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Asymptotic solutions of second-order linear equations with three transition points

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A uniformly valid asymptotic expansion is obtained for the regular solution of a class of second-order linear differential equations with three transition points—a turning point and two regular singular points. The solution is found by matching three different solutions obtained using the Langer Transformation. The matching yields the eigenvalues and the eigenfunctions.

I. INTRODUCTION

We seek asymptotic solutions for large λ to the differential equation

$$\frac{d}{dx}\left((x+a)(b-x)\frac{dy}{dx}\right) + \left(\frac{p(x)}{(x+a)(b-x)} + \lambda x^n g(x)(x+a)^{m+1}(b-x)^{k+1}\right)y = 0 \quad (1)$$

that are regular on the interval $[-a, b]$, where a and b are positive numbers, $p(x)$ and $g(x)$ are regular functions and $g(x) > 0$ on $[-a, b]$, and $n, m,$ and k are integers such that $n \geq 0$, $m \geq -2$, and $k \geq -2$. For $n \neq 0$, Eq. (1) has three transition points—$x=0$ is a turning point and $x=-a$ and $x=b$ are regular singular points.

The special case $p(x) = 2(1-x^2)$, $g(x)=1$, $a=b=1$, $n=-m=-k=1$ describes stationary waves of small amplitude on the surface of a liquid sphere of unit radius whose center of mass is undergoing a constant acceleration.[1] Harper, Chang, and Grube[2] obtained a second-order asymptotic solution to this special case by using the method of matched asymptotic expansions (e.g., Chap. 4 of Ref. 3). The same technique was used by Jeffreys[4] to treat a problem with two simple turning points. The solution was represented by five asymptotic expansions valid on the intervals $[-1, -1+\delta_1]$, $[-1+\delta_1, -\delta_2]$, $[-\delta_2, \delta_3]$, $[\delta_3, 1-\delta_4]$, and $[1-\delta_4, 1]$, where the δ_i are small positive numbers. The five expansions were then matched to determine the eigenvalues and the eigenfunctions.

Nayfeh (Sec. 7.3.3 of Ref. 3) used a combination of the Langer transformation (e.g., Sec. 7.3.2 of Ref. 3) and the method of matched asymptotic expansions to obtain a uniformly valid asymptotic solution to a problem with two simple turning points. Rather than use the procedure of Ref. 2 and represent the solution by five asymptotic expansions, Nayfeh[3] represented the solution by only two expansions. Nayfeh[5] used the method of multiple scales (e.g., Sec. 6.4.4 of Ref. 3) to analyze the case of two simple turning points, while Nayfeh[6] used a combination of the Langer transformation and the method of matched asymptotic expansions to analyze a problem with two transition points—a turning point and a regular singular point of any order.

Problems with multiple transition points were also treated by using the Olver transformation (e.g., Sec. 7.3.2 of Ref. 3). The solution is represented by a single uniformly valid expansion by relating it to the solution of an equation which approximates the original equation. Using this approach, Olver,[7] Moriguchi,[8] and Pike[9] treated problems with two turning points. Problems with several turning points were treated by Evgrafov and Fedoryuk,[10] Hsieh and Sibuya,[11] Sibuya,[12] and Lynn and Keller[13] among others.

In this paper, we determine an asymptotic solution to Eq. (1) by using a combination of the Langer transformation and the method of matched asymptotic expansions. We prefer to use this technique rather than the Olver transformation because there exists no solution yet to the related equation. Thus, we represent the solution of the general problem by three expansions valid on the intervals $[-a, -\delta_2]$, $[-a+\delta_1, b-\delta_4]$, and $[\delta_3, b]$. Then, we match these expansions to determine the eigenvalues.

Before carrying out the expansions, it is more convenient to remove the first derivative in Eq. (1) by introducing the transformation

$$y(x) = u(x)(x+a)^{-1/2}(b-x)^{-1/2}. \quad (2)$$

The result is

$$\frac{d^2u}{dx^2} + \left(\frac{f(x)}{(x+a)^2(b-x)^2} + \lambda q(x)\right)u = 0, \quad (3)$$

where

$$f(x) = p(x) + \tfrac{1}{4}(a+b)^2, \qquad q(x) = x^n g(x)(x+a)^m(b-x)^k. \quad (4)$$

II. AN EXPANSION VALID ON $[-a, -\delta_2]$

To determine an expansion valid near the singular transition point $x=-a$, we note that, as $x \to -a$, Eq. (3) tends to

$$\frac{d^2u}{dx^2} + \left((-1)^n\lambda q_1(x+a)^m + \frac{r_1}{(x+a)^2}\right)u = 0, \quad (5)$$

where

$$q_1 = a^n g(-a)(b+a)^k \quad \text{and} \quad r_1 = f(-a)(b+a)^{-2}. \quad (6)$$

Hence, an asymptotic solution valid near $x=-a$ can be obtained by relating this solution to the solution of the "related" equation

$$\frac{d^2v}{dz^2} + \left((-1)^n\lambda z^m + \frac{r_1}{z^2}\right)v = 0. \quad (7)$$

To relate the solutions of Eq. (3) to the solutions of Eq. (7), we introduce the transformation (e.g., Sec. 7.3.9 of Ref. 3)

$$z = \phi(x), \qquad v = u(x)[\phi'(x)]^{1/2}, \quad (8)$$

$$\beta^{-1}\phi^\beta = \int_{-a}^x [q(-\xi)]^{1/2}\, d\xi = G_1(x)$$

in Eq. (3) and obtain

$$\frac{d^2v}{dz^2}+\left((-1)^n\lambda z^m+\frac{r_1}{z^2}\right)v=F_1(x)v, \tag{9}$$

where primes denote differentiation with respect to x and

$$F_1=\frac{r_1}{\phi^2}-\frac{1}{\phi'^2}\left(\frac{f(x)}{(x+a)^2(b-x)^2}+\frac{3\phi''^2}{4\phi'^2}-\frac{\phi'''}{2\phi'}\right), \tag{10a}$$

$$\beta=(m+2)/2.$$

As $x\to -a$,

$$\beta^{-1}\phi^\beta\to\beta^{-1}q_1^{1/2}(x+a)^\beta \tag{10b}$$

so that

$$\phi\to(q_1)^{1/2\beta}(x+a),\quad \phi'\to(q_1)^{1/2\beta},$$

and

$$F_1=O[(x+a)^{-1}].$$

Hence, a first-approximation to Eq. (9) is given by Eq. (7), whose general solution is

$$v=z^{1/2}[\tilde{c}_1J_\nu(\lambda^{1/2}\beta^{-1}z^\beta)+\tilde{c}_2J_{-\nu}(\lambda^{1/2}\beta^{-1}z^\beta)]\quad \text{for even } n \tag{11}$$

and

$$v=z^{1/2}[\tilde{c}_1I_\nu(\lambda^{1/2}\beta^{-1}z^\beta)+\tilde{c}_2I_{-\nu}(\lambda^{1/2}\beta^{-1}z^\beta)]\quad \text{for odd } n \tag{12}$$

where $\tilde{c}_1$ and $\tilde{c}_2$ are arbitrary constants and

$$\nu=(1-4r_1)^{1/2}/(2+m). \tag{13}$$

In what follows, we restrict our analysis to the case $r_1\leqslant\frac{1}{4}$ so that ν is real.

In order that y be regular at $x=-a$, Eqs. (2), (8), (11), and (12) show that $\tilde{c}_2=0$. Hence,

$$u_1(x)=c_1[G_1(x)]^{1/2}[q(x)]^{-1/4}J_\nu[\lambda^{1/2}G_1(x)][1+o(1)] \quad \text{as } \lambda\to\infty \tag{14}$$

for even n and

$$u_1(x)=c_1[G_1(x)]^{1/2}[q(-x)]^{-1/4}I_\nu[\lambda^{1/2}G_1(x)][1+o(1)] \quad \text{as } \lambda\to\infty \tag{15}$$

for odd n, where c_1 and c_2 are arbitrary constants. These expansions, although valid at $x=-a$, they break down as $x\to 0$ if $n\neq 0$. Thus, they are valid only on the interval $[-a,-\delta_2]$. An expansion valid near $x=0$ is obtained in the next section.

III. AN EXPANSION VALID ON $[-a+\delta_1, b-\delta_4]$

As $x\to 0$, Eq. (3) tends to

$$\frac{d^2u}{dx^2}+\lambda q_0x^nu=0, \tag{16}$$

where $q_0=a^mb^kg(a)$. Therefore, an asymptotic solution to Eq. (3) valid near $x=0$ can be obtained by relating it to the solutions of

$$\frac{d^2v}{dz^2}+\lambda z^nv=0. \tag{17}$$

To do this, we introduce the transformation

$$z=\phi(x),\quad v=u(x)[\phi'(x)]^{1/2},$$
$$\frac{2}{n+2}\phi^{(n+2)/2}=\int_0^x[q(\xi)]^{1/2}d\xi=G_0(x) \tag{18}$$

in Eq. (3) and obtain

$$\frac{d^2v}{dz^2}+\lambda z^nv=F_0v, \tag{19}$$

where

$$F_0=-\frac{1}{\phi'^2}\left(\frac{f(x)}{(x+a)^2(b-x)^2}+\frac{3\phi''^2}{4\phi'^2}-\frac{\phi'''}{2\phi'}\right). \tag{20}$$

As $x\to 0$, $\phi=O(x)$, $\phi'=O(1)$, and $F_0=O(1)$. Since λ is large, v is given approximately by Eq. (17) whose general solution is

$$v=z^{1/2}\left[\tilde{c}_3J_\mu\left(\frac{2\lambda^{1/2}}{n+2}z^{(n+2)/2}\right)+\tilde{c}_4J_{-\mu}\left(\frac{2\lambda^{1/2}}{n+2}z^{(n+2)/2}\right)\right] \tag{21a}$$

for all z if n is even and for $z\geqslant 0$ if n is odd, and

$$v=\xi^{1/2}\left[\tilde{c}_4I_{-\mu}\left(\frac{2\lambda^{1/2}}{n+2}\xi^{(n+2)/2}\right)-\tilde{c}_3I_\mu\left(\frac{2\lambda^{1/2}}{n+2}\xi^{(n+2)/2}\right)\right] \tag{21b}$$

for $z=-\xi<0$ if n is odd. Here, $\tilde{c}_3$ and $\tilde{c}_4$ are arbitrary constants and

$$\mu=(n+2)^{-1}. \tag{21c}$$

Note that the solution (21b) is an analytic continuation of the solution (21a). To see this, we express J_μ and $J_{-\mu}$ in terms of their power series expansions, use Eq. (21c), let $\tau=\lambda^{1/2}/(n+2)$, and rewrite Eq. (21a) as

$$v=\tilde{c}_3\tau^\mu z\sum_{m=0}^\infty\frac{(-1)^m\tau^{2m}z^{m(n+2)}}{m!\,\Gamma(m+\mu+1)} + \tilde{c}_4\tau^{-\mu}\sum_{m=0}^\infty\frac{(-1)^m\tau^{2m}z^{m(n+2)}}{m!\,\Gamma(m-\mu+1)}, \tag{22a}$$

which is an entire function of z, and hence it is defined over the whole complex z plane. Thus, we let $z=-\xi$ in Eq. (22a), use the fact that n is an odd integer, and obtain

$$v=-\tilde{c}_3\tau^\mu\xi\sum_{m=0}^\infty\frac{\tau^{2m}\xi^{m(n+2)}}{m!\,\Gamma(m+\mu+1)} + \tilde{c}_4\tau^{-\mu}\sum_{m=0}^\infty\frac{\tau^{2m}\xi^{m(n+2)}}{m!\,\Gamma(m-\mu+1)}, \tag{22b}$$

which is simply the power series representation of Eq. (21b). Therefore,

$$u_0=[G_0(x)]^{1/2}[q(x)]^{-1/4}\{c_3J_\mu[\lambda^{1/2}G_0(x)] \quad c_4J_{-\mu}[\lambda^{1/2}G_0(x)]\}[1+o(1)] \quad \text{as } \lambda\to\infty. \tag{23a}$$

for all x if n is even and for $x>0$ if n is odd, and

$$u_0=[G_0(-x)]^{1/2}[q(-x)]^{-1/4}\{c_3I_{-\mu}[\lambda^{1/2}G_0(-x)] - c_4I_\mu[\lambda^{1/2}G_0(-x)]\}[1+o(1)] \text{ as } \lambda\to\infty \tag{23b}$$

for $x<0$ if n is odd. Although this expansion is valid at $x=0$, it breaks down as $x\to -a$ or b. Thus, it is valid only on the interval $[-a+\delta_1, b-\delta_4]$. An expansion valid near $x=b$ is obtained in the next section.

IV. AN EXPANSION VALID ON $[\delta_3, b]$

As $x \to b$, Eq. (3) tends to

$$\frac{d^2u}{dx^2} + \left(\lambda q_2(b-x)^k + \frac{r_2}{(b-x)^2}\right)u = 0, \tag{24}$$

where

$$q_2 = b^n g(b)(a+b)^m \quad \text{and} \quad r_2 = f(b)(a+b)^{-2}. \tag{25}$$

Hence, an asymptotic solution for Eq. (3) valid near $x=b$ can be obtained by relating it to the solutions of

$$\frac{d^2v}{dz^2} + \left(\lambda z^k + \frac{r_2}{z^2}\right)v = 0. \tag{26}$$

This is accomplished by using the transformation

$$z = \phi(x), \quad v = u(x)[\phi'(x)]^{1/2}, \quad \beta^{-1}\phi^\beta = \int_x^b [q(\xi)]^{1/2}d\xi = G_2(x), \tag{27}$$

where $\beta = (2+k)/2$.

Introducing the transformation (27) into Eq. (3), we obtain

$$\frac{d^2v}{dz^2} + \left(\lambda z^k + \frac{r_2}{z^2}\right)v = F_2 v, \tag{28}$$

where

$$F_2 = \frac{r_2}{\phi^2} - \frac{1}{\phi'^2}\left(\frac{f(x)}{(x+a)^2(b-x)^2} + \frac{3\phi''^2}{4\phi'^2} - \frac{\phi'''}{2\phi'}\right). \tag{29}$$

As $x \to b$, $\phi = O[(b-x)]$, $\phi' = O(1)$, and $F_2 = O[(b-x)^{-1}]$. Hence, a first approximation to v is given by Eq. (26) whose general solution is

$$v = z^{1/2}[\tilde{c}_5 J_\gamma(\lambda^{1/2}\beta^{-1}z^\beta) + \tilde{c}_6 J_{-\gamma}(\lambda^{1/2}\beta^{-1}z^\beta)], \tag{30}$$

where $\tilde{c}_5$ and $\tilde{c}_6$ are arbitrary constants and

$$\gamma = (1-4r_2)^{1/2}/(2+k). \tag{31}$$

In what follows, we restrict our analysis to the case $r_2 \leq \frac{1}{4}$ so that γ is real.

In order that y be regular at $x=b$, Eqs. (2), (3), and (30) show that $\tilde{c}_6 = 0$. Hence,

$$u_2(x) = c_5[G_2(x)]^{1/2}[q(x)]^{-1/4}J_\gamma[\lambda^{1/2}G_2(x)][1+o(1)] \quad \text{as } \lambda \to \infty \tag{32}$$

Although this expansion is valid at $x=b$, it breaks down as $x \to 0$. In order to obtain a uniformly valid expansion on the interval $[-a, b]$, we match the three expansions obtained in this and the preceding two sections.

V. MATCHING WHEN n IS ODD

Since $u_1(x)$ and $u_0(x)$ are valid over on the interval $-a < -a+\delta_1 \leq x \leq -\delta_2 < 0$, they have a large overlapping region which allows their matching. To match these expansions, we fix x in this overlap interval and expand both u_0 as given by Eq. (23b) and u_1 for large λ. The result is

$$u_1 = c_1(2\pi)^{-1/2}[\lambda q(-x)]^{-1/4}\exp[\lambda^{1/2}G_1(x)][1+o(1)], \tag{33}$$

$$\begin{aligned} u_0 = {} & (2\pi)^{-1/2}[\lambda q(-x)]^{-1/4}\{(c_4-c_3)\exp[\lambda^{1/2}G_0(x)] \\ & + [c_4\exp(i\pi\mu) - c_3\exp(-i\pi\mu)] \\ & \times\exp(-\tfrac{1}{2}i\pi)\exp[-\lambda^{1/2}\tilde{G}_0(x)]\}[1+o(1)], \end{aligned} \tag{34}$$

where

$$\tilde{G}_0(x) = \int_x^0 [q(-\xi)]^{1/2}d\xi = G_0(-x). \tag{35}$$

The expansions (33) and (34) are two WKBJ approximations of $u(x)$ on the same interval $-a+\delta_1 \leq x \leq -\delta_2$. Hence, they must be identical. This is so if, and only if,

$$c_4 = c_3, \quad 2c_3\sin\mu\pi = c_1\exp\left(\lambda^{1/2}\int_{-a}^0 [q(-x)]^{1/2}dx\right). \tag{36}$$

To match u_0 and u_2, we note that they overlap on the interval $[\delta_3, b-\delta_4]$. Thus, we fix x in this overlap interval and expand both u_0 as given by Eq. (23a) and u_2 for large λ. The result is

$$u_0 = 2c_3(2/\pi)^{1/2}[\lambda q(x)]^{-1/4}\cos(\tfrac{1}{2}\pi\mu)\cos[\lambda^{1/2}G_0(x) - \tfrac{1}{4}\pi] \times[1+o(1)], \tag{37}$$

$$u_2 = c_5(2/\pi)^{1/2}[\lambda q(x)]^{-1/4}\cos[\lambda^{1/2}G_2(x) - \tfrac{1}{4}\pi - \tfrac{1}{2}\pi\gamma] \times[1+o(1)]. \tag{38}$$

Since Eqs. (37) and (38) represent u over the same interval $[\delta_3, b-\delta_4]$, they must be identical.

If we let

$$\Delta = \lambda^{1/2}(G_0+G_2) - \tfrac{1}{2}(1+\gamma)\pi = \int_0^b [q(x)]^{1/2}dx - \tfrac{1}{2}(1+\gamma)\pi, \tag{39}$$

then

$$\lambda^{1/2}G_0 - \tfrac{1}{4}\pi = \Delta - (\lambda^{1/2}G_2 - \tfrac{1}{4}\pi - \tfrac{1}{2}\gamma\pi) = \Delta - \alpha. \tag{40}$$

Equating Eqs. (37) and (38), using Eqs. (39) and (40), and equating the coefficients of $\sin\alpha$ and $\cos\alpha$ to zero, we obtain

$$\sin\Delta = 0 \text{ or } \Delta = j\pi, \quad j = 1, 2, 3, \cdots, \tag{41}$$

$$c_5 = 2c_3\cos\tfrac{1}{2}\pi\mu\cos j\pi. \tag{42}$$

Combining Eqs. (39) and (41), we find that the eigenvalues are

$$\lambda = \pi^2\left((j + \tfrac{1}{2} + \tfrac{1}{2}\gamma)/\int_0^b [q(x)]^{1/2}dx\right)^2. \tag{43}$$

For the special case, $a=b=1$, $p(x) = 2(1-x^2)$, $g(x)=1$, $n=-m=-k=1$, Eq. (43) reduces to the first-order solution of Ref. 2. We emphasize again that we represented the solution by only three expansions which were matched, whereas the solution was represented by five expansions in Ref. 2.

VI. MATCHING WHEN n IS EVEN

To match u_1 as given by Eq. (14) with u_0, we fix x in the interval $[-a+\delta_1, -\delta_2]$, expand u_1 and u_0 for large λ, equate the results, and obtain

$$\begin{aligned} c_1\cos(\lambda^{1/2}G_1 - \tfrac{1}{4}\pi - \tfrac{1}{2}\pi\nu) = {} & c_3\cos(\lambda^{1/2}G_0 - \tfrac{1}{4}\pi - \tfrac{1}{2}\pi\mu) \\ & + c_4\cos(\lambda^{1/2}G_0 - \tfrac{1}{4}\pi + \tfrac{1}{2}\pi\mu) \end{aligned} \tag{44}$$

To match u_0 and u_2, we fix x in the interval $[\delta_3, b-\delta_4]$, expand u_0 and u_2 for large λ, equate the results, and obtain

$$\begin{aligned} c_3\cos(\lambda^{1/2}G_0 - \tfrac{1}{4}\pi - \tfrac{1}{2}\pi\mu) + c_4\cos(\lambda^{1/2}G_0 - \tfrac{1}{4}\pi + \tfrac{1}{2}\pi\mu) \\ = c_5\cos(\lambda^{1/2}G_2 - \tfrac{1}{4}\pi - \tfrac{1}{2}\pi\gamma). \end{aligned} \tag{45}$$

If we let

$$\Delta_1 = \lambda^{1/2}(G_1 - G_0) + \tfrac{1}{2}\pi(\mu - \nu), \tag{46}$$

then

$$\lambda^{1/2}G_0 - \tfrac{1}{4}\pi - \tfrac{1}{2}\pi\mu = \alpha - \Delta_1, \tag{47}$$

$$\lambda^{1/2}G_0 - \tfrac{1}{4}\pi + \tfrac{1}{2}\pi\mu = \alpha - (\Delta_1 - \pi\mu), \tag{48}$$

where

$$\alpha = \lambda^{1/2}G_1 - \tfrac{1}{4}\pi - \tfrac{1}{2}\pi\nu. \tag{49}$$

Substituting Eqs. (46)—(49) into Eq. (44) and equating the coefficients of cos α and sin α on both sides, we obtain

$$c_3 \cos\Delta_1 + c_4 \cos(\Delta_1 - \pi\mu) = c_1, \tag{50}$$

$$c_3 \sin\Delta_1 + c_4 \sin(\Delta_1 - \pi\mu) = 0. \tag{51}$$

Similarly, we obtain from Eq. (45) the following relationships:

$$c_3 \cos\Delta_2 + c_4 \cos(\Delta_2 + \mu\pi) = c_5, \tag{52}$$

$$c_3 \sin\Delta_2 + c_4 \sin(\Delta_2 + \mu\pi) = 0, \tag{53}$$

where

$$\Delta_2 = \lambda^{1/2}(G_0 + G_2) - \tfrac{1}{2}\pi - \tfrac{1}{2}\pi(\mu + \gamma). \tag{54}$$

In order that Eqs. (51) and (53) have a nontrivial solution,

$$\sin\Delta_1 \sin(\Delta_2 + \mu\pi) - \sin\Delta_2 \sin(\Delta_1 - \mu\pi) = 0,$$

which gives

$$\sin(\Delta_1 + \Delta_2) = 0.$$

Hence,

$$\Delta_1 + \Delta_2 = j\pi, \quad j = 1, 2, 3, \cdots. \tag{55}$$

Substituting for Δ_1 and Δ_2 from Eqs. (46) and (54) into Eq. (55), then substituting for G_1 and G_2 from Eqs. (8) and (27) into the resulting expression, and solving for λ, we obtain

$$\lambda = \pi^2\left([j + \tfrac{1}{2}(1 + \gamma + \nu)] \Big/ \int_{-a}^{b} [q(x)]^{1/2} dx\right)^2. \tag{56}$$

Once λ is known, we can solve Eq. (53) to determine c_4 as a function of c_3, and then Eqs. (50) and (52) to determine c_1 and c_5 in terms of c_3.

VII. SUMMARY

A general procedure is presented for the determination of approximate solutions of linear differential equations with multiple transition points. The procedure is a combination of the Langer transformation and the method of matched asymptotic expansions. It is applied to a class of second-order differential equations with three transition points—a turning point of any order and two regular singular points.

The solution is represented by three different regular asymptotic expansions. Each expansion is valid on an open interval containing one of the transition points but excluding the other two. These expansions were then matched to relate the arbitrary constants and determine the eigenvalues. Adding these expansions and subtracting their common parts, one can determine a so-called composite expansion, which is a single uniformly valid expansion.

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