Cotangent Schubert Calculus in Grassmannians

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(ABSTRACT)

We find formulas for the Segre-MacPherson classes of Schubert cells in T-equivariant cohomology and the motivic Segre classes of Schubert cells in T-equivariant K-theory. In doing so we look at the pushforward of the projection map from the Bott-Samelson (Kempf-Laksov) desingularization to the Grassmannian. We find that the Segre-MacPherson classes are stable under pullbacks of maps embedding a Grassmannian into a bigger Grassmannian. We also express these formulas using certain Demazure-Lusztig operators that have previously been used to study these classes.

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(GENERAL AUDIENCE ABSTRACT)

Schubert calculus was first introduced in the nineteenth century as a way to answer certain questions in enumerative geometry. These computations relied on the multiplication of Schubert classes in the cohomology ring of Grassmannians, which parameterize k-dimensional linear subspaces of a vector space. More recently Schubert calculus has been broadened to refer to computations in generalized cohomology theories, such as (equivariant) K-theory.

In this dissertation, we study Segre-MacPherson classes and motivic Segre classes of Schubert cells in Grassmannians. Segre-MacPherson classes are related to Chern-Schwartz-MacPherson classes, which are a generalization to singular spaces of the total Chern class of the tangent bundle. Motivic Segre classes are similarly related to motivic Chern classes, which are a K-theory analogue of Chern-Schwartz-MacPherson classes.

This dissertation also studies the relationship between Schubert varieties and their Bott-Samelson desingularizations, specifically their (T-equivariant) cohomology and K-theory rings. Since equivariant cohomology (or K-theory) classes can be represented by polynomials, we can represent the Segre-MacPherson (or motivic Segre) classes as rational functions. Furthermore, we use certain operators that act on such polynomials (or rational functions) to find formulas for the rational function representatives of the aforementioned classes.

Contents

Chapter 1

Introduction

The Chern-Schwartz-MacPherson (CSM) class of a variety is a way to extend the idea of the total Chern class of the tangent bundle to singular spaces. Specifically, it is defined for singular spaces and coincides with the total Chern class of the tangent bundle for smooth spaces. To calculate these classes in the Grassmannian, we first calculate them in a desingularization and push the classes forward using the functoriality properties of CSM classes. As a result, we also look at the pushforward from the desingularization into the Grassmannian. Once we have the CSM class, we can use it to calculate a class dual to the CSM class known as the Segre-MacPherson (SM) class.

There is also a similar story in K-theory, where the analogous class is the motivic Chern class, with its dual class the motivic Segre class. We use the same desingularization and also analyze the pushforward in K-theory.

Topics related to CSM classes and motivic Chern classes have been studied by many people. In [3], Aluffi and Mihalcea find formulas for expressing the CSM classes of Schubert cells in the Grassmannian as a linear combination of Schubert classes. The Bott-Samelson varieties they use are the same kind used here, and this dissertation uses many techniques similar to theirs to expand their work to calculating SM classes in T-equivariant cohomology. They have also studied CSM classes for generalized flag manifolds, of which Grassmannians are a special case, in [2].

In addition to that, they, along with Schürmann and Su, proved some results for CSM and SM classes in generalized flag manifolds, including positivity for CSM classes in [4] and positivity for SM classes in [5].

In [14], Feher and Rimanyi found formulas for the CSM and SM classes of matrix Schubert cells. Their formula for SM classes of matrix Schubert cells coincides with the formula in this dissertation for the non-equivariant SM class of Schubert cells in the Grassmannian, as noted at the end of Chapter 5. One motivation for this research was to recover this formula from the point of view of SM classes of Schubert cells in Grassmannians.

Feher and Rimanyi, along with Weber, also studied motivic Chern classes. In [15], they find formulas for the motivic Chern classes of Schubert cells in partial flag varieties and for matrix Schubert cells.

Aluffi, Mihalcea, Schürmann, and Su also studied motivic Chern classes in [6] and [29] for generalized flag varieties.

In [23], Knutson and Zinn-Justin find combinatorial formulas for the products of motivic Segre classes in partial flag manifolds. Due to the way classes multiply in T-equivariant K-theory, certain coefficients in the multiplication formulas are also the localizations of motivic Segre classes. These can also be obtained from results in this dissertation by applying Lemma 3.4.1 to Theorem 6.0.2.

In [30], Mihalcea, Naruse, and Su use divided difference operators in order to study CSM classes, SM classes, motivic Chern, and motivic Segre classes in generalized flag manifolds. In this dissertation, we use some of these operators to express formulas for SM classes and motivic Segre classes.

1.1 Results

Let $\lambda = (\lambda_1, ..., \lambda_k)$ be a partition, meaning $\lambda_i \geq \lambda_j$ for $i < j$ and $\lambda_i \in \mathbb{Z}_{\geq 0}$ for $1 \leq i \leq k$. Let $N \in \mathbb{N}$ and $n = N + k$. We denote by $\Omega^\lambda \subseteq \text{Gr}(k, n)$ the Schubert variety corresponding to λ as defined in section 2.3. Let $\pi : \mathbb{V}^\lambda \to \Omega^\lambda$ be the Bott-Samelson desingularization defined in section 2.4. In the desingularization, which is a subvariety of $F\ell(1, ..., k; n)$, there are line bundles $\mathcal{L}_i$ over $\mathbb{V}^\lambda$ whose fibers over a point are S_i/S_{i-1}, where $(S_1 \subset ... \subset S_k)$ is the partial flag that is the point in the partial flag variety. In the Grassmannian, there is a tautological sub-bundle denoted by $\mathcal{S}$ whose fiber is $S \subset \mathbb{C}^n$, the k-dimensional subspace of $\mathbb{C}^n$ corresponding to a point in $\text{Gr}(k, n)$. In cohomology, we use $z_i = c_1(\mathcal{L}_i^\vee)$ to express classes in the T-equivariant cohomology ring of $\mathbb{V}^\lambda$ and the Chern roots x_i satisfying $\prod_{i=1}^k (1 + x_i) = c(\mathcal{S}^\vee)$ to express classes in the cohomology ring of $\text{Gr}(k, n)$. In T-equivariant cohomology we use the equivariant Chern classes, so $z_i = c_1^T(\mathcal{L}_i)$ and $\prod_{i=1}^k (1+x_i) = c^T(\mathcal{S}^\vee)$. In T-equivariant cohomology, there are also the equivariant parameters t_i, which are given by $t_i = c_1^T(\mathbb{C}_i)$, where $\mathbb{C}_i = \langle e_i \rangle$, where e_i is the standard unit vector in $\mathbb{C}^n$. In K-theory we use $Z_i = \lambda_{-1}(\mathcal{L}_i)$ for classes in $\mathbb{V}^\lambda$. In the Grassmannian we use X_i which satisfy $\prod_{i=1}^k X_i = \lambda_{-1}(\mathcal{S})$. In T-equivariant K-theory, there are equivariant parameters T_i, which are given by $T_i = \lambda_{-1}^T(\mathbb{C}_i^\vee)$.

In Chapter 3, we examine the pushforwards $\pi_* : H_T^*(\mathbb{V}^\emptyset) \to H_T^*(\text{Gr}(k, n))$ and $\pi_* : K^T(\mathbb{V}^\emptyset) \to K^T(\text{Gr}(k, n))$. In T-equivariant cohomology we have:

Theorem 1.1.1. Let $f \in \mathbb{Z}[z_1, ..., z_k][t_1, ..., t_n]$ represent a class in $H_T^*(\mathbb{V}^\emptyset)$. Then

$$\pi_*(f(z;t)) = \sum_{w \in S_k} \left(\prod_{i=1}^{k-1} \prod_{j=i+1}^{k} \frac{(x_{w(i)} + t_{j-i})}{(x_{w(i)} - x_{w(j)})} f(x_{w(1)}, ..., x_{w(k)}; t) \right).$$

In T-equivariant K-theory we have:

Theorem 1.1.2. Let $F \in \mathbb{Z}[Z_1, ..., Z_k][T_1, ..., T_k]$ represent a class in $K^T(\mathbb{V}^\emptyset)$. Then

$$\pi_*(F(Z;T)) \\ = \sum_{w \in S_k} \left(\prod_{i=1}^{k-1} \prod_{j=i+1}^{k} \frac{(1 - X_{w(j)})(X_{w(i)} + T_{j-i} - X_{w(i)}T_{j-i})}{(X_{w(i)} - X_{w(j)})} F(X_{w(1)}, ..., X_{W(j)}; T) \right).$$

When these are applied to specific polynomials, the result are the factorial Schur functions in cohomology (Theorem 3.2.1) and the factorial Grothendieck polynomials in K-theory (Theorem 3.5.2).

Then we calculate the Segre-MacPherson classes in cohomology and the motivic Chern classes in K-theory.

Theorem 1.1.3. For a partition $\lambda = (\lambda_1, ..., \lambda_k)$ and $N \geq \lambda_1$, the T-equivariant SM class of the Schubert cell in $\mathrm{Gr}(k, N+k)$ is:

$$s^T_{\mathrm{SM}}(\Omega^{\lambda,\circ}) = \pi_*\Big(\prod_{i=1}^{k} \Big(\prod_{j=k+1-i}^{k-i+\lambda_i} (z_i + t_j) \prod_{j=1}^{k+1-i+\lambda_i} \frac{1}{1 + z_i + t_j} \Big) \prod_{i=1}^{k} \prod_{j=i+1}^{k} (1 + z_i - z_j) \cap [\mathbb{V}^\emptyset] \Big).$$

Theorem 1.1.4. For a partition $\lambda = (\lambda_1, ..., \lambda_k)$ with $\lambda_1 \leq N$, the T-equivariant motivic Segre class of the Schubert cell $\Omega^{\lambda,\circ}$ in $\mathrm{Gr}(k, N+k)$ is:

$$mS^T_y(\Omega^{\lambda,\circ} \hookrightarrow \mathrm{Gr}(k, N+k)) = \\ \pi_* \left(\prod_{i=1}^{k} (1+y)[\mathcal{L}_i \otimes \mathbb{C}^\vee_{t_{k+1-i+\lambda_i}}] \prod_{j=k+1-i}^{k-i+\lambda_i} (1 - [\mathcal{L}_i \otimes \mathbb{C}^\vee_{t_j}]) \frac{\prod_{j=i+1}^{k} \lambda^T_y(\mathcal{L}^\vee_j \otimes \mathcal{L}_i)}{\prod_{j=1}^{k+1-i+\lambda_i} \lambda^T_y(\mathbb{C}^\vee_{t_j} \otimes \mathcal{L}_i)} \right).$$

We also prove the stability of the SM class under pullbacks of embeddings of Grassmannians inside of bigger Grassmannians. This stability allows us to express SM classes in an inverse limit of cohomology rings of Grassmannians as $k \to \infty$ and $N \to \infty$.

We can also express the previous formulas in terms of divided difference operators that have been used to study CSM, SM, motivic Chern, and motivic Segre classes in [30].

In cohomology we use the BGG operator

$$\partial_i(f) = \frac{f - s_i(f)}{x_i - x_{i+1}}$$

defined for polynomials $f \in \mathbb{Z}[x_1, ...x_k]$ and where $s_i \in S_k$ is the simple reflection that swaps i and $i+1$. We also use the related operator $T_i = \partial_i + s_i$, meaning

$$T_i(f) = \frac{f - s_i(f)}{x_i - x_{i+1}} + s_i(f).$$

In K-theory we use the Demazure operator

$$\delta_i(f) = \partial_i((1 - X_{i+1})f) = \frac{(1 - X_{i+1})f - (1 - X_i)s_i(f)}{X_i - X_{i+1}}$$

defined for polynomials $f \in \mathbb{Z}[X_1, ...X_n]$ and the related operator $\mathcal{T}_i(f) = \delta_i(1 + y\frac{1-X_i}{1-X_{i+1}})f) - f$, meaning

$$\mathcal{T}_i(f) = \frac{(1 - X_{i+1})f + y(1 - X_i)f}{X_i - X_{i+1}} - \frac{(1 - X_i)s_i(f) + y(1 - X_{i+1})s_i(f)}{X_i - X_{i+1}} - f.$$

All of the operators satisfy the braid relations, so the definitions extend to permutations $w = s_{i_1}...s_{i_\ell}$ by $\partial_w = \partial_{i_1}...\partial_{i_\ell}$. These operators may be realized as generators of certain versions of Hecke algebras and they satisfy the quadratic relations $\partial_i^2 = 0$, $\delta_i^2 = \delta_i$, $T_i^2 = id$, and $(\mathcal{T}_i + id)(\mathcal{T}_i + y) = 0$. When these operators are applied in a certain way to specific polynomials, we can obtain the same formulas for the Segre-MacPherson classes and the motivic Segre classes.

Theorem 1.1.5. For a partition $\lambda = (\lambda_1, ..., \lambda_k)$ with $\lambda_1 \leq N$, the T-equivariant Segre-MacPherson class of the Schubert cell in $\mathrm{Gr}(k, N+k)$ is given by

$$s_M^T(\Omega^{\lambda,\circ}) = \sum_{w \in S_k} T_w \left(\prod_{i=1}^{k} \frac{\prod_{j=1}^{k-i+\lambda_i}(x_i + t_j)}{\prod_{j=1}^{k+1-i+\lambda_i}(1 + x_i + t_j)} \right).$$

Theorem 1.1.6. For a partition $\lambda = (\lambda_1, ..., \lambda_k)$ with $\lambda_1 \leq N$, the T-equivariant motivic Segre class of the Schubert cell in $\mathrm{Gr}(k, N+k)$ is given by

$$\begin{aligned} &mS_y^T(\Omega^{\lambda,\circ} \hookrightarrow \mathrm{Gr}(k, N+k)) \\ &= \sum_{w \in S_k} \mathcal{T}_w \left(\prod_{i=1}^{k} \prod_{j=1}^{k-i+\lambda_i} \frac{(X_i + T_j - X_i T_j)}{1 + y(1 - X_i - T_j + X_i T_j)} \right. \\ &\qquad \left. - \prod_{i=1}^{k} \prod_{j=1}^{k+1-i+\lambda_i} \frac{(X_i + T_j - X_i T_j)}{1 + y(1 - X_i - T_j + X_i T_j)} \right). \end{aligned}$$

These formulas are in terms of Chern roots (in cohomology) of the tautological sub-bundle or their K-theoretic analogue, but another way of expressing these classes is using the Schubert basis, which consists of the fundamental classes of the Schubert varieties. The factorial Schur functions (cohomology) and factorial Grothendieck polynomials (K-theory) give a correspondence between the basis of symmetric functions in x_i or X_i and the Schubert basis, but because these formulas are in terms of rational functions, it is difficult to make the transition. However equivariant localization provides another way to translate from polynomials to Schubert classes.

1.2 Examples

In $\mathrm{Gr}(2,4)$, the Segre-MacPherson class of the Schubert cell for the partition $(1,0)$ is given by Theorem 1.1.3:

$$s^T_{\mathrm{SM}}(\Omega^{\square,\circ}) = \pi_*\left(\frac{(z_1+t_2)(1+z_1-z_2)}{(1+z_1+t_1)(1+z_1+t_2)(1+z_1+t_3)(1+z_2+t_1)}\right).$$

Using Theorem 1.1.1, we can calculate the pushforward:

$$\begin{aligned} s^T_{\mathrm{SM}}(\Omega^{\square,\circ}) = &\frac{(x_1+t_1)(x_1+t_2)(1+x_1-x_2)}{(1+x_1+t_1)(1+x_1+t_2)(1+x_1+t_3)(1+x_2+t_1)(x_1-x_2)} \\ &- \frac{(x_2+t_1)(x_2+t_2)(1+x_2-x_1)}{(1+x_2+t_1)(1+x_2+t_2)(1+x_2+t_3)(1+x_1+t_1)(x_1-x_2)}. \end{aligned}$$

Then to get the localizations, we plug in $x_i = -t_{i+\lambda_{k+1-i}}$, and so expressing a class $\kappa \in H^*_T(\mathrm{Gr}(2,4))$ as

$$\kappa = (\kappa|_\emptyset, \kappa|_{\ydiagram{1}}, \kappa|_{\ydiagram{1,1}}, \kappa|_{\ydiagram{2}}, \kappa|_{\ydiagram{2,1}}, \kappa|_{\ydiagram{2,2}}),$$

we have

$$\begin{aligned} s^T_{\mathrm{SM}}(\Omega^{\square,\circ}) = \Bigg(0, &\frac{t_2-t_3}{1+t_2-t_3}, \frac{t_1-t_3}{(1+t_1-t_3)(1+t_1-t_2)}, \\ &\frac{t_2-t_4}{(1+t_2-t_4)(1+t_3-t_4)}, \frac{t_1-t_4}{(1+t_t-t_4)(1+t_3-t_4)(1+t_1-t_2)}, \\ &\frac{(t_2-t_3)(1+t_2-t_4)(1+t_1-t_3)+(t_1-t_4)}{(1+t_1-t_3)(1+t_1-t_4)(1+t_2-t_3)(1+t_2-t_4)}\Bigg). \end{aligned}$$

Then using the localizations of the Schubert classes, we can find a linear combination of Schubert classes that will match these localizations. Doing so amounts to solving a linear system over rational functions in $t_1, ..., t_4$. Since $[\Omega^\lambda]|_\mu = 0$ if $\lambda \not\leq \mu$, the system is lower triangular and can be solved by forward substitution.

If

$$s^T_{\mathrm{SM}}(\Omega^{\ydiagram{1},\circ}) = c_\emptyset[\Omega^\emptyset]^T + c_{\ydiagram{1}}[\Omega^{\ydiagram{1}}]^T + c_{\ydiagram{1,1}}[\Omega^{\ydiagram{1,1}}]^T + c_{\ydiagram{2}}[\Omega^{\ydiagram{2}}]^T + c_{\ydiagram{2,1}}[\Omega^{\ydiagram{2,1}}]^T + c_{\ydiagram{2,2}}[\Omega^{\ydiagram{2,2}}]^T,$$

we can find the constants c_λ by localizing both sides of the equation at λ provided we have already found the constants c_μ for $\mu \leq \lambda$. The localizations for the Schubert varieties are usually found through interpolation conditions, but we can also find $[\Omega^\lambda]|_\mu$ by plugging in $x_i = -t_{i+\mu_{k+1-i}}$ to the factorial Schur function $s_\lambda(x|t)$.

Then we have $c_\emptyset = 0$ since $s^T_{\mathrm{SM}}(\Omega^{\ydiagram{1},\circ})|_\emptyset = 0$. Since $s_{\ydiagram{1}}(x|t) = x_1 + x_2 + t_1 + t_2$, we have

$$c_{\ydiagram{1}} = \frac{s^T_{\mathrm{SM}}(\Omega^{\ydiagram{1},\circ})|_{\ydiagram{1}}}{[\Omega^{\ydiagram{1}}]^T|_{\ydiagram{1}}} = \left(\frac{t_2 - t_3}{1 + t_2 - t_3}\right)\left(\frac{1}{t_2 - t_3}\right) = \frac{1}{1 + t_2 - t_3}.$$

With this, we can find $c_{\ydiagram{1,1}}$ from the equation

$$s^T_{\mathrm{SM}}(\Omega^{\ydiagram{1},\circ})|_{\ydiagram{1,1}} = c_{\ydiagram{1}}[\Omega^{\ydiagram{1}}]^T|_{\ydiagram{1,1}} + c_{\ydiagram{1,1}}[\Omega^{\ydiagram{1,1}}]^T|_{\ydiagram{1,1}}.$$

With $s^T_{\mathrm{SM}}(\Omega^{\ydiagram{1},\circ})|_{\ydiagram{1,1}} - c_{\ydiagram{1}}[\Omega^{\ydiagram{1}}]^T|_{\ydiagram{1,1}} = -\frac{(t_1-t_2)(t_1-t_3)(2+t_1-t_3)}{(1+t_1-t_2)(1+t_1-t_3)(1+t_2-t_3)}$ and $[\Omega^{\ydiagram{1,1}}]^T|_{\ydiagram{1,1}} = (t_1 - t_2)(t_1 - t_3)$, we have

$$c_{\ydiagram{1,1}} = -\frac{2 + t_1 - t_3}{(1 + t_1 - t_2)(1 + t_1 - t_3)(1 + t_2 - t_3)}.$$

In a similar fashion,

$$c_{\ydiagram{2}} = -\frac{2 + t_2 - t_4}{(1 + t_2 - t_4)(1 + t_3 - t_4)(1 + t_2 - t_3)},$$

$$c_{\ydiagram{2,1}} = \frac{5+6(t_1-t_4)+3(t_2-t_3)+2(t_1-t_4)^2+2(t_1-t_4)(t_2-t_3)}{(1+t_1-t_2)(1+t_1-t_3)(1+t_1-t_4)(1+t_2-t_3)(1+t_2-t_4)(1+t_3-t_4)} + \frac{(t_1-t_3)(t_2-t_4)+(t_1-t_3)(t_1-t_4)(t_2-t_4)}{(1+t_1-t_2)(1+t_1-t_3)(1+t_1-t_4)(1+t_2-t_3)(1+t_2-t_4)(1+t_3-t_4)},$$

and

$$c_{\ydiagram{2,2}} = \frac{-(3+2(t_1-t_4)+(t_2-t_3)+(t_1-t_3)(t_2-t_4))}{(1+t_1-t_2)(1+t_1-t_3)(1+t_1-t_4)(1+t_2-t_3)(1+t_2-t_4)(1+t_3-t_4)}$$

Here we can observe that the signs of the coefficients alternate sign (specifically $(-1)^{|\lambda|-|\mu|}c_\mu$ is positive), and polynomials that appear in t always come in pairs of the form $t_i - t_j$ for $i < j$. We can also set $t_i = 0$ to recover the coefficients for the Segre-MacPherson class in (non-equivariant) cohomology. Doing that for the other partitions in $\mathrm{Gr}(2,4)$ gives:

$$s^T_{\mathrm{SM}}(\Omega^{\emptyset,\circ}) = [\Omega^{\emptyset}] - [\Omega^{\ydiagram{1}}] + [\Omega^{\ydiagram{1,1}}] + [\Omega^{\ydiagram{2}}] - 2[\Omega^{\ydiagram{2,1}}] + [\Omega^{\ydiagram{2,2}}],$$

$$s^T_{\mathrm{SM}}(\Omega^{\ydiagram{1,1},\circ}) = [\Omega^{\ydiagram{1,1}}] - 2[\Omega^{\ydiagram{2,1}}] + 2[\Omega^{\ydiagram{2,2}}],$$

$$s^T_{\mathrm{SM}}(\Omega^{\ydiagram{2},\circ}) = [\Omega^{\ydiagram{2}}] - 2[\Omega^{\ydiagram{2,1}}] + 2[\Omega^{\ydiagram{2,2}}],$$

$$s^T_{\mathrm{SM}}(\Omega^{\ydiagram{2,1},\circ}) = [\Omega^{\ydiagram{2,1}}] - 3[\Omega^{\ydiagram{2,2}}], \text{ and}$$

$$s^T_{\mathrm{SM}}(\Omega^{\ydiagram{2,2},\circ}) = [\Omega^{\ydiagram{2,2}}]$$

Chapter 2

Preliminaries

We first recall some facts about equivariant cohomology and K-theory, some definitions for the Grassmannian and the Bott-Samelson (Kempf-Laksov) desingularization, and some background on Chern-Schwartz-MacPherson classes and motivic Chern classes.

2.1 Equivariant Cohomology

Let X be a complex algebraic variety with a left G-action for G an algebraic group, the G-equivariant cohomology of X is given by

$$H^*_G(X) = H^*(\mathbb{E}G \times^G X),$$

where $\mathbb{E}G$ is a contractible space with a free right G-action, and $\mathbb{E}G \times^G X = \mathbb{E}G \times X/(e \cdot g, x) \sim (e, g \cdot x)$. For more information, refer to [7] and the references therein, but for our purposes here I will just list the needed facts.

In our case the group is $T \cong (\mathbb{C}^*)^n$, the group of invertible diagonal $n \times n$ matrices acting on $\mathbb{C}^n$ in the usual way, with this action extending to Grassmannians by acting on the subspaces. The T-equivariant cohomology of a point is $H^*_T(pt) \cong \mathbb{Z}[t_1, ..., t_n]$, where $t_1, ..., t_n$ are the generators of the weight lattice of T, and $H^*_T(X)$ is an $H^*_T(pt)$-algebra for all spaces X mentioned in this paper.

Given a space X and a closed, irreducible subvariety $Y \subseteq X$ invariant under the T-action, there is an equivariant fundamental class $[Y] \in H_T^{2\text{codim}(Y)}(X)$ associated to Y. Also for any equivariant, proper morphism $f : X \to Y$, there is a pushforward $f_* : H_T^i(X) \to H_T^{i+\dim(Y)-\dim(X)}(Y)$, and for any equivariant morphism, there is a pullback $f^* : H_T^*(Y) \to H_T^*(X)$. In particular for birational morphisms, the pushforward satisfies $\pi_*([Y]) = [\pi(Y)]$. Also for any vector bundle E on a space X, there is an equivariant total Chern class $c^T(E) \in H_T^*(X)$ which satisfies $\pi^*(c^T(E)) = c^T(\pi^*(E))$, where $\pi^*(E)$ is the pullback bundle.

The inclusion map from the set of fixed points X^T into X is equivariant, and so the map induces a pullback map on the equivariant cohomology $\iota^* : H_T^*(X) \to H_T^*(X^T)$. If there are finitely many fixed points, then

$$H_T^*(X^T) \cong \oplus_{x \in X^T} H_T^*(x),$$

since the points will be disconnected from each other. In smooth varieties with finitely many fixed points and finitely many one-dimensional orbits, this map is injective [19, Theorem 1.2.2], and so a class can be identified uniquely by its image under this map. For each $x \in X^T$, the inclusion is equivariant and so induces a pullback map. The image of a class under this map is known as the localization of the class at x and is denoted

$$\iota_x^*(\kappa) = \kappa|_x.$$

For a closed subvariety $Y \subseteq X$ and fixed point $x \in X^T$, we have that $[Y]|_x = 0$ whenever $x \notin Y$. Furthermore, the fundamental classes of the fixed points generate the cohomology in the localization of the ring at $\mathbb{Z}[t_1, ..., t_n]$. In particular, any class can be expressed as

$$\kappa = \sum_{x \in X^T} \kappa|_x \frac{[x]}{[x]|_x}.$$

2.2 Equivariant K-Theory

We recall some basic facts about K-theory, using [10] as a reference. The Grothendieck ring $K^0(X)$ is generated by equivalence classes of vector bundles subject to the relation that whenever there is an exact sequence

$$0 \to E_1 \to E_2 \to E_3 \to 0,$$

the classes of those vector bundles satisfy $[E_1] + [E_3] = [E_2]$. The multiplication of classes is given by the class of the tensor product of the vector bundles. In smooth varieties, a coherent sheaf can be resolved by finitely many vector bundles, and so the class of a coherent sheaf can be expressed in $K^0(X)$ as an alternating sum classes of vector bundles. Any morphism of schemes $f : X \to Y$ induces a pullback map $f^* : K^0(Y) \to K^0(X)$ by $f^*([E]) = [f^*E]$. Since any vector bundle has a sheaf of sections, it can be seen as a sheaf, and so a pushforward can be defined for any proper morphism $f : X \to Y$, $f_* : K^0(X) \to K^0(Y)$ by $f_*([E]) = \sum_{j \geq 0}(-1)^j[R^j f_*(E)]$. There is also a projection formula $f_*((f^*\alpha) \cdot \beta) = \alpha \cdot f_*\beta$ [10, Section 3.3]. Given a vector bundle V, the λ_y class of V is given by

$$\lambda_y(V) = \sum_{p \geq 0} [\Lambda^p V] \cdot y^p.$$

This has the property that for any exact sequence of vector bundles

$$0 \to V_1 \to V_2 \to V_3 \to 0,$$

we have that $\lambda_y(V_2) = \lambda_y(V_1)\lambda_y(V_3)$ [20].

For equivariant K-theory, the equivariant Grothendieck ring $K_T(X)$ is instead generated by equivalence classes of equivariant vector bundles or equivariant coherent sheaves. For equivariant morphisms, there are similarly pushforward and pullbacks maps on the equivariant Grothendieck rings [12, Chapter 5.2].

Like in cohomology, for each T-fixed point $x \in X^T$, the inclusion map $\iota_x : \{x\} \to X$ induces a pullback map on equivariant K-theory $\iota_x^* : K_T(X) \to K_T(x)$. For $\alpha \in K_T(X)$, define the localization of α at the fixed point x as $\alpha|_x = \iota_x^*(\alpha)$. The equivariant K-theory of a point is the representation ring of T, which is isomorphic to the Laurent polynomial ring $\mathbb{Z}[e^{\pm t_1}, ..., e^{\pm t_{N+k}}]$, where e^{t_i} are the characters corresponding to a basis of the Lie algebra of T and $e^{t_i} = [\mathbb{C}_{t_i}]$ [12, 5.2.1]. When there are finitely many fixed points, the localization map $\iota^* : K_T(X) \to K_T(X^T) = \oplus_{x \in X^T} K_T(x)$ is injective by the localization theorem, see [31], so a class can be determined by its localizations. As a result, since the structure sheaf of a fixed point is only supported on that fixed point, we can express any $\kappa \in K_T(X)$ in terms of the structure sheaves of the fixed points by

$$\kappa = \sum_{x \in X^T} \frac{\kappa|_x}{[\mathcal{O}_x]|_x} [\mathcal{O}_x].$$

For a smooth point $x \in X$, we have $[\mathcal{O}_x]|_x = \lambda_{-1}(T_x^* X)$. Then since $\pi_*([\mathcal{O}_x]) = [\mathcal{O}_{\pi(x)}]$, we can use this to calculate the localizations of the pushforward of a class:

$$\pi_*(\kappa)|_x = \sum_{y : \pi(y) = x} \frac{[\mathcal{O}_x]|_x}{[\mathcal{O}_y]|_y} \kappa|_y.$$

2.3 Schubert Cells and Varieties

The Grassmannian of k-planes in $\mathbb{C}^n$ is the set of linear subspaces of $\mathbb{C}^n$:

$$\mathrm{Gr}(k,n) = \{S \subseteq \mathbb{C}^n : \dim(S) = k\}.$$

There is a tautological sequence of vector bundles

$$0 \to \mathcal{S} \to \mathbb{C}^n \to \mathcal{Q} \to 0,$$

where the fiber of the tautological sub-bundle $\mathcal{S}$ at some point $S \in \mathrm{Gr}(k,n)$ is the vector space S, and the fiber of the tautlogical quotient bundle $\mathcal{Q}$ is the quotient $\mathbb{C}^n/S$. Given a partition $\lambda = (\lambda_1 \geq ... \geq \lambda_k \geq 0)$, and $N \geq \lambda_1$, there is a Schubert cell $\Omega^{\lambda,\circ}$ of codimension $|\lambda| = \sum_{i=1}^{k} \lambda_i$ in the Grassmannian $\mathrm{Gr}(k, N+k)$ defined by

$$\{S \in \mathrm{Gr}(k, N+k) : \dim(S \cap F_{N+i-\lambda_i}) = i, \dim(S \cap F_{N+i-1-\lambda_i}) = i-1, 1 \leq i \leq k\}$$

For some complete flag $F_1 \subset ... \subset F_{N+k}$, $\dim(F_i) = i$ for $1 \leq i \leq N+k$. For our purposes, we use the opposite flag, $F_i = \langle e_n, ..., e_{n+1-i} \rangle$. The closure of the Schubert cell $\Omega^{\lambda,\circ}$ is the Schubert variety Ω^λ, which is the disjoint union of Schubert cells

$$\Omega^\lambda = \{S \in \mathrm{Gr}(k, N+k) : \dim(S \cap F_{N+i-\lambda_i}) \geq i, 1 \leq i \leq k\} = \bigcup_{\beta \geq \lambda} \Omega^{\beta,\circ}.$$

The equivariant fundamental classes of the Schubert Varieties form a $H_T^*(pt)$-basis for the equivariant cohomology ring of the Grassmannian.

2.4 Bott-Samelson Varieties

There is also a corresponding Bott-Samelson variety, first used by Kempf and Laksov in [22], in the partial flag manifold $\mathrm{Fl}(1, ..., k; N + k)$,

$$\mathbb{V}^{\lambda} := \{S_1 \subset ... \subset S_k : \dim(S_i) = i, S_i \subseteq F_{N+i-\lambda_i}, 1 \leq i \leq k\}.$$

Define a map $\pi : \mathrm{Fl}(1, ...k; N + k) \to \mathrm{Gr}(k, N + k)$ by $\pi(S_1 \subset ... \subset S_k) = S_k$. By the conditions $S_i \subset S_k$, $\dim(S_i) = i$, and $S_i \subseteq F_{N+i-\lambda_i}$, we have $\dim(S_k \cap F_{N+i-\lambda_i}) \geq i$, and so $\pi(\mathbb{V}^{\lambda}) \subseteq \Omega^{\lambda}$. Then for any $S \in \Omega^{\lambda}$, define $S_i' = S \cap F_{N+i-\lambda_i}$. By the Schubert variety conditions, $\dim(S_i') \geq i$, and so there exist subspaces $S_i \subseteq S_i'$ such that $\dim(S_i) = i$. Then $S_1 \subset ... \subset S_k \in \mathbb{V}^{\lambda}$ and $\pi(S_1, ..., S_k) = S$. With this, $\pi(\mathbb{V}^{\lambda}) = \Omega^{\lambda}$. In addition, for $S \in \Omega^{\lambda,\circ}$, there is a unique $S_1 \subset ... \subset S_k$ such that $\pi(S_1 \subset ... \subset S_k) = S$, specifically $S_i = S \cap F_{N+i-\lambda_i}$, since $\dim(S_i) = i$ in this case by the conditions on $\Omega^{\lambda,\circ}$. So then $\pi(\mathbb{V}^{\lambda}) = \Omega^{\lambda}$, and π is an isomorphism when restricted to $\pi^{-1}(\Omega^{\lambda,\circ})$.

Next we recall that $\mathbb{V}^{\lambda}$ can be constructed as a tower of projective bundles, so it is smooth. Furthermore π is a birational morphism when restricted to $\mathbb{V}^{\lambda}$, so $\mathbb{V}^{\lambda}$ is a desingularization of Ω^{λ}. This construction is similar to the one in [3] but uses different conventions. For this denote by $\mathcal{F}_i$ by the trivial bundle whose fiber is F_i over the relevant space. Start by defining $\mathbb{V}_1^{\lambda} := \mathbb{P}(F_{N+1-\lambda_i})$ with tautological sequence

$$0 \to \mathcal{O}(-1) = \mathcal{L}_1 \to F_{N+1-\lambda_i} \to \mathcal{Q}_1 \to 0.$$

Then for $2 \leq i \leq k$, define the projective bundle

$$p : \mathbb{V}_i^{\lambda} \to \mathbb{V}_{i-1}^{\lambda}$$

as follows: Take the bundles $\mathcal{L}'_j$ and $\mathcal{Q}'_j$ on V^λ_{i-1} for $1 \leq j \leq i-1$, then define

$$\mathbb{V}^\lambda_i = \mathbb{P}((\mathcal{F}_{N+i-\lambda_i}/\mathcal{F}_{N+i-1-\lambda_{i-1}}) \oplus \mathcal{Q}'_{i-1})$$

with tautological sequence

$$0 \to \mathcal{L}_i \to (\mathcal{F}_{N+i-\lambda_i}/\mathcal{F}_{N+i-1-\lambda_{i-1}}) \oplus \mathcal{Q}'_{i-1} \to \mathcal{Q}_i \to 0,$$

and then define

$$\mathcal{Q}_j = p^*(\mathcal{Q}'_j) \text{ and } \mathcal{L}_j = p^*(\mathcal{L}'_j)$$

for $1 \leq j \leq i-1$. Then define $\mathbb{V}^\lambda = \mathbb{V}^\lambda_k$.

With this, for a point $(S_1 \subset ... \subset S_k) \in \mathbb{V}^\lambda$, the fiber of the bundle $\mathcal{Q}_i$ at the point is $F_{N+i-\lambda_i}/S_i$ and the fiber of $\mathcal{L}_i$ is S_i/S_{i-1} for $1 \leq i \leq k$, using $S_0 = \{0\}$.

2.5 Chern-Schwartz-MacPherson and Segre-MacPherson Classes

With X being an algebraic variety over $\mathbb{C}$, denote by $\mathcal{F}(X)$ the group of constructible functions on X:

$$\mathcal{F}(X) = \{\sum_{i=1}^n c_i 1\!\!1_{W_i} : n \in \mathbb{N}, c_i \in \mathbb{Z}, W_i \subseteq X\},$$

where the W_i are locally closed subvarieties of X and $1\!\!1_W$ is the characteristic function of W, meaning $1\!\!1_W(p) = 1$ if $p \in W$ and $1\!\!1_W(p) = 0$ if $p \notin W$. Given a morphism, $f : X \to Y$, one can define a pushforward $f_* : \mathcal{F}(X) \to \mathcal{F}(Y)$ as done on page 7 of [2]. This makes $\mathcal{F}$ a covariant functor, and MacPherson [27] proved a conjecture by Deligne and Grothendieck stating that there is a natural transformation $c_* : \mathcal{F} \to H_*$ such that if X is non-singular, then $c_*(\mathbb{1}_X) = c(T_X) \cap [X]$. With this, we can assign a homology class to a locally closed subvariety W by $c_{\text{SM}}(W) = c_*(\mathbb{1}_W)$. This class coincides with a class defined by Schwartz [32, 33] and so it is called the Chern-Schwartz-MacPherson class.

There is also the Segre-MacPherson class for a constructible subset Y of a smooth variety X:

$$s_{\text{SM}}(Y, X) = c(TX|_Y)^{-1} \cap c_*(\mathbb{1}_Y),$$

as defined on the top of page 2 of [5]. For this paper, we will use $s_{\text{SM}}(Y) = s_{\text{SM}}(Y, X)$, where X is assumed to be a Grassmannian.

For calculating the CSM classes of Schubert cells, we use the fact that for a nonsingular variety X and an open subvariety $W \subseteq X$ such that $X \backslash W$ is a simple normal crossing divisor with components D_i, the CSM class of W is

$$c_{\text{SM}}(W) = \frac{c(T_X)}{\prod_i (1 + D_i)} \cap [X],$$

from [2, Equation (21)].

2.6 Motivic Chern and Motivic Segre Classes

Let $K_0(var/X)$ denote the Grothendieck group of complex algebraic varieties over X, which is generated by isomorphism classes of algebraic morphisms $Y \to X$ with relations

$$[Y \to X] = [Z \to X] + [Y \backslash Z \to X]$$

for $Z \subset Y$ a closed subvariety. The motivic Chern class is given by a transformation $mC_y : K_0(var/X) \to K^0(X)[y]$ as defined by [9]. Then [15] and [6, Theorem 4.2] proved that this definition extends to equivariant K-theory.

Theorem 2.6.1. Let X be a quasi-projective, non-singular, complex algebraic variety with an action of the torus T. There exists a unique natural transformation $mC_y : K_T(var/X) \to K_T(X)[y]$ satisfying the following properties:

1. It is functorial with respect to T-equivariant proper morphisms of non-singular, quasi-projective varieties.

2. It satisfies the normalization condition

$$mC_y^T[id_X : X \to X] = \lambda_y^T(T_X^*) \in K_T(X)[y]$$

The motivic Chern class is given by [28, Proposition 2.2]:

Proposition 2.6.2. Let X be a smooth, complex, algebraic variety, with $D \subset X$ a simple normal crossing divisor, and $i : U = X \backslash D \hookrightarrow X$ the inclusion of the open complement. Then

$$mC_y([i : U \hookrightarrow X]) = [\mathcal{O}_X(-D) \otimes \lambda_y(\Omega_X^1(\log\ D))] \in K_0(X)[y].$$

The motivic Chern class is analogous to the CSM class in homology, and there is a motivic Segre class, analogous to the SM class, obtained by dividing the motivic Chern class by the λ_y class of the cotangent bundle:

$$mS_y(Y \hookrightarrow X) = \frac{mC_y(Y \hookrightarrow X)}{\lambda_y(T_X^*)}.$$

Chapter 3

The Pushforward from the Bott-Samelson Variety to the Grassmannian

In this chapter we calculate the class of $\mathbb{V}^\lambda$ as a subvariety of $\mathbb{V}^\emptyset$ in T-equivariant cohomology and K-theory, calculate the pushforward of the map $\pi : \mathbb{V}^\emptyset \to \mathrm{Gr}(k, N+k)$ for any cohomology class in $\mathbb{V}^\emptyset$, and then show that applying that pushforward to the class of $\mathbb{V}^\emptyset$ gives the factorial Schur functions (in cohomology) and the factorial Grothendieck polynomials (in K-theory) in certain variables. We also give straightening rules in cohomology and K-theory which give relationships between the formula for values of λ which are not partitions and for values of λ that are partitions.

3.1 The Pushforward in Equivariant Cohomology

In this section we calculate the class of $\mathbb{V}^\lambda$ as a subvariety of $\mathbb{V}^\emptyset$ (Lemma 3.1.1) as well as the localizations of the pushforward of any class (Lemma 3.1.2). This leads to calculating the pushforward of any class (Lemma 3.1.3).

The equivariant cohomology ring of $\mathbb{V}^\emptyset$ is generated over $H_T^*(pt)$ by the first Chern classes of the dual line bundles $\mathcal{L}_i^\vee$, which we denote by $z_i = c_1^T(\mathcal{L}_i^\vee)$ for $1 \leq i \leq k$. As a result, any class in $H_T^*(\mathbb{V}^\emptyset)$ can be expressed as a polynomial in $\mathbb{Z}[t_1, ..., t_{N+k}][z_1, ..., z_k]$. One useful thing to know for calculating pushforwards is that

$$\pi_*([\mathbb{V}^\lambda]) = [\Omega^\lambda],$$

since $\mathbb{V}^\lambda$ maps birationally onto Ω^λ under π. The following lemma expresses $[\mathbb{V}^\lambda]$ as a polynomial in the z_i's:

Lemma 3.1.1. For a partition $\lambda = (\lambda_1 \geq ... \geq \lambda_k)$ with $N \geq \lambda_1$,

$$[\mathbb{V}^\lambda] = \prod_{i=1}^{k} \prod_{j=k+1-i}^{k-i+\lambda_i} (z_i + t_j) \cap [\mathbb{V}^\emptyset]$$

Proof. We proceed by induction on n in $\mathbb{V}_n^\lambda$. The base case $n = 0$ is a point, which cannot have nonempty, proper subvarieties, so this holds trivially. The induction hypothesis is that

$$[\mathbb{V}_n^\lambda] = \prod_{i=1}^{n} \prod_{j=k+1-i}^{k-i+\lambda_i} (z_i + t_j) \in H_T^*(\mathbb{V}^\emptyset).$$

If we denote by $\mathcal{S}_i$ the vector bundle on any subvariety of $Fl(1, ..., n; N + k)$ whose fiber is S_i, where $Fl(1, ..., n; N + k) = \{(S_1 \subset ... \subset S_n) : \dim(S_i) = i\}$, then by the construction of $\mathbb{V}^\lambda$, we have that $\mathbb{V}_{n+1}^\lambda = \mathbb{P}(\mathcal{F}_{N+n+1-\lambda_{n+1}}/\mathcal{S}_n)$ as a projective bundle over $\mathbb{V}_n^\lambda$. Then for $\lambda' = (\lambda_1, ..., \lambda_n, 0, ..., 0)$, we have that as a projective bundle over $\mathbb{V}_n^\lambda$, $\mathbb{V}_{n+1}^{\lambda'} = \mathbb{P}(\mathcal{F}_{N+n+1}/\mathcal{S}_n)$. In particular, $\mathbb{V}_{n+1}^\lambda$ is a sub-bundle of $\mathbb{V}_{n+1}^{\lambda'}$ over the same base space. As a result, from [16, B.5.6] (for details, see the proof of Lemma 3.4.2), there is a regular section of $\mathcal{F}_{N+n+1}/\mathcal{F}_{N+n+1-\lambda_{n+1}} \otimes \mathcal{L}_{n+1}^\vee$ over $\mathbb{V}_{n+1}^{\lambda'}$ whose zero locus is $\mathbb{V}^\lambda$. As a result

$$[\mathbb{V}_{n+1}^\lambda] = c_{top}^T(\mathcal{F}_{N+n+1}/\mathcal{F}_{N+n+1-\lambda_{n+1}} \otimes \mathcal{L}_{n+1}^\vee) = \prod_{j=k-n}^{k-n-1+\lambda_{n+1}} (z_{n+1} + t_j) \in H_T^*(\mathbb{V}_{n+1}^{\lambda'}).$$

Now since both $\mathbb{V}^{\emptyset}_{n+1}$ and $\mathbb{V}^{\lambda'}_{n+1}$ are projectivizations of the bundle $\mathcal{F}_{N+n+1}/\mathcal{S}_n$ over their respective base spaces $\mathbb{V}^{\emptyset}_n$ and $\mathbb{V}^{\lambda}_n$, the induction hypothesis implies that

$$[\mathbb{V}^{\lambda'}_{n+1}] = \prod_{i=1}^{n} \prod_{j=k+1-n}^{k-n+\lambda_i} (z_i + t_j) \in H^*_T(\mathbb{V}^{\emptyset}_{n+1}).$$

Then pushing forward along the inclusion maps $\iota_1 : \mathbb{V}^{\lambda}_{n+1} \to \mathbb{V}^{\lambda'}_{n+1}$ and $\iota_2 : \mathbb{V}^{\lambda'}_{n+1} \to \mathbb{V}^{\emptyset}_{n+1}$ gives

$$\begin{aligned}
[\mathbb{V}^{\lambda}_{n+1}] \cap [\mathbb{V}^{\emptyset}_{n+1}] &= (\iota_2)_*([\mathbb{V}^{\lambda}_{n+1}] \cap [\mathbb{V}^{\lambda'}_{n+1}]) \\
&= (\iota_2)_*((\iota_1)_*([\mathbb{V}^{\lambda}_{n+1}] \cap [\mathbb{V}^{\lambda}_{n+1}]) \cap [\mathbb{V}^{\lambda'}_{n+1}]) \\
&= (\iota_2)_* \left(\prod_{j=k-n}^{k-n-1+\lambda_{n+1}} (z_{n+1} + t_j) \cap [\mathbb{V}^{\lambda'}_{n+1}] \right) \\
&= \left(\prod_{i=1}^{n} \prod_{k+1-i}^{k-i+\lambda_i} (z_i + t_j) \right) \prod_{k-n}^{k-n-1+\lambda_{n+1}} (z_{n+1} + t_j) \\
&= \prod_{i=1}^{n+1} \prod_{k+1-i}^{k-i+\lambda_i} (z_i + t_j).
\end{aligned}$$

This completes the induction step and so completes the proof. □

In order to look at the pushforward of classes in $\mathbb{V}^{\emptyset}$, we use the fact that for any $\kappa \in H^*_T(\mathbb{V}^{\emptyset})$,

$$\kappa = \sum_{x \in (\mathbb{V}^{\emptyset})^T} \kappa|_x \frac{[x]}{[x]|_x}.$$

We also know that $\pi_*([x]) = [\pi(x)]$, and so

$$\pi_*(\kappa) = \sum_{x \in (\mathbb{V}^{\emptyset})^T} \kappa|_x \frac{[\pi(x)]}{[x]|_x}.$$

For a smooth point $x \in X$, we have $[x]|_x = c^T_{top}(T_x X)$, the Euler class of the tangent space at x. Then localizing this at some fixed point $y \in (\mathrm{Gr}(k, N+k))^T$ gives

$$\pi_*(\kappa)|_y = \sum_{x:\pi(x)=y} \frac{[y]|_y}{[x]|_x} \kappa|_x.$$

In the Grassmannian, the fixed points are given by the spans of k vectors in the standard basis:

$$e_\lambda = \langle e_{i_1}, ..., e_{i_k} \rangle,$$

where for a partition λ, $i_j = j + \lambda_{k+1-j}$. In $\mathbb{V}^\emptyset$, the fixed points are flags containing vectors in the standard basis and can be parameterized by a partition λ and a permutation $w \in S_k$:

$$e_{\lambda,w} = (\langle e_{i_{w(1)}} \rangle, \langle e_{i_{w(1)}}, e_{i_{w(2)}} \rangle, ..., \langle e_{i_{w(1)}}, ..., e_{i_{w(k)}} \rangle).$$

For certain combinations of λ and w, $S_i \nsubseteq F_{N+i}$ for some $1 \leq i \leq k$, meaning $e_{\lambda,w} \notin \mathbb{V}^\emptyset$, so those are not fixed points in $\mathbb{V}^\emptyset$. With these definitions, $\pi(e_{\lambda,w}) = e_\lambda$ for all appropriate $w \in S_k$ and all $\lambda \leq (N^k)$. Now since $z_i = c_1^T(\mathcal{L}_i^\vee)$, we have that

$$z_j|_{e_{\lambda,w}} = -t_{i_{w(j)}}.$$

With this, we are able to calculate localizations:

Lemma 3.1.2. Given a polynomial $f(z_1, ..., z_k) \in \mathbb{Z}[t_1, ..., t_{N+k}][z_1, ..., z_n]$, representing a class in $H_T^*(\mathbb{V}^\emptyset)$ and a partition $\lambda \leq (N^k)$, the localization of $\pi_*(f)$ at the fixed point corresponding to λ in $\mathrm{Gr}(k, N+k)$ is

$$\pi_*(f(z_1, ..., z_k) \cap [\mathbb{V}^\emptyset])|_{e_\lambda} = \sum_{w \in S_k} \left(\prod_{i=1}^{k-1} \prod_{j=i+1}^{k} \frac{(t_{j-i} - t_{i_{w(i)}})}{(t_{i_{w(j)}} - t_{i_{w(i)}}))} \right) f(-t_{i_{w(1)}}, ..., -t_{i_{w(k)}}),$$

where $i_j = j + \lambda_{k+1-j}$ for $1 \leq j \leq k$.

Proof. Since the localizations of the fixed points are given by the Euler class of the tangent bundle, we look at the tangent bundle in each space. The tangent bundle in the Grassmannian is given by $T_{\mathrm{Gr}(k,N+k)} = \mathcal{S}^\vee \otimes \mathcal{Q}$, where S is the tautological sub-bundle and Q is the tautological quotient bundle. To find the localization of the Euler class of this, we look at the tangent space at a fixed point λ and see how T acts on it. In particular, we know that the Chern roots of $\mathcal{S}^\vee|_{e_\lambda}$ are given by $-t_{i_j}$ for $1 \leq j \leq k$, and the Chern roots of $\mathcal{Q}$ are given by t_j for $j \neq i_p$ for any $1 \leq p \leq k$. With this, we have

$$[e_\lambda]|_{e_\lambda} = c^T_{top}(\mathcal{S}^\vee \otimes \mathcal{Q})|_{e_\lambda} = \prod_{i \in J} \prod_{j \notin J} (t_j - t_i),$$

where $J = \{j + \lambda_{k+1-j} : 1 \leq j \leq k\}$.

The tangent bundle in $\mathbb{V}^\emptyset$ is given by $T_{\mathbb{V}^\emptyset} = \oplus_{i=1}^k (\mathcal{L}_i^\vee \otimes \mathcal{Q}_i)$, since it is a tower of projective bundles and the relative tangent bundle for $p : \mathbb{V}_i^\emptyset \to \mathbb{V}_{i-1}^\emptyset$ is $\mathcal{L}_i^\vee \otimes \mathcal{Q}_i$ (see [16, B.5.8]). At some fixed point $e_{\lambda,w} = (S_1, ..., S_k)$, we have that $\mathcal{L}_i|_{e_{\lambda,w}} = S_i/S_{i-1}$ and $\mathcal{Q}_i|_{e_{\lambda,w}} = F_{N+i}/S_i$. As a result, we have that

$$[e_{\lambda,w}]|_{e_{\lambda,w}} = c^T_{top}(\oplus_{i=1}^k (\mathcal{L}_i^\vee \otimes \mathcal{Q}_i))|_{e_{\lambda,w}} = \prod_{i=1}^k \prod_{j \in J_i} (t_j - t_{i_{w(i)}}),$$

where $J_i = \{j : k+1-i \leq j \leq N+k\} \backslash \{i_{w(j)}, 1 \leq j \leq i\}$, and as before $i_j = j + \lambda_{k+1-j}$. Here note that J_i must have exactly N elements, meaning that $k+1-i \leq i_{w(j)} \leq N+k$ for all $1 \leq j \leq i$ is required for this formula to be correct. This condition is equivalent to the condition that $S_i \subseteq F_{N+i}$, though, which is the condition required for $e_{\lambda,w}$ to be in $\mathbb{V}^\emptyset$, so this will be correct for all valid combinations of w and λ.

To better cancel out terms, we can rewrite the product in $[e_\lambda]|_{e_\lambda}$ as

$$[e_\lambda]|_{e_\lambda} = \prod_{i=1}^{k}\prod_{j\in J_k}(t_j - t_{i_{w(i)}}),$$

since $i_{w(i)}$ will run across all of the indices for the basis vectors in S, and J_k was defined to be all the other basis vectors. With this, we have that

$$\frac{[e_\lambda]|_{e_\lambda}}{[e_{\lambda,w}]|_{e_{\lambda,w}}} = \prod_{i=1}^{k}\frac{\prod_{j\in J_k\setminus J_i}(t_j - t_{i_{w(i)}})}{\prod_{j\in J_i\setminus J_k}(t_j - t_{i_{w(i)}})}$$

To simplify this further, we can specialize to the case where λ satisfies the condition that $e_{\lambda,w}$ is in $\mathbb{V}^\emptyset$ for all $w \in S_k$, which in particular means that i_j satisfies $k \leq i_j \leq N+k$ for all $1 \leq j \leq k$. As a result, $J_k\setminus J_i = \{1, ..., k-i\}$ and $J_i\setminus J_k = \{i_{w(k)}, ..., i_{w(i+1)}\}$. With this, we have that

$$\frac{[e_\lambda]|_{e_\lambda}}{[e_{\lambda,w}]|_{e_{\lambda,w}}} = \prod_{i=1}^{k-1}\prod_{j=i+1}^{k}\frac{(t_{j-i} - t_{i_{w(i)}})}{(t_{i_{w(j)}} - t_{i_{w(i)}})}.$$

With this formula, it happens that if $e_{\lambda,w} \notin \mathbb{V}^\emptyset$, the numerator vanishes. In particular, $e_{\lambda,w} \notin \mathbb{V}^\emptyset$ exactly when there exists an $1 \leq i \leq k$ such that $w(i) < k+1-i$. In such a case, the product $\prod_{j=i+1}^{k}(t_{j-i} - t_{i_{w(i)}})$ vanishes because the t_{j-i} ranges from 1 to $k-i$, which is the possible range for $w(i)$. As a result, we can take the sum over all permutations $w \in S_k$, even if $e_{\lambda,w} \notin \mathbb{V}^\emptyset$, and it will still be correct.

Then we have

$$\begin{aligned}\pi_*(f(z_1,...,z_k)\cap[\mathbb{V}^\emptyset])|_{e_\lambda} &= \sum_{e_{\lambda,w}\in\mathbb{V}^\emptyset}\frac{[e_\lambda]|_{e_\lambda}}{[e_{\lambda,w}]|_{e_{\lambda,w}}}f(-t_{i_{w(1)}},...,-t_{i_{w(k)}})\\ &= \sum_{e_{\lambda,w}\in\mathbb{V}^\emptyset}\prod_{i=1}^{k-1}\prod_{j=i+1}^{k}\frac{(t_{j-i}-t_{i_{w(i)}})}{(t_{i_{w(j)}}-t_{i_{w(i)}})}f(-t_{i_{w(1)}},...,-t_{i_{w(k)}})\\ &= \sum_{w\in S_k}\prod_{i=1}^{k-1}\prod_{j=i+1}^{k}\frac{(t_{j-i}-t_{i_{w(i)}})}{(t_{i_{w(j)}}-t_{i_{w(i)}})}f(-t_{i_{w(1)}},...,-t_{i_{w(k)}}).\end{aligned}$$

□

If we define $x_1,..,x_k$ to be the Chern roots of the dual to the tautological sub-bundle, $\mathcal{S}^\vee$, in the Grassmannian, then for any symmetric polynomial $p(x_1,...,x_k)\in H_T^*(\mathrm{Gr}(k,N+k))$,

$$p(x_1,...,x_k)|_{e_\lambda} = p(-t_{i_1},...,-t_{i_k}),$$

where $i_j = j+\lambda_{k+1-j}$ for $1\le j\le k$. As a result we can modify the localization formula from the above section to express $\pi_*(f(z_1,...,z_k)\cap[\mathbb{V}^\emptyset])$ as a symmetric polynomial in $\mathbb{Z}[t_1,...,t_{N+k}][x_1,...,x_k]^{S_k}$:

Lemma 3.1.3. Given a polynomial $f(z_1,...,z_k;t)\in\mathbb{Z}[t_1,...,t_{N+k}][z_1,...,z_n]$,

$$\pi_*(f(z_1,...,z_k;t)\cap[\mathbb{V}^\emptyset]) = \sum_{w\in S_k}\left(\prod_{i=1}^{k-1}\prod_{j=i+1}^{k}\frac{(t_{j-i}+x_{w(i)})}{(x_{w(i)}-x_{w(j)}))}\right)f(x_{w(1)},...,x_{w(k)};t).$$

Proof. By localizing both sides of the equation at any fixed point, Lemma 3.1.2 gives the same localizations. Since the two classes give the same localization at every fixed point, they are the same class by injectivity of the localization map [19, Theorem 1.2.2]. □

Note that while the expression contains fractions, it is a polynomial. This can be seen by the fact that for any $w \in S_k$,

$$\prod_{i=1}^{k}\prod_{j=i+1}^{k}(x_{w(i)} - x_{w(j)}) = \operatorname{sgn}(w)\prod_{i=1}^{k-1}\prod_{j=i+1}^{k}(x_i - x_j) = \operatorname{sgn}(w)\prod_{1\le i<j\le k}(x_i - x_j).$$

As a result the expression becomes

$$\prod_{1\le i<j\le k}\frac{1}{x_i - x_j}\sum_{w\in S_k}\operatorname{sgn}(w)\left(\prod_{i=1}^{k}\prod_{j=i+1}^{k}(t_{j-i} + x_{w(i)})\right)f(x_{w(1)},...,x_{w(k)}).$$

Since the numerator is an alternating sum over S_k, it is a skew-symmetric polynomial, which means that specializing $x_i = x_j$ for any $i \neq j$ results in 0. This implies $x_i - x_j$ divides it. As a result, the denominator divides the numerator, and so the expression is a polynomial, which will be symmetric because it is the quotient of two skew-symmetric polynomials.

3.2 Factorial Schur Functions

In this section we will use the results of the previous section to show that the factorial Schur functions represent the Schubert classes (Theorem 3.2.1. Specifically, this is the new method by which we re-prove the result that factorial Schur functions represent the equivariant Schubert classes in Grassmannians.

The factorial Schur function is given by [26, eq. (6.3), (6.4)]:

$$s_\lambda(x|a) = \frac{det((x_i|a)^{\lambda_j+\delta_j})_{1\le i,j\le k}}{\prod_{i<j}(x_i - x_j)},$$

where $\delta = (k-1, k-2, ..., 1, 0)$, a is a sequence $(a_i)_{i\in\mathbb{Z}}$, and $(x_i|a)^m = \prod_{j=1}^{m}(x_i + a_j)$. For our purposes, in place of the sequence a we use the sequence $t = (t_i)$, with $t_i = 0$ for $i \le 0$ and $i \ge N + k + 1$ and the usual $t_i = c_1^T(\langle e_i\rangle)$.

We can recover the factorial Schur functions by pushing forward the class of $\mathbb{V}^\lambda$:

Theorem 3.2.1. For any partition $\lambda \leq (N^k)$,

$$[\Omega^\lambda] = \pi_*([\mathbb{V}^\lambda] \cap [\mathbb{V}^\emptyset]) = s_\lambda(x|t).$$

Proof. From the fact that π is a surjective, birational morphism, $\pi_*([\mathbb{V}^\lambda] \cap [\mathbb{V}^\emptyset]) = [\Omega^\lambda]$. Then from Lemma 3.1.1, we have that

$$[\mathbb{V}^\lambda] = \prod_{i=1}^{k} \prod_{k+1-i}^{k-i+\lambda_i} (z_i + t_j).$$

Then we rewrite $\prod_{j=i+1}^{k} (t_{j-i} + x_{w(i)}) = \prod_{j=1}^{k-i} (t_j + x_{w(i)})$, which gives that

$$\left(\prod_{i=1}^{k} \prod_{j=1}^{k-i} (t_j + x_{w(i)})\right) \prod_{i=1}^{k} \prod_{j=k+1-i}^{k-i+\lambda_i} (t_j + x_{w(i)}) = \prod_{i=1}^{k} \prod_{j=1}^{k-i+\lambda_i} (t_j + x_{w(i)}).$$

As a result, plugging $f = \prod_{i=1}^{k} \prod_{j=k+1-i}^{k-i+\lambda_i} (z_i + t_j)$ into Lemma 3.1.3 and simplifying gives

$$\pi_*(\prod_{i=1}^{k} \prod_{j=k+1-i}^{k-i+\lambda_i} (z_i + t_j) \cap [\mathbb{V}^\emptyset]) = \prod_{1 \leq i < j \leq k} \frac{1}{x_i - x_j} \sum_{w \in S_k} \operatorname{sgn}(w) \prod_{i=1}^{k} \prod_{j=1}^{k-i+\lambda_i} (t_j + x_{w(i)}).$$

Then using $\prod_{j=1}^{k-i+\lambda_i} (x_{w(i)} + t_j) = (x_{w(i)}|t)^{k-i+\lambda_i} = (x_{w(i)}|t)^{\delta_i+\lambda_i}$, we obtain

$$\pi_*(\prod_{i=1}^{k} \prod_{j=k+1-i}^{k-i+\lambda_i} (z_i + t_j) \cap [\mathbb{V}^\emptyset]) = \prod_{1 \leq i < j \leq k} \frac{1}{x_i - x_j} \sum_{w \in S_k} \operatorname{sgn}(w) \prod_{i=1}^{k} (x_{w(i)}|t)^{\delta_i+\lambda_i}.$$

By the definition of the determinant $\det(A) = \sum_{w \in S_k} \operatorname{sgn}(w) \prod_{i=1}^{k} a_{w(i),i}$, we have

$$\pi_*(\prod_{i=1}^{k} \prod_{j=k+1-i}^{k-i+\lambda_i} (z_i + t_j) \cap [\mathbb{V}^\emptyset]) = \frac{\det((x_i|t)^{\lambda_j+\delta_j})_{1 \leq i,j \leq k}}{\prod_{1 \leq i < j \leq k} (x_i - x_j)} = s_\lambda(x|t).$$

□

3.3 The Straightening Rule in Cohomology

The formula for factorial Schur functions is expressed in terms of λ where λ is a partition, but the formula can still apply for sequences that are not partitions. The following is a way to express the formula for non-partition sequences in terms of partitions.

Corollary 3.3.1 (The straightening rule)**.** For any sequence of nonnegative integers $\mu = (\mu_1, ..., \mu_k)$, let

$$p_\mu(z,t) = \prod_{i=1}^{k} \prod_{j=k+1-i}^{k-i+\mu_i} (z_i + t_j).$$

Then

1. If μ is a partition, then $\pi_*(p_\mu(z,t) \cap [\mathbb{V}^\emptyset]) = s_\mu(x|t)$.
2. If μ is not a partition and the values $\mu_i + k - i$ are all distinct, then there exists a permutation w and a partition λ such that $\lambda + \delta = (\mu_{w(1)} + \delta_{w(1)}, ..., \mu_{w(k)} + \delta_{w(k)})$, and $\pi_*(p_\mu(z,t) \cap [\mathbb{V}^\emptyset]) = \mathrm{sgn}(w) s_\lambda(x|t)$.
3. If μ is not a partition and the values $\mu_i + k - i$ are not all distinct, then $\pi_*(p_\mu(z,t) \cap [\mathbb{V}^\emptyset]) = 0$.

Proof. Claim (1) is Theorem 3.2.1. By the same process, we have that for any sequence μ,

$$\pi_*(p_\mu(z,t) \cap [\mathbb{V}^\emptyset]) = \frac{\det((x_i|t)^{\mu_j+\delta_j})_{1\le i,j\le k}}{\prod_{1\le i<j\le k}(x_i - x_j)}.$$

From the properties of the determinant, applying a permutation w to the columns of the matrix $((x_i|t)^{\mu_j+\delta_j})_{1\le i,j\le k}$ will multiply the determinant of $\mathrm{sgn}(w)$. In particular, if $\lambda + \delta =$

$w(\mu+\delta)$, then

$$\det((x_i|t)^{\mu_j+\delta_j})_{1\le i,j\le k} = \mathrm{sgn}(w)\det((x_i|t)^{\lambda_i+\delta_i})_{1\le i,j\le k}.$$

If such a permutation w and a partition λ exist, this proves claim (2). Assuming the values $\mu_i+\delta_i$ are all distinct, they can be arranged in descending order, and as a result there exists a permutation w such that $(\mu_{w(1)}+\delta_{w(1)},...,\mu_{w(k)}+\delta_{w(k)})$ is in descending order. Therefore $\mu_{w(i)}+\delta_{w(i)} \ge \mu_{w(i+1)}+\delta_{w(i+1)}+1$, and as a result $\mu_{w(i)}+\delta_{w(i)}-(k-i) \ge \mu_{w(i+1)}+\delta_{w(i+1)}-(k-(i+1))$. So then $\lambda=(\lambda_1,...,\lambda_k)$ with $\lambda_i=\mu_{w(i)}+\delta_{w(i)}-(k-i)$ for $1\le i\le k$ is a partition and $\lambda_i+\delta_i=\mu_{w(i)}+\delta_{w(i)}$, as required. For claim (3), note that if two values of $\mu_i+\delta_i$ are not distinct, then two columns in the matrix are identical, which implies the determinant is 0. □

Claims (2) and (3) together are known as the straightening rule.

Examples 3.3.2.

1. $\pi_*(p_{(1,2)}(z,t)\cap[\mathbb{V}^\emptyset])=0$, since in this case $\mu=(1,2)$, and $\mu+\delta=(1+1,2+0)=(2,2)$, which does not have distinct parts.

2. $\pi_*(p_{(0,2)}(z,t)\cap[\mathbb{V}^\emptyset]) = -s_{(1,1)}(x|-t)$, since in this case $\mu=(0,2)$ and $\mu+\delta=(0,2)+(1,0)=(1,2)$, which can be permuted into $\lambda+\delta=(2,1)$, which results in $\lambda=(2,1)-(1,0)=(1,1)$. Since the permutation required to do this is odd, there is a minus sign.

3.4 The Pushforward in Equivariant K-Theory

This section calculates the class of the structure sheaf of $\mathbb{V}^\lambda$ on $\mathbb{V}^\emptyset$ (Lemma 3.4.3) as well as the localizations of the pushforward of any class (Lemma 3.4.1).

In the Bott-Samelson varieties, the fixed points are

$$e_{\lambda,w} = (\langle e_{i_{w(1)}}\rangle, \langle e_{i_{w(1)}}, e_{i_{w(2)}}\rangle, ..., \langle e_{i_{w(1)}}, ..., e_{i_{w(k)}}\rangle),$$

where $i_j = j + \lambda_{k+1-j}$, for appropriate pairs (λ, w), where $\lambda = (\lambda_1, ..., \lambda_k)$ is a partition with $\lambda_1 \leq N$ and $w \in S_k$. In $\mathrm{Gr}(k, N+k)$, the fixed points are $e_\lambda = \langle e_{i_1}, ..., e_{i_k}\rangle$, where $i_j = j + \lambda_{k+1-j}$. Under the map $\pi : \mathbb{V}^\emptyset \to \mathrm{Gr}(k, N+k)$, $\pi(e_{\lambda,w}) = e_\lambda$ for all $w \in S_k$ such that $e_{\lambda,w} \in \mathbb{V}^\emptyset$. Then, similarly to what was done in Lemma 3.1.2, using the fact that $[\mathcal{O}_{e_\lambda}]|_{e_\lambda} = \lambda_{-1}(T^*_{e_\lambda}\mathrm{Gr}(k, N+k))$ and $[\mathcal{O}_{e_{\lambda,w}}]|_{e_{\lambda,w}} = \lambda_{-1}(T^*_{e_{\lambda,w}}\mathbb{V}^\emptyset)$, we obtain

Lemma 3.4.1. Given a Laurent polynomial in the $\mathcal{L}_i$'s,

$$f(\mathcal{L}_1^{\pm1}, ..., \mathcal{L}_k^{\pm1}; e^{\pm t}) \in \mathbb{Z}[e^{\pm t_1}, ..., e^{\pm t_{N+k}}][\mathcal{L}_1^{\pm1}, ..., \mathcal{L}_k^{\pm1}]$$

representing a class in $K_T(\mathbb{V}^\emptyset)$ and a partition $\lambda \leq (N^k)$, the localization of $\pi_*(f)$ at the fixed point e_λ in $\mathrm{Gr}(k, N+k)$ is

$$\pi_*(f(\mathcal{L}_1^{\pm1}, ..., \mathcal{L}_k^{\pm1}))|_{e_\lambda} = \sum_{w\in S_k} \left(\left(\prod_{i=1}^{k-1} \prod_{j=i+1}^{k} \frac{(1 - e^{t_{i_{w(i)}} - t_{j-i}})}{(1 - e^{t_{i_{w(i)}} - t_{i_{w(j)}}})} \right) f(e^{\mp t_{i_{w(1)}}}, ..., e^{\mp t_{i_{w(k)}}}; e^{\pm t}) \right)$$

for $i_j = j + \lambda_{k+1-j}$.

Proof. Since $[\mathcal{O}_{e_\lambda}]|_{e_\lambda} = \lambda_{-1}(T^\vee_{e_\lambda}(\mathrm{Gr}(k, N+k)))$ and $[\mathcal{O}_{e_{\lambda,w}}]|_{e_{\lambda,w}} = \lambda_{-1}(T^\vee_{e_{\lambda,w}}(\mathbb{V}^\emptyset))$, we have that

$$\frac{[\mathcal{O}_{e_\lambda}]|_{e_\lambda}}{[\mathcal{O}_{e_{\lambda,w}}]|_{e_{\lambda,w}}} = \frac{\prod_{i\in J}\prod_{j\notin J}\lambda_{-1}(\mathbb{C}_{t_i}\otimes\mathbb{C}_{t_j}^\vee)}{\prod_{i=1}^k\prod_{j\in J_i}\lambda_{-1}(\mathbb{C}_{t_{i_{w(i)}}}\otimes\mathbb{C}_{t_j}^\vee)},$$

where $J=\{i_j : 1\le j\le k\}$ and $J_i=\{j : k+1-i\le j\le N+k\}\backslash\{i_{w(j)}, 1\le j\le i\}$ with $i_j=j+\lambda_{k+1-j}$, as in the proof of Lemma 3.1.2. Then with the same cancellations and simplifications from the proof of Lemma 3.1.2, we obtain

$$\frac{[\mathcal{O}_{e_\lambda}]|_{e_\lambda}}{[\mathcal{O}_{e_{\lambda,w}}]|_{e_{\lambda,w}}} = \prod_{i=1}^{k-1}\prod_{j=i+1}^{k}\frac{\lambda_{-1}(\mathbb{C}_{t_{i_{w(i)}}}\otimes\mathbb{C}_{t_j}^\vee)}{\lambda_{-1}(\mathbb{C}_{t_{i_{w(i)}}}\otimes\mathbb{C}_{t_{i_{w(j)}}}^\vee)}.$$

Then from the fact that $\lambda_{-1}(\mathbb{C}_{t_i}\otimes\mathbb{C}_{t_j}^\vee)=1-[\mathbb{C}_{t_i}\otimes\mathbb{C}_{t_j}^\vee]=1-e^{t_i-t_j}$, we obtain

$$\frac{[\mathcal{O}_{e_\lambda}]|_{e_\lambda}}{[\mathcal{O}_{e_{\lambda,w}}]|_{e_{\lambda,w}}} = \prod_{i=1}^{k-1}\prod_{j=i+1}^{k}\frac{(1-e^{t_{i_{w(i)}}-t_{j-i}})}{(1-e^{t_{i_{w(i)}}-t_{i_{w(j)}}})}.$$

Then we have

$$\begin{aligned}\pi_*(f(\mathcal{L}_1^{\pm1},...,\mathcal{L}_k^{\pm1}))|_x &= \sum_{e_{\lambda,w}\in\mathbb{V}^\emptyset}\frac{[\mathcal{O}_{e_\lambda}]|_{e_\lambda}}{[\mathcal{O}_{e_{\lambda,w}}]|_{e_{\lambda,w}}}f(e^{\mp t_{i_{w(1)}}},...,e^{\mp t_{i_{w(k)}}};e^{\pm t})\\ &= \sum_{e_{\lambda,w}\in\mathbb{V}^\emptyset}\prod_{i=1}^{k-1}\prod_{j=i+1}^{k}\frac{(1-e^{t_{i_{w(i)}}-t_{j-i}})}{(1-e^{t_{i_{w(i)}}-t_{i_{w(j)}}})}f(e^{\mp t_{i_{w(1)}}},...,e^{\mp t_{i_{w(k)}}};e^{\pm t})\\ &= \sum_{w\in S_k}\prod_{i=1}^{k-1}\prod_{j=i+1}^{k}\frac{(1-e^{t_{i_{w(i)}}-t_{j-i}})}{(1-e^{t_{i_{w(i)}}-t_{i_{w(j)}}})}f(e^{\mp t_{i_{w(1)}}},...,e^{\mp t_{i_{w(k)}}};e^{\pm t}).\end{aligned}$$

□

To calculate $[\mathcal{O}_{\mathbb{V}^\lambda}]$, we need to use some facts from [16, Appendix B] that can be summarized as the following Lemma.

Lemma 3.4.2. Let X be a projective variety with $p : E, F \to X$ vector bundles and $E\subseteq F$. Then for the quotient bundle $G=F/E$, $[\mathcal{O}_{\mathbb{P}(E)}]=\lambda_{-1}((\mathcal{O}_{\mathbb{P}(F)}(1)\otimes p^*(G))^\vee)\in K_T(\mathbb{P}(F))$

Proof. From the tautological sequence on $\mathbb{P}(F)$, there is a vector bundle map $\mathcal{O}_{\mathbb{P}(F)}(-1) \to p^*F$. When composed with the quotient map $p^*F \to p^*G$, this becomes a map $\mathcal{O}_{\mathbb{P}(F)}(-1) \to p^*G$, and this map is the 0 map exactly when the fiber of $\mathcal{O}_{\mathbb{P}(F)}(-1)$ is included in the fiber of p^*E, which happens when $x \in \mathbb{P}(E) \subseteq \mathbb{P}(F)$. This map corresponds to a section $s \in \mathrm{Hom}(\mathcal{O}_{\mathbb{P}(F)}(-1), p^*(G)) \cong \mathcal{O}_{\mathbb{P}(F)}(1) \otimes p^*G$. This section is regular and its zero locus is $Z(s) = \mathbb{P}(E)$ [16, B.5.6]. Then by [16, B.3.4, (*)], there is an exact sequence

$$0 \to \Lambda^r(\mathcal{O}_{\mathbb{P}(F)}(1) \otimes p^*G)^\vee \to ... \to \Lambda(\mathcal{O}_{\mathbb{P}(F)}(1) \otimes p^*G)^\vee \to \mathcal{O}_{\mathbb{P}(F)} \to \mathcal{O}_{\mathbb{P}(E)} \to 0.$$

This implies

$$[\mathcal{O}_{\mathbb{P}(E)}] - [\mathcal{O}_{\mathbb{P}(F)}] + \sum_{i=1}^{r} (-1)^{i-1} [\Lambda^i(\mathcal{O}_{\mathbb{P}(F)}(1) \otimes p^*G)^\vee] = 0.$$

Then solving for $[\mathcal{O}_{\mathbb{P}(E)}]$ gives

$$\mathcal{O}_{\mathbb{P}(E)}] = \sum_{i=0}^{r} (-1)^i [\Lambda^i(\mathcal{O}_{\mathbb{P}(F)}(1) \otimes p^*G)^\vee] = \lambda_{-1}((\mathcal{O}_{\mathbb{P}(F)}(1) \otimes p^*G)^\vee).$$

□

Lemma 3.4.3. For a partition $\lambda = (\lambda_1, ..., \lambda_k)$ with $\lambda_1 \leq N$,

$$[\mathcal{O}_{\mathbb{V}^\lambda}] = \prod_{i=1}^{k} \prod_{j=k+1-i}^{k-i+\lambda_i} (1 - [\mathcal{L}_i \otimes \mathbb{C}_{t_j}^\vee]) \in K_T(\mathbb{V}^\emptyset).$$

Proof. We proceed by induction on n in $\mathbb{V}_n^\lambda$. The base case, n=0, is a point, which cannot have nonempty, proper subvarieties, so this holds trivially. Suppose

$$[\mathcal{O}_{\mathbb{V}_n^\lambda}] = \prod_{i=1}^{n} \prod_{j=k+1-i}^{k-i+\lambda_i} (1 - [\mathcal{L}_i \otimes \mathbb{C}_{t_j}^\vee]) \in K_T(\mathbb{V}_n^\emptyset).$$

Recall $\mathbb{V}_n^\lambda = \{(S_1, ..., S_n) : \dim(S_i) = i, S_i \subseteq F_{N+i-\lambda_i}\}$, and $\mathbb{V}_{n+1}^\lambda = \mathbb{P}(\mathbb{F}_{N+n+1-\lambda_{n+1}}/(\mathcal{L}_1 \oplus$

$\ldots \oplus \mathcal{L}_n))$, where $\mathcal{F}_{N+n+1-\lambda_{n+1}}$ is the vector bundle on $\mathbb{V}_n^\lambda$ whose fiber is $F_{N+n+1-\lambda_{n+1}}$, and $\mathcal{L}_i$ is the vector bundle whose fiber is S_i/S_{i-1}. Then for $\lambda' = (\lambda_1, \ldots, \lambda_n, 0)$, we have that $\mathbb{V}_{n+1}^\lambda = \mathbb{P}(\mathcal{F}_{N+n+1-\lambda_{n+1}}/(\mathcal{L}_1 \oplus \ldots \oplus \mathcal{L}_n))$ and $\mathbb{V}_{n+1}^{\lambda'} = \mathbb{P}(\mathcal{F}_{N+n+1}/(\mathcal{L}_1 \oplus \ldots \oplus \mathcal{L}_n)$. Since $\mathcal{F}_{N+n+1-\lambda_{n+1}}/(\mathcal{L}_1 \oplus \ldots \oplus \mathcal{L}_n) \subseteq \mathcal{F}_{N+n+1}/(\mathcal{L}_1 \oplus \ldots \oplus \mathcal{L}_n)$ with quotient $G = \mathcal{F}_{N+n+1}/\mathcal{F}_{N+n+1-\lambda_{n+1}}$, by Lemma 3.4.2, we have $[\mathcal{O}_{\mathbb{V}_{n+1}^\lambda}] = \lambda_{-1}((G \otimes \mathcal{L}_{n+1}^\vee)^\vee) \in K_T(\mathbb{V}_{n+1}^{\lambda'})$. Since $F_{N+n+1}/F_{N+n+1-\lambda_{n+1}} = \langle e_{N+k+1-(N+n+1)}, \ldots, e_{N+k-(N+n+1-\lambda_{n+1})}\rangle$, this means

$$[\mathcal{O}_{\mathbb{V}_{n+1}^\lambda}] = \prod_{j=k-n}^{k-n-1+\lambda_{n+1}} (1 - \mathbb{C}_{t_j}^\vee \otimes \mathcal{L}_{n+1}).$$

Then since $\mathbb{V}_n^\lambda \subseteq \mathbb{V}_n^\emptyset$, and both $\mathbb{V}_{n+1}^{\lambda'}$ and $\mathbb{V}_{n+1}^\emptyset$ are projectivizations of the bundle $E = \mathcal{F}_{N+n+1}/(\mathcal{L}_1 \oplus \ldots \oplus \mathcal{L}_n)$ over the previous spaces, $\mathbb{V}_{n+1}^{\lambda'} \subseteq \mathbb{V}_{n+1}^\emptyset$:

$$\begin{array}{ccc} \mathbb{V}_{n+1}^{\lambda'} = \mathbb{P}(E) & \xrightarrow{\iota_2} & \mathbb{V}_{n+1}^\emptyset = \mathbb{P}(E) \\ \downarrow p & & \downarrow p \\ \mathbb{V}_n^\lambda & \xrightarrow{\iota_2} & \mathbb{V}_n^\emptyset \end{array}$$

Furthermore, $[\mathbb{V}_{n+1}^{\lambda'}] = p^*([\mathbb{V}_n^\lambda]) \in K_T(\mathbb{V}_{n+1}^\emptyset)$, where $p : \mathbb{V}_{n+1}^\emptyset \to \mathbb{V}_n^\emptyset$ is the projection map. Now for the inclusion maps $\iota_1 : \mathbb{V}_{n+1}^\lambda \to \mathbb{V}_{n+1}^{\lambda'}$ and $\iota_2 : \mathbb{V}_{n+1}^{\lambda'} \to \mathbb{V}_{n+1}^\emptyset$, we have $\iota = \iota_2 \circ \iota_1$, where $\iota : \mathbb{V}^\lambda \to \mathbb{V}^\emptyset$ is the inclusion map, and so we have

$$\begin{aligned}
\iota_*([\mathcal{O}_{\mathbb{V}_{n+1}^\lambda}]) &= \iota_{2,*}(\iota_{1,*}([\mathcal{O}_{\mathbb{V}_{n+1}^\lambda}])) \\
&= \iota_{2,*}([\mathcal{O}_{\mathbb{V}_{n+1}^{\lambda'}}]\iota_2^*(\prod_{j=k-n}^{k-n-1+\lambda_{n+1}} (1 - \mathbb{C}_{t_j}^\vee \otimes \mathcal{L}_{n+1}))) \\
&= \left(\prod_{i=1}^{n} \prod_{j=k+1-i}^{k-i+\lambda_i} (1 - [\mathcal{L}_i \otimes \mathbb{C}_{t_j}^\vee]) \in K_T(\mathbb{V}_n^\emptyset)\right) \prod_{j=k-n}^{k-n-1+\lambda_{n+1}} (1 - \mathbb{C}_{t_j}^\vee \otimes \mathcal{L}_{n+1}) \\
&= \prod_{i=1}^{n+1} \prod_{j=k+1-i}^{k-i+\lambda_i} (1 - [\mathcal{L}_i \otimes \mathbb{C}_{t_j}^\vee]) \in K_T(\mathbb{V}_{n+1}^\emptyset),
\end{aligned}$$

from the induction hypothesis. This completes the induction step, and so completes the proof. □

3.5 Factorial Grothendieck Polynomials

This section establishes the correspondence between the classes of vector bundles and the variables used in polynomials, uses that correspondence to express the pushforward of any class in terms of polynomials (Lemma 3.5.1), uses that to show that the factorial Grothendieck polynomials represent Schubert classes (Theorem 3.5.2).

Grothendieck polynomials represent the classes of the structure sheaves of Schubert varieties in flag varieties [25]. They are defined recursively using divided difference operators. The symmetric group S_n acts on the polynomial ring $\mathbb{Z}[X_1, ..., X_n]$ by permuting the variables: $w(f(X_1, ..., X_n)) = f(X_{w(1)}, ..., X_{w(n)})$. With $s_i \in S_n$ being the simple reflection which maps i to $i+1$, $i+1$ to i, and fixes all other elements, define the operators ∂_i, δ_i for $1 \leq i \leq n-1$ by

$$\partial_i(f) = \frac{f - s_i(f)}{X_i - X_{i+1}}, \delta_i(f) = \partial_i((1 - X_{i+1})f). \tag{3.1}$$

Then for $w_0 \in S_n$ being the longest permutation, given by $w_0(i) = n + 1 - i$ for $1 \leq i \leq n$, the Grothendieck polynomial for w_0 is $\mathcal{G}_{w_0}(X) = X_1^{n-1} X_2^{n-2} ... X_{n-1}^1$. Then the Grothendieck polynomials for other $w \in S_n$ are defined recursively by $\delta_i \mathcal{G}_w(X) = \mathcal{G}_{ws_i}(X)$ for $ws_i < w$ in the Bruhat order. The equivariant version of these, double Grothendieck polynomials, use two sets of variables, $X_1, ..., X_n$ and $T_1, ..., T_n$, and start with $\mathcal{G}^T_{w_0}(X, T) = \prod_{i+j \leq n}(X_i + T_j - X_i T_j)$, then are recursively defined with the same operators as before, with the permutations acting trivially on the T variables [24, 25]. Note that $\mathcal{G}^T_w(X, 0) = \mathcal{G}_w(X)$ for all $w \in S_n$.

When $w \in S_n$ is k-Grassmannian, meaning $w(i) > w(i+1)$ for all $i \neq k$, it corresponds to the partition

$$\lambda = (w(k) - k, w(k-1) - k + 1, ..., w(1) - 1),$$

and $\mathcal{G}_\lambda^T(X,T)$ is given by [21]:

$$\mathcal{G}_\lambda^T(X,T) = \frac{\det((X_i|T)^{\lambda_j+k-j}(1-X_i)^{j-1})}{\prod_{1\leq i<j\leq k}(X_i - X_j)},$$

where $(X_i|T)^r = \prod_{j=1}^r (X_i + T_j - X_iT_j)$.

To realize the polynomials geometrically, define X_i such that $\lambda_{-1}(\mathcal{S}) = \prod_{i=1}^k X_i$, and T_i as $T_i = 1 - [\mathbb{C}_{t_i}^\vee]$. In the Bott-Samelson variety, we define $Z_i = 1 - [\mathcal{L}_i]$ for $1 \leq i \leq k$. Then, as a K-theory analog to Lemma 3.1.3, we have:

Lemma 3.5.1. For a polynomial $f(Z,T) \in \mathbb{Z}[Z_1, ..., Z_k; T_1, ..., T_n]$ for $n = N + k$ with the above definitions of Z_i and T_i,

$$\pi_*(f(Z,T)) = \sum_{w\in S_k}\left(\prod_{i=1}^{k-1}\prod_{j=i+1}^{k}\frac{(X_{w(i)} + T_{j-i} - X_{w(i)}T_{j-i})}{1-(1-X_{w(i)})(1-X_{w(j)})^{-1}}\right) f(X_{w(1)}, ..., X_{w(k)}; T).$$

Proof. Since $\mathcal{S}|_{e_\lambda} = \langle e_{i_1}, ..., e_{i_k}\rangle$ for $i_j = j + \lambda_{k+1-j}$ for $1 \leq j \leq k$, we have that

$$\prod_{j=1}^k x_j|_{e_\lambda}\lambda_{-1}(\mathcal{S})|_{e_\lambda} = \prod_{j=1}^k (1 - e^{-t_{i_j}}).$$

As a result, we can say $X_j|_{e_\lambda} = 1 - e^{t_{i_j}}$. Now since $T_j = 1 - [\mathbb{C}_{t_j}^\vee]$, we have $T_j|_{e_\lambda} = 1 - e^{-t_j}$. With this,

$$\begin{aligned}(X_{w(i)} + T_{j-i} - X_{w(i)}T_{j-i})|_{e_\lambda} &= (1 - e^{t_{i_{w(i)}}}) + (1 - e^{-t_{j-i}}) - (1 - e^{t_{i_{w(i)}}})(1 - e^{-t_{j-i}}) \\ &= 1 + 1 - 1 - e^{t_{i_{w(i)}}} - e^{-t_{j-i}} + e^{t_{i_{w(i)}}} + e^{-t_{j-i}} - e^{t_{i_{w(i)}}}e^{-t_{j-i}} \\ &= 1 - e^{t_{i_{w(i)}} - t_{j-i}}.\end{aligned}$$

So then the localization of the right side of the equation at a fixed point e_λ is given by

$$\sum_{w\in S_k}\left(\prod_{i=1}^{k-1}\prod_{j=i+1}^{k}\frac{(X_{w(i)}+T_{j-i}-X_{w(i)}T_{j-i})}{1-(1-X_{w(i)})(1-X_{w(j)})^{-1}}\right)f(X_{w(1)},...,X_{w(k)};T)|_{e_\lambda}$$
$$=\sum_{w\in S_k}\left(\prod_{i=1}^{k-1}\prod_{j=i+1}^{k}\frac{(1-e^{t_{i_{w(i)}}-t_{j-i}})}{(1-e^{t_{i_{w(i)}}-t_{i_{w(j)}}})}\right)f(1-e^{t_{i_{w(1)}}},...,1-e^{t_{i_{w(k)}}};1-e^{-t})$$

Using the identification $Z_i=1-[\mathcal{L}_i]$, we have that $f(Z,T)=f(1-[\mathcal{L}_1],...,1-[\mathcal{L}_k],1-e^{-t})$. As a result, we can apply Lemma 3.4.1 to the left side of the equation and obtain

$$\pi_*(f(Z,T))|_{e_\lambda}=\sum_{w\in S_k}\left(\prod_{i=1}^{k-1}\prod_{j=i+1}^{k}\frac{(1-e^{t_{i_{w(i)}}-t_{j-i}})}{(1-e^{t_{i_{w(i)}}-t_{i_{w(j)}}})}\right)f(1-e^{t_{i_{w(1)}}},...,1-e^{t_{i_{w(k)}}};1-e^{-t}).$$

So then the localizations match at every fixed point and so by the injectivity of the localization map, the two classes are the same. □

Since the factorial Grothendieck polynomials represent the classes of the structure sheaves of the corresponding Schubert varieties, we can recover the factorial Grothendieck polynomials as the pushforward of $\mathbb{V}^\emptyset$.

Theorem 3.5.2. For a partition $\lambda=(\lambda_1,...,\lambda_k)$ with corresponding k-Grassmannian permutation w given by $w(i)=i+\lambda_{k+1-i}$ for $1\le i\le k$ and the remaining values arranged in ascending order,

$$[\mathcal{O}_{\Omega^\lambda}]=\pi_*([\mathcal{O}_{\mathbb{V}^\lambda}])=\mathcal{G}_w^T(X;T).$$

Proof. Because Ω^λ has only rational singularities and $\pi:\mathbb{V}^\lambda\to\Omega^\lambda$ is a desingularization, we have that $\pi_*([\mathcal{O}_{\mathbb{V}^\lambda}])=[\mathcal{O}_{\Omega^\lambda}]$. Using Lemma 3.4.3, since $1-[\mathcal{L}_i\otimes\mathbb{C}_{t_j}^\vee]=1-(1-Z_i)(1-T_j)=Z_i+T_j-Z_iT_j$, we have

$$\pi_*([\mathcal{O}_{\mathbb{V}^\lambda}])=\pi_*(\prod_{i=1}^{k}\prod_{k+1-i}^{k-i+\lambda_i}(Z_i+T_j-Z_iT_j)).$$

Then applying Lemma 3.5.1, we have

$$\pi_*([\mathcal{O}_{\mathbb{V}^\lambda}])$$
$$= \sum_{w\in S_k}\left(\prod_{i=1}^{k-1}\prod_{j=i+1}^{k}\frac{(X_{w(i)}+T_{j-i}-X_{w(i)}T_{j-i})}{1-(1-X_{w(i)})(1-X_{w(j)})^{-1}}\right)\left(\prod_{i=1}^{k}\prod_{j=k+1-i}^{k-i+\lambda_i}(X_{w(i)}+T_j-X_{w(i)}T_j)\right).$$

Making the simplification $\frac{1}{1-(1-X_{w(i)})(1-X_{w(j)})^{-1}} = \frac{(1-X_{w(j)})}{(1-X_{w(j)})-(1-X_{w(i)})} = \frac{1-X_{w(j)}}{X_{w(i)}-X_{w(j)}}$ yields

$$\pi_*([\mathcal{O}_{\mathbb{V}^\lambda}]) = \sum_{w\in S_k}\left(\prod_{i=1}^{k-1}\prod_{j=i+1}^{k}\frac{(1-X_{w(j)})(X_{w(i)}+T_{j-i}-X_{w(i)}T_{j-i})}{X_{w(i)}-X_{w(j)}}\right)\cdot \left(\prod_{i=1}^{k}\prod_{j=k+1-i}^{k-i+\lambda_i}(X_{w(i)}+T_j-X_{w(i)}T_j)\right)$$

Reindexing the products to put the $(X_{w(i)}+T_j-X_{w(i)}T_j)$ terms together, we have

$$\pi_*([\mathcal{O}_{\mathbb{V}^\lambda}])$$
$$= \sum_{w\in S_k}\left(\prod_{i=1}^{k-1}\prod_{j=i+1}^{k}\frac{1-x_{w(j)}}{X_{w(i)}-X_{w(j)}}\right)\left(\prod_{i=1}^{k}\prod_{j=1}^{k-i+\lambda_i}(X_{w(i)}+T_j-X_{w(i)}T_j)\right).$$

From here we make several simplifications, starting with

$$\prod_{i=1}^{k-1}\prod_{j=i+1}^{k}\frac{1}{X_{w(i)}-X_{w(j)}} = \operatorname{sgn}(w)\prod_{1\le i<j\le k}\frac{1}{X_i-X_j}.$$

Further,

$$\begin{aligned}\prod_{i=1}^{k-1}\prod_{j=i+1}^{k}(1-X_{w(j)}) &= \prod_{1\le i<j\le k}(1-X_{w(j)})\\ &= \prod_{j=2}^{k}(1-X_{w(j)})^{j-1}\\ &= \prod_{i=1}^{k}(1-X_{w(i)})^{i-1}.\end{aligned}$$

Lastly,

$$\prod_{i=1}^{k}\prod_{j=1}^{k-i+\lambda_i}(X_{w(i)}+T_j-X_{w(i)}T_j)=\prod_{i=1}^{k}(X_{w(i)}|T)^{k-i+\lambda_i}.$$

Putting all of these together gives

$$\pi_*([\mathcal{O}_{\mathbb{V}^\lambda}])=\left(\prod_{1\le i<j\le k}\frac{1}{X_i-X_j}\right)\sum_{w\in S_k}\operatorname{sgn}(w)\prod_{i=1}^{k}(X_{w(i)}|T)^{k-i+\lambda_i}(1-X_{w(i)})^{i-1}.$$

Since $\det(a_{i,j})=\sum_{w\in S_k}\operatorname{sgn}(w)\prod_{i=1}^{k}a_{w(i),i}$, this becomes

$$\pi_*([\mathcal{O}_{\mathbb{V}^\lambda}])=\frac{\det((X_i|T)^{\lambda_j+k-j}(1-X_i)^{j-1})}{\prod_{1\le i<j\le k}(X_i-X_j)},$$

which matches the factorial Grothendieck polynomial. □

3.6 The Straightening Rule in K-Theory

This allows us to deal with $P_\lambda(Z,T)$ when λ is a partition, but if λ is a composition, there is a straightening rule to express the pushforward as a combination of pushforwards of $P_\mu(Z,T)$ for partitions μ. This was proved in the non-equivariant case by Buch [11, Lemma 3.2] and in the equivariant case by Gourbounov and Korff [18, Corollary 2.4].

Corollary 3.6.1. For any composition $\lambda = (\lambda_1, ..., \lambda_k)$, let

$$P_\lambda(Z,T) = \prod_{i=1}^{k} \prod_{j=k+1-i}^{k-i+\lambda_i} (Z_i + T_j - Z_i T_j).$$

Then if λ is a partition with corresponding k-Grassmannian permutation w, $\pi_*(P_\lambda(Z,T)) = \mathcal{G}_w^T(X,T)$. If λ is not a partition, then

$$\pi_*(P_\lambda(Z,T)) = \sum_{j=\lambda_i+1}^{\lambda_{i+1}} \frac{1-T_{j+k-i}}{1-T_{\lambda_{i+1}+k-i}} \pi_*(P_{\mu^{(j)}}(Z,T)) - \sum_{j=\lambda_i+1}^{\lambda_{i+1}-1} \frac{1-T_{j+k-i}}{1-T_{\lambda_{i+1}+k-i}} \pi_*(P_{\nu^{(j)}}(Z,T))$$

in $K_T(\mathrm{Gr}(k,n))$, where $\mu^{(j)} = (\lambda_1, ..., \lambda_{i+1}, j, ..., \lambda_k)$ and $\nu^{(j)} = (\lambda_1, ..., \lambda_{i+1} - 1, j, ..., \lambda_k)$ for any $1 \leq i \leq k$.

Proof. The claim about partitions is Theorem 3.5.2. The other claim boils down to a direct computation with the determinantal formulae. For simplicity, we will look at the computation for partitions with two parts (since the statement only involves switching two parts, the rest of it stays constant). It centers around the identity

$$\begin{aligned}
&\pi_*(P_{(a,b)}(Z,T)) \\
&= \pi_*(P_{(a+1,b)}(Z,T)) + \frac{1-T_{a+2}}{1-T_{b+1}} \pi_*(P_{(b,a+1)}(Z,T)) - \frac{1-T_{a+2}}{1-T_{b+1}} \pi_*(P_{(b-1,a+1)}(Z,T)).
\end{aligned}$$

The indices on the T variables here depend on the power $(X|T)^{\lambda_i+k-i}$ in the determinant, and in this simplified case we have $k=2$ and $i=1$. Therefore when applying this to the general situation, it would need to be $T_{a+1+k-i}$ and T_{b+k-i} instead of T_{a+2} and T_{b+1}. By successively applying this, the result is obtained. The claim is equivalent to

$$\begin{aligned}
&(1-T_{b+1})\pi_*(P_{(a,b)}(Z,T)) + (1-T_{a+2})\pi_*(P_{(b-1,a+1)}(Z,T)) \\
&= (1-T_{b+1})\pi_*(P_{(a+1,b)}(Z,T)) + (1-T_{a+2})\pi_*(P_{(b,a+1)}(Z,T)).
\end{aligned}$$

From Theorem 3.5.2, we have $\pi_*(P_{(\lambda_1,\lambda_2)}(Z,T)) = \frac{det((X_i|T)^{\lambda_j+2-j}(1-X_i)^{j-1})}{X_1-X_2}$, and so by multiplying through by the common denominator the identity is equivalent to

$$\begin{aligned}
&(1-T_{b+1})((X_1|T)^{a+1}(X_2|T)^b(1-X_2)-(X_1|T)^b(X_2|T)^{a+1}(1-X_1))\\
&+(1-T_{a+2})((X_1|T)^b(X_2|T)^{a+1}(1-x_2)-(X_1|T)^{a+1}(X_2|T)^b(1-X_1))\\
&=(1-T_{b+1})((X_1|T)^{a+2}(X_2|T)^b(1-X_2)-(X_1|T)^b(X_2|T)^{a+2}(1-X_1))\\
&+(1-T_{a+2})((X_1|T)^{b+1}(X_2|T)^{a+1}(1-X_2)-(X_1|T)^{a+1}(X_2|T)^{b+1}(1-X_1)).
\end{aligned}$$

For notational simplicity, we will use $G_T(a,b) = (X_1|T)^{a+1}(X_2|T)^b(1-X_2)-(X_1|T)^b(X_2|T)^{a+1}(1-X_1)$, which makes the left side of the equation

$$\begin{aligned}
&(1-T_{b+1})G_T(a,b)+(1-T_{a+2})G_T(b-1,a+1)\\
&=G_T(a,b)-T_{b+1}G_T(a,b)+G_T(b-1,a+1)-T_{a+2}G_T(b-1,a+1).
\end{aligned}$$

Expanding yields

$$\begin{aligned}
&(X_1|T)^{a+1}(X_2|T)^b-X_2(X_1|T)^{a+1}(X_2|T)^b\\
&-(X_1|T)^b(X_2|T)^{a+1}+X_1(X_1|T)^b(X_2|T)^{a+1}-T_{b+1}G_T(a,b)\\
&+(X_1|T)^b(X_2|T)^{a+1}-X_2(X_1|T)^b(X_2|T)^{a+1}\\
&-(X_1|T)^{a+1}(X_2|T)^b+X_1(X_1|T)^{a+1}(X_2|T)^b-T_{a+2}G_T(b-1,a+1).
\end{aligned}$$

Now since $X_i = (X_i+T_j-X_iT_j)-T_j(1-X_i)$, after some cancellations this simplifies to

$$\begin{aligned}
&-(X_1|T)^{a+1}(X_2|T)^{b+1}+T_{b+1}(X_1|T)^{a+1}(X_2|T)^b(1-X_2)\\
&+(X_1|T)^{b+1}(X_2|T)^{a+1}-T_{b+1}(X_1|T)^b(X_2|T)^{a+1}(1-X_1)-T_{b+1}G_T(a,b)\\
&-(X_1|T)^b(X_2|T)^{a+2}+T_{a+2}(X_1|T)^b(X_2|T)^{a+1}(1-X_2)\\
&+(X_1|T)^{a+2}(X_2|T)^b-T_{a+2}(X_1|T)^{a+1}(X_2|T)^b(1-X_1)-T_{a+2}G_T(b-1,a+1)
\end{aligned}$$

Then we have $T_{b+1}G_T(a,b) = T_{b+1}((X_1|T)^{a+1}(X_2|T)^b(1-X_2) - (X_1|T)^b(X_2|T)^{a+1}(1-X_1))$ and $T_{a+2}G_T(b-1,a+1) = T_{a+2}((X_1|T)^b(X_2|T)^{a+1}(1-X_2) - (X_1|T)^{a+1}(X_2|T)^b(1-X_1))$, so these terms cancel and the left side of the equation becomes

$$-(X_1|T)^{a+1}(X_2|T)^{b+1} + (X_1|T)^{b+1}(X_2|T)^{a+1} - (X_1|T)^b(X_2|T)^{a+2} + (X_1|T)^{a+2}(X_2|T)^b.$$

A similar process on the other side gives that the right side of the claim is also equal to this. So then the claim holds, and as a result the proof is complete. □

Chapter 4

Stability of Segre-MacPherson Classes Under Pullbacks

In this chapter we embed Grassmannians into larger Grassmannians and show the Segre-MacPherson classes are stable under pullbacks of these embeddings (Theorem 4.0.4).

For $N_2 \geq N_1$ and $k_2 \geq k_1$, $\mathrm{Gr}(k_1, N_1 + k_1)$ can be embedded into $\mathrm{Gr}(k_2, N_2 + k_2)$ by repeated applications of embeddings of the form

$$\iota_1 : \mathrm{Gr}(k, N + k) \to \mathrm{Gr}(k + 1, N + (k + 1)) \text{ and } \iota_2 : \mathrm{Gr}(k, N + k) \to \mathrm{Gr}(k, (N + 1) + k).$$

To define ι_1, embed $\mathbb{C}^{N+k}$ into $\mathbb{C}^{N+k+1}$ by

$$(x_1, ..., x_{N+k}) \mapsto (x_1, ..., x_{N+k}, 0),$$

then for a subspace $V \in \mathrm{Gr}(k, N + k)$, define

$$\iota_1(V) = V \oplus \langle e_{N+k+1} \rangle,$$

where V is considered as a subspace of $\mathbb{C}^{N+k+1}$ by the above embedding and $e_{N+k+1} = (0, ..., 0, 1)$ is the standard basis vector.

With the embedding $\iota : \mathbb{C}^{N+k} \to \mathbb{C}^{N+k+1}$ given by

$$\iota(x_1, ..., x_{N+k}) = (0, x_1, ..., x_{N+k}),$$

for a subspace $V \in \mathrm{Gr}(k, N+k)$ define

$$\iota_2(V) = \iota(V).$$

For later proofs in this section, we will need to use two different kinds of Schubert cells: $\Omega^{\lambda,\circ}$ and Ω°_λ. The Schubert varieties will always be defined in terms of the opposite flag $F_i = \langle e_{N+k+1-i}, ..., e_{N+k}\rangle$. For a partition λ, $\Omega^{\lambda,\circ}$ is the same as defined in section 1 and has codimension $|\lambda|$, while Ω°_λ has dimension $|\lambda|$ and is defined as follows:

$$\Omega^\circ_\lambda = \{S \in \mathrm{Gr}(k, N+k) : \dim(S \cap F_{\lambda_{k+1-i}+i}) = i, \dim(S \cap F_{\lambda_{k+1-i}+i-1}) = i-1, 1 \leq i \leq k\}.$$

Lemma 4.0.1. With the above definitions, for a paritition $\lambda = (\lambda_1, ..., \lambda_k)$ with $\lambda_1 \leq N$, and the map $\iota_1 : \mathrm{Gr}(k, N+k) \to \mathrm{Gr}(k+1, N+k+1)$,

$$\iota_1(\Omega^\circ_\lambda) = \Omega^\circ_\lambda,$$

where on the right, Ω°_λ is the Schubert cell in $\mathrm{Gr}(k+1, N+k+1)$ for the partition $(\lambda_1, ..., \lambda_k, 0)$.

Proof. Denote by $(E_1, ..., E_{N+k})$ the opposite flag in $\mathbb{C}^{N+k}$ and by $(F_1, ..., F_{N+k+1})$ the opposite flag in $\mathbb{C}^{N+k+1}$. Suppose $S \in \Omega^\circ_\lambda \subseteq \mathrm{Gr}(k, N+k)$. Then $\dim(S \cap E_{\lambda_{k+1-i}+i}) = i$ for $1 \leq i \leq k$. Now observe that $F_{i+1} = E_i \oplus \langle e_{N+k+1}\rangle$ for $1 \leq i \leq k$, since $\langle e_{(N+k+1)-(i+1)}, ..., e_{N+k+1}\rangle =$

$\langle e_{N+k-i}, ..., e_{N+k}\rangle \oplus \langle e_{N+k+1}\rangle$. Then

$$\begin{aligned}\iota_1(S) \cap F_{\lambda_{k+2-i}+i} &= (S \oplus \langle e_{N+k+1}\rangle) \cap (E_{\lambda_{k+1-(i-1)}+(i-1)} \oplus \langle e_{N+k+1}\rangle) \\ &= (S \cap E_{\lambda_{k+1-(i-1)}+(i-1)}) \oplus \langle e_{N+k+1}\rangle.\end{aligned}$$

Therefore $\dim(\iota_1(S) \cap F_{\lambda_{k+2-i}+i}) \geq (i-1)+1 = i$ for $2 \leq i \leq k+1$. For $i = 1$, $\lambda_{k+1} = 0$, and so the condition is $\dim(\iota_1(S) \cap F_1) = 1$, which holds because $(S \oplus \langle e_{N+k+1}\rangle \cap F_1) = F_1$. Therefore $\iota_1(S) \in \Omega_\lambda$. Then $\iota_1(\Omega_\lambda^\circ) \subseteq \Omega_\lambda^\circ$.

Now suppose $S \in \Omega_\lambda^\circ \subseteq \mathrm{Gr}(k+1, N+k+1)$. Therefore $\dim(S \cap F_{\alpha_{k+2-i}+i}) \geq i$, and since $\alpha_{k+1} = 0$, the condition when $i = 1$ gives $\dim(S \cap F_1) = 1$, which implies $F_1 \subseteq S$. Then the dimension of $S' = S \cap \mathbb{C}^{N+k}$ is k and $\iota_2(S') = S$. Now since $E_i = F_{i+1} \cap \mathbb{C}^{N+k}$,

$$S' \cap E_{\lambda_{k+1-1}+i} = (S \cap F_{\lambda_{k+2-(i+1)}+(i+1)}) \cap \mathbb{C}^{N+k}.$$

So then $\dim(S' \cap E_)) = \dim(S \cap F_{\lambda_{k+2-(i+1)}+i+1}) - 1 = i + 1 - 1 = i$, meaning $S' \in \Omega_\lambda^\circ$, and so $S \in \iota_1(\Omega_\lambda^\circ)$, meaning $\Omega_\lambda^\circ \subset \iota_1(\Omega_\lambda^\circ)$. Therefore $\iota_1(\Omega_\lambda^\circ) = \Omega_\lambda^\circ$. □

Lemma 4.0.2. For a partition $\lambda = (\lambda_1, ..., \lambda_k)$ with $\lambda_1 \leq N$, and the map $\iota_2 : \mathrm{Gr}(k, N+k) \to \mathrm{Gr}(k, N+1+k)$,

$$\iota_2(\Omega_\lambda^\circ) = \Omega_\lambda^\circ,$$

where on the right, Ω_λ° is the Schubert cell in $\mathrm{Gr}(k, N+1+k)$ for the partition λ.

Proof. With E_i and F_i denoting the same thing they did in the proof for the previous lemma, suppose $S \in \Omega_\lambda \subseteq \mathrm{Gr}(k, N+k)$. In this case we have $\iota(E_i) = F_i$, and so

$$\iota_2(S) \cap F_{\lambda_{k+1-i}+i} = \iota(S) \cap \iota(E_i) = \iota(S \cap E_{\lambda_{k+1-i}+i}).$$

Therefore $\dim(\iota_2(S)\cap F_{\lambda_{k+1-i}+i}) = \dim(S\cap E_{\lambda_{k+1-i}+i}) = i$ because ι is an embedding and so preserves dimension. So then $\iota_2(S)\in\Omega^\circ_\lambda$.

Now suppose $S\in\Omega^\circ_\lambda\subseteq \mathrm{Gr}(k,N+1+k)$. Since $\lambda_1\leq N$, the Schubert variety condition for $i=k$ is $\dim(S\cap F_{\alpha_1+k})=k$, which implies $S\subseteq F_{\alpha_1+k}$ because S is k-dimensional. Therefore $S\subseteq F_{N+k}$, which mean that $S\subseteq \iota(\mathbb{C}^{N+k})$, and so every element of S is in the image of ι, which means S is in the image of ι_2. Since ι_2 is injective, there is a unique S' such that $\iota_2(S')=S$. Then since

$$\dim(S'\cap E_{\lambda_{k+1-i}+i}) = \dim(\iota(S'\cap E_{\lambda_{k+1-i}+i}) = \dim(S\cap F_{\lambda_{k+1-i}+i}) = i,$$

$S'\in\Omega_\lambda$, and so $S\in\iota_2(\Omega^\circ_\lambda)$. Therefore $\iota_2(\Omega^\circ_\lambda)=\Omega^\circ_\lambda$. □

Lemma 4.0.3. For a partition $\lambda=(\lambda_1,...,\lambda_k)$ with $\lambda_1\leq N$ and the maps

$$\iota_1:\mathrm{Gr}(k,N+k)\to\mathrm{Gr}(k+1,N+k+1) \text{ and } \iota_2:\mathrm{Gr}(k,N+k)\to\mathrm{Gr}(k,N+1+k),$$

we have

$$\iota_1^*(s_{\mathrm{SM}}(\Omega^{\lambda,\circ})) = s_{\mathrm{SM}}(\Omega^{\lambda,\circ}) \text{ and } \iota_2^*(s_{\mathrm{SM}}(\Omega^{\lambda,\circ})) = s_{\mathrm{SM}}(\Omega^{\lambda,\circ}).$$

Proof. [4, Theorem 9.4] implies that

$$\int_{\mathrm{Gr}(k,N)} c_{\mathrm{SM}}(\Omega^\circ_\lambda)\cap s_{\mathrm{SM}}(\Omega^{\mu,\circ}) = \delta_{\lambda\mu}$$

for any partitions $\lambda,\mu\leq(N^k)$. Since the Schubert classes form a basis of the cohomology ring, these conditions are sufficient to determine the SSM class of the Schubert cell. Now for $\lambda \leq (N^k)$, we have

$$\int_{\mathrm{Gr}(k,N+k)} c_{\mathrm{SM}}(\Omega_\mu^\circ) \cap \iota_1^*(\Omega^{\lambda,\circ}) = \int_{\mathrm{Gr}(k+1,N+k+1)} \iota_1^*(c_{\mathrm{SM}}(\Omega_\mu^\circ)) \cap s_{\mathrm{SM}}(\Omega^{\lambda,\circ}).$$

By functoriality,

$$\iota_1^*(c_{\mathrm{SM}}(\Omega_\mu^\circ)) = c_{\mathrm{SM}}(\iota_1(\Omega_\mu^\circ)),$$

and so

$$\int_{\mathrm{Gr}(k,N+k)} c_{\mathrm{SM}}(\Omega_\mu^\circ) \cap \iota_1^*(s_{\mathrm{SM}}(\Omega^{\lambda,\circ})) = \int_{\mathrm{Gr}(k+1,N+k+1)} c_{\mathrm{SM}}(\Omega_\mu^\circ) \cap s_{\mathrm{SM}}(\Omega^{\lambda,\circ}) = \delta_{\mu\lambda}$$

for all $\mu \leq (N^k)$ by Lemma 4.0.1. In a similar fashion by Lemma 4.0.2,

$$\int_{\mathrm{Gr}(k,N+k)} c_{\mathrm{SM}}(\Omega_\mu^\circ) \cap \iota_2^*(s_{\mathrm{SM}}(\Omega^{\lambda,\circ}) = \int_{\mathrm{Gr}(k,N+1+k)} c_{\mathrm{SM}}(\Omega_\mu^\circ) \cap s_{\mathrm{SM}}(\Omega^{\lambda,\circ}) = \delta_{\mu\lambda}.$$

Since these conditions uniquely determine the SSM classes, this gives that $\iota_1^*(s_{\mathrm{SM}}(\Omega^{\lambda,\circ})) = s_{\mathrm{SM}}(\Omega^{\lambda,\circ})$ and $\iota_2^*(s_{\mathrm{SM}}(\Omega^{\lambda,\circ})) = s_{\mathrm{SM}}(\Omega^{\lambda,\circ})$. □

These embeddings ι_1 and ι_2 can be composed with each other to increase k or N by more than one, and it should be noted that the maps commute. For example, when mapping $\mathrm{Gr}(2,4)$ to $\mathrm{Gr}(3,6)$, $\iota_1 \circ \iota_2 = \iota_2 \circ \iota_1$. In particular, each map takes some $V \subset \mathbb{C}^4$ spanned by (x_1, x_2, x_3, x_4) and (y_1, y_2, y_3, y_4) to the space spanned by $(0, x_1, x_2, x_3, x_4, 0)$, $(0, y_1, y_2, y_3, y_4, 0)$, and $(0, 0, 0, 0, 0, 1)$ in $\mathbb{C}^6$.

Theorem 4.0.4. For $k_1, k_2, N_1, N_2 \in \mathbb{N}$ with $k_1 \leq k_2$ and $N_1 \leq N_2$, the embedding $\iota' : \mathrm{Gr}(k_1, N_1 + k_1) \to \mathrm{Gr}(k_2, N_2 + k_2)$ constructed by composing copies of ι_1 and ι_2 satisfies

$$\iota'^*(s_{\mathrm{SM}}(\Omega^{\lambda,\circ})) = s_{\mathrm{SM}}(\Omega^{\lambda,\circ})$$

for all $\lambda \leq (N_1^{k_1})$.

Proof. The mapping is composed of several steps, and at each step, the SSM class is preserved by Lemma 4.0.3. □

Chapter 5

Segre-MacPherson Classes in the Grassmannian

In this chapter we use the properties of the Chern-Schwartz-MacPherson class to calculate it for Schubert cells in the Grassmannian and by extension calculate the Segre-MacPherson class in T-equivariant cohomology (Theorem 5.0.1). Then we use this to calculate the class in ordinary cohomology to note the similarity between this result and a result of Feher and Rimanyi from [14].

By [3, Proposition 2.10], the complement in $\mathbb{V}^\lambda$ of the preimage of the Schubert cell under π is a simple normal crossing divisor whose components are D_i for $1 \leq i \leq k$, and so the CSM class of the Schubert cell embedded in $\mathbb{V}^\lambda$ is

$$c_{\mathrm{SM}}(\pi^{-1}(\Omega^{\lambda,\circ})) = \frac{c^T(T\mathbb{V}^\lambda)}{\prod_{i=1}^k (1+D_i)} \cap [\mathbb{V}^\lambda].$$

Then pushing this forward into the Schubert variety we obtain

$$c^T_{\mathrm{SM}}(\Omega^{\lambda,\circ}) = \pi_*\left(\frac{c^T(T\mathbb{V}^\lambda)}{\prod_{i=1}^k (1+D_i)} \cap [\mathbb{V}^\lambda]\right).$$

Then dividing by the total Chern class of the tangent bundle, the SSM class of the Schubert cell is

$$s^T_{\mathrm{SM}}(\Omega^{\lambda,\circ}) = \frac{1}{c^T(T\mathrm{Gr}(k,N+k))}\pi_*\biggl(\frac{c^T(T\mathbb{V}^\lambda)}{\prod_{i=1}^k(1+D_i)} \cap [\mathbb{V}^\lambda]\biggr).$$

To simplify this, we use

$$c^T(T\mathbb{V}^\lambda) = \prod_{i=1}^k c^T(\mathcal{L}_i^\vee \otimes \mathcal{Q}_i)$$

and $T\mathrm{Gr}(k,N+k) = \mathcal{S}^\vee \otimes \mathcal{Q}$, where $\mathcal{S}$ is the tautological sub-bundle and $\mathcal{Q}$ is the tautological quotient bundle on $\mathrm{Gr}(k,N+k)$. We also have $\pi^*(\mathcal{S}) = \mathcal{J}_k$ and $\pi^*(\mathcal{Q}) = \mathcal{Q}_k$. Then with the projection formula, we obtain

$$s^T_{\mathrm{SM}}(\Omega^{\lambda,\circ}) = \pi_*\biggl(\frac{\prod_{i=1}^k c^T(\mathcal{L}_i^\vee \otimes \mathcal{Q}_i)}{(\prod_{i=1}^k(1+D_i))c^T(\mathcal{J}_k^\vee \otimes \mathcal{Q}_k)} \cap [\mathbb{V}^\lambda]\biggr).$$

Using the exact sequence

$$0 \to \mathcal{J}_i \to \mathcal{F}_{N+i-\lambda_i} \to \mathcal{Q}_i \to 0$$

and tensoring it by $\mathcal{L}_j^\vee$, we obtain

$$c^T(\mathcal{L}_j^\vee \otimes \mathcal{Q}_i) = \frac{c^T(\mathcal{L}_j^\vee \otimes \mathcal{F}_{N+i-\lambda_i})}{c^T(\mathcal{J}_i \otimes \mathcal{L}_j^\vee)}.$$

Then using $c^T(\mathcal{J}_i) = \prod_{j=1}^i c^T(\mathcal{L}_j)$ along with the previous formula gives

$$s^T_{\mathrm{SM}}(\Omega^{\lambda,\circ}) = \pi_*\biggl(\frac{(\prod_{i=1}^k c^T(\mathcal{L}_i^\vee \otimes \mathcal{F}_{N+i-\lambda_i}))(\prod_{i=1}^k\prod_{j=1}^k c^T(\mathcal{L}_j^\vee \otimes \mathcal{L}_i))}{(\prod_{i=1}^k(1+D_i))(\prod_{i=1}^k c^T(\mathcal{L}_i^\vee \otimes \mathcal{F}_{N+k}))(\prod_{i=1}^k\prod_{j=1}^i c^T(\mathcal{L}_i^\vee \otimes \mathcal{L}_j))} \cap [\mathbb{V}^\lambda]\biggr)$$

This simplifies to

$$s^T_{\mathrm{SM}}(\Omega^{\lambda,\circ}) = \pi_*\biggl(\frac{\prod_{i=1}^k\prod_{j=i+1}^k c^T(\mathcal{L}_i^\vee \otimes \mathcal{L}_j)}{\prod_{i=1}^k(1+D_i)c^T(\mathcal{L}_i^\vee \otimes (\mathcal{F}_{N+k}\backslash\mathcal{F}_{N+i-\lambda_i}))} \cap [\mathbb{V}^\lambda]\biggr).$$

In the nonequivariant case, $D_i = z_i$ by [3, Proposition 2.10]. To find D_i in the equivariant case, we use the fact that the divisor is not supported on the fixed point corresponding to the preimage of the Schubert cell, which is given by $S_i/S_{i-1} = \mathcal{F}_{N+i-\lambda_i}/\mathcal{F}_{N+i-\lambda_i-1}$. As a result, we have that $z_i|_{e_{\lambda,\omega}} = -t_{(N+k+1)-(N+i-\lambda_i)} = -t_{k+1-i+\lambda_i}$, where $\omega \in S_k$ is the longest permutation. Therefore

$$D_i = z_i + t_{k+1-i-\lambda_i}.$$

Then using the definition of the z_i's and the fact that the Chern roots of $\mathcal{F}_j$ are $t_{N+k}, ..., t_{N+k+1-j}$, we have

$$s^T_{\mathrm{SM}}(\Omega^{\lambda,\circ}) = \pi_*\left(\frac{\prod_{i=1}^k \prod_{j=i+1}^k (1+z_i-z_j)}{\prod_{i=1}^k ((1+z_i+t_{i+1-i-\lambda_i})\prod_{j=1}^{k-i+\lambda_i}(1+z_i+t_j))} \cap [\mathbb{V}^\lambda]\right).$$

Then using Lemma 3.1.1 and simplifying some, we obtain:

Theorem 5.0.1. For a partition $\lambda = (\lambda_1, ..., \lambda_k)$ and $N \geq \lambda_1$, the T-equivariant SM class of the Schubert cell in $\mathrm{Gr}(k, N+k)$ is:

$$s^T_{\mathrm{SM}}(\Omega^{\lambda,\circ}) = \pi_*\left(\prod_{i=1}^k \Big(\prod_{j=k+1-i}^{k-i+\lambda_i} (z_i+t_j) \prod_{j=1}^{k+1-i+\lambda_i} \frac{1}{1+t_j+z_i}\Big) \prod_{i=1}^k \prod_{j=i+1}^k (1+z_i-z_j) \cap [\mathbb{V}^\emptyset]\right).$$

This, in addition to Lemma 3.1.2 gives that the localization of the SM class is:

$$s^T_{\mathrm{SM}}(\Omega^{\lambda,\circ})|_{e_\mu} = \sum_{w\in S_k} P_w \prod_{i=1}^k \left(\prod_{j=k+1-i}^{k-i+\lambda_i} (t_j - t_{i_{w(i)}}) \prod_{j=1}^{k+1-i+\lambda_i} \frac{1}{1+t_j-t_{i_{w(i)}}} \prod_{j=i+1}^k (1+t_{i_{w(j)}} - t_{i_{w(i)}})\right),$$

where $i_j = j + \mu_{k+1-j}$ for $1 \leq j \leq k$, and

$$P_w = \prod_{i=1}^k \prod_{j=i+1}^k \frac{t_{j-i} - t_{i_{w(i)}}}{t_{i_{w(j)}} - t_{i_{w(i)}}}.$$

Using this formula along with knowledge of the localizations of the equivariant Schubert classes, the SM classes can be expanded in terms of the Schubert basis. Some examples are below.

Examples 5.0.2. In $\mathrm{Gr}(2,4)$ we have:

$$\begin{aligned} s^T_{\mathrm{SM}}(\Omega^{(2,1),\circ}) &= \frac{[\Omega^{(2,1)}]^T}{(1+t_1-t_4)(1+t_3-t_4)(1+t_1-t_2)} \\ &- [\Omega^{(2,2)}]^T \frac{3+2(t_1-t_4)+(t_2-t_3)+(1_1-t_3)(t_2-t_4)}{(1+t_1-t_4)(1+t_3-t_4)(1+t_1-t_2)(1+t_2-t_3)(1+t_2-t_4)(1+t_1-t_3)}. \end{aligned}$$

$$\begin{aligned} s^T_{\mathrm{SM}}(\Omega^{(2),\circ}) &= \frac{[\Omega^{(2)}]^T}{(1+t_2-t_4)(1+t_3-t_4)} \\ &- [\Omega^{(2,1)}]^T \frac{2+(t_1-t_4)}{(1+t_2-t_4)(1+t_3-t_4)(1+t_1-t_2)(1+t_1-t_4)} \\ &+ [\Omega^{(2,2)}]^T \frac{2+(t_1-t_4)}{(1+t_1-t_3)(1+t_2-t_4)(1+t_3-t_4)(1+t_1-t_2)(1+t_1-t_4)}. \end{aligned}$$

By plugging in $t_i = 0$, we can obtain the SM class in ordinary cohomology:

$$s_{\mathrm{SM}}(\Omega^{\lambda,\circ}) = \pi_* \left(\left(\prod_{i=1}^{k} \frac{z_i^{\lambda_i}}{(1+z_i)^{k+1-i+\lambda_i}} \right) \prod_{i=1}^{k} \prod_{j=i+1}^{k} (1+z_i-z_j) \cap [\mathbb{V}^{\emptyset}] \right).$$

Then because $1+z_i-z_j = 1$ when $i=j$, we have $\prod_{i=1}^{k}\prod_{j=i+1}^{k}(1+z_i-z_j) = \prod_{j=1}^{k}\prod_{i=1} 6j(1+ z_i - z_j)$. Furthermore, $\prod_{i=1}^{k}(1+z_i)^{k+1-i} = \prod_{j=1}^{k}\prod_{i=1}^{j}(1+z_i)$, which means the previous formula is equivalent to

$$s_{\mathrm{SM}}(\Omega^{\lambda,\circ}) = \pi_* \left(\left(\prod_{i=1}^{k} \left(\frac{z_i}{1+z_i} \right)^{\lambda_i} \right) \prod_{j=1}^{k} \prod_{i=1}^{j} \frac{1+z_i-z_j}{1+z_i} \right).$$

Then we can construct a system of Grassmannians and their desingularizations $\mathbb{V}^{\emptyset}$ with the embeddings from Chapter 4 with maps $\iota_{1,2} : \mathrm{Gr}(k_1, N_1+k_1) \to \mathrm{Gr}(k_2, N_2+k_2)$ with $k_1 < k_2$

and $N_1 < N_2$. These embeddings extend to embeddings $\iota : \mathbb{V}^{\emptyset}(k_1, k_1+N_1) \to \mathbb{V}^{\emptyset}(k_2, k_2+N_2)$. Then cohomology classes can be pulled back along these embeddings by mapping $z_i \mapsto z_i$ for $i \leq k_1$ and $z_i \mapsto 0$ for $k_1 + 1 \leq i \leq k_2$. This will create an inverse system of cohomology rings over $\mathbb{V}^{\emptyset}$. Then the inverse limit of these cohomology rings can be taken as $k \to \infty$ and as $N \to \infty$. The formula for the SM class still applies, and then with k now meaning the number of nonzero parts of λ, we can define

$$\tilde{s}_\lambda = \pi_* \left(\left(\prod_{i=1}^{k} \left(\frac{z_i}{1+z_i} \right)^{\lambda} \right) \prod_{j=1}^{\infty} \prod_{i=1}^{j} \frac{1+z_i-z_j}{1+z_i} \right)$$

to be the limit of the SM classes of $\Omega^{\lambda,\circ}$ in $\mathrm{Gr}(k, N+k)$ as $k \to \infty$ and $N \to \infty$. Here the pushforward π_* is also a limit of pushforwards and is given by extending the formula in Lemma 3.1.3 to infinitely many variables and setting $t_i = 0$. By Theorem 3.2.1, this map will take a monomial $z^\lambda = \prod_{i=1}^{k} z_i^{\lambda_i}$ to the corresponding Schur function $s_\lambda(x)$, applying the straightening rule (Corollary 3.3.1) if λ is not a partition. This formula happens to exactly match Definition 8.2 from [14], which investigates Segre-MacPherson classes of matrix Schubert cells. Because of this, the coefficients c_μ in the expansion of $\tilde{s}_\lambda$ into Schur functions are the coefficients for the expansion of the Segre-MacPherson class of the Schubert cell into Schubert classes. Since it has been proven in [5, Corollary 4.2] that those coefficients alternate (that $(-1)^{|\mu|-|\lambda|}c_\mu$ is positive), Feher and Rimanyi's conjecture [14, Conjecture 8.4] is proven to be true.

Chapter 6

Motivic Segre Classes in Grassmannians

In this chapter we use the properties of the motivic Chern class to calculate it for Schubert cells in Grassmannians and by extension calculate the motivic Segre class in T-equivariant K-theory (Theorem 6.0.2). Then we use that to calculate the classes in ordinary K-theory.

As with the CSM class, we use the fact that the complement of the preimage of the Schubert cell in the Bott-Samelson variety is a simple normal crossing divisor, so we can get the motivic Chern class in the Grassmannian by calculating it in the Bott-Samelson variety and pushing forward because of functoriality. Using the exact sequence, see, for example, [1, beginning of section 2]

$$0 \to \Omega^1_X \to \Omega^1_X(\log\ D) \to \oplus \mathcal{O}_{D_i} \to 0,$$

and Proposition 2.6.2, we have

$$\begin{aligned} mC_y^T(\pi^{-1}(\Omega^{\lambda,\circ}) \hookrightarrow \mathbb{V}^\lambda) &= [\mathcal{O}_X(-D) \otimes \lambda_y^T(\Omega^1_X(\log\ D))] \\ &= [\mathcal{O}_X(-D)] \cdot \lambda_y^T(\Omega^1_X) \prod_{i=1}^{k} \lambda_y^T(\mathcal{O}_{D_i}). \end{aligned} \tag{6.1}$$

Since Ω^1_X is the cotangent bundle, we have

$$\lambda_y^T(\Omega^1_X) = \prod_{i=1}^{k} \lambda_y^T(\mathcal{L}_i \otimes \mathcal{Q}_i^\vee).$$

Using the exact sequences

$$0 \to \mathcal{J}_i^\vee \to \mathcal{F}^\vee_{N+i-\lambda_i} \to \mathcal{Q}_i^\vee \to 0$$

and tensoring by $\mathcal{L}_i$ gives

$$\lambda_y^T(\mathcal{L}_i \otimes \mathcal{Q}_i^\vee) = \frac{\lambda_y^T(\mathcal{F}^\vee_{N+i-\lambda_i} \otimes \mathcal{L}_i)}{\lambda_y^T(\mathcal{J}_i^\vee \otimes \mathcal{L}_i)} = \frac{\prod_{j=k+1-i+\lambda_i}^{N+k} \lambda_y^T(\mathbb{C}^\vee_{t_j} \otimes \mathcal{L}_i)}{\prod_{j=1}^{i} \lambda_y^T(\mathcal{L}_j^\vee \otimes \mathcal{L}_i)}. \tag{6.2}$$

Now for $\mathcal{O}_{D_i}$, we use the exact sequence

$$0 \to \mathcal{O}_X(-D_i) \to \mathcal{O}_X \to \mathcal{O}_{D_i} \to 0,$$

regarding $\mathcal{O}_X(-D_i)$ as the ideal sheaf. Then

$$\lambda_y^T(\mathcal{O}_{D_i}) = \frac{\lambda_y^T(\mathcal{O}_X)}{\lambda_y^T(\mathcal{O}_X(-D_i))}. \tag{6.3}$$

From doing this in cohomology, we know that $\mathcal{O}_X(-D_i) = \mathcal{L}_i \otimes \mathbb{C}^\vee_{t_{k+1-i+\lambda_i}}$. We also have, since D is a simple normal crossing divisor with components D_i, that

$$[\mathcal{O}_X(-D)] = \prod_{i=1}^{k} [\mathcal{O}_X(-D_i)] = \prod_{i=1}^{k} [\mathcal{L}_i \otimes \mathbb{C}^\vee_{t_{k+1-i+\lambda_i}}]. \tag{6.4}$$

Plugging equations (6.2), (6.3), and (6.4) into equation (6.1), we have

$$mC_y^T(\pi^{-1}(\Omega^{\lambda,\circ}) \hookrightarrow \mathbb{V}^\lambda) = \prod_{i=1}^{k}[\mathcal{L}_i \otimes \mathbb{C}^\vee_{t_{k+1-i+\lambda_i}}] \frac{\lambda_y^T(\mathcal{O}_{\mathbb{V}^\lambda})}{\lambda_y^T(\mathcal{L}_i \otimes \mathbb{C}^\vee_{t_{k+1-i+\lambda_i}})} \frac{\prod_{j=k+1-i+\lambda_i}^{N+k} \lambda_y^T(\mathbb{C}^\vee_{t_j} \otimes \mathcal{L}_i)}{\prod_{j=1}^{i} \lambda_y^T(\mathcal{L}_j^\vee \otimes \mathcal{L}_i)}. \tag{6.5}$$

Since we have a method to calculate the pushforward from $\mathbb{V}^\emptyset$ into $\mathrm{Gr}(k, N+k)$, the best way to get from $\mathbb{V}^\lambda$ to $\mathrm{Gr}(k, N+k)$ is to first go to $\mathbb{V}^\emptyset$.

Applying Lemma 3.4.3 to equation (6), using the fact that $\mathcal{L}_i$ and $\mathbb{C}^\vee_{t_j}$ on $\mathbb{V}^\lambda$ are pullbacks of those bundles in $\mathbb{V}^\emptyset$ for all i and j, we obtain:

Proposition 6.0.1. For $\lambda = (\lambda_1, ..., \lambda_k)$ a partition with $\lambda_1 \leq N$, the equivariant motivic Chern class of the Schubert cell is

$$mC_y^T(\Omega^{\lambda,\circ} \hookrightarrow \mathrm{Gr}(k, N+k)) = \pi_*\left(\prod_{i=1}^{k} \frac{(1+y)[\mathcal{L}_i \otimes \mathbb{C}^\vee_{t_{k+1-i+\lambda_i}}]}{\lambda_y^T(\mathcal{L}_i \otimes \mathbb{C}^\vee_{t_{k+1-i+\lambda_i}})} \prod_{j=k+1-i}^{k-i+\lambda_i}(1 - [\mathcal{L}_i \otimes \mathbb{C}^\vee_{t_j}]) \frac{\prod_{j=k+1-i+\lambda_i}^{N+k} \lambda_y^T(\mathbb{C}^\vee_{t_j} \otimes \mathcal{L}_i)}{\prod_{j=1}^{i} \lambda_y^T(\mathcal{L}_j^\vee \otimes \mathcal{L}_i)}\right).$$

Now to get the motivic Segre class, we need to divide by the λ_y^T class of the cotangent bundle of the Grassmannian. Since $\pi^*(T\mathrm{Gr}(k, N+k)^\vee) = \mathcal{J}_k \otimes \mathcal{Q}_k^\vee$, the projection formula gives us that

$$mS_y(\Omega^{\lambda,\circ} \hookrightarrow \mathrm{Gr}(k, N+k)) = \pi_*\left(\frac{mC_y(\pi^{-1}(\Omega^{\lambda,\circ}) \hookrightarrow \mathbb{V}^\emptyset)}{\lambda_y^T(\mathcal{J}_k \otimes \mathcal{Q}_k^\vee)}\right).$$

Then using the exact sequence

$$0 \to \mathcal{J}_k^\vee \to \mathcal{F}_{N+k}^\vee \to \mathcal{Q}_k^\vee \to 0$$

and tensoring by $\mathcal{L}_i^\vee$, we have

$$\lambda_y^T(\mathcal{J}_k \otimes \mathcal{Q}_k^\vee) = \prod_{i=1}^{k} \frac{\prod_{j=1}^{N+k} \lambda_y^T(\mathcal{L}_i \otimes \mathbb{C}_{t_j}^\vee)}{\prod_{j=1}^{k} \lambda_y^T(\mathcal{L}_i \otimes \mathcal{L}_j^\vee)}$$

because $\mathcal{J}_k = \bigoplus_{i=1}^k \mathcal{L}_i$. Bringing everything together, we obtain:

Theorem 6.0.2. For a partition $\lambda = (\lambda_1, ..., \lambda_k)$ with $\lambda_1 \leq N$, the T-equivariant motivic Segre class of the Schubert cell $\Omega^{\lambda,\circ}$ in $\mathrm{Gr}(k, N+k)$ is:

$$mS_y^T(\Omega^{\lambda,\circ} \hookrightarrow \mathrm{Gr}(k, N+k)) =$$
$$\pi_*\left(\prod_{i=1}^{k}(1+y)[\mathcal{L}_i \otimes \mathbb{C}_{t_{k+1-i+\lambda_i}}^\vee] \prod_{j=k+1-i}^{k-i+\lambda_i}(1-[\mathcal{L}_i \otimes \mathbb{C}_{t_j}^\vee]) \frac{\prod_{j=i+1}^{k} \lambda_y^T(\mathcal{L}_j^\vee \otimes \mathcal{L}_i)}{\prod_{j=1}^{k+1-i+\lambda_i} \lambda_y^T(\mathbb{C}_{t_j}^\vee \otimes \mathcal{L}_i)}\right).$$

To get the class in ordinary cohomology, the bundles $\mathbb{C}_{t_j}^\vee$ become trivial bundles, and the result is

$$mS_y(\Omega^{\lambda,\circ} \hookrightarrow \mathrm{Gr}(k, N+k)) =$$
$$\pi_*\left(\prod_{i=1}^{k}(1+y)[\mathcal{L}_i](1-[\mathcal{L}_i])^{\lambda_i} \frac{\prod_{j=i+1}^{k}(1+y([\mathcal{L}_j^\vee] \otimes [\mathcal{L}_i]))}{\prod_{j=1}^{k+1-i+\lambda_i} 1+y[\mathcal{L}_i]}\right).$$

Using the variables $Z_i = 1 - [\mathcal{L}_i]$ gives

$$mS_y(\Omega^{\lambda,\circ} \hookrightarrow \mathrm{Gr}(k, N+k)) =$$
$$\pi_*\left(\prod_{i=1}^{k}(1+y)(1-Z_i)\frac{Z_i^{\lambda_i}}{(1+y(1-Z_i))^{k+1-i+\lambda_i}} \prod_{j=i+1}^{k}\left(1+y\frac{1-Z_i}{1-Z_j}\right)\right).$$

Then because $1 + y\frac{1-Z_i}{1-Z_j} = 1 + y$ when $i = j$, we have

$$\prod_{i=1}^{k}(1+y) \prod_{j=i+1}^{k}\left(1+y\frac{1-Z_i}{1-Z_j}\right) = \prod_{j=1}^{k}\prod_{i=1}^{j}\left(1+y\frac{1-Z_i}{1-Z_j}\right).$$

Furthermore,

$$\prod_{i=1}^{k} \frac{1}{(1+y(1-Z_i))^{k+1-i}} = \prod_{j=1}^{k}\prod_{i=1}^{j} \frac{1}{1+y(1-Z_i)}.$$

We also have that

$$\frac{1+y\frac{1-Z_i}{1-Z_j}}{1+y(1-Z_i)} = \frac{1-Z_j+y(1-Z_i)}{(1+y(1-Z_i))(1-Z_j)}.$$

These together give that

$$mS_y(\Omega^{\lambda,\circ} \hookrightarrow \mathrm{Gr}(k,N+k)) = \pi_*\left(\prod_{i=1}^{k} \frac{(1-Z_i)Z_i^{\lambda_i}}{(1+y(1-Z_i))^{\lambda_i}} \prod_{j=1}^{k}\prod_{i=1}^{j}\left(\frac{1-Z_j+y(1-Z_i)}{(1+y(1-Z_i))(1-Z_j)}\right)\right).$$

Then taking $k \to \infty$ in a similar manner as in the end of Chapter 5, we would get a K-theory analogue of the $\tilde{s}$ function from [14, Definition 8.2].

Chapter 7

Divided Difference Operators

There are several divided difference operators that are defined for the cohomology and K-theory of the full flag manifold G/B, but they can be defined algebraically on polynomials and used anywhere. In each of the following sections, one such operator will be defined and then used to express a formula from previous chapters.

7.1 Pushforward Formula (Cohomology)

We can express the pushforward in cohomology using the operators defined in [8]. Algebraically they are defined for $1 \leq i \leq k-1$ by

$$\partial_i(f) = \frac{f - s_i(f)}{x_i - x_{i+1}}$$

for a polynomial $f \in \mathbb{Z}[x_1, ..., x_k]$. For any permutation $w \in S_k$ with reduced word decomposition $w = s_{i_1}...s_{i_\ell}$, we define $\partial_w = \partial_{i_1}...\partial_{i_\ell}$. We first establish some computations with the operator.

Lemma 7.1.1. Let $f, g \in \mathbb{Z}[x_1, ...x_k]$

1. If f is symmetric in the variables x_i and x_{i+1}, $\partial_i(fg) = f\partial_i(g)$.

2. For $i \geq q+1$,

$$\partial_{q-1}\left(\frac{1}{x_q - x_i}\right) = \frac{1}{(x_q - x_i)(x_{q-1} - x_i)}.$$

Proof. 1. This follows directly from the definition. If f is symmetric in x_i and x_{i+1}, then $s_i(f) = f$, and so

$$\partial_i(fg) = \frac{fg - s_i(fg)}{x_i - x_{i+1}} = \frac{fg - fs_i(g)}{x_i - x_{i+1}} = f\frac{g - s_i(g)}{x_i - x_{i+1}} = f\partial_i(g).$$

2. Since $i \geq q+1$, $s_{q-1}(x_i) = x_i$. Then by the definition we calculate

$$\begin{aligned}\partial_{q-1}\left(\frac{1}{x_q - x_i}\right) &= \frac{1}{x_{q-1} - x_q}\left(\frac{1}{x_q - x_i} - s_{q-1}\left(\frac{1}{x_q - x_i}\right)\right) \\ &= \frac{1}{x_{q-1} - x_q}\left(\frac{(x_{q-1} - x_i) - (x_q - x_i)}{(x_q - x_i)(x_{q-1} - x_i)}\right) \\ &= \frac{1}{(x_q - x_i)(x_{q-1} - x_i)}.\end{aligned}$$

□

Lemma 7.1.2. For a polynomial $f \in \mathbb{Z}[x_1, ...x_k]$, with w_0 being the longest permutation in S_k defined by $w_0(i) = k+1-i$ for $1 \leq i \leq k$

$$\partial_{w_0}(f) = \sum_{w \in S_k} w\left(f \prod_{1 \leq i < j \leq k} \frac{1}{x_i - x_j}\right).$$

Proof. We proceed by induction on k. The base case could be considered as $k = 1$, which holds trivially because in S_1 the only element is the identity. To make the induction step, we embed S_k into S_{k+1} by taking the permutation $w \in S_k$ and defining a permutation in S_{k+1} by defining $w(k+1) = k+1$. With this, $w_0^{(k+1)} \in S_{k+1}$ can be expressed as $s_1...s_k w_0^{(k)}$, where $w_0^{(k)} \in S_k$ is the longest permutation in S_k. As a result, for the induction step, we apply $\partial_1 ... \partial_k$ to both sides of the equation

$$\partial_{w_0^{(k)}}(f) = \sum_{w \in S_k} w \left(f \prod_{1 \le i < j \le k} \frac{1}{x_i - x_j} \right).$$

For convenience we denote the right side of the previous equation by g. Then we are required to show that

$$\partial_1 ... \partial_k(g) = \sum_{w \in S_{k+1}} w \left(g \prod_{j=1}^{k} \frac{1}{x_j - x_{k+1}} \right).$$

To show this, we use descending induction on q to show that

$$\partial_q ... \partial_k(g) = \sum_{i=q}^{k+1} s_i ... s_k \left(\prod_{j=q}^{k} \frac{1}{x_j - x_{k+1}} \right),$$

where $s_i ... s_k = id$ when $i = k+1$. The base case is when $i = k$, which holds by definition

$$\partial_k(g) = \frac{g}{x_k - x_{k+1}} + s_k \left(\frac{g}{x_k - x_{k+1}} \right).$$

Then for the induction step, we use the fact that g is symmetric in the variables x_1 to x_k. We have by the induction hypothesis that

$$\partial_{q-1} ... \partial_k(g) = \partial_{q-1} \left(\sum_{i=q}^{k+1} s_i ... s_k \left(g \prod_{j=1}^{k} \frac{1}{x_j - x_{k+1}} \right) \right).$$

For the terms in the sum with $i \ge q+1$, we have $s_i ... s_k(g)$ is symmetric with respect to x_{q-1} and x_q because g is symmetric in the first k variables, and so applying the permutation $s_i ... s_k$ will not affect the symmetry of the first s_{i-1} variables. So then we have for $i \ge q+1$,

$$\partial_{q-1} s_i ... s_k \left(g \prod_{j=q}^{k} \frac{1}{x_j - x_{k+1}} \right) = s_i ... s_k \left(g \prod_{j=q+1}^{k} \frac{1}{x_j - x_{k+1}} \right) \partial_{q-1} \left(\frac{1}{x_q - x_i} \right)$$

by part (1) of Lemma 7.1.1. Then applying part (2) of Lemma 7.1.1 yields

$$\partial_{q-1}s_i...s_k\left(g\prod_{j=q}^{k}\frac{1}{x_j-x_{k+1}}\right)=s_i...s_k\left(g\prod_{j=q-1}^{k}\frac{1}{x_j-x_{k+1}}\right).$$

Then applying the definition to the $i=q$ term gives

$$\begin{aligned}\partial_{q-1}s_q...s_k\left(g\prod_{j=q}^{k}\frac{1}{x_j-x_{k+1}}\right)&=\frac{1}{x_{q-1}-x_q}s_q...s_k\left(g\prod_{j=q}^{k}\frac{1}{x_j-x_{k+1}}\right)\\&-\frac{1}{x_{q-1}-x_q}s_{q-1}s_q...s_k\left(g\prod_{j=q}^{k}\frac{1}{x_j-x_{k+1}}\right)\\&=s_q...s_k\left(g\prod_{j=q-1}^{k}\frac{1}{x_j-x_{k+1}}\right)\\&+s_{q-1}...s_k\left(g\prod_{j=q-1}^{k}\frac{1}{x_j-x_{k+1}}\right).\end{aligned}$$

Putting these together gives

$$\partial_{q-1}...\partial_k(g)=\sum_{i=q-1}^{k+1}s_i...s_k\left(g\prod_{j=q-1}^{k}\frac{1}{x_j-x_{k+1}}\right).$$

This completes the induction step on q, which completes the induction step on k, completing the proof. □

With this we are able to show that the pushforward formula (Lemma 3.1.3) can be expressed in terms of these divided difference operators:

Proposition 7.1.3. For $f\in\mathbb{Z}[z_1,...,z_k][t_1,...,t_{N+k}]$, we have

$$\pi_*(f(z;t))=\partial_{w_0}(p_\delta f(x;t)),$$

where $w_0 \in S_k$ is the longest permutation and $p_\delta = \prod_{i=1}^{k-1}\prod_{j=1}^{k-i}(x_i + t_j)$.

Proof. Apply Lemma 3.1.3 on the left side of the equation and Lemma 7.1.2 to the right side of the equation. The result on both sides is the same. □

7.2 Pushforward Formula (K-Theory)

In K-theory, the applicable operator is the Demazure operator [13]. Algebraically they are defined for $1 \leq i \leq k-1$ by

$$\delta_i(f) = \frac{(1-X_{i+1})f - s_i((1-X_{i+1})f)}{X_i - X_{i+1}}$$

for a polynomial $f \in \mathbb{Z}[X_1, ..., X_k]$. Similarly to cohomology, operators are defined for permutations $w \in S_k$ by $\partial_w(f) = \partial_{i_1}...\partial_{i_\ell}(f)$ if $w = s_{i_1}...s_{i_\ell}$ is a reduced decomposition. As before, we get some computations out of the way first.

Lemma 7.2.1. Let $f, g \in \mathbb{Z}[X_1, ..., X_k]$.

1. If f is symmetric in the variables X_i and X_{i+1}, $\delta_i(fg) = f\delta_i(g)$.

2. For $i \geq q+1$,
$$\delta_{q-1}\left(\frac{1-X_i}{X_q - X_i}\right) = \frac{(1-X_i)^2}{(X_{q-1}-X_i)(X_q - X_i)}.$$

Proof. 1. If f is symmetric in X_i and X_{i+1}, then $s_i(f) = f$. From the definition,

$$\begin{aligned}\delta_i(fg) &= \frac{(1-X_{i+1})fg - s_i((1-X_{i+1})fg)}{X_i - X_{i+1}}\\ &= f\frac{(1-X_{i+1})g - s_i((1-X_{i+1})g)}{X_i - X_{i+1}} = f\delta_i(g).\end{aligned}$$

2. Since $i \geq q+1$, $s_{q-1}(X_i) = X_i$. Then applying the definition gives

$$\begin{aligned}
\delta_{q-1}\left(\frac{1-X_i}{X_q-X_i}\right) &= \frac{1}{X_{q-1}-X_q}\left(\frac{(1-X_i)(1-X_q)}{X_q-X_i} - s_{q-1}\left(\frac{(1-X_i)(1-X_q)}{X_q-X_i}\right)\right)\\
&= \frac{1-X_i}{X_{q-1}-X_q}\left(\frac{(1-X_q)(X_{q-1}-X_i)}{(X_q-X_i)(X_{q-1}-X_i)} - \frac{(1-X_{q-1})(X_q-X_i)}{(X_{q-1}-X_i)(X_q-X_i)}\right)\\
&= \frac{1-X_i}{X_{q-1}-X_q}\left(\frac{X_{q-1}-X_i-X_qX_{q-1}+X_qX_i-X_q+X_i+X_qX_{q-1}-X_{q-1}X_i}{(X_{q-1}-X_i)(X_q-X_i)}\right)\\
&= \frac{1-X_i}{X_{q-1}-X_q}\left(\frac{X_{q-1}(1-X_i)-X_q(1-X_i)}{(X_{q-1}-X_i)(X_q-X_i)}\right)\\
&= \frac{(1-X_i)^2}{(X_{q-1}-X_i)(X_q-X_i)}
\end{aligned}$$

□

Lemma 7.2.2. For a polynomial $f \in \mathbb{Z}[X_1, ..., X_k]$, with w_0 being the longest permutation in S_k defined by $w_0(i) = k+1-i$ for $1 \leq i \leq k$,

$$\delta_{w_0}(f) = \sum_{w \in S_k} w\left(f \prod_{1 \leq i < j \leq k} \frac{1-X_j}{X_i - X_j}\right).$$

Proof. We proceed by induction on k, using the fact that $w_0^{(k+1)} = s_1 ... s_k w_0^{(k)}$. The base case $k = 1$ holds trivially, so it required to show that

$$\delta_1 ... \delta_k\left(\sum_{w \in S_k} w\left(f \prod_{1 \leq i < j \leq k} \frac{1-X_j}{X_i - X_j}\right)\right) = \sum_{w \in S_{k+1}} w\left(f \prod_{1 \leq i < j \leq k+1} \frac{1-X_j}{X_i - X_j}\right).$$

For convenience, let $g = \sum_{w \in S_k} w\left(f \prod_{1 \leq i < j \leq k} \frac{1-X_j}{X_i - X_j}\right)$. To compute $\delta_1 ... \delta_k(g)$, we proceed by descending induction on q to show

$$\delta_q ... \delta_k(g) = \sum_{i=q}^{k+1} s_i ... s_k \left(g \prod_{j=q}^{k} \frac{1 - X_{k+1}}{X_j - X_{k+1}} \right),$$

where $s_i ... s_k = id$ when $i = k+1$. The case where $i = k$ holds by definition:

$$\delta_k(g) = \frac{(1 - X_{k+1})g}{X_k - X_{k+1}} + s_k \left(\frac{(1 - X_{k+1})g}{X_k - X_{k+1}} \right).$$

For the induction step, we apply the definition to the $i = q$ term in the sum:

$$\begin{aligned}
\delta_{q-1} s_q ... s_k \left(g \prod_{j=q}^{k} \frac{1 - X_{k+1}}{X_j - X_{k+1}} \right) &= \frac{1 - X_q}{X_{q-1} - X_q} s_q ... s_k \left(g \prod_{j=q}^{k} \frac{1 - X_{k+1}}{X_j - X_{k+1}} \right) \\
&\quad + s_{q-1} \left(\frac{1 - X_q}{X_{q-1} - X_q} s_q ... s_k \left(g \prod_{j=q}^{k} \frac{1 - X_{k+1}}{X_j - X_{k+1}} \right) \right) \\
&= s_q ... s_k \left(g \prod_{j=q-1}^{k} \frac{1 - X_{k+1}}{X_j - X_{k+1}} \right) \\
&\quad + s_{q-1} ... s_k \left(g \prod_{j=q-1}^{k} \frac{1 - X_{k+1}}{X_j - X_{k+1}} \right).
\end{aligned}$$

For the terms in the sum with $i \geq q+1$, we use the symmetry of g and part (1) of Lemma 7.2.1 to get

$$\delta_{q-1} s_i ... s_k \left(g \prod_{j=q}^{k} \frac{1 - X_{k+1}}{X_j - X_{k+1}} \right) = s_i ... s_k(g) \left(\prod_{j=q+1}^{k+1} \frac{1 - X_i}{X_j - X_i} \right) \delta_{q-1} \left(\frac{1 - X_i}{X_q - X_i} \right).$$

Then applying part (2) of Lemma 7.2.1 we have

$$\delta_{q-1}s_i...s_k\left(g\prod_{j=q}^{k+1}\frac{1-X_{k+1}}{X_j-X_{k+1}}\right)=s_i...s_k\left(g\prod_{j=q+1}^{k+1}\frac{1-X_{k+1}}{X_j-X_{k+1}}\right)\left(\frac{(1-X_i)^2}{(X_{q-1}-X_i)(X_q-X_i)}\right)$$
$$=s_i...s_k\left(g\prod_{j=q-1}^{k+1}\frac{1-X_{k+1}}{X_j-X_{k+1}}\right).$$

Then applying these results to the entire sum yields

$$\delta_{q-1}...\delta_k(g)=\sum_{i=q-1}^{k+1}s_i...s_k\left(g\prod_{j=q-1}^{k}\frac{1-X_{k+1}}{X_j-X_{k+1}}\right).$$

This completes the induction step on q, and so we have that as $q\to 1$

$$\delta_1...\delta_k(g)=\sum_{i=1}^{k+1}s_i...s_k\left(g\prod_{j=1}^{k}\frac{1-X_{k+1}}{X_j-X_{k+1}}\right).$$

Then from the fact that $\prod_{i=1}^{k}\frac{(1-X_{k+1})}{X_j-X_{k+1}}$ is symmetric in X_1 to X_k and $\sum_{i=1}^{k+1}s_1...s_k\left(\sum_{w\in S_k}w(f)\right)=\sum_{w\in S_{k+1}}w(f)$, we have

$$\partial_1...\partial_k(g)=\sum_{w\in S_{k+1}}w\left(f\prod_{1\leq i<j\leq k+1}\frac{1-X_j}{X_i-X_j}\right).$$

This completes the induction step on k and the proof. □

With this we are able to show that the pushforward formula (Lemma 3.5.1) can be expressed in terms of the divided difference operators.

Proposition 7.2.3. For $f\in\mathbb{Z}[Z_1,...,Z_k][T_1,...,T_{N+k}]$, we have

$$\pi_*(f(Z;T))=\delta_{w_0}(P_\delta f(X;T)),$$

where $P_\delta = \prod_{i=1}^{k-1}\prod_{j=1}^{k-i}(X_i + T_j - X_iT_j)$.

Proof. Apply Lemma 3.5.1 to the left side of the equation and Lemma 7.2.2 to the right side of the equation. The result on both sides is the same. □

7.3 Segre-MacPherson Classes

In a similar way, we can express Segre-MacPherson classes in terms of a divided difference operator. The operator first appeared in [17] in the context of degenerate Hecke algebras. Later these operators were used to study Chern-Schwartz-MacPherson classes, e.g. [2], [4], [30]. Here we use $T_i = \partial_i + s_i$, and so expanding that definition we have

$$T_i(f) = \frac{f - s_i(f)}{x_i - x_{i+1}} + s_i(f).$$

As before, this definition applies to permutations w by $T_w = T_{i_1}...T_{i_\ell}$ if $w = s_{i_1}...s_{i_\ell}$. We first do some computations.

Lemma 7.3.1. Let $f, g \in \mathbb{Z}[x_1, ..., x_k]$.

1. If f is symmetric in the variables x_i and x_{i+1}, $T_i(fg) = fT_i(g)$.

2. For $i \geq p+1$,

$$T_{p-1}\left(\frac{1}{x_p - x_i}\right) = \frac{1 + x_p - x_i}{(x_p - x_i)(x_{p-1} - x_i)}.$$

Proof. 1. If f is symmetric in x_i and x_{i+1}, then $s_i(f) = f$. Then

$$T_i(f) = \frac{fg - s_i(fg)}{x_i - x_{i+1}} + s_i(fg) = f\left(\frac{g - s_i(g)}{x_i - x_{i+1}} + s_i(g)\right) = fT_i(g).$$

2. Since $i \geq p+1$, $s_{p-1}(x_i) = x_i$. Then

$$\begin{aligned}
T_{p-1}\left(\frac{1}{x_p - x_i}\right) &= \frac{1}{x_{p-1} - x_p}\left(\frac{1}{x_p - x_i} - s_{p-1}\left(\frac{1}{x_p - x_i}\right)\right) + s_{p-1}\left(\frac{1}{x_p - x_i}\right) \\
&= \frac{1}{x_{p-1} - x_p}\left(\frac{(x_{p-1} - x_i) - (x_p - x_i)}{(x_{p-1} - x_i)(x_p - x_i)}\right) + \frac{1}{x_{p-1} - x_i} \\
&= \frac{1}{(x_p - x_i)(x_{p-1} - x_i)} + \frac{1}{x_{p-1} - x_i} \\
&= \frac{1 + x_p - x_i}{(x_p - x_i)(x_{p-1} - x_i)}.
\end{aligned}$$

□

Lemma 7.3.2. For $f \in \mathbb{Z}[x_1, ...x_n]$,

$$\sum_{w \in S_k} T_w(f) = \sum_{w \in S_k} w\left(f \prod_{1 \leq i < j \leq k} \frac{1 + x_i - x_j}{x_i - x_j}\right).$$

Proof. We proceed by induction on k. The base case holds trivially when $k = 1$, since both sides are the identity in that case. To complete the induction step, we use the fact that

$$(1 + T_k + ... + T_1...T_k) \sum_{w \in S_k} T_w(f) = \sum_{w \in S_{k+1}} T_w(f).$$

Then we are required to prove

$$\begin{aligned}
(1 + T_k + ... + T_1...T_k) &\sum_{w \in S_k} w\left(f \prod_{1 \leq i < j \leq k} \frac{1 + x_i - x_j}{x_i - x_j}\right) \\
&= \sum_{w \in S_{k+1}} w\left(f \prod_{1 \leq i < j \leq k+1} \frac{1 + x_i - x_j}{x_i - x_j}\right).
\end{aligned}$$

For convenience we define $g = \sum_{w \in S_k} w\left(f \prod_{1 \leq i < j \leq k} \frac{1+x_i-x_j}{x_i-x_j}\right)$. Then to complete the induction step we show

$$(1+T_k+...T_q...T_k)(g)=\sum_{i=q}^{k+1} s_i...s_k\left(g\prod_{j=1}^{k}\frac{1+x_j-x_{k+1}}{x_j-x_{k+1}}\right)$$

by descending induction on q, where $s_i...s_k=id$ when $i=k+1$. The case case holds trivially for $q=k+1$, since the identity is the identity. To do the induction step, we show

$$\begin{aligned}T_p...T_k(g)=&\sum_{i=p}^{k+1} s_i...s_k\left(g\prod_{j=p}^{k}\frac{1+x_j-x_{k+1}}{x_j-x_{k+1}}\right)\\&-\sum_{i=p+1}^{k+1} s_i...s_k\left(g\prod_{j=p+1}^{k}\frac{1+x_j-x_{k+1}}{x_j-x_{k+1}}\right)\end{aligned}$$

using descending induction on p. Again the base case holds trivially when $p=k+1$. To do the induction step, we apply T_{p-1} to the induction hypothesis. Since the operator is linear, we apply it term-by-term in the two sums. We apply the operator to the $i=p$ term in the first sum to get

$$\begin{aligned}T_{p-1}s_p...s_k\left(g\prod_{j=p}^{k}\frac{1+x_j-x_{k+1}}{x_j-x_{k+1}}\right)=&\frac{1}{x_{p-1}-x_p}s_p...s_k\left(g\prod_{j=p}^{k}\frac{1+x_j-x_{k+1}}{x_j-x_{k+1}}\right)\\&+s_{p-1}\left(\frac{1}{x_{p-1}-x_p}...s_k\left(g\prod_{j=p}^{k}\frac{1+x_j-x_{k+1}}{x_j-x_{k+1}}\right)\right)\\&+s_{p-1}...s_k\left(g\prod_{j=p}^{k}\frac{1+x_j-x_{k+1}}{x_j-x_{k+1}}\right).\end{aligned}$$

Then from the fact that $\frac{1}{x_{p-1}-x_p}=s_p...s_k\left(\frac{1}{x_{p-1}-x_{k+1}}\right)$ and that $\frac{g}{x_{p-1}-x_{k+1}}+g=g\frac{1+x_{p-1}-x_{k+1}}{x_{p-1}-x_{k+1}}$

(and in reverse also that $\frac{g}{x_{p-1}-x_{k+1}} = g\frac{1+x_{p-1}-x_{k+1}}{x_{p-1}-x_{k+1}} - g$), we simplify this to

$$\begin{aligned}
T_{p-1}s_p...s_k\left(g\prod_{j=p}^{k}\frac{1+x_j-x_{k+1}}{x_j-x_{k+1}}\right) &= s_{p-1}...s_k\left(g\frac{1+x_{p-1}-x_{k+1}}{x_{p-1}-x_{k+1}}\prod_{j=p}^{k}\frac{1+x_j-x_{k+1}}{x_j-x_{k+1}}\right)\\
&+ s_p...s_k\left(g\frac{1+x_{p-1}-x_{k+1}}{x_{p-1}-x_{k+1}}\prod_{j=p}^{k}\frac{1+x_j-x_{k+1}}{x_j-x_{k+1}}\right)\\
&- s_p...s_k\left(g\prod_{j=p}^{k}\frac{1+x_j-x_{k+1}}{x_j-x_{k+1}}\right)\\
&= s_{p-1}...s_k\left(g\prod_{j=p-1}^{k}\frac{1+x_j-x_{k+1}}{x_j-x_{k+1}}\right)\\
&+ s_p...s_k\left(g\prod_{j=p-1}^{k}\frac{1+x_j-x_{k+1}}{x_j-x_{k+1}}\right)\\
&- s_p...s_k\left(g\prod_{j=p}^{k}\frac{1+x_j-x_{k+1}}{x_j-x_{k+1}}\right).
\end{aligned}$$

Then for the $i \geq p+1$ terms in the first sum, we have that because g is symmetric in the first k variables, $s_i...s_k g$ is symmetric in the first $i-1$ variables, and so using part (1) of Lemma 7.3.1 gives

$$\begin{aligned}
&T_{p-1}s_i...s_k\left(g\prod_{j=p}^{k}\frac{1+x_j-x_{k+1}}{x_j-x_{k+1}}\right)\\
&\qquad = s_i...s_k\left(g\prod_{j=p+1}^{k}\frac{1+x_j-x_{k+1}}{x_j-x_{k+1}}\right)T_{p-1}\left(\frac{1+x_p-x_i}{x_p-x_i}\right)
\end{aligned}$$

Now since $\frac{1+x_p-x_i}{x_p-x_i} = \frac{1}{x_p-x_i} + 1$, we use part (2) of Lemma 7.3.1 with the fact that $T_i(1) =$

$\partial_i(1) + s_i(1) = 0 + 1 = 1$ to obtain

$$\begin{aligned}
T_{p-1}s_i...s_k&\left(g\prod_{j=p}^{k}\frac{1+x_j-x_{k+1}}{x_j-x_{k+1}}\right)\\
&= s_i...s_k\left(g\prod_{j=p+1}^{k}\frac{1+x_j-x_{k+1}}{x_j-x_{k+1}}\right)\frac{1+x_q-x_i}{(x_q-x_i)(x_{q-1}-x_i)}\\
&+ s_i...s_k\left(g\prod_{j=p+1}^{k}\frac{1+x_j-x_{k+1}}{x_j-x_{k+1}}\right)
\end{aligned}$$

Then applying symmetry to the terms in the second sum, we see that the whole thing is symmetric in x_{p-1} and x_p, and so

$$T_{p-1}s_i...s_k\left(g\prod_{j=p+1}^{k}\frac{1+x_j-x_{k+1}}{x_j-x_{k+1}}\right) = s_i...s_k\left(g\prod_{j=p+1}^{k}\frac{1+x_j-x_{k+1}}{x_j-x_{k+1}}\right).$$

Then combining these terms gives

$$\begin{aligned}
T_{p-1}s_i...s_k&\left(\left(g\prod_{j=p}^{k}\frac{1+x_j-x_{k+1}}{x_j-x_{k+1}}\right)-\left(g\prod_{j=p+1}^{k}\frac{1+x_j-x_{k+1}}{x_j-x_{k+1}}\right)\right)\\
&= s_i...s_k\left(g\prod_{j=p}^{k}\frac{1+x_j-x_{k+1}}{x_j-x_{k+1}}\right)\frac{1}{x_{p-1}-x_i}\\
&+ s_i...s_k\left(g\prod_{j=p+1}^{k}\frac{1+x_j-x_{k+1}}{x_j-x_{k+1}}\right)\\
&- s_i...s_k\left(g\prod_{j=p}^{k}\frac{1+x_j-x_{k+1}}{x_j-x_{k+1}}\right)\\
&= s_i...s_k\left(g\prod_{j=p}^{k}\frac{1+x_j-x_{k+1}}{x_j-x_{k+1}}\right)\left(\frac{1+x_{p-1}-x_i}{x_{p-1}-x_i}-1\right)\\
&= s_i...s_k\left(g\prod_{j=p-1}^{k}\frac{1+x_j-x_{k+1}}{x_j-x_{k+1}}\right)\\
&- s_i...s_k\left(g\prod_{j=p}^{k}\frac{1+x_j-x_{k+1}}{x_j-x_{k+1}}\right)
\end{aligned}$$

Then combining this result with the result for the $i = p$ term in the first sum gives

$$\begin{aligned}
T_{p-1}&\left(\sum_{i=p}^{k+1} s_i \ldots s_k \left(g \prod_{j=p}^{k} \frac{1+x_j-x_{k+1}}{x_j-x_{k+1}}\right)\right. \\
&\left.- \sum_{i=p+1}^{k+1} s_i \ldots s_k \left(g \prod_{j=p+1}^{k} \frac{1+x_j-x_{k+1}}{x_j-x_{k+1}}\right)\right) \\
&= s_{p-1} \ldots s_k \left(g \prod_{j=p-1}^{k} \frac{1+x_j-x_{k+1}}{x_j-x_{k+1}}\right) \\
&+ s_p \ldots s_k \left(g \prod_{j=p-1}^{k} \frac{1+x_j-x_{k+1}}{x_j-x_{k+1}}\right) \\
&- s_p \ldots s_k \left(g \prod_{j=p}^{k} \frac{1+x_j-x_{k+1}}{x_j-x_{k+1}}\right) \\
&+ \sum_{i=p+1}^{k+1} s_i \ldots s_k \left(g \prod_{j=p-1}^{k} \frac{1+x_j-x_{k+1}}{x_j-x_{k+1}}\right) \\
&- \sum_{i=p+1}^{k+1} s_i \ldots s_k \left(g \prod_{j=p}^{k} \frac{1+x_j-x_{k+1}}{x_j-x_{k+1}}\right) \\
&= \sum_{i=p-1}^{k+1} s_i \ldots s_k \left(g \prod_{j=p-1}^{k} \frac{1+x_j-x_{k+1}}{x_j-x_{k+1}}\right) \\
&- \sum_{i=p}^{k+1} s_i \ldots s_k \left(g \prod_{j=p}^{k} \frac{1+x_j-x_{k+1}}{x_j-x_{k+1}}\right).
\end{aligned}$$

This completes the induction step on p, which along with the induction hypothesis on q gives that

$$\begin{aligned}
(1+T_k+...T_{q-1}...T_k)(g) &= \sum_{i=q}^{k+1} s_i...s_k\left(g\prod_{j=q}^{k}\frac{1+x_j-x_{k+1}}{x_j-x_{k+1}}\right)\\
&+\sum_{i=q-1}^{k+1} s_i...s_k\left(g\prod_{j=q-1}^{k}\frac{1+x_j-x_{k+1}}{x_j-x_{k+1}}\right)\\
&-\sum_{i=q}^{k+1} s_i...s_k\left(g\prod_{j=q}^{k}\frac{1+x_j-x_{k+1}}{x_j-x_{k+1}}\right)\\
&=\sum_{i=q-1}^{k+1} s_i...s_k\left(g\prod_{j=q-1}^{k}\frac{1+x_j-x_{k+1}}{x_j-x_{k+1}}\right)
\end{aligned}$$

This in turn completes the induction step on q. Then taking $q\to 1$ we obtain

$$\begin{aligned}
\sum_{w\in S_{k+1}} T_w(f) &= \sum_{i=1}^{k+1} s_i...s_k\left(g\prod_{j=1}^{k}\frac{1+x_j-x_{k+1}}{x_j-x_{k+1}}\right)\\
&=\sum_{i=1}^{k+1} s_i...s_k\left(\sum_{w\in S_k} w\left(f\prod_{1\le i<j\le k+1}\frac{1+x_i-x_j}{x_i-x_j}\right)\right)\\
&=\sum_{w\in S_{k+1}} w\left(f\prod_{1\le i<j\le k+1}\frac{1+x_i-x_j}{x_i-x_j}\right)
\end{aligned}$$

This completes the induction on k and therefore the proof. □

Using this along with Theorem 5.0.1 allows us to express Segre-MacPherson classes in terms of the operators T_w.

Theorem 7.3.3. For a partition $\lambda=(\lambda_1,...,\lambda_k)$ with $\lambda_1\le N$, the T-equivariant Segre-MacPherson class of the Schubert cell in $\mathrm{Gr}(k,N+k)$ is given by

$$s_M^T(\Omega^{\lambda,\circ})=\sum_{w\in S_k} T_w\left(\prod_{i=1}^{k}\frac{\prod_{j=1}^{k-i+\lambda_i}(x_i+t_j)}{\prod_{j=1}^{k+1-i+\lambda_i}(1+x_i+t_j)}\right).$$

Proof. Starting with Theorem 5.0.1, we have

$$s_M^T(\Omega^{\lambda,\circ}) = \pi_*\left(\prod_{i=1}^{k}\left(\prod_{j=k+1-i}^{k-i+\lambda_i}(z_i+t_j)\prod_{j=1}^{k+1-i+\lambda_i}\frac{1}{1+x_i+t_j}\prod_{j=i+1}^{k}(1+z_i-z_j)\right)\cap[\mathbb{V}^{\emptyset}]\right)$$

Applying Lemma 3.1.3 to this yields

$$s_M^T(\Omega^{\lambda,\circ}) = \sum_{w\in S_k} w\left(\prod_{i=1}^{k}\frac{\prod_{j=1}^{k-i+\lambda_i}(x_i+t_j)}{\prod_{j=1}^{k+1-i+\lambda_i}(1+x_i+t_j)}\prod_{1\le i<j\le k}\frac{1+x_i-x_j}{x_i-x_j}\right).$$

Then applying Lemma 7.3.2 to the right side of the claim gives the same result as above, which is equal to the left side. □

7.4 Motivic Segre Classes

For motivic Segre classes, the operator is the Demazure-Lusztig operator from [6], given by $\mathcal{T}_i(f) = \delta_i((1+y\frac{1-X_i}{1-X_{i+1}})f) - f$. Using the definition of δ_i this becomes

$$\mathcal{T}_i(f) = \frac{(1-X_{i+1})f + y(1-X_i)f}{X_i - X_{i+1}} + s_i\left(\frac{(1-X_{i+1})f + y(1-X_i)f}{X_i - X_{i+1}}\right) - f.$$

Similarly to with the other operators, for a permutation w with reduced word decomposition $w = s_{i_1}...s_{i_\ell}$, the operator $\mathcal{T}_w$ is defined as $\mathcal{T}_w = \mathcal{T}_{i_1}...\mathcal{T}_{i_\ell}$. As usual, we do some preliminary computations:

Lemma 7.4.1. Let $f, g \in \mathbb{Z}[X_1, ...X_k]$.

1. If f is symmetric in X_i and X_{i+1}, $\mathcal{T}_i(fg) = f\mathcal{T}_i(g)$.

2. For $i \geq p+1$,

$$T_{p-1}\left(\frac{(1-X_i)+y(1-X_p)}{X_p-X_i}\right)$$
$$= \frac{((1-X_i)+y(1-X_p))((1-X_i)+y(1-X_{p-1})}{(X_p-X_i)(X_{p-1}-X_i)} - y$$
$$- \frac{(1-X_i)+y(1-X_p)}{(X_p-X_i)}$$

Proof. 1. If f is symmetric in X_i and X_{i+1}, then $s_i(f)=f$. Then

$$\mathcal{T}_i(fg) = \delta_i((1+y\frac{1-X_i}{1-X_{i+1}})fg) - fg = f\delta_i((1+y\frac{1-X_i}{1-X_{i+1}})g) - fg = f\mathcal{T}_i(g)$$

using part (1) of Lemma 7.2.1.

2. Since $i \geq p+1$, $s_{p-1}(X_i) = X_i$. Then by definition,

$$\mathcal{T}_{p-1}\left(\frac{(1-X_i)+y(1-X_p)}{X_p-X_i}\right)$$
$$= \frac{((1-X_p)+y(1-X_{p-1}))((1-X_i)+y(1-X_p))}{(X_p-X_i)(X_{p-1}-X_p)}$$
$$+ s_{p-1}\left(\frac{((1-X_p)+y(1-X_{p-1}))((1-X_i)+y(1-X_p))}{(X_p-X_i)(X_{p-1}-X_p)}\right)$$
$$- \frac{(1-X_i)+y(1-X_p)}{X_p-X_i}$$
$$= \frac{(1-X_p)(1-X_i)+y((1-X_p)^2+(1-X_{p-1})(1-X_i))+y^2(1-X_p)(1-X_{p-1})}{(X_p-X_i)(X_{p-1}-X_p)}$$
$$- \frac{(1-X_{p-1})(1-X_i)+y((1-X_{p-1})^2+(1-X_p)(1-X_i))+y^2(1-X_p)(1-X_{p-1})}{(X_{p-1}-X_i)(X_{p-1}-X_p)}$$
$$- \frac{(1-X_i)+y(1-X_p)}{X_p-X_i}.$$

Then combining the fractions with a common denominator, the expression becomes

$$\frac{1}{(X_p - X_i)(X_{p-1} - X_i)(X_{p-1} - X_p)}\Big((1 - X_p)(1 - X_i)(X_{p-1} - X_i)$$
$$- (1 - X_{p-1})(1 - X_i)(X_p - X_i) + y(1 - X_p)^2(X_{p-1} - X_i)$$
$$+ y(1 - X_{p-1})(1 - X_i)(X_{p-1} - X_i) - y(1 - X_{p-1})^2(X_p - X_i)$$
$$- y(1 - X_p)(1 - X_i)(X_p - X_i) + y^2(1 - X_p)(1 - X_{p-1})(X_{p-1} - X_i)$$
$$- y^2(1 - X_p)(1 - X_{p-1})(X_p - X_i)\Big) - \frac{(1 - X_i) + y(1 - X_p)}{X_p - X_i}.$$

then expanding and refactoring, the expression becomes

$$\frac{1}{(X_p - X_i)(X_{p-1} - X_i)(X_{p-1} - X_p)}\Big((1 - X_i)^2(X_{p-1} - X_p)$$
$$y((X_{p-1} - X_p)[(1 - X_pX_{p-1} - 2X_i + X_i(X_p + X_{p-1}))+$$
$$(1 - X_i)(1 - X_p - X_{p-1} + X_i)] + y^2(1 - X_p)(1 - X_{p-1})(X_{p-1} - X_p)\Big)$$
$$- \frac{(1 - X_i) + y(1 - X_p)}{X_p - X_i}.$$

Then with some further simplications, we have

$$\frac{1}{(X_p - X_i)(X_{p-1} - X_i)}\Big((1 - X_i)^2 + y[(1 - X_i)(1 - X_p)+$$
$$(1 - X_i)(1 - X_{p-1}) - (X_p - X_i)(X_{p-1} - X_i)] + y^2(1 - X_p)(1 - X_{p-1})\Big)$$
$$- \frac{(1 - X_i) + y(1 - X_p)}{X_p - X_i}$$
$$= \frac{((1 - X_i) + y(1 - X_p))((1 - X_i) + y(1 - X_{p-1}))}{(X_p - X_i)(X_{p-1} - X_i)} - y$$
$$- \frac{(1 - X_i) + y(1 - X_p)}{X_p - X_i}.$$

□

Lemma 7.4.2. Given a polynomial $f \in \mathbb{Z}[X_1, ..., X_k]$, we have

$$\sum_{w \in S_k} \mathcal{T}_w(f) = \sum_{w \in S_k} w \left(f \prod_{1 \leq i < j \leq k} \frac{1 - X_j + y(1 - X_i)}{X_i - X_j} \right).$$

Proof. We proceed by induction on k. The base case is $k = 1$, which holds trivially.

So assume that

$$\sum_{w \in S_k} \mathcal{T}_w(f) = \sum_{w \in S_k} w \left(f \prod_{1 \leq i < j \leq k} \frac{1 - X_j + y(1 - X_i)}{X_i - X_j} \right).$$

We apply $(1 + \mathcal{T}_k + \mathcal{T}_{k-1}\mathcal{T}_k + ... + \mathcal{T}_1 ... \mathcal{T}_k)$ to both sides to get $\sum_{w \in S_{k+1}} \mathcal{T}_w(f)$ on the left side. Using

$$g = \sum_{w \in S_k} w \left(f \prod_{1 \leq i < j \leq k} \frac{(1 - X_j) + y(1 - X_i)}{X_i - X_j} \right)$$

for convenience, we are required to show that

$$(1 + \mathcal{T}_k + ... + \mathcal{T}_1 ... \mathcal{T}_k)(g) = \sum_{i=1}^{k+1} s_i ... s_k \left(g \prod_{j=1}^{k} \frac{(1 - X_{k+1}) + y(1 - X_j)}{X_j - X_{k+1}} \right),$$

where $s_i ... s_k$ is the identity permutation when $i = k + 1$. We accomplish this by descending induction on q to show

$$(1 + \mathcal{T}_k + ... + \mathcal{T}_q ... \mathcal{T}_k)(g) = \sum_{i=q}^{k+1} s_i ... s_k \left(g \prod_{j=q}^{k} \frac{(1 - X_{k+1}) + y(1 - X_j)}{X_j - X_{k+1}} \right).$$

Then as $q \to 1$, we get the required result. To show this, we again use descending induction on p to show

$$\begin{aligned}\mathcal{T}_p...\mathcal{T}_k(g) = &\sum_{i=p}^{k+1} s_i...s_k \left(g \prod_{j=p}^{k} \frac{(1-X_{k+1}) + y(1-X_j)}{X_j - X_{k+1}} \right) \\ &- \sum_{i=p+1}^{k+1} s_i...s_k \left(g \prod_{j=p+1}^{k} \frac{(1-X_{k+1}) + y(1-X_j)}{X_j - X_{k+1}} \right).\end{aligned}$$

Then as $p \to q$, the result is obtained.

To show this, we look at $\mathcal{T}_{p-1}(\mathcal{T}_p...\mathcal{T}_k(g))$. Since the operators are linear, we look at things term-by-term. For the $i = p$ term in the first sum (which does not have a counterpart in the second sum), we apply the definition for $\mathcal{T}_{p-1}$ to get

$$\begin{aligned}&\mathcal{T}_{p-1} s_p...s_k \left(g \prod_{j=p}^{k} \frac{(1-X_{k+1}) + y(1-X_j)}{X_j - X_{k+1}} \right) \\ &\quad = \frac{(1-X_p) + y(1-X_{p-1})}{X_{p-1} - X_p} s_p...s_k \left(g \prod_{j=p}^{k} \frac{(1-X_{k+1}) + y(1-X_j)}{X_j - X_{k+1}} \right) \\ &\quad + s_{p-1} \left(\frac{(1-X_p) + y(1-X_{p-1})}{X_{p-1} - X_p} s_p...s_k \left(g \prod_{j=p}^{k} \frac{(1-X_{k+1}) + y(1-X_j)}{X_j - X_{k+1}} \right) \right) \\ &\quad - s_p...s_k \left(g \prod_{j=p}^{k} \frac{(1-X_{k+1}) + y(1-X_j)}{X_j - X_{k+1}} \right).\end{aligned}$$

Applying $(s_p...s_k)^{-1} = s_k...s_p$ to $\frac{(1-X_p)+y(1-X_{p-1})}{X_{p-1}-X_p}$ yields

$$\frac{(1-X_p) + y(1-X_{p-1})}{X_{p-1} - X_p} = s_p...s_k \left(\frac{(1-X_{k+1}) + y(1-X_{p-1})}{X_{p-1} - X_{k+1}} \right).$$

As a result, we can simplify the previous expression to

$$\begin{aligned}
&\mathcal{T}_{p-1}s_p...s_k\left(g\prod_{j=p}^{k}\frac{(1-X_{k+1})+y(1-X_j)}{X_j-X_{k+1}}\right)\\
&=s_p...s_k\left(g\prod_{j=p-1}^{k}\frac{(1-X_{k+1})+y(1-X_j)}{X_j-X_{k+1}}\right)\\
&+s_{p-1}s_p...s_k\left(g\prod_{j=p-1}^{k}\frac{(1-X_{k+1})+y(1-X_j)}{X_j-X_{k+1}}\right)\\
&-s_p...s_k\left(g\prod_{j=p}^{k}\frac{(1-X_{k+1})+y(1-X_j)}{X_j-X_{k+1}}\right).
\end{aligned}$$

For the other terms, we recognize that since g is symmetric in the first k variables, $s_i...s_k(g)$ is symmetric in the first $i-1$ variables, and so $\mathcal{T}_{p-1}(s_i...s_k(fg)) = s_i...s_k(g)\mathcal{T}_{p-1}(s_i...s_k(f))$ for any polynomial f as long as $i \geq p+1$. Then in the product $s_i...s_k(\prod_{j=p}^{k}\frac{(1-X_{k+1})+y(1-X_j)}{X_j-X_{k+1}})$, only the $j=p$ term will include X_p or X_{p-1} for $i \geq p+1$, so the others can be pulled out and we have

$$\begin{aligned}
&\mathcal{T}_{p-1}s_i...s_k\left(g\prod_{j=p}^{k}\frac{(1-X_{k+1})+y(1-X_j)}{X_j-X_{k+1}}\right)\\
&\quad=s_i...s_k\left(g\prod_{j=p+1}^{k}\frac{(1-X_{k+1})+y(1-X_j)}{X_j-X_{k+1}}\right)\mathcal{T}_{p-1}\left(\frac{(1-X_i)+y(1-X_p)}{X_p-X_i}\right).
\end{aligned}$$

Now we apply part (2) of Lemma 7.4.1 to calculate this.

Then for the other sum, the terms are $-s_i...s_k(g\prod_{j=p+1}^{k}\frac{(1-X_{k+1})+y(1-X_j)}{X_j-X_{k+1}}$ for $i \geq p+1$. None of these include X_p or X_{p-1} anywhere, and so they are symmetric in those variables and so

$\mathcal{T}_{p-1}$ of these terms is just these terms times $\mathcal{T}_{p-1}(1)$. We have

$$T_{p-1}(1) = \frac{(1-X_p)+y(1-X_{p-1})}{X_{p-1}-X_p} + s_i\left(\frac{(1-X_p)+y(1-X_{p-1})}{X_{p-1}-X_p}\right) - 1$$
$$\frac{(X_{p-1}-X_p)+y(X_p-X_{p-1})}{X_{p-1}-X_p} - 1 = -y.$$

Then using these results, we have

$$\begin{aligned}
&\mathcal{T}_{p-1}\left(\sum_{i=p}^{k+1} s_i \ldots s_k \left(g\prod_{j=p}^{k}\frac{(1-X_{k+1})+y(1-X_j)}{X_j-X_{k+1}}\right)\right.\\
&\left.-\sum_{i=p+1}^{k+1} s_i \ldots s_k \left(g\prod_{j=p+1}^{k}\frac{(1-X_{k+1})+y(1-X_j)}{X_j-X_{k+1}}\right)\right)\\
&= s_p \ldots s_k\left(g\prod_{j=p-1}^{k}\frac{(1-X_{k+1})+y(1-X_j)}{X_j-X_{k+1}}\right)\\
&+ s_{p-1}s_p \ldots s_k\left(g\prod_{j=p-1}^{k}\frac{(1-X_{k+1})+y(1-X_j)}{X_j-X_{k+1}}\right)\\
&- s_p \ldots s_k\left(g\prod_{j=p}^{k}\frac{(1-X_{k+1})+y(1-X_j)}{X_j-X_{k+1}}\right)\\
&\sum_{i=p+1}^{k+1} s_i \ldots s_k\left(g\left(\prod_{j=p+1}^{k}\frac{(1-X_{k+1})+y(1-X_j)}{X_j-X_{k+1}}\right)\right.\\
&\left(\frac{((1-X_{k+1})+y(1-X_p))((1-X_{k+1})+y(1-X_{p-1}))}{(X_p-X_{k+1})(X_{p-1}-X_{k+1})} - y\right.\\
&\left.\left.-\frac{(1-X_i)+y(1-X_p)}{X_p-X_i}\right)\right)\\
&-\sum_{i=p+1}^{k+1} s_i \ldots s_k\left(g\prod_{j=p+1}^{k}\frac{(1-X_{k+1})+y(1-X_j)}{X_j-X_{k+1}}\right)(-y)\\
&=\sum_{i=p-1}^{k+1} s_i \ldots s_k\left(g\prod_{j=p-1}^{k+1}\frac{(1-X_{k+1})+y(1-X_j)}{X_j-X_{k+1}}\right)\\
&-\sum_{i=p}^{k+1} s_i \ldots s_k\left(g\prod_{j=p}^{k+1}\frac{(1-X_{k+1})+y(1-X_j)}{X_j-X_{k+1}}\right).
\end{aligned}$$

This completes the induction step, proving that

$$\begin{aligned}\mathcal{T}_p...\mathcal{T}_k(g) &= \sum_{i=p}^{k+1} s_i...s_k \left(g \prod_{j=p}^{k} \frac{(1-X_{k+1})+y(1-X_j)}{X_j - X_{k+1}} \right) \\ &- \sum_{i=p+1}^{k+1} s_i...s_k \left(g \prod_{j=p+1}^{k} \frac{(1-X_{k+1})+y(1-X_j)}{X_j - X_{k+1}} \right).\end{aligned}$$

Using this gives

$$\begin{aligned}\sum_{p=1}^{k+1} \mathcal{T}_p...\mathcal{T}_k(g) &= \sum_{p=1}^{k+1} \sum_{i=p}^{k+1} s_i...s_k \left(g \prod_{j=p}^{k} \frac{(1-X_{k+1})+y(1-X_j)}{X_j - X_{k+1}} \right) \\ &- \sum_{p=1}^{k+1} \sum_{i=p+1}^{k+1} s_i...s_k \left(g \prod_{j=p+1}^{k} \frac{(1-X_{k+1})+y(1-X_j)}{X_j - X_{k+1}} \right) \\ &= \sum_{i=1}^{k+1} s_i...s_k \left(g \prod_{j=p}^{k} \frac{(1-X_{k+1})+y(1-X_j)}{X_j - X_{k+1}} \right).\end{aligned}$$

Then by the definition of g we have

$$\begin{aligned}&\sum_{i=1}^{k+1} s_i...s_k \left(g \prod_{j=p}^{k} \frac{(1-X_{k+1})+y(1-X_j)}{X_j - X_{k+1}} \right) \\ &= \sum_{i=1}^{k+1} \sum_{w \in S_k} s_i...s_k w \left(f \left(\prod_{1 \le i < j \le k} \frac{(1-X_j)+y(1-X_i)}{X_i - X_j} \right) \prod_{i=1}^{k} \frac{(1-X_{k+1})+y(1-X_i)}{X_i - X_{k+1}} \right) \\ &= \sum_{w \in S_{k+1}} w \left(f \prod_{1 \le i < j \le k} \frac{(1-X_j)+y(1-X_i)}{X_i - X_j} \right).\end{aligned}$$

This completes the proof. □

Using this we can express the Motivic Segre class in terms of this operator.

Theorem 7.4.3. For a partition $\lambda = (\lambda_1, ..., \lambda_k)$ with $\lambda_1 \le N$, the T-equivariant motivic

Segre class of the Schubert cell in $\mathrm{Gr}(k, N+k)$ is given by

$$
\begin{aligned}
&mS_y^T(\Omega^{\lambda,\circ} \hookrightarrow \mathrm{Gr}(k,N+k)) \\
&= \sum_{w \in S_k} \mathcal{T}_w \left(\prod_{i=1}^{k} \prod_{j=1}^{k-i+\lambda_i} \frac{(X_i + T_j - X_i T_j)}{1 + y(1 - X_i - T_j + X_i T_j)} \right. \\
&\qquad \left. - \prod_{i=1}^{k} \prod_{j=1}^{k+1-i+\lambda_i} \frac{(X_i + T_j - X_i T_j)}{1 + y(1 - X_i - T_j + X_i T_j)} \right).
\end{aligned}
$$

Proof. Starting with Theorem 6.0.2, we have

$$
\begin{aligned}
&mS_y^T(\Omega^{\lambda,\circ} \hookrightarrow \mathrm{Gr}(k,N+k)) = \\
&\pi_* \left(\prod_{i=1}^{k} (1+y)[\mathcal{L}_i \otimes \mathbb{C}^\vee_{t_{k+1-i+\lambda_i}}] \prod_{j=k+1-i}^{k-i+\lambda_i} (1 - [\mathcal{L}_i \otimes \mathbb{C}^\vee_{t_j}]) \frac{\prod_{j=i+1}^{k} \lambda_y^T(\mathcal{L}_j^\vee \otimes \mathcal{L}_i)}{\prod_{j=1}^{k+1-i+\lambda_i} \lambda_y^T(\mathbb{C}^\vee_{t_j} \otimes \mathcal{L}_i)} \right).
\end{aligned}
$$

Then using the variables $Z_i = 1 - [\mathcal{L}_i]$ and $T_i = 1 - [\mathcal{C}^\vee_{t_i}]$ this becomes

$$
\begin{aligned}
&\pi_* \left(\prod_{i=1}^{k} (1+y)(1 - Z_i - T_{k+1-i+\lambda_i} + Z_i T_{k+1-i+\lambda_i}) \right. \\
&\qquad \left. \prod_{i=1}^{k} \frac{\prod_{j=k+1-i}^{k-i+\lambda_i} (Z_i + T_j - Z_i T_j)}{\prod_{j=1}^{k+1-i+\lambda_i} (1 + y(1 - Z_i - T_j + Z_i T_j))} \prod_{j=i+1}^{k} \left(1 + y \frac{1 - Z_i}{1 - Z_j} \right) \right).
\end{aligned}
$$

Now applying Lemma 3.5.1 gives

$$
\begin{aligned}
&mS_y^T(\Omega^{\lambda,\circ} \hookrightarrow \mathrm{Gr}(k,N+k)) = \\
&\sum_{w \in S_k} w \left(\left(\prod_{i=1}^{k} (1+y)(1 - X_i - T_{k+1-i+\lambda_i} + X_i T_{k+1-i+\lambda_i}) \right) \right. \\
&\left(\prod_{i=1}^{k} \frac{\prod_{j=1}^{k-i+\lambda_i} (X_i + T_j - X_i T_j)}{\prod_{j=1}^{k+1-i+\lambda_i} (1 + y(1 - X_i - T_j + X_i T_j))} \right) \\
&\left. \left(\prod_{1 \le i < j \le k} \left(1 + y \frac{1 - X_i}{1 - X_j} \right) \left(\frac{1 - X_j}{X_i - X_j} \right) \right) \right).
\end{aligned}
$$

Simplifying $\left(1+y\frac{1-X_i}{1-X_j}\right)\left(\frac{1-X_j}{X_i-X_j}\right)$ gives $\frac{(1-X_j)+y(1-X_i)}{X_i-X_j}$. Then we use the fact that

$$\begin{aligned}(1+y)(1-X_i-T_{k+1-i+\lambda_i}+X_iT_{k+1-i+\lambda_i})\\ &= [1+y(1-X_i-T_{k+1-i+\lambda_i}+X_iT_{k+1-i+\lambda_i}]\\ &- [X_i+T_{k+1-i+\lambda_i}-X_iT_{k+1-i+\lambda_i}]\end{aligned}$$

to simplify things. Then applying Lemma 7.4.2 to the right side of the claim yields the same thing. □

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