{Re}A_j)^2 + (\mathrm{Im}A_j)^2]^{1/2}, \tag{56b}$$

and

$$\beta_j = \tan^{-1}(\mathrm{Im}A_j/\mathrm{Re}A_j). \tag{56c}$$

Substituting for s_1 and s_2 from Eq. (51) into Eqs. (55a) and (55b), we obtain

$$|A_n| = (\delta^2 - 4\Omega)^{-1/2}\{\delta^2 - 4\Omega\cos^2[\tfrac{1}{2}(\delta^2 - 4\Omega)^{1/2}z_1]\}^{1/2}, \tag{57a}$$

$$|A_m| = (n/mk_m)(n^2\pi^2 + k_n k_w) \times \left(\frac{1-(-1)^{m+n}\cos\theta}{\delta^2 - 4\Omega}\right)^{1/2} \sin[\tfrac{1}{2}(\delta^2 - 4\Omega)^{1/2} z_1]. \quad (57b)$$

If these expressions are now substituted into Eq. (36) and then the Poynting vector is evaluated and integrated over the cross section of the guide, one can show that energy is conserved. In other words, the energy of the excitation at $z = 0$ is equal to the sum of the energies of the two interacting modes at any location along the z axis.

IV. CONCLUSIONS

The analysis shows how sinusoidal wall perturbations result in the generation of a propagating mode from an excited mode. The energy is exchanged between the two modes in a manner consistent with conservation of energy. This feature of the problem can be utilized in the design of mode couplers where it is desired to transfer energy from one propagating mode into another whenever $k_m \approx k_n - k_w$.

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