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# Bistability in an Ising model with non-Hamiltonian dynamics

J.R. Heringa
*Laboratorium voor Technische Natuurkunde, Technische Universiteit Delft, P.O. Box 5046, 2600 GA Delft, The Netherlands*

H. Shinkai
*Physics Department, Faculty of Science and Engineering, Waseda University, Shinjuku-ku, Tokyo 169, Japan*

H.W.J. Blöte and A. Hoogland
*Laboratorium voor Technische Natuurkunde, Technische Universiteit Delft, P.O. Box 5046, 2600 GA Delft, The Netherlands*

R.K.P. Zia
*Center for Stochastic Processes in Science and Engineering and Physics Department, Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061*
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We investigate the phenomenon of magnetization bistability in a two-dimensional Ising model with a non-Hamiltonian Glauber dynamics by means of Monte Carlo simulations. This effect has previously been observed in the Toom model, which supports two stable phases with different magnetizations, even in the presence of a nonzero field. We find that such bistability is also present in an Ising model in which the transition probabilities are expressed in terms of Boltzmann factors depending only on the nearest-neighbor spins and the associated bond strengths. The strength on each bond assumes different values with respect to the spins at either of its ends, introducing an asymmetry like that of the Toom model.

Recently, considerable attention has been given to the question of how Ising-like models behave when a preferred direction is introduced through nonequilibrium Glauber[1] dynamics. It has been observed that the two-dimensional Ising universality class is stable against this type of perturbation.[2–5] An example of such a model is the north-east-center (NEC) cellular-automaton model discussed by Toom[6] and later investigated by Bennett and Grinstein.[2] This model consists of Ising spins on a square lattice, which are updated simultaneously. New spin values are assigned with probabilities that depend only on the old value of the "center" spin and on the "north" and "east" neighbors. A second-order phase transition (such as present in the equilibrium Ising model) was found to persist in this model[2] with zero magnetic field. When a magnetic field is applied below the transition temperature, bistability occurs: if the magnetic field does not exceed a certain (temperature-dependent) strength, both a positively and a negatively magnetized state are stable. Thus the first-order line of the equilibrium Ising model splits up in two first-order lines enclosing the region of bistability. On these lines, the magnetic field was observed to be proportional to the third power of the distance to the critical point.[2] Similar behavior was found[3] in various continuum versions of the lattice model.

The purpose of the present work is to investigate whether such bistability occurs also in a similar model, but with a dynamics which "interpolates" between the equilibrium Ising case and and an NEC-like nonequilibrium system. Using a random instead of simultaneous updating of the spins, a perturbation with the same symmetry as the NEC model is introduced via anisotropic transition probabilities of the Yang[7] type. Specifically, the probability $P$ of the new value of a spin $s_{x,y}(=\pm 1)$ with coordinates $(x,y)$ depends on the old values of only its neighboring spins, and is given by

$$P(s_{x,y}) = \frac{\exp[(K^{(1)}_{x,y}s_{x+1,y} + K^{(2)}_{x,y}s_{x,y+1} + K^{(3)}_{x,y}s_{x-1,y} + K^{(4)}_{x,y}s_{x,y-1} + H)s_{x,y}]}{2\cosh(K^{(1)}_{x,y}s_{x+1,y} + K^{(2)}_{x,y}s_{x,y+1} + K^{(3)}_{x,y}s_{x-1,y} + K^{(4)}_{x,y}s_{x,y-1} + H)}, \quad (1)$$

where $H$ is the reduced magnetic field, and the $K^{(i)}_{x,y}$ specify the bond strengths, while the superscript $i = 1,2,3,4$ labels the N,E,S,W bonds, respectively. The following choice is made:

$$K^{(1)}_{x,y} = K^{(2)}_{x,y} = \alpha K, K^{(3)}_{x,y} = K^{(4)}_{x,y} = K.$$

This model is described by three parameters: $H$, $\alpha$, and $K$. For $\alpha$ =1, the model reduces to the equilibrium Ising model. For $\alpha \neq 1$, it can be interpreted in terms of a standard Ising model with a temperature gradient, the effect of which is balanced by inhomogeneous interactions such that the couplings are uniform.[5]

We have investigated the model described by Eq. (1) for a few values of $\alpha \neq 1$, by means of Monte Carlo simulations on the Delft Ising System Processor (DISP), a special-purpose computer for the simulation of Ising models.[8,9] The architecture of this machine is not restricted to the models with equilibrium dynamics; it can also deal with $\alpha \neq 1$ transition rates which depend only on a local spin environment. This environment is coded as a binary number which serves as an address to the look-up table in which the transition rates are stored. A comparison between the transition probability and a pseudorandom number (generated by a binary shift register) then determines the new spin value. Since the look-up table is loaded by means of a software program running on the host computer, no hardware modifications were necessary.

The zero-field critical point of the model described by Eq. (1) has already been investigated[5] for $\alpha$=2 and 4. It was found that the Ising critical point persists for $\alpha \neq 1$; the critical exponents are, within narrow uncertainties, equal to the exactly known values for the two-dimensional equilibrium Ising model. Simulations were performed at three additional coupling ratios: $\alpha$ = 1.2, 1.5, and 8. Similar data are also available for the equilibrium case ($\alpha = 1$, Onsager's model).[10]

Shown in Table I are the results for the critical point, the temperature and the magnetic exponent, and another universal quantity, namely the large-$L$ (system size) limit $Q_c$ of

$$Q_L = < M_L^2 >^2 / < M_L^4 >, \qquad (2)$$

where $< M_L >$ is the magnetization of a system with size $L$. The methods of analysis leading to these results, as well as the results for the cases $\alpha$ = 1, 2, and 4, were already given in Refs. 10 and 5. The results confirm that the model obeys Ising universality. Note that $Q_c$ is nonuniversal in the sense that it depends on the aspect ratio if the system is not square, or similarly on the coupling ratio in two directions if the renormalized Hamiltonian has such anisotropy. Thus, $Q_c$ may depend on $\alpha$. Since we do not need high-precision data for the critical points in our present analysis, we have chosen for simulations with a length up to $10^5$ per data point, which is a factor 10 less than for those reported in Refs. 5 and 10.

Near and below the critical temperatures thus obtained, we have investigated whether bistability occurs.

TABLE I. Critical exponents and dimensionless ratio $Q_c$ for several values of the ratio $\alpha$. For comparison, numerical results (Ref. 10) for the exactly solved case $\alpha$=1 are also included. The exact transition point of this model is $K_c = \ln(1+\sqrt{2})/2 = 0.44069\cdots$, and the exponents are $y_t = 1$ and $y_h = \frac{15}{8}$. The results for $\alpha$ = 2 and 4 are taken from Ref. 5.

| $\alpha$ | $\alpha K_c$ | $y_t$ | $y_h$ | $Q_c$ |
|---|---|---|---|---|
| 1.0 | 0.4408(3) | 0.997(12) | 1.874 (3) | 0.858(7) |
| 1.2 | 0.4820(6) | 0.984(21) | 1.875 (8) | 0.851(7) |
| 1.5 | 0.5353(7) | 0.982(20) | 1.868 (6) | 0.857(3) |
| 2.0 | 0.6102(4) | 0.990(10) | 1.875 (4) | 0.851(3) |
| 4.0 | 0.8130(6) | 0.989(10) | 1.873 (3) | 0.849(3) |
| 8.0 | 1.036 (2) | 0.982(22) | 1.865(10) | 0.845(8) |

Consider a configuration in which the majority of the spins are negative, with only isolated islands of positive spins, in the presence of a positive field. As in the NEC model, straight interfaces between the two ordered states drift under the influence of $H$ and $\alpha$. The drift velocity depends also on the orientation of the interface, such that each island tends to shrink against the action of the field, at least if the field is not too large. As noted in Ref. 3, these islands tend to assume a triangular shape with a north-south, an east-west, and a NW-SE interface. The interface motions are easiest to understand in the zero-temperature limit. A north-south (east-west) interface with the minus spins on its west (south) side drifts in the direction dictated by the field if $H > 2K$. The drift will slow down with decreasing temperature if $H < 2K$. A NW-SE interface with minus spins at its NE side drifts in the SW direction if $H < 2(\alpha-1)K$. This drift will *not* freeze at low temperatures. Therefore, at a sufficiently low temperature, islands having such a triangular shape will shrink if $H$ is smaller than both $2K$ and $2(\alpha-1)K$.

The actual simulations used initial configurations consisting of a $256 \times 256$ lattice with a negatively magnetized background, and a triangularly shaped island of plus spins.[2] After specification of each set of parameters $K$, $\alpha$, and $H$, a Monte Carlo simulation of $5 \times 10^4$ sweeps was performed, and it was observed whether the system had evolved into a positively or a negatively magnetized state. Near the limits of bistability, this procedure was repeated for small increments of $H$, typically some 20 times.

The finite-size dependence of the limits of bistability was investigated by means of simulations of smaller $L \times L$ systems. As in Ref. 2, the finite-size effect was found to be approximately proportional to $1/L$. Thus, we estimate the finite-size corrections to the $L = 256$ data as the differences between the $L = 256$ and the $L = 128$ data.

After application of these corrections, the results for the first-order lines in the $H$-$K$ diagram are shown in Fig. 1. It shows the field $H$ as a function of $K - K_c$ on a logarithmic scale. For all values of $\alpha$ the data points lie close to straight lines, at least if they are close enough to the critical point. The slope of these lines suggests a power-law behavior $H \sim (K-K_c)^p$ with $p = 2.7 \pm 0.2$, which is close to $p = 3.0 \pm 0.4$,[2] as observed for the NEC model. At sufficiently low temperatures, we expect linear behavior instead; indeed, clear deviations from $p = 2.7$ behavior can be observed in Fig. 1.

An interesting question is to what extent renormalization rules for equilibrium systems apply to these dynamic stability phenomena. Parametrizing the system in terms of a temperature field $t \sim K - K_c$, a magnetic field $h$, and a nonequilibrium field $\alpha - 1$, a rescaling transformation may be expressed as

$$(t, h, \alpha-1) \rightarrow (l^{y_t} t, l^{y_h} h, l^{y_\alpha}(\alpha-1)), \qquad (3)$$

where $l$ is the rescaling factor and $y_\alpha$ the renormalization exponent associated with $\alpha - 1$. At the equilibrium fixed point, $y_t = 1$ and $y_h = \frac{15}{8}$. Concerning the value of $y_\alpha$, we will apply the following real-space renormaliza-

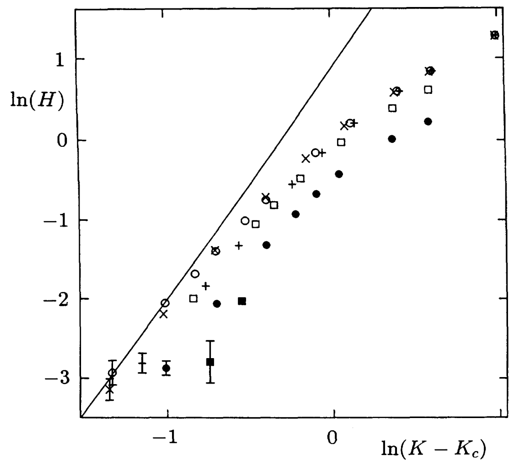


FIG. 1. Limits of bistability in the $H$-$K$ diagram. The maximum value of the field $H$, below which magnetization bistability occurs, is shown as a function of the distance $K - K_c$ to the critical point on a logarithmic scale, for several values of the ratio $\alpha$, namely $\alpha = 1.1$ (■), $\alpha = 1.2$ (•), $\alpha = 1.5$ (□), $\alpha = 2$ (○), $\alpha = 4$ (×), and $\alpha = 8$ (+). The $K_c$ values were taken from Table I, with the exception of the case $K_\alpha = 1.1$. The latter critical point was estimated from the observation (Ref. 5) that $\sinh 2K_c \sinh 2\alpha K_c = 1$ is a good approximation of the critical line, especially near $\alpha = 1$. The slope of the straight line corresponds with $H \sim (K - K_c)^{23/8}$, according to the conjecture described in the text. Error bars are shown only where they exceed the size of the symbols.

tion argument. It is plausible that the dynamic behavior is subject to a linear response of the fluctuation stream velocity proportional to $\alpha - 1$. Under rescaling, the velocity decreases with the scale factor $l$, as measured in the old time units. Since the time scale renormalizes with a factor $l^z$ where $z \approx 2$ is the dynamic exponent,[11] the fluctuation stream velocity as expressed in the new units is faster by a factor $l^{z-1}$. This leads to a relevant exponent $y_\alpha = z - 1$. In order to reconcile this value with the conclusion of Grinstein, Jayaprakash, and He[4] that there are no relevant operators due to nonequilibrium dynamics, we may interpret $\alpha - 1$ in terms of a redundant field.

However, $\alpha - 1$ need not be a pure scaling field; there may be a contribution to an irrelevant field as well. These fields arise naturally under renormalization, thereby enlarging the parameter space with respect to the model defined above. Such an irrelevant scaling field should be treated as dangerous because it leads to bistability. A scaling transformation involving such a field $a \sim \alpha - 1$ with exponent $y_a$ reads

$$(t, h, a) \to (l^{y_t} t, l^{y_h} h, l^{y_a} a). \tag{4}$$

Choosing $l = t^{-1/y_t}$ yields renormalized parameters $(1, t^{-y_h/y_t} h, t^{-y_a/y_t} a)$. At fixed $t \neq 0$ and small $h$ and $\alpha - 1$, we may expect that the limit of bistability is determined by a balancing of the linear responses of both small parameters on the growth rates. Indeed, the results for $\alpha = 1.5$, 1.2, and additional Monte Carlo data taken at 1.1 (see Fig. 1) agree with such a linear dependence. The requirement that small renormalized values of $h$ and $a$ are proportional leads to $h \sim t^{(y_h - y_a)/y_t}$. Assuming furthermore that $y_a$ has the trivial value $-1$, we obtain the conjecture $p = 2\frac{7}{8}$, which is consistent with the numerical results.

In conclusion, we have confirmed that magnetization bistability is present in an Ising model with non-Hamiltonian Glauber dynamics and with the same symmetry as the Toom[6] model.

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