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Analytical solutions of model equations for two phase gas mixtures: transverse velocity perturbations

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Model equations for a dilute binary gas system are derived, using a linear BGK scheme. Complete analytical solutions for the stationary half-space problem are obtained for transverse velocity perturbations. The method of solution relies on the resolvent integration technique.

I. INTRODUCTION

For several years model equations have been proposed for the full Boltzmann equation. The model equations for single gas systems have been studied and solved using, among other techniques, the Case singular eigenfunction approach^{1,2} and more recently the mathematically rigorous resolvent integration technique of Larsen and Habetler.³⁻¹¹

A number of model equations have been suggested to describe a binary gas system. Among these are ones proposed by Gross and Krook,¹² Seriôvich,¹³ Hamel,¹⁴ Oppenheim,¹⁵ Holway,¹⁶ Walker,¹⁷ Boley and Yip,¹⁸ and McCormack.¹⁹ In this text we shall discuss the binary model suggested by Hamel.¹⁴ This model conserves number density, momentum and energy, and takes into consideration velocity and temperature differences between the species. The model is applicable to a binary system in which masses are not greatly different.

Applying the BGK scheme to the Boltzmann equation for a binary gas and linearizing, the model equations decouple into two sets of two coupled equations and one set of four coupled equations. Each of the coupled pairs yields information about perturbations in the transverse velocities, and the coupled quartet of equations leads to information about perturbations in temperature and density.

We restrict our attention to the one-dimensional time-independent problem. Strong evaporation from a planar source into a vacuum has been studied extensively, and analytic solutions of BGK-type equations can be found by a variety of methods.²⁰ However, the physically more relevant problem of evaporation into a gaseous atmosphere has been analyzed only by computational methods. We have begun a rigorous analysis of solutions of such two gas problems. In this paper we derive complete analytical solutions to the plane symmetric half-space problem for the coupled pairs of equations. This provides analytic expressions for the transverse velocity densities. The four coupled equations relating to mass and temperature density will be the subject of a future report.

In Sec. II we shall derive the linearized model equations for the binary gas. In Sec. III we shall use a modification of the Larsen-Habetler technique^{11,21} to study the spectral resolution of the bounded operator $S = (K + iI)^{-1}$, where K is the reduced transport operator. In the following section, the "half-range expansion" is derived. Finally in Sec. V, analytical

solutions of the time-independent problems are presented. These expressions solve the general two-gas boundary value problem for arbitrary incoming flux (for transverse velocity densities), and thus solve a variety of specific physical problems: evaporation into a gaseous atmosphere, the binary gas Kramers problem, etc.

II. DEVELOPMENT OF THE LINEAR MODEL

The Boltzmann equation for a binary mixture is given by

$$\frac{\partial f_i}{\partial t} + \mathbf{v}_i \cdot \nabla f_i = J_{ii} + J_{ij} = J_i, \quad (1)$$

for $i = 1, 2$, $j = 1, 2$, $i \neq j$, where $f_i(t, \mathbf{x}, \mathbf{v})$ represents the distribution function, the collision terms are

$$J_{ij} = \iiint \iiint (f'_i f'_j - f_i f_j) k_{ij} d^2k d^3v_j, \quad (2)$$

$$J_{ii} = \iiint \iiint (f'_i f'_i - f_i f_i) k_{ii} d^2k d^3v_i, \quad (3)$$

and k_{ii} , k_{ij} are appropriate scattering kernels.

Following the work of Hamel¹⁴ the equations for the (nonlinear) BGK binary gas mixture may be derived. The major conditions which lead to this simplification are the following:

- (i) for the two-component gas, self-collisions and cross collisions each return particles to a Maxwellian velocity distribution with an appropriate temperature and velocity;
- (ii) the kinetic model provides conservation of mass, momentum and energy of the total mixture;
- (iii) the kinetic model provides conservation of mass for each separate gas;
- (iv) the kinetic model satisfies the same symmetry property as the full collision integral;
- (v) the collisional transfer of momentum and energy between species is set equal to the value for Maxwellian molecules;
- (vi) the kinetic equation is modeled to a specific intermolecular force law, namely the inverse fifth power law.

We introduce the following notation. The Maxwellian distribution is given by

$$f_{0i}(\mathbf{v}_i) = \frac{n_{0i}}{(2k\pi T_0/m_i)^{3/2}} \exp\left(\frac{-m_i(\mathbf{v}_i - \mathbf{V}_0)^2}{2kT_0}\right), \quad (4)$$

for the gases $i = 1, 2$, where n_{0i} is the number density of gas i , m_i its molecular mass, and V_0 , T_0 are the drift velocity and temperature for the system (assumed the same for both gases). For each species we define the number density

$$n_i = \int f_i d^3\mathbf{v},$$

the drift velocity

$$\mathbf{V}_i = \frac{1}{n_i} \int \mathbf{v}_i f_i d^3\mathbf{v},$$

and the temperature

$$T_i = \frac{1}{3n_i k} \int (\mathbf{v}_i - \mathbf{V}_i)^2 m_i f_i d^3\mathbf{v}.$$

We also introduce the "averaged" quantities

$$\hat{m}_i = m_i / (m_1 + m_2),$$

$$\mathbf{V}_{ij} = \mathbf{V}_i + \hat{m}_j (\mathbf{V}_j - \mathbf{V}_i),$$

$$T_{ij} = T_i + 2\hat{m}_i \hat{m}_j (T_j - T_i)$$

$$+ [\hat{m}_i \hat{m}_j^2 (m_i + m_j) / 3k] (\mathbf{V}_i - \mathbf{V}_j)^2.$$

Writing the intermolecular force law as

$$F_{ij} = \frac{m_i m_j}{(m_i + m_j) r^5} \left(\frac{K_{ij}}{2.66} \right)^2, \quad i \neq j,$$

$$F_{ii} = \frac{m_i}{r^5} \left(\frac{K_{ii}}{2.906} \right)^2,$$

the collision term for the nonlinear BGK model has the form¹⁴

$$J_i = n_i K_{ii} \left[-f_i + \frac{n_i}{(2\pi k T_i / m_i)^{3/2}} \exp\left(\frac{-m_i (\mathbf{v}_i - \mathbf{V}_i)^2}{2k T_i} \right)^2 \right] \\ + n_j K_{ij} \left[-f_i + \frac{n_i}{(2\pi k T_{ij} / m_i)^{3/2}} \right. \\ \left. \times \exp\left(\frac{-m_i (\mathbf{v}_i - \mathbf{V}_{ij})^2}{2k T_{ij}} \right)^2 \right],$$

for $i = 1, 2$, $j = 1, 2$, $i \neq j$.

We now linearize the BGK approximation to the collision terms by expanding about the absolute Maxwellian f_{0i} . Thus we define the perturbed distribution function ϕ_i by

$$f_i(x, y, z, \mathbf{v}_i) = f_{0i}(\mathbf{v}_i) [1 + \phi_i(x, y, z, \mathbf{v}_i)],$$

the perturbations \mathbf{a}_i , b_i , ρ_i in velocity, temperature, and density by

$$\mathbf{V}_i = \mathbf{V}_0 + \mathbf{a}_i (m_i / 2k T_0)^{-1/2},$$

$$T_i = T_0 (1 + b_i),$$

$$n_i = n_{0i} (1 + \rho_i),$$

and the random velocity \mathbf{t}_i by

$$\mathbf{t}_i = (\mathbf{v}_i - \mathbf{V}_i) (m_i / 2k T_0)^{1/2}.$$

Clearly we expect this model to work well when

$$|\phi_i| \ll 1, \quad |(n_i - n_{0i}) / n_{0i}| \ll 1,$$

$$|(T_i - T_0) / T_0| \ll 1, \quad |(\mathbf{V}_i - \mathbf{V}_0) / 2k T_0| \ll 1.$$

Using the above notation and linearizing the right-hand side of Eq. (4) we obtain directly

$$J_i = (n_{0i} K_{ii} + n_{0j} K_{ij}) \{ -f_{0i} (1 + \phi_i) \\ + f_{0i} [1 + \rho_i + 2\mathbf{a}_i \cdot \mathbf{t}_i + b_i (t_i^2 - \frac{3}{2})] \} \\ + \hat{m}_j n_{0j} K_{ij} f_{0i} [-2\mathbf{a}_i \cdot \mathbf{t}_i + 2\mathbf{a}_j \cdot \mathbf{t}_i (m_i / m_j)^{1/2} \\ + 2\hat{m}_i (b_j - b_i) (t_i^2 - \frac{3}{2})]. \quad (5)$$

The model equation now has the form²²

$$\frac{\partial \phi_i}{\partial t} + \mathbf{v}_i \cdot \nabla \phi_i \\ = (n_{0i} K_{ii} + n_{0j} K_{ij}) [-\phi_i + \rho_i + 2\mathbf{a}_i \cdot \mathbf{t}_i + b_i (t_i^2 - \frac{3}{2})] \\ + \hat{m}_j n_{0j} K_{ij} [-2\mathbf{a}_i \cdot \mathbf{t}_i + 2\mathbf{a}_j \cdot \mathbf{t}_i (m_i / m_j)^{1/2} \\ + 2\hat{m}_i (b_j - b_i) (t_i^2 - \frac{3}{2})],$$

$$i = 1, 2, \quad j = 1, 2, \quad i \neq j. \quad (6)$$

This corrects an error in the linearization of Hamel¹⁴ and differs slightly from the equations of Bramlette and Huang.²³

Restricting the study to the one-dimensional time-independent case we assume that

$$\frac{\partial \phi_i}{\partial y} = \frac{\partial \phi_i}{\partial z} = \frac{\partial \phi_i}{\partial t} = 0, \quad i = 1, 2.$$

A change of notation will be made at this time. Quantities with subscript 1 will be replaced by quantities without subscripts and quantities with subscript 2 will be replaced by quantities with superscript *. Defining the constants

$$\beta = (n_0 K_{11} + n_0^* K_{12}) (m / 2k T_0)^{1/2},$$

$$\beta^* = (n_0^* K_{22} + n_0 K_{21}) (m^* / 2k T_0)^{1/2},$$

$$\alpha = \hat{m}^* n_0^* K_{12} (m / 2k T_0)^{1/2}, \quad \alpha^* = \hat{m} n_0 K_{21} (m^* / 2k T_0)^{1/2},$$

$$\gamma = V_{0,x} (m / 2k T_0)^{1/2}, \quad \gamma^* = V_{0,x} (m^* / 2k T_0)^{1/2},$$

and making the change of variables

$$c_x = t_x + \gamma, \quad c_y = t_y, \quad c_z = t_z,$$

$$c_x^* = t_x^* + \gamma^*, \quad c_y^* = t_y^*, \quad c_z^* = t_z^*,$$

the two equations which describe the time-independent, one-dimensional binary system become

$$c_x \frac{\partial \phi}{\partial x} = \beta \left[-\phi + \rho + 2\mathbf{a} \cdot \mathbf{c} + b \left(c^2 - \frac{3}{2} \right) \right] \\ + \alpha [-2\mathbf{a} \cdot \mathbf{c} + 2\mathbf{a} \cdot \mathbf{c} (m / m^*)^{1/2} \\ + 2\hat{m} (b^* - b) (c^2 - \frac{3}{2})] \\ + \beta \gamma [-2a_x + b (2c_x - \gamma)] \\ + \alpha \gamma [-2a_x + 2a_x^* (m / m^*)^{1/2} \\ + 2\hat{m} (b^* - b) (2c_x - \gamma)], \quad (7)$$

and

$$c_x^* \frac{\partial \phi^*}{\partial x} = \beta^* \left[-\phi^* + \rho^* + 2\mathbf{a} \cdot \mathbf{c}^* + b^* \left(c^{*2} - \frac{3}{2} \right) \right] \\ + \alpha^* [-2\mathbf{a}^* \cdot \mathbf{c}^* + 2\mathbf{a} \cdot \mathbf{c}^* (m^* / m)^{1/2} \\ + 2\hat{m}^* (b - b^*) (c^{*2} - \frac{3}{2})] \\ + \beta^* \gamma^* [-2a_x^* + b^* (2c_x^* - \gamma^*)] \\ + \alpha^* \gamma^* [-2a_x^* + 2a_x (m^* / m)^{1/2} \\ + 2\hat{m}^* (b - b^*) (2c_x^* - \gamma^*)]. \quad (8)$$

A linearized approximation for the perturbations in velocity and temperature can be obtained by elementary computations. In the present notation the result is

$$\mathbf{a} = \pi^{-3/2} \int_{-\infty}^{\infty} e^{-c^2} \mathbf{c} \phi d^3c \quad (9)$$

for the velocity,

$$b = \pi^{-3/2} \int_{-\infty}^{\infty} e^{-c^2} \left[\frac{2}{3} \left(c^2 - \frac{3}{2} \right) \right] \phi d^3c \quad (10)$$

for the temperature and the same expressions for \mathbf{a}^* and b^* with ϕ replaced by ϕ^* . For completeness we write the expression for perturbation in density:

$$\rho = \pi^{-3/2} \int_{-\infty}^{\infty} e^{-c^2} \phi d^3c. \quad (11)$$

Following the BGK scheme^{24,25} we shall let

$$\theta(\mathbf{c}) = \pi^{-3/4} \begin{pmatrix} 1 \\ \sqrt{\frac{2}{3}} (c^2 - \frac{3}{2}) \\ \sqrt{2} c_2 \\ \sqrt{2} c_3 \\ \sqrt{2} c_1 \end{pmatrix}, \quad (12)$$

where $\mathbf{c} = (c_1, c_2, c_3)$ and c represents the magnitude of \mathbf{c} . We can then write Eq. (7) as

$$\begin{aligned} c_1 \frac{\partial \phi}{\partial x}(x, \mathbf{c}) = & \beta \left(-\phi(x, \mathbf{c}) + \int_{-\infty}^{\infty} \theta^T(\mathbf{c}) \theta(\mathbf{c}') \phi(x, \mathbf{c}') e^{-c'^2} d^3c' \right) \\ & + \alpha \left\{ - \int_{-\infty}^{\infty} \sum_{i=3}^5 \theta_i(\mathbf{c}) \theta_i(\mathbf{c}') \left[\phi - \left(\frac{m}{m^*} \right)^{1/2} \phi^* \right] (x, \mathbf{c}') e^{-c'^2} d^3c' + 2\hat{m} \int_{-\infty}^{\infty} \theta_2(\mathbf{c}) \theta_2(\mathbf{c}') (\phi^* - \phi)(x, \mathbf{c}') e^{-c'^2} d^3c' \right\} \\ & - (\beta + \alpha) \gamma \sqrt{2} \int_{-\infty}^{\infty} \theta_1(\mathbf{c}) \theta_3(\mathbf{c}') \phi(x, \mathbf{c}') e^{-c'^2} d^3c' \\ & + \frac{2\beta\gamma}{\sqrt{3}} \int_{-\infty}^{\infty} \theta_3(\mathbf{c}) \theta_2(\mathbf{c}') \phi(x, \mathbf{c}') e^{-c'^2} d^3c' - \beta\gamma^2 \left(\frac{2}{3} \right)^{1/2} \int_{-\infty}^{\infty} \theta_1(\mathbf{c}) \theta_2(\mathbf{c}') \phi(x, \mathbf{c}') e^{-c'^2} d^3c' \\ & + \alpha\gamma\sqrt{2} \left(\frac{m}{m^*} \right)^{1/2} \int_{-\infty}^{\infty} \theta_1(\mathbf{c}) \theta_3(\mathbf{c}') \phi^*(x, \mathbf{c}') e^{-c'^2} d^3c' + 2\alpha\gamma\hat{m} \left(\frac{2}{\sqrt{3}} \right) \int_{-\infty}^{\infty} \theta_3(\mathbf{c}) \theta_2(\mathbf{c}') (\phi^* - \phi)(x, \mathbf{c}') e^{-c'^2} d^3c' \\ & - 2\alpha\gamma^2\hat{m} \left(\frac{2}{3} \right)^{1/2} \int_{-\infty}^{\infty} \theta_1(\mathbf{c}) \theta_2(\mathbf{c}') (\phi^* - \phi)(x, \mathbf{c}') e^{-c'^2} d^3c', \end{aligned} \quad (13)$$

with a similar expression for Eq. (8).

Equation (13) is projected along the axes of the subspace spanned by the vectors

$$\sqrt{\pi} g_1 = 1, \quad \sqrt{\pi} g_2 = c_2^2 + c_3^2 - 1, \quad \sqrt{\pi} g_3 = \sqrt{2} c_2, \quad \sqrt{\pi} g_4 = \sqrt{2} c_3,$$

with respect to the inner product $(f, h)_1$ defined by

$$(f, h)_1 = \int_{-\infty}^{\infty} f(c) h(c) \exp(-c_2^2) \exp(-c_3^2) dc_2 dc_3.$$

We have

$$\phi(x, \mathbf{c}) = \sum_{j=1}^4 \psi_j(x, c_1) g_j(c_2, c_3) + \psi_5(x, \mathbf{c}), \quad (14)$$

where

$$\psi_j(x, c_1) = [g_j(c_2, c_3), \phi(x, \mathbf{c})]_1,$$

for $j = 1, 2, 3, 4$ and we have imposed $(\psi_5, g_j)_1 = 0$, $j = 1, 2, 3, 4$. Substituting Eq. (14) into Eq. (13), integrating over dc_2 and dc_3 , and taking the 1-inner product with respect to the four-vector g_j , $j = 1, 2, 3, 4$, we obtain four equations. A similar procedure, starting with Eq. (8), yields four additional equations. A tabulation of the eight equations follows. They have been grouped to show the coupling that results. We obtain two sets of two coupled equations and one set of four coupled equations. The solutions to the coupled pair are related to the perturbation in the transverse speeds a_y, a_y^*, a_z, a_z^* . The solutions to the four coupled equations yield information about perturbation in temperature, density, and longitudinal speed $b, b^*, \rho, \rho^*, a_x$, and a_x^* . The eight equations are:

From $j = 3$,

$$\begin{aligned} c_1 \frac{\partial \psi_3}{\partial x}(x, c_1) = & \beta \left(-\psi_3(x, c_1) + \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \psi_3(x, c'_1) \exp(-c_1'^2) dc'_1 \right) \\ & + \alpha \left\{ - \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left[\psi_3 - \left(\frac{m}{m^*} \right)^{1/2} \psi_3^* \right] (x, c'_1) \exp(-c_1'^2) dc'_1 \right\}. \end{aligned} \quad (15)$$

$$c_1^* \frac{\partial \psi_3^*}{\partial x}(x, c_1^*) = \beta^* \left(-\psi_3^*(x, c_1^*) + \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \psi_3^*(x, c_1') \exp(-c_1'^2) dc_1' \right) + \alpha^* \left\{ -\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left[\psi_3^* - \left(\frac{m^*}{m} \right)^{1/2} \psi_3 \right] (x, c_1') \exp(-c_1'^2) dc_1' \right\}, \quad (16)$$

From $j = 4$,

$$c_1 \frac{\partial \psi_4}{\partial x}(x, c_1) = \beta \left(-\psi_4(x, c_1) + \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \psi_4(x, c_1') \exp(-c_1'^2) dc_1' \right) + \alpha \left\{ -\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left[\psi_4 - \left(\frac{m}{m^*} \right)^{1/2} \psi_4^* \right] (x, c_1') \exp(-c_1'^2) dc_1' \right\} \quad (17)$$

$$c_1^* \frac{\partial \psi_4^*}{\partial x}(x, c_1^*) = \beta^* \left(-\psi_4^*(x, c_1^*) + \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \psi_4^*(x, c_1') \exp(-c_1'^2) dc_1' \right) + \alpha^* \left\{ -\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left[\psi_4^* - \left(\frac{m^*}{m} \right)^{1/2} \psi_4 \right] (x, c_1') \exp(-c_1'^2) dc_1' \right\}, \quad (18)$$

From $j = 1$ and $j = 2$

$$c_1 \frac{\partial \psi_1}{\partial x}(x, c_1) = \beta \left\{ -\psi_1(x, c_1) + \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left[\psi_1(x, c_1') + \frac{2}{3} \left(c_1^2 - \frac{1}{2} \right) \left(c_1'^2 - \frac{1}{2} \right) \psi_1(x, c_1') + 2c_1 c_1' \psi_1(x, c_1') + \frac{2}{3} \left(c_1^2 - \frac{1}{2} \right) \psi_2(x, c_1') \right] \exp(-c_1'^2) dc_1' \right\} + \alpha \left\{ -\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} 2c_1 c_1' \left[\psi_1 - \left(\frac{m}{m^*} \right)^{1/2} \psi_1^* \right] (x, c_1') \exp(-c_1'^2) dc_1' + \frac{4\hat{m}}{3\sqrt{\pi}} \left(c_1^2 - \frac{1}{2} \right) \int_{-\infty}^{\infty} \left[\left(c_1'^2 - \frac{1}{2} \right) (\psi_1^* - \psi_1)(x, c_1') + (\psi_2^* - \psi_2)(x, c_1') \right] \exp(-c_1'^2) dc_1' \right\} + 2\gamma \left[-(\beta + \alpha) + \alpha \left(\frac{m}{m^*} \right)^{1/2} \right] \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} c_1' \psi_1(x, c_1') \exp(-c_1'^2) dc_1' + \beta\gamma \left(\frac{2}{3} \right)^{1/2} (2c_1 - \gamma) \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{2}{3} \left[\left(c_1'^2 - \frac{1}{2} \right) \psi_1(x, c_1') + \psi_2(x, c_1') \right] \exp(-c_1'^2) dc_1' + 2 \left(\frac{2}{3} \right)^{1/2} \alpha\gamma\hat{m} (2c_1 - \gamma) \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{2}{3} \left[\left(c_1'^2 - \frac{1}{2} \right) \times (\psi_1^* - \psi_1)(x, c_1') + (\psi_2^* - \psi_2)(x, c_1') \right] \exp(-c_1'^2) dc_1', \quad (19)$$

$$c_1^* \frac{\partial \psi_1^*}{\partial x}(x, c_1^*) = \beta^* \left\{ -\psi_1^*(x, c_1^*) + \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left[\psi_1^*(x, c_1') + \frac{2}{3} \left(c_1^{*2} - \frac{1}{2} \right) \left(c_1'^2 - \frac{1}{2} \right) \psi_1^*(x, c_1') + 2c_1^* c_1' \psi_1^*(x, c_1') + \frac{2}{3} \left(c_1^{*2} - \frac{1}{2} \right) \psi_2^*(x, c_1') \right] \exp(-c_1'^2) dc_1' \right\} + \alpha^* \left\{ -\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} 2c_1^* c_1' \left[\psi_1^* - \left(\frac{m^*}{m} \right)^{1/2} \psi_1 \right] (x, c_1') \exp(-c_1'^2) dc_1' + \frac{4\hat{m}^*}{3\sqrt{\pi}} \left(c_1^{*2} - \frac{1}{2} \right) \int_{-\infty}^{\infty} \left[\left(c_1'^2 - \frac{1}{2} \right) (\psi_1 - \psi_1^*)(x, c_1') + (\psi_2 - \psi_2^*)(x, c_1') \right] \exp(-c_1'^2) dc_1' \right\} + 2\gamma^* \left[-(\beta^* + \alpha^*) + \alpha^* \left(\frac{m^*}{m} \right)^{1/2} \right] \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} c_1' \psi_1^*(x, c_1') \exp(-c_1'^2) dc_1' + \beta^*\gamma^* \left(\frac{2}{3} \right)^{1/2} (2c_1^* - \gamma^*) \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{2}{3} \left[\left(c_1'^2 - \frac{1}{2} \right) \psi_1^*(x, c_1') + \psi_2^*(x, c_1') \right] \exp(-c_1'^2) dc_1' + 2\alpha^*\gamma^*\hat{m}^* \left(\frac{2}{3} \right)^{1/2} [2c_1^* - \gamma^*] \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{2}{3} \left[\left(c_1'^2 - \frac{1}{2} \right) \times (\psi_1 - \psi_1^*)(x, c_1') + (\psi_2 - \psi_2^*)(x, c_1') \right] \exp(-c_1'^2) dc_1', \quad (20)$$

$$c_1 \frac{\partial \psi_2}{\partial x}(x, c_1) = \beta \left\{ -\psi_2(x, c_1) + \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{2}{3} \left[\left(c_1'^2 - \frac{1}{2} \right) \psi_1(x, c_1') + \psi_2(x, c_1') \right] \exp(-c_1'^2) dc_1' \right\} \\ + \alpha \left\{ \frac{4\hat{m}}{3\sqrt{\pi}} \int_{-\infty}^{\infty} \left[\left(c_1'^2 - \frac{1}{2} \right) (\psi_1^* - \psi_1)(x, c_1') + (\psi_2^* - \psi_2)(x, c_1') \right] \exp(-c_1'^2) dc_1' \right\}, \quad (21)$$

$$c_1^* \frac{\partial \psi_2^*}{\partial x}(x, c_1^*) = \beta^* \left\{ -\psi_2^*(x, c_1^*) + \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{2}{3} \left[\left(c_1'^2 - \frac{1}{2} \right) \psi_1^*(x, c_1') + \psi_2^*(x, c_1') \right] \exp(-c_1'^2) dc_1' \right\} \\ + \alpha^* \left\{ \frac{4\hat{m}^*}{3\sqrt{\pi}} \int_{-\infty}^{\infty} \left[\left(c_1'^2 - \frac{1}{2} \right) (\psi_1 - \psi_1^*)(x, c_1') + (\psi_2 - \psi_2^*)(x, c_1') \right] \exp(-c_1'^2) dc_1' \right\}. \quad (22)$$

We easily see that Eqs. (15) and (16) and Eqs. (17) and (18) form two sets of coupled equations which have the same form. To analyze the equations we only need to study the vector equation

$$\mu \frac{\partial \psi(x, \mu)}{\partial x} + \tilde{\Sigma} \psi(x, \mu) = \tilde{D} \int_{-\infty}^{\infty} \psi(x, s) \exp(-s^2) ds, \quad \mu \neq 0, \quad (23)$$

where

$$\psi(x, \mu) = \begin{bmatrix} \psi_3(x, \mu) \\ \psi_3^*(x, \mu) \end{bmatrix} \quad \text{or} \quad \psi(x, \mu) = \begin{bmatrix} \psi_4(x, \mu) \\ \psi_4^*(x, \mu) \end{bmatrix}, \\ \tilde{\Sigma} = \begin{bmatrix} \beta & 0 \\ 0 & \beta^* \end{bmatrix}, \quad \tilde{D} = \frac{1}{\sqrt{\pi}} \begin{bmatrix} \beta - \alpha & (m/m^*)^{1/2} \alpha \\ (m^*/m)^{1/2} \alpha^* & \beta^* - \alpha^* \end{bmatrix} = [\tilde{D}_{jk}].$$

Equations (19)–(22) constitute a set of four coupled equations which, after some manipulation, can be written in matrix form

$$\mu \frac{\partial \psi}{\partial x}(x, \mu) + \tilde{\Sigma} \psi(x, \mu) = J \int_{-\infty}^{\infty} \psi(x, s) \exp(-s^2) ds + Q(\mu) \int_{-\infty}^{\infty} L(s) \psi(x, s) \exp(-s^2) ds, \quad \mu \neq 0, \quad (24)$$

where

$$\psi(x, \mu) = \begin{bmatrix} \psi_1(x, \mu) \\ \psi_2(x, \mu) \\ \psi_2^*(x, \mu) \\ \psi_1^*(x, \mu) \end{bmatrix}, \quad \tilde{\Sigma} = \begin{bmatrix} \beta & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \beta^* & 0 \\ 0 & 0 & 0 & \beta^* \end{bmatrix},$$

$$J = \frac{1}{\sqrt{\pi}} \begin{bmatrix} \beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta^* \end{bmatrix}, \quad L(s) = \begin{bmatrix} s & 0 & 0 & -(m/m^*)^{1/2} s \\ s^2 - 1/2 & 1 & 0 & 0 \\ 0 & 0 & 1 & s^2 - 1/2 \\ 0 & 0 & 0 & s \end{bmatrix},$$

$$Q(\mu) = \frac{1}{\sqrt{\pi}}$$

$$\times \begin{bmatrix} 2(\beta - \alpha)\mu + 2\gamma[-\beta - \alpha + \alpha(m/m^*)^{1/2}] & \frac{2}{3}(\beta - 2\hat{m}\alpha)[(\mu^2 - \frac{1}{2}) + (\frac{8}{3})^{1/2}\gamma(\mu - \gamma/2)] & (4\hat{m}\alpha/3)[(\mu^2 - \frac{1}{2}) + (\frac{8}{3})^{1/2}\gamma(\mu - \gamma/2)] & 2(m/m^*)^{1/2}[\beta\mu - \beta\gamma - \alpha\gamma + \alpha\gamma(m/m^*)^{1/2}] \\ 0 & \frac{2}{3}(\beta - 2\hat{m}\alpha) & 4\hat{m}\alpha/3 & 0 \\ 0 & 4\hat{m}^*\alpha^*/3 & \frac{2}{3}(\beta^* - 2\hat{m}^*\alpha^*) & 0 \\ 2\alpha^*\mu(m^*/m)^{1/2} & (4\hat{m}^*\alpha^*/3)[(\mu^2 - \frac{1}{2}) + (\frac{8}{3})^{1/2}\gamma^*(\mu - \gamma^*/2)] & \frac{2}{3}(\beta^* - 2\hat{m}^*\alpha^*)[(\mu^2 - \frac{1}{2}) + (\frac{8}{3})^{1/2}\gamma^*(\mu - \gamma^*/2)] & 2\beta^*\mu + 2\gamma^*[-\beta^* - \alpha^* + \alpha^*(m^*/m)^{1/2}] \end{bmatrix}.$$

The perturbation of temperature, velocity and density can be obtained in terms of the ψ_j 's, $j = 1, 2, 3, 4$. Using the 1-inner product we obtain

$$b = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{2}{3} \left[\left(s^2 - \frac{1}{2} \right) \psi_1 + \psi_2 \right] \exp(-s^2) ds, \quad \rho = \frac{1}{\pi} \int_{-\infty}^{\infty} \psi_1 \exp(-s^2) ds,$$

$$a_x = \frac{1}{\pi} \int_{-\infty}^{\infty} \psi_1 s \exp(-s^2) ds, \quad a_y = \frac{1}{\pi\sqrt{2}} \int_{-\infty}^{\infty} \psi_3 \exp(-s^2) ds,$$

$$a_z = \frac{1}{\pi\sqrt{2}} \int_{-\infty}^{\infty} \psi_4 \exp(-s^2) ds.$$

A similar set of expressions hold for b^* , ρ^* , a_x^* , a_y^* , and a_z^* with ψ_j replaced by ψ_j^* for $j = 1, 2, 3, 4$.

III. FULL RANGE EXPANSION

The 4×4 matrix equation will be treated in a subsequent publication.²⁰ Here we wish to study the 2×2 matrix equation representing the coupled pair of equations (15) and (16) [or, equivalently, (17) and (18)].

We define the Banach space

$$X_p = \left\{ \psi = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \middle| \psi_i \in L_p[\exp(-\mu^2)d\mu], \quad i = 1, 2, \quad p > 1 \right\}$$

with

$$\|\psi\|_{X_p}^p = \sum_{i=1}^2 \int_{-\infty}^{\infty} |\mu \psi_i|^p \exp(-\mu^2)d\mu.$$

The transport operator K is defined by

$$(K\psi)(\mu) = -\frac{1}{\mu} \tilde{\Sigma}\psi(\mu) + \frac{1}{\mu} \tilde{D} \int_{-\infty}^{\infty} \psi(s)\exp(-s^2)ds. \quad (25)$$

Since K is a finite-dimensional perturbation of the operator $-(1/\mu)\tilde{\Sigma}$, its spectrum $\sigma(K)$ consists of the real axis along with isolated eigenvalues. We shall see that, in fact, the only eigenvalue is the imbedded eigenvalue at zero.

Following the resolvent integration technique developed for unbounded, noninvertible transport operators,^{11,21} let us define the operator $S = (K + i)^{-1}$. To simplify notation we write

$$\Sigma(\mu) = \frac{1}{\mu} \left(iI - \frac{1}{\mu} \tilde{\Sigma} \right)^{-1} = \begin{bmatrix} 1/i\mu - \beta & 0 \\ 0 & 1/i\mu - \beta^* \end{bmatrix},$$

$$D = \tilde{D} \left(I + \int_{-\infty}^{\infty} \Sigma(s)\tilde{D} \exp(-s^2)ds \right)^{-1}.$$

An easy computation gives

$$S\psi = \mu \Sigma(\mu)\psi - \Sigma(\mu)D \int_{-\infty}^{\infty} s \Sigma(s)\psi(s)\exp(-s^2)ds. \quad (26)$$

Again with some manipulation we can compute the resolvent

$$\begin{aligned} (zI - S)^{-1}\psi(\mu) &= [zI - \mu \Sigma(\mu)]^{-1}\psi(\mu) \\ &\quad - [zI - \mu \Sigma(\mu)]^{-1}\Sigma(\mu)D\tilde{\Lambda}^{-1}(z) \\ &\quad \times \int_{-\infty}^{\infty} s \Sigma(s)[zI - s\Sigma(s)]^{-1}\psi(s)\exp(-s^2)ds, \end{aligned} \quad (27)$$

where the dispersion function $\tilde{\Lambda}(z)$ is

$$\tilde{\Lambda}(z) = I + \int_{-\infty}^{\infty} s \Sigma(s)[zI - s\Sigma(s)]^{-1}\Sigma(s)D \exp(-s^2)ds.$$

For later reference we also define

$$\tilde{\Lambda}(z)\tilde{M}_\psi(z) = \int_{-\infty}^{\infty} s \Sigma(s)[zI - s\Sigma(s)]^{-1}\psi(s)\exp(-s^2)ds.$$

We shall use the notation $\beta = \sigma_1$, $\beta^* = \sigma_2$, recalling that β and β^* are the diagonal entries of the diagonal matrix $\tilde{\Sigma}$. Introducing the complementary error function

$$\operatorname{erfc}(z) = \frac{2z \exp(-z^2)}{\pi} \int_0^{\infty} \frac{\exp(-s^2)}{s^2 + z^2} ds,$$

and utilizing the fact that $\det \tilde{D} > 0$ for all values of the parameters, we can show that

$$I + \int_{-\infty}^{\infty} \Sigma(s)\tilde{D} \exp(-s^2)ds$$

is invertible so long as

$$\begin{aligned} d^{-1} &= 1 - \pi \exp(\sigma_1^2) \operatorname{erfc}(\sigma_1)\tilde{D}_{11} - \pi \exp(\sigma_2^2) \operatorname{erfc}(\sigma_2)\tilde{D}_{22} \\ &\quad + \pi^2 \exp(\sigma_1^2 + \sigma_2^2) \operatorname{erfc}(\sigma_1)\operatorname{erfc}(\sigma_2)(\det \tilde{D}) \neq 0. \end{aligned}$$

Let us write $F(z) = \tilde{\Lambda}(z) - I$, i.e.,

$$F_{jk}(z) = \int_{-\infty}^{\infty} \frac{s}{(is - \sigma_j)[z(is - \sigma_j) - s]} \exp(-s^2)ds D_{jk}$$

for $j = 1, 2$ and $k = 1, 2$. Then

$$\frac{F_{12}(z)}{D_{12}} = \frac{F_{11}(z)}{D_{11}}, \quad \frac{F_{21}(z)}{D_{21}} = \frac{F_{22}(z)}{D_{22}}.$$

Evaluation of $F_{jk}(z)$ at $z = -i$ leads to

$$\frac{F_{jj}}{D_{jj}} = \frac{-\sqrt{\pi}}{\sigma_j} + \pi \exp(\sigma_j^2) \operatorname{erfc}(\sigma_j).$$

A lengthy computation yields

$$\det \tilde{\Lambda}(-i) = 0.$$

The evaluation

$$\frac{d}{dz} \det \tilde{\Lambda}(z) \Big|_{z=-i} = 0$$

is actually very simple because the integral expressions for $(d/dz)F_{jj}(-i)$ vanish immediately. Another tedious computation gives

$$\frac{d^2}{dz^2} \det \tilde{\Lambda}(z) \Big|_{z=-i} = \frac{-d}{\sigma_1^2 \sigma_2^2} (\sigma_1 \alpha^* + \sigma_2 \alpha) \neq 0.$$

Thus $z = -i$ is a double pole for $\det \tilde{\Lambda}(z)$.

Corresponding to the double pole at $z = -i$ are eigenvectors of K with eigenvalue $w = 0$. The derivation of these eigenvectors is similar to the computation of S done previously. The result is an eigenvector

$$\psi^{(1)} = \begin{bmatrix} 1 \\ (m^*/m)^{1/2} \end{bmatrix},$$

and a generalized eigenvector $\psi^{(2)} = \mu \tilde{\Sigma}^{-1} \psi^{(1)}$ with $K\psi^{(2)} = -\psi^{(1)}$.

Let us define the two-dimensional subspace $Y_0 = \operatorname{span}\{\psi^{(1)}, \psi^{(2)}\}$. We seek a space Y_p such that $X_p = Y_p \oplus Y_0$, where Y_p and Y_0 reduce the operators K and S . We define the linear functionals \tilde{l}_1 and \tilde{l}_2 by

$$\tilde{l}_1(\psi) = \frac{2\alpha\sigma_1\sigma_2}{\sqrt{\pi}(\alpha\sigma_1 + \alpha^*\sigma_2)} \left[\frac{\alpha^*}{\alpha} \left(\frac{m}{m^*} \right)^{1/2} \right] \tilde{\Sigma}^{-1}$$

$$\times \int_{-\infty}^{\infty} \psi(s)s^2 \exp(-s^2)ds,$$

$$\tilde{l}_2(\psi) = \frac{2\alpha\sigma_1\sigma_2}{\sqrt{\pi}(\alpha\sigma_1 + \alpha^*\sigma_2)} \left[\frac{\alpha^*}{\alpha} \left(\frac{m}{m^*} \right)^{1/2} \right]$$

$$\times \int_{-\infty}^{\infty} \psi(s)s \exp(-s^2)ds$$

for $\psi \in X_p$. Let $P: X_p \rightarrow Y_0$ be defined by

$$P(\psi) = \tilde{l}_1(\psi)\psi^{(1)} + \tilde{l}_2(\psi)\psi^{(2)},$$

and $Y_p = \ker P$. Since $\tilde{l}_j(\psi^{(k)}) = \delta_{jk}$ by an easy calculation, Y_p and Y_0 reduce K and S .

In order to obtain a spectral resolution of S , it will be necessary to have the boundary values of certain analytic functions. In particular, we are interested in the boundary values A^\pm , M_ψ^\pm of the analytic functions $A(w)$ and $M_\psi(w)$, computed approaching the cut (the real axis) from above and below, where

$$A(w) = I + \int_{-\infty}^{\infty} s\mathcal{Z}(s) \left(\frac{1}{i-w} I - s\mathcal{Z}(s) \right)^{-1} \times \mathcal{Z}(s) D \exp(-s^2) ds,$$

$$A(w)M_\psi(w) = \int_{-\infty}^{\infty} s\mathcal{Z}(s) \left(\frac{1}{i-w} I - s\mathcal{Z}(s) \right)^{-1} \times \psi(s) \exp(-s^2) ds,$$

and $\psi \in H_p^*$, the dense linear manifold of Hölder continuous functions in X_p . We have made the transformation $z = 1/(i-w)$; thus $A(w) = \tilde{A}(z)$.

Using the Plemelj formulas,²⁶ the boundary values of $A(w)M_\psi(w)$ may be obtained.

$$A^\pm(w)M_\psi^\pm(w) = \oint_{-\infty}^{\infty} s\mathcal{Z}(s) \left(\frac{1}{i-w} I - s\mathcal{Z}(s) \right)^{-1} \psi(s) \exp(-s^2) ds \mp i\pi \left(\frac{i-w}{w^2} \right) \tilde{\mathcal{Z}} \mathcal{Z}_{\tilde{\mathcal{Z}}} \left(\frac{1}{w} \right) \begin{bmatrix} \psi_1(\sigma_1/w) \\ \psi_2(\sigma_2/w) \end{bmatrix}, \quad (28)$$

where

$$\mathcal{Z}_{\tilde{\mathcal{Z}}} \left(\frac{1}{w} \right) = \begin{bmatrix} \exp[-(\sigma_1/w)^2] & 0 \\ 0 & \exp[-(\sigma_2/w)^2] \end{bmatrix}.$$

Similarly we can compute

$$A^\pm(w) = I + \oint_{-\infty}^{\infty} s\mathcal{Z}(s) \left(\frac{1}{i-w} I - s\mathcal{Z}(s) \right)^{-1} \mathcal{Z}(s) D \exp(-s^2) ds \mp (i\pi/w) \mathcal{Z}_{\tilde{\mathcal{Z}}}(1/w) D. \quad (29)$$

Hence we may tabulate

$$A^+(w) + A^-(w) = 2 \left[I + \oint_{-\infty}^{\infty} s\mathcal{Z}(s) \left(\frac{1}{i-w} I - s\mathcal{Z}(s) \right)^{-1} \mathcal{Z}(s) D \exp(-s^2) ds \right], \quad (30)$$

$$A^+(w) - A^-(w) = (-2\pi i/w) \mathcal{Z}_{\tilde{\mathcal{Z}}}(1/w) D, \quad (31)$$

$$\begin{aligned} M_\psi^+(w) - M_\psi^-(w) &= (2\pi i/w) A^{-1}(w) \mathcal{Z}_{\tilde{\mathcal{Z}}}(1/w) D A^{+1}(w) \oint_{-\infty}^{\infty} s\mathcal{Z}(s) \left(\frac{1}{i-w} I - s\mathcal{Z}(s) \right)^{-1} \psi(s) \exp(-s^2) ds \\ &\quad - 2\pi i A^{-1}(w) \left[I + \oint_{-\infty}^{\infty} s\mathcal{Z}(s) \left(\frac{1}{i-w} I - s\mathcal{Z}(s) \right)^{-1} \mathcal{Z}(s) D \exp(-s^2) ds \right] \\ &\quad \times A^{+1}(w) \left(\frac{i-w}{w^2} \right) \tilde{\mathcal{Z}} \mathcal{Z}_{\tilde{\mathcal{Z}}} \left(\frac{1}{w} \right) \begin{bmatrix} \psi_1(\sigma_1/w) \\ \psi_2(\sigma_2/w) \end{bmatrix}. \end{aligned} \quad (32)$$

Using the identity

$$\begin{aligned} A^+(w)M_\psi^+(w) - A^-(w)M_\psi^-(w) &= \frac{1}{2} [A^+(w) + A^-(w)] [M_\psi^+(w) - M_\psi^-(w)] \\ &\quad + \frac{1}{2} [A^+(w) - A^-(w)] [M_\psi^+(w) + M_\psi^-(w)], \end{aligned} \quad (33)$$

we can also write

$$\begin{aligned} -2\pi i \left(\frac{i-w}{w^2} \right) \tilde{\mathcal{Z}} \mathcal{Z}_{\tilde{\mathcal{Z}}} \left(\frac{1}{w} \right) \begin{bmatrix} \psi_1(\sigma_1/w) \\ \psi_2(\sigma_2/w) \end{bmatrix} &= \left[I + \oint_{-\infty}^{\infty} s\mathcal{Z}(s) \left(\frac{1}{i-w} I - s\mathcal{Z}(s) \right)^{-1} \mathcal{Z}(s) D \exp(-s^2) ds \right] [M_\psi^+(w) - M_\psi^-(w)] \\ &\quad - (\pi i/w) \mathcal{Z}_{\tilde{\mathcal{Z}}}(1/w) D [M_\psi^+(w) + M_\psi^-(w)]. \end{aligned} \quad (34)$$

The resolution of the identity will be obtained by calculating

$$\frac{1}{2\pi i} \oint_{\Gamma} (zI - S)^{-1} \psi(\mu) dz,$$

for $\psi \in H_p^*$, where Γ is a contour enclosing the spectrum of S , $\sigma(S)$. The spectrum consists only of the closed curve $\{z \in \mathcal{C} \mid (\operatorname{Re} z)^2 + (\operatorname{Im} z + \frac{1}{2})^2 = \frac{1}{4}\}$. We surround $\sigma(S)$ by contours Γ^\pm . In order to study these contours near the pole $z = -i$ and near $z = 0$ (corresponding to $w = \infty$), the contours are deformed, necessitating a fairly detailed geometric description. For example, we may decompose the outer contour into four subcontours; the contour Γ_0^+ consisting of part of the circle centered at $(0,0)$ of radius r_ϵ such that $-\epsilon < \theta < \pi + \epsilon$; the contour Γ_{-i}^+ consisting of the part of the circle centered at $(0, -i)$ of radius t_ϵ such that $\pi + \eta_\epsilon < \theta < 2\pi - \eta_\epsilon$; the contour Γ_μ^+ consisting of part of the circle centered at $z = \mu$ ($\arg \mu = \theta_\mu$) of radius t_ϵ such that $\pi + \eta_\epsilon < \theta < 2\pi - \eta_\epsilon$; and the large subcontour Γ_B^+ consisting of part of the circle centered at $(0, -1/2 - \epsilon)$ of radius $1/2 + \epsilon$ such that $\pi/2 + \theta_\epsilon \leq \theta \leq 3\pi/2 - \epsilon$, $3\pi/2 + \epsilon \leq \theta \leq \theta_\mu - \epsilon$ and $\theta_\mu + \epsilon \leq \theta \leq 5\pi/2 - \theta_\epsilon$. Note that $r_\epsilon = (1 + 2\epsilon) \sin \epsilon$, $t_\epsilon = [1/2 + 2\epsilon^2 - (1/2 - 2\epsilon^2) \cos \epsilon]^{1/2}$ and $\eta_\epsilon, \theta_\epsilon$ are chosen to make the contour continuous. See Ref. 27 for details. The result for the contour $\Gamma_B^+ \cup \Gamma_\mu^+$ using Eq. (27) and the Plemelj formula is

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \frac{1}{2\pi i} \int_{\Gamma_B^+ \cup \Gamma_\mu^+} (zI - S)^{-1} \psi(\mu) dz &= -\frac{1}{2\pi i} \oint_{-\infty}^{\infty} \left(\frac{1}{i-w} I - \mu \Sigma(\mu) \right)^{-1} \psi(\mu) \frac{dw}{(i-w)^2} + \frac{1}{2} \psi(\mu) \\ &+ \frac{1}{2\pi i} \oint_{-\infty}^{\infty} \left(\frac{1}{i-w} I - \mu \Sigma(\mu) \right)^{-1} \Sigma(\mu) D M_\psi^+(w) \frac{dw}{(i-w)^2} \\ &- \frac{1}{2} \Sigma(\mu) \left[\begin{array}{l} D_{11} [M_\psi^+(\sigma_1/\mu)]_1 + D_{12} [M_\psi^+(\sigma_1/\mu)]_2 \\ D_{21} [M_\psi^+(\sigma_2/\mu)]_1 + D_{22} [M_\psi^+(\sigma_2/\mu)]_2 \end{array} \right], \end{aligned}$$

with a similar expression for the contribution from the inner subcontour $\Gamma_B^- \cup \Gamma_\mu^+$. The contour Γ_0^+ yields a nonzero contribution for the term

$$\int_{\Gamma_0^+} [zI - \mu \Sigma(\mu)]^{-1} \Sigma(\mu) D \tilde{M}_\psi(z) dz,$$

which, however, precisely (in the limit $\epsilon \rightarrow 0$) cancels with the similar contribution from the contour Γ_0^- . The evaluation of

$$\int_{\Gamma_\pm^+ \cup \Gamma_\pm^-} [zI - \mu \Sigma(\mu)]^{-1} \Sigma(\mu) D \tilde{M}_\psi(z) dz$$

utilizes residue theory about the double pole of $\det \tilde{A}(z)$ and some tedious algebra. The result is the simple expression

$$\tilde{l}_1(\psi) \psi^{(1)} + \tilde{l}_2(\psi) \psi^{(2)}.$$

Finally, collecting these results, we obtain

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \frac{1}{2\pi i} \oint_{\Gamma} (zI - S)^{-1} \psi(\mu) dz &= \tilde{l}_1(\psi) \psi^{(1)} + \tilde{l}_2(\psi) \psi^{(2)} + \psi(\mu) \\ &- \frac{1}{2} \Sigma(\mu) \left[\begin{array}{l} D_{11} [(M_\psi^+ + M_\psi^-)(\sigma_1/\mu)]_1 + D_{12} [(M_\psi^+ + M_\psi^-)(\sigma_1/\mu)]_2 \\ D_{21} [(M_\psi^+ + M_\psi^-)(\sigma_2/\mu)]_1 + D_{22} [(M_\psi^+ + M_\psi^-)(\sigma_2/\mu)]_2 \end{array} \right] \\ &+ \frac{1}{2\pi i} \oint_{-\infty}^{\infty} \left(\frac{1}{i-w} I - \mu \Sigma(\mu) \right)^{-1} \Sigma(\mu) D [M_\psi^+(w) - M_\psi^-(w)] \frac{dw}{(i-w)^2}. \end{aligned} \quad (35)$$

The expression $(M_\psi^+ + M_\psi^-)(\mu)$ can be eliminated by using the identity, Eq. (34). Define the operator

$$J: X_p \rightarrow X_p \text{ by } [J\psi(\mu)]_j = \psi_j(\mu/\sigma_j), \quad j = 1, 2,$$

and let

$$\Delta(w - 1/\mu) = \begin{bmatrix} \delta(w - \sigma_1/\mu) & 0 \\ 0 & \delta(w - \sigma_2/\mu) \end{bmatrix}.$$

Substituting these into (35) we find the full range expansion formula corresponding to the transport problem (15) and (16) to be

$$\psi(\mu) = \tilde{l}_1(\psi) \psi^{(1)} + \tilde{l}_2(\psi) \psi^{(2)} + \int_{-\infty}^{\infty} \Phi_w(\mu) C_\psi(w) dw, \quad (36)$$

where

$$\Phi_w(\mu) = P [1/(i-w)I - \mu \Sigma(\mu)]^{-1} \Sigma(\mu) [D(i-w)^{-2} - \Delta(w - 1/\mu) Z_{\bar{x}}^{-1} (1/w) \tilde{\Sigma}^{-1} \{ \Lambda^+(w) - \Lambda^-(w) \} / 2], \quad (37)$$

$$\begin{aligned} C_\psi(w) &= \frac{1}{2\pi i} [M_\psi^+(w) - M_\psi^-(w)] = \frac{1}{w} \Lambda^{-1}(w) Z_{\bar{x}} \left(\frac{1}{w} \right) D \Lambda^{+1}(w) \oint_{-\infty}^{\infty} s \Sigma(s) \left(\frac{1}{i-w} I - s \Sigma(s) \right)^{-1} \psi(s) \exp(-s^2) ds \\ &- \Lambda^{-1}(w) \left[I + \int_{-\infty}^{\infty} s \Sigma(s) \left(\frac{1}{i-w} I - s \Sigma(s) \right)^{-1} \Sigma(s) D \exp(-s^2) ds \right] \Lambda^{+1}(w) \left(\frac{i-w}{w^2} \right) \tilde{\Sigma} Z_{\bar{x}} \left(\frac{1}{w} \right) \begin{bmatrix} \psi_1(\sigma_1/w) \\ \psi_2(\sigma_2/w) \end{bmatrix}. \end{aligned} \quad (38)$$

Isolated eigenvalues of S would correspond to zeros of the determinant of the dispersion function; their nonexistence may be derived by a Nyquist diagram analysis. Let us write $F(z) = \det \Lambda(z)$ as

$$F(z) = \det \left(I - \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-x^2) (\tilde{\Sigma} - xz)^{-1} dx \tilde{D} \right) = 1 - \frac{d_{11}}{\sigma_1} f\left(\frac{z}{\sigma_1}\right) - \frac{d_{22}}{\sigma_2} f\left(\frac{z}{\sigma_2}\right) + \frac{\det \tilde{D}}{\sigma_1 \sigma_2} f\left(\frac{z}{\sigma_1}\right) f\left(\frac{z}{\sigma_2}\right),$$

where

$$f(z) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-x^2) (1 - xz)^{-1} dx.$$

Denoting by f_p the integral above for real z , with integration understood in the principal value sense, the boundary values F^\pm of $F(z)$ along the cut $z \in \mathbb{R}$ may be computed:

$$F^\pm(t) = 1 - \frac{d_{11}}{\sigma_1} f_p\left(\frac{t}{\sigma_1}\right) - \frac{d_{22}}{\sigma_2} f_p\left(\frac{t}{\sigma_2}\right) + \frac{\det D}{\sigma_1 \sigma_2} \left[f_p\left(\frac{t}{\sigma_1}\right) f_p\left(\frac{t}{\sigma_2}\right) - \pi \sigma_1 \sigma_2 t^{-2} \exp\left[-(\sigma_1^2 + \sigma_2^2)/t^2\right] \right] \\ \pm i\sqrt{\pi t}^{-1} \left[d_{11} \exp(-\sigma_1^2/t^2) + d_{22} \exp(-\sigma_2^2/t^2) - \det \tilde{D} \left(\frac{f_p(t/\sigma_2)}{\sigma_2} \exp(-\sigma_1^2/t^2) + \frac{f_p(t/\sigma_1)}{\sigma_1} \exp(-\sigma_2^2/t^2) \right) \right] \equiv R(t) \pm i\sqrt{\pi t} J(t).$$

f has the asymptotic behavior $f_p(\pm\infty) = 0$, $f_p(0) = 1^+$, and $f(\pm i\epsilon) \rightarrow 1^-$ as $\epsilon \rightarrow 0$. The conservation laws in gas dynamics transport are represented by the zero $F^\pm(0) = 0$. Moreover, the inequality

$$\beta\beta^* - \alpha\beta^* - \alpha^*\beta > 0,$$

which is trivially established (actually equal to zero for $K_{11} = K_{22} = 0$, which is of course excluded), gives $\det \tilde{D} > 0$. Then either $\text{Im } tF^+(t) > 0$ for all t , or

$$R(t_0) = -\frac{d_{12}d_{21}}{\det \tilde{D}} - \pi t_0^{-2} \exp\left[-(\sigma_1^2 + \sigma_2^2)/t_0^2\right] \det \tilde{D} - \left(\frac{f_p(t_0/\sigma_2)}{\sigma_2} \det \tilde{D} - d_{11}\right)^2 \exp\left[-(\sigma_1^2 - \sigma_2^2)/t_0^2\right] (\det \tilde{D})^{-1} < 0,$$

where t_0 is a zero of $J(t)$. Note that this implies $F(z)$ has no real zeros other than $z = 0$. Collecting the above results and utilizing the symmetry $R(t) = R(-t)$, $J(t) = J(-t)$, $t \in \mathbb{R}$, leads easily to a Nyquist diagram which does not circle the origin.

IV. HALF-RANGE EXPANSION

The expansion of any functions $\psi \in X_p$ in terms of positive eigenmodes only (half-range expansion) depends upon the factorization of the dispersion matrix as a product

$$A(w) = A(w)B(-w),$$

where A and B are analytic matrix valued functions in the right half-plane. The factor matrices satisfy the integral equations²

$$A(w) = \tilde{D}^{-1} - \int_{-\infty}^0 Z_{\bar{x}}\left(\frac{1}{s}\right) DB^{-1}(-s) \frac{ds}{s(s-w)},$$

$\text{Re } w > 0$,

$$B(-w) = D - \int_0^{\infty} A^{-1}(s) Z_{\bar{x}}\left(\frac{1}{s}\right) D \frac{ds}{s(s-w)}, \quad \text{Re } w < 0,$$

and are extended to the left half-plane by

$$A(w) = A(w)B^{-1}(-w), \quad B(w) = A^{-1}(-w)A(w)$$

for $\text{Re } w < 0$.

By the similarity transformation

$$\psi \rightarrow S_1^{-1} \psi, \quad S_1 = \begin{bmatrix} \sqrt{m} & 0 \\ 0 & \sqrt{m^*} \end{bmatrix},$$

the dispersion matrix can be represented with the matrix $\tilde{\Sigma}^{-1} \tilde{D}$ replaced by $S_1^{-1} \tilde{\Sigma}^{-1} \tilde{D} S_1$ which is Markov (except for a factor $\sqrt{\pi}$) and therefore has spectral radius 1.

On the other hand the similarity transformation $\psi \rightarrow S_2^{-1} \psi$,

$$S_2 = \begin{bmatrix} \sqrt{\alpha\sigma_2 m} & 0 \\ 0 & \sqrt{\alpha^*\sigma_1 m^*} \end{bmatrix}$$

represents the dispersion function with a symmetric matrix, with spectral radius 1 by the previous comment (spectral invariance of similarity transformations). The recent work of Greenberg and Van der Mee²⁸ then proves the existence of a unique albedo operator, and hence the existence of a unique solution of the integral equations for A and B . The point is that $\ker S_2^{-1} K S_2$ is degenerate with respect to the indefinite metric obtained from the (signed) measure $s \exp(-s^2) ds$. See

Ref. 28 for details; also Ref. 29, which treats the one-gas problem by a similar Jordan decomposition analysis.

To study half-space problems let us define the space

$$X'_p = [\psi(x, \mu) \in X_p \mid \psi(x, \mu) = 0 \quad \text{a.e. for } \mu < 0].$$

We shall look for an operator

$$E: X'_p \rightarrow X_p$$

such that the full range expansion of E yields a half-range expansion of ψ . In particular we require E to satisfy

(i) $Y'_p = \ker(\tilde{I}_1 \circ E)$ and $Y'_0 = \text{span}(\psi^{(1)})$ reduce E ,

(ii) $\tilde{I}_2(E\psi) = 0$, $\forall \psi \in X'_p$,

(iii) $M_{E\psi}(\lambda)$ is analytic $\forall \lambda \ni \text{Re } \lambda < 0$ and $\det A(\lambda) = 0$,

(iv) $\{[1/(i-w)]I - S\}^{-1}(E\psi)(\mu)$ is analytic for $\text{Re } w < 0$, $\forall \psi \in Y'_p$.

The construction of E follows from standard analyticity arguments (see Refs. 7 and 30). The idea is to study the boundary value behavior of

$$Q(w) = A^{-1}(w)A(w)M_{E\psi}(w)$$

$$- \int_0^{\infty} A_{\bar{x}}^{-1}\left(\frac{1}{s}\right) s \Sigma(s) \left(\frac{1}{i-w} I - s \Sigma(s)\right)^{-1} \\ \times \psi(s) \exp(-s^2) ds,$$

where

$$M_f(w) = A^{-1}(w) \int_0^{\infty} s \Sigma(s) \left(\frac{1}{i-w} I - s \Sigma(s)\right)^{-1} \\ \times f(s) \exp(-s^2) ds.$$

The analyticity properties of $A(w)$ give $Q(w)$ bounded and analytic on the cut plane $\text{Im } w \neq 0$, continuous across the cut, and vanishing at $w = i$. Thus $Q(w) = 0$ and E can be computed from the boundary values of $M_{E\psi}$:

$$(E\psi)(\mu) = \frac{1}{2} \Sigma(\mu) D(\Delta M)_{E\psi}(\mu), \quad \mu < 0,$$

with

$$[(\Delta M)_f(\mu)]_j = [D(M_f^+ + M_f^-)(\sigma_i/\mu)]_j.$$

The result is

$$(E\psi)(\mu) = \psi(\mu), \quad \mu > 0,$$

$$\begin{aligned} \begin{bmatrix} (E\psi)_1(\sigma_1/\mu) \\ (E\psi)_2(\sigma_2/\mu) \end{bmatrix} &= \mu \tilde{\Sigma}^{-1} D B^{-1} (-\mu) \int_0^\infty A^{-1} \left(\frac{1}{s} \right) \\ &\times \begin{bmatrix} 1 & 0 \\ s\mu - \sigma_1 & 1 \\ 0 & s\mu - \sigma_2 \end{bmatrix} s\psi(s) \exp(-s^2) ds, \end{aligned}$$

$\mu < 0$.

The confirmation of (i) and (ii) follows from the evaluation:

$$\begin{aligned} \tilde{l}_1(E\psi) &= -\frac{2\alpha\sigma_1\sigma_2}{\sqrt{\pi}(\alpha\sigma_1 + \alpha^*\sigma_2)} \left[\frac{\alpha^*}{\alpha} \left(\frac{m}{m^*} \right)^{1/2} \right] \tilde{\Sigma} \frac{dA(0)}{dw} \\ &\times \int_0^\infty \frac{ds}{s^3} A^{-1}(s) \tilde{\Sigma} Z_{\tilde{\Sigma}} \left(\frac{1}{s} \right) \begin{bmatrix} \psi_1(\sigma_1/s) \\ \psi_2(\sigma_2/s) \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} \tilde{l}_2(E\psi) &= \frac{2\alpha\sigma_1\sigma_2}{\sqrt{\pi}(\alpha\sigma_1 + \alpha^*\sigma_2)} \left[\frac{\alpha^*}{\alpha} \left(\frac{m}{m^*} \right)^{1/2} \right] \tilde{\Sigma} A(0) \\ &\times \int_0^\infty \frac{ds}{s^3} A^{-1}(s) \tilde{\Sigma} Z_{\tilde{\Sigma}} \left(\frac{1}{s} \right) \begin{bmatrix} \psi_1(\sigma_1/s) \\ \psi_2(\sigma_2/s) \end{bmatrix}, \end{aligned}$$

and the fact that the columns of $\text{cof } A(0)$ [resp. $\text{cof } B(0)^T$] are eigenvectors of $A(0)^T$ [resp. $A(0)$] and can be identified as

$$\tilde{\Sigma} \left(\frac{\alpha^*/\alpha}{(m/m^*)^{1/2}} \right) \left[\text{resp. } D^{-1} \tilde{\Sigma} \left(\frac{1}{(m/m^*)^{1/2}} \right) \right].$$

Thus, the half-range expansion is obtained:

$$\psi(\mu) = \tilde{l}_1(E\psi)\psi^{(1)} + \int_0^\infty \Phi_w(\mu) C_{E\psi}(w) dw$$

for $\psi \in X'_p$.

V. SOLUTION TO THE TIME-INDEPENDENT PROBLEM

The half-range problem

$$\frac{\partial \psi}{\partial x}(x, \mu) - K\psi(x, \mu) = 0, \quad x > 0,$$

with $\psi(0, \mu) = \psi_0(\mu)$, $\mu > 0$, and $\lim_{x \rightarrow \infty} \|\psi(x, \mu)\| < \infty$, describes, for example, a model of a planar evaporation source at $x = 0$ evaporating into a gaseous atmosphere with a fixed evaporation velocity profile $\psi_0(\mu)$. The binary model allows for the evaporation and atmosphere gases to have different masses and collision cross sections, and the boundary value constraint for $\mu > 0$ corresponds to the fact that evaporation occurs only *into* the gaseous atmosphere; i.e., an ingoing flux.

Using the half-range expansion formulas of the preceding section, the solution may be written as

$$\psi(x, \mu) = \int_0^\infty \exp(-x\lambda) \Phi_\lambda(\mu) C_{E\psi_0}(\lambda) d\lambda + \tilde{l}_1(E\psi_0)\psi^{(1)}, \quad (39)$$

where

$$\begin{aligned} C_{E\psi_0}(\lambda) &= (1/2\pi i) [M_{E\psi_0}^+(\lambda) - M_{E\psi_0}^-(\lambda)] = \frac{1}{\lambda} A^{-1}(\lambda) Z_{\tilde{\Sigma}} \left(\frac{1}{\lambda} \right) D A^+(\lambda) \\ &\times \int_{-\infty}^\infty s \mathcal{Z}(s) \left(\frac{1}{i-\lambda} I - s \mathcal{Z}(s) \right)^{-1} E\psi_0(s) \exp(-s^2) ds \\ &- A^{-1}(g) \left[I + \int_{-\infty}^\infty s \mathcal{Z}(s) \left(\frac{1}{i-\lambda} I - s \mathcal{Z}(s) \right)^{-1} \mathcal{Z}(s) D \exp(-s^2) ds \right] \\ &\times A^+(\lambda) \left(\frac{i-\lambda}{\lambda^2} \right) \tilde{\Sigma} Z_{\tilde{\Sigma}} \left(\frac{1}{\lambda} \right) \begin{bmatrix} (E\psi_0)_1(\sigma_1/\lambda) \\ (E\psi_0)_2(\sigma_2/\lambda) \end{bmatrix}. \end{aligned}$$

In more traditional language, the function $E\psi_0$ is the total (incoming plus outgoing) flux at the boundary $x = 0$, and the coefficient $C_{E\psi_0}(\lambda)$ is the Case transform of the total flux $E\psi_0$. Equation (39) provides a solution of the half space problem expressed as a singular eigenfunction expansion.

An essential feature of the method is the construction of the albedo operator E . This was facilitated by the fact that the transport operator K is similar to a self-adjoint operator, and the lack of such a similarity transformation is an added complication in the analysis of the four group problem arising in the mass and temperature density equations.

We have obtained a solution of the transport equations under very general "incoming flux" boundary conditions; in particular, both components of ψ_0 may be nonzero. For example, for an appropriate choice of boundary value we have a solution of the classical Kramers problem for a binary gas. Of course, for the problem of planar evaporations into a gaseous atmosphere, the component of ψ_0 corresponding to the atmosphere gas will vanish (but not the same component of $E\psi_0$ with $\mu < 0$).

Finally, let us note that the evaporation and atmosphere (transverse velocity) densities will converge to the original Maxwellians at infinity if the velocity profile $\psi_0(\mu)$ contains no component in Y_0 . More precisely, this will be the case iff $\tilde{l}_1(E\psi_0) = 0$, and corresponds physically to mass conservation at the boundary.

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