

**A NUMERICAL SOLUTION TO THE BOUNDARY LAYER EQUATIONS  
FOR DISSOCIATED AIR**

by

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TABLE OF CONTENTS

	<u>Page</u>
I. INTRODUCTION .....	8
II. PROPERTIES OF AIR .....	11
A. Thermodynamic Properties .....	11
B. Transport Properties .....	18
III. GOVERNING EQUATIONS .....	23
IV. SOLUTION OF THE BOUNDARY LAYER EQUATIONS .....	27
V. NUMERICAL COMPUTATIONS .....	29
VI. DISCUSSION AND CONCLUSIONS .....	33
VII. ACKNOWLEDGEMENTS .....	35
VIII. BIBLIOGRAPHY .....	36
IX. VITA .....	37

LIST OF TABLES AND FIGURES

		<u>Page</u>
<u>Tables</u>		
TABLE 1	VALUES OF THE DIMENSIONLESS PRODUCT OF VISCOSITY AND DENSITY FOR CONCOMITANT VALUES OF DIMENSIONLESS VELOCITY .....	38
TABLE 2	CONVERSION FACTORS <sup>3</sup> .....	39

		<u>Page</u>
<u>Figures</u>		
FIGURE (1)	PLOT OF $\mu\rho$ VERSUS U .....	40
FIGURE (2)	COMPARISON OF THE COMPUTED SHEAR WITH THAT OBTAINED FROM VAN DRIEST FOR PRANDTL NUMBER OF 0.75 .....	41
FIGURE (3)	COMPARISON OF THE COMPUTED ENTHALPY WITH THAT OBTAINED FROM VAN DRIEST FOR PRANDTL NUMBER OF 0.75 .....	42

LIST OF SYMBOLS

$a$	speed of sound
$a_i, b_i$	stoichiometric coefficients for components $A_i$ and $B_i$
$A_i, B_i$	components of a chemical reaction
$C_p$	specific heat per mol at constant pressure
$C_v$	specific heat per mol at constant density
$D_{ij}$	binary diffusion coefficient for molecules of type $i$ into molecules of type $j$
$e$	base of natural logarithms
$E_i$	energy per mol of component $i$
$E_0$	energy per mol at zero absolute temperature <sup>0</sup>
$g$	shear function
$g_i$	degeneracy of the $i$ th state
$g_n$	degeneracy of the $n$ th electronic scale
$H_i$	enthalpy per mol of component $i$
$h$	specific enthalpy
$J$	rotational quantum number
$k$	Boltzmann constant, also thermal conductivity
$k_0$	reference coefficient of thermal conductivity
$k_r$	partial coefficient of thermal conductivity due to chemical reaction
$K_c$	chemical equilibrium constant for concentration units
$K_p$	chemical equilibrium constant for pressure units

$k_n$	partial coefficient of thermal conductivity due to molecular collisions
$\ln$	logarithm to the base e
$M_i$	weight per mol of component i
$\bar{M}$	mean weight per mol of a gas mixture
$M_o$	weight per mol for undissociated air molecules
n	vibrational quantum number, also electronic quantum number, also concentration in moles per unit volume
N	nitrogen atom, also atoms in general
$N_o$	Avagadro number, molecules per mol
$N_2$	nitrogen molecule
NO	nitric oxide
O	oxygen atom
$O_2$	oxygen molecule
P	pressure
Q	total partition function
$Q_t$	translational partition function
$Q_v$	vibrational partition function
$Q_r$	rotational partition function
$Q_e$	electronic partition function
$Q_c$	total partition function for a standard state of unit concentration
$Q_p$	total partition function for a standard state of unit pressure
R	universal gas constant

$S_0$	collision cross section per undissociated air molecule
$S_{ij}$	collision cross section for particle i with particle j
T	absolute temperature
$u_i$	mean molecular velocity for molecule type i
$u_0$	mean molecular velocity for undissociated air molecules
U	velocity of air
u	velocity component in x direction
v	velocity component in y direction
x	mol fraction
$x_i$	mol fraction of component i
Z	compressibility factor

#### Greek Letter Symbols

$\gamma$	ratio of specific heats
$\epsilon$	fraction of molecules which are dissociated
$\epsilon_i$	energy of the ith state
$\epsilon_n$	energy of the nth electronic state
$\eta$	coefficient of viscosity
$\eta_0$	reference coefficient of viscosity
$\mu$	coefficient of viscosity
$\lambda_i$	mean free path for molecule type i
$\lambda_0$	reference mean free path
$\lambda_t$	total thermal conductivity

$\nu$  vibrational frequency  
 $\rho$  density of air  
 $\rho_i$  density of molecule type  $i$   
 $\rho_0$  reference density  
 $\sigma$  collision diameter

Subscripts

$p$  partial derivative at constant pressure  
 $\rho$  partial derivative at constant density  
 $i, j$  indices referring to molecules type  $i$  and  $j$   
 $t, r, v, e$  indices referring to the contribution of translational, rotational, vibrational, and electronic energy modes respectively  
 $\infty$  free stream conditions  
 $w$  wall conditions  
 $1$  oxygen dissociation reaction  
 $2$  nitrogen dissociation reaction

## I. INTRODUCTION

The solution of the laminar boundary layer equations for the flow of air at high temperatures over a flat plate is complicated by the effects of dissociation. The magnitude of the heat transfer becomes dependent upon atomic diffusion in addition to the thermal conduction characteristic of low speed air flow. Since equilibrium dissociation significantly alters the properties of air, any systematic boundary layer calculations must consider the temperature variation of the Prandtl, Schmidt and Lewis numbers.

The first solution of the laminar boundary layer equations, for a compressible fluid with variable properties including dissociation, was obtained by Moore<sup>6</sup>. His computations were carried out on a differential analyzer for a limited range of wall temperatures and a constant value of free stream temperature. The Hantzsche-Wendt method of transforming the boundary layer equations was used in this method of analysis.

Moore found that for plate temperatures in the range of 400° to 2500° R the heat transfer to the plate was unaffected by dissociation. However, for higher plate temperatures the heat transmission was greatly affected by dissociation.

Hansen<sup>4</sup> subsequently demonstrated that the Prandtl number for dissociated air, employed by Moore, had been miscalculated. Romig and Dore<sup>7</sup> repeated Moore's calculations using a corrected Prandtl



number and a wider range of wall and stream conditions. These investigators found that if the temperature dependent properties, in the perfect gas equations, are based on stream enthalpy instead of stream temperature, the perfect gas equations for drag and heat transfer may be used with a maximum error of fifteen percent.

Unfortunately, the equilibrium air properties employed by Moore, and by Romig and Dore, were based upon an incorrect value for the dissociation energy of molecular nitrogen<sup>4</sup>. Both investigations were based on a Lewis number of unity.

Hassan<sup>5</sup> has presented a method of analyzing the quasi-equilibrium flow of an ideal dissociating gas over a flat plate. This solution is based upon a modification of Crocco's method which takes into consideration the variation of Prandtl, Schmidt and Lewis numbers. Hassan found that for an ideal dissociating gas the shear function is almost identical to that obtained in the classical Blasius solution; and, that the enthalpy distribution is practically independent of the law governing the variation of  $\mu\rho$  with enthalpy.

In order to solve the boundary layer equations, it is necessary that the thermodynamic and transport properties of air be known as functions of temperature. This requirement has been satisfied through the utilization of the closed form expressions evolved by Hansen<sup>3</sup> using approximate partition functions.

The contribution of this thesis will be to present a numerical solution for the boundary layer flow of air past a flat plate using

the thermodynamic and transport properties of air as evaluated by Hansen. The solution will be subjected only to the assumption of thermodynamic equilibrium. The unrestricted variation of Prandtl, Schmidt and Lewis numbers with temperature is contemplated.

This objective will be attained by: (a) developing a program for the IBM 1620 high speed computer, capable of yielding the relevant air properties, for any prescribed temperature, using Hansen's closed form approximations; (b) utilize the evaluated properties of air to numerically solve the transformed boundary layer equations, evolved by Hassan, to include dissociation; and (c) compare the results obtained here with previous solutions.

All calculations will be accomplished through the use of the IBM 1620 high speed computer.

## II. PROPERTIES OF AIR

### A. Thermodynamic Properties<sup>3</sup>

The thermodynamic properties of air are calculated from the partition function under the condition of complete thermal equilibrium, that is

$$Q = \sum_i g_i e^{-\epsilon_i/kT} \quad (1.1)$$

where  $\epsilon_i$  is the energy of the  $i$ th state of the gas particle and  $g_i$  is the degeneracy.

The partition function may also be expressed as:

$$Q = Q_t Q_r Q_v Q_e \quad (1.2)$$

where:  $Q_t$  is the translational partition function,  
 $Q_r$  is the rotational partition function,  
 $Q_v$  is the vibrational partition function, and  
 $Q_e$  is the electronic partition function.

For diatomic molecules, the above products may be expressed in the following forms:

$$Q_t = \left[ \frac{2\pi m kT}{h^2} \right]^{3/2} \frac{RT}{P} \quad (1.3)$$

$$Q_r = \sum_{J=0}^{\infty} (2J+1) e^{-\frac{h^2 J(J+1)}{8\pi^2 I kT}} \quad (1.4)$$

$$Q_v = \sum_{n=0}^{\infty} e^{\frac{-n h\nu}{kT}} = \left[ 1 - e^{\frac{-h\nu}{kT}} \right]^{-1} \quad (1.5)$$

and

$$Q_e = \sum_{n=0}^{\infty} g_n e^{\frac{-\epsilon_n}{kT}} \quad (1.6)$$

It is assumed here that the rotational and vibrational constants, for all excited electronic states, are identical to those of the ground state.

The logarithms of the partition functions are used to calculate the thermal properties directly; thus

$$\ln Q(N_2) = (7/2) \ln T - 0.42 - \ln (1 - e^{-3390/T}) - \ln p \quad (1.7)$$

$$\begin{aligned} \ln Q(O_2) = (7/2) \ln T + 0.11 - \ln (1 - e^{-2270/T}) \\ + \ln (3 + 2e^{-11390/T} + e^{-18990/T}) - \ln p \end{aligned} \quad (1.8)$$

$$\begin{aligned} \ln Q(O) = (5/2) \ln T + 0.50 + \ln (5 + 3e^{-228/T} + e^{-326/T} \\ + 5e^{-22800/T} + e^{-48600/T}) - \ln p \end{aligned} \quad (1.9)$$

and

$$\begin{aligned} \ln Q(N) = (5/2) \ln T + 0.30 + \ln (4 + 10 e^{-27700/T} \\ + 6 e^{-41500/T}) - \ln p \end{aligned} \quad (1.10)$$

where  $T$  = temperatures in degrees Kelvin

and  $p$  = pressure in atmospheres.

Air is assumed, here, to be a mixture of oxygen and nitrogen with ratio of nitrogen to oxygen being four to one. It has been found that neglecting the formation of nitric oxide does not significantly influence the properties of air (see Hansen, ref. 3).

The energy and enthalpy per mol for the components of air, are given by

$$\frac{(E - E_o)}{RT} = T \left[ \frac{\partial}{\partial T} (\ln Q) \right]_p = T \frac{d}{dT} (\ln Q_c) \quad (1.11)$$

$$\frac{H - E_o}{RT} = T \left( \frac{\partial (\ln Q)}{\partial T} \right)_p = T \frac{d(\ln Q_p)}{dT}$$

where  $Q_c = \left( \frac{p}{RT} \right) Q$

and  $Q_p = pQ$

The quantity  $E_o$  represents the energy of the gas at absolute zero temperature, and for diatomic molecules this quantity is presumed to be zero.

The equilibrium mol fractions for the components of air are dependent upon the evaluation of the equilibrium constants for the chemical reactions, thus

$$\sum a_i A_i = \sum b_i B_i .$$

The pressure equilibrium constant, defined in terms of partial pressures, can be expressed as:

$$K_p = \frac{\prod P_{B_i}^{b_i}}{\prod P_{A_i}^{a_i}} \quad (1.13)$$

Now, the equilibrium constants for the dissociation of molecular oxygen, and of molecular nitrogen, are

$$\ln K_p (O_2 \rightarrow 2O) = -\frac{59,000}{T} + 2 \ln Q_p(O) - \ln Q_p(O_2) \quad (1.14)$$

and

$$\ln K_p (N_2 \rightarrow 2N) = -\frac{113,200}{T} + 2 \ln Q_p(N) - \ln Q_p(N_2) \quad (1.15)$$

The concentration equilibrium constant,  $K_c$ , is defined as

$$K_c = \frac{\prod n_{B_i}^{b_i}}{\prod n_{A_i}^{a_i}} \quad (1.16)$$

From the relation  $Q_c = \frac{P}{RT} Q$ , the above equation may be rewritten as

$$K_c = K_p (RT)^{\sum a_i - \sum b_i} \quad (1.17)$$

Taking the logarithm of Eq. (1.17), and then the derivative with respect to temperature (T) yields

$$\frac{Td(\ln K_c)}{dT} = \frac{Td(\ln K_p)}{dT} + (\sum a_i - \sum b_i) \quad (1.18)$$

In the calculation of the equilibrium mol fractions, and their derivatives, two important simplifications can be made. The first, based upon the results of Gilmore<sup>1</sup>, ignores all the component concentrations which are of the order of 0.1 percent or less. The concentration of nitric oxide exceeds this limitation, however Hansen<sup>3</sup> has shown that the nitric oxide does not strongly influence the resulting thermodynamic properties of air. Therefore, the neglecting of this chemical product does not introduce any significant error into the results.

The second simplification, also based upon Gilmore's results, establishes a priority to the process of dissociation. For all practical purposes, the dissociation of oxygen is completed before the dissociation of nitrogen begins. This simplification permits consideration of each of the chemical reactions as independent processes.

The equation of state will be written as

$$\frac{p}{\rho} = \frac{ZRT}{M_0} \quad , \quad (1.19)$$

where  $Z$  is the compressibility factor, and may be expressed as  $M_0/\bar{M}$ . A more usable expression for the compressibility factor is given by

$$Z = 1 + \epsilon_1 + \epsilon_2 \quad (1.20)$$

where  $\epsilon_1$  is the fraction of oxygen molecules which dissociate into oxygen atoms, and  $\epsilon_2$  is the fraction of nitrogen molecules which dissociate into nitrogen atoms.

For values of  $Z$  in the range  $1.0 \leq Z \leq 1.2$ , the mole fractions may be expressed as

$$x(O_2) = \frac{0.2 - \epsilon_1}{1 + \epsilon_1} \quad (1.21)$$

$$x(N_2) = \frac{0.8}{1 + \epsilon_1} \quad (1.22)$$

and

$$x(O) = \frac{2\epsilon_1}{1 + \epsilon_1} \quad (1.23)$$

where

$$\epsilon_1 = \frac{-0.8 + \sqrt{0.64 + 0.8(1 + 4p/K_{p_1})}}{2(1 + 4p/K_{p_1})}$$

and

$$K_{p_1} = \frac{p_o^2}{p_{o_2}}$$

For values of  $\epsilon$  in the range  $1.2 \leq \epsilon \leq 2.0$ , the mole fractions are conveniently given by

$$x(N_2) = \frac{0.8 - \epsilon_2}{1.2 + \epsilon_2} \quad (1.24)$$

$$x(N) = \frac{2\epsilon_2}{1.2 + \epsilon_2} \quad (1.25)$$

$$x(O) = \frac{0.4}{1.2 + \epsilon_2} \quad (1.26)$$



where

$$\epsilon_2 = \frac{-0.4 + \sqrt{0.16 + 3.84(1 + 4p/K_{P_2})}}{2(1 + 4p/K_{P_2})}$$

and

$$K_{P_2} = \frac{P^2 N}{P_{N_2}}$$

The energy, or enthalpy per mol, can be expressed as the weighted sum of component energies or enthalpies. That is

$$E = \sum_i x_i E_i \quad (1.27)$$

or, in dimensionless form,

$$\frac{ZE}{RT} = Z \sum_i x_i \frac{E_i}{RT} \quad (1.28)$$

The dimensionless enghalpy per mol can then be expressed, in terms of the compressibility factor, as

$$\frac{ZH}{RT} = \frac{ZE}{RT} + Z \quad (1.29)$$

The first derivatives of the energy and enthalpy are then given, respectively, by

$$\frac{1}{R} \left[ \frac{\partial(ZE)}{\partial T} \right]_p = \frac{ZC_v}{R} = Z \sum_i x_i \left( \frac{\partial E_i}{R \partial T} \right) + T \sum_i \left( \frac{E_i}{RT} \right) \left[ \frac{\partial(Zx_i)}{\partial T} \right]_p \quad (1.30)$$

and

$$\frac{1}{R} \left( \frac{\partial ZH}{\partial T} \right)_p = \frac{ZC_p}{R} = Z \sum_i x_i \left( \frac{\partial E_i}{R\partial T} + 1 \right) + T \sum_i \left( \frac{E_i}{RT} + 1 \right) \left( \frac{\partial Zx_i}{\partial T} \right)_p . \quad (1.31)$$

By definition the square of the speed of sound is

$$a^2 = \gamma \left( \frac{\partial p}{\partial \rho} \right)_T \quad (1.32)$$

or

$$a^2 = \frac{-\gamma (\partial p / \partial T)_\rho}{(\partial \rho / \partial T)_p} \quad (1.33)$$

which may also be expressed, non-dimensionally, as

$$\frac{a^2 \rho}{p} = \frac{\gamma \left( 1 + (T/Z) \left( \frac{\partial Z}{\partial T} \right)_\rho \right)}{1 + (T/Z) \left( \frac{\partial Z}{\partial T} \right)_p} . \quad (1.34)$$

### B. Transport Properties<sup>3</sup>.

The expression for the viscosity of air, assuming a mixture of hard spherical molecules, is

$$\eta = \frac{5\pi}{32} \sum_i \rho_i u_i \lambda_i \quad (1.35)$$

The viscosity variation with temperature, for undissociated air, utilized as a reference coefficient, can be expressed from Eq. (1.35)

as

$$\eta_o = \frac{5\pi}{32} \rho_o u_o \lambda_o \quad (1.36)$$

Now, the ratio of mean molecular velocities is

$$\frac{u_i}{u_o} = \sqrt{\frac{M_o}{M_i}} \quad (1.37)$$

where  $M_o$  is the weight per mol for undissociated air molecules and  $u_o$  is the mean molecular speed for undissociated air molecules.

In a similar fashion the ratio of densities can be expressed as

$$\frac{\rho_i}{\rho_o} = \frac{M_i}{M_o} x_i \quad (1.38)$$

and the ratio of molecular mean free path lengths is

$$\frac{\lambda_o}{\lambda_i} = \sum_j x_j \frac{s_{ij}}{s_o} \left( \frac{1 + \frac{M_i}{M_j}}{2} \right)^{1/2} \quad (1.39)$$

Using the above relations, the ratio of viscosities can now be expressed as

$$\frac{\eta}{\eta_o} = \sum_i \frac{\rho_i u_i \lambda_i}{\rho_o u_o \lambda_o} = \sum_i \sqrt{\frac{M_i}{M_o}} x_i \frac{\lambda_i}{\lambda_o} \quad (1.40)$$

The coefficient of total thermal conductivity has two components. The first relates to the energy transferred by molecular collisions so that if it is assumed that the internal energy is

distributed among gas particles; independent of their velocity distribution, then the coefficient of thermal conductivity for a gaseous mixture can be expressed as

$$k_n = \frac{5\pi}{32} \sum_i \rho_i u_i \frac{\lambda_i}{M_i} \left( \frac{\partial E}{\partial T} + 9/4 R \right) \quad (1.41)$$

The reference coefficient of thermal conductivity represents the value commensurate with the absence of the vibration, dissociation or ionization degrees of freedom and hence

$$k_o = 19/4 \frac{R}{M_o} \eta_o \quad (1.42)$$

Then, the ratio of the thermal conductivity coefficient to the reference coefficient becomes

$$\frac{k_n}{k_o} = \sum_i \left( \sqrt{\frac{M_i}{M_o}} x_i \frac{\lambda_i}{\lambda_o} \right) \frac{M_o}{M_i} \left[ \frac{4}{19} \left( \frac{\partial E}{R \partial T} \right)_i + \frac{9}{19} \right] \quad (1.43)$$

The second component of the total thermal conductivity relates to the energy transfer which takes place because of the diffusion of the chemical species. This can be given by (see ref. 3)

$$k_r = \frac{R \left( T \frac{d \ln K_p}{dT} \right)^2}{\sum_i \sum_j \frac{8}{3} \frac{a_i}{x_i} \sqrt{\frac{2}{\pi}} \left( \frac{M_i M_j}{M_i + M_j} \right)^{1/2} \frac{N_o}{\sqrt{RT}} S'_{ij} (a_i x_j - a_j x_i)} \quad (1.44)$$

Now, the reference coefficient of thermal conductivity, cited earlier, becomes

$$k_o = \frac{95}{64} \sqrt{\pi} \frac{R}{N_o S_o} \sqrt{\frac{RT}{M_o}} \quad (1.45)$$

Therefore, the ratio of the coefficient of thermal conductivity (due to the chemical reaction) to the reference value, is

$$\frac{k_r}{k_o} = \frac{\left[ (12 \sqrt{2})/95 \right] \left( \frac{Td \ln K_p}{dT} \right)^2}{\sum_i \sum_j \left[ \frac{M_i M_j}{M_o (M_i + M_j)} \right]^{1/2} \frac{S'_{ij}}{S_o} \frac{a_i}{x_i} (a_i x_j - a_j x_i)} \quad (1.46)$$

By neglecting the difference between the mass of oxygen and nitrogen atoms, the double summations in the denominator take on the following simplified forms:

(1) For the oxygen dissociation reaction

$$\sum_i \sum_j = \frac{S'(N_2 - N)}{\sqrt{3} S_o} \left\{ \frac{[x(O) + 2x(O_2)]^2}{x(O) x(O_2)} + \frac{4x(N_2)}{x(O)} \right\} + \frac{x(N_2)}{\sqrt{2x(O_2)}} \quad (1.47)$$

(2) For the nitrogen dissociation reaction

$$\sum_i \sum_j = \frac{S'(N_2 - N)}{\sqrt{3} S_o} \left\{ \frac{[x(N) + 2x(N_2)]^2}{x(N) x(N_2)} + \frac{x(O)}{x(N_2)} \right\} + \frac{S'(N - N)}{S_o} \cdot \frac{2x(O)}{2x(N)} \quad (1.48)$$

Expressing the total coefficient of thermal conductivity as

$$\frac{k}{k_o} = \frac{k_n}{k_o} + \frac{k_r}{k_o} \quad (1.49)$$

then the Prandtl number may be calculated from

$$\text{Pr} = (4/19) \left( \frac{ZC_p}{R} \right) \left( \frac{\eta/\eta_o}{k/k_o} \right) \quad (1.50)$$

where  $\eta/\eta_o$  and  $k/k_o$  are obtained from Eqs. (1.40) and (1.49) respectively.

### III. GOVERNING EQUATIONS

The boundary layer equations for laminar flow over a flat plate may be expressed in the following manner (see ref. 5)

$$\frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial y} (\rho v) = 0 \quad (2.1)$$

$$\rho \left[ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right] = \frac{\partial}{\partial y} \left( \mu \frac{\partial u}{\partial y} \right) \quad (2.2)$$

$$\rho \left[ u \frac{\partial h}{\partial x} + v \frac{\partial h}{\partial y} \right] = \frac{\partial}{\partial y} \left( \lambda_t \frac{dT}{dy} \right) + \mu \left( \frac{\partial u}{\partial y} \right)^2 \quad (2.3)$$

where  $h$  is the specific enthalpy

and  $\lambda_t$  is the total thermal conductivity equal to  $\frac{k}{k_0}$  times a conversion factor (see Table 2).

The velocity of the flow in the boundary layer is chosen as a weighted mean wherein each of the molecules and atoms is given a weight value proportional to its mass. The other (physical) properties of the (air) flow have been defined in the preceding section.

Under the assumption of a constant pressure field, the flow properties become a function of temperature alone. The enthalpy equation, Eq. 1.29, has been defined as

$$h = \sum_i x_i h_i = f(T) \quad (2.4)$$

consequently, the energy equation (2.3) may be rewritten as

$$\rho \left[ u \frac{\partial h}{\partial x} + v \frac{\partial h}{\partial y} \right] = \frac{\partial}{\partial y} \left[ \frac{\lambda_t}{dh/dT} \frac{\partial h}{\partial y} \right] + \mu \left( \frac{\partial h}{\partial y} \right)^2 \quad (2.5)$$

Using Crocco's transformation, which is based on transforming the (x,y) plane to the (x,u) plane, Eqs. 2.1, 2.2, and 2.5, are rewritten in the following forms:

$$u \frac{\partial}{\partial x} \left( \frac{\mu \rho}{\tau} \right) + \frac{\partial^2 \tau}{\partial u^2} = 0 \quad (2.6)$$

$$\frac{\partial}{\partial u} \left( \beta \tau \frac{\partial h}{\partial u} \right) - \left( \frac{\partial \tau}{\partial u} \right) \frac{\partial h}{\partial u} - \left( \frac{\mu \rho u}{\tau} \right) \frac{\partial h}{\partial x} + \tau = 0 \quad (2.7)$$

where

$$\tau = \mu (\partial u / \partial y) \quad (2.7a)$$

and

$$\beta = \frac{\lambda_t}{\mu \, dh/dT} \quad (2.7b)$$

The boundary conditions for the present case are:

$$\tau = \infty \text{ at } x = 0, \text{ for } 0 < u < u_\infty$$

$$\frac{\partial \tau}{\partial u} = 0 \text{ and } h = h_w \text{ at } x > 0, u = 0 \quad (2.8)$$

and

$$\tau = 0 \text{ and } h = h_\infty \text{ at } x > 0, u = u_\infty$$

Now, letting

$$\tau = f(u) \sqrt{2x}, \quad h = h(u) \quad (2.9)$$

Eqs. 2.6 and 2.7 become

$$ff'' + \mu \rho u = 0 \quad (2.10)$$

$$(\beta fh')' - f'h' + f = 0 \quad (2.11)$$



respectively, subject to the boundary conditions:

$$\begin{aligned} f' &= 0, & h &= h_w \text{ at } u = 0 \\ f &= 0, & h &= h_\infty \text{ at } u = u_\infty \end{aligned} \quad (2.12)$$

Equations 2.10 and 2.11 may be expressed in dimensionless form by letting

$$g = \sqrt{\frac{2}{\rho_\infty \mu_\infty u_\infty^3}} f$$

$$\bar{h} = h/h_\infty, \quad \bar{u} = u/u_\infty, \quad \bar{\rho} = \rho/\rho_\infty$$

$$\bar{\mu} = \mu/\mu_\infty \quad \text{and} \quad \bar{\lambda}_t = \lambda t/\lambda t_\infty$$

The resulting dimensionless equations, obtained from Eqs. 2.10 and 2.11 are: (dropping the overscoring bars for convenience)

$$gg'' + 2 \mu \rho u = 0 \quad (2.13)$$

and

$$\left[ \beta f h' \right]' - \sigma f' h' + \sigma \frac{u_\infty^2}{h_\infty} f = 0 \quad (2.14)$$

respectively, where

$$\sigma = \frac{\mu_\infty h_\infty}{T_\infty \lambda_{t_\infty}}$$

subject to the dimensionless boundary conditions:

$$\begin{aligned} g' &= 0, & h &= h_w/h_\infty \text{ at } u = 0 \\ f &= 0, & h &= 1, \quad \text{at } u = 1 \end{aligned}$$

Equations 2.1, 2.2 and 2.5 have been transformed into two simultaneous differential equations. The solution will be obtained

by means of a simultaneous numerical integration of Eqs. 2.13 and 2.14. The physical properties of air, including the effects of dissociation, will be contained in the terms  $\mu\rho$ ,  $\beta$ ,  $\sigma$  and  $u_{\infty}^2/h_{\infty}$ .

The boundary condition,  $g = 0$  at  $u = 1$ , indicates that a singularity exists at the upper limit of integration. It then becomes necessary to develop an additional expression for the shear function, and for the enthalpy, which is valid in the region of the singularity.

IV. SOLUTION OF THE BOUNDARY LAYER EQUATIONS

Equations 2.13 and 2.14 are integrated to yield the following results:

$$g(u) = \int_u^1 du_1 \int_0^{u_1} \frac{2\mu\rho u_2}{g(u)} du_2 \quad (3.1)$$

$$h(u) = \frac{h_w}{h_\infty} + \beta(0) h'(0) \int_0^u \frac{du_1}{\beta(u) \left[ \frac{g(u)}{g(0)} \right]} +$$

$$\sigma \int_0^u \frac{du_1}{\beta(u) \left[ \frac{g(u)}{g(0)} \right]} \int_0^{u_1} \left[ \frac{dh}{du_2} \frac{d}{du_2} \left[ \frac{g(u)}{g(0)} \right] - \frac{u_2^2}{h_\infty} \left[ \frac{g(u)}{g(0)} \right] \right] du_2 \quad (3.2)$$

wherein  $u_1$  and  $u_2$  are dummy variables.

Because of the singularity, due to the boundary condition  $g = 0$  at  $u = 1$ , Eqs. 3.1 and 3.2 are valid only in the range  $0 \leq u \leq 1 - \delta$ , where  $\delta$  is an arbitrarily chosen small number. Thus, in terms of  $\delta$ , an approximate relation analogous to Eq. 3.1, is

$$g(u) = g(1 - \delta) + \int_u^{1-\delta} du_1 \int_0^{u_1} \frac{2\mu\rho u_2}{g(u)} du_2 \quad (3.3)$$

which is valid only over the range

$$0 \leq u \leq 1 - \delta$$

Noting that as  $u \rightarrow 1$ ,  $\mu\rho \rightarrow 1$ , then Eq. 2.13 approaches

$$gg'' + 2 = 0 \quad , \quad (3.4)$$

which can be integrated, in closed form, to yield

$$(1 - u)/g(u) = \frac{\sqrt{\pi}}{2} e^{(g'/2)^2} \left[ 1 - \operatorname{erf} \left( -g'/2 \right) \right] \quad (3.5)$$

valid in the range

$$1 - \delta \leq u \leq 1$$

Assuming  $\beta$  to be constant in the interval  $1 - \delta \leq u \leq 1$ , then

Eq. 2.14 can be integrated to give

$$h = h(1 - \delta) + h'(1 - \delta) \left[ \frac{g(1 - \delta)}{g(0)} \right]^{1 - \operatorname{Pr}} \int_{1 - \delta}^u \left[ \frac{g(u)}{g(0)} \right]^{\operatorname{Pr} - 1} du_1 \quad (3.6)$$

$$- \frac{u^2}{h_\infty} \operatorname{Pr} \int_{1 - \delta}^u \left[ \frac{g(u)}{g(0)} \right]^{\operatorname{Pr} - 1} du_1 \int_{1 - \delta}^{u_1} \left[ \frac{g(u)}{g(0)} \right]^{1 - \operatorname{Pr}} du_2$$

which is valid in the range,  $1 - \delta \leq u \leq 1$ , and where  $\operatorname{Pr} = \sigma/\beta$ .

Crocco found that Eq. 3.5 can be approximated by

$$g(1 - \delta) = \delta \left[ 1 + |g'(1 - \delta)| \left( 0.7328 + 0.0178 |g'(1 - \delta)| \right) \right]^5 \quad (3.7)$$

which is valid for  $g'(1 - \delta)$  between - 2.2 and - 4.0. Equation 3.7 will be used in the computation of  $g(1 - \delta)$ , with  $\delta$  arbitrarily chosen to be 0.01.

## V. NUMERICAL COMPUTATIONS

In order to solve the boundary layer equations, the following values are arbitrarily described:

- (1) the free stream enthalpy
- (2) the wall to free stream enthalpy ratio
- (3) the free stream pressure
- (4) the free stream Mach number .

The enthalpy ratio serves to fix the value of the wall enthalpy. Since the relation between the temperature and the enthalpy is known, then the free stream and wall temperatures can be determined from the concomitant enthalpies.

Given any specified Mach number, the free stream velocity can be evaluated from the relationship:

$$M_{\infty} = \frac{u_{\infty}}{a_{\infty}}$$

where  $a_{\infty}$ , the free stream speed of sound, can be computed as a function of temperature from Eq. 1.34.

The computational procedure is initiated by calculating the boundary layer enthalpy distribution, following Van Driest<sup>8</sup>. The equations used for the calculation of the enthalpy distribution were:

$$h(u) = \frac{h_w}{h_{\infty}} - \left( \frac{h_w}{h_{\infty}} - 1 \right) \theta^I(u) + \frac{u_{\infty}^2}{h_{\infty}} \theta^{II}(u)$$

where

$$\theta^{\text{I}}(u) = \frac{I(u)}{I(1)}$$

and

$$I = \int_0^u \left[ \frac{g}{g(0)} \right]^{\text{Pr}-1} du$$

also

$$\theta^{\text{II}}(u) = \text{Pr} \left[ \theta^{\text{I}}(u) J(1) - J(u) \right]$$

where

$$J = \int_0^u \left[ \frac{g}{g(0)} \right]^{\text{Pr}-1} du_1 \int_0^{u_1} \left[ \frac{g}{g(0)} \right]^{1-\text{Pr}} du_2$$

The quantities  $\theta^{\text{I}}$  and  $\theta^{\text{II}}$  have been evaluated by Van Driest<sup>8</sup>, for various Prandtl numbers, and are available in tabulated form. In the present case, a fixed Prandtl number of 0.75 was used in the initial (or starting) calculations.

The resultant enthalpy distribution was then used to compute an initial boundary layer temperature distribution. This was accomplished by adding successive incremental values of temperature to  $T_w$  and substituting these into Eq. 1.29 until the relationship was satisfied for each value of the enthalpy within the interval  $u = 0.0$  to  $u = 1.0$ .

The variation of the product  $\mu\rho$ , viscosity times density, with the established temperature distribution was calculated using Eq. 1.38 and 1.40, and this in turn was used to solve the shear equation, Eq. 3.3. The Blasius shear distribution was used as an initial estimate, or starting point, in this latter computation.

The function  $\beta$ , (Eq. 2.7b), was then evaluated for the previously determined temperature distribution, and the first approximations

for  $g(u)$  and  $\beta(u)$  were used to resolve for the enthalpy, Eqs. 3.2 and 3.6.

The solution of Eq. 3.2 requires a reasonable approximation for the change of enthalpy with the boundary layer speed  $u$ . This approximation was provided by the following expression from Van Driest<sup>8</sup>:

$$\frac{dh}{du} = -Pr \frac{u_{\infty}^2}{h_{\infty}} \left[ \frac{g}{g(0)} \right]^{Pr-1} \int_0^u \left[ \frac{g}{g(0)} \right]^{1-Pr} du + h'(0) \left[ \frac{g}{g(0)} \right]^{Pr-1}$$

where

$$h'(0) = \left[ 1 - h(0) + Pr \frac{u_{\infty}^2}{h_{\infty}} J(1) \right] \frac{1}{I(1)}$$

Here, also, a fixed Prandtl number value of 0.75 was utilized.

The term  $h'(0)$  was evaluated from Eq. 3.6, employing the boundary condition that  $h = 1$  at  $u = 1$ .

The adjusted, or corrected, enthalpy distribution, derived from the solution of Eq. 3.2 gave rise to a new temperature distribution, which commenced the calculation procedure over again. The subsequent approximation to  $g'(u)$  would yield a new enthalpy distribution,  $h(u)$ , which could in turn be used to compute a new shear distribution. The technique of successive approximations was continued until the solution to the shear and enthalpy equations converged.

In order to reduce the oscillations in the solution, about the exact value of the shear and enthalpy distributions, a shear function,  $g_n$ , was selected as  $\sqrt{g_{n-2} g_{n-1}}$ , and an enthalpy,  $h_n$ , was chosen as

(  $\frac{h_{n-1} + h_{n-2}}{2}$  ). Here the subscripts designate the number of iterations performed; e.g., the subscript,  $n - 1$ , denotes the value of  $h$  and  $g$  obtained in the immediately preceding iteration.

The results obtained in this thesis are presented in Figures (1) to (3) and in Table 1. Figure (1) is a plot of  $\mu\rho$  versus  $u$ . The inflections in this curve are too small to be apparent in the graph; consequently, a tabulation of values of  $\mu\rho$  is presented in Table 1. Figures (2) and (3) compare the calculated shear and enthalpy functions with those obtained from Van Driest<sup>8</sup>. All solutions shown on Figures (1) to (3) were computed for a free stream Mach number of 4.



## VI. DISCUSSION AND CONCLUSIONS

The results of the present investigation show that the shear function, for dissociating air, is very similar to that computed by Van Driest<sup>8</sup> for non-dissociating air. The greatest difference occurs at the wall, where the computed value is 7.1 percent higher than Van Driest's results.

The general shape of the curves, portrayed in Figure (3), exhibit marked dissimilarities. The computed enthalpy distribution attains a maximum value at  $u = 0.975$  whereas the enthalpy distribution obtained from Van Driest attains its maximum value at  $u = 0.85$ . The greatest difference between the two curves is found to occur at  $u = 0.7$  where the Van Driest value is 10.6 percent higher than the corresponding value calculated by the method described here.

It should be noted that for the computed values the Prandtl number oscillates vigorously between the maximum and minimum values of 0.7497 and 0.6074. The ratio of specific heats varies from a maximum value of 1.296 to a minimum value of 1.195, for the range of temperatures under consideration. These variations in flow parameters should be contrasted with Van Driest's assumed constant values of Prandtl number = 0.75 and  $\gamma = 1.40$ .

A study of the present results indicates that the enthalpy distribution is virtually independent of the variation of  $\mu\rho$  with enthalpy. As Crocco<sup>8</sup> pointed out, the enthalpy is dependent upon the ratio  $(\frac{g(u)}{g(0)})$  and variations in viscosity and density would affect  $g(u)$  in the same fashion as they affect  $g(0)$ .

Table 1 shows that for values of  $u$  0.3 the value of the product  $\frac{\mu}{\mu_{\infty}} \cdot \frac{\rho}{\rho_{\infty}}$  is very close to unity. For  $\mu\rho/\mu_{\infty}\rho_{\infty} = 1$  the shear equation is identical to that obtained from the classical Blasius solution. This explains the general resemblance of the shear curve to that obtained from the Blasius solution.

VII. ACKNOWLEDGEMENTS

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VIII. BIBLIOGRAPHY

1. Gilmore, F. R., EQUILIBRIUM COMPOSITION AND THERMODYNAMIC PROPERTIES OF AIR TO 24,000°K, Rand Rep. RM-1543, 1955.
2. HANDBOOK OF SUPERSONIC AERODYNAMICS, Navord Rept. 1488 (Vol. 5), U.S. Govt. Printing Office, August 1953.
3. Hansen, F. C., APPROXIMATIONS FOR THE THERMODYNAMIC AND TRANSPORT PROPERTIES OF HIGH TEMPERATURE AIR, NASA TR R-5, 1959.
4. Hayes, W. D. and Probstein, R. F., HYPERSONIC FLOW THEORY, pp. 284-295, Academic Press, New York, 1959.
5. Hassan, H. A., ON THE LAMINAR FLOW OF AN IDEAL DISSOCIATING GAS, Developments in Mechanics, pp. 409-416, Vol. I, Plenum Press, New York, 1961.
6. Moore, L. L., A SOLUTION OF THE LAMINAR BOUNDARY LAYER EQUATIONS FOR COMPRESSIBLE FLUID WITH VARIABLE PROPERTIES, INCLUDING DISSOCIATION, J. Aero. Sci., Vol. 19, No. 8, pp. 505-518, August 1952.
7. Romig, M. F., and Dore, F. J., SOLUTIONS OF THE COMPRESSIBLE LAMINAR BOUNDARY LAYER INCLUDING THE CASE OF DISSOCIATED FREE STREAM, Convair Rept., No. ZA-7-012, San Diego, Calif., 1954.
8. Van Driest, E. R., INVESTIGATION OF THE LAMINAR BOUNDARY LAYER IN COMPRESSIBLE FLUIDS USING THE CROCCO METHOD, NACA TN 2597, Jan. 1952.

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u	T °K	$\frac{\mu \rho}{\mu_{\infty} \rho_{\infty}}$
0.0	2236	1.71993550
0.1	3094	1.43474500
0.2	3437	1.31022000
0.3	3733	1.21004120
0.4	4018	1.15565380
0.5	4419	1.08755230
0.6	4800	1.04076550
0.7	4890	1.01484610
0.8	4953	1.00639550
0.9	5053	1.01090630
0.925	5065	1.00923630
0.95	5075	1.00784010
0.96	5077	1.00756030
0.97	5078	1.00742050
0.98	5077	1.00756030
0.99	5064	1.00937580
1.00	5000	1.00000000

TABLE 1

VALUES OF THE DIMENSIONLESS PRODUCT OF VISCOSITY AND DENSITY  
FOR CONCOMITANT VALUES OF DIMENSIONLESS VELOCITY

Dimensionless Quantity	Multiply by	Result	Units
$\frac{ZH}{RT}$ or $\frac{ZE}{RT}$	0.0686 T ( $^{\circ}$ K) 1716 T ( $^{\circ}$ R)	Specific enthalpy or energy	cal/gm ft-lb/slug
$\frac{Z C_p}{R}$ $\frac{Z C_v}{R}$	0.0686 1716	Specific heat	cal/gm $^{\circ}$ K ft-lb/slug $^{\circ}$ R
$\frac{\eta}{\eta_0}$	$1.462 \times 10^{-5} T^{1/2} (1 + 112/T)^{-1}$ (T in $^{\circ}$ K) $2.28 \times 10^{-8} T^{1/2} (1 + 202/T)^{-1}$ (T in $^{\circ}$ R)	Coefficient of Viscosity $\mu$	gm/cm-sec lb sec/ft <sup>2</sup>
$\frac{k}{k_0}$	$4.76 \times 10^{-6} T^{1/2} (1 + 112/T)^{-1}$ (T in $^{\circ}$ K) $1.856 \times 10^{-4} T^{1/2} (1 + 202/T)^{-1}$ (T in $^{\circ}$ R)	Coefficient of Thermal Conductivity $\lambda_t$	$\frac{\text{cal}}{\text{cm-sec } ^{\circ}\text{K}}$ $\frac{\text{ft-lb}}{\text{ft-sec } ^{\circ}\text{R}}$

TABLE 2

CONVERSION FACTORS<sup>3</sup>

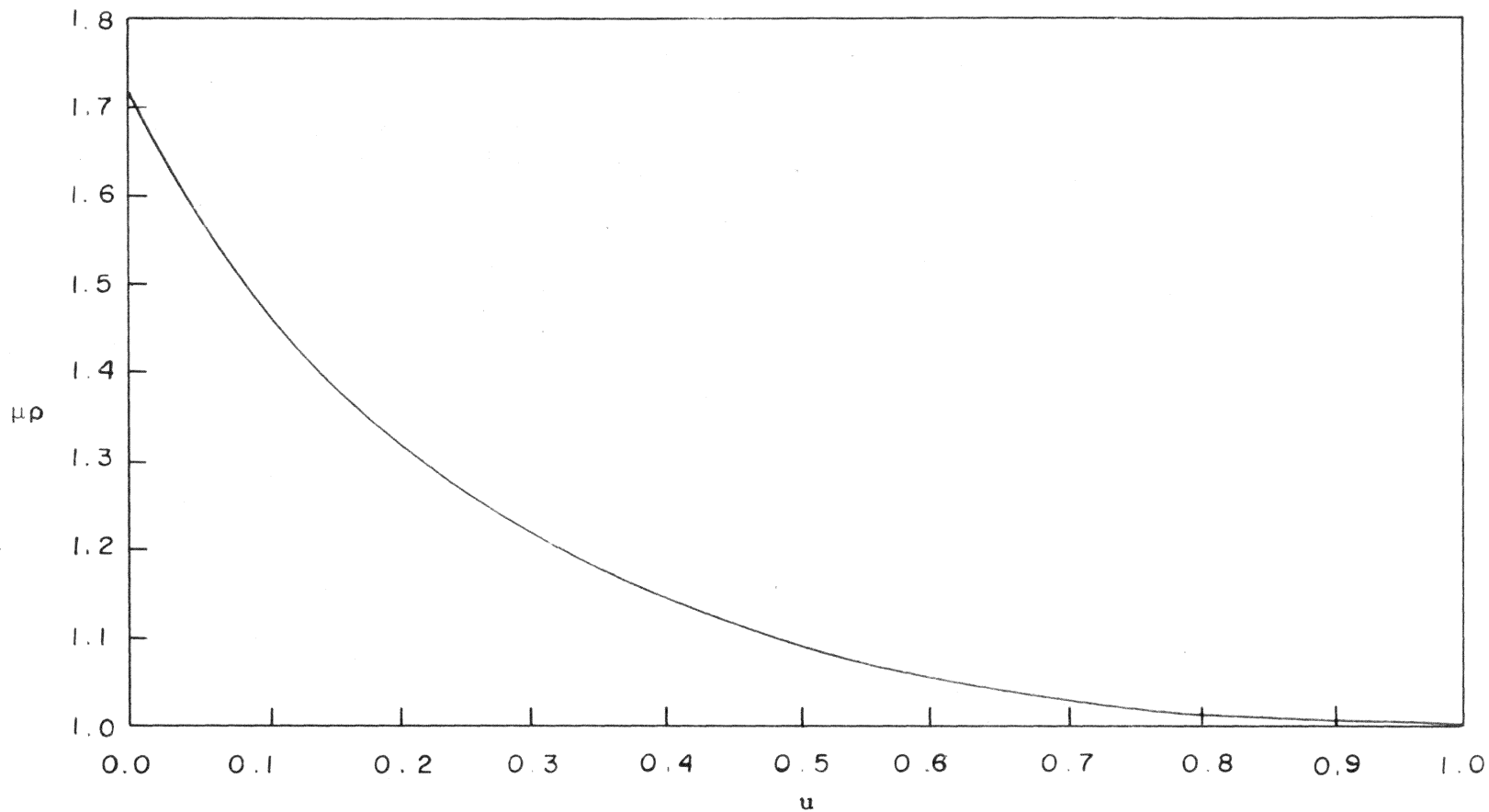


FIGURE (1)  
PLOT OF  $\mu\rho$  VERSUS  $U$



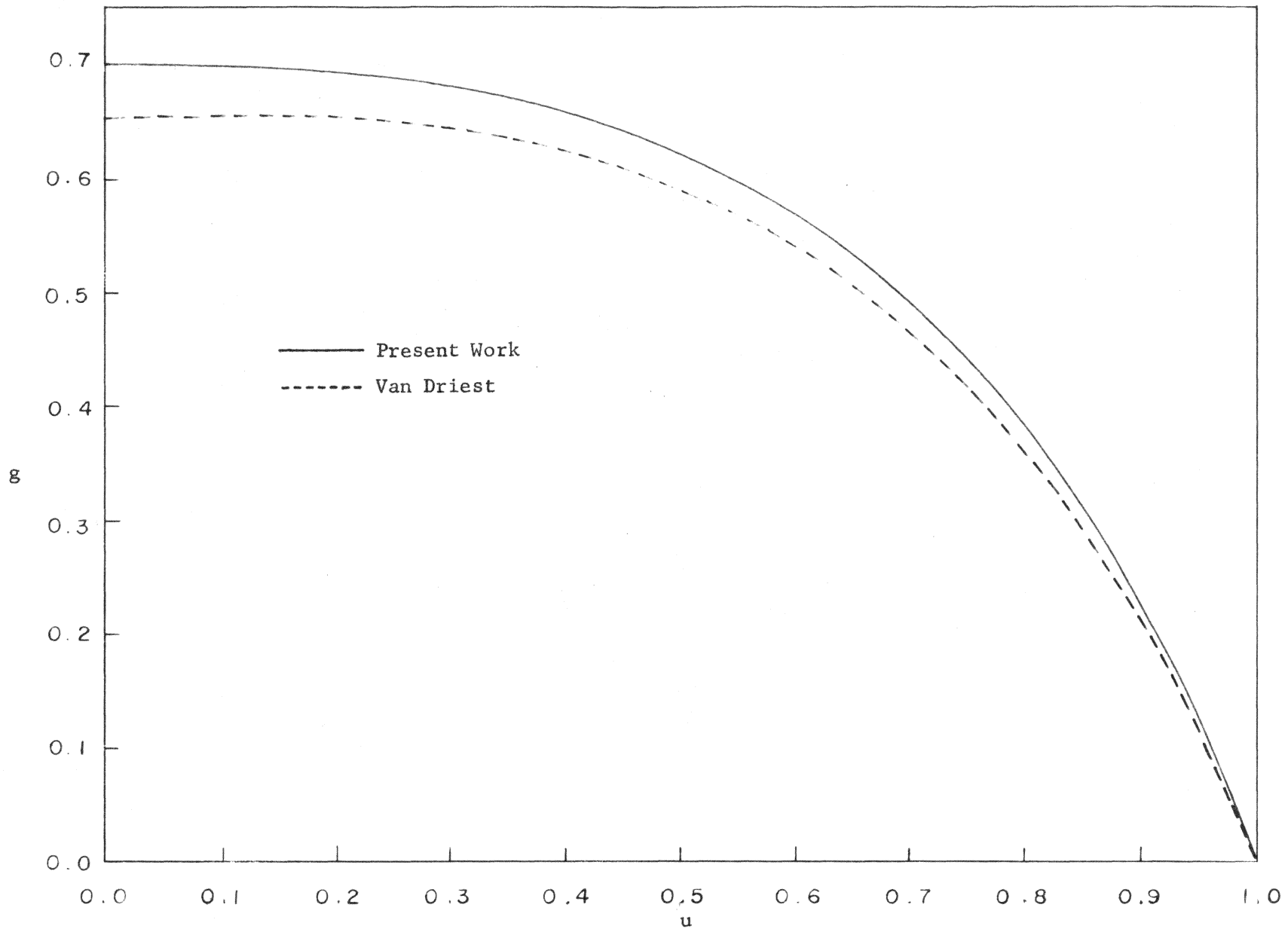


FIGURE (2)

COMPARISON OF THE COMPUTED SHEAR WITH THAT OBTAINED FROM VAN DRIEST FOR  
PRANDTL NUMBER OF 0.75

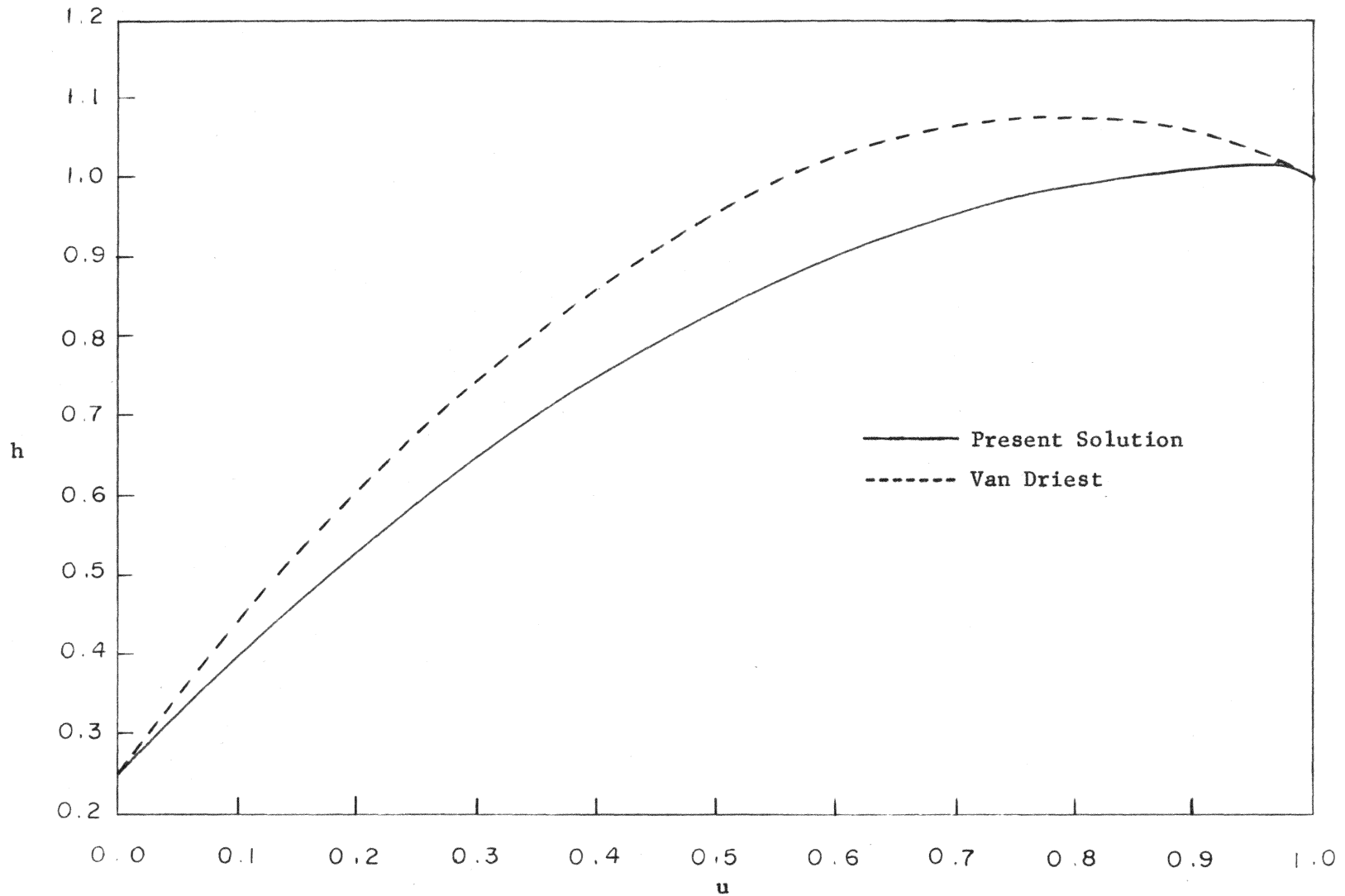


FIGURE (3)

COMPARISON OF THE COMPUTED ENTHALPY WITH THAT OBTAINED FROM VAN DRIEST FOR PRANDTL NUMBER OF 0.75

## ABSTRACT

The boundary layer equations for a quasi-equilibrium flow of dissociating air past a flat plate have been solved on an electronic digital computer. The solution is based upon a modification of Crocco's method, which takes into consideration the variation of the Prandtl, Schmidt and Lewis numbers.

The thermodynamic and transport properties of dissociating air have been evaluated utilizing the closed form expressions evolved by Hansen for approximate partition functions. This method of analysis permits the use of the most valid variations available for dissociating air properties with temperature. Air is treated as a mixture of oxygen and nitrogen in the ratio of one to four.

The calculated shear function for dissociated air exceeds, by a maximum of 7.1 percent, the values obtained by Van Driest for non-dissociating air.

The calculated enthalpy distribution varies considerably from that computed by Van Driest for a Prandtl number of 0.75; the maximum variation being 10.6 percent greater than the calculated value.

Results obtained here indicate that the enthalpy distribution is virtually independent of the variation of the product  $\mu\rho$ , viscosity times density, with enthalpy.