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Citation: Physics of Fluids (1958-1988) **20**, 1296 (1977); doi: 10.1063/1.862000
View online: http://dx.doi.org/10.1063/1.862000
View Table of Contents: http://scitation.aip.org/content/aip/journal/pof1/20/8?ver=pdfcov
Published by the AIP Publishing



# Vlasov theory of plasma oscillations: Linear approximation

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(Received 30 August 1976; final manuscript received 1 April 1977)

A functional analytic approach to the linearized collisionless Vlasov equation is presented utilizing a resolvent integration technique on the resolvent of the transport operator evaluated at a particular point. Formulae for the eigenfunction expansion are found for cases in which the plasma disperion function $\Lambda$ has first and second order zeroes. Special care is taken in the study of real zeroes of $\Lambda$ culminating in new results for this case. For a simple zero of $\Lambda$ with nonvanishing imaginary part the van Kampen–Case discrete modes are reproduced. The results are used to obtain the solution to the initial value problem.

## I. INTRODUCTION

We study the initial value problem for the deviation $f(x, v, t)$ of the electron distribution function in a one-dimensional plasma from its equilibrium value $F(v)$. We assume the positive ions to be immobile and that $f(x, v, t)$ is described by a collisionless Vlasov equation[1,2]

$$\frac{df}{dt}+v\frac{df}{dx}+\frac{e}{m}\frac{d\phi}{dx}\frac{df}{dv}=0, \tag{1a}$$

$$\frac{-d^2\phi}{dx^2}=4\pi Ne\left(1-\int f\,dv\right), \tag{1b}$$

where $N$ is the plasma density, $\phi$ is the electric potential, and $e$ and $m$ are the electron charge and mass, respectively.

Simon and Rosenbluth[3] have recently considered the nonlinear stability of the solutions to Eqs. (1), i.e., the answer to the question: What is the long-term behavior of the solutions given an initial disturbance from equilibrium $f_0(x, v)$? Their procedure was to expand the solution in van Kampen[4]–Case[5] singular eigenmodes. In this approach, one takes the spatial Fourier transform of Eqs. (1) and then expresses the solution to the initial value problem as a sum of exponentials (van Kampen–Case discrete modes) plus an integral over the continuum eigenfunctions. The continuum integral in the linear theory, for quite general equilibrium distributions, decays exponentially for large $t$ and, in fact, the same is true even in nonlinear theory for small enough $f_0(x, v)$.[6] Thus, the question of stability, as is clearly pointed out in Ref. 3, hinges, to a major extent, on the location of the discrete van Kampen–Case eigenvalues in the complex plane. Since the plasma dispersion function whose zeroes define these eigenvalues is a real function of the complex variable $z$, it follows that if any eigenvalues with nonvanishing imaginary part exist, the plasma is, *ipso facto*, unstable. Thus, it is especially important to study the situation that the eigenvalues are imbedded in the continuum $[-\infty, \infty]$. This point is certainly well known, and is stressed in Ref. 3.

Unfortunately, the van Kampen–Case singular eigenmode approach appears to break down for this particular situation. In Ref. 3, the inadequacy of Case's adjoint solution for this case is pointed out, and the adjoint solution is, in fact, modified by the addition of a delta function term. However, there is no way to determine the arbitrary coefficient of the delta function and, even if there were, that approach would not appear to lead to correct results.

In addition, there are serious mathematical difficulties in the usual development of the singular eigenmode approach. This point has been stressed extensively in the context of neutron transport theory,[7,8] and we will not repeat any of the arguments here except to note that the objections carry over into the plasma case essentially unchanged. We, therefore, adapt the resolvent integration technique of Larsen and Habetler[7] to the solution of Eqs. (1). This approach leads to a spectral theorem for the operator $K$ describing the solution $f(x, v, t)$ or rather its Fourier transform $f_k(v, t)$. This spectral theorem or eigenfunction expansion in the sense of Titchmarsh[9] is obtained by integration of the resolvent of $K$ around its spectrum. The operator $K$ is unbounded and its spectrum extends to $\pm\infty$, so that the contour integration cannot be carried out directly. Bareiss[10] encountered this problem in dealing with the neutron transport equation, and resolved it by constructing a sequence of approximating bounded operators. We have avoided this complication by following a suggestion of Larsen,[11] to deal not with $K$ but with the resolvent of $K$ at a particular point and to apply the resolvent integration technique to the resolvent. Then, our method might be called the resolvent-resolvent technique. It is interesting to note that Larsen and Habetler,[7] did exactly this by dealing not with the transport operator, but its inverse (i.e., the resolvent evaluated at $z=0$). We are forced to deal with the resolvent evaluated at a point off the axis, and have chosen $z=i$ arbitrarily. Then, since the point $i$ is in the resolvent set of $K$, $(K-i)^{-1}\equiv S$ is a bounded operator, and the resolvent integration technique of Larsen and Habetler is easily carried out for $S$.

A major advantage of this technique is that no special methods are necessary to deal with eigenvalues imbedded in the continuous spectrum. One merely deforms the contour of integration about the resulting pole of the resolvent of $S$, and evaluates the result by residue theory. We have carried out this computation for both single and double eigenvalues imbedded in the continuous spectrum. Our results differ from those of Refs. 3 and 5.

  

The outline of our paper is as follows. In Sec. II we briefly summarize the results of singular eigenfunction theory, since we will compare our results with these. Then, in Sec. III we compute $S$ and its resolvent and obtain certain boundary values which we utilize in Sec. IV for integrating the resolvent of $S$ about the spectrum of $S$. Our results agree with those of Case except for the single situation noted here. The calculations of Secs. III and IV are carried out in a space of Hölder continuous functions because we must deal with boundary values of analytic functions. In Sec. V we use these results to present the solution to the initial value problem. It is necessary to modify the eigenfunction expansion for $S$ obtained in the previous sections to a similar expansion for $S^{-1}$ since the Vlasov equation involves not $S$ but $K = S^{-1} + i$. This is carried out explicitly.

Finally, we should mention that the resolvent-resolvent technique we introduce here is quite useful for certain problems (critical half-space) in neutron transport theory and in rarefied gas dynamics.

## II. THE SINGULAR EIGENMODES

If Eqs. (1) are Fourier transformed with respect to space, $\phi_k$ eliminated, and terms of order $f^2$ ignored, one arrives at the well-known linearized Vlasov equation which we write in the form

$$\frac{1}{ik}\frac{d}{dt} f_k(v,t) + K_0 f_k(v,t) = 0, \tag{2a}$$

where $K_0$ is the linearization of $K$

$$K_0 f_k(v,t) = v f_k(v,t) + \eta(v)\int_{-\infty}^{\infty} f_k(s,t)\,ds, \tag{2b}$$

$$\eta(v) = (-\omega_p^2/k^2)\frac{dF}{dv}. \tag{2c}$$

Here, $\omega_p^2$ is the plasma frequency

$$\omega_p^2 = 4\pi N e^2/m \tag{3}$$

and $F$ is the equilibrium distribution; we shall assume $\eta(v)$ is threefold continuously differentiable and vanishes exponentially at infinity.

We seek as a solution of Eq. (2a), a differentiable map $f_k\colon \mathrm{R} \to L_p(\mathrm{R})$, $p > 1$. Since the linear operator $K_0$ may be viewed as an operator in $L_p(-\infty,\infty)$, we shall write $f_k(v,t)$ as $f(x)$, where we have also suppressed the index $k$. $K_0$ is a densely defined, closable, unbounded operator in $L_p(-\infty,\infty)$, but, as discussed in Sec. I, we shall deal with $S \equiv (K_0 - i)^{-1}$ which is bounded operator on $L_p(\mathrm{R})$. (We have assumed the point $z = i$ is not in the spectrum of $K$; if it is, some other complex number must be chosen in place of $i$.) In fact, we shall deal not with $S$ but with a restriction of $S$, $S \mid D(S)$ where

$$D(S) = \{f \in L_p(\mathrm{R})\colon\ f \text{ is Hölder continuous on every bounded interval of } \mathrm{R}\}.$$

The singular eigenmode approach of van Kampen and Case is well known.[1,2,4,5] One assumes solutions to Eq. (2a) of the form

$$f(v,t) = \phi_\nu(v)\,e^{-i\omega t} \tag{4}$$

and finds ($\nu = \omega/k$)

$$\phi_\nu(v) = \frac{-\eta(v)}{v-\nu} + \lambda(v)\,\delta(v-\nu). \tag{5}$$

Here,

$$\lambda(v) = \tfrac{1}{2}[\Lambda^+(v) + \Lambda^-(v)], \tag{6a}$$

where $\Lambda$ is the plasma dispersion function

$$\Lambda(z) = 1 + \int_{-\infty}^{\infty}\frac{\eta(s)}{s-z}\,ds. \tag{6b}$$

$\Lambda(z)$ is an analytic function of $z$ on the cut plane $\mathrm{C}/\mathrm{R}$ with continuous boundary values $\Lambda^\pm(v)$ evaluated on the branch cut R from above and below.

The eigenmodes (5) are defined for all values of $\nu \in \mathrm{R}$ and correspond to the continuous spectrum of $K$. In addition, there may be eigenvalues if $\Lambda(\nu) = 8$, $\mathrm{Im}_\nu \neq 0$ (whence $\bar{z}$ is also an eigenvalue) or $\Lambda^+(\nu) = \Lambda^-(\nu) = 0$ in which case we see from

$$\Lambda^\pm(\nu) = \lambda(\nu) \pm \pi i \eta(\nu) \tag{7}$$

that these embedded eigenvalues can be characterized by $\lambda(\nu) = \eta(\nu) = 0$. Case[5] refers to these as "class 1c" modes and in either case we can write the eigenvectors as

$$\phi_i(v) = -\eta(v)/(v-\nu_i). \tag{8}$$

To solve initial value problems in singular eigenmodes, one expands the initial data

$$f_0(v) = \sum_i a_i\phi_i(v) + \int_{-\infty}^{\infty} A(\nu)\,\phi_\nu(v)\,d\nu, \tag{9}$$

where the summation is carried out over all the (discrete) eigenvalues. The expansion coefficients are computed from bi-orthogonality relations. In particular,

$$a_i = \int_{-\infty}^{\infty}\tilde{\phi}_i(v) f_0(v)\,dv\left[\int_{-\infty}^{\infty}\tilde{\phi}_i(v)\,\phi_i(v)\,dv\right]^{-1}, \tag{10a}$$

where the adjoint solution $\tilde{\phi}$ is given by

$$\tilde{\phi}_i(v) = -1/(v-\nu_i). \tag{10b}$$

Equations (9)–(10) represent the orthodox Case method. As Simon and Rosenbluth point out,[3] Eq. (10b) is not well defined if $\nu_i$ is real. In fact, a complete solution to the adjoint equation is

$$\tilde{\phi}_i(v) = -P(-1/(v-\nu_i) + \tilde{\lambda}(v)\,\delta(v-\nu_i), \tag{11}$$

for $\nu_i$ real, with $\tilde{\lambda}(v)$ arbitrary. It appears that $a_i$ is independent of $\tilde{\lambda}(\nu)$ because $\eta(\nu_i) = 0$. As we shall see, however, Eq. (10a) is not valid, with or without the $\tilde{\lambda}$, when $\nu_i$ is real.

It is altogether possible that a real eigenvalue is of second order if we think of a complex conjugate pair coalescing on the real axis for a critical value of the electron density $N$. [A necessary condition would be $\eta'(\nu_i) = 0$.] Although Case states[5] that he considers all eigenvalues to be simple *for simplicity only*, he does not describe how to deal with this important case. In the context of resolvent integration, eigenvalues of any multiplicity on or off the real axis can be dealt with, and we shall present the formulae for single and double eigenvalues.

## III. THE RESOLVENT TRANSFORMATION

A rather elementary calculation of $S=(K_0-i)^{-1}$ gives

$$(Sf)(v)=\frac{f(v)}{v-i}-\frac{\eta(v)}{v-i}\,\frac{1}{\Lambda(i)}\int_{-\infty}^{\infty}\frac{f(s)}{s-i}\,ds, \tag{12}$$

where $\Lambda$ is the dispersion function Eq. (6b), and $K_0$, we recall, is defined by Eq. (2b). We now proceed to obtain an eigenfunction expansion for $S$ which is, of course, a bounded operator on $L_p(\mathrm{R})$. [The assumption that $i$ is not in the spectrum of $K$ corresponds to assuming $\Lambda(i)\neq 0$; note the integral in Eq. (12), for $f\in L_p(\mathrm{R})$, is easily seen to be finite by use of the Hölder inequality.] Again, by a quite analogous calculation, we obtain the resolvent of $S$

$$[(zI-S)^{-1}f](v)=\frac{f(v)}{z-(v-i)^{-1}}-\frac{\eta(v)}{(v-i)}\,\frac{1}{z-(v-i)^{-1}}\,\frac{1}{\Omega(z)}\times\int_{-\infty}^{\infty}\frac{f(s)}{z(s-i)-1}\,ds. \tag{13}$$

Here,

$$\Omega(z)=\Lambda(z^{-1}+i). \tag{14}$$

The spectrum of $S$ corresponds to the singularities of the resolvent viewed as an operator-valued analytic function of $z$. The continuous spectrum consists of the circle

$$C\sigma(s)=\{z:\,|\,z-\tfrac{1}{2}i\,|=\tfrac{1}{2}\} \tag{15a}$$

while the point spectrum occurs at the zeroes of $\Omega(z)$

$$P\sigma(s)=\{z_i:\Omega(z_i)=0,\ i=1,\ldots,n\}. \tag{15b}$$

Real van Kampen–Case eigenvalues fall on the circle $C\sigma(s)$. We note that $\sigma(S)$ is related to $\sigma(K_0)$[5] by the spectral mapping theorem.

For use in later computations we will require the boundary values of

$$M_f(z)\equiv\frac{1}{\Omega(z)}\int_{-\infty}^{\infty}\frac{f(s)}{z(s-i)-1}\,ds. \tag{16}$$

We define $M_f^{\pm}$ as the limiting values of $M_f$ as we approach $C\sigma(S)$ from outside and inside the circle, respectively. These can be obtained from the Plemelj formulas after the change of integration variable $t\to(s-i)^{-1}$:

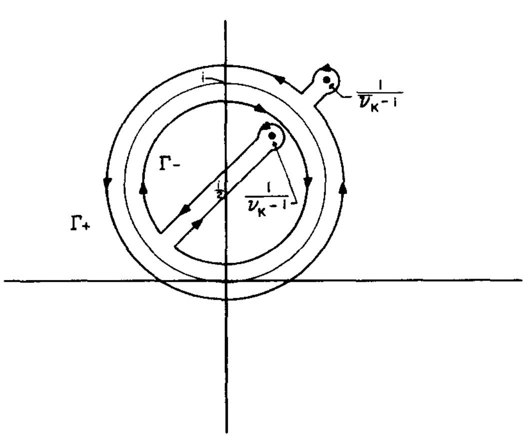


FIG. 1. The contours $\Gamma_+$ and $\Gamma_-$.

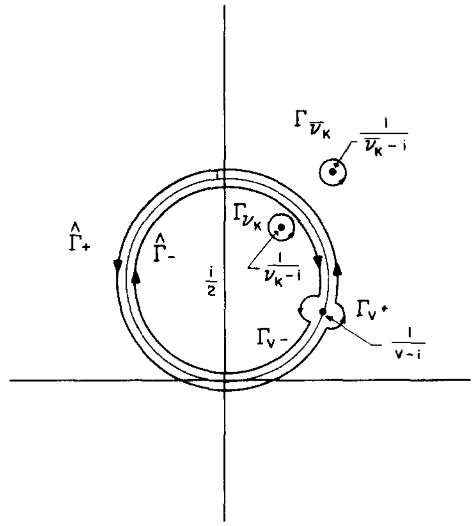


FIG. 2. The contours $\Gamma_\pm$, $\Gamma_{\nu\pm}$, $\Gamma_{\nu k}$, and $\Gamma_{\bar{\nu}k}$.

$$M_f^{\pm}[(v-i)^{-1}]\,\Omega^{\pm}[(v-i)^{-1}]=\pm i\pi f(v)(v-i)+P\int_{-\infty}^{\infty}\frac{f(s)}{z(s-i)-1}\,ds. \tag{17}$$

Using this result and recalling (7) and (14) we obtain, after some algebra,

$$f(v)-\frac{\eta(v)}{v-i}\,\tfrac{1}{2}\{M_f^{+}[(v-i)^{-1}]+M_f^{-}[(v-i)^{-1}]\}=\frac{1}{2\pi i}\,\frac{\lambda(v)}{v-i}\,\{M^{+}[(v-i)^{-1}]-M^{-}[(v-i)^{-1}]\}. \tag{18}$$

In the next section we integrate the resolvent Eq. (13) around the spectrum of $S$ and obtain the eigenfunction expansion for the operator $S$. Equation (18) plays an important role in the analysis.

## IV. INTEGRATION OF THE RESOLVENT

We apply the identity

$$\frac{1}{2\pi i}\oint(zI-s)^{-1}f\,dz=f$$

to Eq. (13), where the contour of integration surrounds the spectrum of $S$. The contour about $\sigma(S)$ can be written as

$$f(v)=\frac{1}{2\pi i}\int_{\Gamma_+}(zI-S)^{-1}f(v)\,dz+\frac{1}{2\pi i}\int_{\Gamma_-}(zI-S)^{-1}f(v)\,dz, \tag{19}$$

where $\Gamma_\pm$ are shown in Fig. 1, and where, for simplicity, we indicate a single isolated pair of eigenvalues. The integral over $\Gamma_-$ is included because, although its contribution is actually zero, its presence simplifies the subsequent formulae. We now deform the contour as shown in Fig. 2, and write

$$f(v)=\frac{1}{2\pi i}\left(\int_{\Gamma_+}+\int_{\Gamma_-}+\int_{\Gamma_{v+}}+\int_{\Gamma_{v^-}}+\sum_k\int_{\Gamma_{v_k}}\right)(zI-S)^{-1}f(v)\,dz, \tag{20}$$

where the sum over $K$ is taken over all conjugate pairs of discrete eigenvalues. We assume for the present that no eigenvalues fall on the circle $C\sigma(S)$ since special evaluation techniques are required for that case.

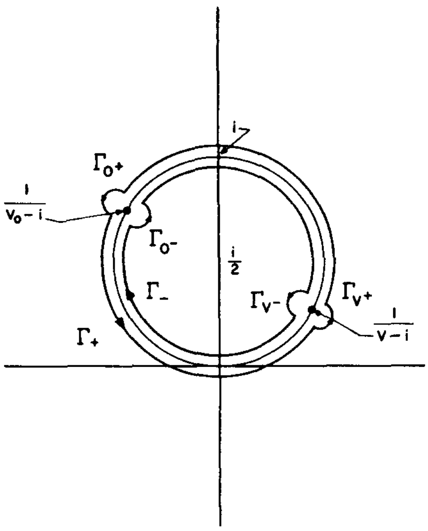


FIG. 3. Contour for eigenvalues imbedded in the continuous spectrum of $S$.

The integrals over the $\Gamma_{\nu_k}$ are simple exercises in residue theory. The results are the following:

(a) Single pole; $\Lambda(\nu_k)=0$, $\Lambda'(\nu_k)\neq 0$; $\mathrm{Im}\,\nu_k \neq 0$.

$$\frac{1}{2\pi i}\int_{\Gamma_{\nu_k}} (zI-S)^{-1} f(v)\,dz$$
$$=\frac{\eta(v)}{v-\nu_k}\int f(s)(s-\nu_k)^{-1}\,ds\left[\int \eta(s)(s-\nu_k)^{-2}\,ds\right]^{-1} \quad (21\text{a})$$
$$=\phi_{\nu_k}(v)\int \bar{\phi}_{\nu_k}(s) f(s)\,ds[\Lambda'(\nu_k)]^{-1}. \quad (21\text{b})$$

This is, of course, the classical Case result.[5]

(b) Double pole; $\Lambda(\nu_k)=\Lambda'(\nu_k)=0$; $\Lambda''(\nu_k)\neq 0$; $\mathrm{Im}\,\nu_k\neq 0$.

$$\frac{1}{2\pi i}\int_{\Gamma_{\nu_k}} (zI-S)^{-1} f(v)\,dz=\eta(v)(v-\nu_k)^{-2}\,2\int f(s)(s-\nu_k)^{-1}$$
$$\times\,ds\,[\Lambda''(\nu_k)]^{-1}+\frac{\eta(v)}{(v-\nu_k)}\left[2\int f(s)(s-\nu_k)^{-2}\,ds\,[\Lambda''(\nu_k)]^{-1}\right.$$
$$\left.-\tfrac{2}{3}\Lambda'''(\nu_k)\int f(s)(s-\nu_k)^{-1}\,ds[\Lambda''(\nu_k)]^{-2}\right]. \quad (22)$$

We recognize [Eq. (8)] that the second term is proportional to the usual Case discrete eigenvector $\phi\nu_k(v)$. The first term is proportional to the generalized eigenvector, since

$$(K_0-\nu_k)\frac{\eta(v)}{(v-\nu_K)^2}=\frac{\eta(v)}{v-\nu_K}. \quad (23)$$

[In constructing solutions to the initial value problem, special care must be used in dealing with the generalized eigenvector (see Sec. V).]

The integration around $C\sigma(S)$ is slightly more complicated to carry out, but the procedure is completely analogous to that of Ref. 7. Denoting

$$\Gamma=\Gamma_+\cup\Gamma_-\cup\Gamma_{v_+}\cup\Gamma_{v_-}$$

we have, using Eq. (13),

$$\frac{1}{2\pi i}\int_\Gamma (zI-S)^{-1}f(v)\,dz=f(v)-\frac{\eta(v)}{v-i}\,\frac{1}{2\pi i}\int_\Gamma \frac{M_f(z)}{z-(v-i)^{-1}}\,dz. \quad (24)$$

The second term can be broken up into its contributions from $\Gamma_{v\pm}$ and $\Gamma_\pm$. We immediately obtain

$$\frac{1}{2\pi i}\int_\Gamma (zI-S)^{-1} f(v)\,dz$$
$$=f(v)-\frac{\eta(v)}{v-i}\,\tfrac{1}{2}[M^+(v-i)^{-1}+M^-(v-i)^{-1}]$$
$$+\frac{\eta(v)}{v-i}\,\frac{1}{2\pi i}P\int_{-\infty}^{\infty}\frac{M^+[(s-i)^{-1}]-M^-[(s-i)^{-1}]}{[(s-i)^{-1}-(v-i)^{-1}]}\,\frac{ds}{(s-i)^2}. \quad (25)$$

Thus, using Eq. (18) we have

$$\frac{1}{2\pi i}\int_\Gamma (zI-S)^{-1} f(v)\,dz$$
$$=\frac{1}{2\pi i}\,\lambda(v)\,\frac{M_f^+[(v-i)^{-1}]-M_f^-[(v-i)^{-1}]}{(v-i)}$$
$$-\frac{1}{2\pi i}\,P\int\frac{\eta(v)}{v-s}\,\frac{M_f^+[(s-i)^{-1}-M_f^-(s-i)^{-1}]}{s-i}\,ds. \quad (26)$$

This can be written as a Case continuum eigenmode expansion as in Eq. (9) by setting

$$A_f(\nu)=\frac{M_f^+[(\nu-i)^{-1}-M_f^-[(\nu-i)^{-1}]}{2\pi i(\nu-i)}. \quad (27)$$

A straightforward computation yields

$$A_f(\nu)=\frac{\eta(v)}{\Lambda^+(\nu)\Lambda^-(\nu)}\int_{-\infty}^{\infty}\bar{\phi}_\nu(s)f(s)\,ds \quad (28)$$

with

$$\bar{\phi}_\nu(v)=P\,\frac{-1}{v-\nu}+\frac{\lambda(v)}{\eta(v)}\,\delta(v-\nu). \quad (29)$$

This is in exact agreement with Ref. 5. Combining the integrals over $C\sigma(s)\cup P\sigma(s)$ yields the van Kampen–Case eigenfunction expansion, Eq. (9).

We now turn to the exceptional case, Case's (1c) mode,[5] where $\Lambda^+(\nu_0)=\Lambda^-(\nu_0)=0$, i.e., there is an eigenvalue $\nu_0$ imbedded in the continuum. The contour $\Gamma_\pm$ must be deformed as shown in Fig. 3. Essentially, we replace the result given in Eqs. (21) and (23) by integrations over the contours $\Gamma_{0\pm}$ as shown in Fig. 3. We do not simply obtain the residue of the second term on the right-hand side of Eq. (13) at $\nu_k$ because both $\Omega'(\nu_k)$ [or $\Omega''(\nu_k)$ if $\nu_k$ is a double pole] and $\int[f(s)/z(s-i)-1]ds$ are discontinuous across the cut, i.e., they have different values on $\Gamma_{0+}$ and $\Gamma_{0-}$. However, one can apply the residue theorem separately to $\Gamma_{0\pm}$ and obtain:

(a) $\nu_0$ a simple pole;

$$\frac{1}{2\pi i}\int_{\Gamma_{0^+}\cup\Gamma_{0^-}}(zI+S)^{-1}f(v)\,ds$$
$$=\frac{\eta(v)}{2(v-\nu_0)}\left\{\left[P\int f(s)(s-\nu_0)^{-1}\,ds+i\pi f(\nu_0)\right][\Lambda'^+(\nu_0)]^{-1}\right.$$
$$\left.+\left[P\int f(s)(s-\nu_0)^{-1}\,ds-i\pi f(\nu_0)\right][\Lambda'^-(\nu_0)]^{-1}\right\}. \quad (30)$$

This coefficient of $\phi_0(v)=-\eta(v)/(v-\nu_0)$ differs from Case's result unless $\Lambda'^+(\nu_0)=\Lambda'^-(\nu_0)$ which occurs if, and only if, $\eta'(\nu_0)=0$. However, there is no reason to assume, in general, that $\eta'(\nu_0)=0$.

As we discussed earlier, there is some physical reason to suspect that double poles may occur in the continuum. The formula corresponding to Eq. (28) is

(b) $\nu_0$ a double pole:

$$\frac{1}{2\pi i}\int_{\Gamma_{0^+}\cup\Gamma_{0^-}}(zI-S)^{-1}f(v)\,dz$$
$$=-A_1\eta(v)(v-\nu_0)^{-1}-A_2\eta(v)(v-\nu_0)^{-2}, \tag{31}$$

where

$$A_1=-\frac{1}{3}\Bigg\{3\left[P\int f(s)(s-\nu_0)^{-2}\,ds+i\pi f'(\nu_0)\right][\Lambda^{+\prime\prime}(\nu_0)]^{-1}$$
$$-\left[P\int f(s)(s-\nu_0)^{-1}\,ds+i\pi f(\nu_0)\right]\Lambda^{+\prime\prime\prime}(\nu_0)[\Lambda^{+\prime\prime}(\nu_0)]^{-2}$$
$$+3\left[P\int f(s)(s-\nu_0)^{-2}\,ds-i\pi f'(\nu_0)\right][\Lambda^{-\prime\prime}(\nu_0)]^{-1}$$
$$-\left[P\int f(s)(s-\nu_0)^{-1}\,ds-i\pi f(\nu_0)\right]\Lambda^{-\prime\prime\prime}(\nu_0)[\Lambda^{-\prime\prime}(\nu_0)]^{-2}$$

and

$$A_2=\left[P\int f(s)(s-\nu_0)^{-1}\,ds+i\pi f(\nu_0)\right][\Lambda^{+\prime\prime}(\nu_0)]^{-1}$$
$$+\left[P\int f(s)(s-\nu_0)^{-1}\,ds-i\pi f(\nu_0)\right][\Lambda^{-\prime\prime}(\nu_0)]^{-1}.$$

Again, see Sec. V for the time dependence of these modes.

## V. SOLUTION OF THE INITIAL VALUE PROBLEM

In the previous section we have obtained the following result, which we state as a theorem.

*Theorem I.* For any function $\psi\in L_p(-\infty,\infty)$ which is Hölder continuous on every bounded interval of R, there exists a function $A$ given by Eq. (20) such that $\psi(v)$ can be written

$$\psi(v)=\int_{-\infty}^{\infty}A(\nu)\,\phi_\nu(v)\,d\nu+\Sigma\,,$$

where $\phi_\nu(v)$ is the van Kampen–Case singular eigenmode defined in Eq. (5). The discrete terms represented by $\Sigma$, are sums over eigenvectors

$$\phi_k^{(1)}(v)=-\eta(v)/(v-\nu_k)$$

and generalized eigenvectors,

$$\phi_k^{(2)}(v)=-\eta(v)/(v-\nu_k)^2$$

with coefficients given by Eqs. (21b), (22), (28), and/or (29) depending upon the multiplicity and location of the eigenvalue. (For eigenvalues of multiplicity higher than two, corresponding expansions could easily be derived, but will not be presented here.)

In order to solve the initial value problem, Eq. (2a), one would expand the initial data $f_0(v)$ by this expansion. Assume that all eigenvalues are simple. Then,

$$f_k(v,t)=\int\exp(-ik\nu t)\,A_{f_0}(\nu)\,\phi_\nu(v)\,d\nu$$
$$+\sum_j\frac{-\eta(v)}{v-\nu_j}\exp(-ik\nu_j t)\,a_j. \tag{32}$$

The coefficients $A_{f_0}(\nu)$ and $a_j$ are presented in the text (see Theorem I for the equation references).

The fact that $f_k(v,t)$ solves Eq. (2a) follows directly because

(i) $f_k(0,t)=f_0(v)$ by direct construction

and

(ii) Equation (32) is a solution of (2a).

Actually, point (ii), while heuristically evident, will be proved explicitly.

The case of a double eigenvalue is more interesting. Here, one must proceed as in Appendix F of Ref. 1. For a double zero of $\Lambda(z)$, call it $\nu_0$, the two linearly independent solutions of Eq. (2a) are found to be

$$\psi_k^{(1)}(v,t)=\frac{-\eta(v)}{(v-\nu_0)}\exp(-ik\nu_0 t), \tag{33a}$$

i.e., just the ordinary eigenvector, while

$$\psi_k^{(2)}(v,t)=\left(\frac{-\eta(v)}{(v-\nu_0)^2}+\frac{ikt\,\eta(v)}{v-\nu_0}\right)\exp(-ik\nu_0 t). \tag{33b}$$

Assume, for simplicity that $\nu_0$ is the only eigenvalue. Then, one expands the initial data

$$f_0(v)=\int A_{f_0}(\nu)\,\phi_\nu(v)\,d\nu+a_1\psi_k^{(1)}(v,0)+a_2\psi_k^{(2)}(v,0) \tag{34}$$

and the coefficients $a_1$, $a_2$ are given by Eq. (22) if $\nu_0$ is off the axis, or Eq. (31) if $\nu_0$ is real. Then,

$$\psi_k(v,t)=\int A_{f_0}(\nu)\,\phi_\nu(v)\exp(-ik\nu t)\,d\nu$$
$$+a_1\psi_k^{(1)}(v,t)+a_2\psi_k^{(2)}(v,t). \tag{35}$$

Obviously, if both single and double eigenvalues occur, the discrete terms of Eqs. (30) and (33) will both occur in the solution.

To verify that these expressions are solutions of the initial value problem with initial data $f_0(v)$, we directly substitute Eq. (35) into (2a) and verify that it is a solution. The terms, $\psi_k^{(1)}(v,t)$ and $\psi_k^{(2)}(v,t)$ are immediately seen to solve Eq. (2a). In fact:

*Lemma* 1. $(1/ik)(d/dt)\psi_k^{(1)}(v,t)+K_0\psi_k^{(1)}(v,t)=0$. That is $\psi_k^{(1)}(v,t)$, as given by Eq. (33a) satisfies the Vlasov equation.

*Proof.* We already noted that $-\eta(v)/(v-\nu_0)$ is an eigenfunction of $K_0$ with eigenvalue $\nu_0$. The result follows by direct calculation and in any event is well known from Ref. 5.

*Lemma* 2. $(1/ik)(d/dt)\psi_k^{(2)}(v,t)+K_0\psi_k^{(2)}(v,t)=0$; that is, $\psi_k^{(2)}(v,t)$ as given by Eq. (33b) satisfies the Vlasov equation.

The proof follows by direct computation, using Eq. (33b). Dealing with the continuum integration, i.e., the first term on the right side of Eq. (32), is a little more difficult. The canonical approach, as developed for the neutron transport equation,[12,13] is to extend the eigenfunction expansion already obtained from functions in $D(S)$ to the complete Banach space $L_p(\mathrm{R})$. Then, a functional calculus is obtained for $S$ which allows one to expand $Kf\equiv(S^{-1}+i)f$ in terms of the eigenfunction expansion, and thus solve the Vlasov equation. (This approach has the additional advantage that initial data are not restricted to the Hölder continuous functions.)

Because this approach involves considerable mathematical manipulation and because we can expect, physically, that initial data be smooth, we shall not use it here. Rather, we shall consider only initial data, $f_0(v)$, which are Hölder continuous and which vanish sufficiently rapidly as $v \to \infty$ that $S^{-1}f \in L_p(\mathrm{R})$. We state,

*Theorem II.* For every $f_0(v)$ in the range of $S\,\psi_k(v,t)$ as given by Eq. (35) is a solution of the Vlasov equation (2a) subject to the initial conditions $\psi_k(v,0)=f_0(v)$.

*Proof.* Setting $t=0$ in Eq. (35), it is clear that the initial conditions are satisfied. Further, lemmas 1 and 2 assure that the terms proportional to $\psi_k^{(1)}(v,t)$ and $\psi_k^{(2)}(v,t)$ in Eq. (35) satisfy Eq. (2a). We therefore need to consider only the continuum term.

Suppose $g \in D(s)$. Then, we can write

$$Sg = \frac{1}{2\pi i}\int_\Gamma S(z-S)^{-1} g(s)\, dz. \qquad (36)$$

By substituting the identity $S = S - z + z$ and utilizing

$$\frac{1}{2\pi i}\int_\Gamma g(s)\, dz = 0,$$

Eq. (30) may be written

$$Sg = \frac{1}{2\pi i}\int_\Gamma z(z-S)^{-1} g(s)\, dz.$$

We can integrate about $\Gamma$ in exactly the same manner as the integration was carried out in Sec. III to conclude

$$Sg = \int (s-i)^{-1} A_g(s)\phi_s(v)\, ds + \Sigma_{sg} \qquad (37)$$

where $\Sigma_{sg}$ represents the discrete terms in the expansion. Now, if $f_0(v)$ satisfies the conditions of the theorem, there exists a $g$, Hölder continuous on compacts, such that

$$Sg = f_0(v).$$

Furthermore, $f_0(v) \in D(S^{-1})$. Thus, $f_0$ can be written

$$f_0(v) = \int A_{f_0}(s)\phi_s(v)\, ds + \Sigma_{f_0}.$$

From a comparison of Eqs. (37) and (38) and an application of Liouville's theorem,[13] it follows that

$$A_{f_0} = (s-i)^{-1} A_g.$$

Thus,

$$S^{-1}\int A_{f_0}(s)\,\phi_s(v)\, ds = \int (s-i) A_{f_0}(s)\,\phi_s(v)\, ds$$

or

$$K_0 \int A_{f_0}\,\phi_s(v)\, ds = \int s A_{f_0}(s)\,\phi_s(v)\, ds.$$

Now, we may obtain

$$\left(\frac{1}{ik}\frac{d}{dt} + K_0\right)\int A_{f_0}(s)\,\phi_s(v)\exp(-ikst)\, ds = 0$$

proving the theorem.

As we have already noted, the assumption $f_0(v) \in D(S^{-1}) = D(K_0)$ implies [see Eq. (2b)] that $|vf_0|$ and $f_0$ are integrable on R. Obviously, a sufficient condition is that $f_0$ vanish sufficiently fast at infinity and that $f_0$ be Hölder continuous on every bounded interval of R.

## VI. CONCLUSIONS

If a stable plasma is subject to a small perturbation, the amplitude of the oscillations induced is given by Eq. (28) or (29) depending upon whether $\nu_0$ is a single or double eigenvalue of $K$. In any event, the amplitude is certainly different from that calculated by Case in Ref. 5. For unstable plasmas, the amplitude of oscillations is obviously immaterial since there should be a divergent term which dominates at long times. Presumably, the Laplace transformation of Landau, as described by Case[5] should give the correct answer, but we have not checked this.

This result is of more or less importance depending upon how seriously one takes a linearized Vlasov model of a plasma, and how much one cares about the amplitudes of stable oscillations. However, the apparent breakdown of the singular eigenfunction method appears to be a first, so it is of some intrinsic interest. More important perhaps will be the application of these techniques of the nonlinear problems treated in Ref. 3.


## ACKNOWLEDGMENTS

This work is based in part on a Ph.D. thesis to be submitted to the graduate school of Virginia Polytechnic Institute by one of us (M.D.A.). We are grateful to our colleagues Dr. R. L. Bowden, Dr. Linda Garbanati, and Mr. W. C. Cameron, and to Dr. E. W. Larsen for helpful discussions.

This work was supported by the National Science Foundation under Grant No. ENG 75-15882.



[1]K. M. Case and P. F. Zweifel, *Linear Transport Theory* (Addison-Wesley, Reading, Mass., 1967), Sec. 10.3.
[2]E. Inönü and P. F. Zweifel, editors, *Developments in Transport Theory* (Academic, New York, 1967), p. 24.
[3]A. Simon and M. N. Rosenbluth, Phys. Fluids 19, 1567 (1976).
[4]N. G. van Kampen, Physica (Utrecht) 21, 949 (1955).
[5]K. M. Case, Ann. Phys. (N.Y.) 7, 3 (1959).
[6]T. O'Neil, Phys. Fluids 8, 2255 (1965).
[7]E. W. Larsen and G. J. Habetler, Commun. Pure Appl. Math. 26, 525 (1973).
[8]R. J. Hangelbroek, Ph.D. thesis, Rijksuniversiteit Groningen, Netherlands (1973).
[9]E. C. Titchmarsh, *Eigenfunction Expansions* (Clarendon, Oxford, 1946), Chap. III.
[10]E. H. Bareiss, J. Math. Anal. Appl. 13 (1966).
[11]E. W. Larsen (private communication).
[12]E. W. Larsen, S. Sancaktar, and P. F. Zweifel, J. Math. Phys. 16, 5 (1975).
[13]W. Greenberg and S. Sancaktar, J. Math. Phys. 17, 2092 (1976).