Some Controllability and Stabilization Problems of Surface Waves on Water with Surface Tension

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Dissertation submitted to the Faculty of the
Virginia Polytechnic Institute and State University
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy
in
Mathematics

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December 18, 2015
Blacksburg, Virginia

Keywords: Kawahara Equation, Contraction Mapping Principle, Boundary Control, Hydrodynamics

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(ABSTRACT)

The thesis consists of two parts. The first part discusses the initial value problem of a fifth-order Korteweg-de Vries type of equation

$$w_t + w_{xxx} - w_{xxxxx} - \sum_{j=1}^{n} a_j w^j w_x = 0\,, \qquad w(x,0) = w_0(x)$$

posed on a periodic domain $x \in [0, 2\pi]$ with boundary conditions $w_{ix}(0,t) = w_{ix}(2\pi,t), i = 0, 2, 3, 4$ and an L^2-stabilizing feedback control law $w_x(2\pi,t) = \alpha w_x(0,t) + (1-\alpha)w_{xxx}(0,t)$ where n is a fixed positive integer, $a_j, j = 1, 2, \cdots, n, \alpha$ are real constants, and $|\alpha| < 1$. It is shown that for $w_0(x) \in H^1_\alpha(0, 2\pi)$ with the boundary conditions described above, the problem is locally well-posed for $w \in C([0,T]; H^1_\alpha(0,2\pi))$ with a conserved volume of w, $[w] = \int_0^{2\pi} w(x,t)dx$. Moreover, the solution with small initial condition exists globally and approaches to $[w_0(x)]/(2\pi)$ as $t \to +\infty$. The second part concerns wave motions on water in a rectangular basin with a wave generator mounted on a side wall. The linear governing equations are used and it is assumed that the surface tension on the free surface is not zero. Two types of generators are considered, flexible and rigid. For the flexible case, it is shown that the system is exactly controllable. For the rigid case, the system is not exactly controllable in a finite-time interval. However, it is approximately controllable. The stability problem of the system with the rigid generator controlled by a static feedback is also studied and it is proved that the system is strongly stable for this case.

The research was partially supported by the National Science Foundation under grant No. DMS-1210979.

Dedication

This thesis is dedicated to my parents and wife.

`For their endless love, support and encouragement`

Acknowledgments

Firstly, I would like to express my sincere gratitude to my advisor Dr. Shu-Ming Sun for the continuous support of my Ph.D study and related research, for his patience, motivation, and immense knowledge. His guidance helped me in all the time of research and writing of this thesis.

I would also like to thank Dr. Jong Kim, Dr. Michael Renardy, Dr. Pengtao Yue and Dr. David Russell for serving on my committee.

I am very thankful to my friends, my teachers and the Virginia Tech Mathematics Department for all their support. I would like to take this opportunity to thank my Master advisor Dr. Zhaosheng Feng for recommending me to the PhD program in Virginia Tech. Finally, I thank my family members for their love, encouragement and support.

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Chapter 1

Introduction

In the first part of this thesis, we study the solutions of the following general nonlinear problem

$$w_t + w_{xxx} - w_{xxxxx} - \sum_{j=1}^{n} a_j w^j w_x = 0\,, \qquad w(x,0) = w_0(x) \tag{1.0.1}$$

posed on a domain $x \in [0, 2\pi]$. Here, n is a positive integer and $a_j, j = 1, 2, \cdots, n$, are real constants. This type of equations have been used to model traveling gravity-capillary waves on a two-dimensional fluid flow of constant depth. In fluid dynamics, gravity waves are those waves generated in a fluid medium or at the interface between two medium when the force of gravity or buoyancy tries to restore equilibrium. An example of such an interface is that between the atmosphere and the ocean which gives rise to wind waves. Wind generated gravity waves on the free surface have a period of between 0.3 and 30 seconds. Shorter waves are also affected by surface tension and are called gravity-capillary waves. When the traveling speed of the wave is near its critical value and the surface tension is not near its critical value, the famous Korteweg-de Vries (KdV) equation

$$w_t + w_{xxx} - aww_x = 0 \tag{1.0.2}$$

can be derived for modeling the propagation of surface waves [23, 48]. However, if the surface tension is also near its critical value, the KdV equation is no longer valid and the equation (1.0.1) with $a_1 \neq 0$ and $a_j = 0, j = 2, 3, \cdots, n$, called a fifth-order KdV equation or Kawahara equation, can be derived [19]. In some special cases, the coefficient a_1 may be zero for two-layer fluid systems and then the equation (1.0.1) with $a_2 \neq 0$ and $a_j = 0, j = 1, 3, 4, \cdots, n$, called a modified fifth-order KdV equation or modified Kawahara equation, can be obtained (see [7] for corresponding modified KdV equation). Here, we are interested in the control and stabilization problems for a general equation (1.0.1) under some boundary conditions. The KdV equation has been studied extensively during last half century (see [5, 6, 18, 20, 41] and references therein) and its control problems have been also discussed since 1990's . One type of control problem for (1.0.2) was first discussed by Russell and Zhang [38, 39] using the boundary conditions

$$w(0,t) = w(2\pi,t)\,, \quad w_{xx}(0,t) = w_{xx}(2\pi,t)\,, \quad w_x(2\pi,t) = \alpha w_x(0,t) \tag{1.0.3}$$

with $|\alpha| < 1$ ($\alpha = -1/2$ is a singular case and was studied in [44]). In control theory, it is called a closed-loop control process that generally refers to control synthesis via some kind of state feedback and is mainly concerned with achieving asymptotic stability of an equilibrium or invariant set. It is straightforward to see that a constant state is an equilibrium state for both (1.0.1) and (1.0.2). In [39], it is shown that the solution of (1.0.2) with boundary conditions (1.0.3) always exists and goes to a constant state $\frac{1}{2\pi}\int_0^{2\pi} w_0(x)dx$ as $t \to +\infty$. Moreover, a singular case was studied in [44] where the method used in [39] fails to produce the asymptotic stability result for (1.0.2) and (1.0.3). Other types of control problems for (1.0.2) can be found in [22, 40, 26, 34]. In particular, a review paper [34] for this type of problems is recommended.

The well-posedness of pure initial value problem for (1.0.1) in $\mathbb{R}$ with $a_j = 0, j = 2, \cdots, n$, i.e., the fifth-order KdV equation, has been discussed in [16, 17]. The controllability problems of (1.0.1) for this case using nonhomogeneous terms in the equation or boundary conditions have been studied in [51, 52]. They showed the boundary smoothing properties for the linear fifth-order KdV equation with boundary conditions [52] and local controllability and stabilization of the nonlinear fifth-order KdV equation on a periodic domain with an internal control [51].

Here, we only study the local and global well-posedness of (1.0.1) and the asymptotic stability of small solutions as $t \to +\infty$ using *a closed-loop point dissipation process* (general discussions on such control problems can be found in [12, 35]). To design a dissipation mechanism for (1.0.1), we multiply both sides of (1.0.1) by $w(x,t)$ and integrate it from zero to 2π. If we impose periodic boundary conditions

$$w_{kx}(0,t) = w_{kx}(2\pi,t) \qquad \text{for} \quad k = 0, 2, 3, 4 \tag{1.0.4}$$

where $w_{kx} = \frac{\partial^k w}{\partial x^k}$, then we can obtain an identity for the solution of (1.0.1),

$$\frac{d}{dt}\left(\int_0^{2\pi} w^2(x,t)dx\right) = \big(w_x(2\pi,t) - w_x(0,t)\big)\cdot\big(w_x(2\pi,t) + w_x(0,t) - 2w_{3x}(0,t)\big). \tag{1.0.5}$$

When we let

$$w_x(2\pi,t) = \alpha w_x(0,t) + (1-\alpha)w_{3x}(0,t)\,, \tag{1.0.6}$$

(1.0.5) implies that

$$\frac{d}{dt}\left(\int_0^{2\pi} w^2(x,t)dx\right) = -\frac{1+\alpha}{1-\alpha}\big(w_x(2\pi,t) - w_x(0,t)\big)^2.$$

For $|\alpha| < 1$, the energy is decreasing as t increases unless $w_x(2\pi, t) = w_x(0, t)$. Therefore, the boundary conditions (1.0.4) and (1.0.6) can be considered as a dissipation mechanism for (1.0.1). Moreover, it is easy to see that the volume

$$[w(x,t)] := \int_0^{2\pi} w(x,t)dx \text{ is conserved and}$$
$$\frac{d}{dt}\left(\int_0^{2\pi} w^2(x,t)dx\right) = \frac{d}{dt}\left(\int_0^{2\pi} \Big(w(x,t) - ([w]/(2\pi))\Big)^2 dx\right). \tag{1.0.7}$$

Thus, because of this boundary dissipation mechanism, it is reasonable to expect that the solution $w(x,t)$ of (1.0.1) goes to $d = [w]/(2\pi)$ as $t \to +\infty$.

To state the result obtained in the dissertation, let

$$Aw = w_{5x} - w_{3x} \tag{1.0.8}$$

with the domain

$$\mathscr{D}(A) = \Big\{w \in H^5(0, 2\pi) \;\Big|\; w_{kx}(0) = w_{kx}(2\pi)\,,\; k = 0, 2, 3, 4\,,$$
$$w_x(2\pi, t) = \alpha w_x(0, t) + (1-\alpha)w_{3x}(0, t)\Big\} \tag{1.0.9}$$

and A^* be its adjoint operator in $L^2 = L^2(0, 2\pi)$. It will be shown that A and A^* have eigenfunctions $\{\phi_k(x) \mid k = 0, \pm1, \pm2, \dots\}$ and $\{\psi_k(x) \mid k = 0, \pm1, \pm2, \dots\}$, respectively, which are complete in L^2, normalized with $\psi_j^*\phi_k = \delta_{kj}$, and form dual Riesz bases for L^2. For any $s \geq 0$ and $p \geq 1$, define

$$H_\alpha^{s,p} = \left\{ w = \sum_{k=-\infty}^{\infty} c_k\phi_k \;\middle|\; \sum_{k=-\infty}^{\infty} (1 + |k|^{ps})|c_k|^p < \infty \right\} \tag{1.0.10}$$

with the norm

$$\|w\|^p_{H_\alpha^{s,p}} = \|w\|^p_{s,p} = \sum_{k=-\infty}^{\infty} (1 + |k|^{ps})|c_k|^p\,. \tag{1.0.11}$$

Let $H_\alpha^s = H_\alpha^{s,2}$ with norm $\|w\|^2_{H_\alpha^{s,2}} = \|w\|_s^2$. Then, our result can be stated as follow.

Theorem 1.0.1. *Assume* $|\alpha| < 1$.

(i) For any $w_0 \in H_\alpha^1$, *there exists a* $T = T(\|w_0\|_1) > 0$ *such that (1.0.1) with (1.0.4) and (1.0.6) has a unique solution* $w(x,t)$ *satisfying* $w \in C([0,T]; H_\alpha^1)$ *and the solution mapping* N *from* H_α^1 *to* $C([0,T']; H_\alpha^1)$ *is Lipschitz continuous for any* $T' < T$.

(ii) Let $\delta > 0$ *be small and* $s_1' > 1 - \frac{3}{p_1}$, $s_2' > 3\left(\frac{1}{2} - \frac{1}{p_2}\right)$ *where* $p_1 = 2n(2+\delta)/\delta$ *and* $p_2 = 2 + \delta$. *Then, for any* $w_0 \in H_\alpha^{s_1',p_1} \cap H_\alpha^{s_2',p_2}$, *there exists a* $T = T(\|w_0\|_{s_1',p_1}, \|w_0\|_{s_2',p_2}) > 0$ *such that (1.0.1) with (1.0.4) and (1.0.6) has a unique solution* $w(x,t)$ *satisfying*

$$w \in Y_T = \left\{u \in C([0,T]; L^2) \cap L^{p_1}([0,T]; H_\alpha^{1/2}) \cap L^{p_2}([0,T]; H_\alpha^1)\right\}$$

and the solution mapping N from $H_\alpha^{s_1',p_1} \cap H_\alpha^{s_2',p_2}$ to $Y_{T'}$ is Lipschitz continuous for any $T' < T$.

(iii) *There exists a $\gamma > 0$ such that for any $w_0 \in H_\alpha^1$ with $\|w_0\|_1 < \gamma$, the corresponding unique solution of (1.0.1) with (1.0.4) and (1.0.6) exists in H_α^1 for $t \in [0, \infty)$ and decays exponentially to $[w_0]/(2\pi)$ as $t \to \infty$, i.e.,*

$$\left\| w(\cdot, t) - \frac{[w_0]}{2\pi} \right\|_{L^2} \leq ce^{-\rho t} \left\| w_0 - \frac{[w_0]}{2\pi} \right\|_{L^2}, \quad t \geq 0, \tag{1.0.12}$$

where $c > 0$ and $\rho > 0$ are independent of w_0 and t.

Note that (i) and (ii) are local well-posedness results and (ii) represents some kind of smoothing property of the solutions since the initial data are in a weak space. (iii) is the global existence and stability results for small initial data.

The idea of the proof for Theorem 1.0.1 is based upon the method in the paper by Russell and Zhang [39]. However, since the equation is fifth-order and the corresponding equations for the eigenfunctions cannot be obtained explicitly, the estimates of the solutions and the derivations of those smoothing properties are more complicated and new techniques have to be introduced to obtain those results. We first show that the operator A generates a C_0-semigroup $S(t)$ and the eigenvalues of A lie in the left half of complex plane except zero. Then, for $|\alpha| < 1$, it is shown that A is a discrete spectral operator whose eigenfunctions form a Riesz basis in L^2. The asymptotic form of the eigenvalues is also derived. These spectral properties of A are essential in obtaining the smoothing properties of the solutions. Then, existence and uniqueness results are deduced from those estimates and smoothing properties using contraction mapping principle. The asymptotic decay of the solutions is then derived by use of Lyapunov techniques based upon the linear operator A and its spectral properties.

In the second part, we mainly focus on the equations of water waves in a rectangular basin [31]. As we know, the water is an inviscid and incompressible fluid. In a constant gravitational field, we consider the water waves generated in a rectangular basin. If we assume the width of this basin is wide enough such that the water waves generated by the left end wave maker are plane waves, we can simplify our model into two-dimensional problem with four boundaries. The boundary $\Gamma = \Gamma_s \cup \Gamma_f \cup \Gamma_1 \cup \Gamma_2$ of our model is given by

- the free surface $\Gamma_s = \{(x, h) | 0 \leq x \leq \pi\}$,
- the bottom $\Gamma_f = \{(x, 0) | 0 \leq x \leq \pi\}$,
- the left end $\Gamma_1 = \{(0, y) | 0 \leq y \leq 1\}$,
- the right end $\Gamma_2 = \{(L, y) | 0 \leq y \leq 1\}$,

where the domain Ω is the rectangle $[0, \pi] \times [0, 1]$. The velocity field at time t is denoted by $V(x, y, t)$ and the corresponding components of the velocity vector are given by (u, w).

Suppose that the density ρ is a constant, the unit vector in the y direction is denoted by j and the gravitational acceleration in the negative y direction by g. Thus the inviscid equations([49], p. 431) are given by

$$\nabla \cdot V = 0, \tag{1.0.13}$$

$$\partial_t V + (V \cdot \nabla)V = -\frac{1}{\rho}\nabla p - gj, \tag{1.0.14}$$

where $V = (u, w)$ is the velocity vector. In the most problems of water waves, the flow is irrotational, i.e. ζ=curl$V = 0$. Therefore (1.0.14) can be rewritten in the form

$$\partial_t V + \nabla\left(\frac{1}{2}V^2\right) = -\frac{1}{\rho}\nabla p - gj. \tag{1.0.15}$$

If a velocity potential ψ is defined by

$$V = \nabla\psi(x, y, t),$$

based upon the assumption of ζ, (1.0.15) can be changed to

$$\frac{p - p_0}{\rho} = -\psi_t - \frac{1}{2}(\nabla\psi)^2 - gy, \tag{1.0.16}$$

where p is a pressure, and p_0 is an arbitrary constant which means the atmospheric pressure. For convenience, let $\eta(x, t)$ be the elevation of a point on the surface with respect to its equilibrium position, i.e. $y = \eta(x, t)$ on Γ_s. If we consider the surface tension, for small deviation $y = \eta(x, t)$ from a plane surface, the net effect is a normal force $T\eta_{xx}$ per unit area. When this is included, the pressure condition at the surface is given by

$$p - p_0 = -\rho T\eta_{xx}. \tag{1.0.17}$$

On the surface boundary Γ_s where $y = \eta(x, t)$, the dynamic condition (1.0.16) is given by

$$T\eta_{xx} = \psi_t + \frac{1}{2}(\nabla\psi)^2 + g\eta. \tag{1.0.18}$$

Besides the dynamic condition, there exists a kinematic condition. In our model, the interface is between atmosphere and water i.e., the surface is free. Let the interface be described by $f(x, y, t) = 0$. Since the fluid does not cross the interface, the normal velocity of a surface $-f/\sqrt{f_x^2 + f_y^2}$ and the normal velocity of the fluid $(uf_x + wf_y)/\sqrt{f_x^2 + f_y^2}$ are equal to each other. Hence, the kinematic condition on the boundary can be defined by

$$f_t + uf_x + wf_y = 0. \tag{1.0.19}$$

Since $u = \psi_x$ and $w = \psi_y$, choosing $f(x, y, t) \equiv \eta(x, t) - y$ in (1.0.19) gives the boundary condition in terms of η

$$\eta_t + \psi_x \eta_x = \psi_y. \tag{1.0.20}$$

Combining (1.0.13), (1.0.18) and (1.0.20), we obtain a wave equation with boundary conditions on the surface. Observe that, the elevation η and potential ψ are both small for small perturbations on water initially at rest. Hence one can linearized (1.0.18) and (1.0.20) in the form

$$\psi_t + g\eta = T\eta_{xx}, \quad \eta_t = \psi_y, \qquad \text{on } \Gamma_s. \tag{1.0.21}$$

Taking the derivative of first equation with respect to t and substituting η_t with second one, η is eliminated which give rise

$$\psi_{tt} + g\partial_y \psi = T\partial_y \psi_{xx}. \tag{1.0.22}$$

This equation is also called surface condition. The boundary condition on the bottom Γ_f directly comes from the fact that normal velocity of the fluid is zero, which is already linear and independent of η. Thus we have the linear problem for ψ alone:

$$\begin{cases} \psi_{xx} + \psi_{yy} = 0, & \text{on } \Omega \times [0, \tau], \\ \psi_{tt} + g\partial_y \psi = T\partial_y \psi_{xx}, & \text{on } \Gamma_s \times [0, \tau], \\ \psi_y = 0, & \text{on } \Gamma_f \times [0, \tau], \\ \psi_x = v, & \text{on } \Gamma_1 \times [0, \tau], \\ \psi_y = 0, & \text{on } \Gamma_2 \times [0, \tau], \\ \psi_{xy}(0) = \psi_{xy}(\pi) = 0, & \text{on } \Gamma_s, \end{cases} \tag{1.0.23}$$

where v represents the velocities of waves generated by wave makers located on the left end of the basin Γ_1. The last boundary condition on Γ_s is called edge condition which is necessary in the proof of controllability. Once the solution for ψ has been found, the elevation of surface is given by

$$\eta = \frac{1}{g}(T\eta_{xx} - \psi_t), \tag{1.0.24}$$

which comes from dynamic condition (1.0.21). The problem in (1.0.23) has to be supplemented by appropriate initial conditions. As we will see in what follows, we specify initial conditions on the surface Γ_s by

$$\psi(0) = \phi_0, \quad \dot{\psi}_0 = \phi_1, \qquad \text{on } \Gamma_s, \tag{1.0.25}$$

where $\phi = \psi|_{\Gamma_s}$.

Practical control problems involving this system without surface tension have been studied, from numerical and experimental point of view. For these aspects we refer to [30, 29, 31]. In this dissertation, we mainly consider the controllability and stabilization problems of (1.0.23) considering the effects of surface tension. Roughly speaking, controllability studies whether it is possible to steer a dynamic system from a given initial state to an arbitrary final state using the set of admissible controls. We are also concerned with achieving asymptotic stability of an equilibrium or invariant set through feedback control law. This is called a closed-loop control process. In particular, for our case, we study the theoretical problem: whether it is possible to find a control $v \in L^2(0, \tau; U)$ in a given space U and a time $\tau > 0$ such that the surface elevation η of the basin is in a given state at time τ. We will consider the strong version of this problems, where the state is to be reached exactly, and the weak version, where the state is reached approximately.

The controllability problems can be categorize into two types depending on state equations and constraints on the control signal. If we concern about infinite dimensional systems, many problems are still unsolved. As we will see in what follows, one can formulate (1.0.23) into the first order system

$$\begin{cases} \dot{\xi} = A\xi + Bv, \\ \xi(0) = \xi_0 \in X, \end{cases} \tag{1.0.26}$$

where $v \in L^2(0, \tau; U)$ is a control input which represents the velocity of the wave maker and U is an admissible control space which could be finite or infinite dimensional. We can show that the operator A generates a strongly continuous semigroup $S(t)$ on a Hilbert space X, and operator B is bounded from U to X. With the semigroup $S(t)$, we define the controllability map $\mathcal{C}_\tau$ of (1.0.25) by

$$\mathcal{C}_\tau v := \int_0^\tau S(\tau - s)Bv(s)ds, \tag{1.0.27}$$

which is bounded from $L^2(0, \tau; U)$ to X, and the controllability gramian on $[0, \tau]$ by

$$L_B := \mathcal{C}_\tau(\mathcal{C}_\tau)^*. \tag{1.0.28}$$

Based upon the definition of controllability map, in the case of infinite dimensional systems, two different types of controllability, exact controllability and approximate controllability, are defined as follow.

Definition 1.0.2. *Exact controllability enables to steer the system to arbitrary final state. In particular,* (1.0.26) *is exactly controllable on* $[0, \tau]$ *if every element of* X *can be reached from the origin at time* τ*, equivalently*

$$ran\ \mathcal{C}_\tau = X.$$

Definition 1.0.3. *Approximate controllability means that system can be steered to arbitrary small neighborhood of final state. In particular,* (1.0.26) *is approximately controllable on* $[0, \tau]$*, if for* $\epsilon > 0$*, it is possible to steer from* ξ_0 *to within a distance* ϵ *from all points in the state space* X *at time* τ*, equivalently*

$$\overline{ran\ \mathcal{C}_\tau} = X.$$

In other words, approximate controllability gives the possibility of steering the system to the states which form the dense subspace in the state space.

There are many methods to prove exact controllability. The classical way is given by Curtain and Zwart in [8].

Theorem 1.0.4. *The linear system* (1.0.26) *is exactly controllable on* $[0, \tau]$ *if and only if one of the following conditions hold for some* $\gamma > 0$ *and every* $\xi \in X$*,*

(i) $\langle L_B \xi, \xi \rangle \geq \gamma \|\xi\|_X^2$,

(ii) $\|\mathcal{C}_\tau^* \xi\|_2^2 := \int_0^\tau \|(\mathcal{C}_\tau^* \xi)(s)\|_{H^2(\Gamma_1)}^2 ds \geq \gamma \|\xi\|_X^2$,

(iii) $\int_0^\tau \|B^* S^*(s) \xi\|_{H^2(\Gamma_1)}^2 ds \geq \gamma \|\xi\|_X^2$,

(iv) *ker* $\mathcal{C}_\tau^* = \{0\}$ *and ran* $\mathcal{C}_\tau^*$ *is closed.*

When we prove approximate controllability, the following theorem [8] will be used.

Theorem 1.0.5. *The linear system* (1.0.26) *is approximately controllable on* $[0, \tau]$ *if and only if one of the following conditions holds for some* $\gamma > 0$ *and every* $\xi \in X$*,*

(i) $L_B > 0$,

(ii) $ker \mathcal{C}_\tau^* = \{0\}$,

(iii) $B^* S^*(s) \xi = 0$ *on* $[0, \tau] \Rightarrow \xi = 0$.

In many papers, the controllability map $\mathcal{C}_\tau$ is considered as surjective, (see, e.g., [2, 33]). However it was pointed out in [45, 46] that if X is an infinite dimensional space and S or B is a compact operator, $\mathcal{C}_\tau$ may not be surjective. This is strongly related to the fact that in infinite dimensional spaces there exist linear subspaces, which are not closed. Therefore the exact controllability is too strong while the approximate controllability is more useful. It should be pointed out that exact and approximate controllability coincide when we consider finite dimensional spaces. However, for infinite dimensional space, sometimes we even cannot obtain the approximate controllability. Therefore we define a generic concept of approximate controllability on $[0, \infty)$. This definition is used by Curtain and Zwart in [8].

Definition 1.0.6. *Let us call* $\mathcal{R}$ *the reachable subspace*

$$\mathcal{R} = \bigcup_{\tau > 0} ran \mathcal{C}_\tau.$$

The system is approximately controllable on $[0, \infty)$ *if* $\overline{\mathcal{R}} = X$.

In this dissertation, the action of generator on the left boundary Γ_1 can produce desired velocity $v(y, t)$ which is considered as a boundary control of our system. To specify our problem, we classify the generators into two types: rigid generators and flexible generators.

The flexible generator means the shape of generator can vary. The rigid generator means the shape is fixed which can be described by a function depending on y. Then, the control input can be written as

$$v(y,t) = f(y)u(t),$$

where $u(t)$ means the small angular velocity of rotation of the generator around an axis located at the bottom of the basin. Here, we will see that exact controllability can eventually be obtained if we consider flexible generators. Indeed, given a flexible generator located on Γ_1, one can show that

Theorem 1.0.7. *For any control input $v \in L^2(0,\tau; H^2(\Gamma_1))$, the system (1.0.23) is exactly controllable on X for any $\tau > 0$.*

In the case where the generators are rigid, one can prove that

Theorem 1.0.8. *Suppose that the shape of the generator $f(y)$ is positive. For any control input $u \in L^2(0,\tau; H^2(\Gamma_1))$, one can show that*

(i) If the shape of the generator $f \in H_c^p(\Gamma_1)$, $p > 3/2$, the system (1.0.23) is not exactly controllable on a finite time interval $[0,\tau]$ for any $\tau > 0$;

(ii) The system (1.0.23) is approximately controllable on $[0,\tau]$ for any $\tau > 0$.

Mottelet studied a similar model with boundary condition $\ddot{\psi} + g\partial_y\psi = 0$ on the surface Γ_s ([29, 15]). In our problem the boundary condition $\psi_{tt} + g\partial_y\psi = T\partial_y\psi_{xx}$ is different due to the surface tension. As a consequence, a different operator $\mathcal{A}$ will be defined later. The associated eigenvalues and eigenfunctions corresponding to $\mathcal{A}$ will be studied. We first show that the operator A in (1.0.26) depending on $\mathcal{A}$ is a Riesz-spectral operator which generates a strongly continuous semigroup $S(t)$ and corresponding eigenvectors form a Riesz basis. Note that two types of generators are considered. For the flexible one, we will show that the system is exactly controllable by proving that

$$\int_0^\tau \|B^*S^*\xi\|^2_{H^2(\Gamma_1)} ds \geq \gamma \|\xi\|^2_X$$

holds for some $\gamma > 0$ and every $\xi \in X$. For the rigid one, from the result in [1], we can show that this system is approximately controllable rather than exactly controllable by proving that the map $B^*S^*\xi$ is injective. In the last section, we consider a feedback stabilization problem of (1.0.23) with a rigid generator. When the elevation of the surface $\eta - \eta_{xx}$ is measured at $x = 0$, we can use a static feedback to control the angular velocity of the generator, $u(t)$. This kind of feedback will require enough regularity for $\dot{\psi}$ on Γ_s. We will focus on the special choice of the shape $f(y)$ such that the system is strongly stable.

The two parts of the dissertation are organized as follows. We consider the fifth order KdV type of equations in Chapters 2 to 4. Chapter 2 discusses how we find the control mechanism and the properties of semi-group generated by the operator A. In chapter 3, we study the spectral properties of the operator A. The estimates of solutions for the linear problem are derived. We also give the local well-posedness of the nonlinear problem. The global well-posedness and decay of the small solutions are presented in Chapter 4. From Chapters 5 to 7, we focus on a two-dimensional physical system, a basin with wave generators. In Chapter 5, we reformulate the original model into an abstract wave equation and analyze the spectral properties of the operator of this abstract system. In Chapter 6, the controllability problems for both rigid and flexible generators are studied . Finally, we give a positive result on the strong stability of the system (1.0.23).

Chapter 2

Stability of Solutions for the Linear Fifth-Order Equation

2.1 Dissipation Mechanism

The equation (1.0.2) is known to possess an infinite set of integral quantities which are conserved. In my study, the application-oriented motivation requires us to find a control law such that the volume

$$[w(\cdot,t)] \equiv \int_0^{2\pi} w(x,t)dx \tag{2.1.1}$$

is conserved. We also refer this quantity as conserved volume or mass of fluid. Therefore it is straightforward to assume that the equilibrium state could be a set of constant states which are

$$w(x,t) = \frac{d}{2\pi}, \tag{2.1.2}$$

where $d = [w(\cdot,0)]$. Moreover, one can observe that

$$\begin{aligned}
& \int_0^{2\pi} \left(w(x) - \frac{d}{2\pi} \right)^2 dx \\
= \; & \int_0^{2\pi} w(x)^2 dx - \frac{d}{\pi} \int_0^{2\pi} w(x) dx + \int_0^{2\pi} \frac{d^2}{4\pi^2} dx \\
= \; & \int_0^{2\pi} w(x)^2 dx - \frac{d^2}{\pi} + \frac{d^2}{2\pi} \\
= \; & \int_0^{2\pi} w(x)^2 dx - \frac{d^2}{2\pi} \\
\geq \; & 0,
\end{aligned}$$

which means

$$\|w(\cdot,t)\|^2_{L^2(0,2\pi)} = \int_0^{2\pi} w(x)^2 dx = \frac{d^2}{2\pi} \quad \text{if and only if} \quad w(x) = \frac{d}{2\pi}.$$

Thus for any $w \in L^2(0, 2\pi)$ with $[w] = d$, the state $w(x) = \frac{d}{2\pi}$ has the smallest norm $\frac{d^2}{2\pi}$ in $L^2(0, 2\pi)$. We may expect to use an appropriate feedback control law such that

$$\| w(\cdot, t) - \frac{1}{2\pi}[w(\cdot, t)] \|_{L^2(0,2\pi)} \to 0 \text{ as } t \to \infty. \tag{2.1.3}$$

On the other hand, for any appropriate smooth solutions of (1.0.1), one can see that

$$\begin{aligned} \frac{d}{dt}\int_0^{2\pi} w(x,t)dx &= \int_0^{2\pi} \frac{\partial w}{\partial t}(x,t)dx \\ &= \int_0^{2\pi} \left(\sum_{j=1}^{n} a_j w^j w_x - w_{xxx} + w_{xxxxx} \right) dx \\ &= \int_0^{2\pi} \sum_{j=1}^{n} \left(\frac{a_j}{j+1} w^{j+1} \right)_x dx - w_{xx}|_0^{2\pi} + w_{xxxx}|_0^{2\pi}. \end{aligned} \tag{2.1.4}$$

In order to conserve the quantity of volume, we only need to impose periodic boundary conditions for $k = 0, 2, 4$. Therefore the first and third order boundary conditions are flexible. This property inspires us that we may obtain asymptotically stable result in (2.1.3) through imposing the feedback control inputs on these two boundaries.

To obtain (2.1.3), let us compute the derivative of energy with respect to time. If we denote $w_{kx} = \frac{\partial^k w}{\partial x^k}$, for any appropriate smooth solution $w(x, t)$, we have

$$\begin{aligned} & \frac{d}{dt}\left(\frac{1}{2}\int_0^{2\pi} w(x,t)^2 dx \right) \\ = & \frac{d}{dt}\left\{ \frac{1}{2}\int_0^{2\pi} \left(w(x,t) - \frac{1}{2\pi}[w(x,t)] \right)^2 dx \right\} \\ = & \frac{d}{dt}\left\{ \frac{1}{2}\left(\int_0^{2\pi} w(x,t)^2 dx - \frac{1}{\pi}[w(x,t)]\int_0^{2\pi} w(x,t)dx + \frac{1}{4\pi^2}[w(x,t)]^2 2\pi \right) \right\} \\ = & \frac{d}{dt}\left\{ \frac{1}{2}\left(\int_0^{2\pi} w(x,t)^2 dx - \frac{1}{\pi}[w(x,t)]^2 + \frac{1}{2\pi}[w(x,t)]^2 \right) \right\} \\ = & \int_0^{2\pi} w(x,t) \left(-w_{3x}(x,t) + w_{5x}(x,t) + \sum_{j=1}^{n} a_j w^j w_x \right) dx \\ = & \int_0^{2\pi} w(x,t)w_{5x}(x,t)dx - \int_0^{2\pi} w(x,t)w_{3x}(x,t)dx + \int_0^{2\pi} \sum_{j=1}^{n} a_j w^{j+1} w_x dx, \end{aligned} \tag{2.1.5}$$

where

$$\begin{aligned} \int_0^{2\pi} w(x,t)w_{3x}(x,t)dx &= w(x,t)w_{2x}(x,t)|_0^{2\pi} - \frac{1}{2}\left(w_x(x,t)\right)^2\Big|_0^{2\pi}, \\ \int_0^{2\pi} w(x,t)w_{5x}(x,t)dx &= w(x,t)w_{4x}(x,t)|_0^{2\pi} + \frac{1}{2}\left(w_{2x}(x,t)\right)^2|_0^{2\pi} - w_x(x,t)w^3(x,t)|_0^{2\pi}. \end{aligned}$$

If we enforce periodic boundary conditions

$$w_{kx}(0,t) = w_{kx}(2\pi, t), \qquad \text{for} \quad k = 0, 2, 3, 4 \tag{2.1.6}$$

on (2.1.5), then we can obtain an identity for the solution of (1.0.1),

$$\begin{aligned}
&\frac{d}{dt}\left\{\int_0^{2\pi}\Big(w(x,t)-\frac{1}{2\pi}[w(x,t)]\Big)^2 dx\right\} \\
=\;& \frac{d}{dt}\Big(\int_0^{2\pi} w^2(x,t)dx\Big) \\
=\;& \big(w_x(2\pi,t)-w_x(0,t)\big)\cdot\big(w_x(2\pi,t)+w_x(0,t)-2w_{3x}(0,t)\big).
\end{aligned} \tag{2.1.7}$$

When we let

$$w_x(2\pi,t)=\alpha w_x(0,t)+(1-\alpha)w_{3x}(0,t)\,, \tag{2.1.8}$$

(2.1.7) implies that

$$\frac{d}{dt}\Big(\int_0^{2\pi} w^2(x,t)dx\Big)=-\frac{1+\alpha}{1-\alpha}\big(w_x(2\pi,t)-w_x(0,t)\big)^2.$$

For $|\alpha|<1$, the energy is decreasing as t increases unless $w_x(2\pi,t)=w_x(0,t)$. Therefore, the boundary conditions (2.1.6) and (2.1.8) can be considered as a dissipation mechanism for (1.0.1). Moreover because of this boundary dissipation mechanism, it is reasonable to expect that the solution $w(x,t)$ of (1.0.1) goes to equilibrium state $d=[w]/(2\pi)$ as $t\to+\infty$.

2.2 The Properties of Semi-group $S(t)$ for Operator A

Define an operator A given by

$$Aw=w_{5x}-w_{3x} \tag{2.2.1}$$

with the domain

$$\begin{aligned}
\mathscr{D}(A)=\Big\{w\in H^5(0,2\pi)\ \Big|\ & w_{kx}(0)=w_{kx}(2\pi)\,,\ k=0,2,3,4\,,\\
& w_x(2\pi,t)=\alpha w_x(0,t)+(1-\alpha)w_{3x}(0,t)\,,\ |\alpha|<1\Big\}.
\end{aligned} \tag{2.2.2}$$

Consider the solution of the problem

$$w_t=Aw\,,\quad w(x,0)=\phi(x)\,,\quad w\in\mathscr{D}(A) \tag{2.2.3}$$

and initial value $\phi(x)\in L^2=L^2(0,2\pi)$. Assume the solution of (2.2.3) is

$$w(x,t)=S(t)\phi(x)\,, \tag{2.2.4}$$

where $S(t)$ is the semi-group generated by A in L^2. Then, we have the following Proposition.

Proposition 2.2.1. *The semigroup $S(t)$ defined in* (2.2.3) *and* (2.2.4) *is a strongly continuous semigroup.*

Proof. First, it is straightforward to show that

$$\begin{aligned} 2Re(Aw,w)_{L^2(0,2\pi)} &= (Aw,w)_{L^2(0,2\pi)} + \overline{(Aw,w)}_{L^2(0,2\pi)} \\ &= (w')\,(\bar{w}')\big|_0^{2\pi} - \bar{w}'w^{(3)}\Big|_0^{2\pi} - w'\bar{w}^{(3)}\Big|_0^{2\pi} \\ &= \frac{-4k}{(k+1)^2}\left(|w'(0)|^2 + |w^{(3)}(0)|^2 - w'(0)\bar{w}^{(3)}(0) - w^{(3)}(0)\bar{w}'(0)\right) \\ &= \frac{-4k}{(k+1)^2}\left(w'(0) - w^{(3)}(0)\right)\left(\bar{w}'(0) - \bar{w}^{(3)}(0)\right) \\ &= \frac{-4k}{(k+1)^2}|w'(0) - w^{(3)}(0)|^2 \\ &\leq 0, \end{aligned} \tag{2.2.5}$$

which implies that A is a dissipative operator. Based on this property, for any $\lambda > 0$ and $w \in \mathscr{D}(A)$, one can prove that

$$\begin{aligned} \| (\lambda I - A)w \|^2 &= \lambda^2 \| w \|^2 - \lambda\left[(w, Aw) + (Aw, w)\right] + \| Aw \|^2 \\ &\geq \lambda^2 \| w \|^2 . \end{aligned} \tag{2.2.6}$$

Thus, for any g in the range of $(\lambda I - A)$ with $\lambda > 0$, one can see that

$$\begin{aligned} \| (\lambda I - A)\cdot(\lambda I - A)^{-1}g \| &\geq \lambda \| (\lambda I - A)^{-1}g \| \\ \| (\lambda I - A)^{-1}g \| &\leq \frac{1}{\lambda} \| g \| \\ \frac{\| (\lambda I - A)^{-1}g \|}{\| g \|} &\leq \frac{1}{\lambda}. \end{aligned}$$

If we denote the resolvent of A by $R(\lambda, A) = (\lambda I - A)^{-1}$, we have

$$\| R(\lambda, A) \| \leq \frac{1}{\lambda}, \qquad \lambda > 0. \tag{2.2.7}$$

According to Lumer-Phillips Theorem [32], we also need to show that for some $\lambda > 0$, the range of $\lambda I - A$ is L^2, i.e., for any $f(x) \in L^2$, there exists $w \in \mathscr{D}(A)$ such that $(\lambda I - A)w = f$. In another word, we need to prove that $R(\lambda, A)$ is defined as a bounded linear operator on $L^2(0, 2\pi)$ for $\lambda > 0$. It is necessarily to establish the explicit form of resolvent using the following equation

$$(\lambda I - A)w = f, \qquad f \in L^2(0, 2\pi). \tag{2.2.8}$$

We then transfer this fifth order equation into the first order system by setting

$$w_0' = w_1, \quad w_1' = w_2, \quad w_2' = w_3, \quad w_3' = w_4.$$

Thus we obtain

$$W' = \begin{pmatrix} w_0' \\ w_1' \\ w_2' \\ w_3' \\ w_4' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \lambda & 0 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -f \end{pmatrix} \equiv F(\lambda)W + \varphi.$$

If we denote μ as the eigenvalue of the matrix $F(\lambda)$, we can see that $\mu^5 - \mu^3 = \lambda$. Assume μ_0, μ_1, μ_2, μ_3, μ_4 are five different eigenvalues of $F(\lambda)$ and $[1, \mu_n, \mu_n^2, \mu_n^3, \mu_n^4]^T$ is the eigenvector of $F(\lambda)$ corresponding to the associated eigenvalue μ_n. Thus one can diagonalize the $F(\lambda)$ with the following transformation

$$W = \begin{pmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ \mu_0 & \mu_1 & \mu_2 & \mu_3 & \mu_4 \\ \mu_0^2 & \mu_1^2 & \mu_2^2 & \mu_3^2 & \mu_4^2 \\ \mu_0^3 & \mu_1^3 & \mu_2^3 & \mu_3^3 & \mu_4^3 \\ \mu_0^4 & \mu_1^4 & \mu_2^4 & \mu_3^4 & \mu_4^4 \end{pmatrix} \cdot \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \equiv V(\mu) \cdot \Xi. \tag{2.2.9}$$

Therefore

$$\begin{aligned} & (V(\mu)\Xi)' = F(\lambda)(V(\mu)\Xi) + \varphi \\ \Longrightarrow \quad & V(\mu)^{-1}V(\mu)\Xi' = V(\mu)^{-1}F(\lambda)V(\mu)\Xi + V(\mu)^{-1}\varphi \\ \Longrightarrow \quad & \Xi' = D(\mu)\Xi + \psi, \end{aligned} \tag{2.2.10}$$

where

$$D(\mu) = \begin{pmatrix} \mu_0 & & & & \\ & \mu_1 & & & \\ & & \mu_2 & & \\ & & & \mu_3 & \\ & & & & \mu_4 \end{pmatrix}.$$

Thus the solution of (2.2.10) can be expressed by

$$\Xi(x) = e^{xD(\mu)}\Xi(0) + \int_0^x e^{(x-\xi)D(\mu)}\psi(\xi)d\xi. \tag{2.2.11}$$

The boundary conditions yield

$$W(2\pi) = \begin{pmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix}(2\pi) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 1-\alpha & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix}(0) \equiv B(\alpha) \cdot W(0).$$

Substituting (2.2.9), one notes that these boundary conditions are equivalent to

$$\begin{aligned} V(\mu)\Xi(2\pi) &= B(\alpha)V(\mu)\Xi(0) \\ \Xi(2\pi) &= V(\mu)^{-1}B(\alpha)V(\mu)\Xi(0). \end{aligned} \tag{2.2.12}$$

Let $x = 2\pi$, (2.2.11) give rise to

$$\begin{aligned} \Xi(2\pi) &= e^{2\pi D(\mu)}\Xi(0) + \int_0^{2\pi} e^{(2\pi-\xi)D(\mu)}\psi(\xi)d\xi \\ V(\mu)^{-1}B(\alpha)V(\mu)\Xi(0) - e^{2\pi D(\mu)}\Xi(0) &= \int_0^{2\pi} e^{(2\pi-\xi)D(\mu)}\psi(\xi)d\xi \\ B(\alpha)V(\mu)\Xi(0) - V(\mu)e^{2\pi D(\mu)}\Xi(0) &= V(\mu)\int_0^{2\pi} e^{(2\pi-\xi)D(\mu)}\psi(\xi)d\xi \\ \underbrace{\left(B(\alpha)V(\mu) - V(\mu)e^{2\pi D(\mu)}\right)}_{U(\mu,\alpha)}\Xi(0) &= V(\mu)\int_0^{2\pi} e^{(2\pi-\xi)D(\mu)}\psi(\xi)d\xi. \end{aligned} \tag{2.2.13}$$

Let

$$\Delta(\mu,\alpha) = detU(\mu,\alpha) = det\left(B(\alpha)V(\mu) - V(\mu)e^{2\pi D(\mu)}\right), \tag{2.2.14}$$

Case 1. If $\Delta(\mu,\alpha) = 0$, there exists a none trivial solution for $U(\mu,\alpha)\Xi(0) = 0$ which yields the eigenfunction,

$$\Xi(\mu,\alpha,x) = e^{xD(\mu)}\Xi(0) \tag{2.2.15}$$

of operator A corresponding to eigenvalue λ. Thus $R(\lambda, A)$ does not exist. However, by inequality (2.2.6), we have shown that $\lambda > 0$ cannot be an eigenvalue of our problem. Thus this case cannot happen.

Case 2. If $\Delta(\mu,\alpha) \neq 0$, $\Xi(0)$ can be solved by multiplying the $U(\mu,\alpha)^{-1}$ for the both sides of (2.2.13), which yields

$$\Xi(0) = \underbrace{\left(B(\alpha)V(\mu) - V(\mu)e^{2\pi D(\mu)}\right)^{-1}}_{U(\mu,\alpha)^{-1}} V(\mu)\int_0^{2\pi} e^{(2\pi-\xi)D(\mu)}\psi(\xi)d\xi. \tag{2.2.16}$$

Substituting (2.2.16) into (2.2.11), one can obtain that

$$\Xi(\mu,\alpha,\psi,x) = e^{xD(\mu)}U(\mu,\alpha)^{-1}V(\mu)\int_0^{2\pi} e^{(2\pi-\xi)D(\mu)}\psi(\xi)d\xi + \int_0^{x} e^{(x-\xi)D(\mu)}\psi(\xi)d\xi, \tag{2.2.17}$$

which shows that $(\lambda I - A)w = f$ is solvable for any $f \in L^2$. Thus, the range of $\lambda I - A$ is L^2 and Lumer-Phillips Theorem concludes that A generates a strongly continuous or C_0-semigroup. □

Proposition 2.2.2. *On the imaginary axis, $\lambda = 0$ is the only point such that $R(\lambda, A)$ fails to exist on L^2.*

Proof. Suppose that there were a point $\lambda = i\omega$, $\omega \neq 0$, such that resolvent $R(\lambda, A)$ fails to exist. Thus, $(A - \lambda I)w = 0$ has nontrivial solutions, i.e., $A - \lambda I$ is not injective. A direct calculation shows that

$$\begin{aligned}&(w, Aw) + (Aw, w) = (w, \lambda w) + (\lambda w, w) = (-4k)(k+1)^{-2}|w'(0) - w^{(3)}(0)|^2 \\ &= \bar{\lambda}(w, w) + \lambda(w, w) = 0, \qquad \text{if} \qquad \lambda = i\omega\end{aligned}$$

or

$$w_x(0) = w_{3x}(0) \qquad \text{and} \qquad w_x(0) = w_x(2\pi). \tag{2.2.18}$$

Hence, the solution w is periodic with period 2π and also satisfies the first equation in (2.2.18). Assume that $w(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$. By the orthogonality condition of e^{ikx} in L^2, we have that the eigenfunction must be of form e^{ikx} with $k^5 + k^3 = \omega$ for some integer k. Moreover, for each $\omega \neq 0$, it can be shown that there is only one integer k satisfying this property. If $k \neq 0$, e^{ikx} does not satisfy the first equation in (2.2.18). Thus, $w(x)$ is trivial if $\omega \neq 0$, i.e., $i\omega I - A$ is one to one and onto. As a consequence, we conclude $R(i\omega, A)$ exists for $\omega \neq 0$. On the other hand, when $\lambda = 0$, $w =$ constant which is nontrivial. So $R(\lambda, A)$ fails to exist only at $\lambda = 0$. □

Finally, by (2.2.5) and Proposition 2.2.2, we conclude that the eigenvalues λ of A satisfy

$$\operatorname{Re}\lambda < 0 \quad \text{or} \quad \lambda = 0\,. \tag{2.2.19}$$

If we define a projection operator Π_P in L^2 by

$$\Pi_P w = \frac{1}{2\pi}\int_0^{2\pi} w(x)dx = \frac{1}{2\pi}[w] \qquad \text{for any } w \in L^2\,,$$

one can show the following results on the resolvent operator $R(\lambda, A) = (\lambda I - A)^{-1}$.

Proposition 2.2.3. *$R(\lambda, A)$ satisfies the following properties: (i). $R(\lambda, A)(I-P)$ is bounded as $\lambda \to 0$; (ii). $R(\lambda, A)$ is uniformly bounded for large λ on the imaginary axis. Indeed, $\|R(iw, A)\| = O(\omega^{-4/5})$ as $|\omega| \to \infty$.*

Proof. For claim (i), at beginning, let us set

$$L_0^2(0, 2\pi) = \left\{ f \in L^2(0, 2\pi) \middle| \int_0^{2\pi} f dx = 0 \right\} \quad \text{and} \quad [L^2](0, 2\pi) = \left\{ f \in L^2(0, 2\pi) \middle| f = \frac{d}{2\pi} \right\},$$

as two orthogonal decompositions of the Hilbert space $L^2(0, 2\pi)$ such that

$$L^2(0, 2\pi) = L_0^2(0, 2\pi) + [L^2](0, 2\pi).$$

Let Π_H and Π_P are two orthogonal projections such that for any $f \in L^2(0, 2\pi)$,

$$\Pi_P f \in [L^2](0, 2\pi), \qquad \Pi_H f \in L_0^2(0, 2\pi).$$

In order to prove claim (i), it is sufficient to show that $R(\lambda, A)$ is bounded on $L_0^2(0, 2\pi)$ as $\lambda \to 0$, i.e.,

$$\| R(\lambda, \tilde{A})f \|_{L_0^2(0,2\pi)} \leq K \| f \|_{L_0^2(0,2\pi)},$$

as $\lambda \to 0$, where $\tilde{A}$ is the restriction of A to $L_0^2(0, 2\pi)$. From (2.2.9) and (2.2.17), one obtains

$$\begin{aligned} W(\lambda, \alpha, f, x) &= V(\mu)e^{xD(\mu)}U(\mu,\alpha)^{-1}V(\mu)\int_0^{2\pi} e^{(2\pi-s)D(\mu)}V(\mu)^{-1}\phi(s)ds \\ &\quad +V(\mu)\int_0^x e^{(x-s)D(\mu)}V(\mu)^{-1}\phi(s)ds. \end{aligned} \tag{2.2.20}$$

Since

$$F(\lambda) = V(\mu)D(\mu)V(\mu)^{-1}, \tag{2.2.21}$$

if we let

$$\begin{aligned} T(\lambda,\alpha) &= U(\mu,\alpha)\cdot V(\mu)^{-1} \\ &= \left(B(\alpha)V(\mu) - V(\mu)e^{2\pi D(\mu)}\right)V(\mu)^{-1} \\ &= B(\alpha) - e^{2\pi F(\lambda)}, \end{aligned} \tag{2.2.22}$$

the equation (2.2.20) can be transformed as follows

$$\begin{aligned} W(\lambda, \alpha, f, x) &= V(\mu)e^{xD(\mu)}V(\mu)^{-1}\underbrace{V(\mu)U(\mu,\alpha)^{-1}}_{T(\lambda,\alpha)^{-1}}V(\mu)\int_0^{2\pi} e^{(2\pi-s)D(\mu)}V(\mu)^{-1}\phi(s)ds \\ &\quad +V(\mu)\int_0^x e^{(x-s)D(\mu)}V(\mu)^{-1}\phi(s)ds \\ &= e^{xF(\lambda)}T(\lambda,\alpha)^{-1}\int_0^{2\pi} e^{(2\pi-s)F(\lambda)}\phi(s)ds + \int_0^x e^{(x-s)F(\lambda)}\phi(s)ds, \end{aligned} \tag{2.2.23}$$

where

$$F(\lambda) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \lambda & 0 & 0 & 1 & 0 \end{pmatrix}, \qquad \phi(s) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -f(s) \end{pmatrix}.$$

For any λ in a neighborhood of zero, we note that both $e^{xF(\lambda)}$ and $\int_0^x e^{(x-s)F(\lambda)}\phi(s)ds$ in (2.2.23) are bounded. However, a direct calculation shows that $T(\lambda,\alpha)^{-1}$ is unbounded, which requires us to check whether or not $T(\lambda,\alpha)^{-1}\int_0^{2\pi} e^{(2\pi-s)F(\lambda)}\phi(s)ds$ is bounded. Using integration by parts, we see that

$$\begin{aligned} &\int_0^{2\pi} e^{(2\pi-s)F(\lambda)}\phi(s)ds \\ =\ & \int_0^{2\pi} e^{(2\pi-s)F(\lambda)} d\int_0^s \phi(\sigma)d\sigma \\ =\ & e^{(2\pi-s)F(\lambda)}\int_0^s \phi(\sigma)d\sigma\Big|_0^{2\pi} - \int_0^{2\pi}\left(\int_0^s \phi(\sigma)d\sigma\right)de^{(2\pi-s)F(\lambda)}. \end{aligned} \tag{2.2.24}$$

Since $f \in L_0^2(0, 2\pi)$, i.e., $\int_0^{2\pi} f(x)dx = 0$,

$$\int_0^{2\pi} \phi(\sigma)d\sigma = \int_0^0 \phi(\sigma)d\sigma = 0. \tag{2.2.25}$$

Thus (2.2.24) is equivalent to

$$\begin{aligned} & -\int_0^{2\pi}\left(\int_0^s \phi(\sigma)d\sigma\right) de^{(2\pi-s)F(\lambda)} \\ = & -\int_0^{2\pi}\left(\int_0^s \phi(\sigma)d\sigma\right) e^{(2\pi-s)F(\lambda)}(-F(\lambda))ds \\ = & \ F(\lambda)\int_0^{2\pi} e^{(2\pi-s)F(\lambda)}\Phi(s)ds, \end{aligned} \tag{2.2.26}$$

where

$$\Phi(s) = \int_0^s \phi(\sigma)d\sigma.$$

Then, we obtain that

$$\begin{aligned} & T(\lambda, \alpha)^{-1}\int_0^{2\pi} e^{(2\pi-s)F(\lambda)}\phi(s)ds \\ = & \ T(\lambda, \alpha)^{-1}F(\lambda)\int_0^{2\pi} e^{(2\pi-s)F(\lambda)}\Phi(s)ds. \end{aligned} \tag{2.2.27}$$

Obviously, $\int_0^{2\pi} e^{(2\pi-s)F(\lambda)}\Phi(s)ds$ is bounded and $T(\lambda, \alpha)^{-1}F(\lambda)$ is also bounded when λ is near zero. As a consequence, we conclude that for any $f \in L_0^2(0, 2\pi)$, $w = R(\lambda, A)f(x)$ is bounded as $\lambda \to 0$. Furthermore, $\| R(\lambda, \tilde{A})f \|_{L_0^2(0,2\pi)} \leq K \| f \|_{L^2(0,2\pi)}$ as $\lambda \to 0$ for some positive K which are independent with f.

For claim (ii), the solution w of (2.2.8) can be rewritten in terms of Green's function

$$w(\lambda, \alpha, x) = \int_0^{2\pi} G(\lambda, x, \xi)f(\xi)d\xi. \tag{2.2.28}$$

We will prove that $G(\lambda, x, \xi)$ is bounded as $\lambda \to \infty$ on the imaginary axis. As we know, $G(\lambda, x, \xi)$ is a solution of $(\lambda I - A)w = 0$ if $x \neq \xi$. Thus $G(\lambda, x, \xi)$ satisfies the following equations

$$G_{3x}(\lambda, x, \xi) - G_{5x}(\lambda, x, \xi) + \lambda G(\lambda, x, \xi) = \delta(x - \xi), \tag{2.2.29}$$

$$G_{kx}(\lambda, 2\pi, \xi) = G_{kx}(\lambda, 0, \xi), \quad k = 0, 2, 3, 4 \tag{2.2.30}$$

$$G_x(\lambda, 2\pi, \xi) = \alpha G_x(\lambda, 0, \xi) + (1 - \alpha)G_{3x}(\lambda, 0, \xi). \tag{2.2.31}$$

If we denote five fifth roots of λ by $\mu_0, \mu_1, \mu_2, \mu_3$, and μ_4, $G(\lambda, x, \xi)$ takes the form

$$\begin{aligned} G(\lambda, x, \xi) = & \ c_0e^{\mu_0(x-\xi)} + c_1e^{\mu_1(x-\xi)} + c_2e^{\mu_2(x-\xi)} + c_3e^{\mu_3(x-\xi)} + c_4e^{\mu_4(x-\xi)} \\ & H(x - \xi)(\hat{c}_0e^{\mu_0(x-\xi)} + \hat{c}_1e^{\mu_1(x-\xi)} + \hat{c}_2e^{\mu_2(x-\xi)} + \hat{c}_3e^{\mu_3(x-\xi)} + \hat{c}_4e^{\mu_4(x-\xi)}), \end{aligned} \tag{2.2.32}$$

where $H(x-\xi)$ is the Heaviside function given by

$$H(x-\xi) = \begin{cases} 1, & x > \xi, \\ 0, & x \le \xi. \end{cases} \tag{2.2.33}$$

Recall that $G(\lambda, x, \xi)$ is also a fundamental solution. If we set

$$G(\lambda, x, \xi) = \begin{cases} G_1(\lambda, x, \xi) & x > \xi, \\ G_2(\lambda, x, \xi) & x \le \xi. \end{cases}$$

Then we have

$$\begin{cases} G_1^{(n)}(\lambda, x, \xi) - G_2^{(n)}(\lambda, x, \xi) = 0 & n = 0, 1, 2, 3, \\ G_1^{(n)}(\lambda, x, \xi) - G_2^{(n)}(\lambda, x, \xi) = 1 & n = 4, \end{cases} \tag{2.2.34}$$

which are equivalent to

$$\begin{aligned} \hat{c}_0 + \hat{c}_1 + \hat{c}_2 + \hat{c}_3 + \hat{c}_4 &= 0, \\ \hat{c}_0\mu_0 + \hat{c}_1\mu_1 + \hat{c}_2\mu_2 + \hat{c}_3\mu_3 + \hat{c}_4\mu_4 &= 0, \\ \hat{c}_0\mu_0^2 + \hat{c}_1\mu_1^2 + \hat{c}_2\mu_2^2 + \hat{c}_3\mu_3^2 + \hat{c}_4\mu_4^2 &= 0, \\ \hat{c}_0\mu_0^3 + \hat{c}_1\mu_1^3 + \hat{c}_2\mu_2^3 + \hat{c}_3\mu_3^3 + \hat{c}_4\mu_4^3 &= 0, \\ \hat{c}_0\mu_0^4 + \hat{c}_1\mu_1^4 + \hat{c}_2\mu_2^4 + \hat{c}_3\mu_3^4 + \hat{c}_4\mu_4^4 &= 1. \end{aligned}$$

Direct computation shows that

$$\begin{aligned} \hat{c}_0 &= \frac{1}{(\mu_1-\mu_0)(\mu_2-\mu_0)(\mu_3-\mu_0)(\mu_4-\mu_0)}, \\ \hat{c}_1 &= \frac{1}{(\mu_0-\mu_1)(\mu_2-\mu_1)(\mu_3-\mu_1)(\mu_4-\mu_1)}, \\ \hat{c}_2 &= \frac{1}{(\mu_0-\mu_2)(\mu_1-\mu_2)(\mu_3-\mu_2)(\mu_4-\mu_2)}, \\ \hat{c}_3 &= \frac{1}{(\mu_0-\mu_3)(\mu_1-\mu_3)(\mu_2-\mu_3)(\mu_4-\mu_3)}, \\ \hat{c}_4 &= \frac{1}{(\mu_0-\mu_4)(\mu_1-\mu_4)(\mu_2-\mu_4)(\mu_3-\mu_4)}. \end{aligned} \tag{2.2.35}$$

In the following, we denote $c = (c_0, c_1, c_2, c_3, c_4)^T$. Through boundary conditions (2.2.30) and (2.2.31), we can obtain the identity

$$U(\mu, \alpha)e^{-\mu\xi}c = a(\mu, \xi) = V(\mu)e^{\mu(2\pi-\xi)}\hat{c}(\mu), \tag{2.2.36}$$

where $e^{\mu x} = (e^{\mu_0 x}, e^{\mu_1 x}, e^{\mu_2 x}, e^{\mu_3 x}, e^{\mu_4 x})$ is a diagonal matrix. Based upon the definition in (2.2.13), $U(\mu, \alpha)$ takes the following form

$$\begin{pmatrix} 1-e^{2\pi\mu_0} & 1-e^{2\pi\mu_1} & 1-e^{2\pi\mu_2} & 1-e^{2\pi\mu_3} & 1-e^{2\pi\mu_4} \\ E_0 & E_1 & E_2 & E_3 & E_4 \\ \mu_0^2(1-e^{2\pi\mu_0}) & \mu_1^2(1-e^{2\pi\mu_1}) & \mu_2^2(1-e^{2\pi\mu_2}) & \mu_3^2(1-e^{2\pi\mu_3}) & \mu_4^2(1-e^{2\pi\mu_4}) \\ \mu_0^3(1-e^{2\pi\mu_0}) & \mu_1^3(1-e^{2\pi\mu_1}) & \mu_2^3(1-e^{2\pi\mu_2}) & \mu_3^3(1-e^{2\pi\mu_3}) & \mu_4^3(1-e^{2\pi\mu_4}) \\ \mu_0^4(1-e^{2\pi\mu_0}) & \mu_1^4(1-e^{2\pi\mu_1}) & \mu_2^4(1-e^{2\pi\mu_2}) & \mu_3^4(1-e^{2\pi\mu_3}) & \mu_4^4(1-e^{2\pi\mu_4}) \end{pmatrix} \tag{2.2.37}$$

where $E_k = (1-\alpha)\mu_k^3 + \mu_k(\alpha - e^{2\pi\mu_k})$. Let $b(\mu, \xi) = (b_0, b_1, b_2, b_3, b_4)^T$, where $b_k(\mu, \xi)$ is the determinant of the matrix obtained from $U(\mu, \alpha)$ by replacing the kth column of $a(\mu, \xi) = V(\mu)e^{\mu(2\pi-\xi)}\hat{c}(\mu)$. By Cramer's rule, we can see that

$$\begin{aligned} U(\mu,\alpha)e^{-\mu\xi}c &= a(\mu,\xi) \\ e^{-\mu\xi}c &= \frac{b(\mu,\xi)}{\Delta(\mu,\xi)} \\ &= \hat{a}(\mu,\xi). \end{aligned} \tag{2.2.38}$$

From (2.2.32) and (2.2.38),

$$\begin{aligned} G(\lambda, x, \xi) &= \epsilon^*(e^{\mu(x-\xi)}c + H(x-\xi)e^{\mu(x-\xi)}\hat{c}(\mu)) \\ &= \epsilon^*(e^{\mu x}\hat{a}(\mu,\xi) + H(x-\xi)e^{\mu(x-\xi)}\hat{c}(\mu)), \end{aligned} \tag{2.2.39}$$

if we choose $\epsilon^* = (1,1,1,1,1)$. Recall that only λ on the imaginary axis are considered. Let $\lambda = i\omega$, where $\omega = \rho^5, \rho > 0$. The eigenvalue μ of (2.2.8) asymptotically goes to $\sqrt[5]{\lambda}$ as $\omega \to \infty$. Therefore we set $\mu_0 \approx i\rho, \mu_1 \approx \rho e^{\frac{9}{10}\pi i}, \mu_2 \approx \rho e^{\frac{13}{10}\pi i}, \mu_3 \approx \rho e^{\frac{1}{10}\pi i}$, and $\mu_4 \approx \rho e^{\frac{17}{10}\pi i}$. Obviously the real part of μ_1 and μ_2 are negative, while μ_3 and μ_4 are positive. Thus the vector $a(\mu, \xi)$ asymptotically approaches

$$\begin{pmatrix} \hat{c}_3 e^{\mu_3(2\pi-\xi)} + \hat{c}_4 e^{\mu_4(2\pi-\xi)} \\ \hat{c}_3\mu_3 e^{\mu_3(2\pi-\xi)} + \hat{c}_4\mu_4 e^{\mu_4(2\pi-\xi)} \\ \hat{c}_3\mu_3^2 e^{\mu_3(2\pi-\xi)} + \hat{c}_4\mu_4^2 e^{\mu_4(2\pi-\xi)} \\ \hat{c}_3\mu_3^3 e^{\mu_3(2\pi-\xi)} + \hat{c}_4\mu_4^3 e^{\mu_4(2\pi-\xi)} \\ \hat{c}_3\mu_3^4 e^{\mu_3(2\pi-\xi)} + \hat{c}_4\mu_4^4 e^{\mu_4(2\pi-\xi)} \end{pmatrix} \tag{2.2.40}$$

as $\rho \to \infty$. Let $\beta = 1 - \alpha$, the determinant of $U(\mu, \alpha)$ is given by

$$\Delta(\mu,\alpha) \approx e^{2\pi(\mu_3+\mu_4)}D(\mu), \tag{2.2.41}$$

where

$$D(\mu) = det \begin{pmatrix} 1-e^{2\pi(\mu_0)} & 1 & 1 & 1 & 1 \\ \beta\mu_0^3 + \mu_0(\alpha - e^{2\pi(\mu_0)}) & \beta\mu_1^3 + \alpha\mu_1 & \beta\mu_2^3 + \alpha\mu_2 & \mu_3 & \mu_4 \\ \mu_0^2(1-e^{2\pi(\mu_0)}) & \mu_1^2 & \mu_2^2 & \mu_3^2 & \mu_4^2 \\ \mu_0^3(1-e^{2\pi(\mu_0)}) & \mu_1^3 & \mu_2^3 & \mu_3^3 & \mu_4^3 \\ \mu_0^4(1-e^{2\pi(\mu_0)}) & \mu_1^4 & \mu_2^4 & \mu_3^4 & \mu_4^4 \end{pmatrix}.$$

We can show that

$$\Delta(\mu,\alpha) = e^{2\pi(\mu_3+\mu_4)}O(\rho^{12}). \tag{2.2.42}$$

Similarly, $b_k, k = 0,1,2,3,4$, are given as follows

$$\begin{aligned} b_0 &= -\hat{c}_3 e^{2\pi(\mu_3+\mu_4)-\xi\mu_3} \underbrace{\left(\beta\mu_3^3 + (\alpha-1)\mu_3\right) D_1(\mu)}_{H_{01}(\mu)} \\ &\quad + \hat{c}_4 e^{2\pi(\mu_3+\mu_4)-\xi\mu_4} \underbrace{\left(\beta\mu_4^3 + (\alpha-1)\mu_4\right) D_2(\mu)}_{H_{02}(\mu)}, \end{aligned} \tag{2.2.43}$$

$$\begin{aligned} b_1 &= -\hat{c}_3 e^{2\pi(\mu_3+\mu_4)-\xi\mu_3} \underbrace{\left(\beta\mu_3^3 + (\alpha-1)\mu_3\right)\left(1-e^{2\pi\mu_0}\right) D_3(\mu)}_{H_{11}(\mu)} \\ &\quad + \hat{c}_4 e^{2\pi(\mu_3+\mu_4)-\xi\mu_4} \underbrace{\left(\beta\mu_4^3 + (\alpha-1)\mu_4\right)\left(1-e^{2\pi\mu_0}\right) D_4(\mu)}_{H_{12}(\mu)}, \end{aligned} \tag{2.2.44}$$

$$\begin{aligned} b_2 &= -\hat{c}_3 e^{2\pi(\mu_3+\mu_4)-\xi\mu_3} \underbrace{\left(\beta\mu_3^3 + (\alpha-1)\mu_3\right)\left(1-e^{2\pi\mu_0}\right) D_5(\mu)}_{H_{21}(\mu)} \\ &\quad + \hat{c}_4 e^{2\pi(\mu_3+\mu_4)-\xi\mu_4} \underbrace{\left(\beta\mu_4^3 + (\alpha-1)\mu_4\right)\left(1-e^{2\pi\mu_0}\right) D_6(\mu)}_{H_{22}(\mu)}, \end{aligned} \tag{2.2.45}$$

$$\begin{aligned} b_3 &= -\hat{c}_3 e^{2\pi(\mu_3+\mu_4)-\xi\mu_3} D(\mu) \\ &\quad + \hat{c}_4 e^{2\pi(\mu_4)-\xi\mu_4} \underbrace{\left(\beta\mu_4^3 + (\alpha-1)\mu_4\right)\left(e^{2\pi\mu_0}-1\right) D_7(\mu)}_{H_{32}(\mu)}, \end{aligned} \tag{2.2.46}$$

$$\begin{aligned} b_4 &= -\hat{c}_3 e^{2\pi(\mu_3)-\xi\mu_3} \underbrace{\left(\beta\mu_3^3 + (\alpha-1)\mu_3\right)\left(e^{2\pi\mu_0}-1\right) D_8(\mu)}_{H_{41}(\mu)} \\ &\quad - \hat{c}_4 e^{2\pi(\mu_3+\mu_4)-\xi\mu_4} D(\mu), \end{aligned} \tag{2.2.47}$$

where $D_k(\mu), k = 1,2,...,8$, are equivalent to $O(\rho^9)$. Thus

$$\hat{a}(\mu,\xi) \approx -\hat{c}_3 e^{-\xi\mu_3} \underbrace{\begin{pmatrix} H_{01}(\mu)/D(\mu) \\ H_{11}(\mu)/D(\mu) \\ H_{21}(\mu)/D(\mu) \\ 1 \\ 0 \end{pmatrix}}_{h_1} + \hat{c}_4 e^{-\xi\mu_4} \underbrace{\begin{pmatrix} H_{02}(\mu)/D(\mu) \\ H_{12}(\mu)/D(\mu) \\ H_{22}(\mu)/D(\mu) \\ 0 \\ -1 \end{pmatrix}}_{h_2}. \tag{2.2.48}$$

In order to estimate $G(\lambda, x, \xi)$ in (2.2.39), we categorize the problems into following two cases:

Case I: For $x \leq \xi$,

$$G(\lambda, x, \xi) = \epsilon^* e^{\mu x} \hat{a}(\mu, \xi),$$

where

$$e^{\mu x}\hat{a}(\mu,\xi) = \begin{pmatrix} -\hat{c}_3 e^{-\xi\mu_3} e^{\mu_0 x} \frac{H_{01}}{D} + \hat{c}_4 e^{-\xi\mu_4} e^{\mu_0 x} \frac{H_{02}}{D} \\ -\hat{c}_3 e^{-\xi\mu_3} e^{\mu_1 x} \frac{H_{11}}{D} + \hat{c}_4 e^{-\xi\mu_4} e^{\mu_1 x} \frac{H_{12}}{D} \\ -\hat{c}_3 e^{-\xi\mu_3} e^{\mu_2 x} \frac{H_{21}}{D} + \hat{c}_4 e^{-\xi\mu_4} e^{\mu_2 x} \frac{H_{22}}{D} \\ -\hat{c}_3 e^{-\xi\mu_3} e^{\mu_3 x} \\ +\hat{c}_4 e^{-\xi\mu_4} e^{\mu_4 x} \end{pmatrix}.$$

Observe that, since the vector h_1 and h_2 are both bounded as $\rho \to \infty$, we can claim that $e^{\mu x}\hat{a}(\mu,\xi)$ is uniformly bounded. From (2.2.35), note that $|\hat{c}_k(\mu)| \approx \frac{1}{\rho^4}$. As a consequence, there exists a constant M, independent of ρ, such that

$$|\epsilon^* e^{\mu x}\hat{a}(\mu,\xi)| \leq \frac{M}{\rho^4}, \quad x \leq \xi. \tag{2.2.49}$$

Case II: For $x > \xi$,

$$G(\lambda, x, \xi) = \epsilon^* e^{\mu(x-\xi)}\hat{c}(\mu) + \epsilon^* \left(-e^{\mu x} e^{-\xi\mu_3}\hat{c}_3(\mu) h_1(\mu) + e^{\mu x} e^{-\xi\mu_4}\hat{c}_4(\mu) h_2(\mu)\right). \tag{2.2.50}$$

For this equation, when $k = 0, 1, 2$, $e^{\mu_k(x-\xi)}$, $e^{\mu_k x}e^{-\xi\mu_3}$ and $e^{\mu_k x}e^{-\xi\mu_4}$ are uniformly bounded for $x > \xi$. When $k = 3$,

$$e^{\mu_3(x-\xi)}\hat{c}_3(\mu) - \left(e^{\mu_3 x}e^{-\xi\mu_3}\hat{c}_3(\mu)\right) = 0.$$

When $k = 4$

$$e^{\mu_4(x-\xi)}\hat{c}_4(\mu) - \left(e^{\mu_4 x}e^{-\xi\mu_4}\hat{c}_4(\mu)\right) = 0.$$

Therefore, $G(\lambda, x, \xi)$ is also bounded with property

$$|G(\lambda, x, \xi)| \leq \frac{\hat{M}}{\rho^4}. \tag{2.2.51}$$

Recall that $\omega = \rho^5$, which implies

$$\|R(i\omega, A)\| = O(\omega^{-4/5}) \quad as \quad |\omega| \to \infty. \tag{2.2.52}$$

□

From a theorem given by Huang [13], one can easily show that the operator $S(t) - P$ has uniform exponential decay property as $t \to +\infty$, i.e.,

$$\| S(t) - P \|_{L^2(0,2\pi)} \leq M e^{-\gamma t}, \quad \gamma > 0. \tag{2.2.53}$$

Furthermore, we can show that

Theorem 2.2.4. *The semigroup $S(t)$ generated by the operator A is also a strongly differentiable semigroup in L^2.*

Proof. In proposition (2.2.3), we have already shown that $S(t)$ is a strongly continuous semigroup which exponential decays to the constant state. Combining with (2.2.52), It is easily to show that

$$\begin{aligned}
&\lim_{|\omega|\to\infty} \sup \log |\omega| \| R(\mu + i\omega : A) \| \\
= &\lim_{|\omega|\to\infty} \frac{\sup \log |\omega|}{O(\omega^{4/5})} \\
= &\lim_{|\omega|\to\infty} \frac{\sup \frac{1}{|\omega|}}{O(4/5\omega^{-1/5})} \\
= &\lim_{|\omega|\to\infty} \sup \frac{O(5/4\omega^{1/5})}{|w|} \\
= &\ 0.
\end{aligned}$$

By Corollary 2.5 in [32], one can prove that $S(t)$ is a strongly differentiable semigroup or C^∞ semigroup. □

Chapter 3

Local Well-Posedness of the Nonlinear Problem

3.1 Spectral Properties of Linear Operator A

Now, we discuss the asymptotic form of the eigenvalues of the operator A defined by (2.2.1) and (2.2.2) and show the corresponding eigenfunctions form a Riesz basis for L^2. The adjoint operator A^* of A takes the form

$$A^* v = -v_{xxxxx} + v_{xxx} \tag{3.1.1}$$

with boundary conditions

$$v_{kx}(2\pi, t) = v_{kx}(0, t),\ k = 0, 2, 3, 4, \quad v'(0, t) = \alpha v'(2\pi, t) + (1 - \alpha)v^{(3)}(2\pi, t)\,. \tag{3.1.2}$$

In the following, we let $\phi \in L^2$ and denote ϕ^* as the corresponding adjoint vector of ϕ in the Hilbert space L^2. For any ϕ and ψ in L^2, we have

$$(\phi, \psi)_{L^2} = \psi^* \phi. \tag{3.1.3}$$

Proposition 3.1.1. *The operator A defined by* (2.2.1) *and* (2.2.2) *with* $|\alpha| < 1$ *is a discrete spectral operator and all but a finite number of eigen-spaces are one-dimensional.*

Proof. First, let us define B_i: a set of linearly independent boundary values for A

$$B_i(f) = \sum_{j=0}^{n-1} \alpha_{ij} f^j(0) + \sum_{j=0}^{n-1} \beta_{ij} f^j(2\pi) \quad i = 1, ..., n.$$

Let $\mathscr{D}(A)$ be the domain of A with

$$\mathscr{D}(A) = \{w \in H^5(0, 2\pi) \;\Big|\; B_j(w) = 0, \quad j = 1, ..., 5\}, \tag{3.1.4}$$

where

$$\begin{aligned} B_1(w) &= w(2\pi) - w(0) = 0, \\ B_3(w) &= w'(2\pi) - \alpha w'(0) - (1-\alpha)w^{(3)}(0) = 0, \\ B_2(w) &= w''(2\pi) - w''(0) = 0, \\ B_4(w) &= w'''(2\pi) - w'''(0) = 0, \\ B_5(w) &= w^{(4)}(2\pi) - w^{(4)}(0) = 0. \end{aligned} \tag{3.1.5}$$

Note that the sum of the orders of $B_j, j = 1, \ldots 5$ is 12. Since the order of the operator A is 5, write $5 = 2v + 1$ with $v = 2$. We denote $w_j, j = 0, ..., 4$ as the fifth-roots of unity with

$$w_0 = 1, \quad w_1 = e^{\frac{2}{5}\pi i}, \quad w_2 = e^{\frac{4}{5}\pi i} \quad w_3 = e^{\frac{6}{5}\pi i}, \quad w_4 = e^{\frac{8}{5}\pi i}. \tag{3.1.6}$$

$\{w_j\}$ are enumerated in such a way that $w_0 = 1$, the imaginary part of w_j is positive for $0 < j \leq 2$ and negative for $2 < j \leq 4$. Let $\mu = \mu(\lambda)$ denote the unique root of $\mu^5 - \mu^3 = \lambda$ which lies in the sector $\pi/10 \geq \arg\mu - \pi > (-\pi)/10$ of the complex plane for λ large. Asymptotically for λ large, define

$$\sigma_k(x,\mu) = \begin{cases} e^{i\mu w_k x} + o(1), & 0 \leq k \leq 2, \\ e^{i\mu w_k (x-2\pi)} + o(1), & 3 \leq k \leq 4, \end{cases} \tag{3.1.7}$$

where $\sigma_0(x, \mu(\lambda)), ..., \sigma_4(x, \mu(\lambda))$ corresponding to five roots of the characteristic equation are a fundamental set of solutions of $Aw = \lambda w$. Denote

$$B_j(\sigma_k(x,\mu)) = M_{jk}(\mu), \quad M(\mu) = \det(M_{jk}(\mu)).$$

It is evident from the form (3.1.5) of B_j and the form (3.1.7) of $\sigma_k(x,\mu)$ that $M_{jk}(\mu)$ has the leading-order terms

$$\begin{aligned} M_{jk}(\mu) &= P_{jk}(\mu) + Q_{jk}(\mu)e^{2\pi i w_k \mu}, \quad 0 \leq k \leq 2, \\ M_{jk}(\mu) &= P_{jk}(\mu) + Q_{jk}(\mu)e^{-2\pi i w_k \mu}, \quad 3 \leq k \leq 4, \end{aligned}$$

where P_{jk} and Q_{jk} are polynomials in μ with orders at most m_j for all $1 \leq j \leq 5, 0 \leq k \leq 4$. If we define

$$\begin{aligned} N_{j0}(\mu) &= M_{j0}(\mu) = P_{j0}(\mu) + Q_{j0}(\mu)e^{2\pi i \mu}, \\ N_{jk}(\mu) &= P_{jk}(\mu), \quad 0 < k \leq 4 \end{aligned}$$

and let $E_k = (1-\alpha)\mu^3 w_k^3 j$ for $j = 0, 1, 2, 3, 4$, a direct computation shows that

$$\begin{aligned} N(\mu) &= \det \begin{pmatrix} e^{i\mu 2\pi} - 1 & -1 & -1 & 1 & 1 \\ i\mu\left(-\alpha + e^{i\mu 2\pi}\right) + E_0 & -\alpha\mu w_1 i + E_1 & -\alpha\mu w_2 i + E_2 & i\mu w_3 & i\mu w_4 \\ \mu^2(1 - e^{i\mu 2\pi}) & \mu^2 w_1^2 & \mu^2 w_2^2 & -\mu^2 w_3^2 & -\mu^2 w_4^2 \\ i\mu^3(1 - e^{i\mu 2\pi}) & i\mu^3 w_1^3 & i\mu^3 w_2^3 & -i\mu^3 w_3^3 & -i\mu^3 w_4^3 \\ \mu^4(e^{i\mu 2\pi} - 1) & -\mu^4 w_1^4 & -\mu^4 w_2^4 & \mu^4 w_3^4 & \mu^4 w_4^4 \end{pmatrix} \\ &\approx \mu^{12} \cdot (-5)(\alpha - 1)(w_1 - 1)^2 w_1^2 e^{2\pi\mu i} + \mu^{12} \cdot (-5)(\alpha - 1)(w_1 - 1)^2 w_1. \end{aligned}$$

Hence

$$\begin{aligned}
\pi_1(\mu) &= \mu^{12}\cdot(-5)(\alpha-1)(w_1-1)^2 w_1^2,\\
\pi_2(\mu) &= \mu^{12}\cdot(-5)(\alpha-1)(w_1-1)^2 w_1,
\end{aligned}$$

which yield that both π_1 and π_2 have the precise order 12 and the boundary conditions $B_j(w), j=1,\cdots,5$ satisfy the first regularity hypothesis ([10] pp. 2336). If we let

$$\begin{aligned}
\hat{N}_{j0}(\mu) &= M_{j0}(\mu)=P_{j0}(\mu)+Q_{j0}(\mu)e^{2\pi i\mu},\\
\hat{N}_{jk}(\mu) &= Q_{jk}(\mu), \quad 0<k\le 4,
\end{aligned}$$

then

$$\begin{aligned}
\hat{N}(\mu) &= \det\begin{pmatrix}
e^{i\mu 2\pi}-1 & 1 & 1 & -1 & -1\\
i\mu\left(-\alpha+e^{i\mu 2\pi}\right)+E_0 & i\mu w_1 & i\mu w_2 & -\alpha\mu w_3 i+E_3 & -\alpha\mu w_4 i+E_4\\
\mu^2(1-e^{i\mu 2\pi}) & -\mu^2 w_1^2 & -\mu^2 w_2^2 & \mu^2 w_3^2 & \mu^2 w_4^2\\
i\mu^3(1-e^{i\mu 2\pi}) & -i\mu^3 w_1^3 & -i\mu^3 w_2^3 & i\mu^3 w_3^3 & i\mu^3 w_4^3\\
\mu^4(e^{i\mu 2\pi}-1) & \mu^4 w_1^4 & \mu^4 w_2^4 & -\mu^4 w_3^4 & -\mu^4 w_4^4
\end{pmatrix}\\
&\approx \mu^{12}\cdot(-5)(\alpha-1)(w_1-1)^2 w_1 e^{2\pi\mu i}+\mu^{12}\cdot(-5)(\alpha-1)(w_1-1)^2 w_1^2 .
\end{aligned}$$

Hence

$$\begin{aligned}
\hat{\pi}_1(\mu) &= \mu^{12}\cdot(-5)(\alpha-1)(w_1-1)^2 w_1 ,\\
\hat{\pi}_2(\mu) &= \mu^{12}\cdot(-5)(\alpha-1)(w_1-1)^2 w_1^2 ,
\end{aligned}$$

which imply that the polynomials $\hat{\pi}_1$ and $\hat{\pi}_2$ have the precise order 12 and the boundary conditions $B_j(w), j=1,\cdots,5$ also satisfy the second regularity hypothesis ([10], pp. 2337). Thus, if $\alpha\neq 1$, both hypotheses 9 and 10 in ([10], pp. 2336) are satisfied (actually, $|\alpha|<1$ here). Following the Theorem 13 ([10], pp. 2341), the operator A is a discrete spectral operator and the eigen-space of A for large eigenvalue λ is one dimensional. □

Proposition 3.1.2. *For $|\alpha|<1$, the resolvents of the operators A, A^* are compact and their eigenfunctions*

$$\{\phi_k \;\big|\; -\infty<k<+\infty\}, \qquad \{\psi_k \;\big|\; -\infty<k<+\infty\},$$

are complete and form dual Riesz bases for L^2, respectively, satisfying

$$(\phi_k,\psi_j)_{L^2}=\psi_j^*\phi_k=\delta_{k,j} .$$

The asymptotic form of the eigenvalues of A is

$$\lambda_k = k^5 i + 5k^2 r, \quad \textit{as} \quad k\to\infty$$

with

$$r=\frac{1+\cos(2\pi/5)}{2\pi}\cdot\frac{5(\alpha+1)}{\alpha-1}<0 .$$

Proof. Obviously, $\lambda = 0$ is an eigenvalue for both cases. Without loss of generality, we can set $\phi_0 \equiv (2\pi)^{-1/2}$, and $\psi_0 \equiv (2\pi)^{-1/2}$ as eigenfunctions of A, A^* respectively, corresponding to $\lambda_0 = 0$. For any eigenvalue λ with $\operatorname{Im}\lambda > 0$, we denote the five roots of $\mu^5 - \mu^3 = \lambda$ by $\mu_0, \mu_1, \mu_2, \mu_3, \mu_4$, where $\pi/2 < \arg(\mu_0) < (3\pi)/5$, i.e., μ_0 has the minimum real part among the five roots and asymptotically

$$\begin{array}{lcll} \mu_1 & \sim & e^{\frac{2}{5}\pi i}\cdot\mu_0 = w_1\mu_0, & \operatorname{Re}\mu_1 < 0, \\ \mu_2 & \sim & w_1^2\cdot\mu_0 = w_2\mu_0, & \operatorname{Re}\mu_2 < 0, \\ \mu_3 & \sim & w_1^3\cdot\mu_0 = w_3\mu_0, & \operatorname{Re}\mu_3 > 0, \\ \mu_4 & \sim & w_1^4\cdot\mu_0 = w_4\mu_0, & \operatorname{Re}\mu_4 > 0. \end{array} \tag{3.1.8}$$

To find the solutions of

$$A\phi(x) \;=\; \lambda\phi(x) \;\text{ or }\; \phi_{xxxxx}(x) - \phi_{xxx}(x) - \lambda\phi(x) = 0\,,$$

note that $\mu_j, j = 0, 1 \ldots 4$ are almost equal to $\sqrt[5]{\lambda}$ as $\lambda \to \infty$ and the general solution for ϕ is

$$\phi(x) = c_0e^{\mu_0x} + c_1e^{\mu_1x} + c_2e^{\mu_2x} + c_3e^{\mu_3(x-2\pi)} + c_4e^{\mu_4(x-2\pi)}\,. \tag{3.1.9}$$

Substituting (3.1.9) into the boundary conditions, since $e^{\mu_1}, e^{\mu_2}, e^{-\mu_3}, e^{-\mu_4} \to 0$ as $\lambda \to \infty$, we obtain a system

$$\underbrace{\begin{pmatrix} e^{\mu_0 2\pi} - 1 & -1 & -1 & 1 & 1 \\ \mu_0\left(\alpha - e^{\mu_0 2\pi}\right) + (1-\alpha)\mu_0^3 & \alpha\mu_1 + (1-\alpha)\mu_1^3 & \alpha\mu_2 + (1-\alpha)\mu_2^3 & -\mu_3 & -\mu_4 \\ \mu_0^2(e^{\mu_0 2\pi} - 1) & -\mu_1^2 & -\mu_2^2 & \mu_3^2 & \mu_4^2 \\ (e^{\mu_0 2\pi} - 1)\mu_0^3 & -\mu_1^3 & -\mu_2^3 & \mu_3^3 & \mu_4^3 \\ (e^{\mu_0 2\pi} - 1)\mu_0^4 & -\mu_1^4 & -\mu_2^4 & \mu_3^4 & \mu_4^4 \end{pmatrix}}_{M_1} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = 0\,.$$

By (3.1.8), we find

$$\begin{aligned} \det(M_1) \approx\; & \mu_0^{12}\cdot 5(\alpha-1)(w_1-1)^2w_1^2e^{2\pi\mu_0} + \mu_0^{12}\cdot 5(\alpha-1)(w_1-1)^2w_1 \\ & + \mu_0^{10}\cdot 5(3+2\alpha)(w_1-1)^2w_1(w_1+1)e^{2\pi\mu_0} + \mu_0^{10}\cdot(-5)(2+3\alpha)(w_1-1)^2w_1(w_1+1). \end{aligned} \tag{3.1.10}$$

Considering the leading-order terms with respect to μ_0, we have

$$\begin{gathered} \mu_0^{12}\cdot 5(\alpha-1)(w_1-1)^2w_1^2e^{2\pi\mu_0} + \mu_0^{12}\cdot 5(\alpha-1)(w_1-1)^2w_1 = 0 \\ \text{or }\; e^{2\pi\mu_0} = -\frac{1}{w_1} \quad\Longrightarrow\quad e^{2\pi\mu_0} = e^{\frac{3}{5}\pi i} \quad\Longrightarrow\quad \mu_{0,k} = i(k + \frac{3}{10})\,. \end{gathered}$$

Denote

$$\hat{\mu}_{0,k} = i(k + \frac{3}{10} + \varepsilon_k)$$

and substitute $\hat{\mu}_{0,k}$ back to (3.1.10) to obtain

$$\begin{aligned} \hat{\mu}_{0,k}^2(\alpha-1)(w_1 e^{2\pi\hat{\mu}_{0,k}}+1) &= (w_1+1)\left((3\alpha+2)-(2\alpha+3)e^{2\pi\hat{\mu}_{0,k}}\right) \\ \Longrightarrow \quad (\alpha-1)(w_1 e^{2\pi\hat{\mu}_{0,k}}+1) &= \frac{1}{\hat{\mu}_{0,k}^2}(w_1+1)\left((3\alpha+2)-(2\alpha+3)e^{2\pi\hat{\mu}_{0,k}}\right). \end{aligned} \tag{3.1.11}$$

Expanding (3.1.11) with respect to ϵ_k near zero, we can obtain

$$\epsilon_k = \frac{5(1+\alpha)(1+e^{2/5\pi i})}{\left[\frac{1}{50}(\alpha-1)(3+10k)^2-4(3+2\alpha)\right]\pi i}\,.$$

Therefore,

$$\begin{aligned} \mathrm{Re}\,(i\epsilon_k) &= \frac{5(1+\alpha)(1+\cos(2\pi)/5)}{\left[\frac{1}{50}(\alpha-1)(3+10k)^2-4(3+2\alpha)\right]\pi} \\ &\approx \frac{1+\cos((2\pi)/5)}{2\pi}\cdot\frac{5(\alpha+1)}{\alpha-1}\cdot\frac{1}{k^2}, \quad \text{as} \quad k\to\infty. \end{aligned} \tag{3.1.12}$$

Substituting (3.1.12) into $\hat{\mu}_{0,k}$, we have that

$$\hat{\mu}_{0,k} \approx ik + r\frac{1}{k^2}$$

where

$$r = \frac{1+\cos((2\pi)/5)}{2\pi}\cdot\frac{5(\alpha+1)}{\alpha-1}. \tag{3.1.13}$$

Since $|\alpha|<1$ and $r<0$, the eigenvalues of A take the form

$$\begin{aligned} \lambda_k &= \hat{\mu}_{0,k}^5 = (k^5+\frac{10r^2}{k}+\frac{5r^4}{k^7})i + (5k^2r-\frac{10r^3}{k^4}+\frac{r^5}{k^{10}}) \\ &\approx k^5 i + 5k^2 r \quad as \quad k\to\infty\,. \end{aligned} \tag{3.1.14}$$

Using Rouché's Theorem, we can establish one to one relationship between the eigenvalues λ_k and the indices k. Using a similar argument, we obtain that $\overline{\lambda_k}$, the eigenvalue of adjoint operator A^*, is the conjugate of λ_k. As a consequence, the eigenfunctions of A corresponding to λ_k take the forms

$$\begin{aligned} &\phi_0(x) = c_0 = (2\pi)^{-1/2}\,, \\ &\phi_k(x) = c_{k,0}e^{\mu_{k,0}x} + c_{k,1}e^{\mu_{k,1}x} + c_{k,2}e^{\mu_{k,2}x} + c_{k,3}e^{\mu_{k,3}(x-2\pi)} + c_{k,4}e^{\mu_{k,4}(x-2\pi)},\ k\neq 0\,. \end{aligned} \tag{3.1.15}$$

From (3.1.14), it is deduced that $\mathrm{Re}\,\lambda_k$ becomes very large as $k\to\infty$. By neglecting exponentially small terms, we can obtain approximate relationships among $(c_0, c_1, c_2, c_3, c_4)$,

$$\begin{aligned}
c_{k,1} &= \frac{4+6e_1+8e_2+5e_3+2e_4}{e_1-e_2-3e_3+3}\cdot c_{k,0}[e^{2\pi\mu_0}-1],\\
c_{k,2} &= -\frac{4+2e_1+5e_2+8e_3+6e_4}{e_1-e_2-3e_3+3}\cdot c_{k,0}[e^{2\pi\mu_0}-1],\\
c_{k,3} &= \frac{2+5e_1+8e_2+6e_3+4e_4}{e_1-e_2-3e_3+3}\cdot c_{k,0}[e^{2\pi\mu_0}-1],\\
c_{k,4} &= -\frac{5+2e_1+4e_2+6e_3+8e_4}{e_1-e_2-3e_3+3}\cdot c_{k,0}[e^{2\pi\mu_0}-1],
\end{aligned}$$

where

$$e_1 = \mu_1/\mu_0, \quad e_2 = \mu_2/\mu_0, \quad e_3 = \mu_3/\mu_0, \quad e_4 = \mu_4/\mu_0\,.$$

Indeed, these relationships are asymptotically valid as $|k| \to \infty$. From these relationships, we conclude that $c_{k,1}, c_{k,2}, c_{k,3}$ and $c_{k,4}$ are uniformly bounded relative to $c_{k,0}$ as $|k| \to \infty$. If we set the eigenfunction of adjoint operator A^* as $\psi_k(x)$, we have that

$$\psi_k(x) = \overline{\phi_k(2\pi - x)}, \quad -\infty < k < \infty\,. \tag{3.1.16}$$

From (3.1.15) and the boundedness of $c_{k,i},\ i = 1,2,3,4$ relative to $c_{k,0}$, we may normalize $\phi_k(x)$ by choosing appropriate coefficient $c_{k,0}$. A similar argument can be applied to $\psi_k(x)$, so that for any $\phi_k, k = 0, 1, \ldots$ there exist unique bi-orthogonal elements ψ_j such that

$$\psi_j^*\phi_k \equiv (\phi_k, \psi_j)_{L^2} = \begin{cases} 1, & k = j, \\ 0, & k \neq j, \end{cases} \qquad -\infty < k, j < \infty\,. \tag{3.1.17}$$

Since both A and its adjoint A^* are discrete spectral operators, based upon the Carleson theory [12], the corresponding eigenfunctions ϕ_k and ψ_k have the uniform l^2-convergence property, i.e., for any square-summable sequence of complex numbers $\{f_k\} \in l^2$ or $\{g_j\} \in l^2$, there is a positive number D independent of the complex coefficient sequence $\{f_k\}$ or $\{g_j\}$ such that

$$\left\| \sum_{k=-\infty}^{\infty} f_k\phi_k \right\|_{L^2}^2 \leq D^2 \sum_{k=-\infty}^{\infty} |f_k|^2, \tag{3.1.18}$$

$$\left\| \sum_{j=-\infty}^{\infty} g_j\psi_j \right\|_{L^2}^2 \leq D^2 \sum_{j=-\infty}^{\infty} |g_j|^2. \tag{3.1.19}$$

By Cauchy-Schwartz inequality, we have

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} |f_k|^2 &= \sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} |f_k|^2(\phi_k, \psi_j)_{L^2} \\
&= \left(\sum_{k=-\infty}^{\infty} f_k\phi_k, \sum_{j=-\infty}^{\infty} f_j\psi_j \right)_{L^2} \leq \left\| \sum_{k=-\infty}^{\infty} f_k\phi_k \right\|_{L^2} \left\| \sum_{j=-\infty}^{\infty} f_j\psi_j \right\|_{L^2}.
\end{aligned} \tag{3.1.20}$$

Replacing g_j by f_j in (3.1.19) and applying it back to (3.1.20), we obtain

$$\left(\sum_{k=-\infty}^{\infty}|f_k|^2\right)^2 \leq \left\|\sum_{k=-\infty}^{\infty} f_k\phi_k\right\|_{L^2}^2 \left\|\sum_{j=-\infty}^{\infty} f_j\psi_j\right\|_{L^2}^2 \leq \left\|\sum_{k=-\infty}^{\infty} f_k\phi_k\right\|_{L^2}^2 D^2 \sum_{j=-\infty}^{\infty}|f_j|^2,$$

which gives

$$\left\|\sum_{k=-\infty}^{\infty} f_k\phi_k\right\|_{L^2}^2 \geq \frac{1}{D^2}\sum_{k=-\infty}^{\infty}|f_k|^2. \tag{3.1.21}$$

A similar argument to $\left\|\sum\limits_{j=-\infty}^{\infty} g_j\psi_j\right\|_{L^2}^2$ yields

$$\left\|\sum_{j=-\infty}^{\infty} g_j\psi_j\right\|_{L^2}^2 \geq \frac{1}{D^2}\sum_{j=-\infty}^{\infty}|g_j|^2. \tag{3.1.22}$$

From (3.1.21) and (3.1.22), we proved that the sequences ϕ_k and ψ_k also have the uniform l^2-independent property. In addition, both ϕ_k and ψ_k are complete in L^2, and thus we can conclude that $\{\phi_k\}$, $\{\psi_j\}$ are two Riesz bases in L^2. □

Next, we consider the similar properties for $\phi_k^{(n)}$ and $\psi_k^{(n)}$. First, recall that a complex sequence $\{f_k\} \in l_n^2$ if $\sum\limits_{k=-\infty}^{\infty}\left((1+|k|^n)|f_k|\right)^2 < \infty$.

Proposition 3.1.3. $\left\{\frac{\phi_k^{(n)}}{k^n}\right\}_{k=-\infty}^{\infty}$ *is uniformly l_n^2-convergent in $L^2(0,2\pi)$, i.e., for any sequence of complex numbers $\{f_k\} \in l_n^2$,*

$$\left\|\sum_{k=-\infty}^{\infty} f_k\phi_k^{(n)}\right\|_{L^2}^2 = \left\|\sum_{k=-\infty}^{\infty} k^n f_k\frac{\phi_k^{(n)}}{k^n}\right\|_{L^2}^2 \leq D_n^2\sum_{k=-\infty}^{\infty}\Big((1+|k|^n)|f_k|\Big)^2. \tag{3.1.23}$$

Proof. From the previous argument, we see $e^{\mathrm{Re}\,(\mu_{k,j})x}$ for $j=0,1,2$ and $e^{\mathrm{Re}\,(\mu_{k,j})(x-2\pi)}$ for $j=3,4$ are bounded and differentiable for all bounded positive or negative x. Since $\mu_{k,j}$ are proportional in magnitude to $|k|$ as $|k|\to\infty$, through Ingham-Komornik result in [21] (or the proof of the result) we have

$$\begin{aligned}\left\|\sum_{k=-\infty}^{\infty} f_k\phi_k^{(n)}\right\|_{L^2}^2 &= \left\|\sum_{k=-\infty}^{\infty} f_k\left(c_{k,0}\mu_{k,0}^n e^{\mu_{k,0}x} + \ldots + c_{k,4}\mu_{k,4}^n e^{\mu_{k,4}(x-2\pi)}\right)\right\|_{L^2}^2 \\ &\leq D_n^2\sum_{k=-\infty}^{\infty}\Big|(1+|k|^n)f_k\Big|^2.\end{aligned}$$

For $\left\{\psi_j^{(n)}\right\}$, we have the similar property

$$\left\|\sum_{j=-\infty}^{\infty} g_j \psi_j^{(n)}\right\|_{L^2}^2 \leq D_n^2 \sum_{j=-\infty}^{\infty} |(1+|j|^n) g_j|^2. \tag{3.1.24}$$

Therefore, both $\left\{\frac{\phi_k^{(n)}}{k^n}\right\}$ and $\left\{\frac{\psi_j^{(n)}}{j^n}\right\}$ are uniform l_n^2-convergent in L^2. □

Proposition 3.1.4. $\left\{\frac{\phi_k^{(n)}}{k^n}\right\}$ *is also uniform l_n^2-independent in L^2 for $n \geq 1$, i.e., there exists a positive $\hat{D}_n^2$ such that for any sequence of complex numbers $\{f_k\} \in l_n^2$,*

$$\left\|\sum_{k=-\infty}^{\infty} f_k \phi_k\right\|_{L^2}^2 + \left\|\sum_{k=-\infty}^{\infty} f_k \phi_k^{(n)}\right\|_{L^2}^2 \geq \hat{D}_n^2 \sum_{k=-\infty}^{\infty} \Big((1+|k|^n)|f_k|\Big)^2.$$

Proof. We first prove the inequality for $n = 4$. Integrating by parts and applying the boundary conditions, we obtain that

$$\begin{aligned}
\left(-\phi_k^{(4)} + \phi_k^{(2)}, \psi_j'\right)_{L^2} &= \int_0^{2\pi} \left(\phi_k^{(5)} - \phi_k^{(3)}\right) \overline{\psi_j} dx + \phi_k^{(2)} \overline{\psi_j}\Big|_0^{2\pi} - \phi_k^{(4)} \overline{\psi_j}\Big|_0^{2\pi} \\
&= \int_0^{2\pi} \left(\phi_k^{(5)} - \phi_k^{(3)}\right) \overline{\psi_j} dx = (A\phi_k, \psi_j) = \lambda_k (\phi_k, \psi_j) = \lambda_k \delta_{k,j}\,,
\end{aligned}$$

which implies that

$$\delta_{k,j} = \frac{k^4 j}{\lambda_k} \left(\frac{-\phi_k^{(4)}}{k^4} + \frac{\phi_k^{(2)}}{k^4}, \frac{\psi_j'}{j}\right)_{L^2} = \frac{j^5}{\lambda_j} \left(\frac{-\phi_k^{(4)}}{k^4} + \frac{\phi_k^{(2)}}{k^4}, \frac{\psi_j'}{j}\right)_{L^2} \qquad k, j \neq 0\,, \tag{3.1.25}$$

Since $|\lambda_k| \sim O(k^5)$ as $k \to \infty$, it is deduced that

$$\begin{aligned}
&\sum_{k=-\infty}^{\infty} |k^4 f_k|^2 = \sum_{k=-\infty, k\neq 0}^{\infty} \sum_{j=-\infty, j\neq 0}^{\infty} k^4 j^4 f_k \bar{f}_j \left(\frac{-\phi_k^{(4)}}{k^4} + \frac{\phi_k^{(2)}}{k^4}, \frac{\psi_j'}{j}\right)_{L^2} \frac{j^5}{\lambda_j} \\
&= \left(\sum_{k=-\infty, k\neq 0}^{\infty} k^4 f_k \left(\frac{-\phi_k^{(4)}}{k^4} + \frac{\phi_k^{(2)}}{k^4}\right), \sum_{j=-\infty, j\neq 0}^{\infty} j^4 f_j \frac{j^5}{\bar{\lambda}_j} \frac{\psi_j'}{j}\right)_{L^2} \\
&\leq \left\|\sum_{k=-\infty, k\neq 0}^{\infty} f_k \phi_k^{(4)}\right\|_{L^2} \left\|\sum_{j=-\infty, j\neq 0}^{\infty} \frac{j^9 f_j}{\bar{\lambda}_j} \frac{\psi_j'}{j}\right\|_{L^2} + \left\|\sum_{k=-\infty, k\neq 0}^{\infty} f_k \phi_k^{(2)}\right\|_{L^2} \left\|\sum_{j=-\infty, j\neq 0}^{\infty} \frac{j^9 f_j}{\bar{\lambda}_j} \frac{\psi_j'}{j}\right\|_{L^2} \\
&= \left(\left\|\sum_{k=-\infty, k\neq 0}^{\infty} f_k \phi_k^{(4)}\right\|_{L^2} + \left\|\sum_{k=-\infty, k\neq 0}^{\infty} f_k \phi_k^{(2)}\right\|_{L^2}\right) \left\|\sum_{j=-\infty, j\neq 0}^{\infty} \frac{j^9 f_j}{\bar{\lambda}_j} \frac{\psi_j'}{j}\right\|_{L^2} \\
&\leq \left(\left\|\sum_{k=-\infty, k\neq 0}^{\infty} f_k \phi_k^{(4)}\right\|_{L^2} + \left\|\sum_{k=-\infty, k\neq 0}^{\infty} f_k \phi_k^{(2)}\right\|_{L^2}\right) C D_4 \left(\sum_{j=-\infty, j\neq 0}^{\infty} |j^4 f_j|^2\right)^{\frac{1}{2}}
\end{aligned}$$

which gives

$$\left\|\sum_{k=-\infty,k\neq 0}^{\infty} f_k\phi_k^{(4)}\right\|_{L^2}^2 + \left\|\sum_{k=-\infty,k\neq 0}^{\infty} f_k\phi_k^{(2)}\right\|_{L^2}^2 \geq \frac{1}{C^2D_4^2}\sum_{k=-\infty,k\neq 0}^{\infty}|k^4f_k|^2. \tag{3.1.26}$$

Since by (3.1.21)

$$\left\|\sum_{k=-\infty,k\neq 0}^{\infty} f_k\phi_k^{(2)}\right\|_{L^2}^2 \leq D_2^2\sum_{k=-\infty,k\neq 0}^{\infty}|k^2f_k|^2 \leq D_2^2\sum_{k=-\infty,k\neq 0}^{\infty}\left(((4\epsilon)^{-1}+\epsilon|k|^4)|f_k|\right)^2,$$

(3.1.26) can be rewritten by

$$c(\epsilon)\left\|\sum_{k=-\infty,k\neq 0}^{\infty} f_k\phi_k\right\|_{L^2}^2 + \left\|\sum_{k=-\infty}^{\infty} f_k\phi_k^{(4)}\right\|_{L^2}^2 \geq \hat{D}_4^2\sum_{k=-\infty}^{\infty}\left|k^4f_k\right|^2.$$

Thus, the case for $n=4$ is proved. Note that both $H^n(0,2\pi)$ and l_n^2 are interpolation spaces. By the Interpolation Theorem [4], we conclude that

$$\left\|\sum_{k=-\infty}^{\infty} f_k\phi_k\right\|_{L^2}^2 + \left\|\sum_{k=-\infty}^{\infty} f_k\phi_k^{(n)}\right\|_{L^2}^2 \geq \hat{D}_n^2\sum_{k=-\infty}^{\infty}\Big((1+|k|^n)|f_k|\Big)^2 \tag{3.1.27}$$

for $n=0,1,2,3,4$. In the remaining part, we will consider if we can extend n to ∞. Using the characteristic function, for any $n\geq 5$, we have

$$\begin{aligned}
\phi^{(n)}-\phi^{(n-2)} &= \lambda\phi^{(n-5)}\\
\sum_{k=-\infty}^{\infty} f_k\phi_k^{(n)} - \sum_{k=-\infty}^{\infty} f_k\phi_k^{(n-2)} &= \sum_{k=-\infty}^{\infty}\lambda_kf_k\phi_k^{(n-5)}\\
\left\|\sum_{k=-\infty}^{\infty} f_k\phi_k^{(n)} - \sum_{k=-\infty}^{\infty} f_k\phi_k^{(n-2)}\right\|_{L^2}^2 &= \left\|\sum_{k=-\infty}^{\infty}\lambda_kf_k\phi_k^{(n-5)}\right\|_{L^2}^2\\
c(\delta)\left\|\sum_{k=-\infty}^{\infty} f_k\phi_k^{(n)}\right\|_{L^2}^2 + \delta\left\|\sum_{k=-\infty}^{\infty} f_k\phi_k^{(n-2)}\right\|_{L^2}^2 &\geq \left\|\sum_{k=-\infty}^{\infty}\lambda_kf_k\phi_k^{(n-5)}\right\|_{L^2}^2\\
\left\|\sum_{k=-\infty}^{\infty} f_k\phi_k^{(n)}\right\|_{L^2}^2 + \left\|\sum_{k=-\infty}^{\infty} f_k\phi_k^{(n-2)}\right\|_{L^2}^2 &\geq \frac{1}{D_n^2}\left\|\sum_{k=-\infty}^{\infty}\lambda_kf_k\phi_k^{(n-5)}\right\|_{L^2}^2.
\end{aligned} \tag{3.1.28}$$

On the other hand, by Ehrling's lemma, we have

$$\left\|\sum_{k=-\infty}^{\infty} f_k\phi_k^{(n-2)}\right\|_{L^2}^2 \leq \left\|\sum_{k=-\infty}^{\infty} f_k\phi_k\right\|_{H^{n-2}}^2 \leq \epsilon\left\|\sum_{k=-\infty}^{\infty} f_k\phi_k\right\|_{H^n}^2 + C(\epsilon)\left\|\sum_{k=-\infty}^{\infty} f_k\phi_k\right\|_{L^2}^2$$

where

$$
\begin{aligned}
\epsilon \left\| \sum_{k=-\infty}^{\infty} f_k \phi_k \right\|_{H^n}^2 &= \epsilon \left\| \sum_{k=-\infty}^{\infty} f_k \phi_k^{(n)} \right\|_{L^2}^2 + \epsilon \left\| \sum_{k=-\infty}^{\infty} f_k \phi_k \right\|_{H^{n-1}}^2 \\
\leq \quad & \epsilon \left\| \sum_{k=-\infty}^{\infty} f_k \phi_k^{(n)} \right\|_{L^2}^2 + \epsilon\delta \left\| \sum_{k=-\infty}^{\infty} f_k \phi_k \right\|_{H^n}^2 + \epsilon \cdot C(\delta) \left\| \sum_{k=-\infty}^{\infty} f_k \phi_k \right\|_{L^2}^2 \\
\leq \quad & \frac{\epsilon}{1-\delta} \left\| \sum_{k=-\infty}^{\infty} f_k \phi_k^{(n)} \right\|_{L^2}^2 + \frac{\epsilon \cdot C(\delta)}{1-\delta} \left\| \sum_{k=-\infty}^{\infty} f_k \phi_k \right\|_{L^2}^2 .
\end{aligned}
$$

Therefore

$$
\left\| \sum_{k=-\infty}^{\infty} f_k \phi_k^{(n-2)} \right\|_{L^2}^2 \leq \frac{\epsilon}{1-\delta} \left\| \sum_{k=-\infty}^{\infty} f_k \phi_k^{(n)} \right\|_{L^2}^2 + \left(C(\epsilon) + \frac{\epsilon \cdot C(\delta)}{1-\delta} \right) \left\| \sum_{k=-\infty}^{\infty} f_k \phi_k \right\|_{L^2}^2 . \tag{3.1.29}
$$

Substituting (3.1.29) back to (3.1.28), we have

$$
\left\| \sum_{k=-\infty}^{\infty} f_k \phi_k^{(n)} \right\|_{L^2}^2 + \left\| \sum_{k=-\infty}^{\infty} f_k \phi_k \right\|_{L^2}^2 \geq \frac{1}{D_n^2} \left\| \sum_{k=-\infty}^{\infty} \lambda_k f_k \phi_k^{(n-5)} \right\|_{L^2}^2 . \tag{3.1.30}
$$

for $n \geq 5$. Thus we can obtain the same result in (3.1.27) for $n = 5, 6, 7, 8, 9$. Similarly, we can use the recurrence to expand n to infinity. Thus, the proof is completed. □

The following corollary will be used in the proof of asymptotic stability of small solutions.

Corollary 3.1.5. *If $f \in L^2$ and $f_k = \int_0^{2\pi} f(x)(\psi_k'(x)/k)dx$, then $\sum_{k\neq 0} |f_k|^2 \leq c\|f\|_{L^2}^2$.*

Proof. Let $e_k(x) = -\frac{\phi_k^{(4)}(x)}{k^4} + \frac{\phi_k^{(2)}(x)}{k^4}, k \neq 0$ and the subspace in L^2 spanned by $\{e_k\}$ with its closure be L_m. Thus, $L^2 = L_m + L_m^\perp$ and for $f \in L^2$, $f = f_m + f_m^\perp$ with $f_m \in L_m$ and $f_m^\perp \in L_m^\perp$. By (3.1.25), $\{e_k(x)\}$ and $\{(k^5\psi_k'(x)/\lambda_k k)\}$ are dual in L_m. Thus,

$$
f_m(x) = \sum_{k\neq 0} g_k e_k(x) , \quad g_k = \int_0^{2\pi} f(x) \frac{k^5 \psi_k'(x)}{\lambda_k k} dx .
$$

Hence, by (3.1.26) and asymptotic form of λ_k,

$$
\begin{aligned}
\|f\|_{L^2}^2 \geq \|f_m\|_{L^2}^2 = & \left\| \sum_{k\neq 0} g_k e_k(x) \right\|_{L^2}^2 \geq \left\| \sum_{k\neq 0} \frac{g_k \phi_k^{(4)}(x)}{k^4} \right\|_{L^2}^2 - \left\| \sum_{k\neq 0} \frac{g_k \phi_k^{(2)}(x)}{k^4} \right\|_{L^2}^2 \\
\geq & c_1 \sum_{k\neq 0} |g_k|^2 - c_2 \left\| \sum_{k\neq 0} \frac{g_k \phi_k^{(2)}(x)}{k^4} \right\|_{L^2}^2 \geq c_1 \sum_{k\neq 0} |f_k|^2 - c_2 \sum_{k\neq 0} \left| \frac{g_k}{k^2} \right|^2 \\
\geq & c_1 \sum_{k\neq 0} |f_k|^2 - c_2 \|f\|_{L^2}^2 \sum_{k\neq 0} \frac{1}{k^4} ,
\end{aligned}
$$

where

$$\left\| \frac{\phi_k^{(2)}(x)}{k^2} \right\|_{L^2}^2 + \left\| \frac{\phi_k'(x)}{k} \right\|_{L^2}^2 \le c$$

derived from (3.1.23) with c independent of k has been used. Thus, $\sum_{k\neq 0} |f_k|^2 \le c\|f\|_{L^2}^2$. $\square$

Definition 3.1.6. *Let*

$$\begin{aligned} H_\alpha^n = \Big\{ w \in H^n[0,2\pi] \;\Big|\; & w \text{ satisfies the boundary conditions,} \\ & w^{(5j+1)}(2\pi,t) = \alpha w^{(5j+1)}(0,t) + (1-\alpha) w^{(5j+3)}(0,t), \\ & w^{(5j+k)}(0,t) = w^{(5j+k)}(2\pi,t), \quad k = 0,2,3,4 \Big\} \end{aligned}$$

where the orders of derivatives are less than n and the norm $\|w\|_{H_\alpha^n} = \|w\|_{H^n}$ is the classical Sobolev norm.

The following is a corollary of Propositions 3.1.3 and 3.1.4 .

Corollary 3.1.7. *A function $w = \sum_{k=-\infty}^{\infty} c_k \phi_k \in L^2$ lies in H_α^n if and only if*

$$\sum_{k=-\infty}^{\infty} \Big| (1+|k|^n) c_k \Big|^2 < \infty.$$

In addition,

$$\|w\|_{H_\alpha^n}^2 \sim \sum_{k=-\infty}^{\infty} \left(|c_k|^2 + |k^n c_k|^2 \right).$$

Based upon this result, we can define a class of Banach spaces $H_\alpha^{s,p}$. Let $\phi_k(x)$ be the Riesz basis of L^2 obtained. For any $s \ge 0$ and $p \ge 1$, define

$$H_\alpha^{s,p} = \left\{ w = \sum_{k=-\infty}^{\infty} c_k \phi_k \;\middle|\; \sum_{k=-\infty}^{\infty} (1+|k|^{ps}) |c_k|^p < \infty \right\}$$

and

$$\|w\|_{H_\alpha^{s,p}}^p \equiv \sum_{k=-\infty}^{\infty} (1+|k|^{ps}) |c_k|^p.$$

This space will be used to study the smoothing properties of the solutions. Notice that using $\ell^p \hookrightarrow \ell^q$ for any $q > p \ge 2$, it is straightforward to show

$$\begin{aligned} H_\alpha^{s,p} &\hookrightarrow H_\alpha^{s',p}, \quad s' < s, \\ H_\alpha^{s,p} &\hookrightarrow H_\alpha^{s,q}, \quad p > q \ge 2\,. \end{aligned}$$

If $s = n$ is an integer and $p = 2$, the space $H_\alpha^{s,p}$ is same as H_α^n defined by Definition 3.1.6. Moreover, since H_α^s for s not an integer is an interpolation space, thus H_α^s is a subspace of H^s as well for all $s \ge 0$. In the remaining part of this paper, we denote $\|\cdot\|_s$ as the norm of H_α^s and $\|\cdot\|_{s,p}$ as the norm of $H_\alpha^{s,p}$.

3.2 Estimates of Solutions for The Linear Problems

In this section, we derive the necessary estimates of solutions for the corresponding linear problems, which will be used later to prove the well-posedness of the nonlinear system.

First, note that if $\phi_k, k \in \mathbb{Z}$ are the eigenfunctions of A and $\psi_k\,, k \in \mathbb{Z}$ are the eigenfunctions of the adjoint operator A^* to ϕ_k, define

$$P_k = \phi_k \psi_k^* : L^2 \to L^2, \quad -\infty < k < \infty,$$

as a projection, which is generally not orthogonal. The resolution of the identity associated with A is of the form

$$I = \sum_{k=-\infty}^{\infty} P_k,$$

which is strongly convergent in $\mathscr{L}(L^2, L^2)$. The corresponding semigroup generated by A is

$$S(t) = \sum_{k=-\infty}^{\infty} e^{\lambda_k t} \phi_k \psi_k^* = \sum_{k=-\infty}^{\infty} e^{\lambda_k t} P_k. \tag{3.2.1}$$

Then, the solution associated with following nonhomogeneous problem

$$\begin{cases} \partial_t w + \partial_x^3 w - \partial_x^5 w = f(x,t), & 0 < x < 2\pi, \quad t \geq 0\,, \\ w(x,0) = w_0(x), & \\ w_x(2\pi,t) = \alpha w_x(0,t) + (1-\alpha) w_{3x}(0,t), & \\ w_{kx}(0,t) = w_{kx}(2\pi,t)\,, & k = 0,2,3,4\,, \end{cases} \tag{3.2.2}$$

can be represented by

$$w(t) = S(t) w_0(x) + \int_0^t S(t-\tau) f(\cdot,\tau) d\tau. \tag{3.2.3}$$

Now, we derive the estimate for $S(t)w_0$.

Proposition 3.2.1. *For any* $s \geq 0$, $\sup_{t \geq 0} \|S(t)w_0\|_s \leq \|w_0\|_s\,.$

Proof. If $w_0 = \sum\limits_{k=-\infty}^{\infty} c_k \phi_k$, then

$$S(t) w_0 = \sum_{k=-\infty}^{\infty} e^{\lambda_k t} c_k \phi_k.$$

From (2.2.19), we know that $\operatorname{Re} \lambda_k \leq 0$ for all k, which implies that for all $t \geq 0$,

$$\begin{aligned} \|S(t)w_0\|_s^2 &= \sum_{k=-\infty}^{\infty} \left(1 + |k|^{2s}\right) |c_k|^2 \left| e^{\lambda_k t} \right|^2 = \sum_{k=-\infty}^{\infty} \left(1 + |k|^{2s}\right) \cdot |c_k|^2 e^{2\operatorname{Re} \lambda_k t} \\ &\leq \sum_{k=-\infty}^{\infty} \left(1 + |k|^{2s}\right) |c_k|^2 = \|w_0\|_s^2\,. \end{aligned}$$

□

Next is analogous to the Kato-type smoothing effect [18, 20].

Proposition 3.2.2. *For $s \geq 0$, $T > 0$ and $w_0 \in H^s_\alpha$,*

$$\int_0^T \|S(t)w_0\|^2_{s+1} dt \leq C_s^2 \|w_0\|_s^2$$

where $C_s > 0$ depends only upon β.

Proof. By (2.2.19) and (3.1.14), there is a positive $\beta > 0$ such that

$$\operatorname{Re}\lambda_k \leq -\beta k^2, \quad \text{for any integer } k \neq 0. \tag{3.2.4}$$

Then, (3.2.1) implies that

$$\begin{aligned}
\int_0^T \|S(t)w_0\|^2_{s+1} dt &= \int_0^T \sum_{k=-\infty}^{\infty} |c_k|^2 \left(1 + |k|^{2(s+1)}\right) |e^{\lambda_k t}|^2 dt \\
&= \int_0^T \sum_{k=-\infty}^{\infty} |c_k|^2 \left(1 + |k|^{2s+2}\right) e^{2\operatorname{Re}\lambda_k t} dt \leq \sum_{k=-\infty}^{\infty} |c_k|^2 \left(1 + |k|^{2s+2}\right) \cdot \int_0^T e^{-2\beta k^2 t} dt \\
&= T|c_0|^2 + \sum_{k=-\infty, k\neq 0}^{\infty} |c_k|^2 \left(1 + |k|^{2s+2}\right) \cdot \frac{1 - e^{-2\beta k^2 T}}{2\beta k^2} \\
&\leq T|c_0|^2 + \sum_{k=-\infty, k\neq 0}^{\infty} |c_k|^2 \left(1 + |k|^{2s+2}\right) \frac{1}{2\beta k^2} \leq T|c_0|^2 + \sum_{k=-\infty, k\neq 0}^{\infty} |c_k|^2 \left(1 + |k|^{2s}\right) \frac{|k|^2}{2\beta k^2} \\
&\leq C_s^2 \sum_{k=-\infty}^{\infty} |c_k|^2 \left(1 + |k|^{2s}\right) = C_s^2 \|w_0\|_s^2,
\end{aligned}$$

where $C_s^2 = \max(T, (1/(2\beta)))$. □

Now, we study the estimates for the nonhomogeneous term $f(x,t)$.

Proposition 3.2.3. *If $s \geq 0$, $T > 0$ and $f \in L^2(0,T; H^s_\alpha)$, then*

$$\sup_{0\leq t\leq T} \left\| \int_0^t S(t-\tau) f(\cdot,\tau) d\tau \right\|_{s+1} \leq C_s \left[\int_0^T \|f(\cdot,\tau)\|_s^2 d\tau \right]^{1/2}$$

where $C_s > 0$ only depends upon β and T.

Proof. Since

$$f(x,t) = \sum_{k=-\infty}^{\infty} f_k(t) \phi_k(x),$$

we have

$$\int_0^t S(t-\tau) f(\cdot,\tau) d\tau = \int_0^t \sum_{k=-\infty}^{\infty} e^{\lambda_k (t-\tau)} f_k(\tau) \phi_k d\tau = \sum_{k=-\infty}^{\infty} \int_0^t e^{\lambda_k(t-\tau)} f_k(\tau) d\tau \phi_k.$$

Thus,

$$
\begin{aligned}
\sup_{0\le t\le T}\left\|\int_0^t S(t-\tau)f(\cdot,\tau)d\tau\right\|_{s+1}^2 &= \sup_{0\le t\le T}\left\|\sum_{k=-\infty}^{\infty}\int_0^t e^{\lambda_k(t-\tau)}f_k(\tau)d\tau\phi_k\right\|_{s+1}^2 \\
&= \sup_{0\le t\le T}\sum_{k=-\infty}^{\infty}\left(1+|k|^{2s+2}\right)\left|\int_0^t e^{\lambda_k(t-\tau)}f_k(\tau)d\tau\right|^2 \\
&\le \sum_{k=-\infty}^{\infty}\sup_{0\le t\le T}\int_0^t\left|e^{\lambda_k(t-\tau)}\right|^2 d\tau\int_0^t|f_k(\tau)|^2 d\tau\left(1+|k|^{2s+2}\right) \\
&= \sum_{k=-\infty}^{\infty}\sup_{0\le t\le T}\int_0^t e^{2\mathrm{Re}\,\lambda_k(t-\tau)}d\tau\int_0^t|f_k(\tau)|^2 d\tau\left(1+|k|^{2s+2}\right) \\
&\le T\cdot\int_0^T|f_0(\tau)|^2 d\tau+\sum_{k=-\infty,k\neq0}^{\infty}\sup_{0\le t\le T}\int_0^t e^{-2\beta k^2(t-\tau)}d\tau\int_0^t|f_k(\tau)|^2 d\tau\left(1+|k|^{2s+2}\right) \\
&\le T\cdot\int_0^T|f_0(\tau)|^2 d\tau+\sum_{k=-\infty,k\neq0}^{\infty}\frac{1+|k|^{2s+2}}{2\beta k^2}\int_0^T|f_k(\tau)|^2 d\tau \\
&\le T\cdot\int_0^T|f_0(\tau)|^2 d\tau+\sum_{k=-\infty,k\neq0}^{\infty}\frac{|k|^2}{2\beta k^2}\left(1+|k|^{2s}\right)\int_0^T|f_k(\tau)|^2 d\tau \\
&\le C_s^2\sum_{k=-\infty}^{\infty}\int_0^T|f_k(\tau)|^2 d\tau\left(1+|k|^{2s}\right)=C_s^2\left[\int_0^T\|f(\cdot,\tau)\|_s^2 d\tau\right],
\end{aligned}
$$

where $C_s^2=\max(T,(1/(2\beta)))$. □

Remark 3.2.4. *If $f_0(t)=0$, we can replace T by ∞.*

Proposition 3.2.5. *For $s\ge0$, there exists a $C_s>0$ such that if $f\in L^\infty(0,\infty;H_\alpha^s)$ with $\int_0^{2\pi}f(x,t)dx=0$, then*

$$\sup_{0\le t\le\infty}\left\|\int_0^t S(t-\tau)f(\cdot,\tau)d\tau\right\|_{s+1}\le C_s\sup_{0\le t\le\infty}\|f(\cdot,t)\|_s .$$

Proof. Define $t_1=\max\{t-1,0\}$ and consider

$$
\begin{aligned}
\int_0^t S(t-\tau)f(\tau)d\tau &= \int_{t_1}^t S(t-\tau)f(\tau)d\tau+\int_0^{t_1}S(t-\tau)f(\tau)d\tau \\
&\equiv I(\cdot,t)+II(\cdot,t).
\end{aligned}
$$

By Proposition 3.2.3 and $t-t_1\le1$, we have

$$\|I(\cdot,t)\|_{s+1}^2\le C_s^2\int_{t_1}^t\|f(\cdot,\tau)\|_s^2 d\tau\le C_s^2\sup_{0\le t<\infty}\|f(\cdot,t)\|_s^2\int_{t_1}^t 1d\tau\le C_s^2\sup_{0\le t<\infty}\|f(\cdot,t)\|_s^2 .$$

If $t\le1$, the proof is completed. If $t>1$,

$$II(\cdot,t)=\int_0^{t-1}S(t-\tau)f(\cdot,\tau)d\tau.$$

Because of $\int_0^{2\pi} f(x,t)dx \equiv 0$, $f_0(\tau) = \int_0^{2\pi} f(x,t)\phi_0 dx = 0$, which implies that

$$f(\cdot,\tau) = \sum_{j=-\infty, j\neq 0}^{\infty} f_j(\tau)\phi_j \quad \text{and} \quad II(\cdot,t) = \sum_{j=-\infty, j\neq 0}^{\infty} \int_0^{t-1} e^{\lambda_j(t-\tau)} f_j(\tau)d\tau\phi_j.$$

For $j \neq 0$ and $\tau \in [0, t-1]$, we have a uniform estimate

$$\begin{aligned}(1+|j|^2)\left|e^{\lambda_j(t-\tau)}\right|^2 &\leq \left(1+|j|^2\right)\left|e^{\lambda_j(t-1-\tau)}\cdot e^{\lambda_j}\right|^2 = \left(1+|j|^2\right)e^{2\mathrm{Re}\,\lambda_j}\left|e^{\lambda_j(t-1-\tau)}\right|^2 \\ &\leq \sup_{j\neq 0}\{\left(1+j^2\right)e^{2\mathrm{Re}\,\lambda_j}\}\cdot\left|e^{\lambda_j(t-1-\tau)}\right|^2 = G^2\left|e^{\lambda_j(t-1-\tau)}\right|^2,\end{aligned}$$

where

$$G^2 = \sup_{j\neq 0}\left\{\left(1+j^2\right)e^{2\mathrm{Re}\,\lambda_j}\right\} < \infty.$$

Thus, by $\beta > 0$ defined in (3.2.4), we obtain

$$\begin{aligned}\|S(t-\tau)f(\cdot,\tau)\|_{s+1}^2 &= \sum_{j=-\infty, j\neq 0}^{\infty}\left(1+|j|^{2s+2}\right)\cdot|e^{\lambda_j(t-\tau)}f_j(\tau)|^2 \\ &\leq \sum_{j=-\infty, j\neq 0}^{\infty}(1+|j|^2)\left(1+|j|^{2s}\right)\cdot|e^{\lambda_j(t-\tau)}|^2|f_j(\tau)|^2 \\ &\leq G^2\left|e^{\lambda_j(t-1-\tau)}\right|^2\sum_{j=-\infty, j\neq 0}^{\infty}\left(1+|j|^{2s}\right)|f_j(\tau)|^2 \\ &\leq G^2 e^{-2\beta j^2(t-1-\tau)}\cdot\|f(\cdot,\tau)\|_s^2 \leq G^2 e^{-2\beta(t-1-\tau)}\cdot\|f(\cdot,\tau)\|_s^2\end{aligned}$$

which gives

$$\begin{aligned}\|II(\cdot,t)\|_{s+1}^2 &= \left\|\sum_{j=-\infty, j\neq 0}^{\infty}\int_0^{t-1} e^{\lambda_j(t-\tau)}f_j(\tau)d\tau\phi_j\right\|_{s+1}^2 \\ &= \sum_{j=-\infty, j\neq 0}^{\infty}\left(1+|j|^{2s+2}\right)\left|\int_0^{t-1} e^{\lambda_j(t-\tau)}f_j(\tau)d\tau\right|^2 \\ &\leq G^2\int_0^{t-1} e^{-2\beta(t-1-\tau)}d(\tau)\sup_{0\leq\tau<t-1}\|f(\cdot,\tau)\|_s^2 \leq \frac{G^2}{2\beta}\sup_{0\leq\tau<t-1}\|f(\cdot,\tau)\|_s^2.\end{aligned}$$

Therefore,

$$\sup_{0\leq t\leq\infty}\left\|\int_0^t S(t-\tau)f(\cdot,\tau)d\tau\right\|_{s+1} \leq \sup_{0\leq t\leq\infty}(\|I(\cdot,t)\|_{s+1} + \|II(\cdot,t)\|_{s+1}) \leq C_s\sup_{0\leq t\leq\infty}\|f(\cdot,t)\|_s$$

and we finish the proof. □

Next proposition gives an estimate which will be used later for weak initial conditions.

Proposition 3.2.6. *Assume that $s \geq 0$ and $p > 2$ satisfying $s + \frac{1}{2} - \frac{3}{p} \geq 0$. For $T > 0$ and $s' > s + \frac{1}{2} - \frac{3}{p}$, there is a constant $C_{s'} > 0$ such that for any $w_0 \in H_\alpha^{s',p}$,*

$$\int_0^T \|S(t)w_0\|_s^p dt \leq C_{s'} \|w_0\|_{s',p}^p \ ,$$

where $C_{s'} \to \infty$, $s' \to s + \frac{1}{2} - \frac{3}{p}$.

Proof. From the definition of $S(t)$ in (3.2.1), by Hölder's inequality, we find that

$$\begin{aligned}
\|S(t)w_0\|_s^p &= \left(\sum_{k=-\infty}^{\infty} \left(1 + |k|^{2s}\right) \left| e^{\lambda_k t} c_k \right|^2 \right)^{\frac{p}{2}} \\
&\leq \left(\sum_{k=-\infty}^{\infty} \left(1 + |k|^{2s}\right) \left| e^{\lambda_k t} c_k \right|^2 (1 + |k|^\epsilon)(1 + |k|^\epsilon)^{-1} \right)^{\frac{p}{2}} \\
&\leq \left(\sum_{k=-\infty}^{\infty} (1 + |k|^{ps}) \left| e^{p\lambda_k t} c_k^p \right| (1 + |k|^\epsilon)^{\frac{p}{2}} \right) \left(\sum_{k=-\infty}^{\infty} (1 + |k|^\epsilon)^{-\frac{p}{p-2}} \right)^{\frac{p-2}{2}} \\
&\leq C_{s'} \sum_{k=-\infty}^{\infty} e^{p \mathrm{Re}\, \lambda_k t} |c_k|^p (1 + |k|^{ps}) (1 + |k|^\epsilon)^{\frac{p}{2}} \ ,
\end{aligned}$$

where

$$C_{s'} = \left(\sum_{k=-\infty}^{\infty} (1 + |k|^\epsilon)^{-\frac{p}{p-2}} \right)^{\frac{p-2}{2}}$$

with $\epsilon = 2(s' - s) + \frac{4}{p} > \frac{p-2}{p}$ and $C_{s'} \to \infty$ as $\epsilon \to \frac{p-2}{p}$. Hence,

$$\begin{aligned}
\int_0^T \|S(t)w_0\|_s^p dt &\leq \int_0^T C_{s'} \sum_{k=-\infty}^{\infty} e^{p \mathrm{Re}\, \lambda_k t} |c_k|^p (1 + |k|^{ps}) (1 + |k|^\epsilon)^{\frac{p}{2}} dt \\
&\leq C_{s'} \left(|c_0|^p T + \sum_{k=-\infty, k \neq 0}^{\infty} |c_k|^p (1 + |k|^{ps}) (1 + |k|^\epsilon)^{\frac{p}{2}} \frac{1}{p |\mathrm{Re}\, \lambda_k|} \right) \\
&\leq C_{s'} \sum_{k=-\infty}^{\infty} |c_k|^p \left(1 + |k|^{p\left(s - \frac{2}{p} + \frac{\epsilon}{2}\right)}\right) = C_{s'} \|w_0\|_{s',p}^p \ ,
\end{aligned}$$

where

$$s' = s - \frac{2}{p} + \frac{\epsilon}{2} > s - \frac{2}{p} + \frac{p-2}{2p} = s + \frac{1}{2} - \frac{3}{p}$$

and $C_{s'} \to \infty$ as $s' \to s + \frac{1}{2} - \frac{3}{p}$. □

3.3 Local Well-Posedness of The Nonlinear Problem

In this section, we discuss the local well-posedness of the IVP for the nonlinear problem

$$\begin{cases} \partial_t w + \partial_x^3 w - \partial_x^5 w = \sum_{j=1}^{n} a_j w^j w_x, & 0 < x < 2\pi, \quad t \geq 0 \\ w(x,0) = w_0(x), & \\ w_x(2\pi,t) = \alpha w_x(0,t) + (1-\alpha) w_{3x}(0,t), & \\ w_{kx}(0,t) = w_{kx}(2\pi,t), & k = 0,2,3,4. \end{cases} \tag{3.3.1}$$

with $|\alpha| < 1$. Here, the local well-posedness means that the existence interval of the solution depends upon the initial condition.

Using Duhamel's principle, we change (3.3.1) to

$$w(\cdot,t) = S(t)w_0 + \int_0^t S(t-\tau) \left(\sum_{j=1}^{n} a_j w^j u_x \right) (\cdot,\tau) d\tau \,, \tag{3.3.2}$$

and define a mapping $F : u(\cdot,t) \to w(\cdot,t)$ by

$$w(\cdot,t) = (Fu)(\cdot,t) := S(t)w_0 + \int_0^t S(t-\tau) \left(\sum_{j=1}^{n} a_j w^j u_x \right) (\cdot,\tau) d\tau. \tag{3.3.3}$$

Therefore, the solution of (3.3.1) becomes a fixed-point of (3.3.3). Now, let us study this equation in the space H_α^1.

Theorem 3.3.1. *If we let*

$$Y_T = \left\{ u \;\middle|\; u \in C([0,T]; H_\alpha^1) \right\} ,$$

then for any $w_0 \in H_\alpha^1$, there exists a $T = T(\|w_0\|_1) > 0$ such that the mapping (3.3.3) *has a unique fixed point $w \in Y_T$ which is the unique solution of* (3.3.1) *where $T \to \infty$ as $\|w_0\|_1 \to 0$. Moveover, for any $T' < T$, there is a neighborhood U of $\tilde{w}_0$ in H_α^1 such that the mapping $N : w_0 \to w(\cdot,t)$ from U to $Y_{T'}$ is Lipschitz continuous.*

Proof. Let

$$B_{T,M} = \left\{ v \in Y_T \;\middle|\; \sup_{0 \leq t \leq T} \|v(\cdot,t)\|_1 \leq M \right\}$$

where T and M will be determined later. In order to prove the existence and uniqueness of the solution, we need to show that the mapping in (3.3.3) defined on $B_{T,M}$ is a contraction by choosing appropriate T and M. First, we show that F maps $B_{T,M}$ to itself.

$$\sup_{0 \leq t \leq T} \|Fv\|_1 \leq \sup_{0 \leq t \leq T} \|S(t)w_0\|_1 + \sup_{0 \leq t \leq T} \left\| \int_0^t S(t-\tau) \left(\sum_{j=1}^{n} a_j v^j v_x \right) (\cdot,\tau) d\tau \right\|_1 .$$

In the following, $\sup_t$ refers to $\sup_{0\le t\le T}$ and $\sup_x$ refers to $\sup_{0\le x\le 2\pi}$. By Proposition 3.2.1, we find that $\sup_t \|S(t)w_0\|_1 \le c\|w_0\|_1$. Proposition 3.2.3 yields

$$\begin{aligned}
&\sup_t \left\| \int_0^t S(t-\tau)\left(\sum_{j=1}^n a_j v^j v_x\right)(\cdot,\tau)d\tau \right\|_1 \\
&\le c\left[\int_0^T \left\|\sum_{j=1}^n a_j v^j v_x\right\|_{L^2}^2 d\tau\right]^{1/2} \le cT^{1/2}\sup_t \left\|\sum_{j=1}^n a_j v^j v_x\right\|_{L^2} \\
&\le cT^{1/2}\sum_{j=1}^n \sup_t \|v^j v_x\|_{L^2} \le cT^{1/2}\sum_{j=1}^n \sup_t \|v\|_1^{j+1} .
\end{aligned}$$

Thus, we have

$$\sup_t \|Fv\|_1 \le c\|w_0\|_1 + cT^{1/2}\sup_t \|v\|_1^2 \left(\sum_{j=1}^n \sup_t \|v\|_1^{j-1}\right) \tag{3.3.4}$$

for some $c > 0$ independent of T and v. If we choose $M = 2c\|w_0\|_1$, i.e.,

$$c\|w_0\|_1 = (M/2)\,, \tag{3.3.5}$$

and $T > 0$ such that

$$cT^{1/2}M^2\left(\sum_{j=1}^n M^{j-1}\right) \le cT^{1/2}M^2\left(\sum_{j=1}^n (j+1)M^{j-1}\right) \le (M/2), \tag{3.3.6}$$

then, from (3.3.4), we can obtain $\sup_t \|Fv\|_1 \le M$, i.e., F is a mapping from $B_{T,M}$ to $B_{T,M}$.

Next, we show that F is a contraction on $B_{T,M}$. Let $w = v_1 - v_2$ for any v_1, $v_2 \in B_{T,M}$.

$$\begin{aligned}
Fv_1 - Fv_2 &= \int_0^t S(t-\tau)\left(\sum_{j=1}^n a_j v_1^j v_{1x}\right)d\tau - \int_0^t S(t-\tau)\left(\sum_{j=1}^n a_j v_2^j v_{2x}\right)d\tau \\
&= \sum_{j=1}^n a_j \int_0^t S(t-\tau)\left(v_1^j v_{1x} - v_2^j v_{2x}\right)d\tau \\
&= \sum_{j=1}^n a_j \left(\underbrace{\int_0^t S(t-\tau)\left(v_1^j - v_2^j\right)v_{2x}d\tau}_{I} + \underbrace{\int_0^t S(t-\tau)\left(v_{1x} - v_{2x}\right)v_1^j d\tau}_{II}\right) .
\end{aligned}$$

Applying Proposition 3.2.3 to I yields

$$\sup_t \left\| \int_0^t S(t-\tau)\left(v_1^j - v_2^j\right) v_{2x} d\tau \right\|_1 \le c\left[\int_0^T \left\|\left(v_1^j - v_2^j\right) v_{2x}\right\|_{L^2}^2 d\tau\right]^{1/2}$$
$$\le c \sup_t \left\|\left(v_1^j - v_2^j\right) v_{2x}\right\|_{L^2} T^{1/2} \le cT^{1/2}\left(\sup_t \left\| w \sum_{\ell=0}^{j-1} v_1^\ell v_2^{j-1-\ell} v_{2x}\right\|_{L^2}\right)$$
$$\le cT^{1/2}\left(\sum_{\ell=0}^{j-1} \sup_t \|v_1\|_1^\ell \sup_t \|v_2\|_1^{j-\ell}\right) \cdot \sup_t \|w\|_1 \le cT^{1/2} j M^j \sup_t \|v_1 - v_2\|_1 . \tag{3.3.7}$$

Similarly, for II, we have

$$\sup_t \left\| \int_0^t S(t-\tau)\left(v_{1x} - v_{2x}\right) v_1^j d\tau \right\|_1 \le c\left[\int_0^T \|w_x v_1^j\|_{L^2}^2 d\tau\right]^{1/2} \le cT^{1/2} \sup_t \|w_x v_1^j\|_{L^2}$$
$$\le cT^{1/2} \sup_t \|w\|_1 \left(\sup_t \|v_1\|_1\right)^j \le cM^j T^{1/2} \sup_t \|v_1 - v_2\|_1 . \tag{3.3.8}$$

Combining (3.3.7) and (3.3.8) with (3.3.6) gives

$$\sup_t \|Fv_1 - Fv_2\|_1 \le cT^{1/2} M \left(\sum_{j=1}^n (j+1) M^{j-1}\right) \cdot \sup_t \|v_1 - v_2\|_1$$
$$\le (1/2) \sup_t \|v_1 - v_2\|_1. \tag{3.3.9}$$

Therefor, F is a contraction defined on $B_{T,M}$ and the unique solution of (3.3.1) exists. Substituting (3.3.5) into (3.3.6), for a given $w_0 \in H_\alpha^1$, we can write T in terms of w_0 as follows,

$$cT^{1/2} \cdot (2c\|w_0\|_1) \cdot \left(\sum_{j=1}^n (j+1)\,(2c\|w_0\|_1)^{j-1}\right) \le (1/2)\,, \quad \text{or}$$
$$T = \rho\left((4c^2\|w_0\|_1) \cdot \left(\sum_{j=1}^n (j+1)\,(2c\|w_0\|_1)^{j-1}\right)\right)^{-2}, \quad \rho \in (0,1)\,.$$

Thus, $T \to \infty$ as $\|w_0\|_1 \to 0$. The existence and uniqueness proof is completed.

To prove the Lipschitz continuity, we note that for any neighborhood U of $\tilde{w}_0$ in H_α^1, there is a $T' < T$ such that the mapping, $N : U \to Y_{T'}$, is well defined. Thus we only need to prove that the mapping N is Lipschitz continuous. For any $w_1, w_2 \in U$, let $u_1 = Nw_1, u_2 = Nw_2$, and $u = u_1 - u_2$. Then,

$$u = S(t)(w_1 - w_2) + \int_0^t S(t-\tau)\left(\sum_{j=1}^n a_j u_1^j u_{1x}\right) d\tau - \int_0^t S(t-\tau)\left(\sum_{j=1}^n a_j u_2^j u_{2x}\right) d\tau.$$

Replacing T with T' in (3.3.9) implies

$$\sup_t \|u(t)\|_1 \leq c\|w_1 - w_2\|_1 + \rho \sup_t \|u_1 - u_2\|_1, \quad 0 < \rho < 1,$$

$$\sup_t \|u_1 - u_2\|_1 \leq \frac{c}{1-\rho}\|w_1 - w_2\|_1.$$

Thus, N is Lipschitz continuous from U to $Y_{T'}$. □

The following Theorem gives a local well-posedness of the nonlinear problem for weaker initial conditions.

Theorem 3.3.2. *Let $\delta > 0$ be small and $p_1 = 2n(2+\delta)/\delta, p_2 = 2+\delta$. Define*

$$Y_T = \left\{u \mid u \in C([0,T]; L^2) \cap L^{p_1}([0,T]; H_\alpha^{1/2}) \cap L^{p_2}([0,T]; H_\alpha^1)\right\}$$

with the norm

$$\|u\|_{Y_T} := \sup_{0\leq t\leq T} \|v(\cdot,t)\|_{L^2} + \left(\int_0^T \|v(\cdot,t)\|_{1/2}^{p_1} dt\right)^{1/p_1} + \left(\int_0^T \|v(\cdot,t)\|_1^{p_2} dt\right)^{1/p_2}.$$

If $s_1' > 1 - \frac{3}{p_1}$ and $s_2' > \frac{3}{2} - \frac{3}{p_2}$ are given, then for any $w_0 \in H_\alpha^{s_1',p_1} \cap H_\alpha^{s_2',p_2}$, there exists a $T = T(\|w_0\|_{s_1',p_1}, \|w_0\|_{s_2',p_2}) > 0$ such that the mapping (3.3.3) has a unique fixed point $w \in Y_T$ which is the unique solution of (3.3.1), where $T \to \infty$ as $\|w_0\|_{s_1',p_1} + \|w_0\|_{s_2',p_2} \to 0$. Moreover, for any $T' < T$, there exists a neighborhood U of $\tilde{w}_0$ in $H_\alpha^{s_1',p_1} \cap H_\alpha^{s_2',p_2}$ such that the mapping $N : w_0 \to w$ from U to $Y_{T'}$ is Lipschitz continuous.

Proof. Define

$$B_{T,M} = \left\{v \in Y_T \;\middle|\; \|v\|_{Y_T} \leq M\right\}$$

where T and M are to be determined. To prove (3.3.3) is a contraction defined on $B_{T,M}$, we first show that F in (3.3.3) maps $B_{T,M}$ to itself, i.e., for any $v \in B_{T,M}$

$$\|Fv\|_{Y_t} = \sup_t \|Fv\|_{L_2} + \left(\int_0^T \|Fv\|_{1/2}^{p_1} dt\right)^{1/p_1} + \left(\int_0^T \|Fv\|_1^{p_2} dt\right)^{1/p_2} \leq M.$$

For any $v \in B_{T,M}$, we have

$$\sup_{0\leq t\leq T} \|Fv\|_{L^2} \leq \underbrace{\sup_{0\leq t\leq T} \|S(t)w_0\|_{L^2}}_{I} + \underbrace{\sup_{0\leq t\leq T} \left\|\int_0^t S(t-\tau)\left(\sum_{j=1}^n a_j v^j v_x\right)(\cdot,\tau)d\tau\right\|_{L^2}}_{II}. \quad (3.3.10)$$

The estimate for I follows from Proposition 3.2.1, i.e.,

$$\sup_{0\leq t\leq T} \|S(t)w_0\|_{L^2} \leq \sup_{0\leq t\leq T} \|w_0\|_{L^2}.$$

Similar argument for II with an application of Minkowski inequality yields

$$\begin{aligned}
\sup_{0\le t\le T}\left\|\int_0^t S(t-\tau)\left(\sum_{j=1}^n a_j v^j v_x\right)(\cdot,\tau)d\tau\right\|_{L^2} &\le \sum_{j=1}^n |a_j|\int_0^T \|v^j v_x\|_{L^2}d\tau \\
&\le \sum_{j=1}^n |a_j|\int_0^T \|v\|_{1/2}^j\|v\|_1 d\tau \le \sum_{j=1}^n |a_j|T^{1/2}\left(\int_0^T \|v\|_{1/2}^{2j}\|v\|_1^2 d\tau\right)^{1/2} \\
&\le \sum_{j=1}^n |a_j|T^{1/2}\left(\frac{1}{q'}\int_0^T \|v\|_{1/2}^{2jq'}d\tau + \frac{1}{q}\int_0^T \|v\|_1^{2q}d\tau\right)^{1/2} \\
&\le \sum_{j=1}^n |a_j|T^{1/2}\left(\frac{1}{q'}\left(\int_0^T \|v\|_{1/2}^{2jq'(n/j)}d\tau\right)^{j/n}\cdot\left(\int_0^T 1^{n/(n-j)}\right)^{(n-j)/n} + \frac{1}{q}\int_0^T \|v\|_1^{2q}d\tau\right)^{1/2} \\
&\le \sum_{j=1}^n |a_j|T^{1/2}\left(\frac{1}{q'}T^{(n-j)/n}\left(\int_0^T \|v\|_{1/2}^{p_1}d\tau\right)^{j/n} + \frac{1}{q}\int_0^T \|v\|_1^{p_2}d\tau\right)^{1/2}, \qquad (3.3.11)
\end{aligned}$$

where $q=(2+\delta)/2, q'=(2+\delta)/\delta$ and $ab\le(1/q)a^q+(1/q')b^{q'}$ have been used. On the other hand, for any $v\in B_{T,M}$ and $m=1,2$,

$$\begin{aligned}
&\left(\int_0^T \|Fv\|_{m/2}^{p_m}dt\right)^{1/p_m} \\
&\le \underbrace{\left(\int_0^T \|S(t)w_0\|_{m/2}^{p_m}\,dt\right)^{1/p_m}}_{I} + \underbrace{\left(\int_0^T \left\|\int_0^t S(t-\tau)\left(\sum_{j=1}^n a_j v^j v_x\right)(\cdot,\tau)d\tau\right\|_{m/2}^{p_m} dt\right)^{1/p_m}}_{II}. \qquad (3.3.12)
\end{aligned}$$

For I, Proposition 3.2.6 gives

$$\left(\int_0^T \|S(t)w_0\|_{m/2}^{p_m}\,dt\right)^{1/p_m} \le c\|w_0\|_{s'_m,p_m},$$

where $s'_m > \frac{m}{2}+\frac{1}{2}-\frac{3}{p_m}$. Applying Proposition 3.2.3 to II yields

$$\begin{aligned}
&\left(\int_0^T \left\|\int_0^t S(t-\tau)\left(\sum_{j=1}^n a_j v^j v_x\right)(\cdot,\tau)d\tau\right\|_{m/2}^{p_m} dt\right)^{1/p_m} \\
&\le T^{1/p_m}\sup_t\left\|\int_0^t S(t-\tau)\left(\sum_{j=1}^n a_j v^j v_x\right)(\cdot,\tau)d\tau\right\|_1 \le cT^{1/p_m}\left(\int_0^T \left\|\sum_{j=1}^n a_j v^j v_x\right\|_{L^2}^2 d\tau\right)^{1/2} \qquad (3.3.13)
\end{aligned}$$

where

$$\left\|\sum_{j=1}^{n} a_j v^j v_x\right\|_{L^2}^2 \le \left(\sum_{j=1}^{n} |a_j| \left\|v^j v_x\right\|_{L^2}\right)^2 \le c\left(\sum_{j=1}^{n} |a_j|^2 \left\|v^j v_x\right\|_{L^2}^2\right)$$
$$\le c\left(\sum_{j=1}^{n} |a_j|^2 \|v\|_{1/2}^{2j} \|v\|_1^2\right) .$$

Using the proof of (3.3.11), we can change (3.3.13) to

$$cT^{1/p_m}\left(\int_0^T \left\|\sum_{j=1}^{n} a_j v^j v_x\right\|_{L^2}^2 d\tau\right)^{1/2} \le cT^{1/p_m}\left(\sum_{j=1}^{n} |a_j|^2 \int_0^T \left(\|v\|_{1/2}^{2j}\|v\|_1^2\right) d\tau\right)^{1/2}$$
$$\le cT^{1/p_m}\left(\sum_{j=1}^{n} |a_j|^2 \left(\frac{1}{q'}T^{(n-j)/n}\left(\int_0^T \|v\|_{1/2}^{p_1} d\tau\right)^{j/n} + \frac{1}{q}\int_0^T \|v\|_1^{p_2} d\tau\right)\right)^{1/2} ,$$

where q, q' are defined in (3.3.11). Therefore, combining (3.3.10) and (3.3.12) yields

$$\|Fv\|_{Y_t} \le \sup_t \|Fv\|_{L_2} + \sum_{m=1}^{2}\left(\int_0^T \|Fv\|_{m/2}^{p_m} dt\right)^{1/p_m}$$
$$\le c\sup_t \|w_0\|_{L^2} + \sum_{j=1}^{n} |a_j| T^{1/2}\left(\frac{1}{q'}T^{(n-j)/n}\left(\int_0^T \|v\|_{1/2}^{p_1} d\tau\right)^{j/n} + \frac{1}{q}\int_0^T \|v\|_1^{p_2} d\tau\right)^{1/2}$$
$$+ c\sum_{m=1}^{2} \|w_0\|_{s'_m, p_m}$$
$$+ \sum_{m=1}^{2} cT^{1/p_m}\left(\sum_{j=1}^{n} |a_j|^2 \left(\frac{1}{q'}T^{(n-j)/n}\left(\int_0^T \|v\|_{1/2}^{p_1} d\tau\right)^{j/n} + \frac{1}{q}\int_0^T \|v\|_1^{p_2} d\tau\right)\right)^{1/2}$$
$$\le c\left(\sup_t \|w_0\|_{L^2} + \sum_{m=1}^{2} \|w_0\|_{s'_m, p_m}\right) + c\cdot \max(T^{1/2}, T^{1/p_1}, T^{1/p_2}) M G(M) ,$$

where

$$G(M) = \left(\sum_{j=1}^{n}\left(T^{(n-j)/n} M^{(p_1 j/n)-2} + M^{p_2-2}\right)\right)^{1/2}$$

and $G(0) = 0$ because $(p_1/n) > 2$ and $p_2 > 2$. If we choose

$$(M/2) = c\left(\sup_t \|w_0\|_{L^2} + \sum_{m=1}^{2} \|w_0\|_{s'_m, p_m}\right)$$

and $T > 0$ such that

$$c\cdot \max(T^{1/2}, T^{1/p_1}, T^{1/p_2}) M G(M) \le (M/2) ,$$

then $\|Fv\|_{Y_t} \leq M$. For any given $w_0 \in H_\alpha^{s_1',p_1} \cap H_\alpha^{s_2',p_2}$, we can write T in terms of w_0 as

$$\max(T^{1/2}, T^{1/p_1}, T^{1/p_2}) = \rho \left(2cG\left(2c\left(\sup_t \|w_0\|_{L^2} + \sum_{m=1}^{2} \|w_0\|_{s_m',p_m}\right)\right)\right)^{-1}, \rho \in (0,1)\,,$$

which shows that $T \to \infty$ when $\sup_t \|w_0\|_{L^2} + \sum_{m=1}^2 \|w_0\|_{s_m',p_m} \to 0$.

Now, we prove that F is a contraction on $B_{T,M}$. For $v_1, v_2 \in B_{T,M}$, define $w = v_1 - v_2$ and

$$Fv_1 - Fv_2 = \sum_{j=1}^{n} a_j \left(\underbrace{\int_0^t S(t-\tau)\left(v_1^j - v_2^j\right) v_{2x} d\tau}_{I} - \underbrace{\int_0^t S(t-\tau)\left(v_{1x} - v_{2x}\right) v_1^j d\tau}_{II} \right).$$

First, let us consider $\sup_t \|Fv_1 - Fv_2\|_{L^2}$. Applying Minkowski inequality and Proposition 3.2.1 to I, we obtain

$$\begin{aligned}
\sup_t \left\| \int_0^t S(t-\tau)\left(v_1^j - v_2^j\right) v_{2x} d\tau \right\|_{L^2} &\leq \sup_t \int_0^t \left\| S(t-\tau)\left(v_1^j - v_2^j\right) v_{2x} \right\|_{L^2} d\tau \\
\leq c \int_0^T \left\| \left(v_1^j - v_2^j\right) v_{2x} \right\|_{L^2} d\tau &\leq cT^{1/2} \left(\int_0^T \left\| \left(v_1^j - v_2^j\right) v_{2x} \right\|_{L^2}^2 d\tau \right)^{1/2} \\
\leq cT^{1/2} \sum_{\ell=0}^{j-1} \left(\int_0^T \left\| w v_1^\ell v_2^{j-1-\ell} v_{2x} \right\|_{L^2}^2 d\tau \right)^{1/2} &\leq cT^{1/2} j \left(\int_0^T \|w\|_{1/2}^2 v_3^{2(j-1)} \|v_{2x}\|_{L^2}^2 d\tau \right)^{1/2}
\end{aligned}$$

where

$$v_3(t) = \max\left(\|v_1\|_{1/2}, \|v_2\|_{1/2}\right).$$

Now, using generalized Hölder's inequality with exponents

$$(q_1, q_2, q_3) = ((2+\delta)n/\delta, (2+\delta)n/(\delta(n-1)), (2+\delta)/2)\,,$$

we can obtain

$$\begin{aligned}
& cT^{1/2} j \left(\int_0^T \|w\|_{1/2}^2 (v_3)^{2(j-1)} \|v_{2x}\|_{L^2}^2 d\tau \right)^{1/2} \\
& \leq cT^{1/2} j \left(\left(\int_0^T \|w\|_{1/2}^{2q_1} d\tau \right)^{1/q_1} \left(\int_0^T (v_3)^{2(j-1)q_2} d\tau \right)^{1/q_2} \left(\int_0^T \|v_{2x}\|_{L^2}^{2q_3} d\tau \right)^{1/q_3} \right)^{1/2} \\
& \leq cT^{1/2} j \left(\int_0^T \|w\|_{1/2}^{p_1} d\tau \right)^{1/p_1} \left(\int_0^T \|v_{2x}\|_{L^2}^{p_2} d\tau \right)^{1/p_2} T^{\frac{(n-j)}{2q_2(n-1)}} \left(\int_0^T v_3^{2(n-1)q_2} d\tau \right)^{\frac{(j-1)}{2q_2(n-1)}} \\
& \leq cT^{1/2} j \left(\int_0^T \|w\|_{1/2}^{p_1} d\tau \right)^{1/p_1} \left(\int_0^T \|v_{2x}\|_{L^2}^{p_2} d\tau \right)^{1/p_2} T^{\frac{(n-j)}{2q_2(n-1)}} \left(\int_0^T v_3^{p_1} d\tau \right)^{(j-1)/p_1} \\
& \leq cT^{\frac{1}{2} + \frac{(n-j)}{2q_2(n-1)}} \left(\int_0^T \|w\|_{1/2}^{p_1} d\tau \right)^{1/p_1} M^j\,.
\end{aligned}$$

Similarly, for II, it is deduced that

$$\begin{aligned}
&\sup_t \left\| \int_0^t S(t-\tau)\,(v_{1x}-v_{2x})\,v_1^j d\tau \right\|_{L^2} \le cT^{1/2} \left(\int_0^T \left\| w_x v_1^j \right\|_{L^2}^2 d\tau \right)^{1/2} \\
&\le cT^{\frac{1}{2}+\frac{(n-j)}{p_1}} \left(\int_0^T \|v_1\|_{1/2}^{p_1} d\tau \right)^{j/p_1} \left(\int_0^T \|w\|_1^{p_2} d\tau \right)^{1/p_2} \le cT^{\frac{1}{2}+\frac{(n-j)}{p_1}} \left(\int_0^T \|w\|_1^{p_2} d\tau \right)^{1/p_2} M^j \,,
\end{aligned}$$

which yield

$$\sup_t \|Fv_1 - Fv_2\|_{L^2} \le c \max_{1\le j\le n} \left(T^{\frac{1}{2}+\frac{(n-j)}{2q_2(n-1)}}, T^{\frac{1}{2}+\frac{(n-j)}{p_1}} \right) M \sum_{j=1}^n M^{j-1} \|v_1 - v_2\|_{Y_T} \,. \tag{3.3.14}$$

Next, consider $\left(\int_0^T \|Fv_1 - Fv_2\|_{m/2}^{p_m} dt \right)^{1/p_m}$ with $m = 1, 2$. Applying Proposition 3.2.3 for I and II and the estimates used for proving (3.3.14), we have

$$\begin{aligned}
&\left(\int_0^T \left\| \int_0^t S(t-\tau)\left(v_1^j - v_2^j\right) v_{2x} d\tau \right\|_{m/2}^{p_m} dt \right)^{1/p_m} \\
&\le cT^{1/p_m} \sup_{0\le t\le T} \left\| \int_0^t S(t-\tau)\left(v_1^j - v_2^j\right) v_{2x} d\tau \right\|_{m/2} \le cT^{1/p_m} \left(\int_0^T \left\| \left(v_1^j - v_2^j\right) v_{2x} \right\|_{L^2}^2 d\tau \right)^{1/2} \\
&\le cT^{\frac{1}{p_m}+\frac{(n-j)}{2q_2(n-1)}} \left(\int_0^T \|w\|_{1/2}^{p_1} d\tau \right)^{1/p_1} M^j
\end{aligned}$$

and

$$\begin{aligned}
&\left(\int_0^T \left\| \int_0^t S(t-\tau) v_1^j\,(v_{1x}-v_{2x})\,d\tau \right\|_{m/2}^{p_m} dt \right)^{1/p_m} \le cT^{1/p_m} \left(\int_0^T \left\| v_1^j\,(v_{1x}-v_{2x}) \right\|_{L^2}^2 d\tau \right)^{1/2} \\
&\le cT^{\frac{1}{p_m}+\frac{(n-j)}{p_1}} \left(\int_0^T \|w\|_1^{p_2} d\tau \right)^{1/p_2} M^j \,,
\end{aligned}$$

which give

$$\left(\int_0^T \|Fv_1 - Fv_2\|_{m/2}^{p_m} dt \right)^{1/p_m} \le c \max_{1\le j\le n} \left(T^{\frac{1}{p_m}+\frac{(n-j)}{2q_2(n-1)}}, T^{\frac{1}{p_m}+\frac{(n-j)}{p_1}} \right) M \sum_{j=1}^n M^{j-1} \|v_1 - v_2\|_{Y_T} \,. \tag{3.3.15}$$

From (3.3.14) and (3.3.15), we have shown that

$$\|Fv_1 - Fv_2\|_{Y_t} \le cK(T) M \sum_{j=1}^n M^{j-1} \|w\|_{Y_t} \le (1/2)\|v_1 - v_2\|_{Y_t} \,, \tag{3.3.16}$$

where

$$K(T) = \max_{1\le j\le n, m=1,2} \left(T^{\frac{1}{p_m}+\frac{(n-j)}{2q_2(n-1)}}, T^{\frac{1}{p_m}+\frac{(n-j)}{p_1}}, T^{\frac{1}{2}+\frac{(n-j)}{2q_2(n-1)}}, T^{\frac{1}{2}+\frac{(n-j)}{p_1}} \right)$$

and T is small such that $cK(T)M\sum_{j=1}^n M^{j-1} \le 1/2$. Thus, the contraction mapping principle yields the existence and uniqueness of solution of (3.3.3) in Y_T.

To prove the Lipschitz continuity, we note that for any neighborhood U of $\tilde{w}_0$ in $L^2 \cap H_\alpha^{s_1',p_1} \cap H_\alpha^{s_2',p_2}$, there is a $T' < T$ such that the mapping, $N : U \to Y_{T'}$, is well defined. Thus, we only need to prove the map N is Lipschitz continuous from U to $Y_{T'}$. For any $w_1, w_2 \in U$, let $u_1 = Nw_1, u_2 = Nw_2$, and $u = u_1 - u_2$. Then,

$$\begin{aligned} u =& S(t)(w_1 - w_2) + \int_0^t S(t-\tau)\left(\sum_{j=1}^n a_j u_1^j u_{1x}\right) d\tau - \int_0^t S(t-\tau)\left(\sum_{j=1}^n a_j u_2^j u_{2x}\right) d\tau \\ =& S(t)(w_1 - w_2) + Fu_1 - Fu_2 \ . \end{aligned}$$

By Proposition 3.2.1 and (3.3.14), we have

$$\begin{aligned} \sup_t \|u\|_{L^2} \leq& \sup_t \|S(t)(w_1 - w_2)\|_{L^2} + \sup_t \|Fu_1 - Fu_2\|_{L^2} \\ \leq& c\|w_1 - w_2\|_{L^2} + c \max_{1\leq j\leq n}\left(T^{\frac{1}{2}+\frac{(n-j)}{2q_2(n-1)}}, T^{\frac{1}{2}+\frac{(n-j)}{p_1}}\right) M \sum_{j=1}^n M^{j-1}\|u_1 - u_2\|_{Y_T} \end{aligned}$$

and from Proposition 3.2.6 and (3.3.15), we obtain that for $m = 1, 2$

$$\begin{aligned} &\left(\int_0^T \|u\|_{m/2}^{p_m} dt\right)^{1/p_m} \leq \left(\int_0^T \|S(t)(w_1 - w_2)\|_{m/2}^{p_m} dt\right)^{1/p_m} + \left(\int_0^T \|Fu_1 - Fu_2\|_{m/2}^{p_m} dt\right)^{1/p_m} \\ &\leq c\|w_1 - w_2\|_{s_m',p_m} + c \max_{1\leq j\leq n}\left(T^{\frac{1}{p_m}+\frac{(n-j)}{2q_2(n-1)}}, T^{\frac{1}{p_m}+\frac{(n-j)}{p_1}}\right) M \sum_{j=1}^n M^{j-1}\|u_1 - u_2\|_{Y_T} \ , \end{aligned}$$

which imply that there is a $\rho \in (0, 1)$ dependent upon $0 < T' < T$ such that

$$\begin{aligned} \|u_1 - u_2\|_{Y_t} &\leq c\sum_{m=1}^2 \|w_1 - w_2\|_{s_m',p_m} + c\|w_1 - w_2\|_{L^2} + \rho\|u_1 - u_2\|_{Y_T} \\ \text{or} \quad \|u_1 - u_2\|_{Y_t} &\leq \frac{c}{1-\rho}\left(\sum_{m=1}^2 \|w_1 - w_2\|_{s_m',p_m} + \|w_1 - w_2\|_{L^2}\right) \end{aligned}$$

and N is Lipschitz continuous from U to $Y_{T'}$ for $T' < T$. □

Finally, consider the regularity of solutions of (3.3.1), i.e., for any given $n > 0$, $u(\cdot, t) \in H_\alpha^n$ if $w_0 \in H_\alpha^n$. Note that from Proposition 3.2.1, $\|S(t)w_0\|_s \leq \|w_0\|_s$ for any $s \geq 0$. However, when we consider the nonlinear case, the problem is more complicated since u is defined in a special space H_α^n which is a Hilbert space inherited from Sobolev space H^n with periodic boundary conditions except at $\partial_x^{5i+1}u$ where $i = 0, 1, 2, ...$ and $5i + 1 < n$. Therefore

$$u \in H_\alpha^n \quad \not\Rightarrow \quad u_x \in H_\alpha^{n-1} \quad n \geq 2.$$

Since the smoothing property in Section 3 is set up on H_α^n, we cannot apply it to (3.3.3) like we did in the proof of Theorem 3.3.1. However, we can see that $\partial_t u$ and $\partial_x^5 u$ are in the same space H^n. This property tells us that we may obtain the regularity of $\partial_t u$ to establish the regularity of u.

First, we consider the estimates for $\dot{f}(\cdot, t) \equiv \partial_t f(\cdot, t)$.

Lemma 3.3.3. *If* $f \in C([0,T]; H^1_\alpha)$ *and* $\dot{f} \in L^2([0,T]; L^2)$*, then*

$$\partial_t \left(\int_0^t S(t-\tau) f(\cdot,\tau) d\tau \right) = S(t) f(\cdot,0) + \int_0^t S(t-\tau) \dot{f}(\cdot,\tau) d\tau$$

and

$$\sup_{0\le t\le T} \left\| \partial_t \int_0^t S(t-\tau) f(\cdot,\tau) d\tau \right\|_1 \le c \|f(\cdot,0)\|_1 + c \left(\int_0^T \|\dot{f}(\cdot,\tau)\|^2_{L^2} dt \right)^{1/2}, \tag{3.3.17}$$

where $c > 0$ *is independent of* T*. If* $\int_0^{2\pi} \dot{f}(x,t)dx \equiv 0$*,*

$$\sup_{0\le t<\infty} \left\| \partial_t \int_0^t S(t-\tau) f(\cdot,\tau) d\tau \right\|_1 \le c \|f(\cdot,0)\|_1 + c \sup_{0\le t<\infty} \|\dot{f}(\cdot,\tau)\|_{L^2}. \tag{3.3.18}$$

Proof. First, we note that

$$u = \int_0^t S(t-\tau) f(\cdot,\tau) d\tau$$

is the solution of (3.2.2) with the initial condition equal to zero. Let $v = \partial_t u$. Thus

$$\begin{aligned} v(\cdot,t) &= \partial_t \int_0^t S_\alpha(t-\tau) f(\cdot,\tau) d\tau \\ &= S_\alpha(t-t) f(\cdot,t) \\ &= S_\alpha(0) \cdot f(\cdot,t) \\ &= f(\cdot,t). \end{aligned}$$

Then, $v(x,0) = f(x,0)$ and $v(x,t)$ satisfies

$$\begin{cases} \partial_t v + \partial_x^3 v - \partial_x^5 v = \dot{f}, & 0 < x < 2\pi, \quad t \ge 0 \\ v(x,0) = f(x,0), & \\ v_x(2\pi,t) = \alpha v_x(0,t) + (1-\alpha) v_{3x}(0,t), & \\ v_{kx}(0,t) = v_{kx}(2\pi,t), & k = 0,2,3,4, \end{cases}$$

or

$$v(\cdot,t) = S(t) f(\cdot,0) + \int_0^t S(t-\tau) \dot{f}(\cdot,\tau) d\tau\,.$$

Propositions 3.2.1 and 3.2.3 yield

$$\begin{aligned} \sup_{0\le t\le T} \|S(t) f(\cdot,0)\|_1 &\le c \|f(\cdot,0)\|_1, \\ \sup_{0\le t\le T} \left\| \int_0^t S(t-\tau) \dot{f}(\cdot,\tau) d\tau \right\|_1 &\le c \left(\int_0^T \|\dot{f}(\cdot,\tau)\|^2_{L^2} dt \right)^{1/2}, \end{aligned}$$

which imply (3.3.17). Similarly, Proposition 3.2.5 gives

$$\sup_{0\le t\le\infty} \left\| \int_0^t S(t-\tau) \dot{f}(\cdot,\tau) d\tau \right\|_1 \le c \sup_{0\le t\le\infty} \|\dot{f}(\cdot,t)\|_{L^2}\,.$$

Therefore, (3.3.18) is obtained. □

Now, we discuss the regularity of (3.3.1) for the initial value

$$w_0 \in Y = \left\{ \phi \in H^6 \cap H^5_\alpha \;\middle|\; \phi^{(5)} - \phi^{(3)} + \sum_{j=1}^{n} a_j \phi^j \phi' \in H^1_\alpha \right\} .$$

Let us define a Banach space Z_T as

$$Z_T = \left\{ u \in C^1(0,T; H^1_\alpha) \;\middle|\; \sup_{0\le t\le T} \|u(t)\|_1 + \sup_{0\le t\le T} \|\dot u(t)\|_1 < \infty \right\}$$

with the norm

$$\|u\|_{Z_T} := \left(\sup_{0\le t\le T} \|u(t)\|_1^2 + \sup_{0\le t\le T} \|\dot u(t)\|_1^2 \right)^{1/2} .$$

If we can show that for any $w_0 \in Y$ there is a unique solution $u \in Z_T$, then

$$\partial_t u = \partial_{5x} u \in C([0,T]; H^1_\alpha) \quad \text{and} \quad u \in C([0,T]; H^6 \cap H^5_\alpha) .$$

Theorem 3.3.4. *For any $w_0 \in Y$, there exists a $T = T(\|w_0\|_Y) > 0$ such that the IBVP (3.3.1) has a unique solution $w \in Z_T$. Moreover, for any $T' < T$, there is a neighborhood U of $\tilde w_0$ in Y such that the mapping $N : w_0 \to w(\cdot,t)$ from U to $Z_{T'}$ is Lipschitz continuous.*

Proof. Let

$$B_{T,M} = \left\{ v \in Z_T \;\middle|\; \|v\|_{Z_T} \le M, v(x,0) = w_0(x) \right\}$$

for some T and M to be determined later. In order to prove the existence and uniqueness, we need to show that the mapping F in (3.3.3) defined on $B_{T,M}$ is a contraction by choosing appropriate T and M. Similar to the proof of Theorem 3.3.1, we first show that for any $v \in B_{T,M}$,

$$\|Fv\|_{Z_T} \le \sup_t \|Fv\|_1 + \sup_t \|\partial_t Fv\|_1 \le M .$$

For any $v \in B_{T,M}$, it is straightforward to see that

$$\begin{aligned}
\sup_{0\le t\le T} \|Fv\|_1 &\le \sup_{0\le t\le T} \|S(t)w_0\|_1 + \sup_{0\le t\le T} \left\| \int_0^t S(t-\tau) \left(\sum_{j=1}^{n} a_j v^j v_x \right)(\cdot,\tau) d\tau \right\|_1 \\
&\le c\|w_0\|_1 + c \left[\int_0^T \left\| \sum_{j=1}^{n} a_j v^j v_x \right\|_{L^2}^2 d\tau \right]^{1/2} \\
&\le c\|w_0\|_1 + cT^{1/2} \sum_{j=1}^{n} \sup_t \|v^j v_x\|_{L^2} \\
&\le c\|w_0\|_1 + cT^{1/2} \left(\sum_{j=1}^{n} \sup_t \|v\|_1^{j+1} \right) .
\end{aligned}$$

Write $\partial_t Fv$ as

$$\partial_t(Fv) = \underbrace{\partial_t\left(S(t)w_0\right)}_{I} + \underbrace{\partial_t \int_0^t S(t-\tau)\left(\sum_{j=1}^n a_j v^j v_x\right)(\cdot,\tau)d\tau}_{II}.$$

For I, since $w = S(t)w_0$ satisfies (3.2.2) with $f = 0$, $v = \partial_t w$ satisfies the following equation

$$\begin{cases} \partial_t v + \partial_x^3 v - \partial_x^5 v = 0, & 0 < x < 2\pi, t \geq 0 \\ v(x,0) = w_t(x,0) = -\partial_x^3 w_0(x) + \partial_x^5 w_0(x), & \\ v_x(2\pi,t) = \alpha v_x(0,t) + (1-\alpha)v_{3x}(0,t), & \\ v_{kx}(0,t) = v_{kx}(2\pi,t), & k = 0,2,3,4, \end{cases}$$

which yields

$$\partial_t(S(t)w_0) = v = S(t)v_0(x) = S(t)\left(-\partial_x^3 w_0(x) + \partial_x^5 w_0(x)\right).$$

For II, by Lemma 3.3.3, we obtain

$$\begin{aligned} &\partial_t \int_0^t S(t-\tau)\left(\sum_{j=1}^n a_j v^j v_x\right)(\cdot,\tau)d\tau \\ &= S(t)\left(\sum_{j=1}^n a_j w_0^j w_{0x}\right) + \int_0^t S(t-\tau)\partial_\tau\left(\sum_{j=1}^n a_j v^j v_x\right)(\cdot,\tau)d\tau. \end{aligned}$$

Thus,

$$\begin{aligned} \sup_t \|\partial_t(Fv)\|_1 \leq & \underbrace{\sup_t \left\| S(t)\left(w_0^{(5)} - w_0^{(3)} + \sum_{j=1}^n a_j w_0^j w_{0x}\right)\right\|_1}_{II_1} \\ & + \underbrace{\sup_t \left\|\int_0^t S(t-\tau)\partial_\tau\left(\sum_{j=1}^n a_j v^j v_x\right)(\cdot,\tau)d\tau\right\|_1}_{II_2}. \end{aligned}$$

Applying Proposition 3.1.1 to II_1 yields,

$$\sup_t \|II_1\|_1 \leq c\left(\|w_0\|_6 + \|w_0\|_4 + \sum_{j=1}^n \|w_0\|_2^{j+1}\right) \leq c\left(\|w_0\|_6 + \sum_{j=1}^n \|w_0\|_2^{j+1}\right)$$

For II_2, Proposition 3.1.3 implies

$$\begin{aligned}\sup_t \left\| \int_0^t S(t-\tau)\partial_\tau \left(\sum_{j=1}^n a_j v^j v_x \right)(\cdot,\tau)d\tau \right\|_1 &\le c \left[\int_0^T \left\| \partial_\tau \left(\sum_{j=1}^n a_j v^j v_x \right) \right\|_{L^2}^2 d\tau \right]^{1/2} \\ \le cT^{1/2} \sup_t \left\| \partial_t \left(\sum_{j=1}^n a_j v^j v_x \right) \right\|_{L^2} &\le cT^{1/2} \sup_t \| \sum_{j=1}^n a_j j v^{j-1} \dot{v} v_x + a_j v^j \dot{v}_x \|_{L^2} \\ \le cT^{1/2} \left(\sum_{j=1}^n \sup_t \|v\|_1^j \|\dot{v}\|_1 \right) &\le cT^{1/2} \left(\sum_{j=1}^n \sup_t \|v\|_1^{2j} + \sup_t \|\dot{v}\|_1^2 \right).\end{aligned}$$

Hence, the estimate for $\sup_t \|\partial_t(Fv)\|_1$ is

$$\sup_t \|\partial_t(Fv)\|_1 \le c \left(\|w_0\|_6 + \sum_{j=1}^n \|w_0\|_2^{j+1} \right) + cT^{1/2} \left(\sum_{j=1}^n \sup_t \|v\|_1^{2j} + \sup_t \|\dot{v}\|_1^2 \right).$$

Since $\sup_t \|v\|_1^{j+1}$ is controlled by $\sup_t \|v\|_1^{2j}$ and $\sup_t \|v\|_1^2$, we obtain

$$\begin{aligned}\|Fv\|_{Z_T} \le & c \left(\|w_0\|_6 + \sum_{j=1}^n \|w_0\|_2^{j+1} \right) + cT^{1/2} \left(\sum_{j=1}^n \sup_t \|v\|_1^{2j} + \sup_t \|\dot{v}\|_1^2 \right) \\ \le & c \left(\|w_0\|_6 + \sum_{j=1}^n \|w_0\|_2^{j+1} \right) + cT^{1/2} \left(\sum_{j=1}^n M^{2j} \right).\end{aligned}$$

If we choose

$$c \left(\|w_0\|_6 + \sum_{j=1}^n \|w_0\|_2^{j+1} \right) = (M/2)$$

and $T > 0$ such that

$$cT^{1/2} \cdot M \max \left(\sum_{j=1}^n M^{2j-2}, \sum_{j=1}^n M^{j-1} \right) \le (1/2),$$

then $\|Fv\|_{Z_T} \le M$.

To show F is a contraction on $B_{T,M}$, let $v_1, v_2 \in B_{T,M}$ and $w = v_1 - v_2$, which gives

$$Fv_1 - Fv_2 = \int_0^t S(t-\tau) \left(\sum_{j=1}^n a_j v_1^j v_{1x} \right) d\tau - \int_0^t S(t-\tau) \left(\sum_{j=1}^n a_j v_2^j v_{2x} \right) d\tau.$$

In (3.3.7) and (3.3.8), we have proved that if $w = v_1 - v_2$,

$$\sup_t \|Fv_1 - Fv_2\|_1 \le cT^{1/2} \left(\sum_{j=1}^n \left(\sup_t \|v_1\|_1^j + \sup_t \|v_2\|_1^j \right) \right) \cdot \sup_t \|w\|_1. \tag{3.3.19}$$

Because of $v_1(\cdot,0) = v_2(\cdot,0)$,

$$\partial_t(Fv_1 - Fv_2) = \int_0^t S(t-\tau)\partial_\tau\left(\sum_{j=1}^n a_j v_1^j v_{1x}\right) d\tau - \int_0^t S(t-\tau)\partial_\tau\left(\sum_{j=1}^n a_j v_2^j v_{2x}\right) d\tau\,.$$

Applying Proposition 3.2.3 to $\partial_t(Fv_1 - Fv_2)$ yields

$$\begin{aligned}
\sup_t \|\partial_t(Fv_1 - Fv_2)\|_1 &\le \sum_{j=1}^n a_j \left[\int_0^T \|\partial_t(v_1^j - v_2^j)v_{2x}\|_{L^2}^2 d\tau\right]^{1/2} \\
&+ \sum_{j=1}^n a_j \left[\int_0^T \|\partial_t(v_{1x} - v_{2x})v_1^j\|_{L^2}^2 d\tau\right]^{1/2} \\
&\le \sum_{j=1}^n cT^{1/2} \sup_t \|\partial_t(v_1^j - v_2^j)v_{2x}\|_{L^2} + \sum_{j=1}^n cT^{1/2} \sup_t \|\partial_t(v_{1x} - v_{2x})v_1^j\|_{L^2} \\
&\le \sum_{j=1}^n cT^{1/2}\left(\underbrace{\sup_t \|(jv_1^{j-1}\dot v_1 - jv_2^{j-1}\dot v_2)\cdot v_{2x}\|_{L^2}}_{III_1} + \underbrace{\sup_t \|(v_1^j - v_2^j)\dot v_{2x}\|_{L^2}}_{III_2}\right) \\
&+ \sum_{j=1}^n cT^{1/2}\left(\underbrace{\sup_t \|(\dot v_{1x} - \dot v_{2x})v_1^j\|_{L^2}}_{III_3} + \underbrace{\sup_t \|(v_{1x} - v_{2x})jv_1^{j-1}\dot v_1\|_{L^2}}_{III_4}\right).
\end{aligned}$$

For III_1,

$$\begin{aligned}
\sup_t \|(jv_1^{j-1}\dot v_1 - jv_2^{j-1}\dot v_2)\cdot v_{2x}\|_{L^2} &= \sup_t \|\left[jv_1^{j-1}(\dot v_1 - \dot v_2) + j\dot v_2(v_1^{j-1} - v_2^{j-1})\right]\cdot v_{2x}\|_{L^2} \\
&\le \sup_t \left[j\|v_1\|_1^{j-1}\|\dot w\|_1 + j\|\dot v_2\|_1\|v_1^{j-1} - v_2^{j-1}\|_1\|v_2\|_1\right] \\
&\le \sup_t j\|v_1\|_1^{j-1}\|\dot w\|_1 + \sup_t j\|\dot v_2\|_1 \sum_{l=0}^{j-2}\|v_1\|_1^l\|v_2\|_1^{j-1-l}\|w\|_1\|v_2\|_1.
\end{aligned}$$

For III_2, we obtain

$$\begin{aligned}
\sup_t \|(v_1^j - v_2^j)\dot v_{2x}\|_{L^2} &= \sup_t \|\sum_{l=0}^{j-1} v_1^l v_2^{j-1-l} w \dot v_{2x}\|_{L^2} \\
&\le \sum_{l=0}^{j-1}\|v_1\|_1^l\|v_2\|_1^{j-1-l}\|w\|_1\|\dot v_2\|_1.
\end{aligned}$$

We can simply estimate III_3 and III_4 to have

$$\begin{aligned}
\sup_t \|(\dot v_{1x} - \dot v_{2x})v_1^j\|_{L^2} &\le \sup_t \|\dot w\|_1\|v_1\|_1^j, \\
\sup_t \|(v_{1x} - v_{2x})jv_1^{j-1}\dot v_1\|_{L^2} &\le \sup_t j\|w\|_1\|v_1\|_1^{j-1}\|\dot v_1\|_1.
\end{aligned}$$

Combining them together yields

$$\begin{aligned}\sup_t \|\partial_t(Fv_1 - Fv_2)\|_1 \leq & cT^{1/2}\left(\sup_t \|\dot{v_1}\|_1 + \sup_t \|\dot{v_2}\|_1\right)\sup_t \|w\|_1 \left(\sum_{j=1}^{n}\sum_{\ell=0}^{j-1} \|v_1\|_1^{\ell}\|v_2\|_1^{j-1-\ell}\right) \\ & + cT^{1/2}\left(\sum_{j=1}^{n}\left(\sup_t \|v_1\|_1^j + \sup_t \|v_2\|_1^j\right)\right)\cdot \sup_t \|\dot{w}\|_1. \qquad (3.3.20)\end{aligned}$$

Combining (3.3.19) and (3.3.20), we have

$$\|Fv_1 - Fv_2\|_{Z_t} \leq cT^{1/2}\left(\sum_{j=1}^{n} M^j\right)(\sup_t \|w\|_1 + \sup_t \|\dot{w}\|_1) \leq (1/2)\|v_1 - v_2\|_{Z_t}\,.$$

Hence, F is a contraction and has a fixed point $u \in Z_T$ which is the unique solution of (3.3.1). Thus, $u_t \in C(0,T;H^1_\alpha)$. Since $u_t = \partial_x^5 u - \partial_x^3 u + \sum_{j=1}^n a_j u^j u_x$, $u \in C(0,T;H^6)$. In addition, the boundary conditions require $u \in C(0,T;H^5_\alpha)$, which implies $u \in C(0,T;H^6 \cap H^5_\alpha)$.

To prove that the mapping N defined in Theorem 3.3.4 is Lipschitz continuous, we let $w_1, w_2 \in U$ and $u_1 = Nw_1, u_2 = Nw_2$ with $u = u_1 - u_2$. Then

$$\|u\|_{Z_T} \leq \|S(t)(w_1 - w_2)\|_{Z_T} + \|Fu_1 - Fu_2\|_{Z_T},$$

where

$$\begin{aligned}\|S(t)(w_1 - w_2)\|_{Z_T} &\leq \sup_t \|S(t)(w_1 - w_2)\|_1 + \sup_t \|\partial_t\left(S(t)(w_1 - w_2)\right)\|_1 \\ &\leq c\|w_1 - w_2\|_1 + \left\|S(t)\partial_x^5(w_1 - w_2) - S(t)\partial_x^3(w_1 - w_2)]\right\|_1 \\ &\leq c\left(\|w_1 - w_2\|_1 + \|w_1 - w_2\|_6 + \|w_1 - w_2\|_4\right) \leq c\|w_1 - w_2\|_6\end{aligned}$$

and

$$\|Fu_1 - Fu_2\|_{Z_T} \leq \rho\|u_1 - u_2\|_{Z_T}\,, \quad \rho \in (0,1),$$

which give

$$\|u_1 - u_2\|_{Z_T} \leq \frac{c}{1-\rho}\|w_1 - w_2\|_6\,.$$

The proof is completed. □

Note that

$$\partial_t(u) = \partial_x^5(u) - \partial_x^3(u) + \sum_{j=1}^{n} \frac{a_j}{j+1}\partial_x u^{j+1}\,.$$

Define a series of differential operators $\{P_k\}, k = 0, 1, ..., m$ as follows. For any $\phi \in H^{5m}$

$$\begin{cases} P_0(\phi) = \phi, \\ P_1(\phi) = \sum_{j=1}^{n} \frac{a_j}{j+1} \partial_x(\phi^{j+1}) + \partial_x^5 \phi - \partial_x^3 \phi \,, \\ \cdots\cdots\cdots \\ P_m(\phi) = \sum_{j=1}^{n} \frac{a_j}{j+1} \partial_x \left((P_{m-1}(\phi))^{j+1} \right) + \partial_x^5 P_{m-1}(\phi) - \partial_x^3 P_{m-1}(\phi) \,. \end{cases}$$

Then, by a similar argument as that for Theorem 3.3.4, we can obtain the following Theorem.

Theorem 3.3.5. *If $m \geq 1$ is given, then for any $w_0 \in H_\alpha^5 \cap H^{5m+1}$ with*

$$P_k(w_0) \in H_\alpha^5 \cap H^{5(m-k)+1}, \quad k = 0, 1, 2, ..., m-1,$$

and $P_m(w_0) \in H_\alpha^1$, there is a $T = T(\|w_0\|_{5m+1}) > 0$ such that a unique solution w of (3.3.1) *exists satisfying*

$$\partial_t^k w \in C([0, T]; H_\alpha^5 \cap H^{5(m-k)+1}), \quad k = 0, 1, 2, ..., m-1,$$

and $\partial_t^m w \in C(0, T; H_\alpha^1)$.

Chapter 4

Global Well-Posedness and Decay of Small Amplitude Solutions

4.1 Global Well-Posedness of The Nonlinear Problem

In Section 4, we proved that the unique solution of (3.3.1) exists in a finite time interval $[0, T)$ where T depends upon the size of initial value w_0. Thus, we can only say that the local solution exists. In this section, we consider whether T can be infinite, called global well-posedness, and the behavior of the solution as $t \to \infty$ if $T = \infty$.

Theorem 4.1.1. *If $w_0 \in H^1_\alpha$, either the solution exists for all time $t > 0$ or there exists a finite T^* such that (3.3.1) has a unique solution $w \in C([0, T^*); H^1_\alpha)$ and $\lim_{t\to T^*} \|w(\cdot, t)\|_1 = \infty$ where T^* is called the lifespan of the solution.*

Proof. Following the proof of Theorem 3.3.1, if $w_0 \in H^1_\alpha$, the solution of (3.3.1) exists as long as $\|w\|_1$ is bounded since we can always extend the solution to a small interval for t. Therefore, either $w \in C([0, \infty); H^1_\alpha)$ or $\|w\|_1$ blows up at finite $T^* > 0$. The proof is complete. □

Theorem 4.1.2. *Let*

$$Y = \left\{ u \;\middle|\; u \in C([0, \infty); H^1_\alpha) \cap L^\infty([0, \infty); H^1_\alpha) \right\} .$$

Then, there is a $\nu > 0$ such that for any $w_0 \in H^1_\alpha$ with $\|w_0\|_1 \le \nu$, (3.3.3) has a unique fixed point $w \in Y$ which is the unique solution of (3.3.1). Moreover, the solution mapping from $w_0 \in H^1_\alpha$ to Y is Lipschitz continuous.

Proof. Let

$$B_M = \left\{ v \in C\left([0, \infty); H^1_\alpha\right) \;\middle|\; \sup_{0 \le t < \infty} \|v(\cdot, t)\|_1 \le M \right\}$$

where $M > 0$ is to be determined. First, let us show that F maps B_M to itself by choosing appropriate M. Propositions 3.2.1 and 3.2.5 yield

$$\sup_{0\le t<\infty}\|Fv\|_1 \le \sup_{0\le t<\infty}\|S(t)w_0\|_1 + \sup_{0\le t<\infty}\left\|\int_0^t S(t-\tau)\left(\sum_{j=1}^n a_j v^j v_x\right)(\cdot,\tau)d\tau\right\|_1$$

$$\le c\|w_0\|_1 + c\sup_{0\le t<\infty}\left\|\sum_{j=1}^n a_j v^j v_x\right\|_{L^2} \le c\|w_0\|_1 + c\sup_t \|v\|_1^2\left(\sum_{j=1}^n \sup_t \|v\|_1^{j-1}\right).$$

If we choose $c\nu \le (M/2)$ and $cM^2\sum_{j=1}^n M^{j-1} \le (M/2)$, then $\sup_{0\le t<\infty}\|Fv\|_1 \le M$ if $\|w_0\|_1 \le \nu$. Next, we show that F is a contraction on B_M. Let $w = v_1 - v_2$ where $v_1, v_2 \in B_M$. Then,

$$\sup_{0\le t<\infty}\|Fv_1 - Fv_2\|_1 \le \sup_{0\le t<\infty}\left\|\int_0^t S(t-\tau)\left(\sum_{j=1}^n \left(v_1^j - v_2^j\right)v_{2x}\right)d\tau\right\|_1$$

$$+ \sup_{0\le t<\infty}\left\|\int_0^t S(t-\tau)\left(\sum_{j=1}^n v_1^j(v_{1x}-v_{2x})\right)d\tau\right\|_1$$

$$\le c\sup_{0\le t<\infty}\left\|\sum_{j=1}^n \left(v_1^j - v_2^j\right)v_{2x}\right\|_{L^2} + c\sup_{0\le t<\infty}\left\|\sum_{j=1}^n v_1^j w_x\right\|_{L^2}$$

$$\le cM\sup_{0\le t<\infty}\|v_1 - v_2\|_1\sum_{j=1}^n M^{j-1} \le (1/2)\sup_{0\le t<\infty}\|v_1 - v_2\|_1.$$

Therefore, F is a contraction on B_M, which yields a fixed point of F in B_M. To prove the Lipschitz continuity, we note that for any neighborhood U of $\tilde{w}_0$ in H_α^1, there is a $T' < T$ such that the mapping, $N : U \to Y_{T'}$, is well defined. Thus we only need to prove that the mapping N is Lipschitz continuous. For any $w_1, w_2 \in U$, let $u_1 = Nw_1, u_2 = Nw_2$, and $u = u_1 - u_2$. Then,

$$u = S(t)(w_1 - w_2) + \int_0^t S(t-\tau)\left(\sum_{j=1}^n a_j u_1^j u_{1x}\right)d\tau - \int_0^t S(t-\tau)\left(\sum_{j=1}^n a_j u_2^j u_{2x}\right)d\tau.$$

Replacing T with T' implies

$$\sup_{0\le t<\infty}\|u(t)\|_1 \le c\|w_1 - w_2\|_1 + \rho\sup_{0\le t<\infty}\|u_1 - u_2\|_1, \quad 0<\rho<1,$$

$$\sup_{0\le t<\infty}\|u_1 - u_2\|_1 \le \frac{c}{1-\rho}\|w_1 - w_2\|_1.$$

Thus, N is Lipschitz continuous from U to $Y_{T'}$. □

4.2 Exponential Decay of Small Amplitude Solution

Now, we prove the exponential decay of small amplitude solutions.

Theorem 4.2.1. *Let $\nu > 0$ be given in Theorem 4.1.2. Then there is a γ satisfying $0 < \gamma \leq \nu$ such that if $w_0 \in H^1_\alpha$ with $\|w_0\|_1 < \gamma$, the unique solution w of (3.3.1) obtained in Theorem 4.1.2 decays exponentially to $[w_0]/(2\pi)$, i.e.,*

$$\left\| w(\cdot,t) - \frac{1}{2\pi}[w_0] \right\|_{L^2} \leq ce^{-\rho t} \left\| w_0 - \frac{1}{2\pi}[w_0] \right\|_{L^2}, \quad t \geq 0, \tag{4.2.1}$$

where $c > 0$ and $\rho > 0$ are independent of w_0 and t.

Proof. The Lyapunov method (see [25, 36, 43]) will be used to prove the theorem. First, let us denote a subspace of L^2 by

$$L^2_0 = L^2_0[0,2\pi] = \left\{ w \in L^2[0,2\pi] \;\middle|\; \int_0^{2\pi} w(x)dx = 0 \right\}$$

and only consider the solution w lying in the $H^1_{\alpha,0}$, which changes (4.2.1) to

$$\|w(\cdot,t)\|_{L^2_0} \leq ce^{-\rho t}\|w_0\|_{L^2_0}, \quad t > 0. \tag{4.2.2}$$

We define

$$Z = \sum_{k\neq 0} Z_k : L^2_0 \to L^2_0$$

as the strongly convergent series operator with $Z_k = \psi_k \psi_k^*$. To show that Z is bounded and positive definite on L^2_0, we write $w \in L^2_0$ as $w = \sum_{j\neq 0} c_j \phi_j$ and obtain that

$$w^* Z w = \left(\sum_{j\neq 0} c_j \phi_j \right)^* \sum_{k\neq 0} \psi_k \psi_k^* \left(\sum_{j\neq 0} c_j \phi_j \right) = \sum_{j\neq 0} c_j^* \phi_j^* \psi_j \psi_j^* c_j \phi_j = \sum_{j\neq 0} |c_j|^2 ,$$

because of $\phi_k^* \cdot \psi_j = \delta_{kj}$. Then, we can see

$$w^* Z w = \sum_{j\neq 0} |c_j|^2 \geq D_0 \left\| \sum_{j\neq 0} c_j \phi_j \right\|^2_{L^2_0} = D_0 \|w\|^2_{L^2_0} .$$

We further define $Y : L^2_0 \to L^2_0$ by

$$Y = \sum_{k\neq 0} \xi_k Z_k, \qquad \xi_k = -(2 \operatorname{Re} \lambda_k)^{-1} .$$

Since $\mathrm{Re}\,\lambda_k \sim -2\beta k^2$ for large k and no eigenvalues with $k \neq 0$ are on the imaginary axis or in right-half complex plane, ξ_k is positive. Therefore, Y is bounded, symmetric and non-negative. Since

$$\begin{aligned}A^*Y + YA + Z &= \sum_{k\neq 0} \left[A^*\xi_k Z_k + \xi_k Z_k A + Z_k\right] \\ &= \sum_{k\neq 0} \left[\xi_k \left((A^*\psi_k)\psi_k^* + \psi_k (A^*\psi_k)^*\right) + \psi_k\psi_k^*\right] \\ &= \sum_{k\neq 0} \left[\xi_k \left((\lambda_k^*\psi_k)\psi_k^* + \psi_k (\lambda_k^*\psi_k)^*\right) + \psi_k\psi_k^*\right] \\ &= \sum_{k\neq 0} \left[\xi_k(\lambda_k^* + \lambda_k) + 1\right]\psi_k\psi_k^* = \sum_{k\neq 0} \left[\xi_k(2\mathrm{Re}\,\lambda_k) + 1\right]\psi_k\psi_k^* = 0,\end{aligned}$$

let $u' = \frac{du}{dx}$, we have

$$\begin{aligned}\frac{d}{dt}\left(w^*Yw\right) &= \left(\frac{d}{dt}w\right)^* Yw + w^*Y\left(\frac{d}{dt}w\right) \\ &= \left[Aw + \sum_{j=1}^{n} a_j w^j w'\right]^* Yw + w^*Y\left[Aw + \sum_{j=1}^{n} a_j w^j w'\right] \\ &= w^*\left(A^*Y + YA\right)w + \left[\left(\sum_{j=1}^{n} a_j w^j w'\right)^* Yw + w^*Y\left(\sum_{j=1}^{n} a_j w^j w'\right)\right] \\ &= -w^*Zw + \left[\left(\sum_{j=1}^{n} a_j w^j w'\right)^* Yw + w^*Y\left(\sum_{j=1}^{n} a_j w^j w'\right)\right].\end{aligned}$$

Let $v = \sum_{j=1}^{n} a_j w^j w'$. For any $d > 0$, we obtain

$$\begin{aligned}&v^*Yw + w^*Yv - d^2 w^*Yw - \frac{1}{d^2}v^*Yv \\ &\qquad = -\left(dw^* - \frac{1}{d}v^*\right)\cdot\left(dYw - \frac{1}{d}Yv\right) = -\underbrace{\left(dw^* - \frac{1}{d}v^*\right)}_{z^*} Y \underbrace{\left(dw - \frac{1}{d}v\right)}_{z} \leq 0\,.\end{aligned}$$

Thus,

$$\begin{aligned}&\left(\sum_{j=1}^{n} a_j w^j w'\right)^* Yw + w^*Y\left(\sum_{j=1}^{n} a_j w^j w'\right) \\ &\qquad\leq d^2 w^*Yw + \frac{1}{d^2}\left(\sum_{j=1}^{n} a_j w^j w'\right)^* Y\left(\sum_{j=1}^{n} a_j w^j w'\right). \qquad (4.2.3)\end{aligned}$$

For the last term of (4.2.3), we can see that

$$\begin{aligned}
&\left(\sum_{j=1}^{n} a_j w^j w'\right)^* Y \left(\sum_{j=1}^{n} a_j w^j w'\right) = \left(\sum_{j=1}^{n} \frac{a_j}{j+1}\left(w^{j+1}\right)'\right)^* \cdot Y \cdot \left(\sum_{j=1}^{n} \frac{a_j}{j+1}\left(w^{j+1}\right)'\right)\\
&= \sum_{k\neq 0} \xi_k \left(\sum_{j=1}^{n} \frac{a_j}{j+1}\left(w^{j+1}\right)'\right)^* \psi_k \psi_k^* \left(\sum_{j=1}^{n} \frac{a_j}{j+1}\left(w^{j+1}\right)'\right)\\
&= \sum_{k\neq 0} \xi_k \left(\psi_k^* \sum_{j=1}^{n} \frac{a_j}{j+1}\left(w^{j+1}\right)'\right)^* \left(\psi_k^* \sum_{j=1}^{n} \frac{a_j}{j+1}\left(w^{j+1}\right)'\right)\\
&= \sum_{k\neq 0} \xi_k \left(\psi_k^* \sum_{j=1}^{n} \frac{a_j}{j+1}\left(w^{j+1}\right)', \psi_k^* \sum_{j=1}^{n} \frac{a_j}{j+1}\left(w^{j+1}\right)'\right)_{L^2}\\
&= \sum_{k\neq 0} \xi_k \left\|\psi_k^* \sum_{j=1}^{n} \frac{a_j}{j+1}\left(w^{j+1}\right)'\right\|_{L^2}^2 .
\end{aligned}$$

Therefor

$$\begin{aligned}
&\left(\sum_{j=1}^{n} a_j w^j w'\right)^* Y \left(\sum_{j=1}^{n} a_j w^j w'\right)\\
&= \sum_{k\neq 0} \xi_k \cdot \left|\int_0^{2\pi} \overline{\psi_k(x)} \left(\sum_{j=1}^{n} \frac{a_j}{j+1}\left(w^{j+1}\right)'\right) dx\right|^2 = \sum_{k\neq 0} \xi_k \left|\sum_{j=1}^{n} \frac{a_j}{j+1} \int_0^{2\pi} \overline{\psi_k(x)} \left(w^{j+1}\right)' dx\right|^2\\
&= \sum_{k\neq 0} \xi_k \left|\sum_{j=1}^{n} \frac{a_j}{j+1} \left(\overline{\psi_k(x)} w(x)^{j+1}\Big|_0^{2\pi} - \int_0^{2\pi} \overline{\psi_k'(x)} w^{j+1} dx\right)\right|^2\\
&= \sum_{k\neq 0} \xi_k \left|\int_0^{2\pi} \overline{\psi_k'(x)} \left(\sum_{j=1}^{n} \frac{a_j}{j+1} w^{j+1}\right) dx\right|^2\\
&= \sum_{k\neq 0} \xi_k \left(\sum_{j=1}^{n} \frac{a_j}{j+1} w^{j+1}\right)^* \psi_k' \psi_k'^* \left(\sum_{j=1}^{n} \frac{a_j}{j+1} w^{j+1}\right)\\
&= \left(\sum_{j=1}^{n} \frac{a_j}{j+1} w^{j+1}\right)^* \hat{W} \left(\sum_{j=1}^{n} \frac{a_j}{j+1} w^{j+1}\right)
\end{aligned}$$

where

$$\hat{W} = \sum_{k\neq 0} \xi_k \psi_k' \psi_k'^* \leq \tilde{D}_2 \sum_{k\neq 0} \frac{1}{4\beta k^2} \psi_k' \psi_k'^* \equiv \tilde{D}_2 W .$$

Since Theorem 4.1.2 implies that $\sup_{0\leq t<\infty} \|w(\cdot,t)\|_1 \leq M$ for $\|w_0\|_1 \leq \beta$, by Corollary 3.1.5, we have

$$
\begin{aligned}
\frac{1}{d^2}\left(\sum_{j=1}^n a_j w^j w'\right)^* Y \left(\sum_{j=1}^n a_j w^j w'\right) &= \frac{1}{d^2}\left(\sum_{j=1}^n \frac{a_j}{j+1} w^{j+1}\right)^* \hat{W} \left(\sum_{j=1}^n \frac{a_j}{j+1} w^{j+1}\right) \\
&\le \frac{\tilde{D}_2}{4\beta d^2}\sum_{k\neq 0}\frac{1}{k^2}\left(\sum_{j=1}^n \frac{a_j}{j+1} w^{j+1}\right)^* \psi_k'\psi_k'^* \left(\sum_{j=1}^n \frac{a_j}{j+1} w^{j+1}\right) \\
&= \frac{\tilde{D}_2}{4\beta d^2}\sum_{k\neq 0}\frac{1}{k^2}\left|\int_0^{2\pi}\left(\sum_{j=1}^n \frac{a_j}{j+1} w^{j+1}(x)\right)\psi_k'(x)dx\right|^2 \\
&\le \frac{c\tilde{D}_2}{4\beta d^2}\sum_{k\neq 0}\sum_{j=1}^n\left(\frac{a_j}{j+1}\right)^2\left|\int_0^{2\pi} w^{j+1}(x)\frac{\psi_k'(x)}{k}dx\right|^2 \\
&= \frac{c\tilde{D}_2}{4\beta d^2}\sum_{j=1}^n\left(\frac{a_j}{j+1}\right)^2\sum_{k\neq 0}\left|\int_0^{2\pi} w^{j+1}(x)\frac{\psi_k'(x)}{k}dx\right|^2 \\
&\le \frac{c\tilde{D}_2}{4\beta d^2}\sum_{j=1}^n\left(\frac{a_j}{j+1}\right)^2 \|w^{j+1}\|_{L_0^2}^2 D_1^2 \le \frac{cD_1^2\tilde{D}_2}{4\beta d^2}\left(\sum_{j=1}^n\left(\frac{a_j}{j+1}\right)^2 M^{2j}\right)\cdot \|w\|_{L_0^2}^2 \le \frac{1}{4}w^*Zw,
\end{aligned}
$$

if we choose M sufficiently small (after d is selected). Hence, (4.2.3) gives

$$
\begin{aligned}
\left(\sum_{j=1}^n a_j w^j w'\right)^* Yw + w^*Y\left(\sum_{j=1}^n a_j w^j w'\right) &\le d^2 w^*Yw + \frac{1}{d^2}\left(\sum_{j=1}^n a_j w^j w'\right)^* Y \left(\sum_{j=1}^n a_j w^j w'\right) \\
&\le d^2 w^*Yw + \frac{1}{4}w^*Zw \le \frac{3}{4}w^*Zw,
\end{aligned}
$$

when d is fixed such that $d^2Y \le (Z/2)$. Thus,

$$
\frac{d}{dt}[w(\cdot,t)^*Yw(\cdot,t)] \le -w(\cdot,t)^*Zw(\cdot,t) + \frac{3}{4}w(\cdot,t)^*Zw(\cdot,t) = -\frac{1}{4}w(\cdot,t)^*Zw(\cdot,t).
$$

Since it is known that $\frac{d}{dt}[w(\cdot,t)^*I_0w(\cdot,t)] \le 0$ where I_0 is the identity map on L_0^2, we have

$$
\begin{aligned}
&\frac{d}{dt}[w(\cdot,t)^*(I_0+Y)w(\cdot,t)] \le -\frac{1}{4}w(\cdot,t)^*Zw(\cdot,t) \\
\Longrightarrow\quad & w(\cdot,t)^*(I_0+Y)w(\cdot,t) \le -\int_0^t \frac{1}{4}w(\cdot,s)^*Zw(\cdot,s)ds + w_0^*(I_0+Y)w_0 \\
\Longrightarrow\quad & \|w(\cdot,t)\|_{L_0^2}^2 \le C_1\|w_0\|_{L_0^2}^2 - \int_0^t C_2\|w(\cdot,s)\|_{L_0^2}^2 ds \\
\Longrightarrow\quad & \|w(\cdot,t)\|_{L_0^2}^2 \le \|w_0\|_{L_0^2}^2 e^{-C_2t},
\end{aligned}
$$

where C_1, C_2 are positive constant and the Gronwall's inequality has been used.

If $[w_0] \neq 0$, then we can let $u = w - ([w]/2\pi)$, where $[w] = [w_0]$ and $u \in L_0^2$. We then consider the equation for u. If $[w_0]$ is small enough, the above argument goes through for the equation of u and (4.2.2) holds for u and u_0. Thus, the proof of the exponential decay is completed. □

Chapter 5

Preliminary Mathematical Analysis of The Basin Problem

5.1 Formulation as a First-Order System

Beginning from this chapter, we consider the controllability and stability problems of following two-dimensional equations (1.0.25) used to describe wave motion in a basin. To simplify the problem, without loss of generality, we choose $g = 1, T = 1, L = \pi, h = 1$. Hence, in the following part, we will consider the simplified equations

$$\begin{cases} \psi_{xx} + \psi_{yy} = 0, & \text{on } \Omega \times [0, \tau], \\ \psi_{tt} + \partial_y \psi = \partial_y \psi_{xx}, & \text{on } \Gamma_s \times [0, \tau], \\ \psi_y = 0, & \text{on } \Gamma_f \times [0, \tau], \\ \psi_x = v, & \text{on } \Gamma_1 \times [0, \tau], \\ \psi_x = 0, & \text{on } \Gamma_2 \times [0, \tau], \\ \psi_{xy}(0) = \psi_{xy}(\pi) = 0, & \text{on } \Gamma_s. \end{cases} \tag{5.1.1}$$

If we define $\varphi = \psi|_{\Gamma_s}$, we may find two corresponding functions Φ and Ψ such that the velocity potential

$$\psi = \Phi + \Psi = D\varphi + Nv, \tag{5.1.2}$$

where D denotes the "Dirichlet map" defined by $D\varphi = \Phi$ such that

$$\begin{cases} \Phi_{xx} + \Phi_{yy} = 0, & \text{on } \Omega, \\ \Phi = \varphi, & \text{on } \Gamma_s, \\ \Phi_y = 0, & \text{on } \Gamma_f, \\ \Phi_x = 0, & \text{on } \Gamma_1, \\ \Phi_x = 0, & \text{on } \Gamma_2, \\ \Phi_{xy}(0) = \Phi_{xy}(\pi) = 0, & \text{on } \Gamma_s, \end{cases} \tag{5.1.3}$$

and N denotes the "Neumann map" defined by $Nv = \Psi$ such that

$$\begin{cases} \Psi_{xx} + \Psi_{yy} = 0, & \text{on } \Omega, \\ \Psi = 0, & \text{on } \Gamma_s, \\ \Psi_y = 0, & \text{on } \Gamma_f, \\ \Psi_x = v, & \text{on } \Gamma_1, \\ \Psi_x = 0, & \text{on } \Gamma_2, \\ \Psi_{xy}(0) = \Psi_{xy}(\pi) = 0, & \text{on } \Gamma_s. \end{cases} \tag{5.1.4}$$

Note that both D and N are continuous

$$D : H^{1/2}(\Gamma_s) \to H^1(\Omega), \quad N : H^{-1/2}(\Gamma_s) \to H^1(\Omega).$$

On the surface Γ_s, from (5.1.1) and (5.1.2), we have

$$\begin{aligned} \Phi_{tt} + \Psi_{tt} + \Phi_y + \Psi_y &= \Phi_{xxy} + \Psi_{xxy} \\ \Longrightarrow \Phi_{tt} + \Phi_y - \Phi_{xxy} &= -\Psi_y + \Psi_{xxy}. \end{aligned}$$

Thus, we can define two operators

$$\begin{aligned} \mathcal{A}\varphi &= \partial_y D\varphi|_{\Gamma_s} - \partial_y D\varphi_{xx}|_{\Gamma_s} = \partial_y \Phi|_{\Gamma_s} - \partial_y \Phi_{xx}|_{\Gamma_s}, \\ \mathcal{B}v &= -\partial_y Nv|_{\Gamma_s} + \partial_y Nv_{xx}|_{\Gamma_s} = -\partial_y \Psi|_{\Gamma_s} + \partial_y \Psi_{xx}|_{\Gamma_s}, \end{aligned}$$

such that the original system (5.1.1) can be reformulated by $\mathcal{A}$ and $\mathcal{B}$ as the following equations

$$\begin{cases} \ddot{\varphi} + \mathcal{A}\varphi = \mathcal{B}v, & \text{on } \Gamma_s \times [0, \tau], \\ \varphi(x, 0) = \varphi_0(x), \\ \dot{\varphi}(x, 0) = \varphi_1(x). \end{cases} \tag{5.1.5}$$

Observe that, the operator $\mathcal{A}$ is a linear operator such that

$$\mathcal{A} : H^{3/2}(\Gamma_s) \to H^{-3/2}(\Gamma_s) \quad \text{and} \quad \mathcal{A} : H^{9/2}(\Gamma_s) \to H^{3/2}(\Gamma_s).$$

Similarly, the operator $\mathcal{B}$ is a linear operator such that

$$\mathcal{B} : H^{3/2}(\Gamma_1) \to H^{-1/2}(\Gamma_1) \quad \text{and} \quad \mathcal{B} : H^{5/2}(\Gamma_1) \to H^{1/2}(\Gamma_1).$$

Using interpolation theory [11], we can show that

$$\mathcal{A} : [H^{3/2}, H^{9/2}]_{1/2} \to [H^{-3/2}, H^{3/2}]_{1/2} \text{ and } \mathcal{B} : [H^{3/2}, H^{5/2}]_{1/2} \to [H^{-1/2}, H^{1/2}]_{1/2},$$

which allow us to conclude the following properties,

1. The operator $\mathcal{A}$ is a linear unbounded operator in $L^2(\Gamma_s)$ with domain $H^3(\Gamma_s)$.
2. The operator $\mathcal{B}$ is a linear bounded operator from $H^2(\Gamma_1)$ into $L^2(\Gamma_s)$.

For reasons that will appear clearly in what follows, we still consider the functions with zero mean. Hence we define

$$\mathscr{D}(\mathcal{A}) \equiv H_0^3(\Gamma_s) = \left\{ \varphi \in H^3(\Gamma_s) \middle| \int_{\Gamma_s} \varphi(x)dx = 0 \right\},$$

and the associated range of $\mathcal{A}$,

$$R(\mathcal{A}) \equiv L_0^2(\Gamma_s) = \left\{ \varphi \in L^2(\Gamma_s) \middle| \int_{\Gamma_s} \varphi(x)dx = 0 \right\}.$$

As what we did in the previous section, we define a pair of orthogonal projections Π_H and Π_R, such that for any $\varphi \in L^2(0, \pi)$,

$$\Pi_R \varphi \in [L^2](0, \pi), \qquad \Pi_H \varphi \in L_0^2(0, \pi),$$

where

$$\begin{aligned} [L^2](0, \pi) &= \left\{ \varphi \in L^2(0, \pi) \middle| \varphi = \frac{1}{\pi} \int_0^\pi \varphi dx \right\}, \\ L_0^2(0, \pi) &= \left\{ \varphi \in L^2(0, \pi) \middle| \frac{1}{\pi} \int_0^\pi \varphi dx = 0 \right\}. \end{aligned}$$

As a consequence, for any $\varphi \in L^2(0, \pi)$, $\varphi = \Pi_R \varphi + \Pi_H \varphi$. Substituting φ into (5.1.5) yields

$$\Pi_H \ddot{\varphi} + \Pi_R \ddot{\varphi} = -\mathcal{A}(\Pi_H \varphi + \Pi_R \varphi) + (\Pi_H \mathcal{B} v + \Pi_R \mathcal{B} v).$$

We may simply choose

$$\begin{aligned} \Pi_H \ddot{\varphi} &= -\mathcal{A} \Pi_H \varphi + \Pi_H \mathcal{B} v, \\ \Pi_R \ddot{\varphi} &= -\mathcal{A} \Pi_R \varphi + \Pi_R \mathcal{B} v = \Pi_R \mathcal{B} v. \end{aligned}$$

Then, we adopt the following abstract first-order formulation of the original system (5.1.5)

$$\partial_t \underbrace{\begin{pmatrix} \Pi_H \varphi \\ \Pi_H \dot{\varphi} \\ \Pi_R \dot{\varphi} \end{pmatrix}}_{\xi} = \underbrace{\begin{pmatrix} \Pi_H \dot{\varphi} \\ -\mathcal{A} \Pi_H \varphi \\ 0 \end{pmatrix}}_{A\xi} + \underbrace{\begin{pmatrix} 0 \\ \Pi_H \mathcal{B} v \\ \Pi_R \mathcal{B} v \end{pmatrix}}_{Bv} \tag{5.1.6}$$

with initial condition $\xi(0) = \xi_0 \in X$. Note that since $\eta = \eta_{xx} - \psi_t$ on Γ_s, if we define the observation operator $C = -(0, 1, 1)$, the elevation of surface

$$\eta = \eta_{xx} - (\Pi_H \dot{\varphi} + \Pi_R \dot{\varphi}) \Longrightarrow \eta = \eta_{xx} + C\xi.$$

As what we will see in the following, the operator $\mathcal{A}^{1/2}$ is crucial when we define the inner product in the energy space X. Results of interpolation theory allow us to assume that the domain of $\mathcal{A}^{1/2}$ is given by

$$\mathscr{D}(\mathcal{A}^{1/2}) = [\mathscr{D}(\mathcal{A}), R(\mathcal{A})]_{1/2} = [H_0^3(\Gamma_s), L_0^2(\Gamma_s)]_{1/2} = H_0^{3/2}(\Gamma_s).$$

However, in order to guarantee this result, we are required to prove

Proposition 5.1.1. *The operator $\mathcal{A}$ is strictly positive, self-adjoint, and $R(\lambda I + \mathcal{A}) = R(\mathcal{A})$ for $\lambda > 0$.*

Proof. Based on definition of $\mathcal{A}$,

$$\begin{aligned} < \mathcal{A}\varphi, \varphi > &= < \partial_y D\varphi\Big|_{\Gamma_s} - \partial_y D\varphi_{xx}\Big|_{\Gamma_s}, \varphi >= \int_{\Gamma_s} (\partial_y D\varphi - \partial_y D\varphi_{xx})\, D\bar{\varphi} d\Gamma_s \\ &= \underbrace{\int_{\Gamma_s} \partial_y D\varphi \cdot D\bar{\varphi} d\Gamma_s}_{I} - \underbrace{\int_{\Gamma_s} \partial_y D\varphi_{xx} \cdot D\bar{\varphi} d\Gamma_s}_{II}. \end{aligned}$$

For part I,

$$\begin{aligned} \int_\Omega \triangle\, D\varphi \cdot D\bar{\varphi} d\Omega &= \int_\Omega Div(\nabla D\varphi \cdot D\bar{\varphi}) d\Omega - \int_\Omega \nabla D\varphi \cdot \nabla D\bar{\varphi} d\Omega \\ &= \int_{\partial\Omega} \frac{\partial D\varphi}{\partial n} \cdot D\bar{\varphi} d\partial\Omega - \int_\Omega \nabla D\varphi \cdot \nabla D\bar{\varphi} d\Omega \\ &= \int_{\Gamma_s} \partial_y D\varphi \cdot D\bar{\varphi} d\Gamma_s - \int_\Omega \nabla D\varphi \cdot \nabla D\bar{\varphi} d\Omega = 0. \end{aligned}$$

Therefore

$$\int_{\Gamma_s} \partial_y D\varphi \cdot D\bar{\varphi} d\Gamma_s = \int_\Omega \nabla D\varphi \cdot \nabla D\bar{\varphi} d\Omega.$$

For part II, we have

$$\begin{aligned} \int_\Omega \triangle\, D\varphi_{xx} \cdot D\bar{\varphi} d\Omega &= \int_\Omega Div(\nabla D\varphi_{xx} \cdot D\bar{\varphi}) d\Omega - \int_\Omega \nabla D\varphi_{xx} \cdot \nabla D\bar{\varphi} d\Omega \\ &= \int_{\partial\Omega} \frac{\partial D\varphi_{xx}}{\partial n} \cdot D\bar{\varphi} d\partial\Omega - \int_\Omega \nabla D\varphi_{xx} \cdot \nabla D\bar{\varphi} d\Omega \\ &= \int_{\Gamma_s} \partial_y \Phi_{xx} \bar{\Phi} dx + \int_{\Gamma_f} \partial_y \Phi_{xx} \bar{\Phi} dx + \int_{\Gamma_1} \partial_x \Phi_{xx} \bar{\Phi} dy \\ &\quad + \int_{\Gamma_2} \partial_x \Phi_{xx} \bar{\Phi} dy - \int_\Omega \nabla \Phi_{xx} \cdot \nabla \bar{\Phi} d\Omega = 0. \end{aligned}$$

Obviously

$$\int_{\Gamma_f} \partial_y \Phi_{xx} \bar{\Phi} dx = 0.$$

Considering $\int_{\Gamma_1} \partial_x \Phi_{xx} \bar{\Phi} dy$ and $\int_{\Gamma_2} \partial_x \Phi_{xx} \bar{\Phi} dy$, one can show that

$$\begin{aligned}
\int_{\Gamma_1} \partial_x \Phi_{xx} \bar{\Phi} dy &= -\int_{\Gamma_1} \Phi_{xyy} \bar{\Phi} dy \\
&= -\left(\Phi_{xy} \bar{\Phi}|_0^1 - \int_{\Gamma_1} \Phi_{xy} \bar{\Phi}_y dy \right) \\
&= -\left(\Phi_{xy} \bar{\Phi}|_0^1 - \Phi_x \bar{\Phi}_y|_0^1 + \int_0^1 \Phi_x \bar{\Phi}_{yy} dy \right) \\
&= 0,
\end{aligned}$$

and

$$\begin{aligned}
\int_{\Gamma_2} \partial_x \Phi_{xx} \bar{\Phi} dy &= -\int_{\Gamma_2} \Phi_{xyy} \bar{\Phi} dy \\
&= -\left(\Phi_{xy} \bar{\Phi}|_0^1 - \int_{\Gamma_1} \Phi_{xy} \bar{\Phi}_y dy \right) \\
&= -\left(\Phi_{xy} \bar{\Phi}|_0^1 - \Phi_x \bar{\Phi}_y|_0^1 + \int_0^1 \Phi_x \bar{\Phi}_{yy} dy \right) \\
&= 0.
\end{aligned}$$

Therefore

$$\int_{\Gamma_s} \partial_y D\varphi_{xx} \cdot D\bar{\varphi} d\Gamma_s = \int_{\Omega} \nabla D\varphi_{xx} \cdot \nabla D\bar{\varphi} d\Omega.$$

Furthermore,

$$\begin{aligned}
\int_{\Omega} \nabla D\varphi_{xx} \cdot \nabla D\bar{\varphi} d\Omega &= \int_0^1 (\nabla D\varphi)_x \cdot \nabla D\bar{\varphi} \Big|_0^{\pi} dy - \int_0^1 \int_0^{\pi} (\nabla D\varphi)_x \cdot (\nabla D\bar{\varphi})_x dx dy \\
&= \int_0^1 (\Phi_{xx} \cdot \bar{\Phi}_x + \Phi_{xy} \cdot \bar{\Phi}_y)|_0^{\pi} dy - \int_0^1 \int_0^{\pi} (\nabla \Phi)_x \cdot (\nabla \bar{\Phi})_x dx dy \\
&= -\int_{\Omega} (\nabla D\varphi)_x \cdot (\nabla D\bar{\varphi})_x d\Omega.
\end{aligned}$$

Thus one can obtain

$$< \mathcal{A}\varphi, \varphi >= \|\nabla D\varphi\|^2_{L^2(\Omega)} + \|\nabla D\varphi_x\|^2_{L^2(\Omega)} = \|\nabla D\varphi\|^2_{H^1(\Omega)} \geq 0. \tag{5.1.7}$$

Similarly, using the same way, we see

$$< \varphi, \mathcal{A}\varphi >= \|\nabla D\varphi\|^2_{H^1(\Omega)}. \tag{5.1.8}$$

Therefore $\mathcal{A} = \mathcal{A}^*$ and $\mathcal{A}$ is a self-adjoint operator. If $\mathcal{A}\varphi = 0$, $\partial_y D\varphi - \partial_y D\varphi_{xx} = 0$ on Γ_s. Let $u = \partial_y D\phi$. Substituting u gives rise to

$$\begin{cases} u - u_{xx} = 0, \\ u_x(0,1) = u_x(\pi,1) = 0. \end{cases} \tag{5.1.9}$$

Multiplying by u and taking integral of both sides, we obtain

$$\begin{aligned} \int_0^\pi u_{xx} \cdot u dx - \int_0^\pi u^2 dx &= u_x u\Big|_0^\pi - \int_0^\pi (u_x)^2 dx - \int_0^\pi u^2 dx \\ &= -\int_0^\pi (u_x)^2 dx - \int_0^\pi u^2 dx = 0, \end{aligned}$$

which yields $\int_0^\pi (u_x)^2 dx = 0$ and $\int_0^\pi u^2 dx = 0$. Thus $u = 0$ and $u_x = 0$. Following this result, Φ in (5.1.3) satisfies

$$\begin{cases} \Phi_{xx} + \Phi_{yy} = 0, & \\ \Phi_y = 0, & \text{on } \Gamma_s, \\ \partial_n \Phi = 0, & \text{on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_f. \end{cases}$$

Using the separation of variables, if $\Phi(x,y) = X(x) \cdot Y(y)$, we can obtain two equations

$$X''(x) - \delta X(x) = 0, \quad -Y''(y) - \delta Y(y) = 0,$$

with four boundary conditions

$$X_x(0) = 0, \quad X_x(\pi) = 0, \quad Y_y(0) = 0, \quad Y_y(1) = 0.$$

If $\delta \neq 0$, it is easy to solve that

$$\begin{aligned} X_k(x) &= \cos kx, \\ Y_k(y) &= c_k \cosh ky + d_k \sinh ky, \end{aligned}$$

where c_k and d_k are both constants. Therefore

$$\Phi(x,y) = \sum_{k=1}^{\infty} \cos kx \cdot (c_k \cosh ky + d_k \sinh ky).$$

Applying the boundary conditions, we have $\Phi(x,y) \equiv 0$. If $\delta = 0$, we can see that $\Phi = c$ is another solution. However, according to the special choice of $\mathscr{D}(\mathcal{A})$, for any $\varphi \in \mathscr{D}(\mathcal{A})$, $\int_{\Gamma_s} \varphi(x) dx = 0$. Therefore $\varphi(x)$ can only be zero. We conclude that $\mathcal{A}\varphi = 0$ only if $\varphi = 0$, i.e., $\mathcal{A}$ is injective and strictly positive. Furthermore, for any $\lambda > 0$, the operator $\lambda I + \mathcal{A}$ is trivially a bijection. By interpolation Theorem [11], $\lambda I + \mathcal{A}$ is well defined from H_0^3 to L_0^2 and $R(\lambda I + \mathcal{A}) = R(\mathcal{A})$. □

This Theorem can guarantee us to define an inner product based on $\mathcal{A}^{1/2}$. For any φ and ψ in $H_0^{3/2}(\Gamma_s)$, one may define

$$< \mathcal{A}\varphi, \psi >_{H_0^{-3/2}, H_0^{3/2}} = \left\langle \mathcal{A}^{1/2}\varphi, \mathcal{A}^{1/2}\psi \right\rangle = \int_\Omega \nabla D\varphi \cdot \nabla D\psi + \int_\Omega \nabla D\varphi_x \cdot \nabla D\psi_x \qquad (5.1.10)$$

rather than finding the explicit representation of $\mathcal{A}^{1/2}$. From (5.1.10), we observe that $\Pi_H \varphi$ is required to be defined in $\mathscr{D}(\mathcal{A}^{1/2})$. Furthermore, (5.1.6) requires that the domain of $\Pi_H \dot{\varphi}$ should be in $\mathscr{D}(\mathcal{A})$. As a consequence, the state space X of ξ in (5.1.6) is defined as follows:

$$X = \mathscr{D}(\mathcal{A}^{1/2}) \times R(\mathcal{A}) \times R = H_0^{3/2}(\Gamma_s) \times L_0^2(\Gamma_s) \times R.$$

On the other hand, since $\xi_t = A\xi + Bv$, $\mathscr{D}(A) = \{\xi | \xi \in X, A\xi \in X\}$ is given by

$$\mathscr{D}(A) = \mathscr{D}(\mathcal{A}) \times \mathscr{D}(\mathcal{A}^{1/2}) \times R = H_0^3(\Gamma_s) \times H_0^{3/2}(\Gamma_s) \times R.$$

Thus, the inner product in the energy space X can be defined by

$$\begin{aligned} < \xi, \zeta >_X &= \left\langle \mathcal{A}^{1/2}\xi_1, \mathcal{A}^{1/2}\zeta_1 \right\rangle + \langle \xi_2 + \xi_3, \zeta_2 + \zeta_3 \rangle \\ &= \int_\Omega \nabla D\xi_1 \cdot \nabla D\zeta_1 + \int_\Omega \nabla D\xi_{1x} \cdot \nabla D\zeta_{1x} + \int_{\Gamma_s} (\xi_2 + \xi_3)(\zeta_2 + \zeta_3), \quad (5.1.11) \end{aligned}$$

where $\xi = < \xi_1, \xi_2, \xi_3 >^T$ and $\zeta = < \zeta_1, \zeta_2, \zeta_3 >^T$. The associated norm $\| \cdot \|_X$ is defined by

$$\begin{aligned} \|\xi\|_X^2 &= |\mathcal{A}^{1/2}\xi_1|^2 + |\xi_2 + \xi_3|^2 \\ &= \int_\Omega |\nabla D\xi_1|^2 + \int_\Omega |\nabla D\xi_{1x}|^2 + \int_{\Gamma_s} |\xi_2 + \xi_3|^2. \end{aligned}$$

The reason why we define the norm $\| \cdot \|_X$ is given by the following proposition.

Proposition 5.1.2. *Suppose that the control input $v = 0$, the natural energy $E(\psi, \dot{\psi})$ is related in some sense to the norm $\| \cdot \|_X$ in terms of the original potential ψ, indeed*

$$\frac{1}{2}\|\xi\|_X^2 = E(\psi, \dot{\psi}).$$

Meanwhile the norm $\| \cdot \|_X$ is coercive on X, i.e., $\|\xi\|_X^2 = 0$ if and only if $\xi = 0$.

Proof. Through (5.1.4), we see $\Psi = 0$ if $v = 0$ which yields

$$\begin{aligned} &\int_\Omega \nabla D\varphi \cdot \nabla Nv = \int_\Omega (\Phi_x, \Phi_y) \cdot (\Psi_x, \Psi_y) = \int_\Omega \Phi_x \Psi_x + \Phi_y \Psi_y \\ = \; &\int_0^1 \left(\Phi_x \Psi|_0^\pi - \int_0^\pi \Phi_{xx} \Psi dx \right) dy + \int_0^\pi \left(\Phi_y \Psi|_0^1 - \int_0^1 \Phi_{yy} \Psi dy \right) dx = 0. \end{aligned}$$

Similarly, we can show that $\int_\Omega \nabla D\varphi_x \cdot \nabla N v_x = 0$ too. Based upon these conditions, one can show that

$$\begin{aligned}\int_\Omega |\nabla\psi|^2 &= \int_\Omega |\nabla D\varphi + \nabla N v|^2 \\ &= \int_\Omega |\nabla D\varphi|^2 + \int_\Omega |\nabla N v|^2 + \int_\Omega 2|\nabla D\varphi \nabla N v|^2 \\ &= \int_\Omega |\nabla D\varphi|^2,\end{aligned}$$

and $\int_\Omega |\nabla\psi_x|^2 = \int_\Omega |\nabla D\varphi_x|^2$. Thus, we obtain

$$\begin{aligned}\|\xi\|_X^2 &= \int_\Omega |\nabla D\varphi|^2 + \int_\Omega |\nabla D\varphi_x|^2 + \int_{\Gamma_s} |\Pi_H\dot\varphi + \Pi_R\dot\varphi|^2 \\ &= \int_\Omega |\nabla\psi|^2 + \int_\Omega |\nabla\psi_x|^2 + \int_{\Gamma_s} |\dot\psi|^2,\end{aligned}$$

which is equivalent to the double of the natural energy

$$E(\psi,\dot\psi) = \frac{1}{2}\int_\Omega |\nabla\psi|^2 + \frac{1}{2}\int_\Omega |\nabla\psi_x|^2 + \frac{1}{2}\int_{\Gamma_s} |\dot\psi|^2 = \frac{1}{2}\|\xi\|_X^2.$$

Note that $\|\xi\|_X^2 = 0$ if and only if $\xi = 0$. Indeed, $\|\xi\|_X^2 = 0$ yields $\nabla D\xi_1 = 0$, $\nabla D\xi_{1x} = 0$ and $\xi_2 + \xi_3 = 0$. Since ξ_2 and ξ_3 are orthogonal to each other, $\xi_2 = \xi_3 = 0$. $\xi_1 = c$ is the only point such that $\|\xi\|_X^2 = 0$. However $\xi_1 \in \mathscr{D}(\mathcal{A})$ yields $\xi_1 = 0$ only. So we conclude that the norm $\|\cdot\|_X$ is coercive on X. Whereas, the natural energy is not coercive on the original energy space. Note that initial conditions

$$\psi(0) = c, \quad \dot\psi(0) = 0, \quad \text{on } \Gamma_s,$$

yield $\psi = c$ which stays invariant. However, all solutions of the type

$$\psi = c, \quad t \in [0,\infty),$$

verify $E(\psi,\dot\psi) = 0$. Therefore the system is not controllable under such energy space. □

5.2 Spectral Properties and Semigroup Generation

In this part, we will discuss the spectral properties of the operator A defined in (5.1.6) which can be rewritten as follows

$$A\begin{pmatrix}\xi_1\\ \xi_2\\ \xi_3\end{pmatrix} = \begin{pmatrix}0 & 1 & 0\\ -\mathcal{A} & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\\ \xi_3\end{pmatrix} = \begin{pmatrix}\xi_2\\ -\mathcal{A}\xi_1\\ 0\end{pmatrix}. \tag{5.2.1}$$

At beginning, we try to find out the eigenvalues λ_k and associated eigenfunctions w_k of the operator $\mathcal{A}$ for $k > 0$. Let $w_k = W_k|_{\Gamma_s}$. Through the definition of $\mathcal{A}$, we have

$$\mathcal{A}W_k = \partial_n W_k - \partial_n W_{kxx}, \quad \text{on } \Gamma_s.$$

To obtain λ_k and w_k, let us consider the following system

$$\begin{cases} \triangle W_k = 0, & \text{in } \Omega, \\ \partial_n W_k - \partial_n W_{kxx} = \lambda_k W_k, & \text{on } \Gamma_s, \\ W_{kx}(0, y) = 0, & \text{on } \Gamma_1, \\ W_{kx}(\pi, y) = 0, & \text{on } \Gamma_2, \\ W_{ky}(x, 0) = 0, & \text{on } \Gamma_f, \\ W_{kxy}(0, 1, t) = W_{kxy}(\pi, 1, t), & \text{on } \Gamma_s. \end{cases} \tag{5.2.2}$$

Let $W(x, y) = X(x) \cdot Y(y)$. One can show that

$$\begin{aligned} & X''(x)Y(y) + X(x)Y''(y) = 0 \\ \Longrightarrow \quad & \frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \delta. \end{aligned}$$

Therefore, we can obtain two equations

$$X''(x) - \delta X(x) = 0, \quad -Y''(y) - \delta Y(y) = 0, \tag{5.2.3}$$

with respect to X and Y respectively. From boundary conditions on Γ_1 and Γ_2, we get

$$X'(0) = 0, \quad X'(\pi) = 0.$$

Considering the equations about X,

$$\begin{cases} X''(x) = \delta X(x), \\ X'(0) = 0, \\ X'(\pi) = 0. \end{cases} \tag{5.2.4}$$

If $\delta > 0$, choosing $a = \sqrt{\delta} > 0$ yields $X(x) = c_1 e^{ax} + c_2 e^{-ax}$. The boundary conditions give rise to $X(x) \equiv 0$. If $\delta = 0$, $X(x)$ will be a constant. Similarly if $\delta < 0$, it is easy to solve that

$$X_k(x) = \cos kx \quad \text{and} \quad \delta_k(x) = -k^2 \quad \text{for } k = 1, 2, 3, ...$$

Substituting δ_k into (5.2.3) yields

$$\begin{aligned} & -Y''(y) - \delta Y(y) = 0 \\ \Longrightarrow \quad & Y_k''(y) - k^2 Y_k(y) = 0, \\ \Longrightarrow \quad & Y_k(y) = c_1 e^{ky} + c_2 e^{-ky}, \\ \Longrightarrow \quad & Y_k(y) = c_k \cosh ky + d_k \sinh ky, \end{aligned}$$

where $c_k = c_1 + c_2, \quad d_k = c_1 - c_2$. Thus

$$W_k(x,y) = X_k(x)Y_k(y) = \cos kx \cdot (c_k \cosh ky + d_k \sinh ky), \quad \text{for } k = 1, 2, ... \tag{5.2.5}$$

The general solution is the superposition of the eigen-modes

$$W(x,y) = \sum_{k=1}^{\infty} W_k(x,y) = \sum_{k=1}^{\infty} \cos kx \cdot (c_k \cosh ky + d_k \sinh ky). \tag{5.2.6}$$

Applying the boundary condition on the floor Γ_f in (5.2.2) yields

$$W_{ky}(x,0) = \cos kx \cdot (c_k k \sinh(0) + d_k k \cosh(0)) = 0 \Longrightarrow d_k = 0.$$

Thus one can obtain that

$$W_k(x,y) = \alpha \cos kx \cosh ky, \quad \text{for } k = 1, 2, ..., \tag{5.2.7}$$

where $\alpha = c_k$ is a non-zero constant. If we choose $\alpha = 1/\cosh k$, we have

$$w_k(x) = W_k(x,1) = \cos kx, \quad \text{for } k = 1, 2, ... \tag{5.2.8}$$

Since $w_k(x)$ is an even function, the similar results are obtained for $k < 0$. On the other hand, the eigenvalue λ_k can be solved from the boundary condition on the surface Γ_s

$$\begin{aligned} & \partial_y W_k - \partial_y W_{kxx} = \lambda_k W_k \\ \Longrightarrow \quad & (k + k^3)\alpha \cos kx \sinh ky = \lambda_k \alpha \cos kx \cosh ky, \\ \Longrightarrow \quad & \lambda_k = (k + k^3) \tanh k \quad \text{for } k = 1, 2, 3, ... \end{aligned} \tag{5.2.9}$$

Note that λ_k, for $k = 1, 2, ...$, are always positive. Furthermore, for $k = 0$, $\lambda_0 = 0$. However, we see that the associated eigenfunction $w_0 = 1$ is not in $\mathscr{D}(\mathcal{A})$ since the special choice of $\mathscr{D}(\mathcal{A})$. So it is not the eigen-pair of $\mathcal{A}$. But we will consider it in the following.

Next, the operator A defined in (5.2.1) will be considered. Let μ_k be denoted as the eigenvalue of A. Observe that the characteristic equation $|\mu I - A| = \mu(\mu^2 + \lambda) = 0$ yields

$$\mu_k = \begin{cases} i\sqrt{\lambda_k} = i\omega_k, & \text{for } k > 0, \\ 0, & \text{for } k = 0, \\ -i\sqrt{\lambda_{-k}} = i\omega_k, & \text{for } k < 0. \end{cases}$$

Note that $\omega_k = \sqrt{\lambda_k} > 0$ for $k > 0$ and $\omega_k = -\sqrt{\lambda_{-k}} < 0$ for $k < 0$. If we let φ_k denote the associated eigenvectors of A, we can see that

$$\varphi_0 = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{\pi}} \end{pmatrix} \quad \text{for } \mu_0 = 0,$$

$$\varphi_k = \frac{1}{\sqrt{\pi}} \begin{pmatrix} \frac{1}{\mu_k} w_k \\ w_k \\ 0 \end{pmatrix} \quad \text{for } \mu_k, \text{ where } k \in Z^*.$$

Based on these results, we can show the following proposition,

Proposition 5.2.1. *The operator A is the infinitesimal generator which yields a strongly continuous semigroup $S(t)$ on $X = H_0^{3/2}(\Gamma_s) \times L_0^2(\Gamma_s) \times R$, where $S(t)$ is given by*

$$S(t)\xi = \sum_{k \in Z} e^{i\omega_k t} \langle \xi, \varphi_k \rangle_X \varphi_k.$$

Proof. Considering the energy space X, one can see that $\{\varphi_k\}$ is a basis of X. Recall that $w_k(x) = \cos kx$. We can also show that $\{\varphi_k\}$ is an orthonormal basis in X, see

$$\begin{aligned} < \varphi_k, \varphi_j >_X &= \frac{1}{\pi} \left\langle \mathcal{A}^{1/2} \frac{1}{\mu_k} w_k, \mathcal{A}^{1/2} \frac{1}{\mu_j} w_j \right\rangle + \frac{1}{\pi} < w_k, w_j > \\ &= \frac{1}{\pi \mu_k \mu_j} \left(\int_\Omega \nabla D w_k \nabla D w_j + \int_\Omega \nabla D w_{kx} \nabla D w_{jx} \right) + \frac{1}{\pi} \int_{\Gamma_s} w_k w_j \\ &= \frac{1}{\pi \mu_k \mu_j} \int_{\Gamma_s} (\partial_n D w_k - \partial_n D w_{kxx}) D w_j d\Gamma_s \\ &= \frac{1}{\pi \mu_k \mu_j} \int_{\Gamma_s} \lambda_k w_k w_j d\Gamma_s = 0, \end{aligned}$$

and

$$\begin{aligned} < \varphi_k, \varphi_k >_X &= \frac{1}{\pi} \left\langle \mathcal{A}^{1/2} \frac{1}{\mu_k} w_k, \mathcal{A}^{1/2} \frac{1}{\mu_k} w_k \right\rangle + \frac{1}{\pi} < w_k, w_k > \\ &= \frac{1}{\pi |\mu_k^2|} \int_{\Gamma_s} (\partial_n D w_k - \partial_n D w_{kxx}) D w_k dx + \frac{1}{\pi} \int_{\Gamma_s} w_k^2 dx \\ &= \frac{1}{\pi |\mu_k^2|} \int_{\Gamma_s} \lambda_k w_k^2 dx + \frac{1}{2} = 1. \end{aligned}$$

Therefore, $\|\varphi_k\|_X = 1$. By a theorem in ([8], p. 38), $\{\varphi_k\}$ is a Riesz basis of X. Thus we can conclude that the operator A is a Riesz-spectral operator by the following definition in [8],

Definition 5.2.2. *A is a Riesz-spectral operator if A is a linear, closed on a Hilbert space, X, with simple eigenvalues $\mu_n, n \geq 1$, where the closure of $\{\mu_n, n \geq 1\}$ is totally disconnected, and the corresponding eigenvectors $\varphi_n, n \geq 1$ form a Riesz basis in X. By totally disconnected we mean that no two points $\mu_1, \mu_2 \in \overline{\{\mu_n, n \geq 1\}}$, can be joined by a segment lying entirely in $\overline{\{\mu_n, n \geq 1\}}$.*

Meanwhile, for the operator A, a direct calculation shows that

$$\begin{aligned}\langle A\xi,\xi\rangle_X &= \int_\Omega \nabla D\xi_2 \nabla D\bar{\xi}_1 + \int_\Omega \nabla D\xi_{2x}\nabla D\bar{\xi}_{1x} + \int_{\Gamma_s} -\mathcal{A}\xi_1\overline{(\xi_2+\xi_3)} \\ &= \int_{\Gamma_s}\mathcal{A}\bar{\xi}_1\xi_2 - \int_{\Gamma_s}\mathcal{A}\xi_1\bar{\xi}_2 - \int_{\Gamma_s}\mathcal{A}\xi_1\bar{\xi}_3 \\ &= \int_{\Gamma_s}\mathcal{A}\bar{\xi}_1\xi_2 - \int_{\Gamma_s}\mathcal{A}\xi_1\bar{\xi}_2.\end{aligned}$$

Similarly, we have $<\xi,-A\xi>_X = -\int_{\Gamma_s}\mathcal{A}\xi_1\bar{\xi}_2 + \int_{\Gamma_s}\mathcal{A}\bar{\xi}_1\xi_2$. Therefore A is a skew-adjoint operator in X, i.e.,

$$<A\xi,\xi> = <\xi, A^*\xi> = <\xi,-A\xi>.$$

Based on this property, the following identity

$$<A\xi,\xi> = <\xi,-A\xi> = -<\xi,A\xi> = -\overline{<A\xi,\xi>},$$

gives rise to $2Re<A\xi,\xi>=0$. On the other hand, since $A^*=-A$, the corresponding eigenvector ψ_n of adjoint operator A^* is same as φ_n. Meanwhile $\{\varphi_n\}$ and $\{\psi_m\}$ are complete and form dual Riesz bases for X. Thus the proposition directly follows from

Theorem 5.2.3. *Suppose that A is a Riesz spectral operator with simple eigenvalues $\{\mu_n, n\geq 1\}$ and corresponding eigenvectors $\{\varphi_n, n\geq 1\}$. Let $\{\psi_n, n\geq 1\}$ be the eigenvectors of A^* such that $<\varphi_n,\psi_m> = \delta_{n,m}$. Then A is the infinitesimal generator of a C_0-semigroup if and only if $sup_{n\geq 1} Re\mu < \infty$ and the semigroup $T(t)$ is given by*

$$S(t)\xi = \sum_{k\in Z} e^{\mu_k t}\langle\xi,\psi_k\rangle_X\,\varphi_k.$$

(see [8], Theorem 2.3.5). □

Chapter 6

Controllability Problems of The Basin Model

6.1 Introduction

In this chapter, we will study the controllability problems for the following abstract system, posed on the free surface Γ_s,

$$\begin{cases} \dot{\xi} = A\xi + Bv, \\ \xi(0) = \xi_0, \end{cases} \tag{6.1.1}$$

where ξ is related to the original velocity potential φ by

$$\xi = \begin{pmatrix} \Pi_H \varphi \\ \Pi_H \dot{\varphi} \\ \Pi_R \dot{\varphi} \end{pmatrix}.$$

If we consider the flexible generators, exact controllability can eventually be obtained. In the case where the generators are rigid, only approximate controllability can eventually be obtained. Note that, there are two unknowns of this system $(\psi, \dot{\psi}|_{\Gamma_s})$ which are expressed in the natural energy norm

$$E(\psi, \dot{\psi}) = \frac{1}{2}\int_\Omega |\nabla\psi|^2 + \frac{1}{2}\int_\Omega |\nabla\psi_x|^2 + \frac{1}{2}\int_{\Gamma_s} |\dot{\psi}|^2. \tag{6.1.2}$$

As what we mentioned, the exact controllability means for any given target state

$$(\psi_T, \dot{\psi}_T|_{\Gamma_s}) = (\psi_T, -(\eta_T - \eta_{Txx})), \tag{6.1.3}$$

a control input v can be found such that at a given time τ, we can exactly steer the system to the target state $(\psi_T, \dot{\psi}_T|_{\Gamma_s})$, i.e.,

$$E\left(\psi_\tau - \psi_T, \dot{\psi}(\tau)|_{\Gamma_s} - \dot{\psi}_T|_{\Gamma_s}\right) = 0. \tag{6.1.4}$$

In terms of the abstract system, the target $(\psi_T, \dot{\psi}_T|_{\Gamma_s})$ corresponds to the abstract target

$$\xi = \begin{pmatrix} \Pi_H \psi_T|_{\Gamma_s} \\ \Pi_H(-\eta_T + \eta_{Txx}) \\ \Pi_R(-\eta_T + \eta_{Txx}) \end{pmatrix}.$$

Note that, we have also shown that $\frac{1}{2}\|\xi\|_X^2 = E(\psi, \dot{\psi}|_{\Gamma_s})$. Hence (6.1.4) is equivalent to

$$\frac{1}{2}\left\| \begin{pmatrix} \Pi_H(\psi_\tau - \psi_T)|_{\Gamma_s} \\ \Pi_H(\dot{\psi}_\tau|_{\Gamma_s} + \eta_T - \eta_{Txx}) \\ \Pi_R(\dot{\psi}_\tau|_{\Gamma_s} + \eta_T - \eta_{Txx}) \end{pmatrix} \right\|_X = 0.$$

Similarly, the approximate controllability means for any given target state

$$(\psi_T, \dot{\psi}_T|_{\Gamma_s}) = (\psi_T, -(\eta_T - \eta_{Txx})), \tag{6.1.5}$$

we can find a control input v such that at a given time τ, the final state $(\psi_\tau, \dot{\psi}_\tau|_{\Gamma_s})$ lies in the closure of the target state $(\psi_T, \dot{\psi}_T|_{\Gamma_s})$ in case of the energy norm (6.1.2), i.e.,

$$E\left(\psi_\tau - \psi_T, \dot{\psi}(\tau)|_{\Gamma_s} - \dot{\psi}_T|_{\Gamma_s}\right) \le \epsilon, \tag{6.1.6}$$

which is equivalent to show that

$$\frac{1}{2}\left\| \begin{pmatrix} \Pi_H(\psi_\tau - \psi_T)|_{\Gamma_s} \\ \Pi_H(\dot{\psi}_\tau|_{\Gamma_s} + \eta_T - \eta_{Txx}) \\ \Pi_R(\dot{\psi}_\tau|_{\Gamma_s} + \eta_T - \eta_{Txx}) \end{pmatrix} \right\|_X \le \epsilon.$$

However, we should notice that, in proposition (5.1.2), $\frac{1}{2}\|\xi(\tau)\|_X^2 = E(\psi(\tau), \dot{\psi(\tau)}|_{\Gamma_s})$ only if v is in the admissible control space $L^2(0, \tau; H^2(\Gamma_1))$ and $v(\tau) = 0$.

In the remaining part of this chapter, we categorize the generators into 2 types: flexible and rigid. For the flexible generator, exactly controllable can be showed by proving that the following inequality

$$\int_0^\tau \|B^* S^*(s)\xi\|_U^2 ds \ge \gamma \|\xi\|_X^2$$

holds for some positive γ. In case of the rigid generator, we will show that the system is not exactly controllable, but it is approximately controllable by proving that the operator $B^* S(t)^* \xi$ is an injective operator.

6.2 Exact Controllability for Flexible Generator

Now, we will prove that the abstract system

$$\begin{cases} \dot{\xi} = A\xi + Bv, \\ \eta = \eta_{xx} + C\xi, \\ \xi(0) = \xi_0 \in X, \end{cases} \tag{6.2.1}$$

with flexible generator $v \in L^2(0,\tau;H^2(\Gamma_1))$ is exactly controllable by Theorem (1.0.4). There are many methods to prove exact controllability. The classical way [8, 27] is to show that for every $\xi \in X$, there exists a $\gamma > 0$ such that

$$\int_0^\tau \|B^*S^*(s)\xi\|^2_{H^2(\Gamma_1)}ds \geq \gamma\|\xi\|^2_X. \tag{6.2.2}$$

Generally speaking, if $H^2(\Gamma_1)$ (the domain of control v) is a finite dimensional space and B is a bounded operator, the controllability map $\mathcal{C}_\tau$ defined by (1.0.27) will be compact. Therefore the system is not exactly controllable. In our case, we will show that there exists a positive γ such that the inequality (6.2.2) holds for every $\xi \in X$.

Theorem 6.2.1. *For given control $v \in L^2(0,\tau;H^2(\Gamma_1))$ created by a flexible generator located on Γ_1, the system* (6.2.1) *is exactly controllable on* $[0,\tau]$ *for any* $\tau > 0$.

Proof. To show this theorem, it is necessary to find out the explicit form of the operator $\mathcal{B}^*$, which is the adjoint of $\mathcal{B}$ defined by $\mathcal{B}v = -\partial_y Nv|_{\Gamma_s} + \partial_y Nv_{xx}|_{\Gamma_s}$. In another word, we are required to determine $\mathcal{B}^*\theta$ for $\theta \in L^2(\Gamma_s)$. For this purpose, let us define the auxiliary function Θ which satisfies

$$\begin{cases} \triangle\,\Theta = 0, & \text{in } \Omega, \\ \Theta = \theta, & \text{on } \Gamma_s, \\ \partial_n\Theta = 0, & \text{on } \Gamma_f \cup \Gamma_1 \cup \Gamma_2, \\ \Theta_{xy}(0) = \Theta_{xy}(\pi) = 0, & \text{in } \Gamma_s. \end{cases} \tag{6.2.3}$$

Consider

$$\begin{aligned} < \mathcal{B}v, \theta > &= < -\partial_y Nv|_{\Gamma_s} + \partial_y (Nv)_{xx}|_{\Gamma_s}, \theta > \\ &= \underbrace{\int_{\Gamma_s} -\partial_y Nv \cdot \theta dx}_{I} + \underbrace{\int_{\Gamma_s} \partial_y (Nv)_{xx}\theta dx}_{II}. \end{aligned} \tag{6.2.4}$$

For part (I), direct computation shows that

$$\begin{aligned} <\triangle\, Nv, \Theta> &= \int_\Omega \triangle\, Nv \cdot \Theta d\Omega \\ &= \int_\Omega Div(\nabla Nv \cdot \Theta) d\Omega - \int_\Omega \nabla Nv \cdot \nabla\Theta d\Omega \\ &= \int_{\partial\Omega} \partial_n Nv \cdot \Theta d\partial\Omega - \int_\Omega \nabla Nv \cdot \nabla\Theta d\Omega = 0. \end{aligned}$$

Therefore

$$\int_{\Gamma_s} \partial_y Nv \cdot \theta dx + \int_{\Gamma_1} v \cdot \Theta|_{\Gamma_1} dy = \int_\Omega \nabla Nv \cdot \nabla\Theta d\Omega. \tag{6.2.5}$$

Meanwhile, since $\partial_n\Theta = 0$ on $\Gamma_f \cup \Gamma_1 \cup \Gamma_2$ and $Nv = 0$ on Γ_s, through the construction of $\triangle\,\Theta = 0$, one can show that

$$\int_\Omega \nabla Nv \cdot \nabla\Theta d\Omega = \int_{\partial\Omega} \partial_n\Theta \cdot Nv d\partial\Omega = 0.$$

Thus

$$-\int_{\Gamma_s} \partial_y Nv \cdot \theta dx = \int_{\Gamma_1} v \cdot \Theta|_{\Gamma_1} dy. \tag{6.2.6}$$

For part (II), we see that

$$\begin{aligned}
<\triangle\, (Nv)_{xx}, \Theta> &= \int_\Omega \triangle\, (Nv)_{xx} \cdot \Theta d\Omega \\
&= \int_\Omega Div(\nabla (Nv)_{xx} \cdot \Theta) d\Omega - \int_\Omega \nabla (Nv)_{xx} \cdot \nabla \Theta d\Omega \\
&= \int_{\partial\Omega} \partial_n (Nv)_{xx} \cdot \Theta d\partial\Omega - \int_\Omega \nabla (Nv)_{xx} \cdot \nabla \Theta d\Omega \\
&= \int_{\Gamma_s} \partial_y (Nv)_{xx} \cdot \Theta|_{\Gamma_s} dx + \int_{\Gamma_1} \partial_x (Nv)_{xx} \cdot \Theta|_{\Gamma_1} dy \\
&\quad + \int_{\Gamma_2} \partial_x (Nv)_{xx} \cdot \Theta|_{\Gamma_2} dy - \int_\Omega \nabla (Nv)_{xx} \cdot \nabla \Theta d\Omega = 0.
\end{aligned}$$

Since

$$\begin{aligned}
\int_\Omega \nabla (Nv)_{xx} \cdot \nabla \Theta d\Omega &= \int_{\partial\Omega} \partial_n \Theta \cdot (Nv)_{xx} d\partial\Omega \\
&= \int_{\Gamma_s} \partial_y \Theta \cdot (Nv)_{xx} dx \\
&= 0,
\end{aligned}$$

we have

$$\int_{\Gamma_s} \partial_y (Nv)_{xx} \cdot \Theta|_{\Gamma_s} dx = \underbrace{-\int_{\Gamma_1} \partial_x (Nv)_{xx} \cdot \Theta|_{\Gamma_1} dy}_{III} \underbrace{- \int_{\Gamma_2} \partial_x (Nv)_{xx} \cdot \Theta|_{\Gamma_2} dy}_{IV}. \tag{6.2.7}$$

From (5.1.4), we observe the following relationship

$$\begin{aligned}
\Psi_{xx} &= -\Psi_{yy} \\
\Longrightarrow \Psi_{xxx} &= -\Psi_{xyy} \\
\Longrightarrow \Psi_{xxx} &= -v_{yy}, \quad \text{on } \Gamma_1.
\end{aligned}$$

Thus the part (III) is equivalent to

$$\int_{\Gamma_1} v_{yy} \cdot \Theta|_{\Gamma_1} dy = (\Theta|_{\Gamma_1} \cdot v_y)|_0^1 - (\Theta_y|_{\Gamma_1} \cdot v)|_0^1 + \int_0^1 v \cdot \Theta_{yy}|_{\Gamma_1} dy. \tag{6.2.8}$$

Furthermore, from Neumann map defined in (5.1.4), we have boundary conditions $v(1) = 0$ and $v_y(0) = 0$ on Γ_1. Meanwhile, two more boundary conditions $\Theta(0,1) = 0$ and $\Theta_y(0,0) = 0$ are given by (6.2.3). Therefore, (6.2.8) is equivalent to

$$\int_{\Gamma_1} v_{yy} \cdot \Theta|_{\Gamma_1} dy = \int_0^1 v \cdot \Theta_{yy}|_{\Gamma_1} dy. \tag{6.2.9}$$

For part (IV), we can show that

$$\begin{aligned}
& -\int_{\Gamma_2} \partial_x (Nv)_{xx} \cdot \Theta|_{\Gamma_2} dy \\
= & \int_0^1 \Psi_{yyx}|_{\Gamma_2} \cdot \Theta|_{\Gamma_2} dy \\
= & (\Psi_{yx}|_{\Gamma_2} \cdot \Theta|_{\Gamma_2})|_0^1 - \int_0^1 \Psi_{yx}|_{\Gamma_2} \cdot \Theta_y|_{\Gamma_2} dy \\
= & (\Psi_{yx}|_{\Gamma_2} \cdot \Theta|_{\Gamma_2})|_0^1 - (\Psi_x|_{\Gamma_2} \cdot \Theta_y|_{\Gamma_2})|_0^1 + \int_0^1 \Psi_x|_{\Gamma_2} \cdot \Theta_{yy}|_{\Gamma_2} dy = 0. \qquad (6.2.10)
\end{aligned}$$

Thus (6.2.7) yields

$$\int_{\Gamma_s} \partial_y (Nv)_{xx} \cdot \Theta|_{\Gamma_s} dx = \int_{\Gamma_1} v \cdot \Theta_{yy}|_{\Gamma_1} dy. \qquad (6.2.11)$$

Based on these calculations, the equation (6.2.4) gives rise to

$$\begin{aligned}
\int_{\Gamma_s} -\partial_n N v \cdot \theta dx + \int_{\Gamma_s} \partial_y (Nv) xx \cdot \theta dx &= \int_{\Gamma_1} v \cdot \Theta|_{\Gamma_1} dy + \int_{\Gamma_1} v \cdot \Theta_{yy}|_{\Gamma_1} dy \\
< \mathcal{B} v, \theta > &= < v, \Theta|_{\Gamma_1} + \Theta_{yy}|_{\Gamma_1} > \\
< v, \mathcal{B}^* \theta > &= < v, \Theta|_{\Gamma_1} + \Theta_{yy}|_{\Gamma_1} > \\
\mathcal{B}^* \theta &= \Theta|_{\Gamma_1} + \Theta_{yy}|_{\Gamma_1}. \qquad (6.2.12)
\end{aligned}$$

Next, we try to identify the adjoint operator B^*. Using the definition of inner product $< \cdot, \cdot >_X$ given by (5.1.11), for any $\xi \in X$ and $v \in H^2(\Gamma_1)$, we can define the operator B by $\mathcal{B}^*$ as follows,

$$\begin{aligned}
< Bv, \xi >_X &= \left\langle \begin{pmatrix} 0 \\ \Pi_H \mathcal{B} v \\ \Pi_R \mathcal{B} v \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} \right\rangle \\
&= < \mathcal{A}^{1/2} 0, \mathcal{A}^{1/2} \xi_1 > + < \Pi_H \mathcal{B} v + \Pi_R \mathcal{B} v, \xi_2 + \xi_3 >_X \\
&= \int_\Omega \nabla D 0 \nabla D \xi_1 + \int_\Omega \nabla D 0_x \nabla D \xi_{1x} + < \mathcal{B} v, \xi_2 + \xi_3 >_{L^2(\Gamma_s)} \\
&= < \mathcal{B} v, \xi_2 + \xi_3 >_{L^2(\Gamma_s)} \\
&= < v, \mathcal{B}^* (\xi_2 + \xi_3) >_{L^2(\Gamma_1)} .
\end{aligned}$$

Thus the operator $B^* : L^2(\Gamma_s) \to H^2(\Gamma_1)$ is given by

$$B^* \xi = \mathcal{B}^* (\xi_2 + \xi_3) \qquad (6.2.13)$$

for any $\xi \in X$. Since $\{\varphi_k\}$ is an orthonormal basis in X (see proposition (5.2.1)), ξ can be written into the series form $\xi = \sum_{n\in Z} c_n\varphi_n$. Then we can show that

$$\begin{aligned} B^*S^*(t)\xi &= \sum_{k\in Z} e^{-i\omega_k t} B^* \langle \xi, \varphi_k\rangle \varphi_k \\ &= \sum_{k\in Z} e^{-i\omega_k t} B^* \left\langle \sum_{n\in Z} c_n\varphi_n, \varphi_k \right\rangle \varphi_k \\ &= \sum_{k\in Z} c_k e^{-i\omega_k t}\cdot B^*\varphi_k \\ &= \sum_{k\in Z} c_k e^{-i\omega_k t}\cdot \mathcal{B}^* \frac{1}{\sqrt{\pi}}(w_k + 0) \\ &= \frac{1}{\sqrt{\pi}} \sum_{k\in Z} c_k \mathcal{B}^* w_k e^{-i\omega_k t}, \end{aligned} \tag{6.2.14}$$

where w_k is the k^{th} eigenfunction of $\mathcal{A}$. On the other hand, W_k and $w_k = W_k|_{\Gamma_s}$ defined by (5.2.2) satisfy

$$\begin{cases} \triangle W_k = 0, & \text{in } \Omega, \\ W_k = w_k, & \text{on } \Gamma_s, \\ \partial_n W_k = 0, & \text{on } \Gamma_f \cup \Gamma_1 \cup \Gamma_2. \end{cases}$$

Combining (6.2.12) and (6.2.14) yields

$$B^*S^*(t)\xi = \frac{1}{\sqrt{\pi}} \sum_{k\in Z} c_k e^{-i\omega_k t} \left(W_k|_{\Gamma_1} + W_{kyy}|_{\Gamma_1}\right). \tag{6.2.15}$$

From (5.2.7) one obtains that

$$W_k = \frac{\cosh ky \cos kx}{\cosh k} \quad \text{and} \quad W_{kyy} = \frac{k^2 \cosh ky \cos kx}{\cosh k}.$$

Thus, on the left-end-side Γ_1 of the basin

$$W_k|_{\Gamma_1} = \frac{\cosh ky}{\cosh k} \quad \text{and} \quad W_{kyy}|_{\Gamma_1} = \frac{k^2\cosh ky}{\cosh k}.$$

Directly computation shows that

$$\begin{aligned} \int_0^\tau \|B^*S^*\xi\|^2_{H^2(\Gamma_1)} dt &= \int_0^\tau \Big\|\frac{1}{\sqrt{\pi}} \sum_{k\in Z} c_k \mathcal{B}^* w_k e^{-i\omega_k t}\Big\|^2_{H^2(\Gamma_1)} dt \\ &= \underbrace{\frac{1}{\pi}\int_0^\tau\int_0^1 \left|\sum_{k\in Z} c_k \mathcal{B}^* w_k e^{-i\omega_k t}\right|^2 dydt}_{V} + \underbrace{\frac{1}{\pi}\int_0^\tau\int_0^1 \left|\sum_{k\in Z} c_k (\mathcal{B}^* w_k)_y e^{-i\omega_k t}\right|^2 dydt}_{VI} \\ &\quad + \underbrace{\frac{1}{\pi}\int_0^\tau\int_0^1 \left|\sum_{k\in Z} c_k (\mathcal{B}^* w_k)_{yy} e^{-i\omega_k t}\right|^2 dydt}_{VII}. \end{aligned}$$

Note that, the eigenvalues $\{\omega_k\}$ have the following property

$$\begin{aligned}
|\omega_k - \omega_n| &= \left|[(k^3+k)\tanh k]^{1/2} - [(n^3+n)\tanh n]^{1/2}\right| \\
&\geq \left|(\tanh n)^{1/2}(\sqrt{k^3+k} - \sqrt{n^3+n})\right| \\
&= \left|(\tanh n)^{1/2} \cdot \frac{k^3+k-n^3-n}{\sqrt{k^3+k}+\sqrt{n^3+n}}\right| \\
&= \left|\frac{(\tanh n)^{1/2} \cdot (k^2+nk+n^2+1)}{\sqrt{k^3+k}+\sqrt{n^3+n}} \cdot (k-n)\right|.
\end{aligned}$$

Thus, there exists a positive number $\alpha \leq \frac{(\tanh n)^{1/2} \cdot (k^2+nk+n^2+1)}{\sqrt{k^3+k}+\sqrt{n^3+n}}$, such that

$$|\omega_k - \omega_n| \geq \alpha |k-n|.$$

Therefore, due to Ingham's Theorem [14], for all $\epsilon > 0$, there exist two positive constants C_1, C_2 such that

$$C_2 \sum_{n=1}^{\infty} |x_n|^2 \geq \int_0^{2\pi/\alpha+\epsilon} |\sum_{n=1}^{\infty} x_n e^{-it\omega_n}|^2 dt \geq C_1 \sum_{n=1}^{\infty} |x_n|^2.$$

Based on this inequality, one can show that

$$\begin{aligned}
V &= \frac{1}{\pi}\int_0^1 \int_0^{\tau} \left|\sum_{k\in Z} c_k e^{-i\omega_k t}\left(\frac{\cosh ky}{\cosh k} + \frac{k^2 \cosh ky}{\cosh k}\right)\right|^2 dtdy \\
&\geq \lim_{n\to\infty} \frac{1}{\pi}\int_0^1 \sum_{k=1}^{n} \left|c_k\left(\frac{\cosh ky}{\cosh k} + \frac{k^2 \cosh ky}{\cosh k}\right)\right|^2 dy \\
&\geq \lim_{n\to\infty} \frac{c}{\pi}\int_0^1 \sum_{k=1}^{n} |c_k|^2 \left(\left|\frac{\cosh ky}{\cosh k}\right|^2 + \left|\frac{k^2 \cosh ky}{\cosh k}\right|^2\right) dy \\
&= \lim_{n\to\infty} \frac{c}{\pi} \sum_{k=1}^{n} |c_k|^2 \cdot (1+k^4)\frac{k \cdot (sechk)^2 + \tanh k}{2k},
\end{aligned}$$

$$\begin{aligned}
VI &= \frac{1}{\pi}\int_0^1 \int_0^{\tau} \left|\sum_{k\in Z} c_k e^{-i\omega_k t}\left(\frac{k\sinh ky}{\cosh k} + \frac{k^3 \sinh ky}{\cosh k}\right)\right|^2 dtdy \\
&\geq \lim_{n\to\infty} \frac{1}{\pi}\int_0^1 \sum_{k=1}^{n} \left|c_k\left(\frac{k\sinh ky}{\cosh k} + \frac{k^3 \sinh ky}{\cosh k}\right)\right|^2 dy \\
&\geq \lim_{n\to\infty} \frac{c}{\pi}\int_0^1 \sum_{k=1}^{n} |c_k|^2 \left(\left|\frac{k\sinh ky}{\cosh k}\right|^2 + \left|\frac{k^3 \sinh ky}{\cosh k}\right|^2\right) dy \\
&= \lim_{n\to\infty} \frac{c}{\pi} \sum_{k=1}^{n} |c_k|^2 \cdot (k^2+k^6)\frac{-k \cdot (sechk)^2 + \tanh k}{2k},
\end{aligned}$$

$$\begin{aligned}
VII &= \frac{1}{\pi}\int_0^1\int_0^\tau \left|\sum_{k\in Z} c_k e^{-i\omega_k t}\left(\frac{k^2\cosh ky}{\cosh k}+\frac{k^4\cosh ky}{\cosh k}\right)\right|^2 dtdy \\
&\geq \lim_{n\to\infty}\frac{1}{\pi}\int_0^1\sum_{k=1}^{n}\left|c_k\left(\frac{k^2\cosh ky}{\cosh k}+\frac{k^4\cosh ky}{\cosh k}\right)\right|^2 dy \\
&\geq \lim_{n\to\infty}\frac{c}{\pi}\int_0^1\sum_{k=1}^{n}|c_k|^2\left(\left|\frac{k^2\cosh ky}{\cosh k}\right|^2+\left|\frac{k^4\cosh ky}{\cosh k}\right|^2\right) dy \\
&= \lim_{n\to\infty}\frac{c}{\pi}\sum_{k=1}^{n}|c_k|^2\cdot(k^4+k^8)\frac{k\cdot(sechk)^2+\tanh k}{2k}.
\end{aligned}$$

Note that $\|\xi\|_X^2=\sum_{n=1}^\infty |c_k|^2$. Hence for fixed $\tau>0$ one has

$$\frac{\int_0^\tau \|B^*S^*\xi\|_{H^2(\Gamma_1)}^2 dt}{\|\xi\|_X^2}\to\infty \quad \text{as} \quad k\to\infty.$$

Thus, there exists at least one $\gamma>0$ such that the inequality (6.2.2) holds for any $\xi\in X$, which implies that the system (6.2.1) with flexible generator is exactly controllable. □

6.3 Approximate Controllability for Rigid Generator

In this section, we will discuss the controllability problem of (6.2.1) with a rigid generator. As what we mentioned in the first chapter, the rigid generator means that the shape of generator is fixed. Then we can define the control v by

$$v(y,t)=f(y)u(t),$$

where $u(t)$ is the small angular velocity and $f(y)$ is the shape of generator which depends on y. Let $\Lambda=\mathcal{B}f(y)$. We can define a new operator $B_f : L^2(0,\tau)\to L^2(\Gamma_s)$ by

$$B_f=\begin{pmatrix}0\\ \Pi_H\mathcal{B}f(y)\\ \Pi_R\mathcal{B}f(y)\end{pmatrix}=\begin{pmatrix}0\\ \Pi_H\Lambda\\ \Pi_R\Lambda\end{pmatrix}. \tag{6.3.1}$$

Thus the abstract system (6.2.1) is changed to

$$\begin{cases}\dot{\xi}=A\xi+B_f u,\\ \eta=\eta_{xx}+C\xi,\\ \xi(0)=\xi_0\in X,\end{cases} \tag{6.3.2}$$

only with a scalar control $u(t)$. One can show that this system is approximately controllable for any $u\in L^2(0,\tau)$. Indeed, we will show that the adjoint of controllability map $C_\tau^*:=B_f^*S^*$ is injective, i.e.,

$$B_f^*S^*(t)\xi=0 \quad \text{on } [0,\tau]\Rightarrow \xi=0. \tag{6.3.3}$$

Theorem 6.3.1. *For any given* $u \in L^2(0,\tau)$ *created by a rigid generator located on* Γ_1*, suppose that the shape of the generator* $f(y) > 0$*. Then the system* (6.3.2) *is approximately controllable on* $[0,\tau]$ *for any* $\tau > 0$*.*

Proof. Based on Theorem (1.0.5), let us consider

$$\begin{aligned} B_f^* S^*(t)\xi &= \sum_{k\in Z} B_f^* \langle \xi, \varphi_k \rangle_X \varphi_k e^{-i\omega_k t} \\ &= \sum_{k\in Z} B_f^* \left\langle \sum_{n\in Z} c_n\varphi_n, \varphi_k \right\rangle_X \varphi_k e^{-i\omega_k t} \\ &= \sum_{k\in Z} B_f^* \varphi_k \cdot c_k e^{-i\omega_k t}. \end{aligned} \tag{6.3.4}$$

From (5.2.9), recall that

$$\omega_k = [(k^3+k)\tanh k]^{1/2} \to k^{3/2}, \quad \text{as } k \to \infty. \tag{6.3.5}$$

Therefore, for any ω_k, one can show that

$$\begin{aligned} |\omega_{k+1} - \omega_k| &= \left| \sqrt{(k+1)^3} - \sqrt{k^3} \right| \\ &= \left| \frac{(k+1)^3 - k^3}{\sqrt{(k+1)^3} + \sqrt{k^3}} \right| \\ &= \left| \frac{3k^2+3k+1}{\sqrt{(k+1)^3} + \sqrt{k^3}} \right| \to \infty, \text{as } k \to \infty. \end{aligned}$$

Obviously, there exists a gap between ω_k and ω_{k+1}, i.e., $\{e^{-i\omega_k t}\}$ is not a basis of $L^2(0,\tau)$. However, one can add some new sequence $\{e^{-i\omega'_k t}\}$ to form a basis $\{e^{-iv_k t}\}$ of $L^2(0,\tau)$. Thus according to the Theorem, every function f in $L^2(0,\tau)$ has a unique nonharmonic Fourier series expansion

$$f(t) = \sum_{k=1}^{\infty} c_k e^{-iv_k t}, \quad \text{with} \quad \sum_{k=1}^{\infty} |c_k|^2 \leq \infty.$$

In our case, the coefficients of $\{e^{i\omega'_k t}\}$ are all zero. Thus

$$B_f^* S^*(t)\xi = 0$$

requires the coefficients $c_k B_f^* \varphi_k = 0$ for all $k \in Z$. Observe that

$$\begin{aligned} < B_f u, \xi > &= \int_0^\tau < B_f u, \xi >_X dt \\ &= \int_0^\tau u < B_f, \xi >_X dt \\ &= < u, < B_f, \xi >_X >_{L^2(0,\tau)} \\ &= < u, B_f^* \xi >_{L^2(0,\tau)} . \end{aligned} \tag{6.3.6}$$

Therefore

$$
\begin{aligned}
B_f^*\xi =< B_f, \xi >_X &= \left\langle \begin{pmatrix} 0 \\ \Pi_H \mathcal{B} f \\ \Pi_R \mathcal{B} f \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} \right\rangle \\
&= < \mathcal{A}^{1/2} 0, \mathcal{A}^{1/2} \xi_1 > + < \Pi_H \mathcal{B} f + \Pi_R \mathcal{B} f, \xi_2 + \xi_3 > \\
&= \int_\Omega \nabla D0 \nabla D\xi_1 + \int_\Omega \nabla D0_x \nabla D\xi_{1x} + < \mathcal{B} f, \xi_2 + \xi_3 > \\
&= < \mathcal{B} f, \xi_2 + \xi_3 >_{L^2(\Gamma_s)}, \qquad (6.3.7)
\end{aligned}
$$

which yields

$$
B_f^* \varphi_k = \frac{1}{\sqrt{\pi}} < \mathcal{B} f, w_k >_{L^2(\Gamma_s)} = \frac{1}{\sqrt{\pi}} < f, \mathcal{B}^* w_k >_{L^2(\Gamma_1)} . \qquad (6.3.8)
$$

Observe that

$$
\begin{aligned}
& < f, \mathcal{B}^* w_k >_{L^2(\Gamma_1)} \\
=& \int_0^1 f(y) \left(W_k|_{\Gamma_1} + W_{kyy}|_{\Gamma_1} \right) dy \\
=& \int_0^1 f(y) \frac{\cosh ky}{\cosh k} dy + \int_0^1 f(y) \frac{k^2 \cosh ky}{\cosh k} dy \\
=& \frac{1 + k^2}{\cosh k} \int_0^1 f(y) \cosh kydy.
\end{aligned}
$$

Thus one may have

$$
B_f^* \varphi_k = \frac{1 + k^2}{\sqrt{\pi} \cosh k} \int_0^1 f(y) \cosh kydy. \qquad (6.3.9)
$$

Note that $B_f^* \varphi_k > 0$ when $f(y) > 0$. In papers [31, 29], they call the shape $f(y)$ strategic. Therefore, $B_f^* \varphi_k c_k = 0$ if and only if $c_k = 0$ for all $k \in Z$, i.e., $\xi = 0$. Finally, we can claim that the controllability map $C_\tau^* := B_f^* S^*$ is injective. Using Theorem (1.0.5), the system (6.3.2) is approximately controllable on X.

□

6.4 Lack of Exact Controllability for Rigid Generator

Theorem 6.4.1. *For any given $u \in L^2(0,\tau)$ created by a rigid generator located on Γ_1, suppose that the shape of the generator $f(y) \in H_c^p(\Gamma_1)$, $p > 3/2$ and $f(y) > 0$. Then the system* (6.3.2) *is not exactly controllable on* $[0,\tau]$ *for any* $\tau > 0$.

Proof. In fact, the system (6.3.2) is not approximately controllable on $[0,\tau]$ for any $\tau > 0$. Let us define a Sobolev space

$$
H_c^p(\Gamma_1) = \{ f \in H^p(\Gamma_1),\ f^{(q)}(1) = f^{(q)}(0) = 0,\ q = 0, ..., p-1 \}.
$$

Observe that for any $f(y) \in H_c^p(\Gamma_1)$, we can show that

$$
\begin{aligned}
|B_f^*\varphi_k| &= \left|(1+k^2)\int_0^1 \frac{\cosh ky}{\cosh k} f(y)dy\right| \\
&= \begin{cases} \left|\dfrac{1+k^2}{k^p}\displaystyle\int_0^1 f^{(p)}\frac{\cosh ky}{\cosh k}dy\right|, & \text{if } p \text{ is even,} \\ \left|\dfrac{1+k^2}{k^p}\displaystyle\int_0^1 f^{(p)}\frac{\sinh ky}{\cosh k}dy\right|, & \text{if } p \text{ is odd.} \end{cases}
\end{aligned}
$$

Furthermore, we note that

$$
\begin{aligned}
& \left|\frac{1+k^2}{k^p}\int_0^1 f^{(p)}\frac{\sinh ky}{\cosh k}dy\right| \\
\leq & \left|\frac{1+k^2}{k^p}\left(\int_0^1 (f^{(p)}(y))^2 dy\right)^{1/2} \cdot \left(\int_0^1 \left(\frac{\sinh ky}{\cosh k}\right)^2 dy\right)^{1/2}\right| \\
\leq & \left|\frac{1+k^2}{k^p}\left(\int_0^1 (f^{(p)}(y))^2 dy\right)^{1/2} \cdot \frac{1}{k^{1/2}}\left(\int_0^k \left(\frac{\sinh x}{\cosh k}\right)^2 dx\right)^{1/2}\right| \\
\leq & \left|\left(\frac{C}{k^{p+1/2}} + \frac{C}{k^{p-2+1/2}}\right)\sqrt{\frac{1}{2}\left(\frac{-k}{\cosh^2 k} + \tanh k\right)}\right| \\
\to & \frac{1}{\sqrt{2}}\left(\frac{C}{k^{p+1/2}} + \frac{C}{k^{p-2+1/2}}\right) \\
\to & \frac{C}{k^{p-3/2}} \quad \text{as } k \to \infty.
\end{aligned}
$$

If p is even, we have a similar result

$$
\left|\frac{1+k^2}{k^p}\int_0^1 f^{(p)}\frac{\cosh ky}{\cosh k}dy\right| \leq \frac{C}{k^{p-3/2}} \quad \text{as } k \to \infty.
$$

Next, we will give a counterexample to show that the system is not exactly controllable. One considers the sequence

$$
\xi^{(k)} = \varphi_k, \quad k > 0,
$$

where φ_k is the kth eigenfunction of A. This sequence has the property $\|\xi^{(k)}\|_X^2 = 1$. From

(6.3.4) and (6.3.9), one can easily show that

$$\begin{aligned}
\int_0^\tau \|B_f^* S^*(s)\xi^{(k)}\|^2_{H^2(\Gamma_1)} dt &= \int_0^\tau \|c_k B_f^* \varphi_k e^{-i\omega_k t}\|^2_{H^2(\Gamma_1)} dt \\
&= \int_0^\tau \|c_k \frac{1+k^2}{\cosh k} \int_0^1 f(y) \cosh ky dy e^{-i\omega_k t}\|^2_{H^2(\Gamma_1)} dt \\
&= \int_0^\tau |c_k (1+k^2) \int_0^1 f(y) \frac{\cosh ky}{\cosh k} dy e^{-i\omega_k t}|^2 dt \\
&\leq \left| c_k (1+k^2) \int_0^1 f(y) \frac{\cosh ky}{\cosh k} dy \right|^2 \\
&\leq \left| \frac{c_k}{k^{p-3/2}} \right|^2 .
\end{aligned}$$

Note that, for any fixed $\tau > 0$, one has

$$\lim_{k\to\infty} \frac{\left|\frac{c_k}{k^{p-3/2}}\right|^2}{|c_k|^2} = 0 \quad \text{if } p > 3/2. \tag{6.4.1}$$

Therefore, we cannot find a positive γ such that

$$\int_0^\tau \|B_f^* S^*(s)\xi^{(k)}\|^2_{H^2(0,\tau)} \geq \gamma \|\xi\|^2_X .$$

By Theorem (1.0.4), the system (6.3.2) is not exactly controllable with rigid operator. □

Chapter 7

Stabilization Problems of The Basin Model

7.1 Ad hoc Energy Space

In this chapter, we will discuss the stabilization problems of the basin model

$$\begin{cases} \psi_{xx} + \psi_{yy} = 0, & \text{on } \Omega \times [0, \tau], \\ \psi_{tt} + \partial_y \psi = \partial_y \psi_{xx}, & \text{on } \Gamma_s \times [0, \tau], \\ \psi_y = 0, & \text{on } \Gamma_f \times [0, \tau], \\ \psi_x = v, & \text{on } \Gamma_1 \times [0, \tau], \\ \psi_y = 0, & \text{on } \Gamma_2 \times [0, \tau], \\ \psi_{xy}(0) = \psi_{xy}(\pi) = 0, & \text{on } \Gamma_s. \end{cases} \tag{7.1.1}$$

Note that we only consider the rigid generator located on the left hand side of the basin. Recall that the boundary condition on Γ_1 is given by

$$\partial_n \psi = v(t) = f(y)u(t),$$

where $f(y) \in H^2(\Gamma_1)$ is the shape of generator which is supposed to be positive on Γ_1 and $u(t) \in L^2(0, \tau)$ is the angular velocity of generator. We can observe this system through measuring the elevation of the surface at Γ_1 $(x = 0)$. Note that, on the surface Γ_s,

$$\eta(0, t) - \eta_{xx}(0, t) = -\dot{\psi}(0, 1, t).$$

Physically, this requires a sensor installed on the generator itself. We may simply choose the angular velocity of the generator

$$u(t) = \eta(0, t) - \eta_{xx}(0, t) = -\dot{\psi}(0, 1, t) \tag{7.1.2}$$

to construct a simple feedback control. This kind of feedback will require enough regularity for $\dot{\psi}$ on Γ_s. We are interested by the choices of $f(y)$ which will make the system strongly stable. In terms of the original system (7.1.1), this feedback leads to the following equations:

$$\begin{cases} \psi_{xx} + \psi_{yy} = 0, & \text{on } \Omega \times [0,\tau], \\ \psi_{tt} + \partial_y \psi = \partial_y \psi_{xx}, & \text{on } \Gamma_s \times [0,\tau], \\ \psi_y = 0, & \text{on } \Gamma_f \times [0,\tau], \\ \psi_x + f(y)\dot{\psi}(0,1) = 0, & \text{on } \Gamma_1 \times [0,\tau], \\ \psi_y = 0, & \text{on } \Gamma_2 \times [0,\tau], \\ \psi_{xy}(0) = \psi_{xy}(\pi) = 0, & \text{on } \Gamma_s. \end{cases}$$

At beginning, on surface Γ_s, we may reformulate (7.1.1) with a single unknown $\varphi = \psi|_{\Gamma_s}$, which is the solution of

$$\ddot{\varphi} + \mathcal{A}\varphi = \Lambda(y)u(t).$$

Combining with (7.1.2) yields

$$\ddot{\varphi} + \mathcal{A}\varphi + \Lambda\dot{\varphi}(0,1,t) = 0 \quad \text{on } \Gamma_s \times [0,\infty), \tag{7.1.3}$$

with initial conditions

$$\varphi(0) = \varphi_0, \quad \dot{\varphi}(0) = \varphi_1. \tag{7.1.4}$$

Recall that the operator $\mathcal{A}$ is defined by $\mathcal{A}\varphi = \partial_n D\varphi|_{\Gamma_s} - \partial_n D\varphi_{xx}|_{\Gamma_s}$ and the function $\Lambda \in H^2(\Gamma_s)$ is defined by $\Lambda = \mathcal{B}f = -\partial_n Nf|_{\Gamma_s} + \partial_n Nf_{xx}|_{\Gamma_s}$. Therefore the original potential ψ is related to φ and f by

$$\psi = D\varphi + Nv = D\varphi + Nf(y)u(t) = D\varphi - Nf(y)\dot{\varphi}(0,1,t). \tag{7.1.5}$$

In order to exhibit very general aspect of the method that we will use to obtain our result, we define an observation operator C such that the feedback control $u(t) = \eta(0,t) - \eta_{xx}(0,t)$ is obtained by $C\xi$. Note that the feedback control is given by

$$u = \eta(0,t) - \eta_{xx}(0,t) = -\dot{\psi}(0,1,t), \quad \text{on } \Gamma_s.$$

Then the observation operator C can be defined as follows

$$C\xi = u(t) = -\dot{\varphi}(0,1,t) = -\left(\Pi_H\dot{\varphi}(0,1,t) + \Pi_R\dot{\varphi}(0,1,t)\right) = -\left(\xi_2(0,1,t) + \xi_3\right). \tag{7.1.6}$$

By the definition, the abstract system (5.1.6) is equivalent to

$$\dot{\xi} = A\xi + Bv = A\xi + Bf(y)u(t) = A\xi + B_f u(t) = A\xi + B_f C\xi = (A + B_f C)\xi.$$

Thus, the system (7.1.1), in terms of the new unknown ξ, can be changed to the problem

$$\begin{cases} \dot{\xi} = (A + B_f C)\xi, \\ \xi(0) = \xi_0, \end{cases} \tag{7.1.7}$$

which appears as a perturbation of the original open-loop system. In the following, we define $A_f = A + B_f C$. Note that, the initial value

$$\psi(0) = c, \quad \dot{\psi}(0) = 0, \quad \text{on } \Gamma_s,$$

will stay invariant, i.e.,

$$\psi = c, \quad \text{in } \Omega \times [0, \infty),$$

is the solution of equation (7.1.1). Moveover, $\psi = c$ is also the zero of the natural energy

$$E(\psi, \dot{\psi}) = \frac{1}{2}\int_{\Omega} |\nabla\psi|^2 + \frac{1}{2}\int_{\Omega} |\nabla\psi_x|^2 + \frac{1}{2}\int_{\Gamma_s} |\dot{\psi}|^2$$

of the system. In order to have a correct framework for the stability analysis of the system (7.1.7), we will eliminate such case by choosing a particular f to guarantee the regularity of Λ.

Without considering the surface tension, the stability problems have been already studied in [29] by showing that the eigenvalues of A_f have strictly negative real parts. However, the result on the real parts of A_f is not enough to claim that the system is strongly stable, because we do not know whether the eigenfunctions of A_f can form a Riesz basis. If we can find out the explicit form of the eigenvalues of A_f, the result of Triggiani [47, 24] can help. But we recognize that, this is not easy. In order to eliminate the initial data we discussed, in the following part, we will show that the system (7.1.7) is strongly stable by means of an ad hoc energy, and we will have to restrict ourselves to a particular choice of f.

The ad hoc energy actually is a bilinear form coming from classical inner product

$$< \xi, \zeta >_X = < \mathcal{A}^{1/2}\xi_1, \mathcal{A}^{1/2}\zeta_1 > + < \mathcal{A}^{1/2}\xi_{1x}, \mathcal{A}^{1/2}\zeta_{1x} > + < \xi_2 + \xi_3, \zeta_2 + \zeta_3 >,$$

which is defined by

$$H(\xi, \zeta) = \sum_{k \in Z} d_k < \xi, \varphi_k >_X \overline{< \zeta, \varphi_k >_X}, \tag{7.1.8}$$

where d_k is the weight and defined by

$$d_k = -\frac{C\varphi_k}{< \varphi_k, B_f >_X},$$

and φ_k denotes the kth eigenfunction of A. Recall that $C = -(0, 1, 1)$, $\varphi_k = (\varphi_{k_1}, \varphi_{k_2}, \varphi_{k_3})^T$ and $B_f = (0, \Pi_H\Lambda, \Pi_R\Lambda)^T$. Thus

$$< \varphi_k, B_f >_X = \left\langle \begin{pmatrix} \varphi_{k_1} \\ \varphi_{k_2} \\ \varphi_{k_3} \end{pmatrix}, \begin{pmatrix} 0 \\ \Pi_H\Lambda \\ \Pi_R\Lambda \end{pmatrix} \right\rangle_X = \langle \varphi_{k_2} + \varphi_{k_3}, \Pi_H\Lambda + \Pi_R\Lambda \rangle_{L^2(\Gamma_s)}.$$

Therefore from (5.1.11) and (6.3.8), for any $k > 0$,

$$\begin{aligned} < \varphi_k, B_f >_X &= \frac{1}{\sqrt{\pi}} < w_k, \Lambda > \\ &= \frac{1}{\sqrt{\pi}} < w_k, \mathcal{B}f >_{L^2(\Gamma_s)} \\ &= \frac{1}{\sqrt{\pi}} < \mathcal{B}^* w_k, f >_{L^2(\Gamma_1)} \\ &= \frac{1+k^2}{\sqrt{\pi}\cosh k} \int_0^1 \cosh ky \cdot f(y) dy. \end{aligned} \tag{7.1.9}$$

By the hypothesis, $f(y) > 0$ on $[0, 1]$ applies that $< \varphi_k, B_f >_X$ is positive for any $k > 0$. Meanwhile $< \varphi_k, B_f >_X = < \varphi_{-k}, B_f >_X$ since $w_k = \cos kx$. As a consequence, we can extend this result to any $k \in Z$. Thus

$$d_k = \frac{\varphi_{k_2}(0) + \varphi_{k_3}(0)}{< \varphi_{k_2} + \varphi_{k_3}, \Lambda >} = \frac{\frac{1}{\sqrt{\pi}} w_k(0)}{< \frac{1}{\sqrt{\pi}} w_k, \Lambda >} = \frac{1}{< w_k, \Lambda >} \quad \text{for } k \in Z. \tag{7.1.10}$$

On the other hand

$$\begin{aligned} < \xi, \zeta >_X &= \left\langle \sum_k < \xi, \phi_k >_X \phi_k, \sum_k < \zeta, \phi_k >_X \phi_k \right\rangle \\ &= \sum_k < \xi, \phi_k >_X \overline{< \zeta, \phi_k >_X} < \phi_k, \phi_k >_X \\ &= \sum_k < \xi, \phi_k >_X \overline{< \zeta, \phi_k >_X}. \end{aligned}$$

Thus, the positivity of d_k allows us to claim that the bilinear form $H(\xi, \xi)$ defines a scalar product and

$$\begin{aligned} H(\xi, \xi) &= \sum_{k \in Z} d_k < \xi, \phi_k >_X \overline{< \xi, \phi_k >_X} \\ &= \sum_{k \in Z} d_k | < \xi, \phi_k >_X |^2 \end{aligned}$$

can be used as a norm on the related ad hoc energy space, which is to be determined. Observe that, the weight d_k depends on $< w_k, \mathcal{B}f >$. The regularity of f will obviously determine the energy space in which the stability result will hold. From an engineering point of view, the generator is considered as a plane rotating generator which is given by $f(y) = y$, because this corresponds to the most widely used device for wave generators. However if we choose

$f(y) = y$, one can see that $f(y)$ is not in $H^{1/2}$ by showing

$$\begin{aligned} & \sum_{n=1}^{m} (1+n) \left|c_n\right|^2 \\ = & \sum_{n=1}^{m} (1+n) \left| \int_0^1 y \cdot e^{iny} dy \right|^2 \\ = & \sum_{n=1}^{m} (1+n) \left| \frac{1}{in} \left(y e^{iny}|_0^1 - \int_0^1 e^{iny} dy \right) \right|^2 \\ = & \sum_{n=1}^{m} (1+n) \frac{1}{n^4} \left| (in-1) e^{in} + 1 \right|^2 \\ = & \sum_{n=1}^{m} (1+n) \frac{1}{n^4} \left| (1 - \cos n - n \sin n) + (n \cos n - \sin n) i \right|^2 \\ \sim & \sum_{n=1}^{m} \frac{1}{n}, \end{aligned}$$

which is not convergent as $m \to \infty$. Thus, there exists $\epsilon > 0$ such that $f(y) = y$ is in $H^{1/2-\epsilon}$ rather than in $H^{7/2}$. In our case, the energy space can be identified as

$$\mathscr{D}(A) = \mathscr{D}(\mathcal{A}) \times \mathscr{D}(\mathcal{A}^{1/2}) \times R = H_0^3(\Gamma_s) \times H_0^{3/2}(\Gamma_s) \times R,$$

which requires $\mathcal{B}f \in H^{3/2}(\Gamma_s)$. On the other hand, $\mathcal{B}f = -\partial_n N f|_{\Gamma_s} + \partial_n N f_{xx}|_{\Gamma_s}$ yields $Nf \in H^5(\Gamma_s)$. Thus, from Neumann map N defined in (5.1.4), it is necessary to find a shape $f(y)$ which is at least in $H^{7/2}$. In the follows, we will focus on the generators with the shape given by

$$f(y) = \begin{cases} -\dfrac{5}{2\epsilon^7} y^8 + \dfrac{10}{\epsilon^6} y^7 - \dfrac{14}{\epsilon^5} y^6 + \dfrac{7}{\epsilon^4} y^5, & 0 \le y \le \epsilon, \\ y - \dfrac{\epsilon}{2}, & \epsilon \le y \le 1-\epsilon, \\ C_8 (y-1)^8 + C_7 (y-1)^7 - C_6 (y-1)^6 - C_5 (y-1)^5 + C_5 (y-1)^4, & 1-\epsilon \le y \le 1. \end{cases}$$

where $C_8 = -\frac{5(-14+13\epsilon)}{2\epsilon^8}$, $C_7 = -\frac{10(-16+15\epsilon)}{\epsilon^7}$, $C_6 = -\frac{14(-20+19\epsilon)}{\epsilon^6}$, $C_5 = -\frac{7(-32+31\epsilon)}{\epsilon^5}$ and $C_4 = -\frac{70(-1+\epsilon)}{\epsilon^4}(y-1)^4$. We can see that $f(y)$ is arbitrary close to the original shape. For this choice of $f(y)$, we have $f(y) \in H^8(\Gamma_1)$. From (6.3.8), one can show that

$$\begin{aligned} < w_k, \Lambda > & \to \frac{840}{k^3 \epsilon^8 \cosh k} \left\{ 2\epsilon^4 \sinh k - 2\epsilon^5 \sinh k \right\} \\ & \to \frac{1680}{k^3 \epsilon^4} \frac{\sinh k}{\cosh k}, \end{aligned} \tag{7.1.11}$$

which gives

$$k^3 < w_k, \Lambda > \to \frac{1}{\epsilon^4}, \quad k \to \infty. \tag{7.1.12}$$

From (7.1.10), one can see that $\frac{k^3}{d_k} \to \frac{1}{\epsilon^4}$ as $k \to \infty$. Since $\lambda_k = (k^3 + k)\tanh k$, one can easily show that

$$\lim_{k\to\infty} \frac{\lambda_k}{d_k} = \frac{1}{\epsilon^4}, \quad k \to \infty. \tag{7.1.13}$$

Considering the Ad hoc energy, we can obtain

$$\begin{aligned} H(\xi,\xi) &= \sum_{k\in z} d_k | < \xi, \varphi_k >_X |^2 \\ &= d_0 | < \xi, \varphi_0 >_X |^2 + \sum_{k\neq 0} d_k | < \xi, \varphi_k >_X |^2 \\ &= d_0 \left| \int_{\Gamma_s} (\xi_2 + \xi_3) \cdot \frac{1}{\sqrt{\pi}} \right|^2 \\ &+ \sum_{k\neq 0} \frac{1}{\pi} \cdot d_k \left| \frac{1}{\mu_k} \left(\int_\Omega \nabla D\xi_1 \cdot \nabla D w_k + \int_\Omega \nabla D\xi_{1x} \cdot \nabla D w_{kx} \right) + \int_{\Gamma_s} (\xi_2 + \xi_3) w_k \right|^2 . \end{aligned}$$

Directly computation shows that

$$d_0 \left| \int_{\Gamma_s} (\xi_2 + \xi_3) \cdot \frac{1}{\sqrt{\pi}} \right|^2 = d_0 \frac{1}{\pi} \left| \int_{\Gamma_s} \xi_2 + \xi_3 dx \right|^2 = d_0 \frac{1}{\pi} \left| \int_{\Gamma_s} \dot{\varphi} dx \right|^2 = d_0 \pi \xi_3^2,$$

and

$$\begin{aligned} & \sum_{k\neq 0} \frac{1}{\pi} \cdot d_k \left| \frac{1}{\mu_k} \left(\int_\Omega \nabla D\xi_1 \cdot \nabla D w_k + \int_\Omega \nabla D\xi_{1x} \cdot \nabla D w_{kx} \right) + \int_{\Gamma_s} (\xi_2 + \xi_3) w_k \right|^2 \\ = & \sum_{k\neq 0} \frac{1}{\pi} d_k \left| \frac{1}{\mu_k} \left(\int_{\Gamma_s} \frac{\partial}{\partial y} w_k \xi_1 - \int_{\Gamma_s} \frac{\partial}{\partial y} w_{kxx} \xi_1 \right) + < \xi_2, w_k > \right|^2 \\ = & \sum_{k\neq 0} \frac{1}{\pi} d_k \left| \frac{\lambda_k}{\sqrt{\lambda_k} i} < w_k, \xi_1 > + < \xi_2, w_k > \right|^2 \\ = & \sum_{k\neq 0} \frac{1}{\pi} d_k \left(\lambda_k < w_k, \xi_1 >^2 + < \xi_2, w_k >^2 \right). \end{aligned}$$

Therefore, the bilinear form is given by

$$H(\xi,\xi) = d_0 \pi \xi_3^2 + \frac{1}{\pi} \sum_{k\neq 0} \left(d_k \lambda_k < w_k, \xi_1 >^2 + d_k < \xi_2, w_k >^2 \right). \tag{7.1.14}$$

Considering the limit in (7.1.13) yields that (7.1.14) is equivalent to

$$\begin{aligned} H(\xi,\xi) &= d_0 \pi \xi_3^2 + \frac{1}{\pi} \sum_{k\neq 0} \left(\lambda_k^2 < w_k, \xi_1 >^2 + \lambda_k < w_k, \xi_2 >^2 \right) \\ &= d_0 \pi \xi_3^2 + \frac{1}{\pi} \sum_{k\neq 0} \left(< \lambda_k w_k, \xi_1 >^2 + < \lambda_k^{1/2} w_k, \xi_2 >^2 \right). \end{aligned}$$

Thus $H(\xi, \xi)$ will be convergent provided $\xi_1 \in \mathscr{D}(\mathcal{A})$ and $\xi_2 \in \mathscr{D}(\mathcal{A}^{1/2})$. Then the energy space defined by (7.1.14) is

$$X_f = \mathscr{D}(\mathcal{A}) \times \mathscr{D}(\mathcal{A}^{1/2}) \times R = H_0^3(\Gamma_s) \times H_0^{3/2}(\Gamma_s) \times R,$$

and the domain of A_f is defined by

$$\mathscr{D}(A_f) = \{\xi \in X_f, A_f \xi \in X_f\},$$

which is

$$\mathscr{D}(A_f) = \mathscr{D}(\mathcal{A}^{3/2}) \times \mathscr{D}(\mathcal{A}) \times R = H_0^{9/2}(\Gamma_s) \times H_0^3(\Gamma_s) \times R.$$

In the remaining part, the space X_f will be endowed with the norm

$$\|\xi\|_f^2 \equiv H(\xi, \xi),$$

where $H(\cdot, \cdot)$ is defined by (7.1.8). For any ξ, $\zeta \in X_f$, the associated inner product will

$$< \xi, \zeta >_f \equiv H(\xi, \zeta).$$

7.2 Strong Stability

In the last part, we will show that the system (7.1.7) is strongly stable in X_f. The proof relies on the following observation.

Lemma 7.2.1. *Considering the system* (7.1.7)*, if we denote B_f^* as the adjoint operator of B_f with respect to $< .,. >_f$, we have the following relationship*

$$B_f^* = -C, \quad \text{in } X_f,$$

where C is the observation operator defined in (7.1.6)*.*

Proof. From (6.3.8) and (7.1.9), we have

$$B_f^* \xi = < \xi, B_f >_f .$$

Since $< B_f, \phi_k >$ is real, we can also obtain that

$$\begin{aligned} < \xi, B_f >_f &= \sum_{k \in Z} d_k < \xi, \varphi_k >_X \overline{< B_f, \varphi_k >}_X \\ &= -\sum_{k \in Z} \frac{C\varphi_k}{< B_f, \varphi_k >} < \xi, \varphi_k >_X \overline{< B_f, \varphi_k >}_X \\ &= -\sum_{k \in Z} C\varphi_k < \xi, \varphi_k > \\ &= -C\xi. \end{aligned}$$

□

This property is called "collocation" of the sensor and the actuator (see [9]). There are many examples in the literature where such a property corresponds to a realizable actuator-sensor device. With this lemma, the system (7.1.7) is equivalent to

$$\begin{cases} \dot{\xi} = (A - B_f B_f^*)\xi, & t > 0, \\ \xi(0) = \xi_0, \end{cases} \tag{7.2.1}$$

with initial data ξ_0 in X_f. Finally, we will show that

Theorem 7.2.2. *The system* (7.2.1) *is strongly stable in* X_f*, i.e.,* $\lim_{t\to\infty} \|\xi(t)\|_f = 0$.

Proof. We will use the result in [3, 8] to show this Theorem. At beginning, note that, since $\|\cdot\|_f$ is a weighted norm of $\|\cdot\|_X$ with weight d_k, from proposition 5.2.1, we can see that

$$< A\xi, \xi >_f = < \xi, A^*\xi >_f = < \xi, -A\xi >_f,$$

which implies that $Re < A\xi, \xi >_f = 0$. Meanwhile, we have shown that A generates a strongly continuous semigroup in Proposition 5.2.1. On the other hand,

$$\begin{aligned} Re < (A - B_f B_f^*)\xi, \xi >_f &= Re < A\xi, \xi >_f - < B_f B_f^*\xi, \xi >_f \\ &= -|B_f^*\xi|^2 \leq 0, \end{aligned} \tag{7.2.2}$$

which means $A - B_f B_f^*$ is dissipative. Since B is bounded, using the following Theorem in Curtain's book [8],

Theorem 7.2.3. *Suppose that A is a closed, densely defined, linear operator on the Hilbert space Z such that*

$$\begin{aligned} Re < Az, z > &\leq 0 \quad \textit{for } z \in \mathscr{D}(A) \\ Re < A^*z, z > &\leq 0 \quad \textit{for } z \in \mathscr{D}(A^*). \end{aligned}$$

Suppose that B is bounded from U to Z, where U is a Hilbert space. Then $A - BB^*$ generates a contraction semigroup.

We can show that $A - B_f B_f^*$ generates a contraction semigroup too. Furthermore for any $\lambda > 0$ and $\xi \in \mathscr{D}(A)$, we have

$$\begin{aligned} \|(\lambda I - A)\xi\|_f^2 &= \lambda^2\|\xi\|^2 - \lambda[< \xi, A\xi >_f + < A\xi, \xi >_f] + \|A\xi\|_f^2 \\ &\geq \lambda^2\|\xi\|_f^2, \end{aligned}$$

which means

$$\|(\lambda I - A)^{-1}\|_f \leq \frac{1}{\lambda}.$$

Therefore the resolvent is bounded. Meanwhile through embedding Theorem, the injection of $\mathscr{D}(A_f) = H_0^{9/2}(\Gamma_s) \times H_0^3(\Gamma_s) \times R$ into $X_f = H_0^3(\Gamma_s) \times H_0^{3/2}(\Gamma_s) \times R$ can be easily shown to be compact. Thus the resolvent $(\lambda I - A)^{-1} : X_f \to \mathscr{D}(A_f)$ is compact. In Chapter 6.5, we have shown that the operator pair (A, B_f) is approximately controllable if $f > 0$. Then the result directly comes from the following Theorem 7.2.4:

Theorem 7.2.4. *Let A be the infinitesimal generator of a strongly continuous semigroup in X_f and A have a compact resolvent. Then the operator $A - B_f B_f^*$ generates a strongly stable semigroup provided that the pair (A, B_f) is weakly approximately controllable.*

□

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