

Experimental and Analytical Examination of Golf Club Dynamics

by

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Thesis submitted to the faculty of

Virginia Polytechnic Institute and State University

in partial fulfillment of the requirements for the degree of

Master of Science

in

Mechanical Engineering

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December 11, 1998
Blacksburg, Virginia

Keywords: Finite Element Analysis, Experimental Modal Analysis, Golf

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(Abstract)

To provide the average golfer with more consistent results, manufacturers have continued to improve the available equipment. This has led to larger club-heads, with larger “sweet spots”, different shaft thickness for different swing styles, and the use of advanced materials, such as graphite and titanium, for the construction.

The development of improved equipment, which utilizes advanced materials, has spurred the need for advanced scientific analysis using a variety of techniques. Among the most prevalent of these methods are finite element analysis and experimental modal analysis, and use of these techniques in examining a golf club is the focus of this work.

The primary goals of this work are the development and correlation of an appropriate finite element model, the characterization of the hands-free boundary condition and the examination of the club golf dynamic response. To accomplish these objectives, the physical parameters of the golf club are determined to develop the finite element model. The analysis of natural frequencies and mode shapes correlate well with the results extracted from experimental modal analysis for the free-free and clamped-free boundary conditions. With the correlation established, a third boundary condition, hands-free, is tested experimentally to ascertain the effects of the golfer’s grip on the boundary conditions. With the FEA model confirmed, a nonlinear dynamic response of the club during the down-swing is investigated using the nonlinear solver in Algor, and the club-head position relative to the shaft is predicted.

Acknowledgements

Dr. Charles Knight

I would like to thank Dr. Knight for all of his help, patience and guidance. Under his tutelage, I have enhanced my knowledge of finite elements and their applications, and I have grown in my development as an engineer. I owe him my utmost gratitude.

Dr. Reginald Mitchiner
and Dr. Alfred Wicks

I would like to thank the members of my committee for all their assistance and valuable advice. Their knowledge of mechanics of materials, vibrations and modal analysis proved to be an invaluable asset.

Jim Neighbors

I would like to thank Jim for all his help, especially with the modal analysis. His assistance was instrumental to the completion of this project, and his patience as an office mate and friend will not be forgotten.

Dan Clatterbuck, Steve
Weidmann and Zachary Kitts

I would like to thank these three individuals for their entertainment, friendship and support. They have made my time at Virginia Tech a memorable period.

Karen Kowalski

I would like to thank Karen for all her love, patience and unwavering support. She has truly been my guiding light.

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Chapter 1

Introduction and Literature Review

The game of golf is an enigma. Shots, which are solidly hit straight and true on one day, may slice, fade or hook on the next, and for years, golfers of all ages have attempted to conquer, or at least understand, the mysterious nature of this game. To provide the average player with more consistent results, manufacturers have continued to improve the available equipment. This has led to larger club-heads, with larger corresponding “sweet spots”, different shaft thickness for different swing styles, and the use of advanced materials, such as graphite and titanium, for the construction.

The development of improved equipment, which utilizes advanced materials, has spurred the need for advanced scientific analysis using a variety of techniques. Among the most prevalent of these methods are finite element analysis and experimental modal analysis, and use of these techniques in examining a golf club is the focus of this research.

The finite element method is a numerical method for analyzing structures and system that are typically too complex to be analyzed through standard analytical

techniques. In this technique, a structure is divided piecewise into elements, and the performance of each element is then simply characterized. The elements are then connected, and the resulting algebraic equations are simultaneously solved utilizing computational capabilities.

Experimental modal analysis has emerged as an extremely useful procedure. Performed under controlled conditions, it encompasses excitation of a structure, or component; acquisition of data; and the subsequent analysis of the response. The uses of modal analysis are varied and range from determination of natural frequencies and damping factor to full development of a mass-spring-damper model of a particular system.

For this work, a finite element model, which uses beam elements, is developed from analytically and experimentally determined data and is used to examine the dynamic response of the golf club. The number of elements is increased until convergence, and resulting eigenvalues and eigenvectors are subsequently correlated with results from experimental modal analysis.

Two boundary conditions, free-free and clamped-free, are employed for both techniques since these conditions are among easiest and most widely used for both methods. Also, an additional case, hands-free, is investigated solely using modal analysis to ascertain the boundary conditions of a typical grip.

The experimental modal analysis is performed on a golf club, and the mode frequencies and mode shapes are then extracted for three different boundary conditions. The first two boundary conditions, free-free and clamped-free, are used to compare with

the finite element model, while the final case, hands-free, is employed to simulate the conditions due the golfer gripping the club.

The finite element analysis (FEA) and experimental modal analysis (EMA) results are correlated to determine the suitability of the finite element model, and with this established, the nonlinear dynamic response of the club during the down-swing is investigated using FEA in the software package Algor[®]. To accomplish this task, swing shaft strain data is first calibrated and converted to produce an input moment curve, which is then approximated in the software. This information is input into the software, which automatically converges the model at the desired time steps, and the results are then examined to determine shaft and club-head response, with special interest in the club-head position at impact with the ball.

Literature Review

The earliest scientific analysis of the golf-swing and club performance is contained in Cochran and Stobbs' ^[1] groundbreaking work, *The Search for the Perfect Swing*. Since its publication in 1968, numerous scientific articles concerning golf have been published, and three scientific congresses on the game have been held.

Determination of the Club-Head Inertia Properties

The center of gravity of a structure is traditionally determined by suspending an object at several different points, noting the position and using the principles of statics. To determine the club-head center of gravity, Thomas, Deiter and Best ^[2] utilize a similar technique. The club-head is suspended from a string and photographed with the appropriate scale, and the center of gravity is then determined from the scales and

photographs. This method can be quite cumbersome, so Twigg and Butler^[3] utilize an “analytical balance” with a sliding beam balance and the sum of moments principle about a pivot to determine the center of gravity.

The club-head moments of inertia are generally determined using two different approaches. The first method, utilized by Thomas et al^[2] and Johnson^[4], uses a pendulum to determine the moment of inertia about three axes. The values are obtained for a second coordinate system, and the principle moments of inertia are then determined using the direction cosines between the two coordinate systems or by solving the inertia matrix. For the other method, used by Twigg and Butler, the moment of inertia is obtained by spinning the object on a spring attached to a drill motor, and “as the system reaches steady-state”, the object will align itself with a principle moment axis.

Modal Analysis and Finite Element Analysis

The use of modal analysis and finite element analysis has increased in the determination of shaft response. The correlation of FEA and EMA is examined in a number of articles, with most focusing on the correlation of results for particular boundary conditions and some attempting to characterize the hands-free case as either free-free or clamped-free.

In their work, Swider and Ferraris^[5] model the shaft and club-head using plate elements but consider only a clamped-free boundary condition for the club. Examination of the results reveals “good” correlation between the experimental and finite element mode shapes and frequencies, and this suggests that an appropriate FEA model can be developed for the club.

Okubo and Simada ^[6] consider the three boundary conditions – free-free, clamped-free and hands. In this case, tests are performed solely on 1 wood, and the EMA results are compared with “the vibration and strain mode shapes generated using computer aided engineering.” Based upon their results, the authors suggest that the boundary conditions due to gripping vary through the swing, with the conditions similar to clamped-free at lower frequencies and closer to the free-free case at higher frequencies.

Thomas, Deiter and Best ^[2] model the shaft using tapered beam elements, and the club-head is modeled as a lumped mass at the center of gravity. Three boundary conditions are again employed, but for the finite element analysis, the hands-free case is assumed to be a clamped-free case using springs to represent the hands. Additionally, only the FEA and EMA frequency results are compared, with no comparison of mode shapes included.

Friswell, Smart and Mottershead ^[7] also utilize beam elements, but only the clamped-free boundary condition is analyzed to avoid any uncertainty that may be associated with the hands-free boundary condition. The club-head mass moments of inertia are initially determined from a computer-aided design (CAD) package and then updated in their model.

Experimental frequencies are determined directly from the frequency response functions (FRFs) and then compared with the initial and updated FEA results. Friswell, Smart and Mottershead suggest that mode shapes are difficult to obtain “due the mass loading of the accelerometer”, and thus the typical “modal analysis techniques, using a roving accelerometer or roving hammer excitation, are impractical on the golf club.”

Dynamic Response and Simulation

Cochran and Stobbs provide the earliest analysis of the golf swing and the shaft response. In the work, Cochran and Stobbs model the golfer and the golf club as a two-pendulum system, with the upper pendulum constituting the golfer's shoulders and arms. With the wrists as a pivot, the lower pendulum consists of the club, hands and wrist.

Using this double pendulum model, Budney and Bellows^[8] develop a dynamic model of the club to relate the forces and torques that are present during the swing. Expanding on this, a kinetic analysis of the golf swing is performed, which establishes the force curve for the swing.^[9]

While the double pendulum model is the most commonly held, the swing can also be modeled as a cantilever beam that is attached to a rotating hub. Using this method, the natural frequencies have been determined by Schilansl^[10], Rao and Carnegie^[11] and Pnuelli^[12].

The dynamic deflections are determined by both Christensen and Lee^[13] and Yoo, Ryan and Scott^[14]. Christensen and Lee utilize a Newton-Raphson method to solve a nonlinear finite element model, which considers both axial and transverse deflections. Meanwhile, Yoo, Ryan and Scott use the Raleigh-Ritz method to solve a set of linearized equations, which account for the both axial and transverse deflections.

The effects of tip mass are introduced in the works of Bhat^[15], Hoa^[16], Lee^[17], Putter and Mannor^[18] and Winfield and Soriano^[19]. In Bhat's approach, the mode frequencies and mode shapes are determined by the Raleigh-Ritz method, while the remaining approaches use the finite element method to determine the natural frequencies.

In their work, Winfield and Soriano determine “the dynamic response of the beam due to a specified hub rotation”, and the results of this analysis suggest that the club-head “kicks” or “springs back” just before impact with the ball. In other words, the club-head position is behind the shaft through the majority of the swing, but just prior to impact, the club-head quickly moves to a position just in front of the shaft. This sudden change of club-head position, or “kick”, is then thought to impart greater momentum on the ball.

This “kick” phenomenon has also been the focus of other works, and there appears to be a slight controversy on the actual club-head position just prior to impact. On one side, Horwood^[20], Thomas et al^[2], Masuda and Korjima^[21] all believe that the club-head “springs back” just beforehand. On the other side, Milne^[22] and Milne and Davis^[23] believe the club-head lags just before impact.

Chapter 2

Finite Element Theory

The finite element method is a numerical method for analyzing structures which are usually too complicated to be solved through standard analytical techniques. In this method, a structure is divided piecewise into elements, and the response of each element is simply characterized. The elements are connected, and the resulting algebraic equations are simultaneously solved utilizing computational capabilities.

The finite element method is utilized in a wide range of applications including, heat transfer, fluid mechanics, acoustics, electromagnetism, and structural mechanics, and the desired field quantity is particular to each area of interest. The primary interest of this work is the area of structural dynamics, and the desired field quantities are the natural frequencies and displacement response.

A brief discussion of the element formulation and solution techniques is discussed in the subsequent sections. The formulation of the stiffness matrix for beam elements is first considered, and the various types of dynamic analysis and the formulations of the mass and damping matrices follow.

Stiffness Matrix

The stiffness matrix relates nodal displacements to nodal forces. There are three basic methods used to determine the stiffness matrix – the direct method, the variational method, and the weighted residual method.

The direct method, based on physical understanding, is limited to simple elements, but is helpful in understanding the finite element method. In this technique, force components and general displacements are related by the following equation,

$$\mathbf{k}\mathbf{d} = \mathbf{f} \quad (2.1)$$

$\mathbf{k}$ is the element stiffness matrix, $\mathbf{d}$ is the associated nodal displacement vector, and $\mathbf{f}$ is the internal force component vector. Considering the physical characteristics of the element, the stiffness matrix is produced from the superposition of simple element solutions. Applying a unit displacement to one component while the remaining components remain zero, the magnitude of the force required to maintain the displacement state is evaluated. The procedure is repeated for the remaining components, and the values are recorded in matrix form.

The development of the stiffness matrix is first examined for a bar element and the simple beam element. Using superposition, the 2-D plane beam and 3-D beam element are developed from these two elements and basic beam theory.

Bar Element

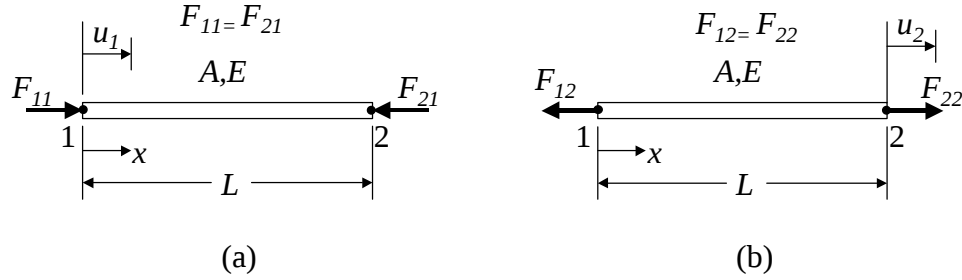


Figure 2.1 Nodal Forces associated with deformation of a two-node bar element. (a) Node 1 displaced u_1 units. (b) Node 2 displaced u_2 units. (From Reference 24)

The uniform prismatic bar in Figure 2.1 has a length (L), modulus of elasticity (E) and a cross-sectional area (A). With nodes located at the ends, the displacement is only allowed in the axial direction. A unit displacement is applied to one component while the remaining components are held zero; the magnitude of the force required to maintain the displacement state is then calculated. These forces are easily determined from the basic formula for bar displacement.

$$d = \frac{FL}{AE} \quad (2.2)$$

Solving the preceding for F , the results for the respective cases where $d = u_1$ and $d = u_2$, in Figure 2.1 are

$$F_{11} = F_{21} = \frac{AE}{L}u_1 \text{ and } F_{12} = F_{22} = \frac{AE}{L}u_2 \quad (2.3)$$

F_{ij} is the force at node i associated with the node j displacement, and these results can then be written in matrix form. Using the sign convention that force and displacements are positive to the right, the matrix form is thus,

$$\begin{bmatrix} F_{11} & -F_{12} \\ -F_{21} & F_{22} \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \text{ or } \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \quad (2.4)$$

Here, F_1 and F_2 are the resultant, internal forces applied to the bar at nodes 1 and 2 respectively.

Simple Beam

With the bar element completed, a simple beam element is considered. It is prismatic with a modulus of elasticity, E , and a centroidal area moment of inertia, I , of its cross-sectional area. From Euler beam theory, the centerline, lateral displacement, $v=v(x)$, is a cubic polynomial in x for a uniform prismatic beam with loads only at the ends. (Figure 2.2) The associated degrees of freedom are two lateral translations, v_1 and v_2 , and two rotations parallel to the z -axis, θ_{z1} and θ_{z2} .

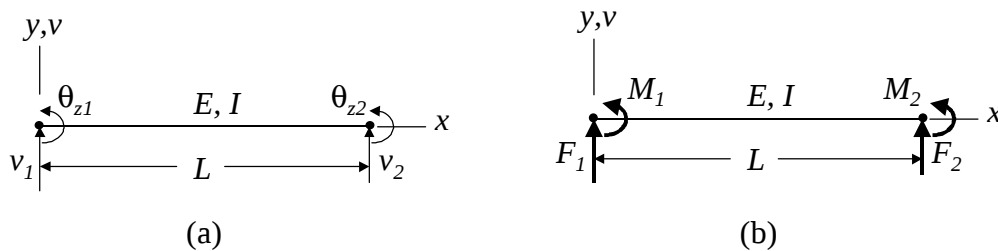


Figure 2.2 (a) Simple plane beam element and its nodal d.o.f. (b) Nodal loads associated with the d.o.f. (From Reference 24)

Using a similar procedure to the bar element, the stiffness matrix for the simple beam element can be constructed column by column using the principles of beam theory and superposition. As before, a unit displacement is applied to one component while the remaining components remain zero, and the magnitude of the force and moments required to maintain the displacement state must then be evaluated.

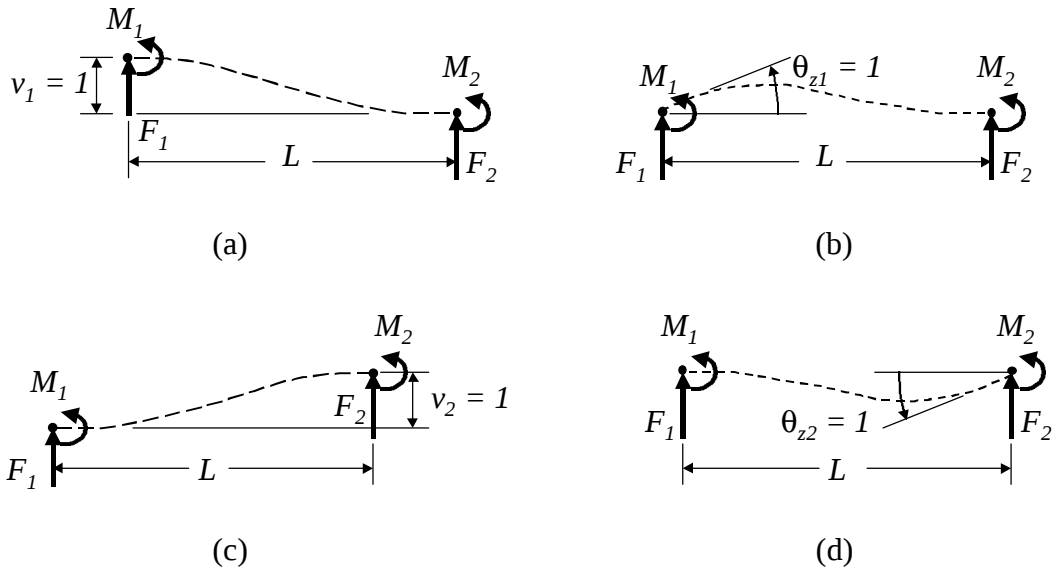


Figure 2.3 (a-d) Deflected shapes associated with activation of each d.o.f. in turn.
(From Reference 24)

As an example, the development of the first column of the stiffness matrix is considered. Using the element in Figure 2.2, a unit vertical displacement, $v_1 = 1$, is applied with the remaining values (v_2 , q_{z1} and q_{z2}) held to zero, and the depicted deflection of Figure 2.3a results. To produce this deflection, the appropriate nodal forces and moments are superposed, and the element equations expressed in matrix form are,

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{Bmatrix} \quad (2.5)$$

From the preceding, the following relationships can be defined,

$$\begin{aligned} k_{11} &= F_1 \\ k_{21} &= M_1 \\ k_{31} &= F_2 \\ k_{41} &= M_2 \end{aligned} \quad (2.6)$$

These nodal forces and moments are then related to deflection and rotation using superposition and beam equations,

$$\begin{aligned} v_1 = 1 &= \frac{F_1 L^3}{3EI} - \frac{M_1 L^2}{2EI} \\ \theta_{z1} = 0 &= \frac{F_1 L^2}{2EI} - \frac{M_1 L}{EI} \end{aligned} \quad (2.7)$$

Solving the preceding simultaneously yields,

$$\begin{aligned} F_1 &= \frac{12EI}{L^3} \\ M_1 &= \frac{6EI}{L^2} \end{aligned} \quad (2.8)$$

The values for the F_2 and M_2 are then determined using the principles of statics,

$$F_2 = -\frac{12EI}{L^3}$$

$$M_2 = \frac{6EI}{L^2}$$
(2.9)

Using the relationships in equation (2.6), the values for the first column of the stiffness matrix are thus determined.

Using similar procedures, the remaining columns are determined, and the resulting stiffness matrix, **k**, is

$$\mathbf{k} = \begin{bmatrix} \frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}$$
(2.10)

This is the stiffness matrix associated with $\mathbf{d} = [v_1 \quad q_{z1} \quad v_2 \quad q_{z2}]^T$, and this formulation provides an exact representation of the beam using traditional beam theory for loads applied at the nodes. For load distributed across the span, the solution is inexact but approaches the exact solution with increased numbers of element. Therefore, careful modeling should be employed when using this formulation.

2 Dimensional Beam Element

While the foregoing formulation is useful for simple beams, another formulation is desired when axial loads exist as well. The 2-D beam element, also called a plane frame element, considers axial loads, shear force and rotation in one direction, and it is formed by the superposition of the simple bar element and the simple beam. Combining equations (2.4) and (2.10), the resulting stiffness matrix along the x-axis is

$$\mathbf{k} = \begin{bmatrix} \frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \quad (2.11)$$

The matrix superposes the stiffness matrices of the bar and beam, and hence, the d.o.f vector is $\mathbf{d} = [u_1 \quad v_1 \quad q_{z1} \quad u_2 \quad v_2 \quad q_{z2}]^T$.

As with all element formulations, care must be taken when employing this element. For small displacements, this superposition of elements “will be accurate, however there is an interaction that occurs between axial and lateral loading on beams.”^[25] The effects of a tensile axial load tend to attenuate the effect of lateral loads. Conversely, a compressive axial load tends to magnify the effect of the lateral loads.

3 Dimensional Beam Element

The beam element most often utilized in general finite element codes has 3-dimensional capability and is also termed a “space beam” element. To develop this element, the formulation must include “the capability for torsional loads about the axis of the line element as well as flexural loads acting in the x - z plan” ^[25] First, a local coordinate system, xyz , is established for a beam element in the global, XYZ coordinate system. (See Figure 2.4) The x -axis is along the line of the element, the y -axis is one lateral direction, and the z -axis, along the orthogonal lateral direction, completes the right-hand coordinate system.

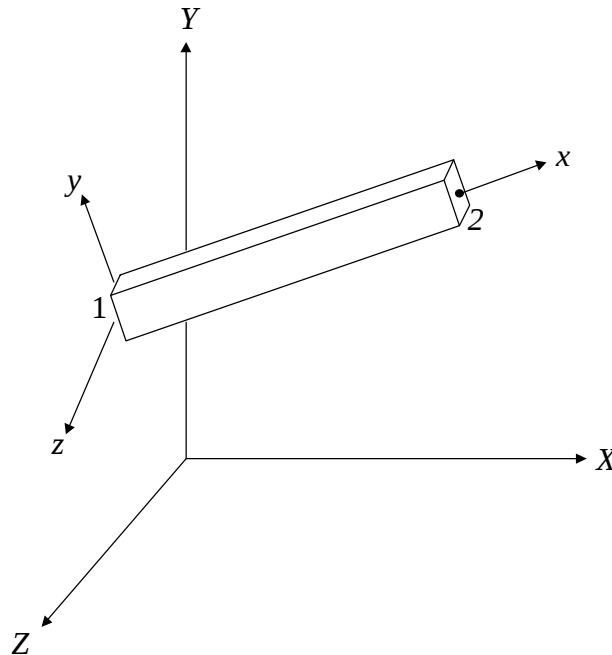


Figure 2.4 3-D beam element oriented with respect to local and global coordinate systems.

Using the relationship of torque and angle of twist from basic mechanics of materials, the effects of torsion are added through superposition. For a two-node element, the relationship is given by following equation in matrix notation.

$$\begin{bmatrix} \frac{JG}{L} & -\frac{JG}{L} \\ -\frac{JG}{L} & \frac{JG}{L} \end{bmatrix} \begin{Bmatrix} \mathbf{f}_{xi} \\ \mathbf{f}_{xj} \end{Bmatrix} = \begin{Bmatrix} T_i \\ T_j \end{Bmatrix} \quad (2.12)$$

J is a torsional constant about the x -axis, but for a beam with circular cross-section, it is the polar moment of inertia. G is the modulus of rigidity, L is the element length, and $\mathbf{f}_{xi}$ and $\mathbf{f}_{xj}$ are the nodal d.o.f. associated with the angle of twist at each node about the x -axis. Finally, T_i and T_j are the torques or moments about the x -axis for the individual nodes.

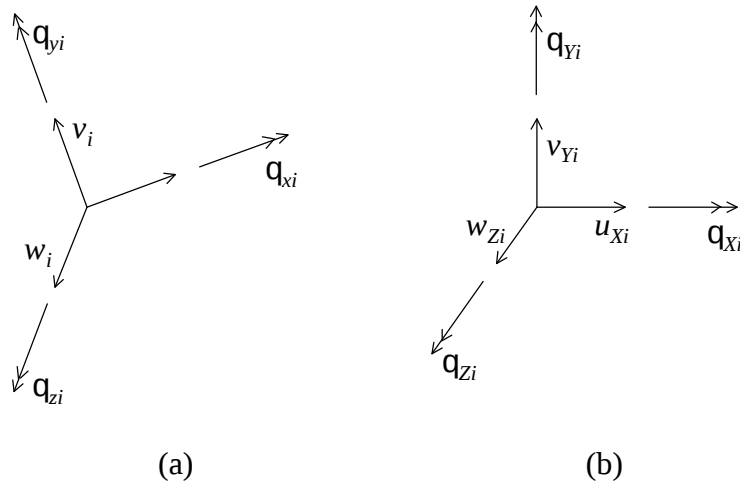


Figure 2.5 3-D beam element nodal degrees of freedom expressed in (a) local coordinates and (b) global coordinates. (From Reference 24)

With the addition of flexure in the x-z plane, another stiffness matrix, similar to the one in equation (2.11), is added. In this case, the area moment of inertia is about the y-axis and passes through the cross-sectional, neutral axis.

These two matrices are then superimposed with the stiffness matrix from equation (2.13), and this yields the 12 x 12 stiffness matrix in equation.

$$\begin{bmatrix}
 \frac{AE}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{AE}{L} & 0 & 0 & 0 & 0 & 0 \\
 0 & \frac{12EI_z}{L^3} & 0 & 0 & 0 & \frac{6EI_z}{L^2} & 0 & -\frac{12EI_z}{L^3} & 0 & 0 & 0 & \frac{6EI_z}{L^2} \\
 0 & 0 & \frac{12EI_y}{L^3} & 0 & -\frac{6EI_y}{L^2} & 0 & 0 & 0 & -\frac{12EI_y}{L^3} & 0 & -\frac{6EI_y}{L^2} & 0 \\
 0 & 0 & 0 & \frac{JG}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{JG}{L} & 0 & 0 \\
 0 & 0 & -\frac{6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & 0 & 0 & 0 & \frac{6EI_y}{L^2} & 0 & \frac{2EI_y}{L} & 0 \\
 0 & \frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{4EI_z}{L} & 0 & -\frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{2EI_z}{L} \\
 -\frac{AE}{L} & 0 & 0 & 0 & 0 & 0 & \frac{AE}{L} & 0 & 0 & 0 & 0 & 0 \\
 0 & -\frac{12EI_z}{L^3} & 0 & 0 & 0 & -\frac{6EI_z}{L^2} & 0 & \frac{12EI_z}{L^3} & 0 & 0 & 0 & -\frac{6EI_z}{L^2} \\
 0 & 0 & -\frac{12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & 0 & 0 & 0 & \frac{12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & 0 \\
 0 & 0 & 0 & -\frac{JG}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{JG}{L} & 0 & 0 \\
 0 & 0 & -\frac{6EI_y}{L^2} & 0 & \frac{2EI_y}{L} & 0 & 0 & 0 & \frac{6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & 0 \\
 0 & \frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{2EI_x}{L} & 0 & -\frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{4EI_z}{L}
 \end{bmatrix}
 \tag{2.13}$$

This is the stiffness matrix in the local coordinate system in Figure 2.4, and its corresponding displacement vector,

$$d = \{u_1 \quad v_1 \quad w_1 \quad q_{x1} \quad q_{y1} \quad q_{z1} \quad u_2 \quad v_2 \quad w_2 \quad q_{x2} \quad q_{y2} \quad q_{z2}\}^T.$$

To formulate this matrix for the global coordinate system, a coordinate transformation must be performed.

Since this element is based upon conventional beam theory, this element is limited by the fundamental assumptions of conventional beam and torsion theories, and the results can only be as accurate as the theory. Also, this formulation fails to couple the lateral and axial loading, and nonlinear coupling does exist between the two. The analysis also does not account for stress concentration factors at changes in cross-section nor point load applications.

Dynamic Analysis

To account for inertial effects, dynamic analysis utilizes mass and damping matrices as well as the stiffness matrix. For a structure modeled by a finite number of nodal d.o.f., the set differential equations of motion for dynamic analysis can be expressed in matrix form.

$$\mathbf{M}\ddot{\mathbf{D}} + \mathbf{C}\dot{\mathbf{D}} + \mathbf{K}\mathbf{D} = \mathbf{F} \quad (2.14)$$

Here, $\mathbf{M}$ is mass matrix of the structure, $\ddot{\mathbf{D}}$ is the nodal acceleration vector, $\mathbf{C}$ is the damping matrix of the structure, and $\dot{\mathbf{D}}$ is the nodal velocity vector. $\mathbf{K}$ is again the stiffness matrix of the structure, $\mathbf{D}$ is the nodal displacement vector, and $\mathbf{F}$ is the time-variable nodal load vector.

Equation (2.14) is the general equation for all dynamic problems, but these problems can be classified as either wave propagation or structural dynamics. Wave propagation problems typically have impact or blast loading. The response is generally replete with high frequencies, and effects of stress waves are the primary interest. In structural dynamics, the frequency of excitation is generally in the same range as the

lowest natural frequencies of the structure. Structural dynamics is usually subdivided into eigenvalue analysis and time-history analysis. In eigenvalue analysis, the natural frequency and the corresponding mode shapes of a structure are desired, while in time-history analysis, the movements of the structure under prescribed conditions are sought. This category includes frequency response analysis and transient response analysis. In frequency response analysis, the steady state response of the structure to harmonic force input is also harmonic; while the loading is an arbitrary time dependent function for transient response analysis.

Structural Dynamics

For this work, the natural frequencies, mode shapes and response under designated conditions are the primary interests. Therefore, the two structural dynamics subdivisions will be briefly examined here, while a thorough discussion of wave propagation problems, or shock loading problems is included in references [24] and [26].

Undamped Free Vibration

In structural dynamics analysis, the eigenvalue analysis is the most typical form. Here, the undamped free vibration of a system by an initial disturbance is the focus, and the natural frequencies and corresponding mode shapes are primarily sought. To accomplish this, the applied forces, $\mathbf{F}$, and damping matrix, $\mathbf{C}$, in equation (2.14) are set to zero. The motion of each node is then assumed to be a “sinusoidal function of the peak displacement amplitude for that node” ^[25], and the displacement vector is also sinusoidal. The system can then be expressed as,

$$(\mathbf{K} - \omega^2 \mathbf{M})\mathbf{A} = \mathbf{0} \quad (2.15)$$

ω^2 is the eigenvalue, and for the trivial case, equation (2.15) has the solution $\mathbf{D} = \mathbf{0}$. The nontrivial solutions are the primary interest and are the same number as the nodal degrees of freedom.

For a nontrivial case, the determinant of the $(\mathbf{K} - \omega^2 \mathbf{M})$ matrix must be zero, and solution yields the characteristic equation where the roots are the eigenvalues - ω^2 . The natural frequencies are then calculated, and the corresponding eigenvector - ϕ , a set of relative node displacements, can be determined. This is accomplished by substitution of ω^2 into equation (2.15) and solving for $\mathbf{A}$. $\mathbf{A}$ is then normalized, and the result is the eigenvector, or normal mode shape, for a particular eigenvalue.

Although the solution of the eigenproblem yields a natural frequency for each d.o.f., the nature of the finite element approximation yields high inaccuracies in the higher eigenvalues and eigenvectors. Only the lowest eigenvalues of a model are generally considered because the primary response typically involves only the lowest natural frequencies.

Transient Response Analysis

Transient response analysis is performed when arbitrary time-dependent loading function is used rather than a harmonic load. Two general approaches, direct integration and modal superposition, are used to solve this type of problem. In the direct integration, the system equations are first analyzed using finite elements and subsequently integrated in the time for the velocity and acceleration component. While the preceding technique requires considerable computing resources, the modal superposition technique alleviates some of the burden. The approach assumes that the superposition of lower frequency

mode shapes sufficiently represents the actual dynamic response of the system. Basically, the node displacement coordinates are changed to a set of modal coordinates; therefore, the set of system equations is reduced from one per nodal d.o.f. to a set of modal equations for the desired number of modes. Several solution algorithms have been developed for the two techniques, and a few are discussed in reference [26].

Formulation of the Mass and Damping Matrices

For the various dynamic analysis techniques examined previously, the solution includes the mass and damping matrices, but the formation of each has yet to be discussed. Analogous to the formulation of the stiffness matrix, the matrices are first formulated on the elemental level, the structural level matrices are then developed.

Mass Matrix Formulation

The formulation of the mass matrix begins by employing D'Alembert principle to develop body forces and then uses a work equivalent distribution. The general form of mass matrix, $\mathbf{m}$, is thus,

$$\mathbf{m} = \int \mathbf{N}^T \mathbf{N} r dV \quad (2.16)$$

$\mathbf{N}$ is the element shape function matrix, r is the mass density, and dV is the incremental volume. Since this mass matrix uses the same shape functions as the stiffness matrix, it is commonly called the “consistent” mass matrix. When applied to the bar element (Figure 2.1), the matrix is

$$\mathbf{m} = \begin{bmatrix} \frac{rAL}{3} & \frac{rAL}{6} \\ \frac{rAL}{6} & \frac{rAL}{3} \end{bmatrix} \quad (2.17)$$

The corresponding acceleration vector is $\ddot{\mathbf{d}} = [\ddot{u}_1 \quad \ddot{u}_2]^T$. For the simple beam element (Figure 2.2 and Figure 2.3), the matrix becomes.

$$\mathbf{m} = \frac{rAL}{420} \begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix} \quad (2.18)$$

In this case, the acceleration vector includes rotational components and is

$\ddot{\mathbf{d}} = [\ddot{v}_1 \quad \ddot{q}_{z1} \quad \ddot{v}_2 \quad \ddot{q}_{z2}]^T$. Carefully considering the appropriate d.o.f. and coordinate system, these can be used to formulate the mass matrix for 2-D beam element, and using a similar technique as the stiffness matrix formulation, $\mathbf{m}$ can be constructed for a 3-D beam element.

Damping Matrix Formulation

If damping is truly viscous, damping forces would be directly proportional to velocity, and a technique similar to the mass matrix formulation could be utilized. Unfortunately, for structural dynamic problems, the sources of damping are not easily measured and difficult to represent mathematically, and these sources include internal

friction and Coulomb friction in connections. Although it is difficult to represent, the damping forces in many cases are small – less than 10% of the $\mathbf{KD}$, $\mathbf{M}\ddot{\mathbf{D}}$, and $\mathbf{F}$ – and can be idealized as viscous damping. Two typical methods of including viscous damping in the model are proportional damping and modal damping.

In proportional damping, also called Rayleigh Damping, the damping matrix is arbitrarily defined as,

$$\mathbf{C} = a\mathbf{M} + b\mathbf{K} \quad (2.19)$$

The proportionality constants, a and b , are then selected by the analyst and are typically based on experimental data measured from a similar material and structure. For modal damping the, both sides of equation (2.14) are divided by the mass matrix:

$$\ddot{\mathbf{d}} + \left(\frac{\mathbf{c}}{\mathbf{m}} \right) \dot{\mathbf{x}} + \left(\frac{\mathbf{k}}{\mathbf{m}} \right) \mathbf{x} = \left(\frac{\mathbf{F}_0}{\mathbf{m}} \right) \sin \omega t \quad (2.20)$$

Here,

$$\frac{\mathbf{c}}{\mathbf{m}} = 2\zeta\omega_n \quad (2.21)$$

$$\frac{\mathbf{k}}{\mathbf{m}} = \omega_n^2 \quad (2.22)$$

ζ is the selected damping ratio, and ω_n is the natural frequency of the structure.

Both of the preceding techniques attempt to relate the enigmatic nature of damping to the finite element formulation, and for structures with relatively low damping, each appears to perform reasonably well. For further information regarding

these methods, consult references [24], [25], [26] and [27] as well as structural dynamics books.

Chapter 3

Modal Theory

In the study of structural dynamics, experimental modal analysis has emerged as an extremely useful procedure. Performed under controlled conditions, it encompasses excitation of a structure, or component; acquisition of data; and the subsequent analysis of the response. The uses of modal analysis are varied and range from determination of natural frequencies and damping factor to full development of a mass-spring-damper model of a particular system.

The development of the model contains four primary tasks – identification of measurement locations and transducers, data acquisition, modal parameter extraction and modal parameter application. In the first task, excitation and response locations are based on the number of desired modes and analogous points in an analytical or finite element model, while the structure and nature of the test govern transducer selection. The acquisition of data includes the frequency range selection and choice of excitation method. Additionally, this task includes the selection of frequency response estimation technique, number of averages, and windowing method. Modal parameter extraction

includes the determination of natural frequencies, damping, and eigenvectors. This is accomplished by implementing either time domain or frequency domain techniques for the appropriate single-degree of freedom (SDOF) or multi-degree of freedom (MDOF) model. Finally, the modal parameters are used to correlate the analytical and modal model, update or modify the analytical model, or develop design properties.

Theoretical Background

To develop an effective modal model, the fundamental theories of this technique must first be examined. For the sake of brevity, the two theoretical models used, the single-degree of freedom model and the multi-degree of freedom model, will be briefly covered here, while a more detailed explanation is contained in [28] and [29].

Single-degree of Freedom Model

Although limited in the number of practical applications, the single degree of freedom model is an important foundation of modal theory. The equation of motion for the system is again given by,

$$\mathbf{m}\ddot{\mathbf{d}} + \mathbf{c}\dot{\mathbf{d}} + \mathbf{k}\mathbf{d} = \mathbf{f} \quad (3.1)$$

Here, $\mathbf{m}$ is mass of the structure, $\ddot{\mathbf{d}}$ is the acceleration, $\mathbf{c}$ is the damping value of the system, $\dot{\mathbf{d}}$ is the velocity, $\mathbf{k}$ is again the stiffness, $\mathbf{d}$ is the displacement, and $\mathbf{f}$ is the time-variable force.

If the force is harmonic, the displacement and force relationship can be expressed as,

$$\frac{\mathbf{d}}{\mathbf{f}} = \mathbf{H}(\omega) = \frac{1}{(-\mathbf{m}\omega^2 + \mathbf{k}) + i\omega\mathbf{c}} \quad (3.2)$$

$\mathbf{H}(\omega)$ is the frequency response function (FRF), or receptance of this system, and is only a function of the frequency and the physical parameters of the system.

If normalized with respect to one physical parameter, equation (3.2) can be expressed solely in terms of frequency. Dividing $\mathbf{m}$ and $\mathbf{c}$ by $\mathbf{k}$ yields,

$$\frac{\mathbf{m}}{\mathbf{k}} = \frac{1}{\omega_n^2} \quad (3.3)$$

$$\frac{\mathbf{c}}{\mathbf{k}} = \frac{2\zeta}{\omega_n} \quad (3.4)$$

If the two preceding equations are substituted into equation (3.2), the magnitude of $\mathbf{H}(\omega)$ is then

$$|H(\omega)| = \frac{1}{\sqrt{\left(\left[1 - \left(\frac{\omega}{\omega_n} \right)^2 \right]^2 + \left[2\zeta \left(\frac{\omega}{\omega_n} \right) \right]^2 \right)}} \quad (3.5)$$

Again, ω is the particular frequency, and ω_n is the natural frequency. The corresponding phase angle is then,

$$j = \tan^{-1} \left[\frac{2x \left(\frac{w}{w_n} \right)}{1 - \left(\frac{w}{w_n} \right)^2} \right] \quad (3.6)$$

Equation (3.5) is the receptance of the system and relates displacement to force.

Multi-degree of Freedom Model

The single-degree of freedom serves as a theoretical base and an introduction to important modal concepts, but its limited application necessitates the development of a more general model – the multi-degree of freedom model. In this model, systems are characterized by N differential equations, which must be solved simultaneously. These equations are given by the following general equation,

$$\mathbf{M}\ddot{\mathbf{D}} + \mathbf{C}\dot{\mathbf{D}} + \mathbf{K}\mathbf{D} = \mathbf{F} \quad (3.7)$$

Again, $\mathbf{M}$ is the mass matrix of the system, $\ddot{\mathbf{D}}$ is the acceleration vector, $\mathbf{C}$ is the damping matrix, $\dot{\mathbf{D}}$ is the velocity vector, $\mathbf{K}$ is the stiffness matrix, $\mathbf{D}$ is the nodal displacement vector, and $\mathbf{F}$ is the forcing function vector.

The MDOF systems are typically divided into three classes – undamped, proportionally damped, and generally damped. The undamped case, as the name implies, does not include damping and is the simplest case. The remaining cases attempt to relate the complex mechanisms of damping by utilizing either viscous or structural damping techniques. In proportional damping, the damping matrix is related to the mass and stiffness matrices, while the third case considers a general solution to equation (3.7).

Undamped Case

The undamped multi-degree of freedom system is the simplest form, and equation (3.7) can be expressed as

$$\mathbf{M}\ddot{\mathbf{D}} + \mathbf{K}\mathbf{D} = \mathbf{F} \quad (3.8)$$

To establish the natural modal properties, the free vibration problem is first examined.

The displacement is assumed harmonic, and employing typical vibration analysis techniques, the eigenvalues and eigenvectors are determined by,

$$\mathbf{K}\mathbf{A}_n = \omega_n^2 \mathbf{M}\mathbf{A}_n \quad n = 1, 2, \dots, N \quad (3.9)$$

The natural frequency, ω_n , is obtained from the eigenvalue $-\omega_n^2$, and $\mathbf{A}_n$ is the associated eigenvector of the system.

While the eigenvalues, ω_n^2 , are unique, the corresponding eigenvectors are not unique; only the eigenvector amplitude is unique. If $\mathbf{A}_n$ is normalized to the mass matrix, the resulting mass-normalized mode shapes, ϕ_n , are

$$\phi_n = \frac{\mathbf{A}_n}{\mathbf{A}_n^T \mathbf{M} \mathbf{A}_n} \quad n = 1, 2, \dots, N \quad (3.10)$$

This normalized eigenvector is related to the stiffness matrix by the following,

$$\phi_n^T \mathbf{K} \phi_n = \omega_n^2 \quad (3.11)$$

The preceding demonstrates an important property of the undamped system – orthogonality. When weighted by the mass or stiffness matrices, the eigenvectors are orthogonal.

$$\mathbf{A}_n \mathbf{M} \mathbf{A}_m = 0 \quad \mathbf{n} \neq \mathbf{m} \quad \mathbf{n}, \mathbf{m} = 1, 2, \dots, N \quad (3.12)$$

$$\mathbf{A}_n \mathbf{K} \mathbf{A}_m = 0 \quad \mathbf{n} \neq \mathbf{m} \quad \mathbf{n}, \mathbf{m} = 1, 2, \dots, N \quad (3.13)$$

Thus the mode shapes are orthogonal when weighted by either the stiffness or mass matrices.

With the natural frequency and normal mode shapes determined, the focus shifts to the response and development of the frequency response function for the undamped case. Assuming a harmonic excitation and response, the FRF is then,

$$\mathbf{H}(\omega) = \mathbf{\Phi} \left[\text{Diag}(\omega_n^2 - \omega^2) \right] \mathbf{\Phi}^T \quad (3.14)$$

Here $\text{Diag}(\omega_n^2 - \omega^2)$ is a diagonal matrix, and $\mathbf{\Phi}$, the mass-normalized modal matrix, contains the individual mass-normalized mode shapes, ϕ , as its columns.

The preceding demonstrates an important property of the FRF matrix – symmetry. That is,

$$H_{jk}(\omega) = \frac{D_j}{F_k} = H_{kj}(\omega) = \frac{D_k}{F_j} \quad (3.15)$$

This is the principle of reciprocity, and combined with equation (3.14), it allows any individual FRF parameter to be determined.

$$H_{jk}(w) = \sum_{n=1}^N \frac{f_{jn} f_{kn}}{w_n^2 - w^2} \quad (3.16)$$

$$H_{jk}(w) = \sum_{n=1}^N Z_{kn} f_{jn} \quad (3.17)$$

Here,

$$Z_{kn} = \frac{f_{kn}}{w_n^2 - w^2} \quad (3.18)$$

Z_{kn} is the modal participation factor of the n th mode due to forcing at k . From a modal analysis perspective, a more practical form is,

$$H_{jk}(w) = \sum_{n=1}^N \frac{A_{jk}^n}{w_n^2 - w^2} \quad (3.19)$$

Here, A_{jk}^n , the modal constant, relates coordinates j and k at mode n and is often called the residue, while the corresponding natural frequency is termed the pole.

Damped Models

The undamped system provides an important foundation for modal theory, but most structures and systems exhibit the complicated mechanism of damping, which tends to couple the equations of motion. To account for damping, two cases, proportionally damped and generally damped, are developed to uncouple the equations of motion, and for this research, proportional damping is utilized. Since the focus of this work is the application of modal theory rather than its development, the reader is encouraged to examine [28] for a detailed investigation.

General FRF Formulation

In the previous sections, the FRF was developed from the equations of motion using an assumed harmonic forcing-function and corresponding response, but the FRF is an intrinsic property of the system and independent of excitation form. It is also important to note that the FRF is defined in the frequency domain, and thus, the utilized excitation forms should be expressed in the frequency domain.

Since there are a variety of excitation types, a more general definition of the FRF is needed. To accomplish this task, brief derivations of the FRF are examined for two cases – impulse response function and random vibrations, which are characteristic of typical modal excitation techniques

Impulse Response

In the first case, the frequency response function is developed using a impulse response function, otherwise known as a Dirac delta function, and the input force is assumed to be

$$f(t) = d(t - t) \quad (3.20)$$

t is the time, t is the time shift, and $d(t - t)$ is zero everywhere except at $t = t$. Here, the magnitude is infinite with a zero duration and unit area.

Employing the convolution integral and taking the Fourier transform, the frequency response function, $H(w)$, be expressed as,

$$H(w) = \frac{D(w)}{F(w)} \quad (3.21)$$

Here, $D(w)$ and $F(w)$ are the Fourier transform of the response and input respectively, and as before, the FRF is complex-valued function, which has both magnitude and phase.

Random Vibration

Random vibration signals unfortunately cannot be analyzed using the preceding approach because random signals are not periodic, and their inherent properties violate the Dirichlet condition, which is necessary to perform the Fourier transform. To circumvent this problem, a practical approach, which utilizes signal-processing techniques, is applied.

For a random signal, $f(t)$, Fourier transform is performed on the auto-correlation function, R_{ff} , and S_{ff} , the power spectral or auto-spectral density function, results. This is a real and even function of frequency and forms the Fourier transform pair with the R_{ff} .

$$S_{ff}(w) = |H(w)|^2 S_{xx}(w) \quad (3.22)$$

Utilizing a similar approach, the cross-correlation, R_{xf} , and cross-spectral density, S_{xf} , relate a pair of random function $x(t)$ and $f(t)$.

$$S_{xf}(w) = H(w) S_{xx}(w) \quad (3.23)$$

Through similar manipulations, the following is developed

$$S_{ff}(w) = H(w) S_{fx}(w) \quad (3.24)$$

Here, S_{ff} and S_{xx} are the input and output dual-sided, auto-spectral densities, and S_{xf} and S_{fx} are the dual-sided cross-spectral densities.

Examination of the preceding equations provides important insight. In equation (3.22), no FRF phase information is retained, while equations (3.23) and (3.24) retain both the magnitude and phase of the frequency response function. The preceding equations form the basis for signal processing of random signals and are utilized in typical signal processors.

Modal Model Formulation

With the theoretical background established, the four main components of the model modal development can then be examined. These primary tasks include: identification of measurement locations and transducers, data acquisition, modal parameter extraction and modal parameter application.

Test Configuration and Transducer Location

In the first task, the test configuration and nature of the test are established, and the location and type of transducers are determined. The structure and nature of the test govern transducer selection, while excitation and response locations are based on the number of desired modes and analogous points in an analytical or finite element model.

The structure and test configurations are an integral component in the selection of transducers. Since the structure, test configuration, and transducers all interact, the test configuration and transducer selection should be carefully determined. Utilizing appropriate boundary conditions, test configuration and transducer influence on the tested structure should be minimized to ensure the proper system or structure is tested.

Various transducers are used to measure vibration or motion of the system, and the number of modes and corresponding points in an analytical model generally

determine the location of excitation and response transducers. While various transducers and measuring methods exist, piezoelectric sensing transducers, either force or acceleration or both, and strain gauges are typically used.

Data Acquisition

The second component of the modal model encompasses the excitation of the structure and the analysis of the response. Using an exciter, the system is vibrated, and the data is captured using a data acquisition system and subsequently analyzed using signal processing techniques.

Excitation is provided to a structure by various methods, but electromagnetic, also called electrodynamic, shakers and modal impact hammers are typical techniques. In the shaker, a supplied input signal is converted to a mechanical response that is introduced via a connector to the structure. The modal impact hammer is another popular method of exciting a structure. In this procedure, an impact hammer, which contains a force transducer in the tip, strikes the structure, and a transducer measures the response.

Input excitation and response output are captured using a data acquisition system, which includes conversion, amplification and conditioning of the signal. The analog input signal is first conditioned and amplified. It is then changed to a digital one using an analog to digital (A/D) converter and subsequently transformed using either a Discrete Fourier Transform (DFT) or the Fast Fourier Transform (FFT). These, as well as a signal generator, are included in one commonly used device – the digital signal analyzer. It is an important tool and allows the examination of actual signal processing and FRF estimation rather than the idealized ones presented previously.

Signal Analysis

In previous sections, signal processing and frequency response formulations were examined for idealized cases. In reality, the process of measuring true excitation and response does not exist. Uncorrelated signals, or noise, affect both the input and output, and three different cases have been developed to consider the effects of noise. The first case assumes the uncorrelated noise only influences the input, while the noise is presumed to affect only the output in the second case. The final, considered subsequently, assumes both the input and output are influenced by noise.

In the final case, the noise is assumed to affect both the input and output and is generally a more realistic estimated of the frequency response. The values $f(t)$ and $x(t)$ are the measured for the input and output, respectively, and given by

$$f(t) = u(t) \pm m(t) \quad (3.25)$$

$$x(t) = v(t) \pm n(t) \quad (3.26)$$

Here, $u(t)$ and $v(t)$ are the true input and output of the signal, and $m(t)$ and $n(t)$ are the uncorrelated noise signals (See Figure 3.1).

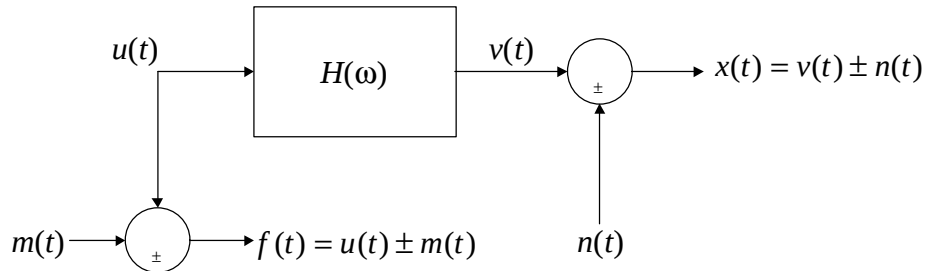


Figure 3.1 Linear model with noise $m(t)$ and $n(t)$ at input and output

In the previous sections, S_{xx} and S_{xf} , equations (3.22) and (3.23), were defined as the dual-sided, auto-spectral and cross-spectral density, respectively, of the time functions. This is because the Fourier transform is defined for the range $-\infty$ to $+\infty$, but in practice, the time records generally start at $t = 0$. To account for this, the single-sided auto-spectral, G_{xx} , and cross-spectral, G_{xf} , are defined from 0 to $+\infty$, and the FRFs can be stated in practical terms.

H₁ Estimator

Using the preceding equations and the FRF derived for random noise (equations (3.22) and (3.23)), the FRF could be expressed as,

$$H_1(\omega) = \frac{G_{fx}}{G_{ff}} \quad (3.27)$$

$$H_1(\omega) = \frac{G_{uv}}{G_{uu} + G_{mm}} \quad (3.28)$$

Here, the $H_1(\omega)$ is the FRF estimator in terms of the single-sided cross-spectrum of the signals $u(t)$ and $v(t)$ and the auto-spectrum of the input and corresponding noise. If the noise is zero, G_{mm} , the auto-spectrum of the noise, is then zero, and $H_1(\omega)$ gives an unbiased estimate of the frequency response function.

Coherence Function

To relate the input and output signals and the FRFs, the coherence function, based upon the concepts of the correlation coefficient, is examined. Defined in the frequency domain, the coherence function measures the linear relationship between the output and input and can be defined as

$$g^2(w) = \frac{|G_{fx}(w)|^2}{G_{ff}(w)G_{xx}(w)} \quad (3.29)$$

Where, $g^2(w)$ is the coherence function relating the input and output and is defined over the range $0 \leq g^2(w) \leq 1$.

Parameter Extraction

The third component of the modal model formulation is the extraction of modal parameters and the subsequent determination of natural frequencies, damping, and eigenvectors. Parameter extraction, or estimation, methods are available for both the frequency and time domains and utilize SDOF or MDOF curve-fitting techniques. In SDOF methods, the closest mode is assumed to dominate the region around resonance, and these techniques “work adequately for structure whose FRF exhibit well-separated modes.” [28] The MDOF methods consider the response contributions due to other modes and are necessary for closely space modes.

SDOF Curve-fitting

In the single-degree of freedom techniques, each FRF of the structure is curve-fit individually, and resonance response is presumably the result of one mode. Various methods are available and examined in more detail in [28], but the simplest is called the peak-amplitude method. This is accomplished by assuming the natural frequency, w_n , of a particular mode occurs at the corresponding resonance, and the maximum value of the frequency response function, $\hat{a}$, is then determined. The frequency bandwidth, (Δw) , is

determined for a half-power level, $\hat{a}/\sqrt{2}$, and the two points that enclose the band are the “half-power points”, w_b and w_a . (See Figure 3.2)

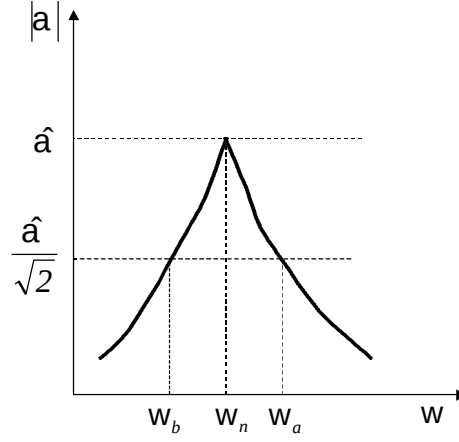


Figure 3.2 Half-power points and natural frequency for the Peak Amplitude Method (From Reference 28)

The damping of the particular mode is then determined by one of the following equations ^[28],

$$h_n = \frac{(w_a^2 - w_b^2)}{w_n} \quad (3.30)$$

$$x_n = 2h_n \quad (3.31)$$

Here, h_n and x_2 and the corresponding damping values of the particular mode. The modal constant, or residue, is then given by the following.

$$A^n = |\hat{a}| w_n^2 h_n \quad (3.32)$$

Here, A^n is the modal constant for a specific mode, and the total resonant region response is attributed to the mode in question.

MDOF Methods

While SDOF are useful for particular circumstances, the MDOF methods are applicable to a broader range and attempt to ameliorate the limitations of the single-degree of freedom models. To accomplish this, two different methods have been developed; the first method is merely an extension of the SDOF methods, while the second approach is a general curve-fitting approach.

SDOF Extension Curve-fitting

The first approach is an extension of the SDOF methods, but the effects of other modes are not assumed constant near the resonance of a particular mode. By utilizing the results of previous SDOF analysis, the FRF can be then be determined for a particular mode. Thus, the effects of additional modes on a specific mode are considered through an iterative examination of each mode.

General Curve-fitting

The second approach, which considers the modes simultaneously, is a general curve-fitting procedure that utilizes the advancements in numerical methods and computational capabilities. In this method, the error between the curve-fit of theoretical and experimental frequency response function is given by, ^[28]

$$\mathbf{e}_i = E_{jk}(\mathbf{w}_i) - H_{jk}(\mathbf{w}_i) \quad i = 1, 2, \dots, m \quad (3.33)$$

For m modes determined experimentally, E_{jk} and H_{jk} are the experimental and theoretical FRFs, respectively, and e_i is the error in fit for a given frequency. Although complex in form, the error can be expressed as a scalar,

$$e_i = \sum_{i=1}^m |e_i|^2 \quad (3.34)$$

This could be further generalized by adding a weighting factor, w_i , to each point of interest,

$$E = \sum_{i=1}^m w_i e_i \quad (3.35)$$

E is the total, weighted error, and the modal parameters are then estimated by minimizing the total error. This is accomplished by differentiating with respect to E ,

$$\left(\frac{\partial E}{\partial q} \right) = 0 \quad q = w_1, w_2, \dots, h_1, h_2, \dots, A_{jk}^1, A_{jk}^2 \quad (3.36)$$

The preceding results in equations that are linear in A_{jk}^n but nonlinear w_n and h_n . To solve these, various algorithms, which make individual assumptions, are then employed.

Modal Parameter Correlation

The final task of the model modal development is the correlation of parameters. In this procedure, the extracted modal parameter, natural frequencies and mode shapes of the structure or system, are compared with the corresponding values in the theoretical or analytical methods. Natural frequencies are generally compared by inspection, but to

provide more insight, a linear least-square curve-fitting of the two methods may be applied. While mode shapes can be compared by visual inspection, the Modal Assurance Criterion is generally employed for comparison.

To correlate the mode shapes, the Modal Assurance Criterion, or MAC is utilized,

$$MAC(a,b) = \frac{\left| \sum_{j=1}^J (\mathbf{f}_a)_j (\mathbf{f}_b)_j^* \right|^2}{\left(\sum_{j=1}^J (\mathbf{f}_a)_j (\mathbf{f}_a)_j^* \right) \left(\sum_{j=1}^J (\mathbf{f}_b)_j (\mathbf{f}_b)_j^* \right)} \quad (3.37)$$

Here $\mathbf{f}$ are the eigenvectors, j is the spatial location, and a and b refer to the compared methods. The values for the MAC tend toward unity for good correlation, while poor correlation tends toward zero.

Chapter 4

Model Development and Experimental Setup

The development of appropriate FEA and EMA models requires knowledge of the physical parameters of the system, and a coordinate system must be established. For this work, a coordinate system based upon the club-head geometry is established, and the physical parameters include location of the club-head center of gravity and mass moments of inertia along with shaft area moments of inertia and elastic modulus.

Coordinate System Development and Club-Head Characterization

A coordinate system for the club-head and shaft must be established before these characteristics can be determined, and the shaft and club-head can then be positioned with respect to this system. The z-axis is defined as the center-line of the hosel and shaft, and the xy-plane is placed at the end of the hosel. The x-axis is then defined as a line in the xy-plane parallel to score lines on the club-face, and the y-axis is defined normal to x and z in a right-hand orientation.

To position the club-face in the coordinate system, a three-axis table and dial indicator are employed. The hosel of the club-head is secured in a fixture that is subsequently fastened in the work-plane of the mill by a vise, and the values are measured from a starting point. (See Figure 4.1)

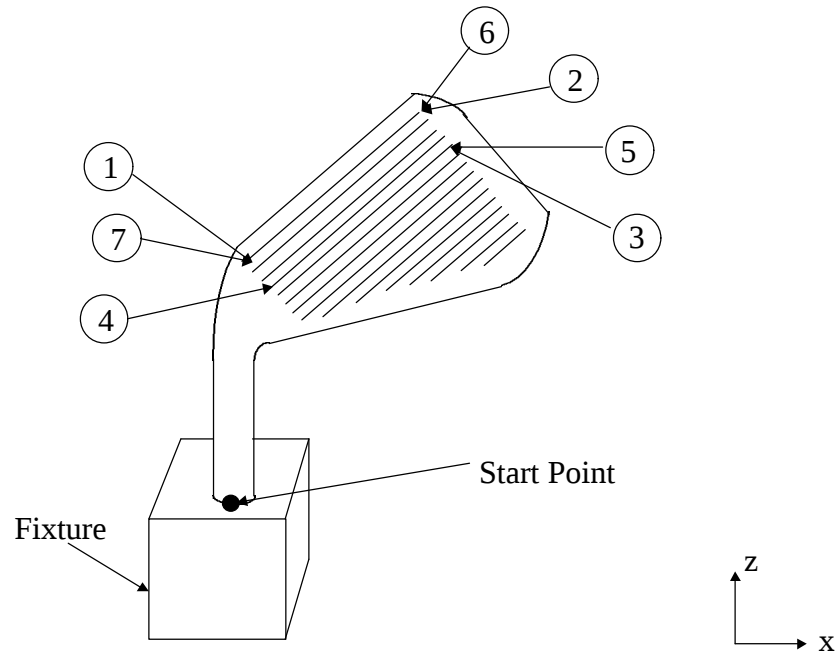


Figure 4.1 Measured Points using a three axis table and dial indicator.

Using the vertical slide, the locations in the z-direction are determined in reference to the starting point a fixed distance from the hosel end. Employing the horizontal slide of the mill, the x-coordinate for each point is then measured from the centerline of the hosel. The y-coordinate for each point is then determined cross-slide in the mill. The number of points characterized is limited by the adjustments of the mill and the dial indicator. To ensure repeatability in the results, several points are measured more than once, and all these values are included in Table 4.1.

Table 4.1 : Coordinates for the Club-Head in the Desired System

Point	x-coordinate (in)	y-coordinate (in)	z-coordinate (in)
1	0.232	0.214	1.712
2	2.127	0.204	2.838
3	2.361	0.433	2.47
4	0.47	0.378	1.448
5	2.378	0.444	2.456
6	2.132	0.202	2.843
7	0.2316	0.212	1.715

Lie and Loft

The characterization of the club head establishes the coordinates of the club-face geometry in the desired system, and the club-face geometry can now be expressed in terms of two common golf terms – lie and loft. These values help relate the position of the shaft, club-head and the ground and are used to relate a global coordinate system and a club-face based system in the modal parameter extraction discussed subsequently.

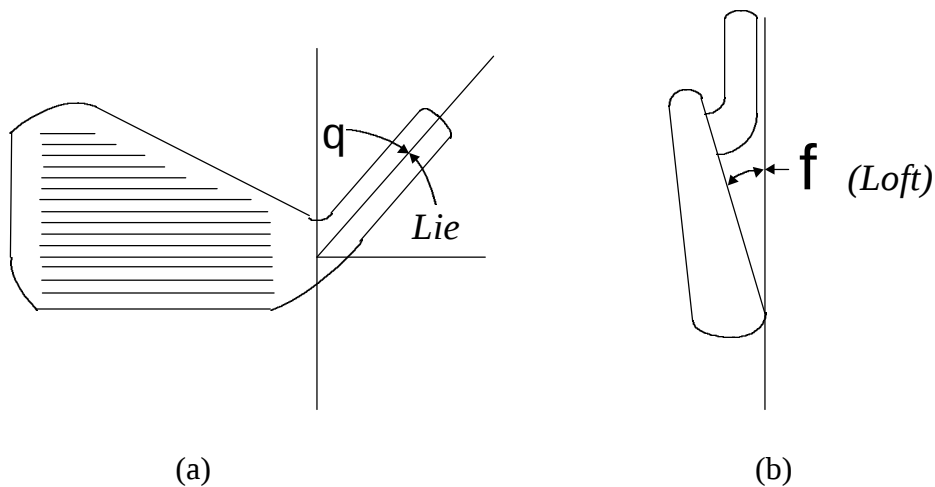


Figure 4.2 Club-head Lie and Loft (a) Lie Angle. (b) Loft Angle

The lie of the club, the angle formed by the shaft and the ground, is first determined. Using the x and z coordinates for points 2 and 3 (Figure 4.1 and Table 4.1), the value for q , 32.451 degrees, is determined through simple geometry. (See Figure 4.2a) The lie is $90 - q$ or 57.549 degrees.

The loft of the club is the angle formed by the club-face and a shaft plane perpendicular to the ground, and it is determined using translational and rotational transformations. Transforming points 2 and 3, the value for f is then calculated to be 27.709 degrees. (See Figure 4.2(b)) These values are then compared with typical five iron lie and loft angles of 60 and 28 degrees, respectively, and the resulting discrepancies are approximately four and one percent, respectively.

Determination of the Center of Gravity

With the locations of various positions on the club-face determined, the center of gravity of the club head is then found. Using an electronic balance, the mass of the club is first measured to be 253.5 grams. The hosel is then supported on a shaft, a pointed support is then placed at point 3 of the club-face (See Figure 4.3a), and the resultant force is determined. Using the mass of the club-head, the location of point 3, and the resultant force due to the club-head at the point support, the moments about the center are balanced. A line of action 1.597 inches relative to the hosel center-line (See Figure 4.3a) is determined for the center of gravity.

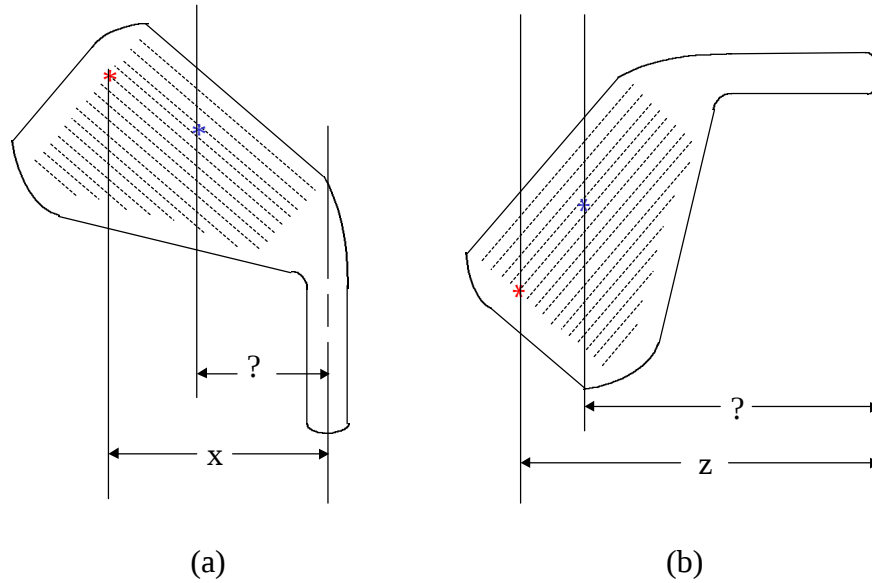


Figure 4.3 Determination of x and z coordinates of c.g.

With the distance from the z -axis (hosel center-line) determined, the direction from the x -axis can then be determined using a similar procedure. The hosel end is rotated 90 degrees using a fixture (Figure 4.8a) and again supported on the shaft. Using a pointed support and the principles of statics as before, another line of action 2.562 inches relative to the plane of the hosel end is calculated for the center of gravity. (Figure 4.3b)

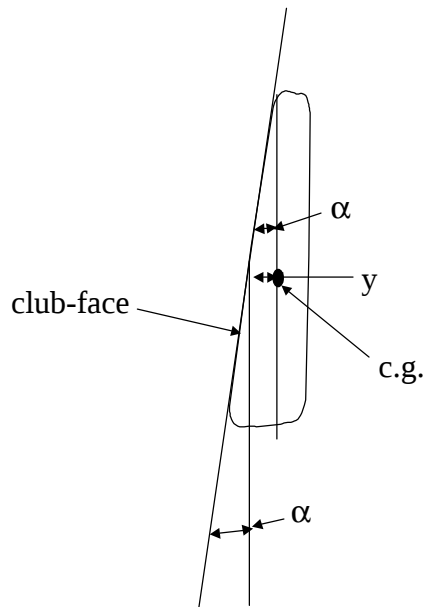


Figure 4.4 Determination of the y-coordinate of the center of c.g.

The location of the center of gravity relative to the hosel coordinate system is now known, and the y-coordinate distance to the center of gravity is determined. The hosel is again placed on a shaft and supported (See Figure 4.4). Since the force of gravity acts in the vertical direction, the distance can be related to a vertical plane by an angle. Using a standard, adjustable drafting triangle, the value of the angle was determined, and the distance from the club-face is then computed to be 0.126 inches using standard geometrical techniques.

Moment of Inertia

The mass moments of inertia of the club-head are determined using the compound pendulum method developed by S.H. Johnson ^[4]. In this technique, a rigid body is placed on a rotating shaft and allowed to swing with small amplitudes. The moment of inertia about the axis of rotation is then given by the following equation.

$$I = \frac{Img}{w^2} \quad (4.1)$$

I is the mass moment of inertia in $lb\text{-in-sec}^2$, l is distance from the center of gravity to the axis of rotation, and g is acceleration due to gravity in in/sec^2 . The mass, m , is in $lb\text{-sec}^2/\text{in}$, and w is the undamped natural frequency in rad/sec . With the moment of inertia about the axis of rotation determined, the parallel- axis theorem can then be applied to determine the moment of inertia about an axis through the center of gravity.

Using the apparatus in Figure 4.5, the moments of inertia with respect to the three axes at the hosel end – I_{xx} , I_{yy} and I_{zz} – can be determined. Then applying the parallel axis theorem, the moments of inertia with respect to a coordinate system at the cg can then be determined. The determined values are not generally the principal values, and the products of inertia – I_{xy} , I_{xz} and I_{yz} – are not necessarily zero.

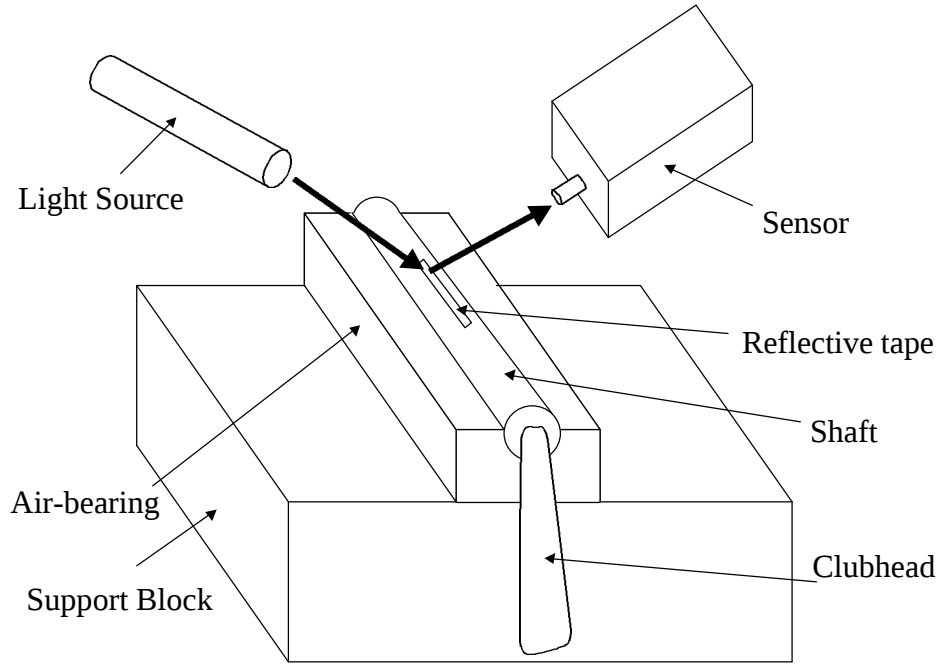


Figure 4.5 Experimental Setup for the Determination of Club-Head Mass Moment of Inertia

Products of inertia cannot be directly determined using the previously discussed procedure. However, these values can be determined with respect to an axis system at the center of rotation, xyz , and another coordinate system also at the center of rotation, $x'y'z'$. This second system results from a 45° rotation about the original z axis, and using the preceding direct method, the moments of inertia of this new coordinate system can then be determined.

The product of inertia, I_{xy} , can then be computed from the moments of inertia in a two-axis system ^[30],

$$I_{xy} = I_{x'x'} - \frac{(I_{xx} + I_{yy})}{2}, \quad I_{xy} = \frac{(I_{xx} + I_{yy})}{2} - I_{y'y'} \quad (4.2)$$

Another coordinate system, $x''y''z''$, can then be created with the 45° rotation about the original y axis, and the product of inertia, I_{xz} , can then be determined.

$$I_{xz} = I_{x''z''} - \frac{(I_{xx} + I_{zz})}{2}, \quad I_{xz} = \frac{(I_{xx} + I_{zz})}{2} - I_{z''z''} \quad (4.3)$$

If a third coordinate system is established by a 45° rotation about the original x axis, the product of inertia, I_{yz} , can then be determined.

$$I_{yz} = \frac{(I_{yy} + I_{zz})}{2} - I_{y'''z'''}, \quad I_{yz} = I_{z'''z'''} + \frac{(I_{yy} + I_{zz})}{2} \quad (4.4)$$

Using the preceding direct method, moments of inertia can be determined for the $I_{x'x'}$, $I_{y'y'}$, $I_{x''x''}$, $I_{z''z''}$, $I_{y'''y'''}$ and $I_{z'''z'''}$.

Moment of Inertia Setup

To determine the undamped natural frequency, the experimental setup depicted in Figure 4.5 is utilized. Ball-bearings were initially used, but examination of the of the time data reveals a highly damped response, and to circumvent this problem, an air-bearing is manufactured.

In this technique, the shaft is supported by a thin layer of air and allowed to rotate freely, and examination of the time data reveals an undamped response. The supporting shaft is coated with a matte black paint, and a reflective strip is placed on the top of the shaft. The reflected light is detected by the optical sensor, and the time response is recorded by a HP Digital Signal Analyzer. Ten averages are recorded for each axis of

rotation, and the *freqtest.m* program (Appendix B) then determines the frequency of the collected data.

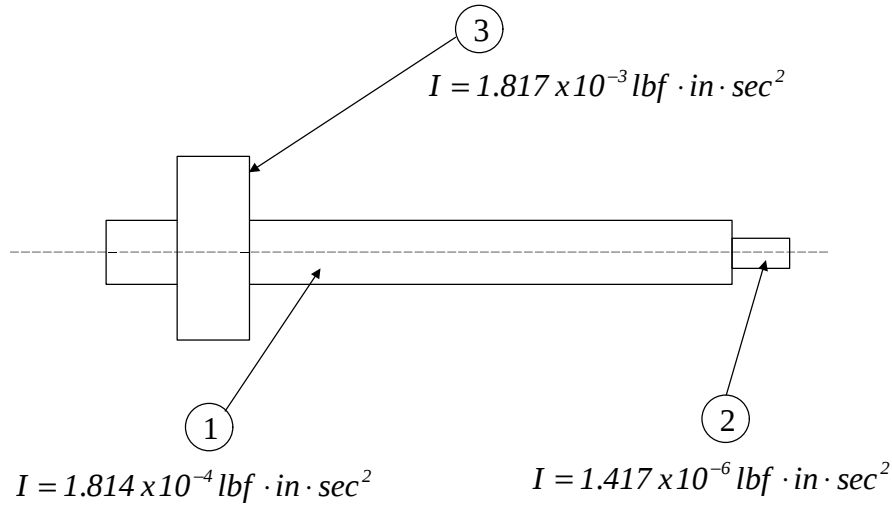


Figure 4.6 Setup Components

This frequency includes the contributions due to mass moments of inertia of the setup components (Figure 4.6); therefore, the values of these components have to be determined and subtracted from the measured moment of inertia. The setup includes a solid steel shaft (1 and 2) and an aluminum cylinder (3) to balance the weight of the test specimen – the club-head. For the purposes of analysis, the shaft has been examined as two separate parts. The masses of the components are calculated using a balance, and the dimensions of the components are measured a micrometer and dial caliper. With the masses and dimensions determined, the centroids of each of the components are calculated using standard theoretical techniques, and the setup centroid is subsequently determined. The moments of inertia of each of the components, and subsequently the setup, are then computed.

Setup Calibration

A test specimen with the proceeding characteristics is machined (See Figure 4.7) in order to test the experimental apparatus. The length of the specimen (L), the width (w), the thickness (t), the radius (r) and the offset (o) are then measured using calipers, while the mass is determined to be 290.7 grams using the balance. The centroids and theoretical moments of inertia are then calculated for each of the axes.

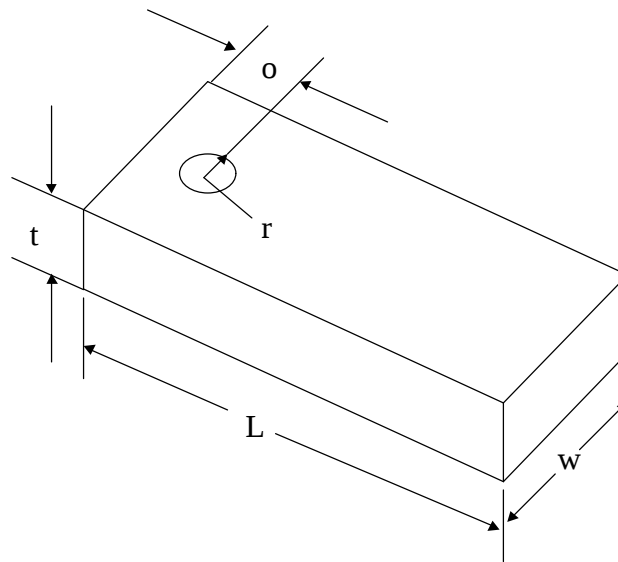


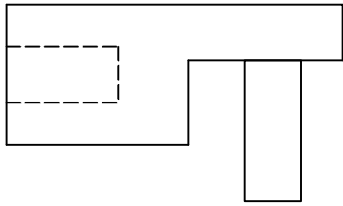
Figure 4.7 Test Specimen Used for Calibration

The undamped frequency for the test specimen and the shaft setup are determined for the experimental setup (Figure 4.5). Using the techniques described previously, the mass moments of inertia for the combined setup and club-head system are determined, and the values for the setup are subsequently subtracted to determine effects of the club-head alone. From these values, the value for the moments of inertia about a parallel axis

through the center of gravity is then computed. These values are then compared with theoretical values, and the values are within 6 percent.

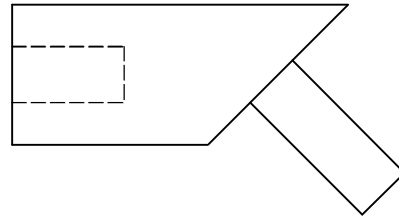
Determination of Club-Head Moments of Inertia

With the experimental setup calibrated, the moments of inertia of the club-head are then determined with respect to each axis of the coordinate system using the shaft and the 90 degree connector. The 45 degree connector is then used, and the values for the products are determined. These values are used in the lumped mass approximation in the finite element model.



$$I = 4.005 \times 10^{-5} \text{ lbf} \cdot \text{in} \cdot \text{sec}^2$$

(a)



$$I = 1.520 \times 10^{-5} \text{ lbf} \cdot \text{in} \cdot \text{sec}^2$$

(b)

Figure 4.8 (a) 90 Degree Connector (b) 45 Degree Connector

Flexural Stiffness

The finite element method requires the knowledge of the physical parameters of the desired structure to relate a force input to a displaced output. Since the finite element model for this work utilized beam elements, its formulation is governed by the principles of beam theory. Namely, the deflection is related by the product of the modulus of elasticity and moment of inertia, EI , also known as the flexural stiffness.

For objects with constant properties across a prescribed cross-section, such as the steel shaft, these parameters can be easily determined from the dimensions of the structure and material property tables. For those objects with variable properties across a cross-section, experimental techniques must be utilized to determine these parameters and relationship.

The moments of inertia are first determined for the steel shaft based upon the physical dimensions of the shaft. The shaft is cut based upon the length of the tapered section, and the length of each section is then measured. The outer and inner radii are measured at each end of the selected segment, and the corresponding outer and inner radii for other points of the segment are interpolated. From this information, the moment of inertia of the shaft can be calculated, while the modulus is obtained from material property tables.

For the graphite shaft, the materials vary across the cross-section, and a different method must be utilized. In this case, I is calculated from the shaft geometry, but determination of E requires proprietary knowledge from the manufacturer. For modeling convenience, the modulus of elasticity can be assumed constant, and the moment of inertia allowed to vary.

Experimental Determination of EI

Since the EI values for the steel shafts can be determined from dimensions and tables, the experimental technique is calibrated using the steel shaft. The shaft blank is placed in a three-point bend apparatus in an Instron tester, which applies an increasing load and records the corresponding displacement. This data is curve-fit using a linear least squares method to determine the relationship between displacement and force.

Using this information, the value of EI is determined using traditional beam deflection theory.

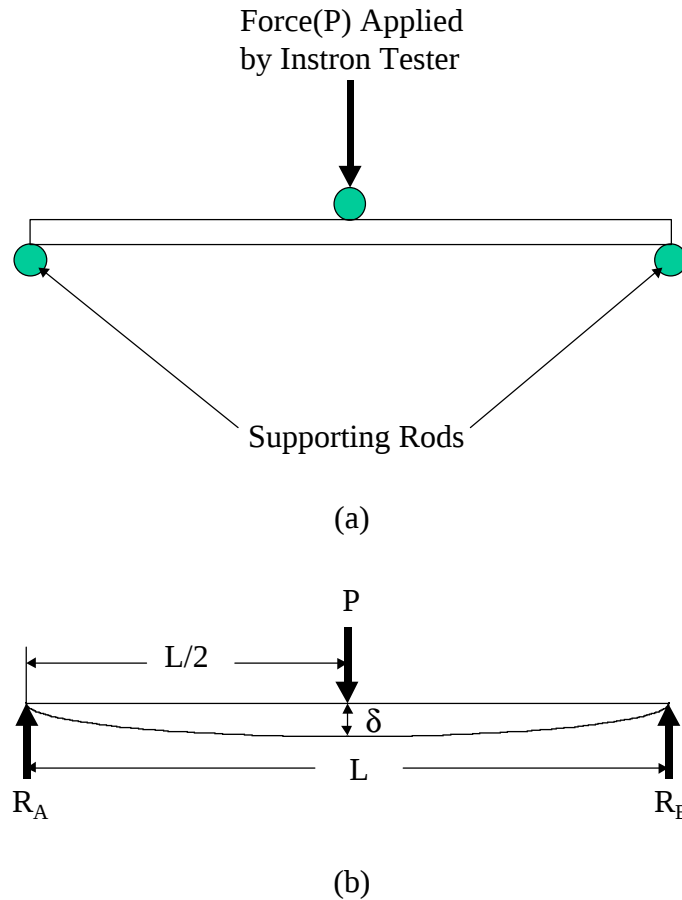


Figure 4.9 (a) Concentrated Load Applied by Instron Tester. (b) Corresponding Theoretical Representation of Applied Load and Supports

Originally, a concentrated load with simple support (Figure 4.9) is utilized, but a comparison of the results with the theoretical values, determined by the area-moment method, reveals a significant discrepancy over most of the shaft. In an effort to explain the discrepancies, a finite element model using shell elements of the shaft in a three-point bending is analyzed. From this analysis, an ovalization of the shaft is revealed near the

application of the load. In other words, the concentrated load causes localized deflections that are greater than expected, and the shaft ovalizes similar to Figure 4.10

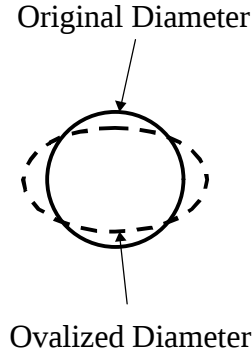
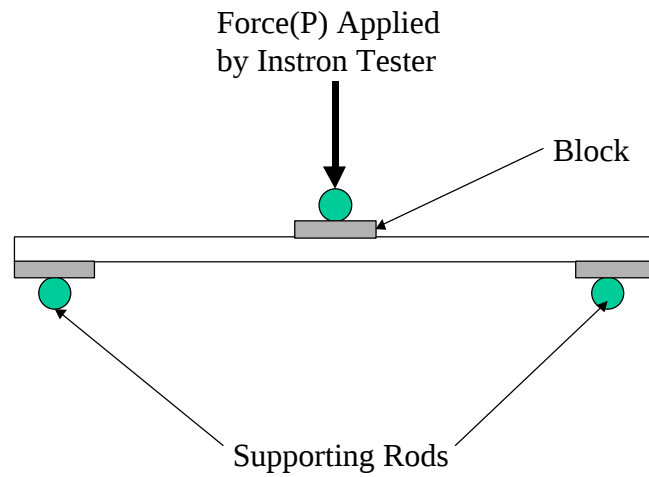


Figure 4.10 The Ovalization of the Shaft Due to Localized Deflection by Concentrated Load

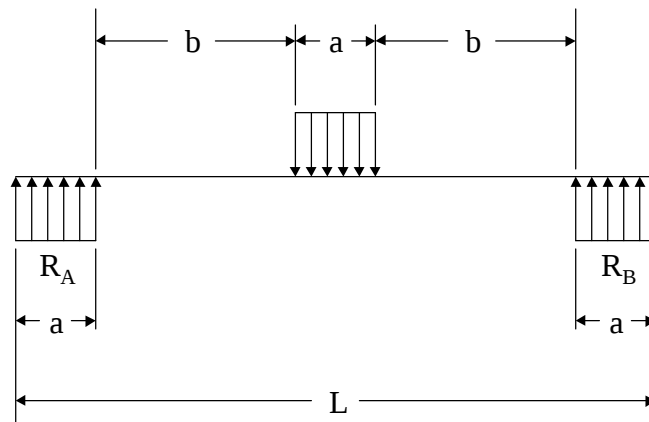
Since the concentrated loading and support did not yield the desired results, two additional cases are examined using FEA to determine an appropriate testing procedure. The first case, a simply-supported distributed load, exhibits improvement from the first case, but the values still differ significantly from the expected values. Therefore, a second case, a distributed load with distributed supports (Figure 4.11) is examined. Examination of the FEA results reveals significant improvement, and this case is tested using three one inch blocks loosely attached to the shaft using electrical tape. The deflection is then compared with theoretical values determined by the area-moment method.

$$y = \frac{wa}{192EI} \left[8a^3 + 45a^2 \left(b + \frac{a}{2} \right) + 96a \left(b + \frac{a}{2} \right)^2 + 32 \left(b + \frac{a}{2} \right)^3 \right] \quad (4.5)$$

Here, y is the deflection, w is the distributed load, a is the length of the applied load, and b is length between the loads. E is the modulus of elasticity, and I is the moment of inertia of the shaft.



(a)



(b)

Figure 4.11 (a) Distributed Load Applied by Instron Tester. (b) Corresponding Theoretical Representation of Applied Load and Supports

The preceding experimental technique calibration establishes the validity for use on the graphite shaft. Also, for the graphite shaft, the distribution of the mass must be determined experimentally. To accomplish this, the graphite shaft is first cut into segments of lengths based upon the taper, and then the mass of each shaft segment is then measured using the electronic balance. To determine the volume, the outer and inner radii are again measured at each end of the selected segment, and the corresponding values for the remaining points of the segment are interpolated. The volume of each segment is calculated based upon a frustum assumption.

$$V = \frac{1}{3}\pi l^2 \left[(a^2 + ab + b^2) - (c^2 + cd + d^2) \right] \quad (4.6)$$

Here, V is the volume of the segment, and l is the length of the segment. a and b are the outer and inner radii, respectively, for one end, while c and d are the corresponding values for the opposite end. With the mass and volume of the segment calculated, the mass density, important for the representation of mass in the finite element analysis, can then be determined.

Modal Analysis

Three modal cases are examined for the golf club to compare the boundary conditions and correlate with the finite element models. For the first case, free-free boundary condition is simulated, while the clamped-free boundary case is examined for the second. In the final case, the club is hand-gripped to simulate the conditions of a golf swing.

Modal Model

As stated previously, the development of the model contains four primary tasks – identification of measurement locations and transducers, data acquisition, modal parameter extraction and modal parameter application. In the first task, the number of desired modes and analogous points in analytical or finite element model govern excitation and response locations, while the structure and nature of the test influence transducer selection. The acquisition of data includes the frequency range selection and choice of excitation method. Additionally, this task includes the selection of frequency response estimation technique, number of averages, and windowing method. Modal parameter extraction includes the determination of natural frequencies, damping, and eigenvectors. This is accomplished by implementing either time domain or frequency domain techniques for the appropriate single-degree of freedom (SDOF) or multi-degree of freedom (MDOF) model. Finally, the modal parameters are used to correlate the analytical and modal model, update or modify the model, or develop design properties.

For the first task, the tests are configured to simulate the desired boundary conditions, and accelerometers are used to measure the output response for each test at specific spatial locations along the shaft and the face of the club-head. Used to measure response in both the swing and droop planes, the accelerometers are spaced two inches along the length of the shaft and correspond to the locations of the *EI* testing. To measure the input force, a force transducer is used for the free-free boundary, while a modal impact hammer, which contains a force transducer, is used for the clamped-free and hands cases.

The second component of the modal model encompasses the excitation of the structure and the analysis of the response. Excitation is introduced to golf shaft using a shaker for the free-free condition, while a modal impact hammer is used for the clamped-free and hands-free cases. To acquire the necessary frequency response, the HP Digital Signal Analyzer, 800 spectral lines, is utilized, and the H_1 estimator is employed.

Since the golfer does not typically detect higher frequencies^[31], the frequency range of primary interest is 0 to 800 Hz. To ensure consistency in the response, the number of averages is set for each case, and the coherence between averages is examined. Additionally, the power spectrums are examined to ensure the appropriate input and output for the system, and windows are employed if necessary. For this work, windows are not used for the clamped-free and free-free cases, while a force-exponential window is employed for the hands-free case. These general data acquisition settings are summarized in Table 4.2.

Table 4.2 : Summary of modal test settings

Parameter	Setting
Frequency Range	0 to 800 Hz
Frequency Resolution	800 spectral Lines
FRF Estimator	H_1
Window	None for Clamped-Free and Free-Free Forced Exponential for Hands

Modal parameter extraction, the third component of the modal model, includes the determination of natural frequencies, damping, and eigenvectors and either SDOF or MDOF techniques are employed. Since the modes are too closely spaced, the SDOF

curve-fitting techniques cannot be employed, and a multi-degree of freedom technique is utilized. To accomplish this, CADA PC[®], general MDOF curve-fitting technique by LMS[®], is used to determine the natural frequencies, damping and mode shapes, and these values are then compared and correlated in the final task of the modal model. Finally, the frequencies and mode shapes of the hands case are compared with the clamped-free and free-free conditions to ascertain any similarities, and these are then compared with the finite element results.

Boundary Conditions

Free-Free Boundary Condition

The free-free boundary condition is typically employed in modal analysis since it is among the easiest to simulate and correlate. To simulate this condition, the club is suspended by a rubber band from a beam clamped to a support table. The HP signal analyzer produces a burst-chirp source signal that is amplified and supplied to a shaker, which is suspended by bungee cords and attached to a force transducer. The force transducer is connected via a stinger to a clamp, which positioned on the grip one inch from the end of the shaft. To measure the response, a roving accelerometer is employed at determined spatial locations along the shaft and club-head, and the acquired data is analyzed using CADA PC[™].

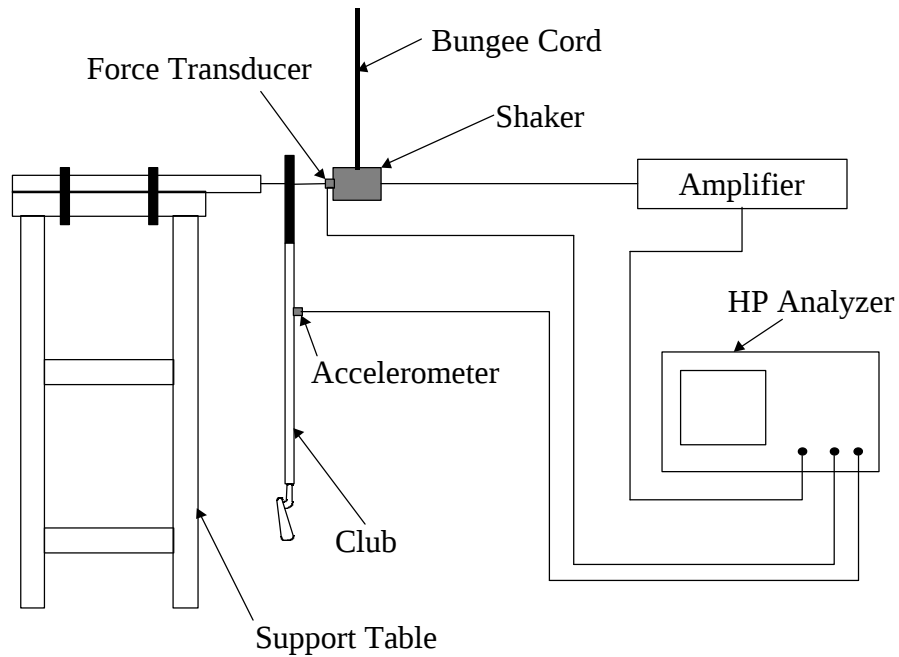


Figure 4.12 Modal Setup for the Free-Free Boundary Conditions

Clamped-Free

The clamped-free case is the one most often employed in finite element analysis, and other analytical methods, since it is easy to model. To compare these FEA results with the experimental data, the club grip is secured in a vise grip, which is attached to a support table. A modal impact hammer is then used to excite the club on the face of the club-head, while the response for the swing and droop plane is measured by a roving accelerometers.

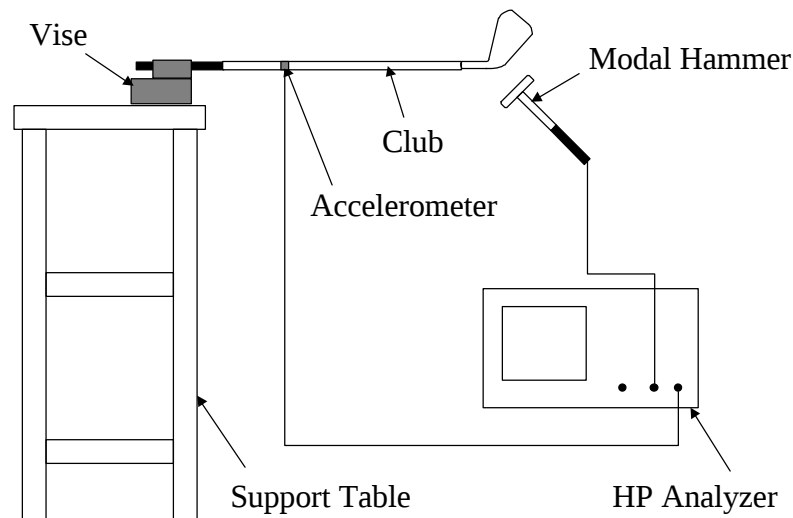


Figure 4.13 Modal Setup for the Clamped-Free

Hands-Free

In the final case, the club is hand-gripped to simulate the conditions of a golf swing, while the club-face is struck again by a modal impact at location midway along the first score line. The response is again measured using a roving accelerometer, but the spatial locations are limited by the addition of hands and by the 1 inch diameter tip used for the modal hammer. These values are then analyzed and compared with the clamped-free and free-free boundary conditions.

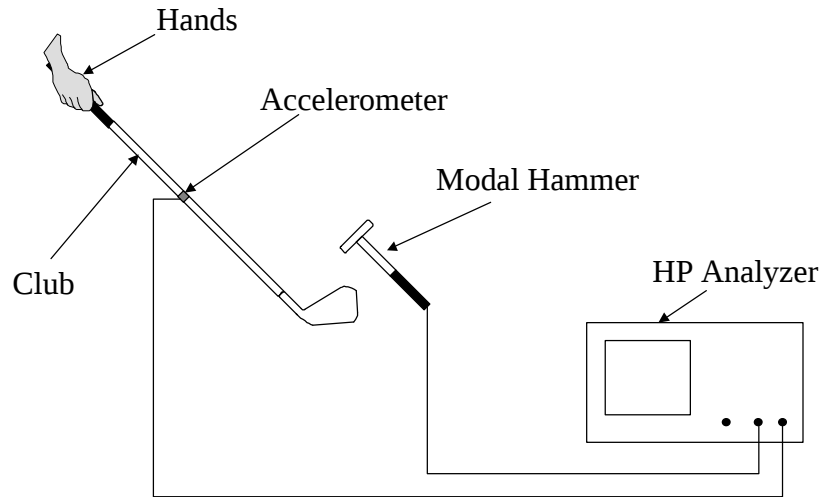


Figure 4.14 Modal Setup for Hands-Free

Finite Element Analysis

Finite element analysis is employed in a variety of applications from acoustics to heat transfer. The primary focus of this research, however, is structural dynamics, and the desired output is the frequencies and corresponding mode shapes. Once these values compare with the modal analysis to establish the suitability of the model, the nonlinear dynamic response of the shaft is investigated using the nonlinear dynamic solver in Algor[®].

Eigenvalue Analysis

One of the primary objectives of this research is the examination of eigenvalues and eigenvectors for two clubs – a steel-shafted five iron and a graphite-shafted club. They are evaluated for two boundary conditions – free-free and clamped-free. These

cases are examined because both are typically employed in the finite element modeling of a golf club and represent two typical ways to emulate the effects of the hands.

Development of the Model

Since the shaft is intrinsically a beam, beam elements can be employed to model the shaft, while the head can be represented as a lumped mass with corresponding moments of inertia. This reduces the complexity of the model, and thus the computational requirements.

The club is geometrically represented as two components – the shaft and a connecting link to the lumped mass. The shaft is divided into beam elements of desired length, and the physical parameters of each element, which include moment of inertia and polar moment of inertia are then input. To account for the mass effects of the grip, the grip is divided into segments and the mass of each section is measured with electronic balance, and these values are then applied as a lumped mass to appropriate nodes of the shaft.

Although a rigid link is more desirable, it is difficult to implement in Algor[®], and a stiff beam is used instead to connect the shaft end to the cg location of the lumped mass, which represents the club-head. To simulate the effects of a rigid link, the values for E and I are increased by an order of magnitude, while the area, which is used with weight density to determine the element mass, is decreased. The lumped mass is then applied to the end of this “rigid” beam, and the experimentally determined mass moment of inertia properties are used.

Steel Shaft

As previously mentioned, the physical properties of the steel shaft are constant along the cross-section. Hence, modulus of elasticity and mass density remain constant, while the moment of inertia, the polar moment of inertia and the area vary along the length of the shaft. These values are entered, and the number of elements is increased until convergence.

Graphite

The development of a FEA model for the graphite shaft is a little more involved than the steel shaft model because the physical parameters of the graphite are not constant across the cross-section. Therefore, the interrelation of the physical and geometrical parameters is used, and the parameters are adjusted to produce constant properties over the element length and cross-section.

Using the experimentally determined EI data, the modulus of elasticity is held at a constant value, while the I is allowed to vary down the length of the shaft in agreement with the EI data. The area of each element must be adjusted so the appropriate mass is included in the eigenvalue solution.

Non-Linear Dynamics

To determine the suitability of the FEA model, it is correlated with modal analysis, and once this has been accomplished, non-linear dynamic analysis is performed using the nonlinear dynamic solver in Algor[®]. This package allows the user to simulate an event, part of the golf swing in this case, over a desired interval for a prescribed event

curve, the moment applied at the grip for this application. The goal of this analysis is to determine the position of the club-head with respect the shaft through the swing.

Event Simulation

The software simulates the desired event for a certain time interval based upon a known, or assumed, event curve. Since part of the golf swing is desired, the time interval and strain curve are determined from previous work on shaft strain. The data, obtained using strain gauges placed *14* inches from the grip end of the shaft, is first converted to a curve of the input moment by calibrating the strain gauges. With the data converted, an event curve for the swing is constructed by selecting a few representative points from the moment curve. In other words, the moment curve is simulated using a number of linear approximations, which are successively increased to better represent the curve.

The finite element model is created, and the event is simulated while the software automatically converges each step of the model. Once the model has been converged, the dynamic performance of the shaft and club-head is examined over the prescribed event curve and time interval, and this response is then investigated to determine the club-head position relative to the shaft through the swing as expected.

Chapter 5

Results and Conclusions

The primary goals of this work are the development and correlation of an appropriate finite element model, the characterization of the hands-free boundary condition and the examination of the club golf dynamic response. To accomplish these objectives, the physical parameters of the club golf club must first be determined to develop the finite element model, which is then analyzed. The mode frequencies and mode shapes are then correlated with results extracted from modal analysis for the free-free and clamped-free modal cases. With the values correlated, a third modal case, hands-free, is employed to ascertain the effects of the grip on the boundary conditions. With the appropriateness of the FEA model confirmed, the nonlinear dynamic response of the club during part of the down-swing is investigated using the nonlinear solver in Algor[®], and the possibility of club-head “lag” is then examined.

Club-Head Mass Moment of Inertia

The values for the moments of inertia are determined using the procedure discussed in the previous chapter. The time response of the club-head and shaft setup is first determined using the experimental setup, and the frequency is then determined using

the *freqtest.m* program. From this frequency, the moment of inertia of the entire system is then determined, and the value for the club-head is determined by subtracting the values of the shaft setup. The values for the moments of the original axis system are then,

$$\begin{aligned} I_{xx} &= 1.656 \times 10^{-3} \text{ lb} \cdot \text{in} \cdot \text{sec}^2 \\ I_{yy} &= 2.839 \times 10^{-3} \text{ lb} \cdot \text{in} \cdot \text{sec}^2 \\ I_{zz} &= 2.824 \times 10^{-3} \text{ lb} \cdot \text{in} \cdot \text{sec}^2 \end{aligned} \quad (5.1)$$

These are the values for the original coordinate system, and these are the only moment values employed to characterize the lumped mass in the Algor software. For other software packages, the products of inertia are required, and these are determined by rotating the original axis system about each axis.

A 45-degree rotation about the original z-axis yields a new coordinate system – $x'y'z'$, and the value of the moment inertia about the y' -axis yields the following,

$$I_{y'y'} = 6.083 \times 10^{-3} \text{ lb} \cdot \text{in} \cdot \text{sec}^2 \quad (5.2)$$

From this value, the value for the I_{xy} is then determined using Equation (4.2).

$$I_{xy} = -3.836 \times 10^{-3} \text{ lb} \cdot \text{in} \cdot \text{sec}^2 \quad (5.3)$$

Using a 45 degree rotation about the original y axis, another coordinate system, $x''y''z''$, is then created, and the moment about the z'' -axis is then,

$$I_{z''z''} = \quad . \quad ^2 \quad (5.4)$$

$$I_{xz} \quad (4.3).$$

$$I_{xz} = -4.034 \times 10^{-3} \text{ lb} \cdot \text{in} \cdot \text{sec}^2 \quad (5.5)$$

The final product of inertia is determined by a 45° rotation of the original x -axis. From this new coordinate system – $x'''y'''z'''$, the moment of inertia about the y''' -axis is then determined.

$$I_{y'''} = 14.025 \times 10^{-3} \text{ lb} \cdot \text{in} \cdot \text{sec}^2 \quad (5.6)$$

Using Equation (4.4), the value for the remaining product of inertia, I_{yz} , is then computed.

$$I_{yz} = -11.194 \times 10^{-3} \text{ lb} \cdot \text{in} \cdot \text{sec}^2 \quad (5.7)$$

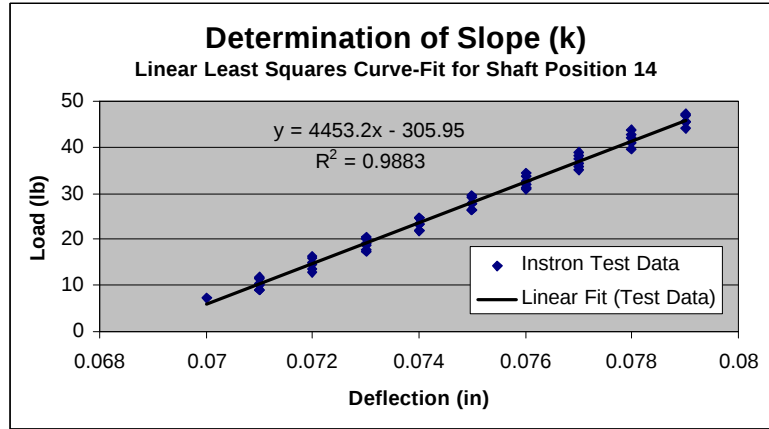
The values for moments of inertia are now determined for the original coordinate system.

Shaft EI

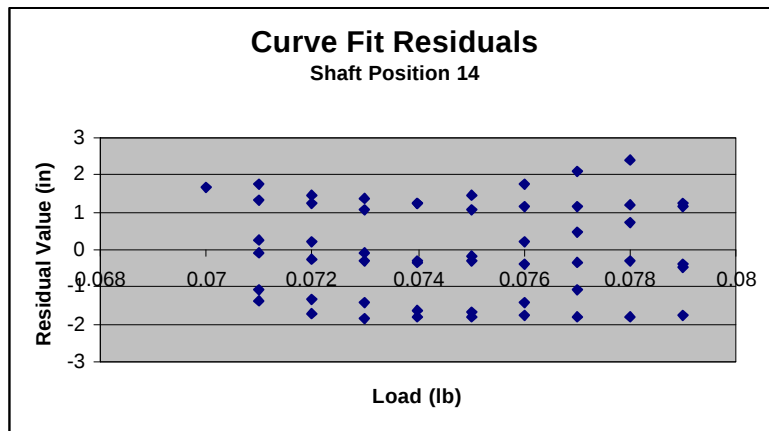
The area moments of inertia of the shaft are experimentally determined using a three-point bend apparatus and an Instron testing machine. To calibrate the experimental shaft EI method, a steel shaft is examined, and the values are compared with the theoretical values determined by carefully measuring the shaft.

Originally, a concentrated load is applied mid-span on a simply supported beam, and the deflection versus load data is curve-fit using a linear least squares technique - Figure 5.1. The correlation coefficient, R^2 equals 0.9883, and the well-scattered residuals

indicate a “good” fit of the experimental data, and the data can then be compared with the theoretical values using an assumed load value.



(a)



(b)

Figure 5.1 (a) Determination of k Using a Linear Least Squares Technique (b) Curve-Fit Residuals

Assuming a 25 pound load, the deflection of the shaft at each position is compared with the theoretical values using the deflection formulas for a beam. (See

Figure 5.2) From the figure, it is apparent that the values differ significantly towards the butt end of the shaft and become closer towards tip.

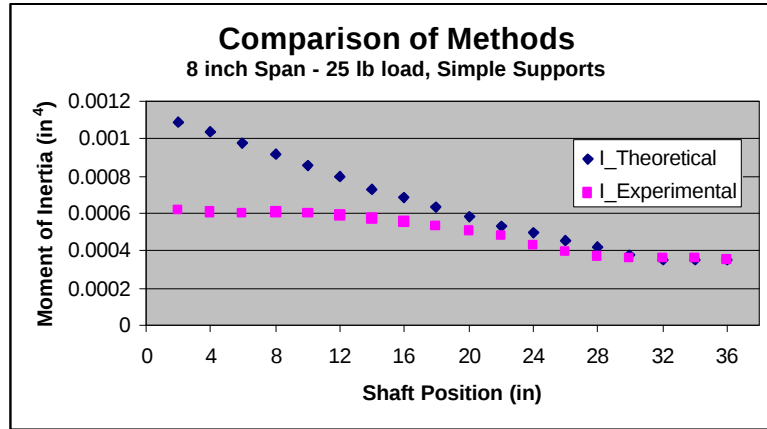


Figure 5.2 Comparison of Theoretical and Experimental Techniques for the Determination of the Moment of Inertia for an 8 inch Beam with a Mid-Span Load and Simple Supports

This suggests a problem with the technique, and finite element analysis is used to examine the problem. Performed at five positions on the shaft, FEA results demonstrated ovalization of the shaft at the application of the load. This is most likely due to localized deflection, especially for positions with thin cross-sections compared with the outer diameter. (See Figure 5.3)

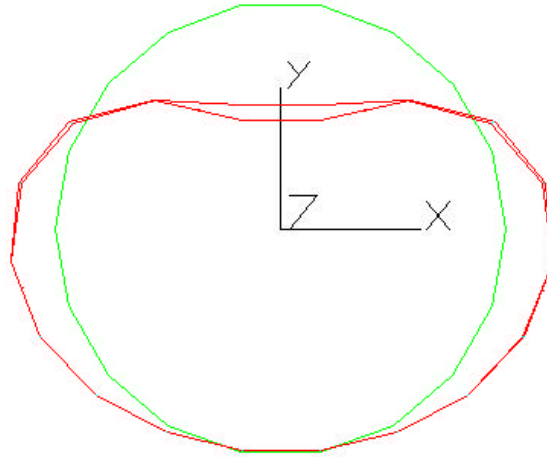


Figure 5.3 Exaggerated Ovalization of the Shaft at the Point of Load Application.

With the finite element analysis performed, the FEA results are then compared with the theoretical and experimental values for deflection. (See Figure 5.4) It is apparent that the deflection values for experimental and finite element are greater than values determined by theory. Additionally, the deflections of the experimental and FEA are similar. Thus, “ovalization” of the shaft is considered a major contributor to large discrepancies, and additional load conditions are considered using finite element analysis of the shaft sections.

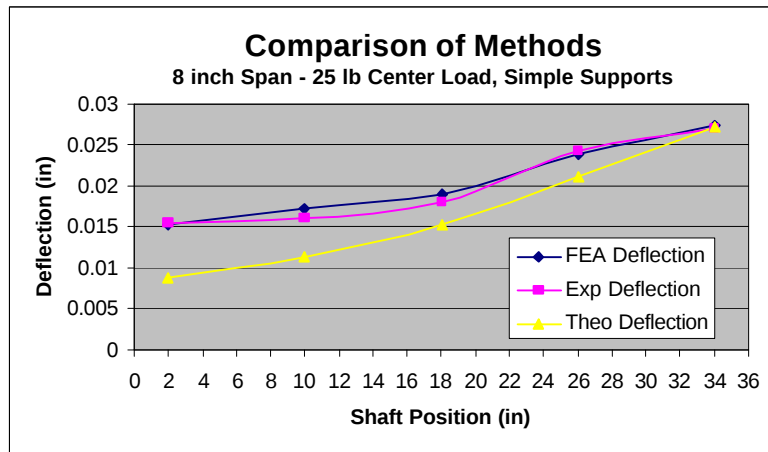


Figure 5.4 Comparison of Deflection Values for Finite Element, Experimental, and Theoretical Methods

The large discrepancy of deflection values establishes the need to evaluate additional cases, and supplementary loading conditions are examined using FEA. In the hopes that a longer span would improve results, the span length is increased from 8 inches to 12 inches for the first case. An examination of the percent discrepancies in Figure 5.5 reveals an improvement with the discrepancies ranging from 25% at the butt end to 4 % at the tip. Although extension of the span length has reduced the discrepancies, the values still differ significantly from the theoretical values, and the effects of localized deflection are still evident. Therefore, other loading conditions must be examined.

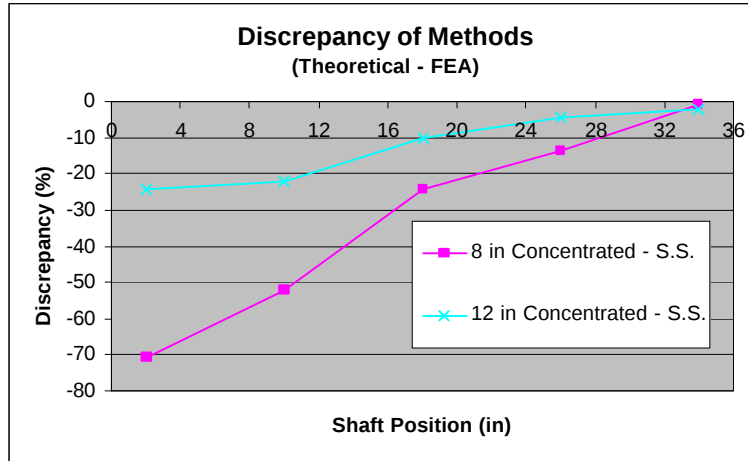


Figure 5.5 Comparison of the percent discrepancies of 8 and 12 inch simply-supported span with a 25 lb load applied mid-span.

To circumvent the localized deflection of the concentrated load, distributed loading cases are examined. A distributed loading case, 25 lb/inch over 1 inch, with simple supports is first considered, but the discrepancy values are only slightly better than the 12 inch concentrated load with simple supports. (See Figure 5.6) A distributed load with 1 inch distributed support is then considered, and the discrepancies are less than the simply-supported distributed case and range from 4.5% at the butt end to 2.7% at the tip of the shaft.

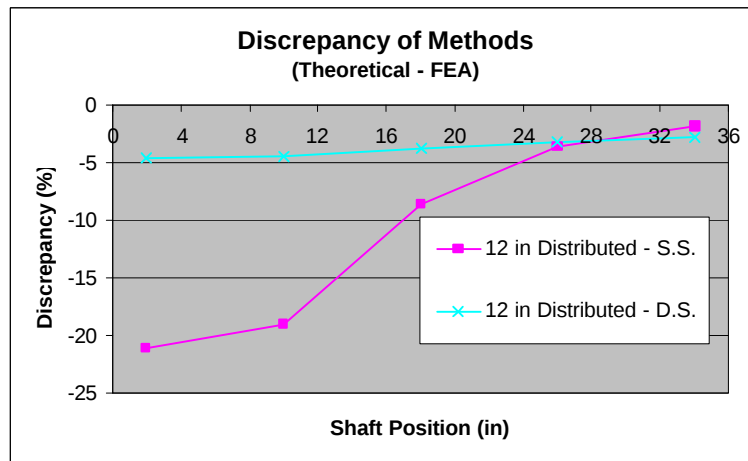


Figure 5.6 Comparison of the percent discrepancies of a 12 inch under a distributed load with simply supported and distributed supports.

Based upon the FEA results, experimental testing with the Instron tester is again performed on the steel shaft, but this time, three one inch blocks are also employed to simulate the distributed load and support. The applied load and corresponding deflection is again curve-fit using a least squares technique, and the deflection is compared with the theoretical values determined using Equation (4.5) on page 58. From Figure 5.7, it is apparent that the deflections are similar for the majority of the shaft but differ at the ends, especially the butt end of the shaft. Since the shaft hangs in the three-point bend apparatus, the force of gravity on the overhanging portion may provide slightly higher deflections at the butt end and slightly lower deflections at the tip. Unfortunately, tests could not be performed to confirm these suspicions, but investigation of this possibility is recommended for further work in the experimental determination of the shaft.

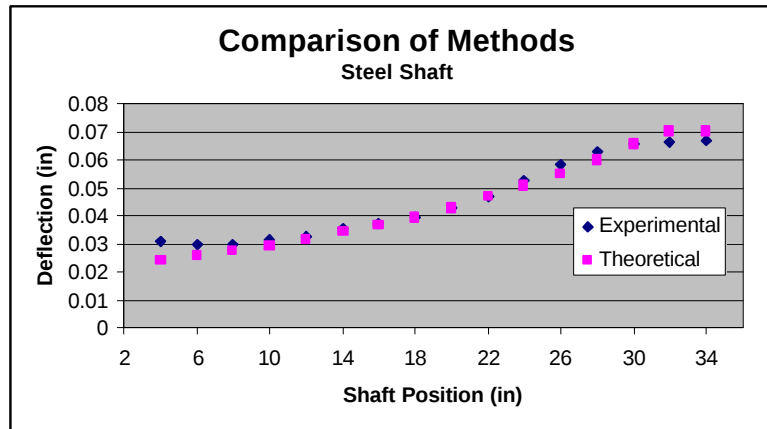


Figure 5.7 Comparison of experimental and theoretical deflections for steel shaft.

The flexural stiffness of the shaft is then determined by rearranging and solving Equation (4.5) on page 58. Since the cross-section material of the steel shaft is homogeneous, the moment of inertia is easily determined and compared with the theoretical values. From Figure 5.8, it is apparent that the results are similar for most of the shaft.

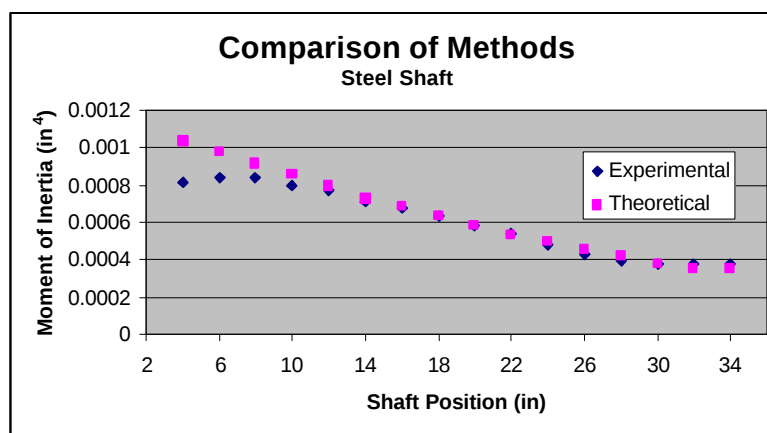


Figure 5.8 Comparison of experimental and theoretical moments of inertia.

The material properties of the graphite shafts cannot be determined by simply measuring the shaft diameter, and the preceding experimental testing technique, therefore, is particularly useful. Using the preceding experimental procedure, the flexural stiffness curve is determined and compared with the steel shaft. (See Figure 5.9)

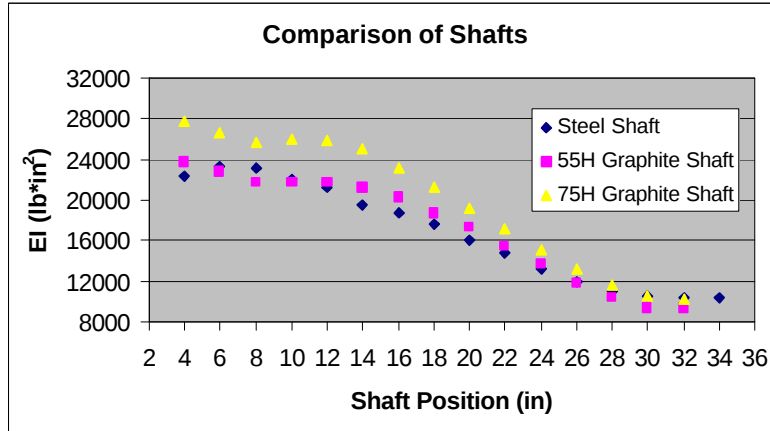
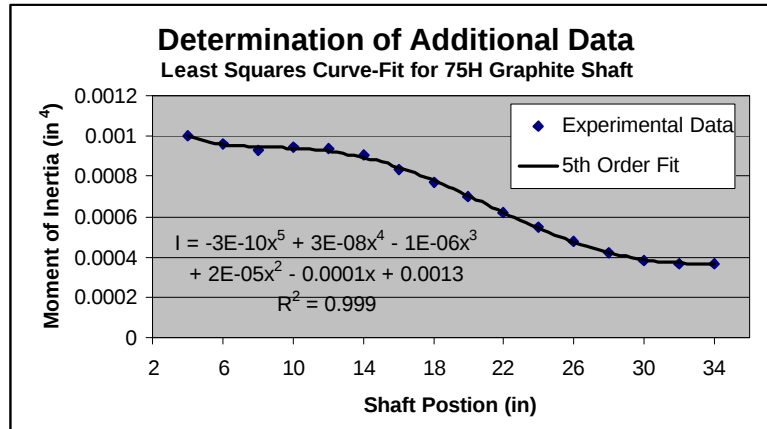


Figure 5.9 Comparison of the EI curves for the steel and graphite shafts.

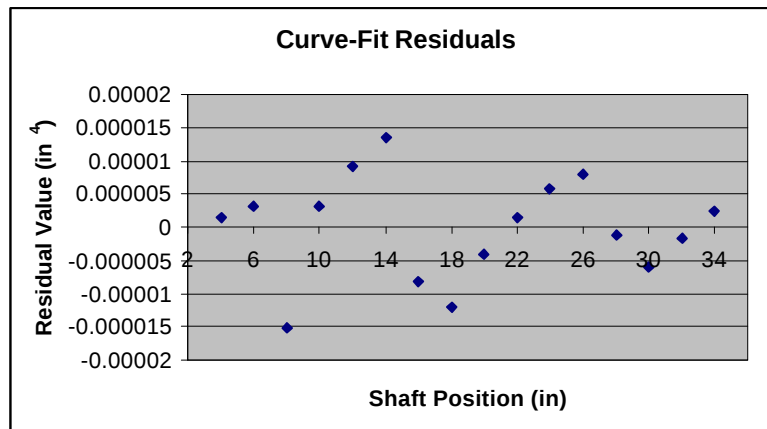
The modulus of elasticity and moment of inertia are thus related by the preceding flexural stiffness curves, and one value is obtained by assuming the other constant across the cross-section. In this case, E is assumed to be constant, while the moment of inertia is allowed to vary along the length of the shaft, and these values can now be used in the development of the finite element model.

This experimental data is used in the development of the finite element model, but the number of spatial locations used in the testing limits the data. To account for this, additional data is obtained through least squares curve-fitting or an interpolation technique. Least squares curve-fitting is first employed, but the residuals of the fit suggest a characteristic behavior even for a 5th order curve-fit. (See Figure 5.10) Since a

“good” curve-fit is unavailable, the additional data is merely interpolated using the experimentally measured positions.



(a)



(b)

Figure 5.10 (a) Fifth order least-squares curve-fit of the experimental moment of inertia data for the 75H graphite shaft. (b) Corresponding curve-fit residuals.

With the determination of EI, the remaining physical parameters of the shaft are then determined. As previously stated, each shaft is divided into sections based upon the taper of the shaft, and the mass and physical dimensions are measured. The mass is measured using an electronic balance, while the length and diameters of the sections are

determined using the calipers. The inner and outer diameters are measured at each section end using dial calipers, while the values for the interior positions of each section are then interpolated. Assuming the section is a frustum, the volume of each section is calculated using Equation (4.6) on page 60, and the weight density is obtained by dividing the weight by the volume.

Finally, the grip is characterized to account for the effects of its mass on the system. Using a similar procedure as the shaft, the grip is divided into sections, and the mass of each section is measured using the electronic balance. The value for the mass at corresponding shaft locations is then interpolated and used as lumped mass in the finite element modeling of the system.

Dynamic Analysis Results

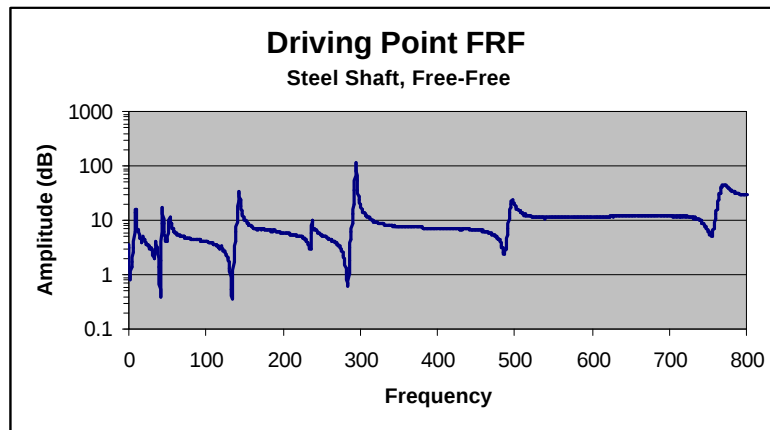
The finite element and modal models are developed according to the procedures outlined in the previous chapters, and the requisite eigenvalues and eigenvectors are obtained for the desired boundary conditions. Free-free and clamped-free cases are examined for both, and the mode shapes and frequencies are compared to determine the degree of correlation, if any between the two methods. Also, an additional case, hands-free, is investigated solely using modal analysis to ascertain the boundary conditions of a typical grip.

Once the appropriateness of the FEA is established through correlation, the dynamic effects of the shaft are examined using the nonlinear dynamic solver in Algor[®]. To accomplish this task, previously acquired swing data is calibrated, and the down-swing portion of the moment curve is then interpolated by a series of lines. These lines

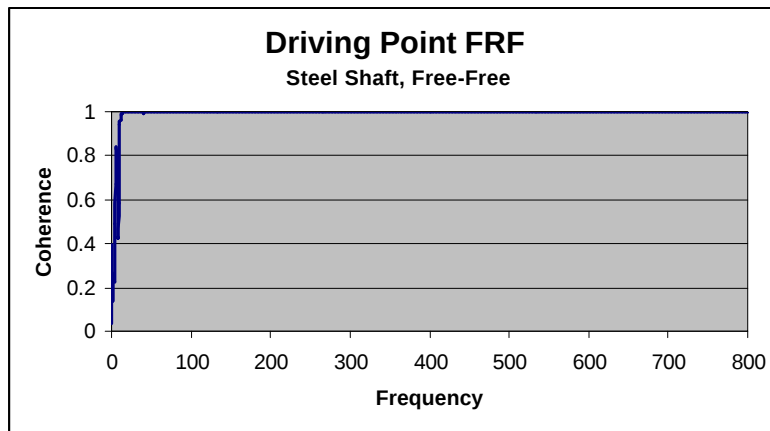
are then input into the Accupak utility, and the dynamic response of the club is determined for the club model.

Modal Data

The coherence is first examined to determine the causality between the input and output at each spatial position. As stated previously, the coherence function measures the linear relationship between the output and input and is defined between 0 and 1, with unity being total correlation. Figure 5.11b is the driving point coherence for free-free, steel shaft test and is representative of the coherences. It approaches unity for most frequencies, but is low at the extremely low frequencies. Since the HP analyzer uses the H^1 estimator, low coherence can occur at anti-resonances, which is the case here. Examination of the driving point FRF, Figure 5.11a, reveals an anti-resonance at the extremely low frequency, which is most likely due to rigid body modes.



(a)

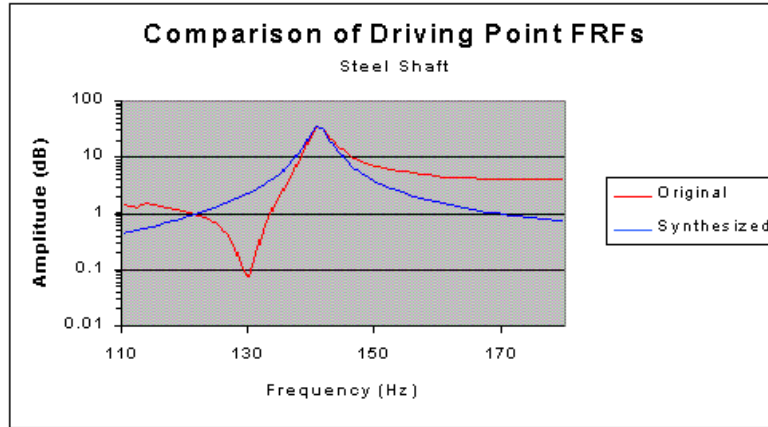


(b)

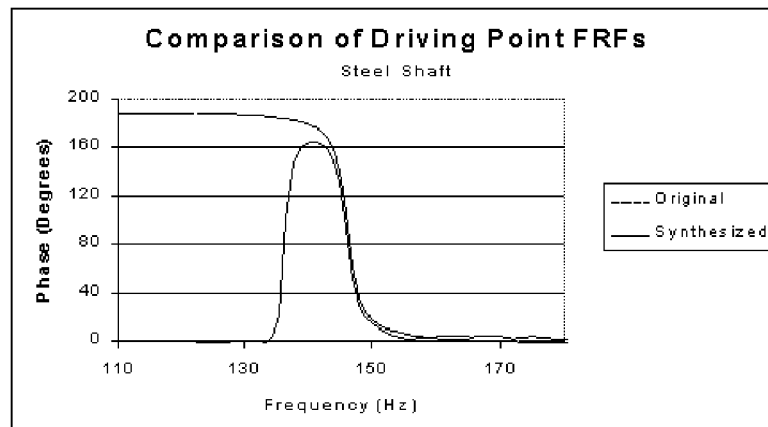
Figure 5.11 Steel shaft, free-free driving point FRF. (a) Amplitude (b) Coherence

With the frequency response functions acquired, modal data is input into CADA PCTM, and the modal parameters are extracted using a MDOF curve-fitting technique in the software package. The poles, mode frequencies, are then selected, and the corresponding mode shapes are estimated for each mode. To ensure proper curve-fitting, an FRF is synthesized by the software and then compared with the original. From Figure

5.12, it is apparent that the amplitude and phase of the synthesized FRF are similar to the original, and therefore, a good fit is inferred.



(a)



(b)

Figure 5.12 Comparison of driving point FRFs for steel shaft – 2nd swing plane mode.
(a) Amplitude (b) Phase

Comparison and Correlation

Perhaps the most important component of any analysis is the comparison of the obtained results. After the appropriate values have been garnered, the mass-normalized mode shapes and frequencies are compared for the desired boundary conditions. This is first accomplished with a comparison of the mode frequencies and then inspection of the mode shapes.

Frequency Comparison

The contrasting of frequencies is among the simplest forms of comparing different analysis methods and provides insight into correlation, or lack thereof, between the techniques. Thus, if two methods are truly similar, the results will agree within a relatively small discrepancy, while dissimilar values will have larger discrepancies.

Steel Shaft

The finite element and modal frequencies are first compared for the steel shaft. The finite element and modal frequencies are compared for the free-free and clamped-free boundary conditions. Examining Table 5.1, it is apparent that the mode frequencies are similar for both the free-free and clamped-free cases, with the largest discrepancy occurring at the third droop-plane mode.

Table 5.1 : Mode frequencies for steel shaft. (a) Free-Free (b) Clamped-Free

Mode	FEA (Hz)	Modal (Hz)	Damping %	Plane	% discrepancy (FEA-Modal)
1	42.13	39.73	0.19	Droop	5.69
2	47.05	46.01	1.12	Swing	2.21
3	126.92	119.62	0.46	Droop	5.75
4	143.91	144.01	0.13	Swing	-0.07
5	261.02	243.02	1.16	Droop	6.90
6	287.98	290.99	1.90	Swing	-1.05

(a)

Mode	FEA (Hz)	Modal (Hz)	Damping %	Plane	% discrepancy (FEA-Modal)
1	5.90	5.70	0.84	Droop	3.35
2	5.90	5.87	1.17	Swing	0.58
3	58.01	59.61	1.76	Torsional	-2.77
4	65.82	63.43	0.13	Droop	3.64
5	83.38	88.94	1.29	Swing	-6.67
6	186.77	169.89	1.90	Droop	9.04
7	231.92	231.83	1.13	Swing	0.04

(b)

Additionally, another case, hands-free in the swing-plane only, is examined using modal analysis alone since the exact mechanisms for simulating the finite element boundary conditions are unknown for this case, and the results are listed in Table 5.2. The hands-free frequencies are compared against the swing-plane values determined for the other two boundary cases.

Table 5.2 : Mode frequencies for the hands-free boundary condition for the steel shaft.

Mode	Modal (Hz)	Damping %
1	50.11	6.36
2	148.02	3.37
3	281.56	0.87

Inspecting Table 5.3, it appears that the hands-free value correspond better with the free-free frequencies than the clamped-free frequencies. The percent discrepancy between the hands-free and free-free is less than 9% for the modes in question, while the clamped-free case has discrepancies ranging from 21.5% to 760%.

Table 5.3 : Mode frequency comparison between hands-free and the free-free and clamped-free cases

Mode	Hands (Hz)	Free (Hz)	Clamped (Hz)	% discrepancy (Free-Hands)	% discrepancy (Clamped-Hands)
1	50.11	46.01	5.84	-8.91	-758.05
2	148.02	144.01	88.94	-2.78	-66.43
3	281.56	290.99	231.83	3.24	-21.45

Graphite Shaft

With the steel shaft complete, the FEA and modal frequencies are next compared for the graphite shaft. Examination of Table 5.4 reveals the frequency correlation between the two analysis techniques, and similar to the steel shaft, the largest discrepancies occur at the third droop-plane.

Table 5.4 : Mode frequencies for graphite shaft. (a) Free-Free (b) Clamped-Free

Mode	FEA (Hz)	Modal (Hz)	Damping %	Plane	% discrepancy (FEA-Modal)
7	48.01	46.38	1.13	Droop	3.39
8	52.61	50.99	0.97	Swing	3.09
9	136.62	128.57	0.54	Droop	5.89
10	155.81	156.04	0.29	Swing	-0.15
11	282.91	262.68	1.05	Droop	7.15
12	309.56	313.22	0.50	Swing	-1.18

(a)

Mode	FEA (Hz)	Modal (Hz)	Damping %	Plane	% discrepancy (FEA-Modal)
1	5.90	5.76	0.58	Droop	2.35
2	5.91	5.84	0.59	Swing	1.12
3	58.32	53.69	1.19	Torsional	7.94
4	69.56	67.17	1.57	Droop	3.44
5	91.63	94.66	0.74	Swing	-3.31
6	209.40	190.35	1.41	Droop	9.10
7	257.70	262.00	0.73	Swing	-1.67

(b)

The hands-free case is also examined for the graphite shaft, and the resulting frequency values, Table 5.5, are once more compared with the free-free and clamped-free boundary conditions. From Table 5.6, it once again appears that the free-free values appear to correlate better with the hands-free frequencies than with the clamped-free frequencies.

Table 5.5 : Mode frequencies for the hands-free boundary condition for the steel shaft.

Mode	Modal (Hz)	Damping %
1	54.01	5.71
2	158.02	3.37
3	302.65	0.74

Table 5.6 : Mode frequency comparison between hands-free and the free-free and clamped-free cases

Mode	Hands (Hz)	Free (Hz)	Clamped (Hz)	% discrepancy (Free-Hands)	% discrepancy (Clamped-Hands)
1	54.01	50.99	5.84	-5.92	-824.83
2	158.02	156.04	94.66	-1.27	-66.93
3	302.65	313.22	262.00	3.37	-15.52

Mode Shape

With the mode frequency comparison complete, the mode shapes of the different techniques are contrasted. This is first accomplished by visual inspection of the mode shapes for the plane of interest, the droop and swing planes in this case. With visual similarities established, the eigenvectors are then compared numerically using the Modal Assurance Criterion, which is discussed in Chapter 3.

Since the mode shapes are similar for both the graphite and steel shafts, a detailed examination of the steel shaft is discussed in the ensuing sections. Conversely, the graphite shaft will only be covered tersely in the subsequent discussion of results, but all mode shapes are included in the appendix of this work.

Steel Shaft

Free-free and clamped-free cases are examined using both modal and finite element analysis, and the mode shapes and frequencies are compared to determine the degree of correlation, if any between the two methods. Also, an additional case, hands-free, is investigated solely using modal analysis to ascertain the boundary conditions of a typical grip.

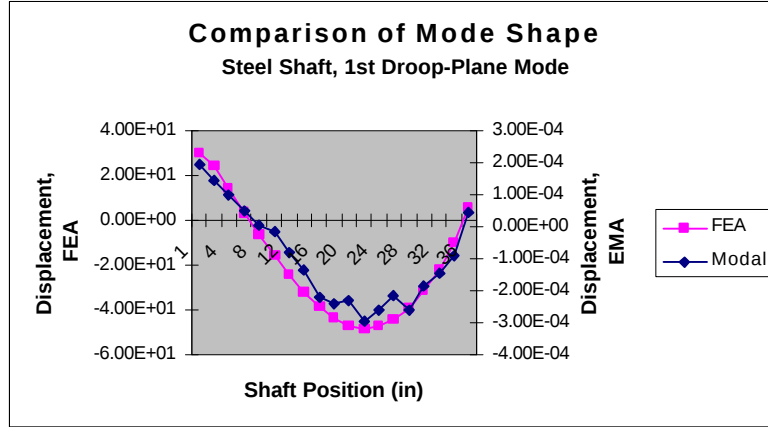
The examination of mode shapes is first accomplished with a visual inspection of the mode shapes. When visual similarity has been established, the mode shapes are then numerically compared using the MAC, and the values range from 0 to 1, with unity again being total correlation and zero representing no correlation.

Free-Free

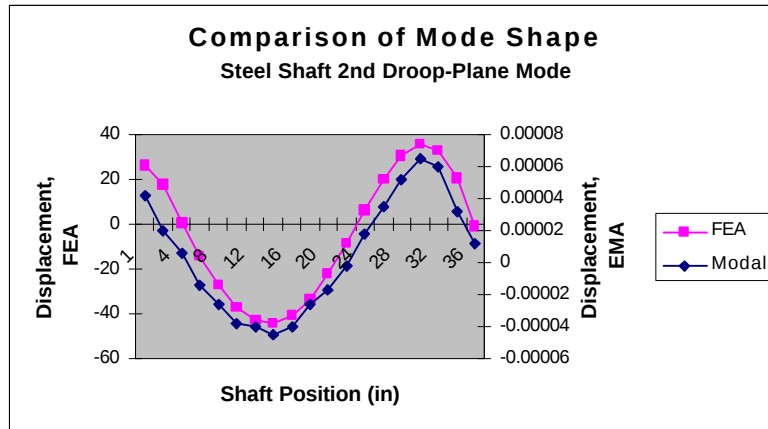
The free-free boundary condition is typically utilized in structural dynamic analysis since it is among the easiest to simulate in finite element and modal analysis. Therefore, it is the first case examined by visual inspection and the MAC.

Visual inspection of the droop (Figure 5.13) and swing (Figure 5.14) planes reveals similarities between the finite element and EMA mode shapes. Additionally, closer examination of the figures reveals particularly close visual correlation for the second swing mode (Figure 5.14b), while the second droop (Figure 5.13b) and the third swing mode (Figure 5.14c) exhibit a greater discrepancy.

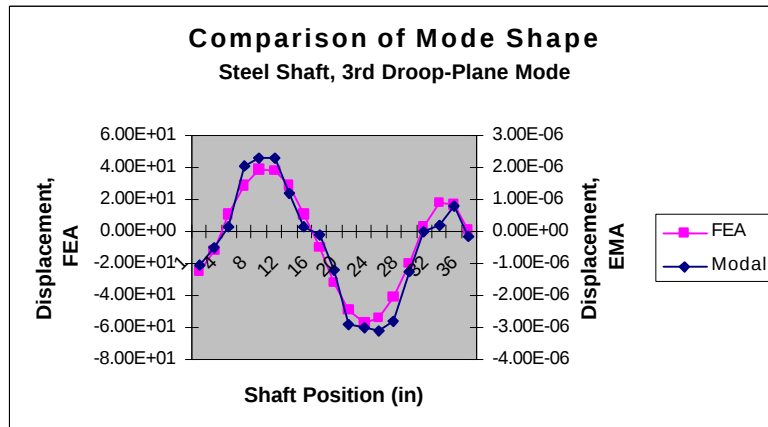
This slight discrepancy is to be expected due to the nature of the analysis methods. The finite element analysis is analytical technique and therefore yields the smoothed mode shapes exhibited in the two figures. Conversely, the modal analysis is an experimental technique, and the eigenvectors are obtained by curve-fitting data obtained through testing. Thus, the results are not the “smoothed” mode shapes predicted by analytical and theoretical techniques.



(a)

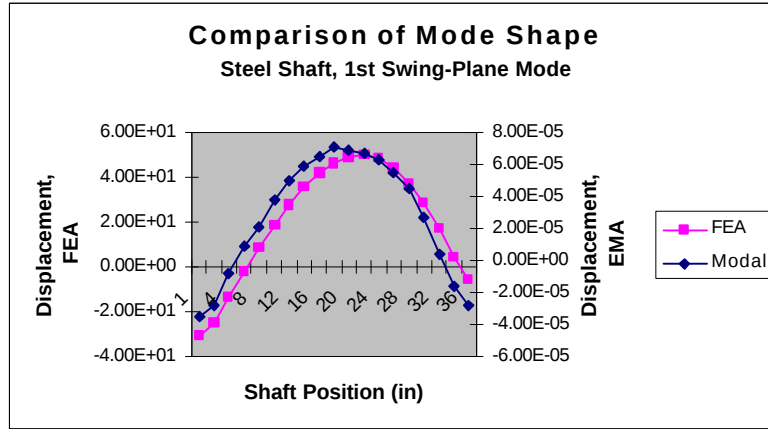


(b)

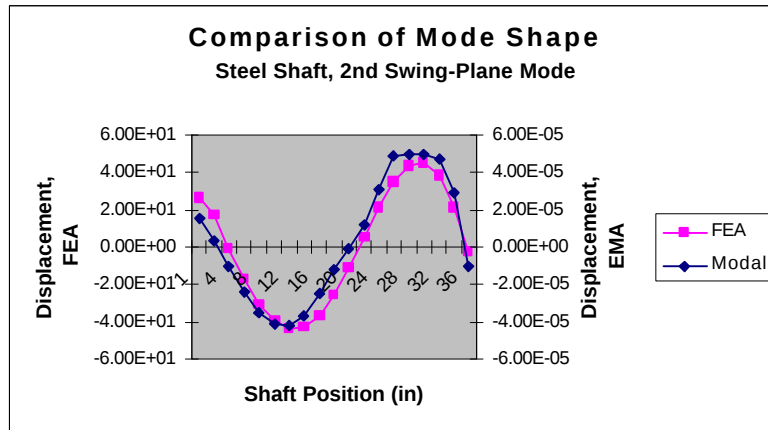


(c)

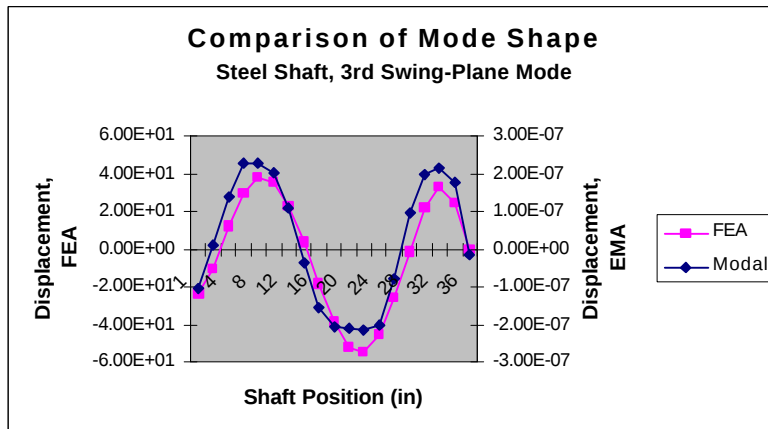
Figure 5.13 Comparison of the mass-normalized mode shapes for the droop-plane of the steel shaft with free-free boundary conditions. (a) First Mode (b) Second Mode (c) Third Mode



(a)



(b)



(c)

Figure 5.14 Comparison of the mass-normalized mode shapes for the swing-plane of the steel shaft with free-free boundary conditions. (a) First Mode (b) Second Mode (c) Third Mode

With the visual correlation established, the mode shapes are then compared numerically using the modal assurance criterion. The diagonal values are the primary interest of the comparison since a value close to unity implies “good” correlation between the compared eigenvectors, while the off-diagonal terms compare the relationship between different modes.

Examination of the MACs (Table 5.7) reveals diagonal values of 0.9 or greater with the exception of the third swing mode, which has a value of 0.88. Using the visual method (Figure 5.14c), it is apparent that the displacements are less than expected for a few spatial locations (positions 20 and 22) near the middle of the club. The exact nature of this discrepancy is unknown, and it may just be the nature of the test and the curve-fitting. To examine these possibilities, future work, with focus on the curve-fitting and data acquisition, must be performed.

Although not unity, these diagonal MAC values suggest “good” correlation between the finite element and modal mode shapes even for the third swing mode. It is also notable that the first and third droop planes exhibit some similarity, roughly 0.25 or 25 percent, and visual inspection implies a small degree of likeness as well.

Table 5.7 : Free-Free MAC for steel shaft (a) Droop-plane (b) Swing Plane

Modal FEA		Mode		
		1	2	3
M o d e	1	0.9667	0.0244	0.2671
	2	0.0003	0.9038	0.0632
	3	0.2577	0.0624	0.9559

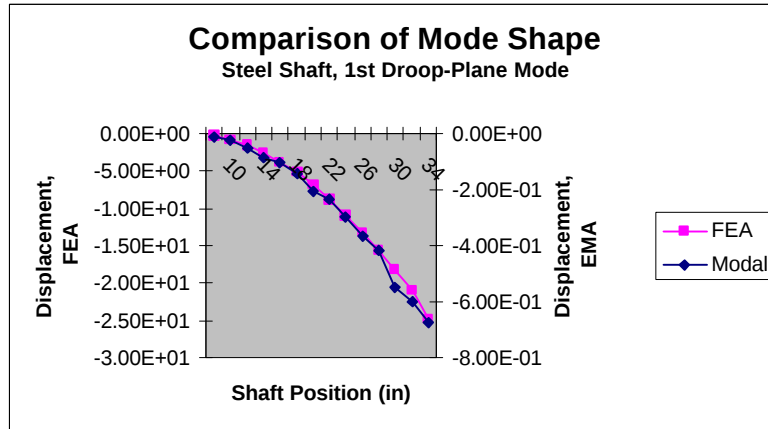
(a)

Modal FEA		Mode		
		1	2	3
M o d e	1	0.9469	0.0718	0.1652
	2	0.0224	0.9227	0.0539
	3	0.0836	0.0037	0.8873

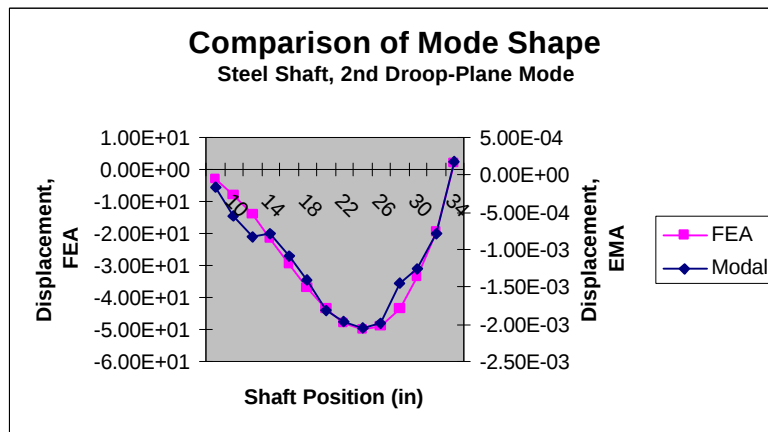
(b)

Clamped-Free

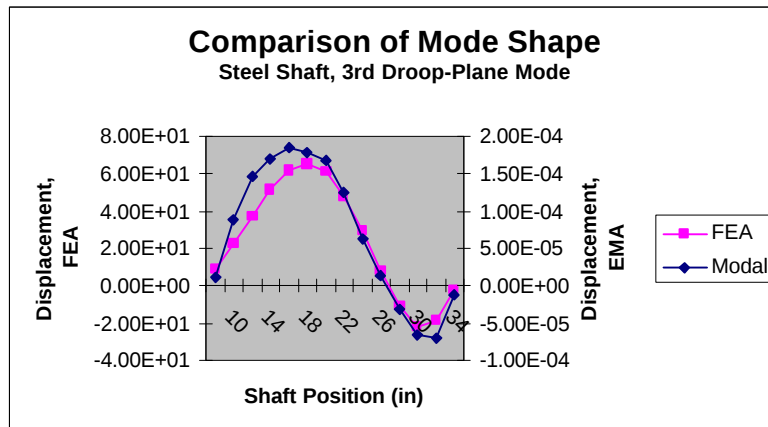
The clamped-free is the one most often employed in finite element analysis of the golf club, and it is typically used to simulate the effects of the players grip in dynamic analysis. Since one of the objectives of this work is to establish the boundary conditions of the hands, visual inspection and the use of the MAC are again utilized to contrast the FEA and EMA mode shapes. Visual inspection of the two planes (Figure 5.15 and Figure 5.16) reveals similarities between the finite element and modal mode shapes, particularly for the first droop and swing plane modes



(a)

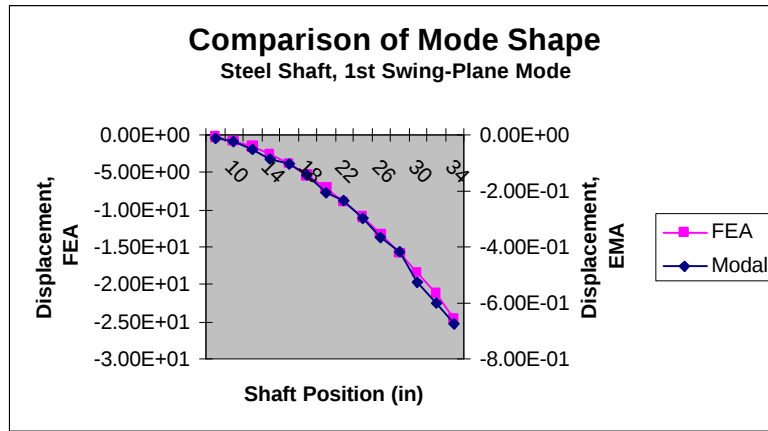


(b)

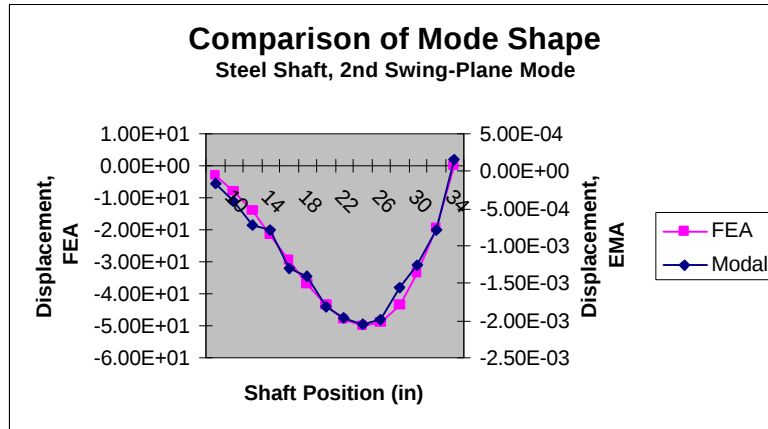


(c)

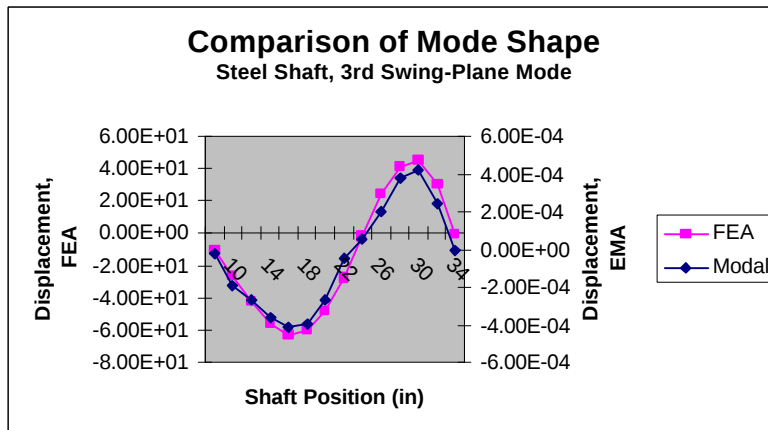
Figure 5.15 Comparison of the mass-normalized mode shapes for the droop-plane of the steel shaft with clamped-free boundary conditions. (a) First Mode (b) Second Mode (c) Third Mode



(a)



(b)



(c)

Figure 5.16 Comparison of the mass-normalized mode shapes for the droop-plane of the steel shaft with clamped-free boundary conditions. (a) First Mode (b) Second Mode (c) Third Mode

The mode shapes are then compared using the MAC, and for both planes, examination of the MAC (Table 5.8) reveals “high” correlation between the finite element analysis and modal analysis. In fact, the diagonal values are above 0.97 or 97 % for the droop plane, while 0.946, or 94.6%, is the lowest for the swing-plane. Upon first glance, these values seem unreasonably “high”, especially after the visual comparison has been completed, but after further consideration, these values appear more feasible.

The two lowest diagonal values of the MAC occur at the third mode for both droop and swing planes, and these mode shapes are more closely examined. From Figure 5.15c and Figure 5.16c, it first appears as though the two mode shapes differ significantly, but the scale of the figures is different from the free-free mode shape scale. For the free-free mode shapes, spatial locations from 4 to 34 are examined, while the number of location is limited to spatial location from 10 to 34 because the grip is clamped in the vise. Thus, a small difference in mode shape visually appears to be a larger one.

Table 5.8 : Clamped-Free MAC for steel shaft (a) Droop-plane (b) Swing Plane

Modal FEA		Mode		
		1	2	3
M o d e	1	0.9976	0.6812	0.0380
	2	0.6701	0.9903	0.4355
	3	0.0190	0.3341	0.9782

(a)

Modal FEA		Mode		
		1	2	3
M o d e	1	0.9986	0.6860	0.0316
	2	0.6744	0.9955	0.0769
	3	0.1353	0.0022	0.9464

(b)

It is also interesting to note the “high” degree of correlation, greater than 67%, between the first and second modes for both planes. Although this seems unusual, visual inspection of the FEA mode shapes reveals the similarity. From Figure 5.17, it is apparent that the two modes have comparable shapes, particularly for spatial location’s 4 to 20.

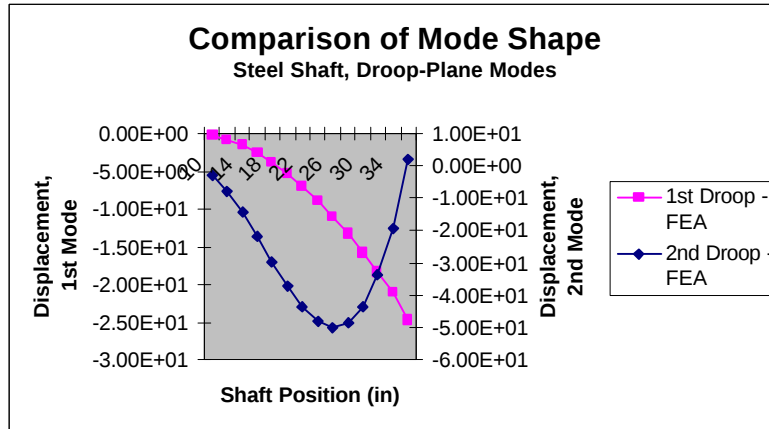


Figure 5.17 Comparison of first and second droop modes for the steel shaft.

Comparison of Hands-Free versus Free-Free and Clamped-Free

The hands-free is the final case examined and is employed to simulate the boundary conditions due to the gripping of the golf club. In this case, the club is gripped, the face is struck by a modal impact hammer, and the corresponding response is captured. The resulting FRFs are then curve-fit to determine the frequencies and mode shapes, and the eigenvalues and eigenvectors are then compared with the free-free and clamped-free cases to determine any similarities.

The hands-free case is first visually compared with the free-free case to examine any correlation. (See Figure 5.18) Although the number of spatial locations is limited by the grip of the hands, it is apparent that the hands-free and free-free mode shapes are similar, especially for the first and second modes.

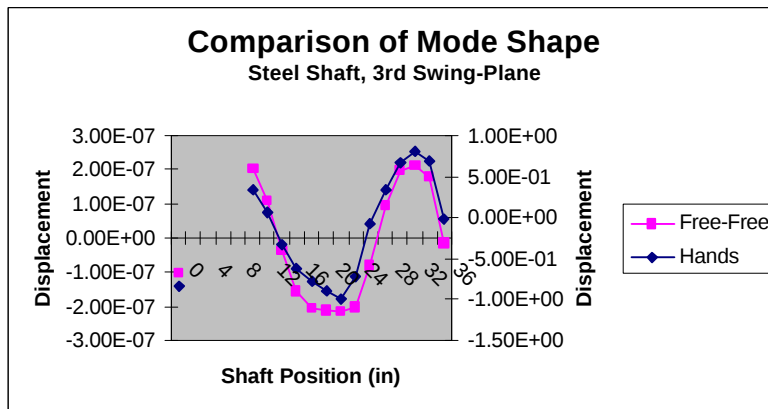
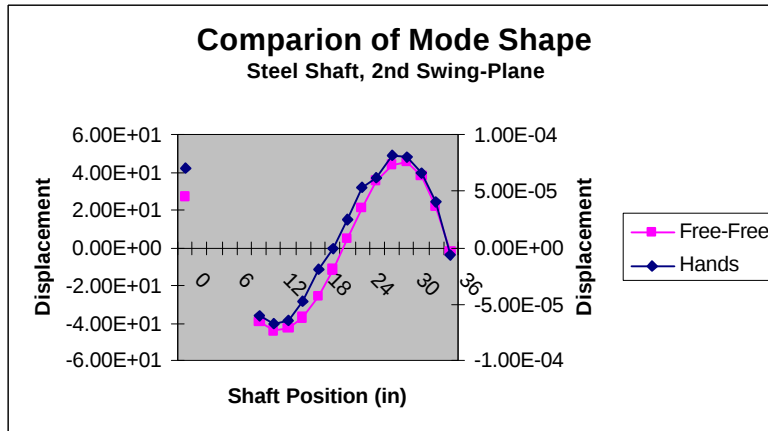
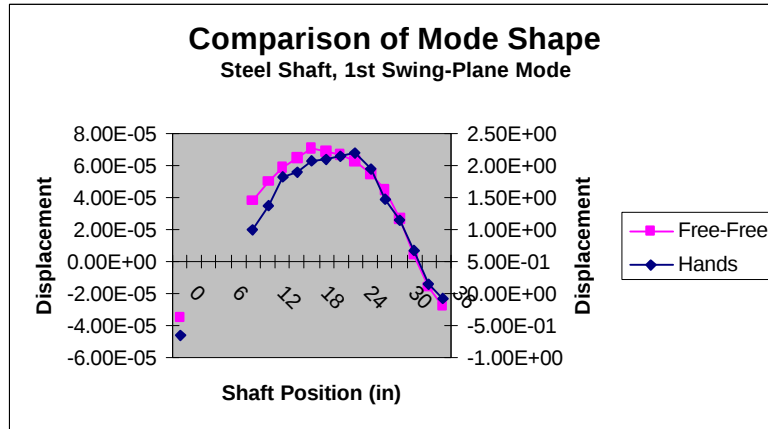


Figure 5.18 Comparison of the mass-normalized mode shapes for the swing-plane of the steel shaft with free-free versus hands-free boundary conditions. (a) First Mode (b) Second Mode (c) Third Mode

After the free-free and hands-free cases have been compared, the relationship between the clamped-free and hands-free is then investigated. (See Figure 5.19) The mode shapes may initially appear to be alike, but closer inspection, especially at the first spatial location, reveals the dissimilarity between the mode shapes.

Examining the first mode shape, it is apparent the mode shapes are different at the spatial locations for the upper (butt end) half of the shaft. This is particularly evident for the first spatial location, which differs significantly between the two modes. For the clamped-free case, the first spatial location is obviously clamped and thus has zero displacement, while the corresponding location in the hands-free case obviously has a relative displacement.

The second and third modes also appear to have an initial resemblance, especially for the second half (tip end) of the shaft, but further examination, with particular attention to the first spatial location, exposes the differences between the two boundary conditions. As previously noted, the clamped-case limits the first location displacement to zero, but the relative displacements of the corresponding location are obviously for the hands-free mode.

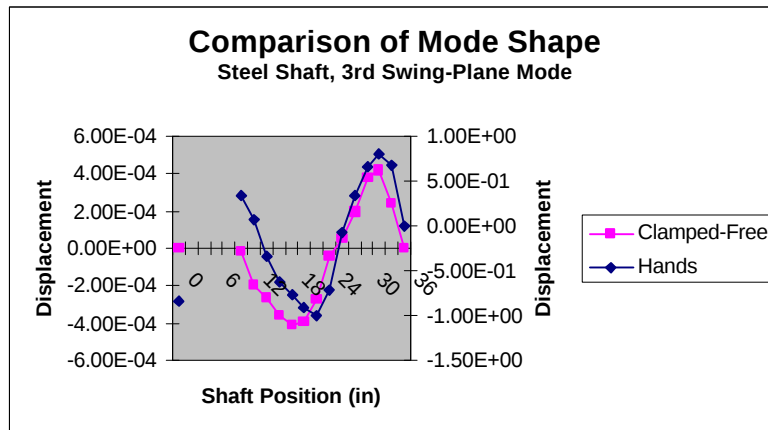
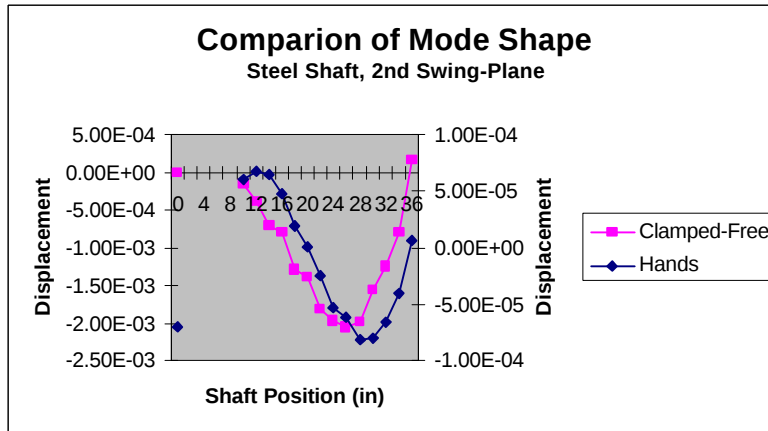
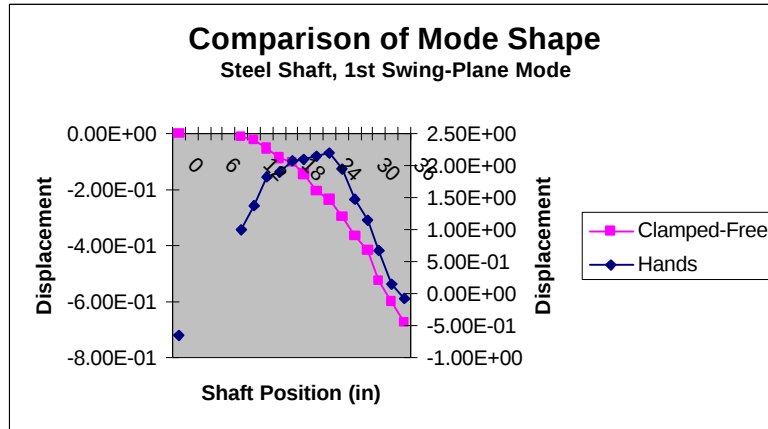


Figure 5.19 Comparison of the mass-normalized mode shapes for the swing-plane of the steel shaft with clamped-free versus hands-free boundary conditions. (a) First Mode (b) Second Mode (c) Third Mode

The visual inspection of the mode shapes establishes the similarity of the free-free and hands-free mode shapes and exposes the disparity between the clamped-free and hands-free eigenvectors. Therefore, the free-free and hands-free are only compared using the modal assurance criterion, which confirms the high degree of correlation.

Examining the MAC – Table 5.9, the “high” correlation between the two cases is confirmed by the diagonal values, which are greater than 0.9 or 90%, and the comparatively low off-diagonal terms. These values, combined with the frequency comparison, indicate strong similarity between the free-free and hands-free case; hence it can be confidently inferred that gripping of the club produces a boundary condition that is intrinsically a free-free case.

Table 5.9 : Free-free versus hand-free MAC for steel shaft

		Mode		
		1	2	3
M o d e	1	0.9603	0.0036	0.1066
	2	0.0023	0.9896	0.0061
	3	0.2460	0.0251	0.9071

Graphite Shaft

Since the mode shapes are similar for both the graphite and steel shafts, the steel shaft is examined in detail in the previous sections, and the graphite results are now

briefly examined. The free-free and clamped-free are again analyzed using both FEA and EMA, and EMA is again employed to analyze the hands-free conditions.

The eigenvectors are again determined using a MDOF curve-fitting technique, and the resulting mode shapes are correlated using the same procedure as the steel shaft. The resulting MACs are included in the subsequent sections, but for the sake of brevity, the visual comparisons of the mode shapes are included in the appendix of this work.

Free-Free

The free-free eigenvectors are again contrasted visually (See Appendix.) to determine similarity. With the visual comparison complete, the droop and swing plane mode shapes are then contrasted numerically using the modal assurance criterion.

Examination of the MACs (Table 5.10) reveals diagonal values of 0.9 or greater with the exception of the third swing mode, which has a value of 0.887 or 87%. This correlation value is similar to the value determined for the steel shaft, and thus reinforces the concerns, and need for future investigation, that are noted in the foregoing steel shaft section.

Table 5.10 : Free-Free MAC for graphite shaft. (a) Droop Plane (b) Swing Plane

Modal FEA		Mode		
		1	2	3
M o d e	1	0.9655	0.2623	0.0853
	2	0.0696	0.9628	0.0589
	3	0.2530	0.0614	0.9591

(a)

Modal FEA		Mode		
		1	2	3
M o d e	1	0.9469	0.0718	0.1652
	2	0.0224	0.9227	0.0539
	3	0.0836	0.0037	0.8873

(b)

Clamped-Free

The clamped-free is typically employed in finite element of the shaft, and the resulting mode shapes are once more compared by visual inspection (See Appendix). This reveals similarities between the finite element and modal mode shapes, and the mode shapes are thus compared using the MAC.

Inspection of the MAC (Table 5.11) reveals a “high” correlation between the finite element analysis and modal analysis mode shapes. While not as “high” as the steel shaft, the diagonal values are above 0.978 or 97.8% for the droop plane, and 0.945, or 94.5%, is the lowest for the swing-plane. These values may once again seem unreasonably “high” after the visual inspection, but as previously noted, the visual

discrepancy appear larger because of the different scaling for the clamped-free and free-free boundary conditions.

Table 5.11 : Clamped-Free MAC for graphite shaft. (a) Droop Plane (b) Swing Plane

Modal FEA		Mode		
		1	2	3
M o d e	1	0.9949	0.8689	0.2057
	2	0.7225	0.9780	0.6411
	3	0.2418	0.6176	0.9872

(a)

Modal FEA		Mode		
		1	2	3
M o d e	1	0.9972	0.7981	0.0029
	2	0.7216	0.9914	0.2389
	3	0.0278	0.0621	0.9455

(b)

Comparison of Hands-Free versus Free-Free and Clamped-Free

With the prevalence of graphite shafts in the market, the hands-free is also investigated for this shaft material and is utilized to simulate the boundary conditions due to the gripping of the golf club. The testing and parameter extractions are the same as previously stated, and the resulting eigenvalues and eigenvectors are again contrasted with the free-free and clamped-free case to investigate any similarities.

The hands-free case is first visually compared with the free-free and clamped-free cases to examine any correlation between the boundary conditions. (See Appendix), and

as expected, the correlation is similar to the steel shaft. Examining the figures, it is also evident that the hands-free correlates “better” with the hands-free case than the clamped-free case, and therefore, only the free-free and hands-free are compared using the MAC.

Inspecting the MAC –Table 5.12, the “high” correlation between the two cases is confirmed by the diagonal values, which are greater than 0.91 or 91%, and the comparatively low off-diagonal terms. These values, combined with the foregoing frequency comparison, reinforce the inference made for the steel shaft. Namely, the gripping of the club produces a boundary condition that is essentially a free-free case.

Table 5.12 : Free-free versus hand-free MAC for graphite shaft

		Mode		
		1	2	3
M o d e	1	0.9728	0.0668	0.2437
	2	0.0100	0.9114	0.0001
	3	0.2394	0.1068	0.9154

Nonlinear Dynamic Analysis

With the finite element and modal analysis correlated, the large displacement nonlinear dynamic response of the shaft is then investigated using Algor[®]. Previously acquired strain gage data has been taken during an actual club swing. The moment strain in the shaft at the gage location may be assumed to be an approximation of the applied moment from the golfer’s hands. The moment strain may be calibrated to get the applied moment on the shaft. The moment curve for the down swing portion is then interpolated by a series of line segments. These lines are then input into the nonlinear solver, and the dynamic response of the club is determined for the club model.

Determination of Moment Curve

The input moment curve for the swing is determined by calibrating strain, obtained using strain gauges, and converting the data. Using previously acquired strain data, the strain gauge is first calibrated for a flexible and stiff steel shaft (Figure 5.20), and the input moment curve is then obtained by converting strain data for a typical swing curve (Figure 5.21).

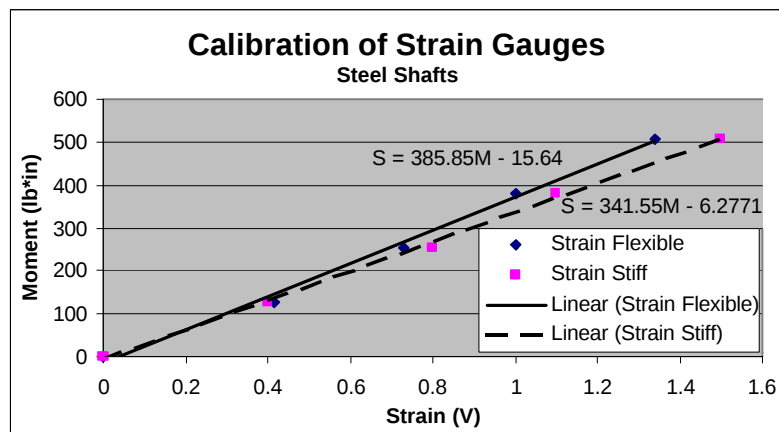


Figure 5.20 Calibration of strain gauges for two steel shafts.

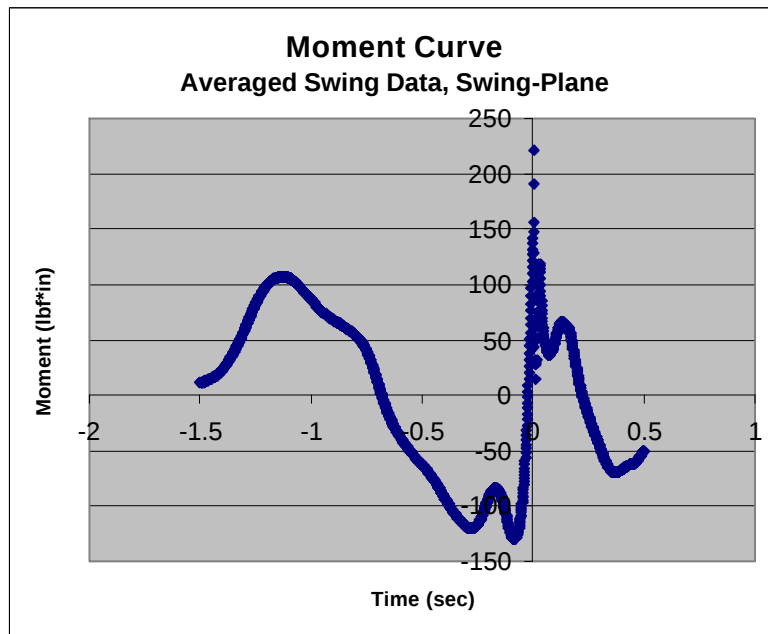


Figure 5.21 Moment for curve for the swing-plane.

With the data converted, a piecewise linear curve for the swing is then constructed by selecting a number representative points from the moment curve to linearly approximate the moment curve. To accomplish this task, the curve is initially simulated using eight points, while the second, and final, approximation employs 14 points. (See Figure 5.22) These event curves are then input into the Accupak software utility, and the finite element is then solved until the software determines convergence.

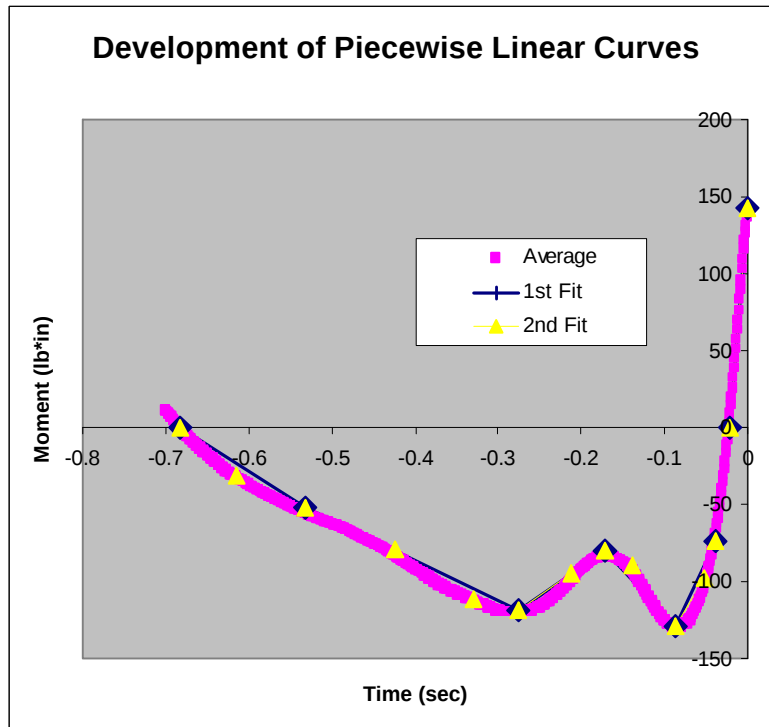


Figure 5.22 Development of Accupak event curve using consecutive linear approximations.

Dynamic Response

The model is solved until convergence has been attained for the desired time steps, and the response of the golf shaft is then investigated. Examining Figure 5.23, it is interesting to note the shaft displacement at each time step during the swing. The shaft and club-head are initially inline, but the club-head soon starts to “lag” the shaft. Although this “lagging” continues through the downswing, the head begins to “catch” the shaft midway through the down-swing and barely “lags” the shaft at impact.

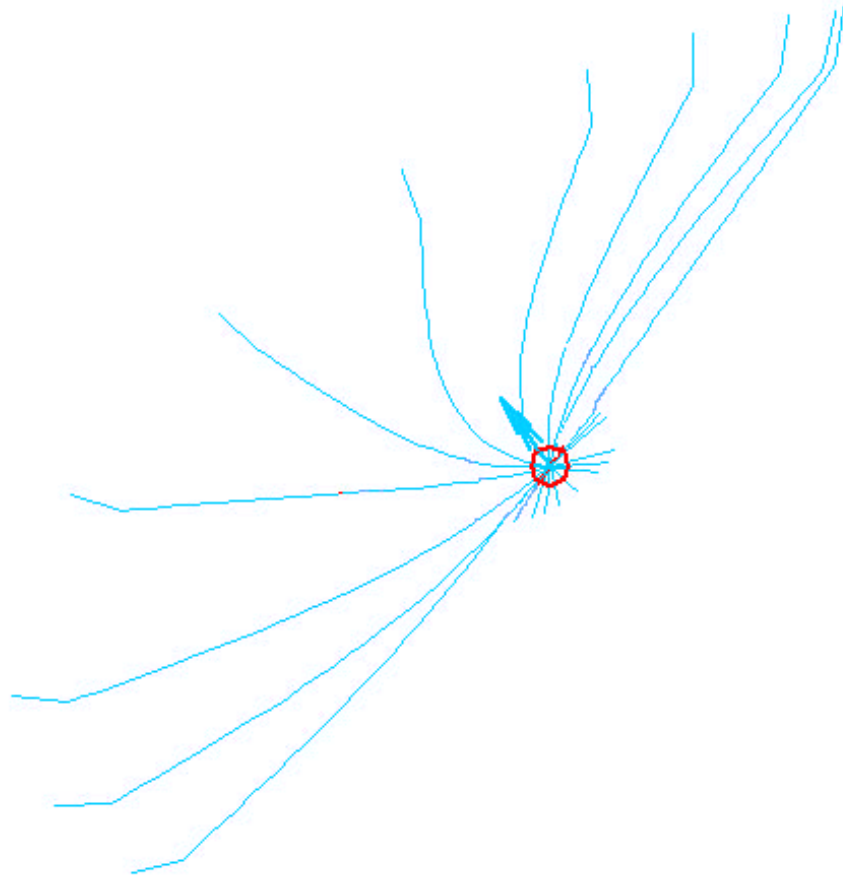


Figure 5.23 Dynamic swing results obtained using Accupak software utility in Algor.

Summary of Results and Conclusions

The primary objectives of this research are three-fold: 1) the development and correlation of an appropriate finite element model, 2) the characterization of the hands-free boundary condition and 3) the examination of the golf club swing dynamics. A finite element model is developed the using experimentally determined physical parameters. The model is then solved and correlated with results from the experimental modal analysis, while a separate modal case is used to determine the boundary conditions

caused by the gripping of the club. With the modes correlated, the shaft down-swing is simulated on the computer.

Summary of Results

The physical parameters of the club-head and golf shaft are determined experimentally, and then used to develop the finite element model. The club-head center of gravity is determined using elementary statics, and using the pendulum technique developed by Johnson (2), the moments of inertia are calculated for the axes of interest.

For the golf shaft, the desired physical parameters are determined through measurement of the shaft dimensions and three-point bend testing on the Instron tester. For the Instron testing, it is important to note the effects of localized deflection (See Figure 5.3) for the concentrated load case with simple supports. The applied load causes the shaft to ovalize at the point of application and causes the results to differ significantly from the theoretical.

Finite element analysis, using plate elements, is then performed to examine alternative testing techniques. The effects of increasing span length, as well as different loading conditions and supports, are examined. Examining the results, the closest testing results to theory came by using the 12-inch span beam with a distributed load and distributed supports. The discrepancies for this case are relatively low, less 5%.

This case is therefore simulated using three 1-inch blocks in the three-point bend apparatus and Instron tester, and the resulting experimental deflections are comparable to theory. The EI curves can then be determined, and the values for either the moment of inertia or modulus of elasticity are determined by holding one constant while the other

varies. Here, E is assumed constant for both the graphite and steel shafts, while the moment of inertia is allowed to vary along the length of the shaft.

Using these physical parameters, the finite element model is analyzed for two boundary conditions – free-free and clamped-free. These cases are solved until convergence, and the mode frequencies and shapes are then visually and numerically correlated with the values extracted from the modal analysis. Visually comparing the mode shapes (Figure 5.13 to Figure 5.19), it is evident that the finite element and experimental mode shapes are quite similar. This is further confirmed by the diagonal values of the MACs, which are about 0.9% or better.

One additional modal case, the hands-free, is performed to ascertain the boundary conditions due to gripping of the club. When this case is visually compared with the free-free and clamped-free cases, the hands-free mode shapes correspond with the free-free case, rather than the clamped-free case. This is confirmed by the values of the MAC, and it can then be inferred that the hands-free is close to a damped free-free boundary condition.

With the FEA and modal analysis correlated, the dynamic response of the club during the down-swing is then computed by analyzing finite element model with the Accupak software. By converting previously acquired strain data, a moment curve is determined, and an event curve is established by linearly approximating the moment curve.

Examining Figure 5.23, it is interesting to note the shaft displacement during the displayed swing positions. Initially, the shaft and club-head are inline, but club-head soon starts to “lag” the shaft. This “lagging” continues throughout the downswing, but

the head begins to “catch” the shaft midway through the down-swing and “lags” the shaft slightly at impact.

Future Work

This thesis is the culmination of a great amount of research and analysis, but it is not the completion of this research. It is merely the beginning, and opportunities abound for continued work for this project, especially in a few key areas.

The finite element model of the club-head is the first area of opportunity for continued work. In this research, the club-head was modeled as a lumped mass at the center of gravity, and the solver did not consider the moments of inertia. To better characterize the club and its mass distribution, a solver that considers the moments of inertia must be used, or the club-head could be model using either shell or solid elements.

Additionally, modeling the club-head with either shell or solid elements allows for the comparison of the club-face mode shapes with the results from experimental analysis. This would aid in the examination of the club-face node line and the relationship of the “sweet spot” with the center of gravity.

The characterization of the shaft flexural stiffness is another opportunity for continued research. In this project, the flexural stiffness was determined at discrete points using a distributed load simulated by a three-point apparatus and three 1-inch bars, but this technique limits the number of points examined. To better represent the flexural stiffness, alternative loading techniques may need to be examined.

A third area of future work concerns the analysis of the golf club during the backswing and downswing. For this work, only the down-swing of golf club was

simulated using a nonlinear dynamic solver, and the analysis was based upon an event curve of a limited number of points and was not correlated with experimental results.

To improve the analysis, experimental model analysis must be performed on the club during the swing and through the impact. To accomplish this, different response transducers and bonding techniques must be used. Additionally, the number of event curve points must be increased to better represent the actual swing in the dynamic simulation.

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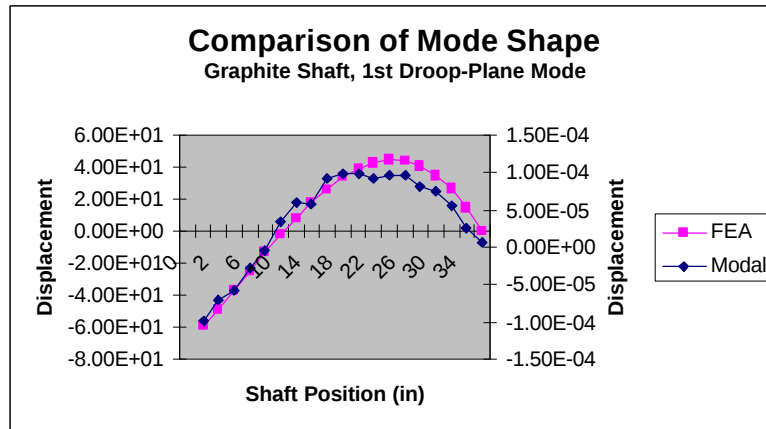
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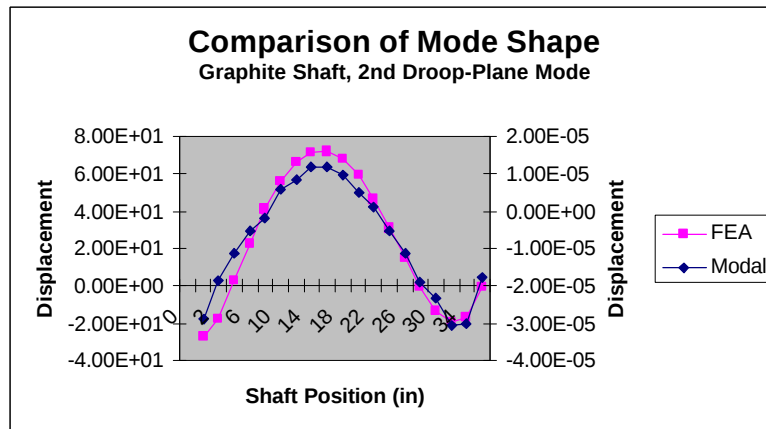
Appendix A - Graphite Shaft Results

Free-Free Boundary Conditions

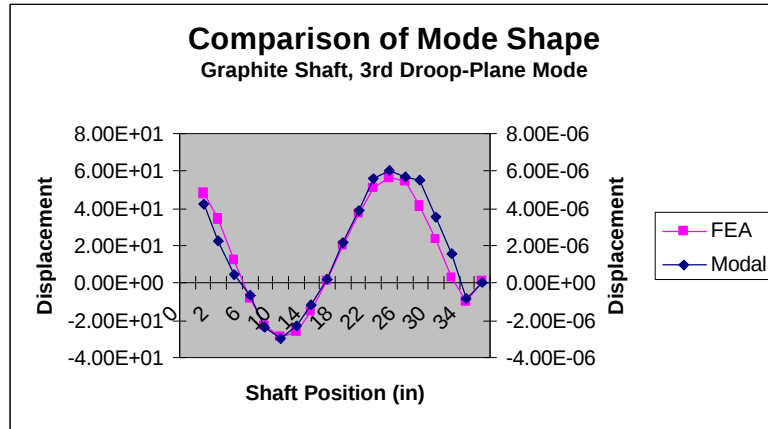
Droop Plane



(a)



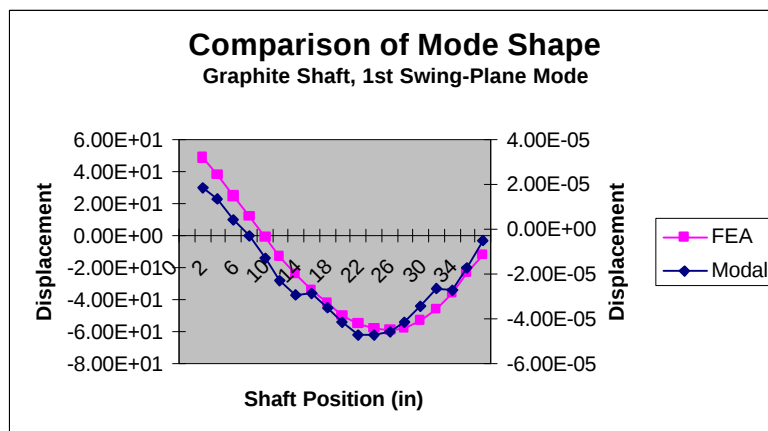
(b)



(c)

Figure A.1 Comparison of the mass-normalized mode shapes for the droop-plane of the graphite shaft with free-free boundary conditions. (a) First Mode (b) Second Mode (c) Third Mode

Swing Plane



(a)

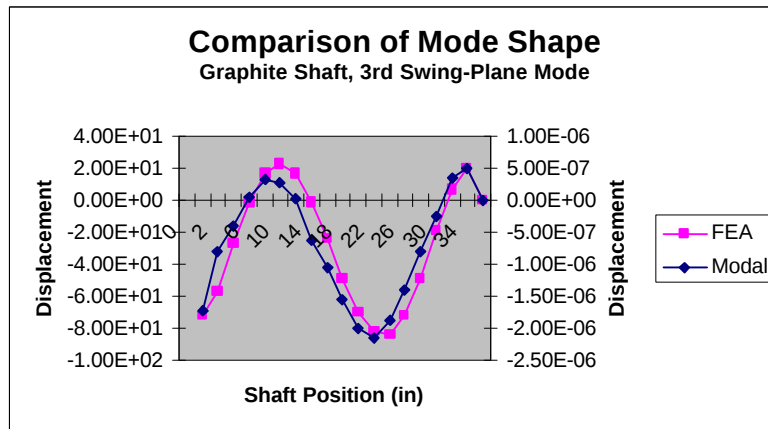
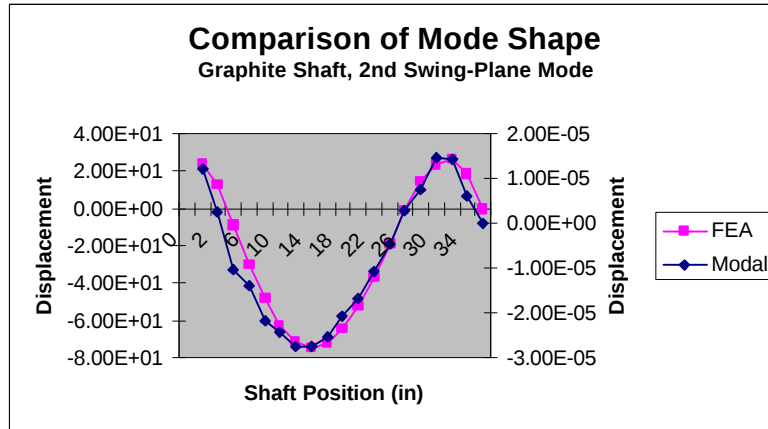
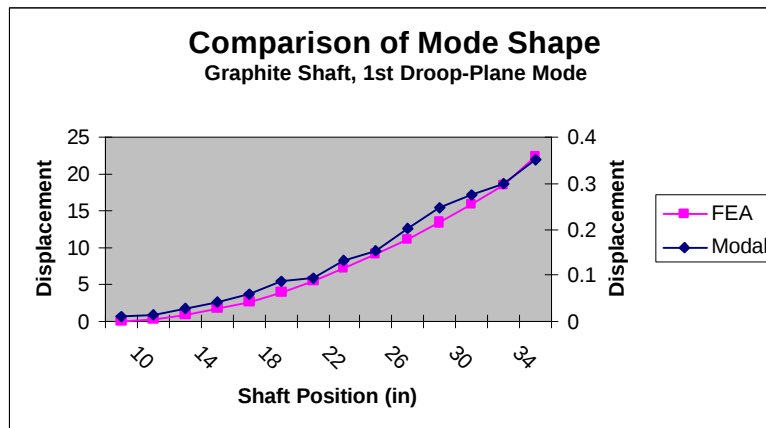


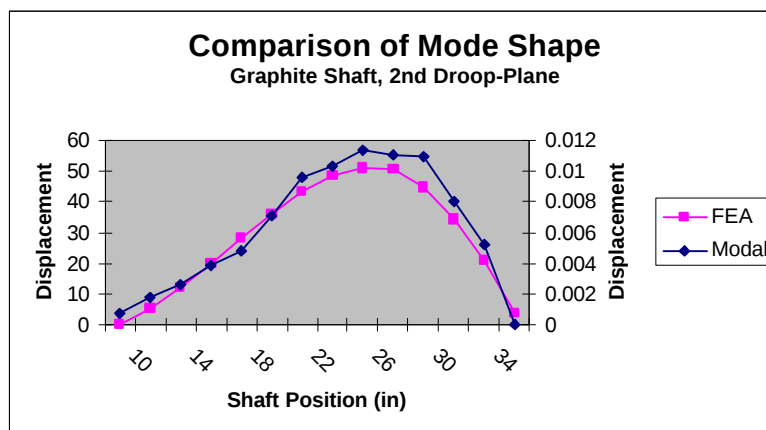
Figure A.2 Comparison of the mass-normalized mode shapes for the swing-plane of the graphite shaft with free-free boundary conditions. (a) First Mode (b) Second Mode (c) Third Mode

Clamped-Free

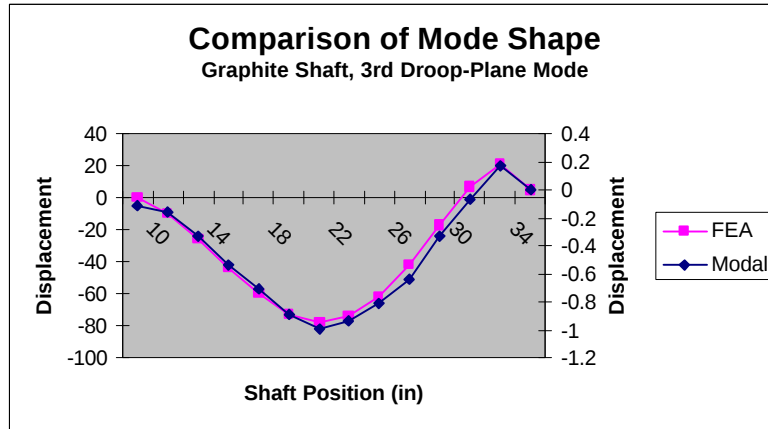
Droop-Plane



(a)



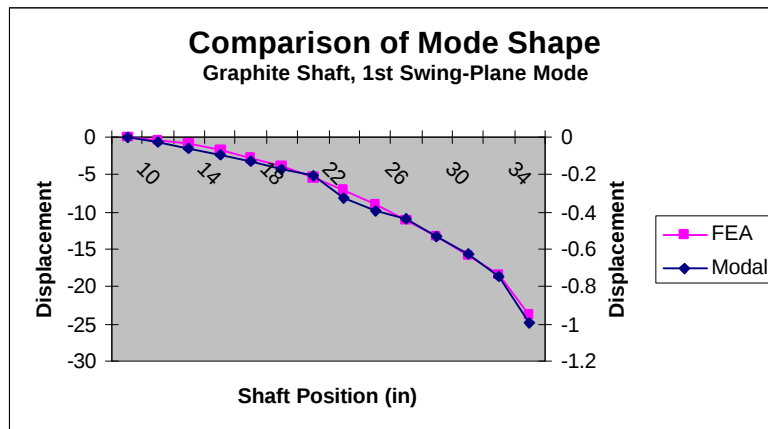
(b)



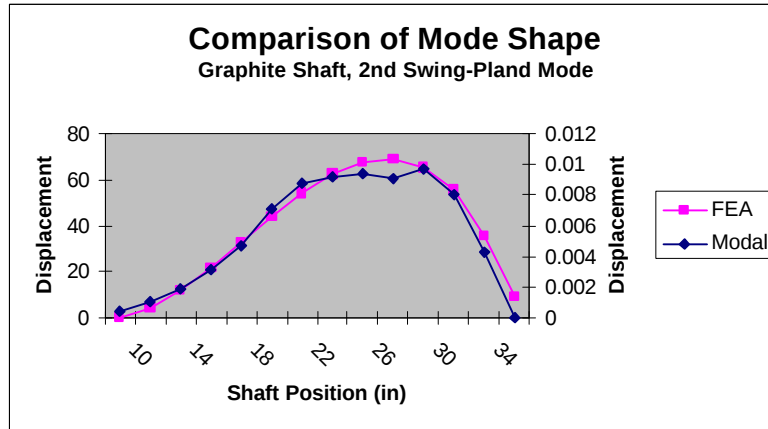
(c)

Figure A.3 Comparison of the mass-normalized mode shapes for the droop-plane of the graphite shaft with clamped-free boundary conditions. (a) First Mode (b) Second Mode (c) Third Mode

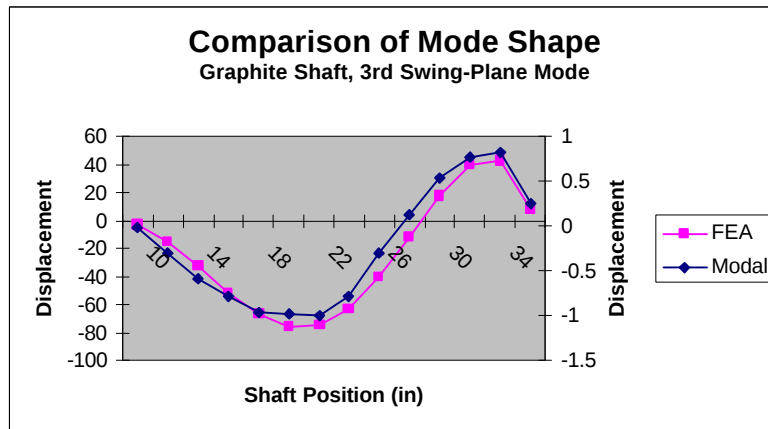
Swing-Plane



(a)



(b)

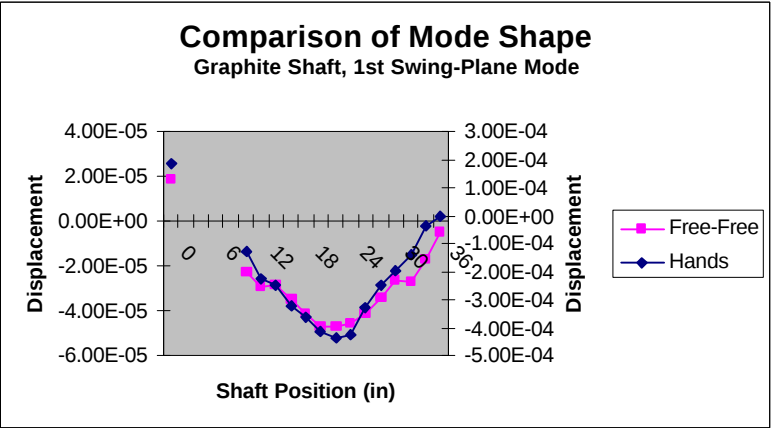


(c)

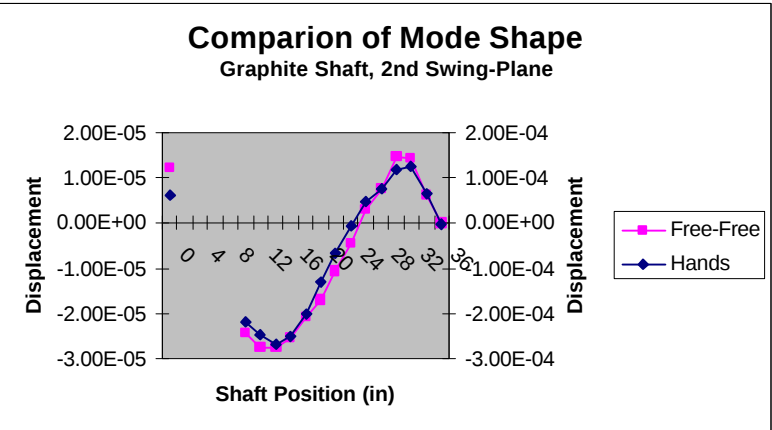
Figure A.4 Comparison of the mass-normalized mode shapes for the swing-plane of the graphite shaft with clamped-free boundary conditions. (a) First Mode (b) Second Mode (c) Third Mode

Comparison of Hands-Free versus Free-Free and Clamped-Free

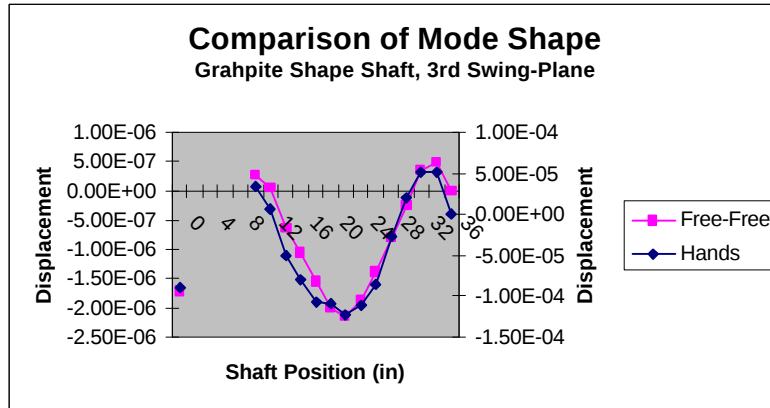
Hands-Free vs. Free-Free



(a)



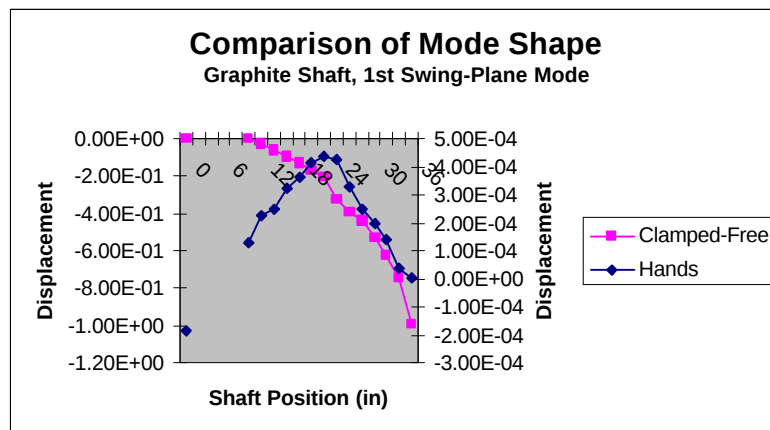
(b)



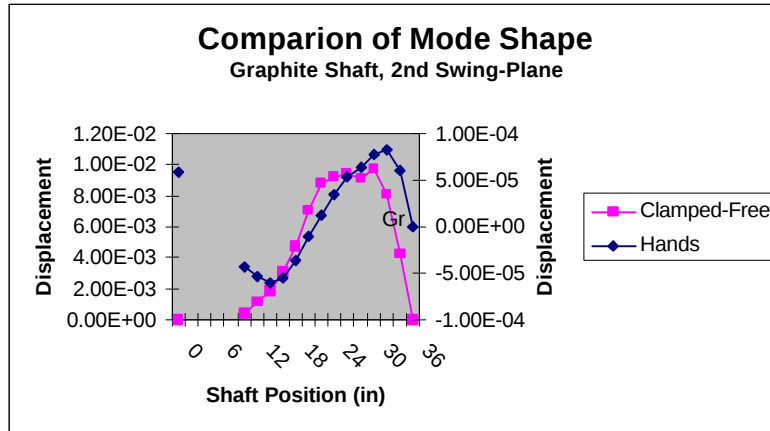
(c)

Figure A.5 Comparison of the mass-normalized mode shapes for the swing-plane of the graphite shaft with free-free versus hands-free boundary conditions. (a) First Mode (b) Second Mode (c) Third Mode

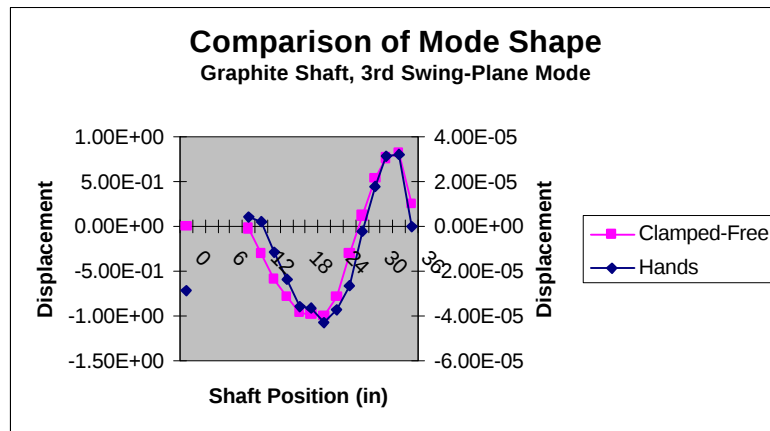
Hands-Free vs. Clamped-Free



(a)



(b)



(c)

Figure A.6 Comparison of the mass-normalized mode shapes for the swing-plane of the graphite shaft with clamped-free versus hands-free boundary conditions. (a) First Mode (b) Second Mode (c) Third Mode

Appendix B – Computer Programs

Freqtest.m

The following program is used to determine the frequency of the combined experimental setup and club-head. The collected data is first loaded and normalized, the zero-crossings of each signal are determined, and the period is calculated for each set. These values are then averaged, and traditional statistical techniques are used for comparison.

```
clear all
format long
home

% Selects the location of the data files

[a,path]=uigetfile('*.asc','Select File',0,0);
cd(path(1:length(path)-1));

% Prompts user to enter filename, extension and number of files

prompt={'Enter Filename Without Extension:','Enter file extension:',...
        'Number of Files', 'To plot data and splines, type y'};

def={'',''.asc','','y'};
xtitle='Natural Frequency';
lineNo=1;
answer=inputdlg(prompt,xtitle,lineNo,def);
rootnamef=char(answer(1));
ext=char(answer(2));
r=str2num(char(answer(3)));
c=char(answer(4));
res = 0.0001;

% Loads data and Creates time and amplitude vectors for each data set

time=0;
amp=0;
newamp=0;
```

```

for p=1:r;
    [a,b]=loaddata(rootnamef,ext,p);
    newamplitude=b-((max(b)-min(b))/2)-min(b); % meantot=(maxtot-
mintot)/2;
    if time == 0;
        time=a;
    else
        time=[time,a];
    end
    if amp==0;
        amp=b;
    else
        amp=[amp,b];
    end
    if newamp==0;
        newamp=newamplitude;
    else
        newamp=[newamp,newamplitude];
    end
end

% Plots the Mean-Zero Data of Time Output 1

figure(1)
plot(time(:,1),newamp(:,1)),grid;
title(['Mean-Zero Time Output ',num2str(1)]);
xlabel('Time (s)');
ylabel('Amplitude (V)');

% User selects the inputs for the ranges

disp('pick once to the left and right');
startpoint1 = ginput(1); % User selects start point
endpoint1 = ginput(1); % User selects end point

disp('pick once to the left and right');
startpoint2 = ginput(1);
endpoint2 = ginput(1);

% Splines the data and determines the Zero-Crossing by finding the
% index where the sign of the absolute value of the derivative changes

for n=1;
    xi = time(:,n);
    yi = newamp(:,n);

    [timeilow,timeihigh]=freqperi(xi,yi,startpoint1,endpoint1,startpoint2,
endpoint2,res,c,n);

    peri = timeihigh - timeilow; % Period of the first data
                                % set
end

```

```

for n=2:r;

    xi = time(:,n);
    yi = newamp(:,n);

    [timeilow,timeihigh]=freqperi(xi,yi,startpoint1,endpoint1,startpoint2,e
ndpoint2,res,c,n);
    peri = [peri,(timeihigh - timeilow)];
end

% Outputs the period, average period, standard deviation and variance
for analyzed data

period = peri'                                % Vector of period data
avgper = mean(period)                         % Average of the period data
standard_deviation = std(period)              % Standard deviation of
                                              % period vector
variance = std(period)^2                     % Variance of the period

```

Vita

Paul R. Braunwart was born November 1, 1973 in Bronx, New York and spent his early childhood there until his family moved across the Hudson River to Rockland County, New York. Located in the scenic Hudson Valley Region, Rockland County offered outdoor escapes that complemented the cultural and athletic opportunities offered in New York City, and these complementary forces help shape his childhood.

While a young boy, Paul joined a Cub Scout Pack in Thiells, New York, and this formed the basis of his involvement in the scouting movement that continues to this day. The appreciation of the outdoors and scouting skills that he developed in the Cubs blossomed in Boy Scouts, and in April of 1989, Paul received the Eagle Scout Award, the highest award in scouting. During his trail towards eagle, Paul was elected to scouting's service organization of honor campers, the Order of the Arrow, and in 1991, he received the highest honor of this organization, the Vigil Honor.

Although Scouting was an influential part of his life, Paul also sought the refuge of the athletic fields, and as a youth, he participated in little league, basketball and recreation soccer. At Don Bosco Prep High School in Ramsey, New Jersey, Paul participated in cross-country and developed his love for the game of lacrosse. After lettering at Don Bosco, he played for four years at Manhattan College, but he returned to Don Bosco as assistant lacrosse coach for one season before pursuing his graduate studies.

Scholastically, Paul developed his academic foundation at St. Gregory Barbarigo Grammar School in Garneville, New York and continued his academic pursuits at Don Bosco Prep High School. At Don Bosco, he developed a love of both the arts and

literature and the sciences, and his appreciation of both pointed him toward Manhattan College. Returning to the borough of his birth, Paul pursued his bachelor of science in mechanical engineering while writing for the school paper and hosting a radio program on the school station. After completing his bachelor's degree in 1996, Paul moved to Blacksburg, Virginia to pursue his master of science in mechanical engineering, and his studies has focused in experimental and analytical examination of vibrations.

The relocation to Virginia also affected him personally with one person in particular has significantly impacting his life for the better. While attending a Christmas party at a friend's home, he became enraptured with a fellow displaced New Yorker. The two soon learned that they had numerous interests in common, and love soon blossomed. The two quickly realized that the lives would be forever intermingled and have set August 7, 1999 as their wedding date.