Essays in Decision Theory

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(ABSTRACT)

This dissertation studies decision theories for both individual and interactive choice problems. This thesis proposes three non-standard models that modify assumptions and settings of standard models. Chapter 1 provides an overview of this dissertation.

In the second chapter I present a model of decision-making under uncertainty in which an agent is constrained in her cognitive ability to consider complex acts. The complexity of an act is identified by the corresponding partition of state space. The agent ranks acts according to the expected utility net of complexity cost. I introduce a new axiom called *Aversion to Complexity*, that depicts an agent's aversion to complex acts. This axiom, together with other modified classical expected utility axioms characterizes a *Complexity Aversion Representation*. In addition, I present applications to competitive markets with uncertainty and optimal contract design.

The third Chapter discusses how a complexity averse agent measures the complexity cost of an act after she receives new information. I propose an updating rule for the complexity cost function called *Minimal Complexity Updating*. The idea is that if the agent is told that the true state must belong to a particular event, she needs not consider the complexity of an act outside of this event. The main result characterizes axiomatically the *Minimal Complexity Aversion Representation*. Lastly, I apply the idea of *Minimal Complexity Updating* to the theory of rational inattention.

The last chapter deals with a variant model of fictitious play, in which each player has a perturbation term that measures to what extent his rival will stick to the rules of traditional fictitious play. I find that the empirical distribution can converge to a pure Nash equilibrium if the perturbation term is bounded. Furthermore, I introduce an updating rule for the perturbation term. I prove that if the perturbation term is updated in accordance with this rule, then play can converge to a pure Nash equilibrium.

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(GENERAL AUDIENCE ABSTRACT)

This dissertation explores how people make decisions in various situations. It presents three new models that offer different perspectives from traditional models. The aim is to better understand decision-making processes and their applications in different areas.

In the second chapter, the focus is on decision-making when there is uncertainty or limited information. The model considers how people think about complex options and their preferences. It introduces the idea that people tend to avoid complex choices, and this influences their decision-making. The model has implications for understanding competitive markets and designing optimal contracts.

The third chapter explores how people update their thinking when they receive new information. It proposes a rule called *Minimal Complexity Updating* which suggests that people only consider the complexity of options within certain events or situations. This has implications for understanding how people pay attention to different information.

The final chapter introduces a variation of a common strategic decision-making model called fictitious play. It takes into account how individuals may deviate from the traditional rules of the model. The findings show that if these deviations are limited, the outcomes can still converge to a desirable solution.

Overall, this dissertation offers new insights into how people make decisions in differ-

ent scenarios. It provides practical implications for understanding decision-making under uncertainty, updating beliefs with new information, and considering deviations from traditional models.

Dedication

To my wife, Kaili Qi. I couldn't have done this without her love and support.

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Chapter 1

Overview

In economics, decision-making refers to the process by which individuals make choices among alternative courses of action to allocate scarce resources (physical resources or cognitive resources) in order to achieve their goals. Decision-making in economics can occur at various levels, from the individual choices made by consumers and workers, to the strategic decisions made by firms and the policy choices made by governments. Understanding the decision-making process is crucial to understanding economic behavior and outcomes. This study is conducted to analyze and explain how individuals make decisions in different situations.

1.1 Decision Making with Complexity Cost

In the second chapter I present a model of decision-making under uncertainty in which an agent is constrained in her cognitive ability to consider complex acts. The complexity of an act is identified by the corresponding partition of state space. A partition of state space is called subjective information (Ellis (2018)), subjective contingencies (Minardi and Savochkin (2019)) or conjectures (Gilboa et al. (2022)). There are several studies which provide axiomatic foundations for representations in which an agent has to face the trade-off between complex choices and saving on the cognitive resources of doing so (Puri (2018), Ortoleva (2013), Ergin and Sarver (2010), Valenzuela-Stookey (2020)).

In this model, a complexity averse agent ranks acts according to the expected utility net of complexity cost. I introduce a new axiom called Aversion to Complexity, that depicts an agent's aversion to complex acts. This axiom, together with other modified classical expected utility axioms characterize a Complexity Aversion Representation. My model explains broad experimental and empirical evidence. For example, some empirical studies find that insurance or bond agents tend to advise customers to purchase dominated products. In addition, I consider several applications of the model. For example, the application to optimal contract design shows that if the principal is complexity averse, he tends to design a simple optimal wage scheme.

1.2 Updating Complexity Cost

The third Chapter discusses how a complexity averse agent measures the complexity cost of an act after she receives new information. Here I am not discussing belief updating. Instead, I focus on what kinds of conditional cost function she will use to measure the complexity cost of an act after the occurrence of new information.

I propose an updating rule for the complexity cost function called Minimal Complexity Updating. The minimal complexity updating rule requires that the agent measures the complexity cost of an act conditional on the new information by using the cost of another act that gives exactly the same partition on the new information, but has lowest unconditional complexity cost. The idea is that if the agent is told that the true state must belong to a particular event, she needs not consider the complexity of an act outside of this event. To characterize the behavior of minimal complexity updating, I propose a novel axiom: Minimal Complexity Updating. Moreover, to allow the violation of Dynamic Consistency (See Tversky and Kahneman (1974), Ghirardato (2002), Grether (1992) for more discussions), I suggest a novel axiom, called Dynamic Complexity Aversion. It can be viewed as an introspective reaction when the agent is self-aware of the updating of complexity cost function. The main result characterizes axiomatically the Minimal Complexity Aversion Representation. Lastly, I apply the idea of Minimal Complexity Updating to the theory of rational inattention.

1.3 Learning in Games with Belief Perturbations

The last chapter deals with a variant model of fictitious play, in which each player has a perturbation term that measures to what extent his rival will stick to the rules of traditional fictitious play. Fictitious play is a process of Bayesian learning, in which the behavior of players is assumed stable. First introduced by Brown (1951), more modified models are discussed by Fudenberg and Kreps (1993) and Benaïm et al. (2009), and Hopkins (2002).

I find that unlike the traditional fictitious play, the empirical distribution is not guaranteed to converge to the strict Nash equilibrium according to the model of fictitious play with perturbation. Fortunately, with appropriate restrictions on assessment rules, I find that the empirical distribution can converge to a pure Nash equilibrium if the perturbation term is bounded. Furthermore, I introduce an updating rule for the perturbation term. I prove that if the perturbation term is updated in accordance with this rule, then play can converge to a pure Nash equilibrium.

Chapter 2

Complexity Aversion

2.1 Introduction

2.1.1 Motivation

An underlying assumption of classical expected utility theory, that is not explicitly formed as an axiom in the literature (e.g., Savage (1972), Anscombe and Aumann (1963)), is that there is no contemplation cost when an agent considers an act. However, in reality, there is often a trade-off between choosing a more complex act that leads to more distinct consequences and saving on the cognitive resources (or physical resources) of doing so. To account for this complexity aversion, I propose an axiomatic model that incorporates the complexity costs of acts into the decision-making process.

To illustrate the idea, take the decision problem of individual investment as an example. An investor tries to figure out the amount of investment in each stock in the market. Since there are tens of thousands of stocks in the market, it requires an extremely high level of cognitive efforts to deal with that number of stocks or lots of money paid to consulting managers for professional suggestions. At the other extreme, she only considers acts that determine the same amount of investment on all stocks. This requires a much lower level of cognitive resources. Or, instead, she chooses to focus on several stocks that she is familiar with or has invested in before. This corresponds to acts that determine different levels of investment on selected stocks and no investment on all others. Those kinds of acts need cognitive resources in between the former two cases. It is obvious that there are numerous acts with different level of complexity she can choose. Moreover, after receiving new information[1], the investor might know better which stocks might be worth investing in. Then she can focus on these stocks and is able to choose more complex acts than before.

In this paper, I model a decision maker making a choice under uncertainty who is constrained in her cognitive ability to consider complex acts. I identify to what extent the agent is constrained. The agent is able to identify the complexity of an act based on the corresponding partition of the state space. In the literature, a partition of the state space is described as an information structure[2] (e.g., Ellis (2018)), or an agent's understanding of uncertainty (e.g., Ahn and Ergin (2010), Minardi and Savochkin (2019)). Here, we call it complexity. The complexity costs can be identified from act-choice data.

To illustrates how the complexity costs of acts can play a critical role in decision-making, especially when the consequences of different actions are uncertain or difficult to predict, let's consider a more detailed example of individual investment. Consider an investor who wants to decide her investment plan. A state of the world indicates the state of the economy. Each investment plan is an act, which attaches a payoff to each state. However, there are substantive economic indicators for both microeconomics and macroeconomics. It is very difficult for individual investors to figure out how the economy will perform. As a result, when the investor considers an investment

[1] In this paper, new information is referred to as an event.

[2] It is widely discussed in the rational inattention literature. Our paper also motivated by studies of rational inattention.

plan, she may be limited in her ability to consider every possible outcome associated with each state of the world that the act gives. Even if a sophisticated investment plan offers different outcomes for each state, the investor may be unwilling to choose it because of the high complexity cost of this plan. Consequently, she may choose a simpler investment plan, such as putting money in the bank.

How does an agent like the investor I am describing choose acts? Naturally, when the agent chooses an act, she identifies the complexity of an act f by $\sigma(S^f)$ which is the σ-algebra generated by act f, where S^f is the partition induced by f. f is more complex than g if $\sigma(f)$ is finer than $\sigma(g)$. In words, the agent considers f is more complex than g if she has to consider more distinct outcomes that f gives than g does. Thus, if f is too complex compared with g, she may choose g even when f yields higher expected utility than g.

Example 2.1. To fix the idea, suppose uncertainty is modeled by the state space $\Omega = \{\omega_1, \omega_2, \omega_3\}$, where ω indicates the state of the economy. There are two investment plans available which can be represented as vectors of state-contingent payoffs as follows:

Ω	ω_1	ω_2	ω_3	S
f	1	3	4	$S^f = \big\{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}\big\}$
g	2	2	2	$S^h = \{\Omega\}$

Table 2.1: Investment Example.

Her prior is $\mu = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. It is easy to see that f is more complex than g and f gives higher expected utility than g. However, suppose we observe that the investor chooses g, which is not the choice predicted by SEU. We can infer that the investor chooses g because she cannot bear the complexity cost of f.

I propose an axiomatic model of an agent who is constrained in her cognitive resources to consider complex acts, like the investor above. The primitive is a collection of preferences relations on acts. One of the key aspects of the model is that the agent chooses an act that maximizes expected utility net of complexity cost.

I introduce a new axiom, called Aversion to Complexity, that depicts the agent's aversion to complex acts. Suppose for any acts f, g and constant outcome x, we observe the following preference

$$\alpha x + (1-\alpha)f \sim \beta x + (1-\beta)f,$$

for any $\alpha, \beta \in (0,1)$ with $\alpha > \beta$. That is the agent is indifferent to acts x and f if the agent does not have to consider the complexity cost of f. Since x is the simplest act, if the agent is indeed complexity averse, we anticipate that she will prefer $\lambda x + (1-\lambda)g$ to the randomization over f and g if $\lambda f + (1-\lambda)g$ is more complex than g, i.e., her preference should be

$$\lambda x + (1-\lambda)g \succsim \lambda f + (1-\lambda)g$$

for any $\lambda \in (0,1)$. In other words, since $\lambda f + (1-\lambda)g$ causes at least as high complexity cost as $\lambda x + (1-\lambda)g$, the agent has to devote more cognitive efforts to process it. However, an SEU agent is indifferent between $\lambda x + (1-\lambda)g$ and $\lambda f + (1-\lambda)g$.

The above axiom, together with other modified classical SEU axioms characterizes the representation

$$V(f) = \int_{\omega \in \Omega} u(f(\omega))\mu(d\omega) - \mathcal{C}\big(\sigma(f)\big)$$

where $\mathcal{C}$ is the function used to measure the complexity cost of any act f,[3] u is a utility index and μ is a probability measure over state space. The representation suggests that the agent ranks acts according to expected utility net of complexity cost. Since the agent is self-aware of her cognitive constraint, she compares acts by considering the complexity cost which can be identified by the σ-algebra generated by acts. The formal definition of cost function is given in Definition 2.1. One key feature of this representation is that I do not assume any specific forms of cost function except for monotonicity.

In the above example, I discuss how this model can be used to describe those kind of contexts in which an agent is aware of better choices but constrained in her cognitive ability. This model does not require the observer to know an agent's complexity cost function — which can be identified by the act-choice data.

2.1.2 Experimental and Empirical Evidence

In this subsection, I provide evidence for complexity aversion. Section 2.2 introduces the formal setup. Section 2.3 presents the general model: Axioms that define preference relations and the representation theorem. Section 2.4 includes the discussion of the comparison between different agents. Section 2.5 develops two applications to optimal contract design and competitive markets with uncertainty. Section 2.6 discusses related theoretical literature. Section 2.7 concludes.

Indeed, there are lots of experimental evidence suggesting that agents are complexity averse. Early studies, focusing on testing SEU theories, show that if lotteries have the same number of outcomes, then the experimental data can be well explained

[3]See Section 2.3.2 for more discuss of properties of the complexity cost function.

by SEU. If lotteries have different numbers of outcomes, most of the choices violate SEU (e.g., Conlisk (1989), Harless and Camerer (1991), and Sopher and Gigliotti (1993)). Moffatt et al. (2015) document the evidence that agents do exhibit complexity aversion, but the degree of complexity aversion is decreasing with more rounds of experiment, where complexity is defined in terms of the number of different outcomes in the lottery.[4] In this paper, I introduce an updating rule of complexity cost function which can be used to explain the reduction of complexity aversion. Other similar studies also use the above definition of complexity, such as Huck and Weizsäcker (1999), Sonsino et al. (2002), and Sonsino and Mandelbaum (2001). Instead of lotteries, our model studies preferences over acts.

Regarding evidence out of the lab. Garrod et al. (2008) find evidence that customers' loyalty to one brand may be induced by complexity aversion to various products. Anagol et al. (2017) find that life insurance agents tend to suggest customers buy a dominated product such as whole life insurance instead of a combination of investments. Egan (2019) finds similar evidence in the bond market. They show that brokers advise consumers to purchase dominated bonds.

2.2 Setup

The setup that I adopt is the classical setting of Anscombe and Aumann (1963). The uncertainty is depicted by Ω which is a (nonempty) finite set of states of the world. Elements E, E' in $\Sigma = 2^{\Omega}\backslash\{\emptyset\}$, referred to as *events*. Let $\Delta(\Omega)$ denote the set of all probability measures on Ω. The $\mu \in \Delta(\Omega)$ are called *beliefs*. Let X denote the set of *consequences*, which is assumed to be a convex subset of a vector space (see

[4] There is also other definition of complexity. For example, Mador et al. (2000) and Puri (2018) define complexity as the size of a lottery's support.

Maccheroni et al. (2006)). For example, this is the case if X is the set of all lotteries on a set of prizes.

The following notations are about information partition. Let $\mathbb{P}$ denote the set of partitions of Ω, denoting $S \in \mathbb{P}$ for a partition of Ω, i.e. $S = \{s_1, s_2, \cdots, s_L\}$ with $\cup s_l = \Omega$, $s_l \neq \emptyset$ for all $l \in \{1, \cdots, L\}$, and $s_i \cap s_j = \emptyset$ for all $i \neq j \in \{1, \cdots, L\}$. For any partitions $S, S' \in \mathbb{P}$, I denote by $\sigma(S)$ the algebra generated by partition $S \in \mathbb{P}$, and I say partition S is finer than S' if $\sigma(S') \subset \sigma(S)$.

Let $\mathcal{F}$ denote the set of functions $f : \Omega \to X$, which are referred to as *acts*. That is, an act is a function attaching a consequence to each state of the world, e.g., an amount of money. Let $\mathcal{F}^S$ denote the set of acts that respect the partition S, i.e, $\sigma(f) = \sigma\big(\{f^{-1}(x) : x \in f(\Omega)\}\big) = \sigma(S^f)$ where S^f is the partition corresponding to act f.[5] That is, for any $f \in \mathcal{F}^S$, and any $\omega, \omega' \in E \in S^f$, we have $f(\omega) = f(\omega')$. So we obtain $\mathcal{F} = \bigcup_{S \in \mathbb{P}} \mathcal{F}^S$. Furthermore, for any $f, g \in \mathcal{F}$, if $\sigma(g) \subset \sigma(f)$, I say f is more complex than g. Following a standard abuse of notation, I denote by $x \in \mathcal{F}$ the *constant act* yielding $x \in X$ in every state. $\overline{\mathcal{F}}$ denotes the set of all constant acts. The linear structure of X allows mixtures to be defined as following: for any $f, g \in \mathcal{F}$, and $\alpha \in [0, 1]$, a state-wise mixture of two acts $f, g \in \mathcal{F}$ is $\alpha f + (1 - \alpha)g$ which is identified as $(\alpha f + (1 - \alpha)g)(\omega) := \alpha f(\omega) + (1 - \alpha)g(\omega)$.

[5] With σ-algebra, the model can be extended to a more generalized setting where the state space is infinite.

2.3 Foundations and Representations

2.3.1 Foundations

In this section I introduce behavioral axioms. I start by presenting classical axioms that are widely used in the model of subjective expected utility.

Axiom 2.1 (Weak order)**.** $\succsim$ is reflexive, transitive and complete.

In order to represent the agent's preferences with a utility function, it is essential that her preferences exhibit transitivity and completeness. These properties do not necessarily require the agent to have a full understanding of acts as functions that attach outcomes to states. The Weak Order axiom simply requires that the agent will make a certain choice and will not randomly make such decisions.

Axiom 2.2 (Continuity)**.** For any $f, g, h \in \mathcal{F}$, $\alpha \in [0,1]$, and $\sigma(\alpha f+(1-\alpha)g) = \sigma(f)$, the following sets are closed:

$$\{\alpha \in [0,1] : \alpha f + (1-\alpha)g \succsim h\} \quad \text{and} \quad \{\alpha \in [0,1] : h \succsim \alpha f + (1-\alpha)g\}.$$

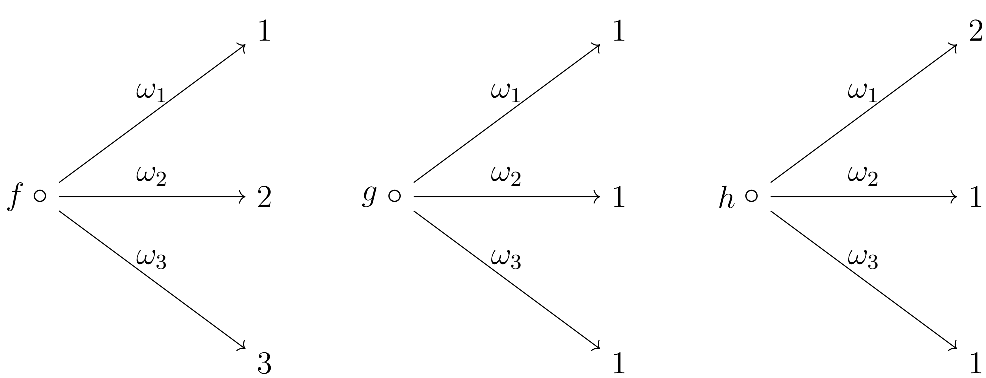

Figure 2.1: Three acts f, g, and h.

Full continuity is too strong in my setting. For example, consider three acts in Figure 2.1. Let's construct two acts $f_n = \frac{1}{n}h + (1 - \frac{1}{n})f$ and $g_n = \frac{1}{n}h + (1 - \frac{1}{n})g$ such that $f_n \succsim g_n$ for all $n \in \mathbb{N}$.[6] It is easy to see that $f_n \to f$ and $g_n \to g$ as $n \to \infty$. If we don't restrict the complexity level of f_n and g_n, it is possible that the agent prefers g to f when $\mathcal{C}(f) \gg \mathcal{C}(g)$. The reason behind this is that the complexity level of the mixture of two acts is ambiguous. In this case, her choice violates continuity.

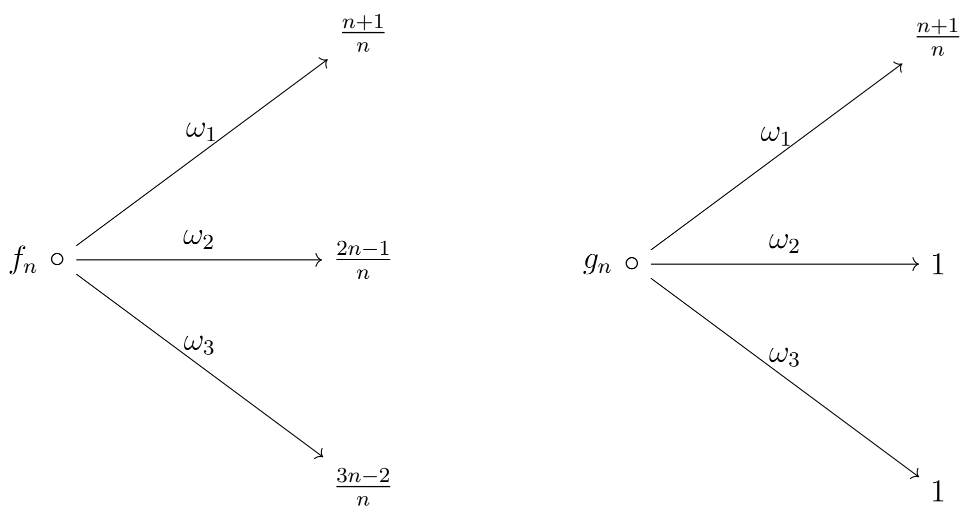

Figure 2.2: Two acts f_n and g_n.

Axiom 2.3 (Weak Certainty Independence)**.** For any $f, g \in \mathcal{F}$, $x, x' \in \overline{\mathcal{F}}$, and $\alpha \in (0, 1)$,

$$\alpha f + (1-\alpha)x \succsim \alpha g + (1-\alpha)x \Longleftrightarrow \alpha f + (1-\alpha)x' \succsim \alpha g + (1-\alpha)x'.$$

See Maccheroni et al. (2006).[7] Before illustrating this axiom, I introduce the

[6] With three states, the complexity cost is $\mathcal{C} = \{\mathcal{C}(\{\Omega\}), \mathcal{C}(\{\{\omega_1, \omega_2\}, \{\omega_3\}\}), \mathcal{C}(\{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}\})\}$. Then the costs of acts are defined trivially.

[7] De Oliveira et al. (2017) and Ergin and Sarver (2010) also discuss this axiom for preferences defined on menus.

standard independence axiom:

Axiom I (Independence). For any $f, g, h \in \mathcal{F}$, and $\alpha \in (0,1)$,

$$f \succsim g \Longleftrightarrow \alpha f + (1-\alpha)h \succsim \alpha g + (1-\alpha)h.$$

Axiom I is too restrictive in my setting. To see this, let's revisit the investment example. Except for f and g, we have another act $h = (3,1,0)$. Suppose we observe $g \succsim f$. By Axiom I, we expect to observe $\alpha g + (1-\alpha)h \succsim \alpha f + (1-\alpha)h$ for any $\alpha \in (0,1)$. However, this is not the case in my model. Let $\alpha = \frac{1}{2}$, then the corresponding partition of $\alpha g+(1-\alpha)h$ is $\{\{\omega_1\},\{\omega_2\},\{\omega_3\}\}$. Similarly, the corresponding partition of $\alpha f + (1-\alpha)h$ is $\{\Omega\}$. It is easy to see that $\alpha f + (1-\alpha)h$ gives higher expected utility than $\alpha g + (1-\alpha)h$ and is less complex than $\alpha g + (1-\alpha)h$. Hence, the agent prefers $\alpha f + (1-\alpha)h$ to $\alpha g + (1-\alpha)h$, which violates Independence. Weak Certainty Independence is more compatible with the agent's behavior in my setting. By the definition of $\sigma(f)$, $\sigma(\alpha f + (1-\alpha)x)$ and $\sigma(f)$ have the same level of complexity for any $f \in \mathcal{F}$ and any $\alpha \in (0,1)$.

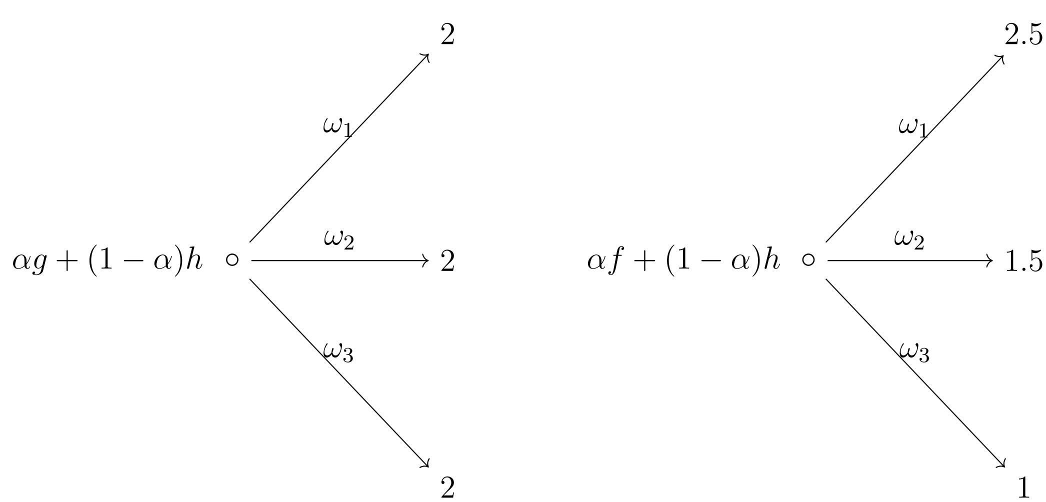

Figure 2.3: Violation of Independence.

Axiom 2.4 (Weak Monotonicity)**.** For any $f, g \in \mathcal{F}$, if $\sigma(f) \subset \sigma(g)$ and $f(\omega) \succsim g(\omega)$ for all $\omega \in \Omega$, then $f \succsim g$.

For the monotonicity axiom, I put an additional restriction compared to the traditional definition of monotonicity. To illustrate the idea, consider the following case. If f is more complex than g, the cost of f is greater than g, thus, $f(\omega) \succsim g(\omega)$ cannot make sure $f \succsim g$. Therefore, I require weak monotonicity. First, if f and g have the same level of complexity, then $f(\omega) \succsim g(\omega)$ for all $\omega \in \Omega$ directly implies $f \succsim g$. Second, if $\sigma(f) \subset \sigma(g)$, that is g is more complex than f, the cost of f is less than g, thus we can apply the weak monotonicity to this case. The violation of monotonicity is documented in many studies such as Birnbaum (2008), and Gneezy et al. (2006)). They show that agents prefer a dominated lottery. This phenomenon cannot be explained by standard expected utility theory. In contrast, the behavior of a complexity averse agent may violate monotonicity and can be characterized by this model.

Notation 2.1. For any act $g \in \mathcal{F}$, let $\mathcal{F}^c(g) = \{f : \sigma(g) \subset \sigma(\lambda f + (1-\lambda)g)$ for any $\lambda \in (0,1)\}$.

Axiom 2.5 (Aversion to Complexity)**.** For any $\alpha, \beta, \lambda \in (0,1)$ with $\alpha > \beta$, any $x \in \overline{\mathcal{F}}$, $g \in \mathcal{F}$, and $f \in \mathcal{F}^c(g)$ we have

$$\alpha x + (1-\alpha)f \sim \beta x + (1-\beta)f \implies \lambda x + (1-\lambda)g \succsim \lambda f + (1-\lambda)g.$$

The challenge of characterizing the agent's attitude to complex acts is to use appropriate axiom to depict her considerations of the complexity cost of acts. In my framework, since the agent has limited cognitive ability to process complex acts, she has to balance the desire for complex acts and the cost of complexity. For any

$\alpha, \beta \in (0,1)$ with $\alpha > \beta$, $\alpha x + (1-\alpha)f \sim \beta x + (1-\beta)f$ implies that the agent is indifferent between x and f if she does not have to consider the complexity cost of f. Then, if she has to choose an act between $\lambda x + (1-\lambda)g$ and $\lambda f + (1-\lambda)g$, she will choose the former one since $\lambda f+(1-\lambda)g$ is more complex than $\lambda x+(1-\lambda)g$. However, an SEU agent would be indifferent between these two acts, if she is indifferent between x and f.

To illustrate this axiom, consider the following modified investment example in Figure 2.4. Now let $f = (1,2,3)$ and $g = (2,2,2)$. It is obvious that acts f and g have the same expected utility given $\mu = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. In other words, the agent is indifferent between f and g if she does not have to consider the complexity cost of f. Suppose she has two more options: $\lambda f+(1-\lambda)h$ and $\lambda g+(1-\lambda)h$, where $h = (4,4,6)$. Again, we know that $\lambda f+(1-\lambda)h$ has the same expected utility as $\lambda g+(1-\lambda)h$. Moreover, it is easy to see that $\sigma(\lambda f + (1-\lambda)h)$ is finer than $\sigma(g)$ for all $\lambda \in (0,1)$. Thus, if the agent is complexity averse, she prefers $\lambda g + (1-\lambda)h$ to $\lambda f + (1-\lambda)h$.[8]

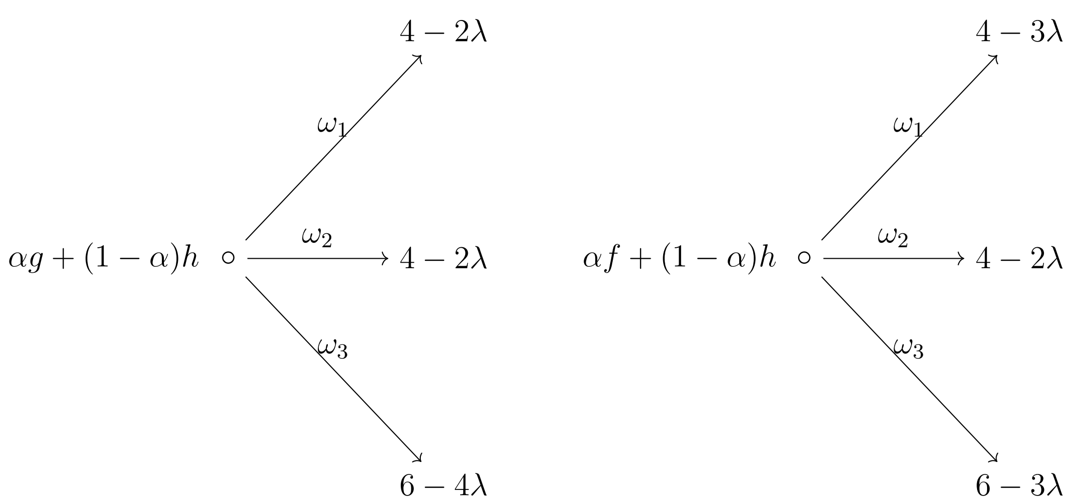

Figure 2.4: An Example illustrating Aversion to Complexity.

[8]By the definition of $\sigma(f)$, $\sigma(\alpha f + (1-\alpha)x) = \sigma(f)$ for any $f \in \mathcal{F}$ and any $\alpha \in (0,1)$.

Axiom 2.6 (Unboundedness)**.** There exist outcomes x and y, with $x \succ y$, such that for any $\alpha \in (0,1)$, there exists an outcome z such that either $y \succ \alpha z + (1-\alpha)x$ or $\alpha z + (1-\alpha)y \succ x$.

Unboundedness axiom implies that preferences over outcomes are unbounded (the range of $u(\cdot)$ is $\mathbb{R}$). $y \succ \alpha z + (1-\alpha)x$ implies $u(y) \geq u(\alpha z + (1-\alpha)x) = \alpha u(z) + (1-\alpha)u(x)$, and $\alpha z + (1-\alpha)y \succ x$ implies $u(\alpha z + (1-\alpha)y) = \alpha u(z) + (1-\alpha)u(y) \geq u(x)$. When α is very close to zero, to keep $x \succ y$, $u(z)$ should be $-\infty$ or ∞. It is used to prove Lemma 2.

Axiom 2.2-2.4 are the assumptions of continuity, independence, and monotonicity, which are the classic conditions paving the way to the subjective expected utility (see Anscombe and Aumann (1963)). Note that Axiom 2.4 and 2.5 require that the agent is able to have a full understanding of acts as functions that attach outcomes to states. With this assumption, the agent is able to identify the complexity of all acts. With Axiom 2.1-2.6, we define the preference on acts in this model.

2.3.2 The Representation

Definition 2.1 (Complexity Aversion Representation)**.** An agent admits a *Complexity Aversion Representation* if there exist

- an unbounded affine utility function $u : X \to \mathbb{R}$;
- a probability measure $\mu \in \Delta(\Omega)$; and
- a complexity cost function $\mathcal{C} : \{\sigma(S) : S \in \mathbb{P}\} \to [0, \infty)$, $\sigma(S) \subset \sigma(S')$ implies $\mathcal{C}(\sigma(S)) \leq \mathcal{C}(\sigma(S')$, and $\mathcal{C}(\{\Omega, \emptyset\}) = 0$.

Such that: $\succsim$ is represented by

$$V(f) = \int_{\omega \in \Omega} u(f(\omega))\mu(d\omega) - \mathcal{C}(\sigma(f)).$$

I refer to the Complexity Aversion Representation as a tuple $\langle u, \mu, \mathcal{C} \rangle$. The utility function u and belief μ have the same interpretations as in the model of SEU. The complexity cost function $\mathcal{C}$ maps each act to an extended real number measuring the cost that the agent has to bear if she chooses that act. If the σ-algebra generated by the corresponding partition of act f is finer than that of act g, then f has higher complexity cost than g.

I am now ready to state the main result : the representation theorem of complexity aversion.

Theorem 2.1. *A preference relation $\succsim$ satisfies Axioms 2.1-2.6 if and only if it admits a Complexity Aversion Representation.*

Theorem 2.1 provides a behavioral foundation for the Complexity Aversion representation. A noteworthy remark on Theorem 2.1 before proceeding. I do not assume any other properties about the complexity cost function except for monotonicity. Some other properties can be imposed on the complexity cost function like convexity. By doing this, appropriate behavioral axioms should be characterized.

The proof is in Appendix A.1. Here I only present a sketch and some discussions of the proof. First, it is obvious that if the expected utility of act f is smaller than that all constant acts, then the agent definitely will not prefer f to x because x is the simplest act that she can choose. Thus, the agent only considers acts that might be preferred to some constant acts. Only for those kind of acts, complexity cost is meaningful. By this observation, I can prove that there exists certainty equivalence for such acts. Then Axiom 2.3 and Axiom 2.5 shows that there exist an affine utility function u with unbounded range and a prior probability measure μ over Ω such that $U(f) = \int_\Omega u(f(\omega))\mu(d\omega)$.

Second, I prove the existence of such complexity cost function $\mathcal{C}$ by construction. We can observe that $f \sim g$ does not mean that the two acts have the same expected utility. Then we can construct an act g by mixing a constant act and act f', where $u(f'(\omega)) = 2u(f(\omega))$. By the definition of σ-algebra, we observe that $\sigma(f) = \sigma(g)$. Therefore, by rearranging the expected utility of g, we construct such complexity cost function $\mathcal{C}$ for act f.

The last step is to show the monotonicity of complexity cost function $\mathcal{C}$. Suppose $\alpha x + (1-\alpha)f \sim \beta x + (1-\beta)f$ for any $\alpha, \beta \in (0,1)$. It is easy to see that x and f have the same expected utility. Then we observe that $\lambda x + (1-\lambda)g$ and $\lambda f + (1-\lambda)g$ also have the same expected utility. By Axiom 2.5, $\lambda x + (1-\lambda)g \succsim \lambda f + (1-\lambda)g$ if $\sigma(g) \subset \sigma(\lambda f + (1-\lambda)g)$ implies $\mathcal{C}(\sigma(g)) \leq \mathcal{C}(\sigma(\lambda f + (1-\lambda)g))$.

2.3.3 Identification

In this subsection, I discuss the uniqueness of a Complexity Aversion Representation. Moreover, I give an example to illustrate the identification of complexity cost function $\mathcal{C}$.

At a glance, the cost function $\mathcal{C}$ seems not be unique. For example, one may say that $\mathcal{C} + c$ ($c \in \mathbb{R}$ is a constant) might represent the same preference relations if it does not change the ordinal ranking over acts. Actually, if $\mathcal{C}$ is defined as in Definition 2.1, it is unique. To see this, for any act $\bar{f} \in \mathcal{F}$, consider a modified act f'

such that $f'(\omega) = \bar{f}(\omega) - \epsilon$ with a very small but positive ϵ for all $\omega \in \Omega$. Suppose we can find another complexity cost function $\mathcal{C}'$ that represents the same preference. The only difference between $\mathcal{C}$ and $\mathcal{C}'$ is that $\mathcal{C}'\big(\sigma(f)\big) > \mathcal{C}\big(\sigma(f)\big)$ for any act $f \in \mathcal{F}^{S^{\bar{f}}}$. Since $\sigma(\bar{f}) = \sigma(f')$, we have $\mathcal{C}\big(\sigma(\bar{f})\big) = \mathcal{C}\big(\sigma(f')\big)$. Moreover, suppose $f' \sim g$ with complexity cost function $\mathcal{C}$. It is easy to see that $\bar{f} \succ g$ with complexity cost function $\mathcal{C}$. Then it is impossible to have $f' \sim g$ with complexity cost function $\mathcal{C}'$. In this case, two complexity cost functions are only different on partition S^f. If two complexity cost functions vary considerably, the preference relations will also vary considerably. Therefore, we have the following uniqueness corollary of complexity cost function $\mathcal{C}$.

Corollary 2.1. *Let $\succsim$ be a complexity aversion preference represented by $\langle u, \mu, \mathcal{C}\rangle$. Then the complexity cost function, defined in Definition 2.1, is unique.*

Corollary 2.2. *If $\langle u, \mu, \mathcal{C}\rangle$ and $\langle u', \mu', \mathcal{C}'\rangle$ represent the same preferences relations, then u' is a positive affine transformation of u, $\mu = \mu'$ and $\mathcal{C}' = \alpha\mathcal{C}$ for some $\alpha > 0$.*

Corollary 2.2 establishes that the agent's utility function, prior, and complexity cost function are unique. It is a standard practice to identify the agent's utility. To identify the cost function, consider two acts $f \in \mathcal{F}$ and $z \in \overline{\mathcal{F}}$ such that $u\big(f(\omega)\big) = u(x) > u(z)$ for all $\omega \in s_1$, $u\big(f(\omega)\big) = u(y) < u(z)$ for all $\omega \in s_2$ and $\{s_1, s_2\} \in \mathbb{P}$. Then the agent prefers f to z whenever $u(f) - \mathcal{C}\big(\sigma(f)\big) > u(z) - \mathcal{C}\big(\sigma(z)\big)$. Suppose $u(y)$ is very close to 0 that is y is a significantly unpreferred outcome. Then she prefers f to z if $u(x) - \mathcal{C}\big(\sigma(f)\big) > u(z) - \mathcal{C}\big(\sigma(z)\big)$. Therefore, we can identify $\mathcal{C}\big(\sigma(f)\big)$ uniquely by $min\{u(x) - u(z) : x \in \overline{\mathcal{F}}\}$ given z.

2.4 Comparative Statics

2.4.1 Comparing the Degree of Complexity Aversion

For the preference $\succsim$, the agent faces a trade-off between her desire for complex acts and their costs. To formalize the notion, let $\langle u^1, \mu^1, \mathcal{C}^1 \rangle$ and $\langle u^2, \mu^2, \mathcal{C}^2 \rangle$ represent two agents' preferences $\succsim^1$ and $\succsim^2$. To compare the cost function, let $(u^1, \mu^1) = (u^2, \mu^2)$.

Definition 2.2. $\succsim^1$ has *a lower degree of complexity aversion* than $\succsim^2$ if for any $f \in \mathcal{F}$ and $x \in \overline{\mathcal{F}}$, $x \succ^1 f$ implies $x \succ^2 f$.

The definition says that the DM1 has a lower degree of complexity aversion than DM2 if: DM1 strictly prefers a constant act x to an act f implies DM2 also strictly prefers x than f. The following result shows that DM1 has a lower degree of complexity aversion than DM2 in terms of the parameters of the Complexity Aversion Representation.

Theorem 2.2. $\succsim^1$ *has a lower degree of complexity aversion than* $\succsim^2$ *if and only if DM1 has lower complexity costs than DM2, that is* $\mathcal{C}^1 \leq \mathcal{C}^2$.

Theorem 2.2 characterizes that if DM1 has a lower degree of complexity aversion than DM2, then her complexity cost $\mathcal{C}^1$ is smaller than $\mathcal{C}^2$. In other words, DM1 needs less efforts to process every act than DM2.

2.4.2 Comparing the Capacity for Complex Acts

The second comparison considers two agents' capacities for complex acts. For the preference $\succsim$, except for the trade-off between her desire for complex acts and their costs, the agent also faces the trade-off between her desire for complex acts and her aversion to randomized acts. Consider two randomized acts $\alpha f + (1-\alpha)g$ and $\alpha f + (1-\alpha)x$. It is easy to understand that act g is more complex than act x, while the former is more random than the latter. Therefore, the DM1 has a higher capacity for complex acts than DM2 if DM1 is more capable of exploring complex acts than DM2.

Definition 2.3. $\succsim^1$ has *a higher capacity for complex acts* than $\succsim^2$ if for any $f, g \in \mathcal{F}$, $x \in \overline{\mathcal{F}}$, and $\alpha \in (0,1)$, $\alpha f + (1-\alpha)x \succ^1 \alpha f + (1-\alpha)g$ implies $\alpha f + (1-\alpha)x \succ^2 \alpha f + (1-\alpha)g$.

This definition says that the DM1 has a higher capacity for complex acts than DM2 if: DM1 strictly prefers $\alpha f+(1-\alpha)x$ to $\alpha f+(1-\alpha)g$ implies DM2 also strictly prefers $\alpha f + (1-\alpha)x$ to $\alpha f + (1-\alpha)g$. The following result characterizes that DM1 has a higher capacity for complex acts than DM2 in terms of the parameters of the Complexity Aversion Representation.

Theorem 2.3. $\succsim^1$ *has a higher capacity for complex acts than* $\succsim^2$ *if and only if* $supp(\mathcal{C}^1) \subset supp(\mathcal{C}^2)$.

$supp(\mathcal{C})$ is the set of acts that the agent would choose. $supp(\mathcal{C}^1) \subset supp(\mathcal{C}^2)$ indicates that DM1 has a a higher capacity to exclude coarser acts than DM2.

2.5 Applications

2.5.1 Design of Contracts Under Moral Hazard

In this section, I discuss an application to optimal contract design of wage scheme. I show that complexity aversion may profoundly change the principals' choice of optimal wage scheme.

The risk neutral principal tends to design an incentive contract to hire a manager (the agent) for a specific project. The agent is risk averse and receives a utility $u(W)$ given wage $W \geq 0$ net of the cost $c(e)$ of effort e. The agent chooses her effort level $e \in \{0, 1\}$. For simplicity, let $c(1) = c > c(0) = 0$. The utility function $u : \mathbb{R}^+ \to \mathbb{R}$ is given by $u(W) = \sqrt{W}$.

The wage scheme $W : \Omega \to \mathbb{R}^+$ is designed based on the agent's performance data, which is modeled as a finite state space $\Omega = \{\omega_1, \omega_2, \omega_3\}$. The probability of observing performance data ω_1 is given by $\mu_e(\omega_1) > 0$, where $e \in \{0, 1\}$. It is naturally to think of these probabilities as being affected by efforts. Note that $\sum_\omega \mu_e(\omega) = 1$ for each effort level e.

I discuss the setting in which the agent's choice of effort cannot be observed by the principal (I refer to Holmström (1979) for more discussion of this setting). The principal keeps seeking the wage scheme that minimize the incentive cost plus the complexity cost. In other words, the principal faces the trade-off between designing a complex contract and saving the design cost. Moreover, suppose the complexity cost of W is given by $\mathcal{C}(\sigma(W)) = 0$ for any W such that $S^W = \{\Omega\}$, and $\mathcal{C}(\sigma(W)) = \delta|S^W|$ otherwise. $|S^W|$ measures the number of elements in the partition induced by W, and $\delta > 0$ measures the degree of complexity aversion.

Since the agent's effort level cannot be observed, the principal must make sure that the optimal wage scheme leads the agent to voluntarily choose desired effort level. Thus, if the principal wishes to induce the agent to exert high effort, the problem is

$$\min_W \sum_\omega \mu_1(\omega)W(\omega) + \delta|S^W| \quad \text{subject to}$$

$$\sum_\omega \mu_1(\omega)\sqrt{W(\omega)} - c \geq \bar{u}, \text{ and} \tag{2.1}$$

$$\sum_\omega \mu_1(\omega)\sqrt{W(\omega)} - c \geq \sum_\omega \mu_0(\omega)\sqrt{W(\omega)}. \tag{2.2}$$

The constraint (2.1) requires that the wage scheme yields the agent at least her reservation utility $\bar{u}$. The constraint (2.2) ensures that the effort level that the principal intends to induce is the same as that actually chosen by the agent. Let $\mu(\omega) = 1 - \frac{\mu_0(\omega)}{\mu_1(\omega)}$, constraint (2.2) can be rewritten as $\sum_\omega \mu_1(\omega)\mu(\omega)\sqrt{W(\omega)} \geq c$.

A complexity averse principal takes two steps to determine the optimal wage scheme. First, he chooses a contract that minimizes the cost for each complexity level. There are three types of contracts in this setting:

Type	contract
Simple contract	$W(\omega) = W \ \forall \omega \in \Omega$
Moderate complex contract	$W(\omega_i) = W(\omega_j) \neq W(\omega_k) \ \forall i, j, k \in \{1, 2, 3\};$
Complex contract	$W(\omega_1) \neq W(\omega_2) \neq W(\omega_3)$

Then, he determines the optimal contract which gives the minimal cost.

Proposition 2.1. *Suppose for any $e \in \{0, 1\}$, $\mu_e(\omega) \neq \mu_e(\omega')$ for any $\omega \in \Omega$ and $\mu_1(\omega) \neq \mu_0(\omega)$ for at least one $\omega \in \Omega$. An optimal wage scheme that induces high* *effort*[9] *exists and*

(1) If $\delta > \left[\frac{\mu_1(\omega_k)(1-\mu_1(\omega_k))}{\mu^2(\omega_k)} + 1\right]c^2$*, the principal chooses an optimal contract that is moderate complex.* $W^*(\omega_i) = W^*(\omega_j) \neq W^*(\omega_k)\ \forall i, j, k \in \{1, 2, 3\}$ *and*

$$\big(W^*(\omega_i), W^*(\omega_k)\big) = \big((\bar{u} - c\mu_1(\omega_k)/\mu(\omega_k))^2, (\bar{u} + c(1 - \mu_1(\omega_k))/\mu(\omega_k))^2\big)$$

(2) If $\delta < \left[\frac{\mu_1(\omega_k)(1-\mu_1(\omega_k))}{\mu^2(\omega_k)} + 1\right]c^2$*, the principal chooses an optimal contract that is complex.* $W(\omega_1) \neq W(\omega_2) \neq W(\omega_3)$*,*

$$W^*(\omega) = \left[\bar{u} - c\mu(\omega)\left(\sum_{\omega} \mu_0(\omega)\mu(\omega)\right)^{-1}\right]^2 \quad \forall \omega \in \Omega.$$

Proposition 2.1 shows that if the principal is complexity averse, he may not choose a complex wage scheme which gives each possible state of performance a different wage. In other words, if the principal is constrained in his ability to analyze all possible states of agent's performance, he tends to design and offer moderate complex wage scheme. Moreover, it is easy to see that as δ increases, the principal is more likely to choose a moderate complex wage scheme.

Proposition 2.2. *Suppose a principal with the degree of complexity aversion* δ *chooses an optimal contract that is moderate complex, then so is one with the degree of complexity aversion* δ' *if* $\delta' \geq \delta$*.*

2.5.2 Competitive Markets with Uncertainty

In this section, I show how the model is embedded in the general equilibrium framework. The model provides an interpretation for extreme prices. Following the gen-

[9]It is easy to see that a simple contract can only induce low effort.

eral equilibrium literature, I represent uncertainty by assuming that endowments and preferences depend on the state of the world.

Formally, consider an economy consisting of I consumers (a typical consumer is denoted $i \in \{1, \cdots, I\}$), and L goods (indexed by $l = 1, \cdots, L$). Again, here we take Ω to be a finite set of states of the world, a typical element is denoted $\omega \in \{\omega_1, \cdots, \omega_n\}$. Let μ^i_ω denote the probability of the state ω (which could be objective or subjective). Then a state-contingent consumption vector of consumer i is specified by

$$c^i = (c^i_{1\omega_1}, \cdots, c^i_{L\omega_1}, \cdots, c^i_{1\omega_n}, \cdots, c^i_{L\omega_n}).$$

It is clearer if we rewrite c^i as $c^i = (c^i_{\omega_1}, \cdots, c^i_{\omega_n})$, where $c^i_{\omega_1} = (c^i_{1\omega_1}, \cdots, c^i_{L\omega_1})$, i.e., c^i_ω is the consumer i's consumption vector under state ω. Similarly, we have consumer i's endowment vector

$$e^i = (e^i_{1\omega_1}, \cdots, e^i_{L\omega_1}, \cdots, e^i_{1\omega_n}, \cdots, e^i_{L\omega_n}),$$

A complexity averse consumer's utility of the consumption vector $c^i \in \mathbb{R}^{LS}_+$ is

$$U(c^i) = \sum_{\omega \in \Omega} u^i_\omega(c^i_\omega)\mu^i_\omega - \mathcal{C}^i(c^i),$$

where $\mathcal{C}^i(\cdot)$ is consumer i's complexity cost function. Here a consumption vector can be view as an act. Preference relations are defined on consumption vectors as what we discussed in section 2.3. With complexity cost function, a complexity averse consumer i is unable to choose every consumption vector that a standard consumer can choose. Note that if we let $\mathcal{C}^i(c^i) = 0$ for all $c^i \in \mathbb{R}^{LS}_+ = C$ and $i \in \{1, \cdots, I\}$, then the classical model can be viewed as a special case in my setting. This also embeds CCE (Gul et al. (2017)) as a special case, where $\mathcal{C}^i(c^i) = 0$ for all crude consumption vectors and $\mathcal{C}^i(c^i) = \infty$ for all others.

Definition 2.4. Given an economy $\mathcal{E}$ specified by $\{u^i, e^i, \mathcal{C}^i, \mu^i\}_{i=1}^I$, an allocation $(\hat{c}^1, \cdots, \hat{c}^I)$, and a price vector $p = (p_{1\omega_1}, \cdots, p_{L\omega_n})$ constitute a complexity averse competitive equilibrium (CACE) if

(1) For every i, $\hat{c}^i \in C^i$ is maximal for $\succsim^i$ in the budget set

$$\{c^i \in C^i : p \cdot c^i \leq p \cdot e^i\};$$

(2) $\sum_i \hat{c}^i = \sum_i e^i$.

Example 2.2. To illustrate the idea, consider the following example of consumption under uncertainty. Consider an economy $\mathcal{E}$ with $I = 3,\ L = 1$, and $\Omega = \{\omega_1, \omega_2, \omega_3\}$. Suppose we have $e^1 = (1, 0, 0),\ e^2 = (0, 2, 0),\ e^3 = (0, 0, 3)$, and utility index of the form $u^i = \ln c^i_\omega$. Thus, an agent's utility of her consumption vector c^i is $U(c^i) = \mu^i_{\omega_1} \ln c^i_{\omega_1} + \mu^i_{\omega_2} \ln c^i_{\omega_2} + \mu^i_{\omega_3} \ln c^i_{\omega_3}$, where $(\mu^i_{\omega_1}, \mu^i_{\omega_2}, \mu^i_{\omega 3}) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ are the agent i's subjective probabilities of the three states.

Since there are three states, consumers could have five different information structures

$$(\{1, 2, 3\}), (\{1\}, \{2\}, \{3\}), (\{1, 2\}, \{3\}), (\{1\}, \{2, 3\}), (\{1, 3\}, \{2\}).$$

Thus, every agent has three types of consumption vectors $C \in \{C_1, C_2, C_3\}$. C_1 means that the agent chooses to make a most complex consumption vector where $c^i_\omega \neq c^i_{\omega'}$ for any $\omega, \omega' \in \Omega$ corresponding to information structure $(\{\omega_1\}, \{\omega_2\}, \{\omega_3\})$;

C_2 is the coarsest consumption vector where $c^i_\omega = c^i_{\omega'}$ for any $\omega, \omega' \in \Omega$ corresponding to information structure $(\{\omega_1, \omega_2, \omega_3\})$; C_3 indicates that the agent is able to make some complex consumption vectors where $c^i_\omega = c^i_{\omega'} \neq c^i_{\omega''}$ for any $\omega, \omega', \omega'' \in \Omega$. Preferences are well defined, so the agent is capable of identifying the cost for all consumption vectors. By assumption, we have $\mathcal{C}(\hat{c}^{ij}) = \mathcal{C}(\hat{c}^{ik})$ $\forall \hat{c}^{ij}, \hat{c}^{ik} \in C_1$, $\mathcal{C}(\bar{c}^{ij}) = \mathcal{C}(\bar{c}^{ik})$ $\forall \bar{c}^{ij}, \bar{c}^{ik} \in C_3$, and $\mathcal{C}(\tilde{c}^{ij}) = \mathcal{C}(\tilde{c}^{ik}) = 0$ $\forall \hat{c}^{ij}, \tilde{c}^{ik} \in C_2$. Moreover, $\mathcal{C}(\hat{c}^i) > \mathcal{C}(\bar{c}^i) > \mathcal{C}(\tilde{c}^i)$ for any $\hat{c}^i \in C_1, \bar{c}^i \in C_3, \tilde{c}^i \in C_2$.

If all consumers choose consumption vectors in C_1, this is the standard competitive equilibrium (SCE), we have $c^* = (c^i_1, c^i_2, c^i_3) = (\frac{1}{3}, \frac{2}{3}, 1)$ for all $i \in I$, $\delta = (1, 0, 0)$, and the prices are given by $p^* = (p_1, p_2, p_3) = (1, \frac{1}{2}, \frac{1}{3})$. Note that when consumer 1 chooses a plan in C_1, her utility is $\frac{1}{3} \ln\left(\frac{(p_1)^2}{27 p_2 p_3}\right) - \mathcal{C}(C_1)$. If consumer 1 chooses a consumption vector where $c^1_1 = c^1_2 \neq c^1_3$, it is easy to verify that her utility is $\frac{1}{3} \ln\left(\frac{4(p_1)^3}{27(p_1+p_2)^2 p_3}\right) - \mathcal{C}(C_3)$. If we let $\mathcal{C}(C_1) - \mathcal{C}(C_3) > \frac{1}{3} \ln(\frac{(p_1+p_2)^2}{4 p_1 p_2})$, then (c^*, p^*) is not an equilibrium any more. Suppose consumer 1 is complexity averse and her best choice is to pick consumption vectors in C_3 with specific cost function $\mathcal{C}^1$, and consumer 2 and 3 are standard agents. Then, we can find a CACE $(\hat{c}, \hat{p}) = \Big(\big((\frac{1}{2}, \frac{1}{2}, 1.2), (\frac{2}{9}, \frac{2}{3}, \frac{4}{5}), (\frac{5}{18}, \frac{5}{6}, 1)\big); \big(1, \frac{1}{3}, \frac{5}{18}\big)\Big)$. There is a significant difference between SCE price p^* and CACE price $\hat{p}$. We can see that $\frac{p_1^*}{p_3^*} < \frac{\hat{p}_1}{\hat{p}_3}$, which indicates that CACE prices might be more extreme than SCE prices.

2.6 Related Theoretical Literature

There are several related strands of theoretical literature. The first are studies which provide axiomatic foundations for representations in which an agent has to face the trade-off between complex choices and saving on the cognitive resources of doing so.

In Puri (2018), an agent dislikes a lottery with more outcomes. Ortoleva (2013) models the behavior of an agent who dislikes lotteries of menus of objects that have a larger number of menus because of the cost of thinking involved in choosing from them. The agent of Ergin and Sarver (2010) considers the cost of contemplation before choosing an object from a menus. Instead of lotteries, I model preferences over acts. Valenzuela-Stookey (2020) introduces a model of preferences inspired by similar considerations to ours. The complexity of an act is measured by the cardinality of the partition induced by the act. In Valenzuela-Stookey's Simple Bounds representation, not well-understood acts are mapped to the set of well-understood acts and then compared by their expected utilities. An act is called well-understood if the number of elements of it's partition equals to a cut-off. The key difference between Valenzuela-Stookey (2020) and this paper is that this paper does not restrict agents' understanding of acts. Instead, in my model, an agent evaluates an act using it's expected utility net of the complexity cost. To illustrate the difference of behavioral implications, consider two acts f and g. f is well understood and its' corresponding partition has two elements; g is not well understood and its' corresponding partition is the state space. In Valenzuela-Stookey (2020), if the simple greatest lower bound of f is preferred to the simple least upper bound of g, then f is preferred to g. However, in my model, the agent might prefer g to f because of the complexity cost of f.

Second are papers which focus on the interactive decision making situation. Neyman (1985) studies the finitely repeated game in which only strategies that use a bounded number of states in the automaton available to players. Ben-Porath (1993), and Megiddo and Wigderson (1986) follow this approach. Instead of limiting the set of strategies, Abreu and Rubinstein (1988) assume that more complex strategies means higher costs. Rubinstein (1986), and Abreu and Rubinstein (1988) restrict players' strategies in a repeated game to those implementable by finite state automata. Mengel (2012) studies the learning process of two players who face many games. Since it requires too much cognitive resources to distinguish all games, players choose to partition the set of all games into categories. My setting is closest to Abreu and Rubinstein (1988).

I use the corresponding partition of an act to measure complexity which is interpreted as coarse understanding of the state space. However, the agent in my model fully understands the state space. Here, the coarseness is induced by her constraints in cognitive ability to consider the complexity of acts. I discuss two strands of related studies on coarse contingencies and show how these models differ from my model.

The first are studies which focus on coarse understanding and ambiguity. The works most relevant to my model are Ahn and Ergin (2010) and Epstein et al. (2007).[10]

Ahn and Ergin (2010) proposes a model of decision making under uncertainty in which the primitive is a class of preference relations indexed by partitions of the state space. If two acts respect the same partition, then the agent ranks the two acts by their expected utilities based on her partition-dependent belief. The agent in my model behaves similarly but with respect to partition-independent belief (prior). How does the agent in the model of Ahn and Ergin (2010) ranks two acts that respect different partitions, e.g., two acts f and g in the investment example? They define that the agent compares f and g by computing their expected utilities based on the coarsest common refinement of S^f and S^g. In the investment example, the coarsest common refinement of S^f and S^g is S^f. If the corresponding partition-dependent belief is $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, she prefers f to g. However, in my model, a complexity averse

[10]For more details about subjective contingencies and ambiguity, see Dekel et al. (2001); Dillenberger et al. (2014); Ghirardato (2001); Minardi and Savochkin (2019); Mukerji (1997); and Saponara (2022).

agent prefers g to f. From above example, we can conclude two main different aspects between my model and theirs. First, a complexity averse agent's belief has nothing to do with partitions. Second, a complexity averse agent compares two acts by computing expected utility net of complexity cost. She does not try to identify two acts in the same partition.

Epstein et al. (2007) models an agent who forms some contingencies and is self-aware of the coarseness of these contingencies. They show that coarse contingencies induce a preference for hedging, as in the ambiguity aversion studies.[11] The key difference between my model and Epstein et al. (2007) is that in the latter paper the coarse contingencies are exogenous. In this model, the coarseness is endogenous and is induced by her aversion to complexity. Another noteworthy remark is that, complexity aversion does not induce a preference for hedging. In my setting, the mixture of two acts might increase or decrease the complexity level, thus, the agent in this model does not exhibit uncertainty aversion (the key axiom in the model of Gilboa and Schmeidler (1989)).

Second are papers about rational inattention which is introduced by (Sims, 1998, 2003). In this model, attention cost is interpreted as the expected difference between the prior uncertainty about the state and the posterior uncertainty. And this theory has been applied to many economic problems.[12] However, I am more interested in axiomatic models of inattention. De Oliveira et al. (2017) provide an axiomatic characterization of rationally inattentive preferences over menus. Ellis (2018) introduces a representation of preferences similar to theirs but takes a choice correspondence as a primitive. Other related studies are Dillenberger et al. (2014); Lu (2016). However,

[11]For more studies about ambiguity aversion, see Gilboa and Schmeidler (1989); Schmeidler (1989).

[12]For instance, there are many studies that apply to consumption-savings problems: Sims (2006); Maćkowiak and Wiederholt (2015). There are also many studies that apply the model to the theory of price setting: Luo (2008); Stevens (2012); Maćkowiak and Wiederholt (2009).

no studies discuss the situation in which the agent reallocates her attention after the arrival of new information. This paper fills this gap.

There are two key differences between complexity aversion and rational inattention. First, in my model, the corresponding partition of an act is only used to measure the complexity cost of this act. A complexity averse agent has full attention and directly chooses acts both at ex-ante and ex-post stage. Instead, an optimal inattentive agent chooses what to pay attention to at the ex-ante stage, she is unable to make the decision before the arrival of information. Second, the corresponding partition of an act, chosen by a complexity averse agent, is not the same information partition that an optimal inattentive agent will choose. To see this, suppose that a complexity averse agent chooses an act f at ex-ante stage such that the corresponding partition is $\{\{\omega_1\}, \{\omega_2, \omega_3, \omega_4\}, \{\Omega_5\}\}$. But, an inattentive agent might have a more accurate understanding of f, since $\{\{\omega_1\}, \{\omega_2, \omega_3\}, \{\omega_4\}, \{\Omega_5\}\}$ gives the same expected utility with f. In other words, a complexity averse agent does not try to find an optimal partition.

2.7 Conclusion

Since Simon (1978), there are plenty of studies on the theory of bounded rationality. All axiomatic models take acts as cost free choice over all stages of a decision problem. However, numerous evidence shows that the agent has limited cognitive resources to measure acts.

Based on this motivation, I discuss, and axiomatically characterize a model of decision-making under uncertainty in which an agent is constrained in her cognitive ability to consider complex acts. I identify the complexity of an act according to the corresponding partition of state space. The agent ranks acts according to the expected utility net of complexity cost. The main result shows the equivalence of the Complexity Aversion Representation with the novel behavioral axiom, Aversion to Complexity, together with other classical SEU axioms.

Moreover, I provide comparative analysis of the model and show that an agent with lower complexity cost has a lower degree of complexity aversion. Finally, I conclude by presenting some possible applications. In section 2.5.1, I apply the theory to optimal contract design. It shows that if the principal is constrained in his ability to analyze all possible states of agent's performance, he tends to design and offer simple wage scheme. But for a standard principal, who does not have to consider the complexity cost, he will offer the most complex contract that gives each possible performance data a different wage. This application shows that the complexity aversion model provides insights to help understand why many employers try to sort employees' performance data into categories, and offer matched wage contracts. Besides what we discussed in section 2.5.2, the model could be applied to trading in financial markets (e.g., the investment example). However, more assumptions or restrictions of complexity cost function are needed for further discussion. Finally, a challenging but interesting application is to study the strategic interaction between complexity averse players. It is complicated because the cost function is payoff-related private information.

Chapter 3

Minimal Complexity Updating

3.1 Introduction

Suppose an investor who wants to decide her investment plan. A state of the world indicates the state of the economy. Each investment plan is an act, which attaches a payoff to each state. However, there are substantive economic indicators for both microeconomics and macroeconomics. It is very difficult for individual investors to figure out how the economy will perform. Thus, a complexity averse investor is constrained in her ability to consider complex acts that give many different outcomes. Based on the model that is discussed in Chapter 2, a complexity averse investor prefers g to f in the investment example. Now, suppose that the investor will receive a report about the state of the economy which indicates that the true state must lie in E.[1] How should the investor measure the complexity cost of an act after the arrival of new information? This is the problem that I will address in this chapter.

In this chapter, I model how the complexity averse agent measures the cost of acts after the reduction of uncertainty. I propose a novel conditional complexity cost function and provide axiomatic foundations for this updating rule of the complexity cost function. In my model, preference reversal is not explained by belief updating

[1] Note that we assume the analyst (or we call observer, consulting manger) and the agent (the investor) have the same and correct understanding of the report.

but instead by the agent's updated complexity cost function. When the agent is told that the true state must belong to a particular event, she needs not consider the complexity of an act outside of this event. Thus, if the complexity of two acts differs only outside of an event, then the complexity costs are the same for the two acts after the arrival of new information. If an agent measures the complexity cost of an act by means of a conditional complexity cost function (conditional on the new information), we say that she is sensitively complexity averse. Otherwise, although she discards all irrelevant states after receiving the new information, she still uses the same level of cognitive efforts to process acts. Her choice under this circumstance can be viewed as arising from cost-measuring biases.

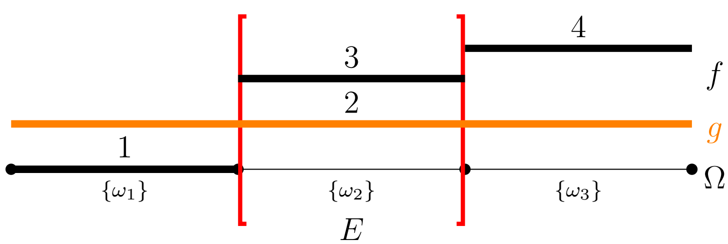

Figure 3.1: Revisiting Investment Example 2.1.

Revisiting Example 2.1 Let's consider the Example in Figure 3.1. Acts f and g are the same as in Investment Example 2.1. But here, suppose that the investor receives a report about the state of the economy which indicates that the true state must lie in $E = \{\omega_2\}$. Now, after the arrival of this new information, we may observe that f is chosen. Therefore, we observe preference reversal which cannot be explained by SEU. It is obvious that $S^{f|E} = S^{g|E} = \{\omega_2\}$, where $S^{f|E}$ is the partition on E induced by f. Thus, we can infer that the investor uses a new cost function to measure the complexity cost of acts after receiving the new information. That is, with the reduction of uncertainty, she is able to process some complex acts that she could not deal with before.

To model the investor's ex post choice, I propose a possible updating procedure to deal with incoming information. The agent has to decide how to measure the complexity cost of acts after receiving the new information. In my setting, the agent precisely knows that all states outside of an event E can be discarded after the occurrence of E.[2] Then, it is intuitive to reason that the agent does not have to consider the complexity of an act f outside of E. From this point of view, given $E \in \Sigma'$ and $\mu \in \Delta(\Omega)$, we suggest the following conditional complexity cost function:

$$\mathcal{C}_{E,\mu}\big(\sigma(f)\big) = min\big\{\mathcal{C}\big(\sigma(h)\big)/\mu(E) : h \in \mathcal{F}, \text{ and } \sigma(h|E) = \sigma(f|E)\big\},$$

where $\mathcal{F}$ is the set of all acts and $\sigma(f|E)$ is the σ-algebra generated by act $f \in \mathcal{F}$ on E. $\mathcal{C}$ is the complexity cost function that is defined in Definition 2.1. If an agent's conditional complexity cost function is formed as above, I say that she is a minimal complexity updater. The minimal complexity updating rule indicates that the agent measures the complexity cost of an act f conditional on E by using the cost of act h that gives exactly the same partition as f on E, but has lowest cost on Ω.

Then, the conditional preference $\succsim_E$ is represented by

$$V(f|E) = \int_{\omega\in\Omega} u(f(\omega))\mu_E(d\omega) - \mathcal{C}_{E,\mu}\big(\sigma(f)\big),$$

where μ_E is the Bayesian update of μ conditional on E and $\mathcal{C}_{E,\mu}$ is the conditional cost function we discussed above. The agent constructs her conditional preferences by two updating rules. First, she updates her prior beliefs by Bayes' rule. Second, she updates her measure of complexity costs of acts by minimal complexity updating

[2] This is stated as an axiom in the standard expected utility theory, which is well known as Consequentialism.

rule.

To identify the behavior of a complexity averse agent after the arrival of new information, I introduce the following key axiom, called Minimal Complexity Updating. Suppose for any non-null event E, $f \in \mathcal{F}$ and $x, z \in \bar{\mathcal{F}}$, we observe the following preference

$$fEz \succsim_E x,$$

That is the agent prefers fEz to x conditional on E. Then given another act fEz', where $\sigma(fEz) \subset \sigma(fEz')$. If the agent is a minimal complexity updater, we anticipate that her preference between fEz' and x should be

$$fEz' \succsim_E x.$$

The two acts fEz and fEz' are the same on E, but fEz' is more complex than fEz on Ω. If the agent uses a minimal complexity cost function to measure the complexity cost of an act after the arrival of new information, the above two acts should be assigned the same complexity cost.

A stream of literature that is strongly related to our paper is the research on preference reversal and dynamic consistency. Most of the theoretical literature focuses on belief-based biases. All of these papers suggest different limited-Bayesian updating rules. In the model of Epstein (2006) and Epstein et al. (2008) the primitive is a class of preferences over menus. Their model incorporates the idea of temptation and self-control ((Gul and Pesendorfer, 2001, 2004)). The behavioral implication is that the agent is eager to maximize his expected utility using the Bayesian update of his new prior at the interim stage. To resist this temptation, he behaves as though using the Bayesian update of a compromise measure lies between priors before and after receiving the signal. Caplin and Leahy (2001) consider a two-period model in which the agent's utility is defined over both prizes and psychological states and studies the role of anticipatory feelings. Ortoleva (2012) presents a decision model in which agents' preferences admit *hypothesis testing representation*. Before the arrival of new information, the agent chooses the prior according to the likelihood of the prior over priors. After the arrival of new information, he compares the probability of the new information to a threshold. If the probability is below the threshold she will update her prior over priors conditional on the new information; then chooses the new prior to which the updated prior over priors assigns the highest likelihood. There are also lots of experimental evidence suggesting dynamic inconsistent behavior, such as Kahneman et al. (1982), Griffin and Tversky (1992), Grether (1992), Holt and Smith (2009). And some of the documented violations provide evidence that agents significantly overweight the new information of low probabilities and underweight the new information of high probabilities. For example, Grether (1992) and Holt and Smith (2009). This paper introduces an updating rule on cost function that explains the behavior of dynamic inconsistency.

It is well known that Consequentialism and Dynamic Consistency imply Bayes' rule. However, the minimal complexity updating rule suggests that the agent's behavior will depart from Dynamic Consistency. Thus, instead of Dynamic Consistency, we suggest a novel axiom, called Dynamic Complexity Aversion.

Given any act g, we can construct $h = gEx$ such that $\mathcal{C}(h) \leq \mathcal{C}(gEx')$ for any constant outcome x'. Suppose we observe the agent's preference to be

$$fEx \succsim gEx,$$

where gEx is the act g outside of E but yields x for all $\omega \in E$. *Dynamic Complexity Aversion* requires that she would not prefer g to f if E actually happens, i.e. her conditional preference should be

$$f \succsim_E g.$$

Therefore, Dynamic Complexity Aversion allows for violations of Dynamic Consistency.

A noteworthy remark is that, I can impose different structures on the conditional complexity cost function. By doing this, appropriate behavioral properties (axioms) should be formalized.

In the above example, I discuss how my model can be used to describe those kind of contexts in which agents are aware of better choices after receiving the new information. The example shows the importance of new information that enables a complexity averse agent remeasures the complexity of an act by means of a conditional complexity cost function.

Outline In section 3.2, I introduces the formal setup. section 3.3 proposes a possible conditional complexity cost function and presents the general model: the axioms that define preference relations and the representation theorem. section 3.4, I apply the model to rational inattention. section 3.5 concludes.

3.2 Setup

There is a (nonempty) finite set Ω of states of the world, and elements E, E' in $\Sigma = 2^{\Omega} \backslash \{\emptyset\}$, referred to as *events*. I denote by $\Delta(\Omega)$ the set of all probability measures on Ω. The $\mu \in \Delta(\Omega)$ are called *beliefs*. Let X denote the set of *consequences*, which is assumed to be a convex subset of a vector space (see Maccheroni et al. (2006)). For example, this is the case if X is the set of all lotteries on a set of prizes (this is the classical setting of Anscombe and Aumann (1963)).

The following notations are about information partition. Let $\mathbb{P}$ denote the set of partitions of Ω, denoting $S \in \mathbb{P}$ for a partition of Ω, i.e. $S = \{s_1, s_2, \cdots, s_L\}$ with $\cup s_l = \Omega$, $s_l \neq \emptyset$ for all $l \in \{1, \cdots, L\}$, and $s_i \cap s_j = \emptyset$ for all $i \neq j \in \{1, \cdots, L\}$. For any partitions $S, S' \in \mathbb{P}$, I denote by $\sigma(S)$ the algebra generated by partition $S \in \mathbb{P}$, and I say partition S is finer than S' if $\sigma(S') \subset \sigma(S)$.

Let $\mathcal{F}$ denote the set of functions $f : \Omega \to X$, which are referred to as *acts*. That is, an act is a function attaching a consequence to each state of the world, e.g., an amount of money. Let $\mathcal{F}^S$ denote the set of acts that respect the partition S, i.e, $\sigma(f) = \sigma\big(\{f^{-1}(x) : x \in f(\Omega)\}\big) = \sigma(S^f)$ where S^f is the partition corresponding to act f.[3] That is, for any $f \in \mathcal{F}^S$, and any $\omega, \omega' \in E \in S^f$, we have $f(\omega) = f(\omega')$. So we obtain $\mathcal{F} = \bigcup_{S \in \mathbb{P}} \mathcal{F}^S$. Furthermore, for any $f, g \in \mathcal{F}$, if $\sigma(g) \subset \sigma(f)$, I say f is more complex than g. Similarly, for any $f \in \mathcal{F}$, and any $E \in \Sigma$, I use $\sigma(f|E)$ to denote the σ-algebra that is generated by act f conditional on the event E, and denote by $S^{f|E}$ the corresponding partition of act f conditional on E. Following a standard abuse of notation, I denote by $x \in \mathcal{F}$ the *constant act* yielding $x \in X$ in every state. $\overline{\mathcal{F}}$ denotes the set of all constant acts. For any two acts $f, g \in \mathcal{F}$, and for any event $E \in \Sigma$, let fEg denote the act that returns $f(\omega)$ for every $\omega \in E$, and returns $g(\omega)$ for every $\omega \in \Omega \backslash E$. The linear structure of X allows mixtures to be defined as following: for any $f, g \in \mathcal{F}$, and $\alpha \in [0, 1]$, a state-wise mixture of two acts $f, g \in \mathcal{F}$ is $\alpha f + (1-\alpha)g$ which is identified as $(\alpha f + (1 - \alpha)g)(\omega) := \alpha f(\omega) + (1 - \alpha)g(\omega)$. Additionally, for

[3] With σ-algebra, the model can be extended to a more generalized setting where the state space is infinite.

any $f \in \mathcal{F}$ and $E \in \Sigma$, let $f(E) := \{x : f(\omega) = x \text{ and } \omega \in E\}$.

The primitive is a class of preference relations $\{\succsim_E\}_{E\in\Sigma}$ on $\mathcal{F}$. The agent observes the realization of event $E \in \Sigma$, then chooses an act that maximizes the conditional expected utility. Correspondingly, $\succsim_\Omega$ (the ex ante preference relation) is the case when the agent receives no information, or before he receives it.

3.3 Foundations and Representations

3.3.1 Foundations

In this section, I discuss the behavioral implications of an agent who is a minimal complexity updater. The first axiom, Conditional CAR, consists of standard complexity aversion axioms, which are all discussed in Section 2.3.1, Chapter 2.

Axiom 3.1 (Conditional CAR)**.** For all $E \in \Sigma$, $f, g, h \in \mathcal{F}$ and $x, x' \in \bar{\mathcal{F}}$.

(i) **(Weak Order)** $\succsim_E$ is reflexive, transitive and complete.

(ii) **(Continuity)** For any $\sigma(\alpha f + (1-\alpha)g|E) = \sigma(f|E)$, the following sets are closed:

$$\{\alpha \in [0,1] : \alpha f + (1-\alpha)g \succsim_E h\} \quad \text{and} \quad \{\alpha \in [0,1] : h \succsim_E \alpha f + (1-\alpha)g\}.$$

(iii) **(Weak Certainty Independence)** For any $\alpha \in (0,1)$,

$$\alpha f + (1-\alpha)x \succsim_E \alpha g + (1-\alpha)x \Longleftrightarrow \alpha f + (1-\alpha)x' \succsim_E \alpha g + (1-\alpha)x'.$$

(iv) **(Weak Monotonicity)** If $\sigma(f|E) \subset \sigma(g|E)$ and $f(\omega) \succsim_E g(\omega)$ for all $\omega \in \Omega$, then $f \succsim_E g$.

(v) **(Aversion to Complexity)** For any $\alpha, \beta, \lambda \in (0,1)$ with $\alpha > \beta$, and $\sigma(g|E) \subset \sigma(\lambda f' + (1-\lambda)g|E)$, we have

$$\alpha x + (1-\alpha)f' \sim_E \beta x + (1-\beta)f' \implies \lambda x + (1-\lambda)g \succsim_E \lambda f' + (1-\lambda)g.$$

(vi) **(Unboundedness)** There exist outcomes x and y, with $x \succ y$, such that for any $\alpha \in (0,1)$, there exists an outcome z such that either $y \succ \alpha z + (1-\alpha)x$ or $\alpha z + (1-\alpha)y \succ x$.

Notice that for parts (ii), (iv) and (v), the agent compares the complexity of two acts conditional on the new information E. However, since constant acts have zero complexity cost, it is unnecessary to consider the new information when the agent ranks two constant acts.

In next part, I discuss a possible "updating rule" that an agent could use when she receives the new information. To avoid confusion, I have to point out that I am not discussing belief ($\mu \in \Delta(\Omega)$) updating. Instead, I focus on what kinds of conditional cost function she will use to measure the complexity cost of an act when E occurs.

Recall the standard definition of a *null* event: for any preference relation $\succsim$, an event $E \subseteq \Omega$ is called $\succsim$-null if $fEg \sim g$ for any acts $f, g \in \mathcal{F}$. In the expected utility framework, null events have zero probability. Since the way of a complexity averse agent ranking acts is different from an SEU agent, here I present a new version of the definition of a *null* event: for any preference relation $\succsim$, an event $E \subseteq \Omega$ is called $\succsim$-null if there exist no $f \in \mathcal{F}$ and $x, y \in \overline{\mathcal{F}}$ such that $xEf \succ yEf$. This definition is more compatible with my setting because constant acts x and y have the same level of complexity on E.

The following three axioms impose dynamic properties on the agent's preference relations which are related to how the agent processes the non-null events in Σ'. First, I introduce Consequentialism. In the SEU setting, Consequentialism is the axiom that guarantees that the preference conditional on $E \in \Sigma'$ does not depend on how act f behaves outside of E. In other words, the agent believes that the true state must lie in E and she is indifferent between two acts that differ only outside of E.

Axiom 3.2 (Consequentialism)**.** For any $E \in \Sigma'$, and any $f, g \in \mathcal{F}$, if $f(\omega) = g(\omega)$ for all $\omega \in E$, then $f \sim_E g$.

To some extend, this standard axiom is enough to characterize the behavior of agents in this model. To see this, consider two acts f and g such that f is more complex than g on Ω. Since $f(\omega) = g(\omega)$ for all $\omega \in E$, we have $\sigma(f|E) = \sigma(g|E)$. Thus, if the agent measures the cost of f and g on E instead of Ω after the occurrence of E, that is, the conditional complexity cost of f is the same as that of g, then she is indifferent between f and g conditional on the realization of E. I discuss the conditional complexity cost function below.

Before that, I turn to discuss the relations between ex ante preferences and ex post preferences. In the standard setting, the preference is required to be dynamically consistent.

Axiom DC (Dynamic Consistency). For any $E \in \Sigma'$, and any $f, g \in \mathcal{F}$, we have $f \succsim_E g \iff fEg \succeq g$.

Dynamic Consistency requires that the ex ante preference over acts implies the agent's ex post preference. In particular, if the agent prefers f in E to g before the arrival of new information, then if E happens, the agent still prefers f to g.[4] However,

[4] Ghirardato (2002)) provides more discussion of this axiom and its implications. He proves that Consequentialism and Dynamic Consistency imply that the agent is a Bayesian updater.

if the agent is complexity averse, things are different. For example, consider two acts f and g in $\mathcal{F}$, such that $\int_{\omega\in\Omega} u(f(\omega))\mu(d\omega) \leq \int_{\omega\in\Omega} u(g(\omega))\mu(d\omega)$ and $\sigma(f) \subset \sigma(g)$. Suppose she prefers f to g, which means although g gives higher expected utility than f, the complexity cost of act g is too high. Then, after the arrival of new information, it is possible that g still gives higher expected utility than f on E, but with lower complexity cost on E. Under this circumstance, she prefers g to f after she receives the new information. Thus, standard dynamic consistency will not apply. Before I introduce the axiom, consider the following definition of conditional complexity cost function.

Definition 3.1. Given $E \in \Sigma'$ and $\mu \in \Delta(\Omega)$, a conditional complexity cost function $\mathcal{C}_{E,\mu} : \{\sigma(f) : f \in \mathcal{F}\} \longrightarrow \mathbb{R}_+ \cup \{\infty\}$ is *minimal* if $\mathcal{C}_{E,\mu}\big(\sigma(f)\big) = min\{\mathcal{C}\big(\sigma(h)\big)/\mu(E) : h \in \mathcal{F},\ \text{and}\ \sigma(h|E) = \sigma(f|E)\}$.[5]

By Axiom 3.2, we know that when the agent receives the new information $E \in \Sigma'$, she believes that the true state must lie inside E. Then she does not have to care about the complexity of an act outside of E. Therefore, after the arrival of E, she can choose an act that gives the same $\sigma(f|E)$ as f on E but with lowest costs on Ω, and use $\mathcal{C}\big(\sigma(h)\big)$ to measure the cost of $\mathcal{C}_{E,\mu}(\sigma(f))$.

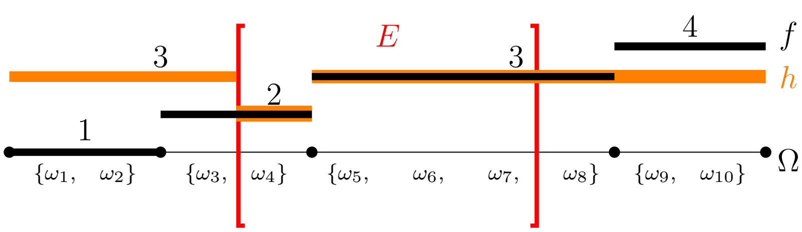

Figure 3.2: Example of Minimal Complexity Updating.

Example 3.1. Consider the following example in Figure 3.2. Act f gives 4 distinct outcomes (represented by black dashes) for all states in Ω. Now suppose event E is

[5]Here, $\mu(E)$ is a technique term that plays a role in the proof of Theorem 3.1.

realized, so we can see that f gives 2 distinct outcomes for states in E. Let $S^{f|E}$ denote the partition generated by f on E, i.e., $S^{f|E} = \{\{\omega_4, \{\omega_5, \omega_6, \omega_7\}\}\}$. It is obvious that element $\{\omega_5, \omega_6, \omega_7\}$ has more states than any any other elements in E (top right black dash in E). If the agent uses a minimal conditional complexity cost function, then she would choose $h = fEx$ where $x \in f(E)$ and $f(\omega) = x$ for all $\omega \in \{\omega_5, \omega_6, \omega_7\}$ (represented by orange dashes) and use use $\mathcal{C}\big(\sigma(h)\big)$ to measure the cost of $\mathcal{C}_E(\sigma(f))$.

Notation 3.1. For any act $f \in \mathcal{F}$ and $E \in \Sigma'$, let $\mathcal{F}^{min}(f, E) = \big\{h : h \in \mathcal{F}(f, E)$ and $\sigma(h) \subset \sigma(h')$ for all $h' \in \mathcal{F}(f, E)\big\}$, where $\mathcal{F}(f, E) = \{h : h \in \mathcal{F}$ and $\sigma(h|E) = \sigma(f|E)\}$.

The following axiom describes the behavior of a complexity averse agent after the arrival of new information.

Axiom 3.3 (Minimal Complexity Updating)**.** Given any $E \in \Sigma'$, $f \in \mathcal{F}$, and $x, z \in \overline{\mathcal{F}}$. For any $z' \in \overline{\mathcal{F}}$ such that $\sigma(fEz) \subset \sigma(fEz')$, we have

$$fEz \succsim_E x \Longrightarrow fEz' \succsim_E x.$$

This axiom concerns the agent's attitude toward the complexity cost of an act f outside of event E. To see the behavioral implications of minimal conditional complexity cost function, consider an act $f \in \mathcal{F}$. If the agent believes that the true state must lies in E, she is indifferent between fEz and fEz' where $\sigma(fEz|E) = \sigma(fEz'|E)$ for any $z, z' \in \overline{\mathcal{F}}$. As such, if we observe the preference $fEz \succsim_E x$ where $x \in \overline{\mathcal{F}}$, then we would expect that a more complex (on Ω) act fEz' is still preferred to x.

I am now ready to be back to discuss the violation of Dynamic Consistency. To illustrate this behavioral postulate, consider the following two cases for $f, g \in \mathcal{F}$ and $x \in \overline{\mathcal{F}}$:

$$case1 : f \succsim g \text{ and } fEx \succsim gEx,$$
$$case2 : g \succ f \text{ and } fEx \succsim gEx.$$

In case 1, the preference between f and g is consistent with the preference f and g on E. However, in our model, case 1 requires more specific structures of cost function, e.g., $\mathcal{C}\big(\sigma(g)\big) - \mathcal{C}\big(\sigma(f)\big) = \mathcal{C}\big(\sigma(gEx)\big) - \mathcal{C}\big(\sigma(fEx)\big)$. Case 2 violates Dynamic Consistency (e.g., the investment example). Even with a minimal conditional complexity cost function, we cannot exclude case 2 in this model. Instead of Dynamic Consistency, I introduce an axiom called Dynamic Complexity Aversion that aligns with an agent's minimal conditional complexity cost function.

Axiom 3.4 (Dynamic Complexity Aversion)**.** For any $E \in \Sigma'$, $f, g \in \mathcal{F}$, and $h \in \mathcal{F}^{min}(g, E)$, there exists $x \in g(E)$ such that $h = gEx$. We have:

$$fEx \succsim gEx \Longrightarrow f \succsim_E g.$$

Here I restrict gEx in $\mathcal{F}^{min}(g, E)$, in words, gEx is the simplest act that gives the same $\sigma(g|E)$ as g on E. If we observe the preference $fEx \succsim gEx$, by axiom A8, the agent will use $\mathcal{C}\big(\sigma(fEx_f)\big)$ $\big($ where $fEx_f \in \mathcal{F}^{min}(f, E)\big)$ to measure the cost of $\mathcal{C}_{E,\mu}(\sigma(f))$ when E is realized. Since $\sigma(fEx_f) \subset \sigma(fEx)$, fEx must be more costly than fEx_f, we must have that $f \succsim_E g$. Dynamic Complexity Aversion can be viewed as an introspective reaction when the agent is self-aware of the updating of complexity cost function.[6] Broadly speaking, Dynamic Complexity Aversion is neither weaker nor stronger than Dynamic Consistency. Although it allows for some violations of Dynamic Consistency, it restricts agents' behavior according to minimal complexity updating.

3.3.2 The Representation

Definition 3.2 (Minimal Complexity Aversion Representation)**.** An agent admits a *Minimal Complexity Aversion Representation* if there exist

- an unbounded affine utility function $u : X \to \mathbb{R}$;
- a probability measure $\mu \in \Delta(\Omega)$; and
- a complexity cost function $\mathcal{C} : \{\sigma(S) : S \in \mathbb{P}\} \to [0, \infty)$, $\sigma(S) \subset \sigma(S')$ implies $\mathcal{C}(\sigma(S)) \leq \mathcal{C}(\sigma(S')$, and $\mathcal{C}(\{\Omega, \emptyset\}) = 0$;
- a minimal conditional complexity cost function $\mathcal{C}_{E,\mu}$ defined in Definition 3.1.

Such that:

(i) $\succsim$ is represented by

$$V(f) = \int_{\omega \in \Omega} u\big(f(\omega)\big)\mu(d\omega) - \mathcal{C}\big(\sigma(f)\big);$$

[6] Dynamic Complexity Aversion indicates that the agent is introspective. She knows that a new complexity cost function will be used after the arrival of E. Thus, she pretends that she will measure the complexity cost of an act by using conditional complexity cost function at ex ante. If she finds that she prefers fEx to gEx, then she learns that she will still prefer f to g after the occurrence of E.

(ii) For any $E \in \Sigma'$, $\succsim_E$ is represented by

$$V(f|E) = \int_{\omega \in \Omega} u(f(\omega))\mu_E(d\omega) - \mathcal{C}_{E,\mu}\big(\sigma(f)\big).$$

$\langle u, \mu, \mathcal{C}, \{\mathcal{C}_{E,\mu}\}_{E \in \Sigma'} \rangle$ is referred to as the Minimal Complexity Aversion Representation. The utility function u, belief μ, and the complexity cost function $\mathcal{C}$ have the same interpretations as in the model of CAR. The conditional complexity cost function $\mathcal{C}_{E,\mu}$ measures the cost of acts that the agent has to bear after the occurrence of E. If the σ-algebra generated by the corresponding partition of act f conditional on E is finer than that of act g, then f has higher conditional complexity cost than g. Above three axiom, along with conditional CAR axioms that are discussed in Section 2.3.1, leads to the following representation theorem.

Theorem 3.1. *An ex ante preference relation $\succsim$ and a collection $\{\succsim_E\}_{E \in \Sigma'}$ of conditional preference relations jointly satisfy Axioms 3.1-3.4 if and only if they admit a Minimal Complexity Aversion Representation.*

Theorem 3.1 provides a behavioral foundation for the Minimal Complexity Aversion Representation. Note that minimal complexity updating does not make further assumptions on the agent's understanding of acts and uncertainty. The standard axiom Consequentialism has already characterized the agent's attitude to the new information E.

3.3.3 Uniqueness

I show that the complexity cost function $\mathcal{C}$ is unique in Section 2.3.3. Furthermore, the following corollary shows that the conditional complexity cost function $\mathcal{C}_{E,\mu}$ shares the same properties as $\mathcal{C}$ and is also unique.

Corollary 3.1. *Let* $(\succsim, \succsim_E)_{E\in\Sigma'}$ *be a minimal complexity aversion preference represented by* $\langle u, \mu, \mathcal{C}, \mathcal{C}_{E,\mu}\rangle$*. The complexity cost function* $\mathcal{C}$ *satisfies (i)* $\mathcal{C}(\{\Omega, \emptyset\}) = 0$*; (ii) monotonicity. Then* $\mathcal{C}_{E,\mu}$*, defined as in Definition 3.1, also satisfies above two properties and is unique.*

Corollary 3.2. *If* $\langle u, \mu, \mathcal{C}, \{\mathcal{C}_{E,\mu}\}_{E\in\Sigma'}\rangle$ *and* $\langle u', \mu', \mathcal{C}', \{\mathcal{C}'_{E,\mu'}\}_{E\in\Sigma'}\rangle$ *represent the same preferences relations, then* u' *is a positive affine transformation of* u*,* $\mu = \mu'$*,* $\mathcal{C}' = \alpha\mathcal{C}$ *and* $\mathcal{C}'_{E,\mu'} = \alpha\mathcal{C}_{E,\mu}$ *for some* $\alpha > 0$*.*

Corollary 2.3 shows that the agent's utility function, prior, the complexity cost function, and the conditional complexity cost function are unique.

3.4 Application to Attention Reallocation

3.4.1 Introduction

There are two types of axiomatic models of inattention: signal-based model and partition-based model. In the signal-based model (e.g., Caplin and Dean (2015), De Oliveira et al. (2017), Matějka and McKay (2015)), the agent has a prior that represents her initial beliefs. Each possible realization of the signal induces a corresponding posterior belief via Bayes' rule. That is, each signal induces a distribution over posteriors. The characterization of the preference requires that the agent chooses a signal that maximizes the expected utility net of attention cost. It means that the agent chooses a signal that determines what she will choose in each state. In the partition-based model the agent has to deal with a two-stage decision problem. First, she chooses what information she tends to pay attention to (Ellis (2018) calls this *subjective information*). Then, after the arrival of new information, she chooses the act that maximizes her expected utility conditional on the realized part of her subjective information. The two types of models have a key distinction that indicates different behavior implications. In the signal-based model, the agent is constrained to choose signals. Once a signal is chosen, the agent's decision is determined. In contrast, the agent of Ellis (2018) cannot information. She chooses an information partition that determines her decision of acts. However, no research discusses the situation in which the agent reallocates her attention after the arrival of new information.

If we say that the act chosen conditional on the new information is the "correct" one, then because of limited attention, the behavior of choosing act conditional on the realized part of subjective information can be regarded as making mistakes. In this paper, we suggest that the agent is self-aware that herself has an attention constraint. Thus, when new information arrives, although she cannot improve her cognitive ability in a short period of time to use the new information to make a decision, she can update her subjective state space, which leads to a new set of subjective information that is more compatible with the arrival information. I discuss the theory of attention reallocation under the setting of Ellis (2018). I extend the existing studies in following ways. I propose a modified framework of choice under uncertainty to model inattentive decision maker where the agent reallocates her attention after she receives new information. She is unable to precisely observe the new information, however, she could use some identification strategies to keep or discard possibilities. Then, she can pay attention to a finer information partition with the decreasing of uncertainty. Therefore, she can make a decision closer to the "correct" one.

3.4.2 Optimal Inattention

In this section, I introduce the model of optimal inattention and discuss how can we incorporate the idea of minimal complexity updating into this model.

The agent's choices are defined by *a conditional choice correspondence*. The agent chooses any acts in $c(B|E)$ when the new information is E. Ellis (2018) characterizes axiomatically the following Optimal Inattention Representation

$$\hat{S}^B \in \arg\max_{S \in \mathbb{P}} \left[\sum_{E \in S} \mu_\Omega(E) \max_{f \in B} \int_{\omega \in \Omega} u(f(\omega)) \mu_E(d\omega) - \mathcal{C}(\sigma(S)) \right] \tag{3.1}$$

$$c(B|E) = \arg\max_{f \in B} \int_{\omega \in \Omega} u(f(\omega)) \mu_{\hat{S}^B(\succsim_E)}(d\omega) \tag{3.2}$$

where choice problem $B \subset \mathcal{F}$ is a finite subset of acts, $\hat{S}^B \in \mathbb{P}$ is her *subjective information* when facing the decision problem B given $\mu \in \Delta(\Omega)$. The agent faces a two-stage decision problem. At the first stage (Equation 3.1), the agent chooses her *subjective information* $\hat{S}^B$. After the realization of event E, she subjectively chooses the set of possible states $\hat{S}^B(\succsim_E)$ based on her *subjective information*. Then she makes the decision conditional on $\hat{S}^B(\succsim_E)$.

The way an inattentive agent processes new information is different from an agent who has full attention. Namely, when an event occurs, an agent who has no inattention problem could discard all other states that are not in the new event, and choose an act conditional on this event. However, an inattentive agent could not pay full attention to the new event, she might only ambiguously perceive some elements in her subjective information that contains the new event. An inattentive agent can be regarded as myopic, unable to observe the new event clearly. To illustrate the idea, consider the following investment example 3.

Example 3.2. Suppose there are 8 possible states of the economy. Assume that the agent chooses her *subjective information* $\hat{S}^B = \{s_1, s_2, s_3, s_4, s_5,\}$ at the first stage. If the agent is a standard expected utility maximizer, then after the arrival of new information $E = \{\omega_3, \omega_4, \omega_5\}$, she could make her decision conditional on the event E. If the agent is inattentive, she actually could not pay full attention to the event E. The best she can do is to discard s_1 and s_5. Therefore, the main difficulty here is to depict the agent's behaviors about choosing what is possible after the occurrence of the new event.

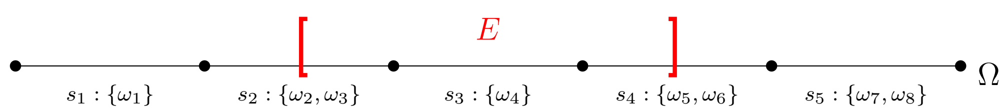

Figure 3.3: Investment Example 3.2.

Before discussing the strategies that the agent will use to choose what is possible after the arrival of new information, I introduce the following notation. Recall that I denote by $\sigma(S)$ the algebra generated by partition $S \in \mathbb{P}$, that is, given $S \in \mathbb{P}$, $\sigma(S)$ is the family of events obtained by taking unions of elements in S.

Definition 3.3. For any $B \in \mathcal{F}$ and $E \subseteq \Omega$, given $\hat{S}^B$, let $\hat{S}^B(\succsim_E) \in \sigma(\hat{S}^B)$ denote the set of states that the agent treats possible after the realization of E. A possibility selection rule is inattentive if $\hat{S}^B(\succsim_E) \in argmin\{|S^B(\succsim_E)| : S^B(\succsim_E) \in \sigma(\hat{S}^B), E \subseteq S^B(\succsim_E)\}$.

In the structure of SEU, $\hat{S}^B(\succsim_E) = E$ for any $E \in \Sigma$, the agent treats a state ω as impossible after the occurrence of event E if $\omega \notin E$. However, the inattentive agent might treat subjectively a state ω as possible after the occurrence of event E, even if ω is not in E objectively. For instance, $\hat{S}^B(\succsim_E)$ can be $s_1 \cup s_2 \cup s_3 \cup s_4$ or

$s_2 \cup s_3 \cup s_4 \cup s_5$ in Figure 3.3 example. This is the key difference between an inattentive agent and SEU agent, which implies how an inattentive agent processes an event E.

A noteworthy remark is that, here I do assume a possibility selection rules. The notation imposes a particular structure on the agent's behavior when choosing what is possible after the arrival of new information. It requires that the agent will discard all irrelevant cells of her subjective information. I argue that this does make sense, since I assume that the agent is self-aware that she has an attention constraint.

Continue Example 3.2. After the occurrence of event E, the agent treats $\hat{S}^B(\succsim_E) = \{\omega_2, \omega_3, \omega_4, \omega_5, \omega_6\}$ as possible. Since states ω_1 and ω_7 are regarded as impossible, her new subjective state space is $\{\omega_2, \omega_3, \omega_4, \omega_5, \omega_6\}$. It is reasonable to infer that she will try to reallocate her attention on her new subjective state space. Consider another information partition $\bar{S}^B = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4, \omega_5, \omega_6\}, \{\omega_7\}, \{\omega_8\}\}$ which is finer than $\hat{S}^B$ on Ω, but has the same level of complexity as $\hat{S}^B$ on E. If she reallocate her attention to $\bar{S}^B$, $\bar{S}^B(\succsim_E) = \{\omega_3, \omega_4, \omega_5, \omega_6\}$, which means she can take another step and keep discarding ω_2. In other words, it is possible that an inattentive agent could keep discarding state ω_6 by reallocating her attention and get $S^B(\succsim_E) = E$.

In next section, I formalize the attention reallocation model and present two main results. The first result shows that an inattentive agent can have a more accurate reasoning about the event E by reallocating her attention to a new subjective information. The second result shows that under some circumstances, it is impossible to discard any states by attention reallocation.

3.4.3 Model and Results

In the model of Minimal Attention Reallocation, an inattentive agent faces a T-stage decision problem.

- At time $t = 1$, she chooses her *subjective information* S_1^B (the same as the behavior of agents in the model of optimal inattention, here I rewrite Equation 3.1 as Equation 3.3 but with the same meaning).

$$S_1^B \in \arg\max_{S \in \mathbb{P}} \left[\sum_{E \in S} \mu_\Omega(E) \max_{f \in B} \int_{\omega \in \Omega} u(f(\omega)) \mu_E(d\omega) - \mathcal{C}(\sigma(S)) \right] \tag{3.3}$$

At time $T = 2$, she receives the new information E and selects $S_1^B(\succsim_E)$ as the set of possible states. Then, she can choose a new subjective information to pay attention to. Before proceeding, we need to figure out two things. First, does she still choose subjective information in $\mathbb{P}$? Second, how does she measure the attention cost after the arrival of new information? To answer the first question, consider the following definition. $\mathbb{P}(\hat{S}^B(\succsim_E))$ requires that she will only consider information partitions that are compatible with what she considers possible after the realization of E.

Definition 3.4. For any $B \in \mathcal{F}$ and $E \subseteq \Omega$, given $\hat{S}^B$, let $\mathbb{P}(\hat{S}^B(\succsim_E)) \in \mathbb{P}$ denote the set of partitions that the agent might choose after the realization of E. An attention reallocation rule is adaptive if for all $S^B \in \mathbb{P}(\hat{S}^B(\succsim_E))$, $\cup s = \hat{S}^B(\succsim_E)$ for some cells s in S^B.

The minimal conditional attention cost function deals with the second question. Since all states outside of $S_1^B(\succsim_E)$ are regarded as impossible, she does not have to consider the attention cost of those states for the new subjective information. Therefore, when she try to measure her new subjective information S, she can choose an information partition R that gives the same information structure as S on $S_1^B(\succsim_E)$ but with lowest attention costs on Ω.

Definition 3.5. Given $E \in \Sigma'$ and $\mu \in \Delta\Omega$, for any $B \in \mathcal{F}$, a conditional attention cost function $\mathcal{C}_{E,\mu} : \{\sigma(S) : S \in \mathbb{P}\} \longrightarrow [0, +\infty)$ is *minimal* if $\mathcal{C}_{E,\mu}\big(\sigma(S)\big) = min\big\{\mathcal{C}\big(\sigma(R)\big)/\mu\big(S(\succsim_E)\big) : R \in \mathbb{P},\text{ and } \sigma\big(R|S(\succsim_E)\big) = \sigma\big(S|S(\succsim_E)\big)\big\}$

- At time $t = 2$, she chooses S_2^B as her stage2 subjective information (Equation 3.4).

$$S_2^B \in \arg \max_{S\in\mathbb{P}(S_1^B(\succsim_E))} \left[\sum_{E\in S} \mu_{S_1^B(\succsim_E)}(E) \max_{f\in B} \int_{E\in\Omega} u(f(\omega))\mu_E(d\omega) - \mathcal{C}_{E,\mu}(\sigma(S))\right] \quad (3.4)$$

where $\mu_{S_1^B(\succsim_E)}(E) = \frac{\mu(E\cap S_1^B(\succsim_E))}{\mu(S_1^B(\succsim_E))}$, and $\mathcal{C}_{E,\mu}\big(\sigma(S)\big) = min\big\{\mathcal{C}\big(\sigma(R)\big)/\mu\big(S_B^1(\succsim_E)\big) : R \in \mathbb{P},\text{ and } \sigma\big(R|S_B^1(\succsim_E)\big) = \sigma\big(S|S_B^1(\succsim_E)\big)\big\}$.

- Based on S_2^B, she selects $S_2^B(\succsim_E)$ as the set of possible states for stage2. The agent will keep repeat this procedure until no states can be discarded. Suppose she will stop at stage$T$ (Equation 3.5).

$$S_T^B \in \arg \max_{S\in\mathbb{P}(S_{T-1}^B(\succsim_E))} \left[\sum_{E\in S} \mu_{S_{T-1}^B(\succsim_E)}(E) \max_{f\in B} \int_{E\in\Omega} u(f(\omega))\mu_E(d\omega) - \mathcal{C}_{E,\mu}(\sigma(S))\right] \quad (3.5)$$

- Finally, she makes the decision conditional on $S_T^B(\succsim_E)$ (Equation 3.6).

$$c(B|E) = \arg \max_{f\in B} \int_{\omega\in\Omega} u(f(\omega))\mu_{S_T^B(\succsim_E)}(d\omega) \quad (3.6)$$

Revisiting Example 3.2. I illustrate the attention reallocation rule and minimal conditional attention cost function by Example 3.2.

Recall that at stage1, the agent chooses $\hat{S}^B$. After the occurrence of event $E = \{\omega_3, \omega_4, \omega_5\}$, the agent selects $\hat{S}^B(\succsim_E) = \{\omega_2, \omega_3, \omega_4, \omega_5, \omega_6\}$ as possible. Then she chooses to reallocate her attention to information partitions in $\mathbb{P}(\hat{S}^B(\succsim_E))$, e.g., $\{\{\omega_1, \omega_7, \omega_8\}, \{\omega_2\}, \{\omega_3, \omega_4, \omega_5\}, \{\omega_6\}\}$ and $\{\{\omega_1, \omega_7\}, \{\omega_2, \omega_3\}, \{\omega_4, \omega_6\}, \{\omega_5\}, \{\omega_8\}\}$. At stage2, suppose she reallocates her attention to information partition $\bar{S}^B = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4, \omega_5, \omega_6\}, \{\omega_7\}, \{\omega_8\}\}$. The attention cost of $\bar{S}^B$ is measured by information partition $R = \{\{\omega_1, \omega_4, \omega_5, \omega_6, \omega_7, \omega_8\}, \{\omega_2\}, \{\omega_3\}\}$. By Definition 3.3, she selects $\bar{S}^B(\succsim_E) = \{\omega_3, \omega_4, \omega_5, \omega_6\}$ as possible. Compared to $\hat{S}^B(\succsim_E)$, she achieves a more accurate understanding of event E by reallocating her attention to a new subjective information. She repeats the process until no states can be discarded.

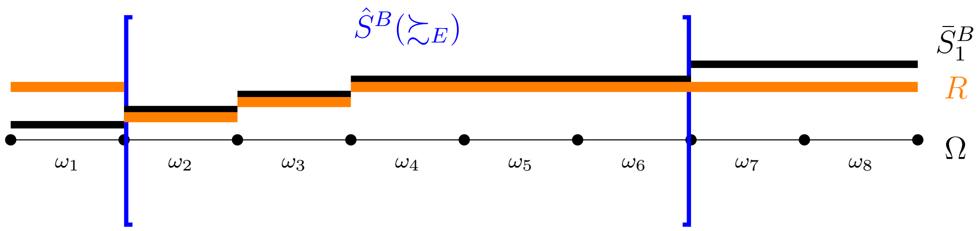

Figure 3.4: Revisiting Investment Example 3.2.

The first result shows that the agent can discard more irrelevant states by attention reallocation under some appropriate decision rules. Proposition 3.1 indicates that an inattentive agent is able to achieve a more sophisticated understanding of new information by attention reallocation. The key difference between the Attention Reallocation Representation and the Optimal Inattention representation is that subjective information is not just an information partition but a “belief” for the agent with Attention Reallocation Representation. She can use the new information to update her subjective information. In the model of optimal inattention, the agent makes a decision based on her subjective information. There is actually no “belief” updating, the agent does not try to have a more accurate understanding of the state space. This is not reasonable, because if we all assume that inattention stems from redundant information, the agent must be self-aware of her inattention problem. Then why she does not try to do something to overcome the problem after the arrival of new information? My model provides an updating rule so that the agent can update her subjective information to try to overcome the inattention problem.

Proposition 3.1 (Possibility)**.** *Given attention cost function, if i) the possibility selection rule is inattentive; ii) the attention reallocation rule is adaptive; and iii) the conditional attention cost function is minimal, then there exists some $B \in \mathcal{F}$, $E \in \Sigma'$ and $\mu \in \Delta(\Omega)$, such that $S_T^B(\succsim_E) \subset S_1^B(\succsim_E)$.*

Proof. Suppose for some $B \in \mathcal{F}$, $E \in \Sigma'$ and $\mu \in \Delta(\Omega)$, her stage1 subjective information is

$$S_1^B = \{\{\omega_1\}, \{\omega_2, \omega_3\}, \cdots, \{\omega_l\}, \cdots\},\ l \in \{5, \cdots, n\},$$

where n is the number of states. Let $E = \{\omega_3, \cdots, \omega_{n-1}\}$, then her possibility set is

$$S_1^B(\succsim_E) = \{\omega_2, \cdots, \omega_{n-1}\}.$$

Consider an information partition

$$S_2^B = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3, \cdots, \omega_{n-1}\}, \{\omega_n\}\}.$$

The agent uses the attention cost of $R = \{\{\omega_2\}, \{\omega_1, \omega_3, \cdots, \omega_n\}\}$ to measure the cost of S_2^B, the cost of which is very close to zero. That is

$$\mathcal{C}_{E,\mu}(S_1^B) \gg \mathcal{C}_{E,\mu}(S_2^B).$$

Therefore, the agent might choose S_2^B at the stage2. So we have $S_2^B(\succsim_E) \subset S_1^B(\succsim_E)$.

□

The following proposition provides some insights about under what circumstances, it is impossible to discard any states by attention reallocation. First, it shows that if the degree of uncertainty is lower, then attention reallocation cannot induce a more precise understanding of new information. Second, if the new information cannot help the agent discard any irrelevant states, the agent is unable to reallocate her attention to a new subjective information after the arrival of E.

Proposition 3.2 (Impossibility)**.** *Given $B \in \mathcal{F}$, for any $E \in \Sigma'$ and $\mu \in \Delta(\Omega)$, if i) $n \leq 3$ or ii) $S_1^B = \{\Omega\}$, then we must have $S_T^B(\succsim_E) = S_1^B(\succsim_E)$.*

Proof. It is obvious for *ii*). For *i*), it is easy to see the case when $n \leq 2$. Consider the case $n = 3$, that is $\Omega = \{\omega_1, \omega_2, \omega_3\}$. Suppose her stage1 subjective information is

$$S_1^B = \{\{\omega_1, \omega_2\}, \{\omega_3\}\}.$$

Let $E = \{\omega_2\}$. It is obvious that she will discard ω_3. So we have

$$\mathbb{P}(S_1^B(\succsim_E)) = \{S_1^B, S_1^2\} = \Big\{\{\{\omega_1, \omega_2\}, \{\omega_3\}\}, \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}\}\Big\}.$$

Then all she has to do is to compare the value of the two information partitions by Equation 3.4. By Definition 3.3, we have $\mathcal{C}_{E,\mu}(\sigma(S_1^B)) = \mathcal{C}(\sigma(\{\Omega\})) = 0$ and $\mathcal{C}_{E,\mu}(\sigma(S_2^B)) = \mathcal{C}(\sigma(S_1^B))$. Let $V^i(S)$ denote the value of the information partition

$S \in \mathbb{P}$ at stage$i$ for all $i \in \{1, \cdots, T\}$. Thus, we have

$$\begin{aligned}
V^2(S_1^B) - V^2(S_2^B) &= \max_{f \in B} u(f(\omega_1)) \frac{\mu(\omega_1)}{\mu(\{\omega_1, \omega_2\})} + \max_{f \in B} u(f(\omega_2)) \frac{\mu(\omega_2)}{\mu(\{\omega_1, \omega_2\})} - \mathcal{C}_{E,\mu}(\sigma(S_1^B)) \\
&- \frac{\mu(\omega_1)}{\mu(\{\omega_1, \omega_2\})} \max_{f \in B} u(f(\omega_1)) - \frac{\mu(\omega_1)}{\mu(\{\omega_2, \omega_2\})} \max_{f \in B} u(f(\omega_2)) + \mathcal{C}_{E,\mu}(\sigma(S_2^B)) \\
&= \max_{f \in B} u(f(\omega_1)) \frac{\mu(\omega_1)}{\mu(\{\omega_1, \omega_2\})} + \max_{f \in B} u(f(\omega_2)) \frac{\mu(\omega_2)}{\mu(\{\omega_1, \omega_2\})} \\
&- \frac{\mu(\omega_1)}{\mu(\{\omega_1, \omega_2\})} \max_{f \in B} u(f(\omega_1)) - \frac{\mu(\omega_1)}{\mu(\{\omega_2, \omega_2\})} \max_{f \in B} u(f(\omega_2)) + \mathcal{C}(\sigma(S_1^B)) \\
&= V^1(S_1^B) - V^1(S_2^B) + 2\mathcal{C}(\sigma(S_1^B)) > 0
\end{aligned}$$

Therefore, the agent will still choose S_1^B at the second stage. □

3.5 Conclusion

In this paper, I propose and axiomatically characterize an updating rule for the complexity cost function called Minimal Complexity Updating. The minimal complexity updating rule requires that the agent measures the complexity cost of an act conditional on the new information by using the cost of another act that gives exactly the same partition as that act on new information, but has lowest unconditional complexity cost. The idea is that if the agent is told that the true state must belong to a particular event (the new information), she does not have to consider the complexity of an act outside of this event. With this updating rule, the agent is able to choose a more accurate act and make fewer mistakes. Agents' choices at all stages are consistent and preferences at every stage are all well defined over acts.

The main result shows the equivalence of the Minimal Complexity Aversion Representation with two novel behavior axioms, Minimal Complexity Updating and Dynamic Complexity Aversion, together with other CAR axioms. Moreover, I apply our theory to the model of optimal inattention and show that show that with our conditional attention cost function, an inattentive agent can reallocate her attention after the occurrence of an event.

Chapter 4

Fictitious Play with Belief Perturbations

4.1 Introduction

The basic solution concept for strategic games, Nash equilibrium, and its refinements such as perfection, are well understood and widely applied. Then the most important question is when and why the strategies played will correspond to one of these equilibria and what these equilibria are. Based on the assumptions that the environment of the game (including belief formation rules, the set of actions of players, and the payoff functions of players) and the rationality of players are all common knowledge, the outcome will correspond to an equilibrium. However, these theories entail a number of problems.

First, it is common for a game to have multiple equilibria, but a crucial question arises: Do players expect the same equilibrium? If they do not, the strategies played may not correspond to any of the equilibria. One possible solution is to restrict players to follow a common selection procedure (Harsanyi et al. (1988)). But again, does this procedure become common knowledge? Second, rational decisions and behavior are not guaranteed in many games. Individuals tend to follow some simple adaptive rules or heuristics rather than to engage in perfectly rational behavior. Lastly, equilibrium theory does a poor job in explaining the shift of play from non-equilibrium to equilibrium in repeated games or in the long-run.

Learning theory offers a different approach to study behavior in games, particularly in games that are played repeatedly. It assumes that players form beliefs of what strategies their rivals will choose based on some specific rules regarding past plays, and these beliefs are then used to maximize current payoffs. However, several questions arise. First, does play converge in the long-run? Second, in some cases, the time needed for convergence is quite long so cannot be empirically observed, or the learning processes need not converge in other cases. Thus, the question arises of how learning theory can address these problems. (For example, Börgers and Sarin (2000).) Third, because of the existence of perturbations and mutations, even when the learning processes converge to a Nash equilibrium, can it be stable for a long time? (Fudenberg and Levine (1994); Monderer et al. (1997)).

Cournot (1838) and Bertrand (1883) duopoly suggest specific learning rules in static simultaneous move games. Brown (1951) introduces a model, which is known as fictitious play. In this kind of game, the basic assumption is that the distribution of opponents' strategies is stable but unknown. At the beginning of the game, each player makes an arbitrary choice, with no learning taking place. Subsequently, players can observe and keep track of the frequency of different strategies played by their opponents, and choose their actions in each period to maximize that period's expected payoff given their assessment of the distribution of opponents' actions in that period. This approach has been applied to a wide range of games and has provided useful insights into the dynamics of strategic interactions.

There are considerable literature on the convergence properties and modifications of fictitious play. Miyasawa (1961) and Robinson (1951) prove that fictitious play always converges for any 2×2 zero-sum game. Fudenberg and Kreps (1988) study fictitious play in extensive form games. Fudenberg and Kreps (1993) introduce payoff shocks into the model to explain why mixed behavior occurs. Fudenberg and Levine (1995) propose a variation of fictitious play, where the probability of an action is proportional to an exponential function of that action's utility against the historical frequency of opponents' play.

Fictitious play is a process of Bayesian learning, in which the behavior of players is assumed stable. Milgrom and Roberts (1991) introduce the concept of "adaptive learning". They show that the sequence of plays converges to Nash equilibrium if and only if strategies that were not played have zero probability. Kalai and Lehrer (1993) show that if players assign positive probability to rivals' true strategies, then Bayesian Updating will always lead to precise prediction of future play.

All existing models of fictitious play do not allow belief perturbations. The behavior in the learning process is assumed stable. However, the true behavior is not stationary. Even though some models have introduced a degree of randomness, for example, Fudenberg and Kreps (1993) introduce payoff perturbations (Smooth Fictitious Play), players actually do not learn to what extent their rivals will stick to the rules when the game is repeated.[1]

This paper proposes a variant model of fictitious play that allows players to have a prior regarding rivals' rationality. I extend the traditional model in two ways:

1) I discuss some potential extensions of the behavior rule and assessment rule. For traditional fictitious play, the behavior rule is expected utility maximization and players are assumed to be rational and mutually aware of this fact (Aumann and

[1] More researches about Smooth Fictitious Play can be found here: Benaım and Hirsch (1999), Benaïm et al. (2009), Ellison and Fudenberg (2000), Hofbauer (2001), Hofbauer and Hopkins (2005), Hofbauer and Sandholm (2002), Hopkins (1999), and Hopkins (2002).

Brandenburger (1995)). Meanwhile, the assessment rule determines the belief updating process, which assumes no irrationality exists. There are no perturbations in the behavior rule and assessment rule. However, in this paper, I assume that not all players are fully rational, and this is common knowledge. (Mutual knowledge may be enough. Needs to be verified.) Take a two-player game as example, $player1$ knows that not both players are rational, $player2$ also knows this, they both know the other one knows that, and so on ad infinitum. Then irrationality is introduced into the model as common knowledge. Every player will form a prior to what extent his opponent will deviate from these rules. This approach enables me to examine how players with imperfect rationality interact and how they form beliefs about their opponents' behavior. Moreover, it allows us to study the impact of deviations from the expected behavior rule and their effects on the convergence of the game to a Nash equilibrium.

2) My analysis reveals that when perturbations are introduced, the convergence properties of fictitious play are significantly altered compared to traditional fictitious play. To address this issue, I propose certain conditions for the behavior rule and assessment rule, and I also construct an updating rule for perturbations. By implementing these modifications, players can learn both pure and mixed strategy Nash equilibria, thereby facilitating convergence in the game. My approach enables me to explore how the presence of perturbations affects the stability of Nash equilibria and how players can adapt their strategies to overcome these perturbations.

I adopt some definitions and notions introduced by Fudenberg and Kreps (1993). In their paper, they focus on introducing a stronger criterion of convergence of play for the traditional fictitious play. Different from their paper, I am interested in whether play can converge when irrationality is introduced into the model. They further introduce payoff perturbations to try to explain why players learn mixed strategies. I do analyze if belief perturbations will lead players to learn mixed strategy profiles, but I do not focus on Nash equilibria. I am rather interested in what players actually learn when the game is repeated under the model of fictitious play with belief perturbations.

Esponda and Pouzo (2016) propose a framework for modeling agents with misspecified models. They assume that every player has a subjective model relative to the objective game. My model appears to be a special case of that framework. But actually I discuss very different conditions here. I do not assume any true distributions over perturbations and learning rules. That is I do not assume that absence of perturbations is the true environment, and what the true perturbations are. Rivals' rationality is not presumed fixed. Players can learn what the perturbation term will be only when the game is repeated. My model allows for more flexibility in modeling players' beliefs and updating processes, and does not require any assumptions about the true environment.

Section 4.2 introduces the general form of the learning model in a normal-form game. The basic assumption is that players play a finite normal-form game repeatedly and will observe their opponents' strategies after each round.

In Section 4.3, I propose the modified model of fictitious play with perturbations, and discuss the convergence of the empirical distribution briefly. For traditional fictitious play, if the empirical distributions converge to some (mixed) strategy profile, then this limit is a (mixed) Nash equilibrium (Fudenberg and Kreps (1993)). However, Propositions 4.1 and 4.2 show that this conclusion is no longer valid for the modified model. If there exist perturbations, convergence yielding a Nash equilibrium is not guaranteed.

In Section 4.4, I discuss the convergence to pure strategy Nash equilibria. I present more general conditions for the behavior rule and assessment rule that yield convergence. (i) The behavior rule is asymptotically myopic if players maximize their immediate expected payoffs (Fudenberg and Kreps (1993)). (ii) The assessment rule is adaptive if players assign almost zero probability to strategies that have not been played for a long time (Milgrom and Roberts (1991)). In Proposition 4.3, I show that the assumption of adaptive assessment can be relaxed for the convergence to a pure strategy profile of fictitious play with perturbations. However, it takes a very long time to converge with strictly positive perturbation term. To increase the rate of convergence, we allow players to update their perturbations.

In Section 4.5, I discuss the convergence to mixed strategy Nash equilibria. I first study the cycles of fictitious play with perturbations. For traditional fictitious play, the empirical distributions converge to the Nash equilibria if the game is zero-sum (Robinson (1951)), or is 2×2 (Miyasawa (1961)). However, convergence is not guaranteed for more general games (see Shapley (1964)). Specifically, Krishna and Sjöström (1998) prove that fictitious play cannot converge cyclically to a mixed strategy Nash equilibrium if players use more than two strategies. Furthermore, I show that fictitious play with perturbations cannot converge cyclically to a mixed strategy Nash equilibrium even if players use two pure strategies. Finally, I show that perturbations can help players to learn the mixed strategy Nash equilibrium, but only for some specific games.

4.2 Setup

The game is in strategic form. The players are indexed $i = 1, 2, \ldots, I$, and $-i$ denotes all players except i. S^i is the finite pure strategy space for player i, and $u^i : S \to \mathfrak{R}$

is the payoff function of player i. Also, $S = S^1 \times \cdots \times S^I$ is the set of pure-strategy profile. $\Sigma = \Sigma^1 \times \cdots \times \Sigma^I$ denotes the mixed strategy profiles where Σ^i is the set of probability distributions on S^i, with generic elements $\sigma^i \in \Sigma^i$. Moreover, let S^{-i} denote $\Pi_{j \neq i} S^j$, and denote Σ^{-i} as the set of probability distributions over S^{-i}. Further denote $u^i(s^i, \sigma^{-i})$ as player $i's$ expected utility when he chooses pure strategy $s^i \in S^i$ and his opponents act according to the distribution $\sigma^{-i} \in \Sigma^{-i}$.

I assume that the game is repeatedly played, denoting time $t = 1, 2, \ldots$ as the round of play. After each round, players can observe the realized actions chosen by their opponents. The history of play from time 1 to $t-1$ is denoted by ζ_t, $\zeta_t = (s_1, \ldots, s_{t-1})$. The set of all such histories is denoted by $\mathcal{L}_t$. A model of learning consists of mainly two parts: behavior rules and assessment rules (Fudenberg and Kreps (1993)). There are some misleading notions for behavior rules. I provide clearer notions. For traditional fictitious play, the behavior rules and assessment rules are defined as follows.

1) **Assessment rules:** $\mu_t^i : \mathcal{L}_t \to \Sigma^{-i}$, which expresses a player's belief on what his opponents will choose. And I denote $\Sigma^{-i}(\mu_t^i)$ as the set of probabilities that player i assigns to player $-i's$ all strategies. Here a player is allowed to form an assessment on the joint behavior of his opponents which may be correlated. This is different from the independence assumption in *Behavior rules*: For *Behavior rules*, at every time t strategies chosen are not allowed to be correlated.

2) **Behavior rules based on assessment rules:** $\phi_t^i : \Sigma^{-i}(\mu_t^i) \to S^i$. Which is related to how players choose strategies based on assessments. For example, maximizing expected utility at each stage is a rule to choose strategies. Maximizing total expected utility for all stages is another rule to choose strategies.

$2'$) **Behavior rules based on histories:** $\varphi_t^i : \mathcal{L}_t \to \Sigma^i$. Which is related to how players choose strategies based on histories. We can form a probability distribution based on $\mathcal{L}_t$. Then the conditional probabilities of strategies at time t can be achieved. Furthermore, I construct the joint probability that player 1 plays s^1, player i plays s^i, and so on. I denote the joint probability as $\mathbf{P}(\cdot)$, and I make the following assumption:

$$\mathbf{P}(s_t = (s^1, \ldots, s^I)|\zeta_t) = \varphi_t^1(\zeta_t)(s^1) \times \cdots \times \varphi_t^I(\zeta_t)(s^I)$$

$P(\cdot)$ reflects the history of play. I do not allow correlations between strategies chosen by all players.

4.3 The Model Based on Fictitious Play

In the traditional model of fictitious play (Brown (1951)), there are two players, that is here $I = 2$. And I denote $-i$ as not i. Moreover, I denote $S_L^1 = \{s_1, s_2, \cdots, s_L\}$ and $S_K^2 = \{s_1, s_2, \cdots, s_K\}$ as the sets of available pure strategies to player 1 and player 2 with $L, K \in \mathbb{N}^+$. Other details are similar to what I discussed in Section 4.2.

(a) Player i has an exogenous initial weight function $\eta_0^i : S^{-i} \to \mathfrak{R}_+$ such that $\sum_{s^{-i} \in S^{-i}} \eta_0^i(s^{-i}) > 0$. And it is updated by

$$\eta_t^i(\zeta_t)\left(s^{-i}\right) = \eta_{t-1}^i(\zeta_t)\left(s^{-i}\right) + \begin{cases} 1 - \varepsilon^i \text{ if } s_{t-1}^{-i} = s^{-i} \\ \varepsilon_{s_{t-1}^{-i}}^i \text{ if } s_{t-1}^{-i} \neq s^{-i} \end{cases}$$

where

$$\sum_{\substack{s_{t-1}^{-i} \neq s^{-i} \\ s_{t-1}^{-i} \in S^{-i}}} \varepsilon_{s_{t-1}^{-i}}^i = \varepsilon^i, \ \varepsilon^i \in [0, 1]$$

Here ε^i is player i's prior perturbation term, which measures to what extend player j deviates from the rules. Given ε^i, at any time t, player i assigns $\varepsilon^i_{s^j}$ to each of player j's strategies except for the strategy that is played at time $t-1$, for example, player i assigns $\varepsilon^1_{s^2_K}$ to player 2's strategy s^2_K if s^2_K is not played in the previous round. For traditional fictitious play, $\varepsilon^i = 0$ for all $i, j \in I$.

ε^i is supposed fixed at the very beginning. I discuss how it updates in Section 4.4.

(b) The assessment rule μ^i_t can be given as:

$$\mu^i_t(\zeta_t)\left(s^{-i}\right) = \frac{\eta^i_t(\zeta_t)\left(s^{-i}\right)}{\sum_{s' \in S^{-i}} \eta^i_t(\zeta_t)(s')}.$$

It is the probability that player i assigns to player $-i$ playing s^{-i} at time t.

(c) The behavior rule is that at each time t, player i maximizes:

$$\sum_{s^{-1} \in S^{-1}} u^i\left(s^i, s^{-i}\right))\mu^i_t(\zeta_t)\left(s^{-i}\right).$$

for all $s^i \in S^i$. The maximizer is denoted $\phi^i_t(\zeta_t)$. Here the behavior rule is relative to assessments.

With the perturbation term, I introduce irrationality into the model. Each player forms a prior about perturbations of other players, which means each player holds the belief that his opponents' behaviors will not fully stick to the rules of traditional fictitious play.

Example 4.1. To help understand the model, consider the following normal form game with two players. Suppose the game is played using the method of fictitious play, with initial weight:

$$\eta_0^1 = (1, \sqrt{2}) \text{ and } \eta_0^2 = (\sqrt{2}, 1).$$

Here the initial weights are written as row vectors. For η_0^1, the first number is the initial weight for strategy A of player 2, the second number is the initial weight for strategy B of player 2. So for the η_0^2 as well.

		Player 2	
		A	B
Player 1	A	$(0, 0)$	$(1, 1)$
	B	$(1, 1)$	$(0, 0)$

Table 4.1: A Simple Normal Form Game in Example 4.1.

In Table 4.2, in the first round, player 1 has initial weight $(1, \sqrt{2})$ (Column 2), that is he believes that player 2 will choose A with probability $\frac{1}{1+\sqrt{2}}$, and B with probability $\frac{\sqrt{2}}{1+\sqrt{2}}$. Thus, for fictitious play without perturbations, player 1 chooses A or B to maximize:

$$\frac{1}{1+\sqrt{2}}u^1(s^1, A^2) + \frac{\sqrt{2}}{1+\sqrt{2}}u^1(s^1, B^2),$$

Here player 1 will choose A since it yields higher expected utility. So for player 2, he will choose B. (A, B) is chosen for round $t = 1$.

Then for the second round, since player 2 played B in $t = 1$, player 1's belief of player 2 plays B is increased by 1 (Thus, $\eta_2^1 = (1, \sqrt{2} + 1)$). For player 2, $\eta_2^2 = (1 + 1, \sqrt{2})$. (A, B) is chosen for round $t = 2$. It is obvious that player 1 will always choose A and player 2 will always choose B, (A, B) will be the steady state of fictitious play and also be a Nash equilibrium.

	"Beliefs"	Expected Payoffs	Choice of Action
		t=1	
Player1	$(1, \sqrt{2})$	$(\frac{\sqrt{2}}{1+\sqrt{2}}, \frac{1}{1+\sqrt{2}})$	A
Player2	$(\sqrt{2}, 1)$	$(\frac{1}{1+\sqrt{2}}, \frac{\sqrt{2}}{1+\sqrt{2}})$	B
		t=2	
Player1	$(1, \sqrt{2}+1)$	$(\frac{\sqrt{2}+1}{2+\sqrt{2}}, \frac{1}{2+\sqrt{2}})$	A
Player2	$(1+\sqrt{2}, 1)$	$(\frac{1}{2+\sqrt{2}}, \frac{\sqrt{2}+1}{2+\sqrt{2}})$	B

Table 4.2: Traditional Fictitious Play in Example 4.1.

However, if we put a perturbation term into the model (still with the same expected utility form), we will have Table 4.4. In Table 4.4, in the second round, player 1 has initial weight $(1+\varepsilon, \sqrt{2}+1-\varepsilon)$ (Based on our model), that is he believes player 2 will play A with probability $\frac{1+\varepsilon}{2+\sqrt{2}}$, and B with probability $\frac{\sqrt{2}+1-\varepsilon}{2+\sqrt{2}}$. Thus, for fictitious play with perturbation, player 1 chooses A or B to maximize:

$$\frac{1+\varepsilon}{2+\sqrt{2}}u^1(s^1, A^2) + \frac{\sqrt{2}+1-\varepsilon}{2+\sqrt{2}}u^1(s^1, B^2),$$

Here when $\varepsilon > \frac{\sqrt{2}}{2}$, player 1 will choose A giving higher expected utility (Column 3). So for player 2, he will choose B under this condition. And so on. Thus, the strategy profile will be $((A, B), (B, A), (A, B), (B, A), \cdots)$, that is the empirical frequencies of players' choices converge to $(\frac{1}{2}, \frac{1}{2})$, which is very different from the former one. (Which also means no pure NE is reached.) Only when $\varepsilon < \frac{\sqrt{2}}{2}$, we can still have (A, B) in round 2; and further with $\varepsilon < \frac{\sqrt{2}+1}{4}$, we can still have (A, B) in round 3, and so on. The perturbation term should shrink in every round, so we can have the same solution as the former one.

Therefore, unlike the traditional fictitious play, strict Nash equilibria are not absorbing for play according to the model of fictitious play with perturbation.

Proposition 4.1. *Suppose that s_* is played at time t in the process of fictitious play with perturbations, s_* is a strict Nash equilibrium and there is no weakly dominant strategy. Then there exists a perturbation term $\varepsilon^i \in (0,1]$, for all $i = 1, 2$, such that $s_\tau \neq s_*$ for some $\tau > t$.*

Proof. See the Appendix C.1. □

That is to say, even when the Nash equilibrium is played at some date t, with the existence of perturbations, it is not stable. Furthermore, we have the following proposition.

Proposition 4.2. *Suppose that a pure strategy profile $\hat{s}$ is played at time τ in the process of fictitious play with perturbation, and also played for all $t > \tau$. Then the strategy profile $\hat{s}$ is not guaranteed to be a Nash equilibrium.*

Proof. I prove this proposition by a counter example.

Example 4.2. The following game provides a counter example.

In this game, B is strictly dominated by T for player 1, so player 1 will always choose T if he maximizes his immediate expected payoff. The only pure Nash equilibrium is (T, L). Consider the initial weight for both players: $\eta_0^i = (1, 2)$. For $t = 1$, $\eta_1^i = \eta_0^i$ and player 2's expected payoff of choosing R is greater than that of choosing L ($2 > \frac{2}{3}$). For $t = 2$, $\eta_2^2 = (2 - \varepsilon, 2 + \varepsilon)$ and player 2's expected payoff of choosing R is greater than that of choosing L ($\frac{6+3\varepsilon}{4} > \frac{6-\varepsilon}{4}$). Since player 1 will always choose

		Player 2	
		R	L
Player 1	T	$(1, 0)$	$(1, 2)$
	B	$(0, 3)$	$(0, 1)$

Table 4.3: A Counter Example Demonstrating Proposition 4.2.

T, we have $\eta_n^2 = (n - (n-1)\varepsilon, 2 + (n-1)\varepsilon)$ for $t = n$. As long as $\varepsilon > \frac{1}{2}$, player 2 will keep choosing R. Therefore, (T, R) is played for all times t. However, it is not a Nash equilibrium. □

From the above two propositions, we can find that fictitious play with perturbations does not have the nice convergence properties of the traditional model.[2] Because of the perturbation term, the strategy profile is easy to switch. Then two cases need to be considered: First, switch might lead to cycles, a game might can converge to a pure strategy Nash equilibrium, but with cycles, it might converge to a mixed strategy Nash equilibrium; Second, the empirical distributions are unlikely to converge because of switches. Shapley (1964) shows that in a modified version of Rock-Scissors-Paper, fictitious play does not converge. Krishna and Sjöström (1998) prove that fictitious play cannot converge cyclically to a mixed strategy equilibrium in games where players' available pure strategies are more than two.

[2]For traditional fictitious play, if the empirical distributions converge to a (mixed) strategy profile, then this limit is a (mixed) Nash equilibrium (Fudenberg and Kreps (1993)). The proposition can be found in their paper.

	"Beliefs"	Expected Payoffs	Choice of Action
		t=1	
Player1	$(1, \sqrt{2})$	$(\frac{\sqrt{2}}{1+\sqrt{2}}, \frac{1}{1+\sqrt{2}})$	A
Player2	$(\sqrt{2}, 1)$	$(\frac{1}{1+\sqrt{2}}, \frac{\sqrt{2}}{1+\sqrt{2}})$	B
		t=2	
Player1	$(1+\varepsilon, \sqrt{2}+1-\varepsilon)$	$(\frac{\sqrt{2}+1-\varepsilon}{2+\sqrt{2}}, \frac{1+\varepsilon}{2+\sqrt{2}})$	B if $\varepsilon > \frac{\sqrt{2}}{2}$; A if $\varepsilon < \frac{\sqrt{2}}{2}$
Player2	$(1+\sqrt{2}-\varepsilon, 1+\varepsilon)$	$(\frac{1+\varepsilon}{2+\sqrt{2}}, \frac{\sqrt{2}+1-\varepsilon}{2+\sqrt{2}})$	A if $\varepsilon > \frac{\sqrt{2}}{2}$; B if $\varepsilon < \frac{\sqrt{2}}{2}$
		t=3	
Player1	$(2, \sqrt{2}+1)$	$(\frac{\sqrt{2}+1}{3+\sqrt{2}}, \frac{2}{3+\sqrt{2}})$	A if $\varepsilon > \frac{\sqrt{2}}{2}$
Player2	$(1+\sqrt{2}, 2)$	$(\frac{2}{3+\sqrt{2}}, \frac{\sqrt{2}+1}{3+\sqrt{2}})$	B if $\varepsilon > \frac{\sqrt{2}}{2}$
		or	
Player1	$(1+2\varepsilon, \sqrt{2}+2-2\varepsilon)$	$(\frac{\sqrt{2}+2-2\varepsilon}{3+\sqrt{2}}, \frac{1+2\varepsilon}{3+\sqrt{2}})$	A if $\varepsilon < \frac{\sqrt{2}}{2}; \varepsilon < \frac{\sqrt{2}+1}{4}$
Player2	$(\sqrt{2}+2-2\varepsilon, 1+2\varepsilon)$	$(\frac{1+2\varepsilon}{3+\sqrt{2}}, \frac{\sqrt{2}+2-2\varepsilon}{3+\sqrt{2}})$	B if $\varepsilon < \frac{\sqrt{2}}{2}; \varepsilon < \frac{\sqrt{2}+1}{4}$
		t=4	
Player1	$(2+\varepsilon, \sqrt{2}+2-\varepsilon)$	$(\frac{\sqrt{2}+2-\varepsilon}{4+\sqrt{2}}, \frac{2+\varepsilon}{4+\sqrt{2}})$	B if $\varepsilon > \frac{\sqrt{2}}{2}$
Player2	$(2+\sqrt{2}-\varepsilon, 2+\varepsilon)$	$(\frac{2+\varepsilon}{4+\sqrt{2}}, \frac{\sqrt{2}+2-\varepsilon}{4+\sqrt{2}})$	A if $\varepsilon > \frac{\sqrt{2}}{2}$
		or	
Player1	$(1+3\varepsilon, \sqrt{2}+3-3\varepsilon)$	$(\frac{\sqrt{2}+3-3\varepsilon}{4+\sqrt{2}}, \frac{1+3\varepsilon}{4+\sqrt{2}})$	A if $\varepsilon < \frac{\sqrt{2}}{2}; \varepsilon < \frac{\sqrt{2}+2}{6}$
Player2	$(\sqrt{2}+3-3\varepsilon, 1+3\varepsilon)$	$(\frac{1+3\varepsilon}{4+\sqrt{2}}, \frac{\sqrt{2}+3-3\varepsilon}{4+\sqrt{2}})$	B if $\varepsilon < \frac{\sqrt{2}}{2}; \varepsilon < \frac{\sqrt{2}+2}{6}$

[1] Here we suppose $\varepsilon^i_{sj} = \varepsilon \ \forall i, j$

Table 4.4: Fictitious Play with Belief Perturbations in Example 4.1.

4.4 Convergence to a Pure Strategy Profile

In Section 4.3, I introduce two simple propositions about the convergence of fictitious play with perturbations, which are based on two basic assumptions. Players choose strategies according to the assessment rules, and the behavior rule is set to maximize the immediate payoff for each stage. In this section, I specify more general conditions for the two rules that reveal the properties of convergence of fictitious play with perturbations.

First, I discuss the behavior rule. Immediate expected payoff maximization for each stage means that players are focusing on the short term (exactly one stage) payoff and have no incentive to influence the future play of other players. The idea and definition is introduced by Townsend (1978) and Fudenberg and Kreps (1993).

Definition 4.1. Given an assessment rule $\mu^i = (\mu^i_1, \mu^i_2, \ldots)$ for player i, the behavior rule $\phi^i = (\phi^i_1, \phi^i_2, \ldots)$ for i is called *myopic* relative to μ^i, if for every t and ζ_t, $\phi^i_t(\zeta_t)$ maximizes $i's$ immediate expected payoff:

$$u^i(\phi^i_t(\zeta_t), \mu^i_t(\zeta_t)) = \max_{s^i \in S^i} u^i(s^i, \mu^i_t(\zeta_t)).$$

By relaxing the rigid requirement for the use of optimal pure strategies, I obtain the notion of asymptotically myopic:

Definition 4.2. Given an assessment rule $\mu^i = (\mu^i_1, \mu^i_2, \ldots)$ for player i, the behavior rule $\phi^i = (\phi^i_1, \phi^i_2, \ldots)$ for i is called *asymptotically myopic* relative to μ^i, if for some $\delta_t > 0$ with $\lim_{t\to\infty} \delta_t \to 0$ and for every t and ζ_t, $\phi^i_t(\zeta_t)$ maximizes $i's$ immediate expected payoff:

$$u^i(\phi_t^i(\zeta_t), \mu_t^i(\zeta_t)) + \delta_t \geq \max_{s^i \in S^i} u^i(s^i, \mu_t^i(\zeta_t)).$$

The asymptotically myopic behavior rule allows players use suboptimal strategies, Radner (1986) also suggests the same idea which is defined as ε-equilibrium.

It is obvious that the behavior rule of fictitious play with perturbations is asymptotically myopic. However, since players only focus on the maximization of expected payoff at each stage, the assumption of asymptotically myopia means that players have no incentive to influence the future play of other players. Will this be a problem? Fudenberg and Kreps (1993) provide two justifications to defend the property of asymptotically myopic: First, people intend to discount the future, so any big influence will be small in the long term; Second, a player's action is not influenced by his opponent if his rival plays a fixed strategy asymptotically.

However, different from the traditional fictitious play, where players believe that their rivals will play a fixed strategy profile asymptotically, here players all have the prior that each player may deviate from his past play. What renders the myopia valid is that each player believes that his rivals' perturbations are not randomly determined for each stage.

Then, I discuss the property of the assessment rule. Milgrom and Roberts (1991) suggest the concept of *adaptive learning*. The idea is that the pure strategy profile (s_t^i) is consistent with adaptive learning if player i chooses almost best replies to the distribution of rivals' past play, and assigns almost zero probability to the strategies that were not played for a long time. Fudenberg and Kreps (1993) use this idea defining the adaptive assessments.

Definition 4.3. The assessment rule μ^i is *adaptive* if for every $\delta > 0$ and for every t, there is some T such that for all $\bar{t} > T$ and histories $\zeta_{\bar{t}}$, $\mu^i_{\bar{t}}(s^{-i}_{\bar{t}}) \leq \delta$ for all $s^{-i}_{\bar{t}} \notin \zeta_{t'}$ with $t \leq t' \leq \bar{t}$.

That is, the player assigns very small probability to the strategies that were not played for a sufficiently long time. We can see that asymptotically myopic behavior and adaptive assessments are all derived from the idea of *adaptive learning*, and are well defined to describe the two rules of fictitious play.

With both *asymptotically myopic behavior* and *Adaptive Assessment*, if $s_t = s_* \forall t \geq T$ in the process of traditional fictitious play, then s_* must be a Nash equilibrium of the repeated game (Fudenberg and Kreps (1993)). However, for the extended model I proposed, the property of adaptive assessments may be violated and no Nash equilibrium will be reached.

Example 4.3. A case of violating adaptive assessments:

		Player 2	
		$C1$	$C2$
Player 1	$R1$	$(1, 0)$	$(3, 2)$
	$R2$	$(2, 1)$	$(4, 0)$

Table 4.5: Two-person Game in Example 4.3.

In this game, $R1$ is dominated by $R2$ for player 1. So player 1 will choose $R2$ if his behavior is myopic for any assessment rule. Eventually, player 2 will choose $C1$. However, for my model, consider the initial weight for both players: $\eta^i = (2, 1)$. Then I have the repeated game below (Table 4.6).

	"Beliefs"	Expected Payoffs	Choice of Action
		t=1	
Player1	$(2, 1)$	$R2$ *dominate* $R1$	R2
Player2	$(2, 1)$	$(\frac{1}{3}, \frac{4}{3})$	C2
		t=2	
Player1	$(2 + \varepsilon^1, 2 - \varepsilon^1)$	$R2$ *dominate* $R1$	R2
Player2	$(2 + \varepsilon^2, 2 - \varepsilon^2)$	$(\frac{2-\varepsilon^2}{4}, \frac{4+2\varepsilon^2}{4})$	C2
		and so on until with $(R2, C2)$	
		t=5	
Player1	$(2 + 4\varepsilon^1, 5 - 4\varepsilon^1)$	$R2$ *dominate* $R1$	R2
Player2	$(2 + 4\varepsilon^2, 5 - 4\varepsilon^2)$	$(\frac{5-4\varepsilon^2}{7}, \frac{4+8\varepsilon^2}{7})$	C2 if $\varepsilon^2 > \frac{1}{12}$
		t=n	
Player1	$(2 + (n-1)\varepsilon^1, n - (n-1)\varepsilon^1)$	$R2$ *dominate* $R1$	R2
Player2	$(2 + (n-1)\varepsilon^2, n - (n-1)\varepsilon^2)$	$(\frac{n-(n-1)\varepsilon^2}{2+n}, \frac{4+2(n-1)\varepsilon^2}{2+n})$	C2 if $\varepsilon^2 > \frac{n-4}{3n-3}$

Table 4.6: Fictitious play with Belief Perturbations in Example 4.3.

That is, if $\varepsilon^2 > \frac{1}{3}$, then player 2 will always play $C2$; and player 1 always chooses $R2$ as $n \to \infty$. And it is obvious that $(R2, C2)$ is not a Nash equilibrium. Only when $\varepsilon^2 \to 0$, this mutation effects will vanish. Obviously, here the assessment rule is not adaptive, since when $\varepsilon^2 > \frac{n-4}{3n-3}$, player 2 gives higher probability to $R1$ than $R2$, but $R1$ is never played by player 1. But if $\varepsilon^2 < \frac{1}{3}$, then with some $t' \geq T$, the game will eventually convergence to a pure strategy Nash equilibrium, and ε^2 need not to be very small.

Definition 4.4. The assessment rule μ^i satisfies *bounded adaptiveness* if there exists some positive $\bar{\delta} > 0$ such that for every t, there is some T such that for all $\bar{t} > T$ and histories $\zeta_{\bar{t}}$, $\mu^i_{\bar{t}}(s^{-i}_{\bar{t}}) \leq \delta$ for all $\delta < \bar{\delta}$ and $s^{-i}_{\bar{t}} \notin \zeta_{t'}$ with $t \leq t' \leq \bar{t}$.

Proposition 4.3. *Let $s^i_t \in \phi^i_t(\zeta_t)$ $\forall t = 1, 2, \ldots$ and $\forall i = 1, 2$. There exists $\bar{\varepsilon}^i \in [0, 1]$ $\forall i = 1, 2$, such that for some $s_* \in S$ and T, $s_t = s_*$ for all $t \geq T$ in the process of fictitious play with perturbations. If behavior rules ϕ^i are asymptotically myopic relative to assessment rules, and assessment rules satisfy bounded adaptiveness, that is $0 < \varepsilon^i < \bar{\varepsilon}^i$ $\forall i = 1, 2$, then s_* is a Nash equilibrium of the repeated game.*

Proof. First, I prove that if a Nash equilibrium s_* is played at time T, then $\exists \bar{\varepsilon}^i \in (0, 1)$ $\forall i = 1, 2$ such that s_* will be played for all $t \geq T$.

Let $x_{lk} = u^1(s^1_l, s^2_k)$, $y_{lk} = u^2(s^1_l, s^2_k)$, and $\mu^1_t(s^2_k) = \alpha_k / \sum_{\kappa=1}^{K} \alpha_\kappa$, $\forall k = 1, 2, \ldots, K$, where $\eta^1_t = (\alpha_1, \ldots, \alpha_k, \ldots, \alpha_K)$ is the belief of player 1 at time t.

Suppose $s_* = (s^1_1, s^2_1)$ played at time T is a Nash equilibrium. Then if s_* is played at time $T + 1$, we have:

$$\mu^1_{T+1}(s^2_1) = \frac{\alpha_1 + 1 - \varepsilon^1}{\sum_{\kappa=1}^{K} \alpha_\kappa + 1}, \ \mu^1_{T+1}(s^2_k) = \frac{\alpha_k + \varepsilon^1_k}{\sum_{\kappa=1}^{K} \alpha_\kappa + 1} \ \forall\ k = 2, 3, \ldots, K, \text{ and}$$

$$u(s_1^1, \mu_{T+1}^1(s^2)) \geq u(s_l^1, \mu_{T+1}^1(s^2)) \ \forall l = 2, \ldots, L, \text{ that is}$$

$$x_{11}\frac{\alpha_1 + 1 - \varepsilon^1}{\sum\limits_{\kappa=1}^{K}\alpha_\kappa + 1} + \sum_{k=2}^{K} x_{1k}\frac{\alpha_k + \varepsilon_k}{\sum\limits_{\kappa=1}^{K}\alpha_\kappa + 1} \geq x_{l1}\frac{\alpha_1 + 1 - \varepsilon^1}{\sum\limits_{\kappa=1}^{K}\alpha_\kappa + 1} + \sum_{k=2}^{K} x_{lk}\frac{\alpha_k + \varepsilon_k^1}{\sum\limits_{\kappa=1}^{K}\alpha_\kappa + 1},$$

$$(x_{11} - x_{l1})(\alpha_1 + 1 - \varepsilon^1) \geq \sum_{k=2}^{K}(x_{lk} - x_{1k})(\alpha_k + \varepsilon_k^1)$$

If $\sum\limits_{k=2}^{K}(x_{1k} - x_{lk})(\alpha_k + \varepsilon_k^1) \geq 0$, no need to be discussed. So here we consider the case $\sum\limits_{k=2}^{K}(x_{1k} - x_{lk})(\alpha_k + \varepsilon_k^1) < 0$.

Since $\sum\limits_{k=2}^{K}\varepsilon_k^1 = \varepsilon^1$, let $\varepsilon_k^1 = c_k\varepsilon^1$, where $c_k \geq 0$ and $\sum\limits_{k=2}^{K}c_k = 1$. Then it can be simplified as

$$(x_{11} - x_{l1})\alpha_1 - \sum_{k=2}^{K}(x_{1k} - x_{lk})(\alpha_k) + (x_{11} - x_{l1}) \geq \varepsilon^1[(x_{11} - x_{l1}) + \sum_{k=2}^{K}(x_{lk} - x_{1k})c_k].$$

Thus, we have the condition:

$$\varepsilon^1 \leq \frac{(x_{11} - x_{l1})\alpha_1 - \sum\limits_{k=2}^{K}(x_{lk} - x_{1k})\alpha_k + (x_{11} - x_{l1})}{(x_{11} - x_{l1}) + \sum\limits_{k=2}^{K}(x_{lk} - x_{1k})c_k}$$

For time $T + 2$, we have:

$$\varepsilon^1 \leq \frac{(x_{11} - x_{l1})\alpha_1 - \sum\limits_{k=2}^{K}(x_{lk} - x_{1k})\alpha_k + 2(x_{11} - x_{l1})}{2(x_{11} - x_{l1}) + \sum\limits_{k=2}^{K}(x_{lk} - x_{1k})2c_k}$$

Let $a = (x_{11} - x_{l1})\alpha_1 - \sum\limits_{k=2}^{K}(x_{lk} - x_{1k})\alpha_k$, $b = x_{11} - x_{l1}$, and $c = (x_{11} - x_{l1}) +$

$\sum_{k=2}^{K}(x_{jk} - x_{1k})c_k$. Then for all $T + n$ we have:

$$\varepsilon^1 \leq \frac{a + nb}{nc} = \frac{a}{nc} + \frac{b}{c}$$

Thus, there exist $\bar{\varepsilon}^1 \in (0, 1)$, such that $\lim_{n\to\infty} \frac{a}{nc} + \frac{b}{c} = \frac{b}{c} = \frac{x_{11}-x_{l1}}{(x_{11}-x_{l1})+\sum_{k=2}^{K}(x_{jk}-x_{1k})c_k} < \bar{\varepsilon}^1$.

Second, a strategy profile played for time $t' \leq t < T$ is not a Nash equilibrium. Since the first part of the proof is valid, there must exist $\bar{\varepsilon}^1 > 0$ such that the strategy will switch to a Nash equilibrium, and it will be played at all times $t \geq T$. □

The proposition shows that assessment rules do not have to be fully adaptive for fictitious play with perturbations. The empirical distribution can converge to a Nash equilibrium even when players are allowed to deviate from the assessment rules. However, we can also see from the proof that if the prior of perturbation is not small, the equilibrium strategy profile can be played for a sufficient long time (See the Example 4.3). Moreover, although the empirical distribution eventually may converge to a Nash equilibrium, players do not learn and update their beliefs (They update the assessments but not perturbations). Thus, the assessment rules should allow the updating of beliefs about perturbations based on the past play.

Let $\#(\zeta_t)(s) : \zeta_t \to \mathfrak{R}^+$ give the number of times that the player switches his strategies from time 1 to $t - 1$. Then $\#^i(\zeta_t)(s^j)$ gives the number of times player i observes that player j switches his strategies from time 1 to $t - 1$.

ε-Updating Rule. Given $t = 1, 2, \ldots,$ and player i's prior of the perturbation term of player j, $\varepsilon^i \in [0,1]$. The perturbation term is updated by

$$\varepsilon^i(t) = [\varepsilon^i]^{t-1-\#^i(\zeta_t)(s^j)}, \text{ with } t \geq 3.$$

The ε-updating rule determines how players update the perturbation term based on the past play. The idea is that if player i observes that player j does not intend to switch his strategies as the game is repeated, player i will learn that his opponent has very low tendency to deviate from the rule of traditional fictitious play. Since $\varepsilon^i \in [0,1]$, then if player j has not switched for a very long time, then player i will update ε^i to a very small value. And it is easy to see that with the ε-updating rule, the assessment rule will be adaptive when the perturbation term converges to 0 as t goes to infinity.

Proposition 4.4. *Let $s_t^i \in \phi_t^i(\zeta_t)$ $\forall t = 1, 2, \ldots$ and $\forall i = 1, 2$, such that for some $s_* \in S$ and T, $s_t = s_*$ for all $t \geq T$ in the process of fictitious play with perturbations. If behavior rules ϕ^i are asymptotically myopic relative to assessment rules, and perturbations of assessment rules are updated in accordance with the ε-updating rule, then s_* is a Nash equilibrium of the stage game.*

Proof. See the Appendix C.2. □

Proposition 4.4 indicates that even though the perturbation term is very large, as long as the perturbation term is updated in accordance with the ε-updating rule, the play can converge to a pure Nash equilibrium. In Example 4.3, since player 1 always chooses $R2$, player 2 learns that player 1 will not deviate from the rule, thus the perturbation term will shrink to a small value in a short time, which means player 2 will choose $C1$ eventually.

4.5 Convergence to a Mixed Strategy Profile

This section analyses the convergence to a mixed strategy profile. I first show that if the perturbation term is large and not updated, then fictitious play with perturbations cannot converge cyclically to a mixed strategy equilibrium. Furthermore, I show that even for some 2×2 matrix games, the empirical frequency does not converge to a mixed strategy equilibrium even it converges together with players' assessments.

4.5.1 The perturbation term is fixed

Section 4.3 shows that fictitious play can converge to a pure strategy Nash equilibrium with some perturbations, but if the perturbations are very large and fixed, the empirical distribution cannot converge to a a pure strategy Nash equilibrium. Next I proceed to discuss convergence to a mixed strategy Nash equilibrium. Firstly, I study if fictitious play with perturbations can converge cyclically to a mixed strategy Nash equilibrium.

Before we go any further, a cycle should be defined. From the assessment rule and behavior rule, we can define $BR(\mu_t^i)$ and $BR(\mu_t^{-i})$ as the best responses to assessments for all $t = (t_0, t_1, t_2, \cdots, t_n, \cdots)$, with $t_0 = 1$. And the strategy profile switches at time t_n for all $n \geq 1$. Since $BR(\cdot)$ is singleton for all t, we have:

$$(s_{t_n}^i, s_{t_n}^{-i}) \equiv (BR(\mu_t^i), BR(\mu_t^{-i})),\ \forall t \in [t_{n-1}, t_n)$$

That is, the best response to assessments in time period $[t_{n-1}, t_n)$ are the same. If the strategy profile switches at t_n, then we have $(s_{t_{n+1}}^i, s_{t_{n+1}}^{-i}) \equiv (BR(\mu_t^i), BR(\mu_t^{-i})),\ \forall t \in (t_n, t_{n+1})$. The repeated game follows a cycle of length H if the sequence of strategy profile

$$(s_1^i, s_1^{-i}), (s_2^i, s_2^{-i}), \cdots, (s_H^i, s_H^{-i})$$

is played over and over again with the fixed order in the history of fictitious play. The cycle can be denoted as

$$c = ((s_1^i, s_1^{-i}), (s_2^i, s_2^{-i}), \cdots, (s_H^i, s_H^{-i})).$$

Example 4.4. Game with no pure strategy Nash equilibrium:

		Player 2	
		R	L
Player 1	T	$(2,0)$	$(0,2)$
	B	$(0,1)$	$(1,0)$

Table 4.7: Game with no Pure Strategy Nash Equilibrium in Example 4.4.

Suppose the game is played under traditional fictitious play. Consider the initial weight for each player: $\eta_0^1 = (1,2)$ and $\eta_0^2 = (2,1)$. For $t = 1$, (B, L) is played and it is played until player 2 switches to R at time $t = 4$ with $\eta_3^2 = (2,4)$. (B, R) is played until player 1 switches to T at time $t = 6$. (T, R) is played until player 2 switches to L at time $t = 8$. And (T, L) is played until player 1 switches to B at time $t = 13$. So here, the switching times are $(t_0, t_1, t_2, t_3, t_4, t_5, \dots) = (1, 4, 6, 8, 13, \dots)$. And here the cycle is:

$$c = ((B, L), (B, R), (T, R), (T, L))$$

It is played over and over again, but the length of time spent playing each of the 4 strategy profiles is different for each round.

For traditional fictitious play, if players use only 2 pure strategies, the empirical distribution can converge cyclically to a mixed strategy equilibrium with any initial weights. However, under fictitious play with perturbations, a cycle is not guaranteed even for 2×2 games.

Proposition 4.5. *For almost all games, $\exists \eta_0 \in \Re^n$ and $\exists \varepsilon \in [0, 1]$, such that fictitious play with perturbations cannot follow a cycle c.*

The proof uses the idea of Proposition 4.3. If the sequence of play follows a cycle $c = ((s_1^i, s_1^{-i}), (s_2^i, s_2^{-i}), \cdots, (s_H^i, s_H^{-i}))$, then by definition, c is played over and over again. However, Proposition 4.3 shows that we can always find some $\varepsilon \in [0, 1]$ such that the cycle c breaks at some time T, and strategy profile (s_h^i, s_h^{-i}) will be played for all $t \geq T$, especially for games that have no pure strategy Nash equilibrium. In example 4.4, let the game played under fictitious play with perturbations. For $t = 1$, (B, L) is played. However, when $t \geq 2$, if $\varepsilon > \frac{1}{3}$, player 1 will always choose T instead of B, And player 2 will always play L as long as $\varepsilon < \frac{1}{3} + \frac{4}{3t-9}$ for all $t \geq 8$. Therefore, if $\eta_0^1 = (1, 2)$, $\eta_0^2 = (2, 1)$, and $\varepsilon = \frac{1}{3} + \frac{3}{3t-9}$, then the empirical distribution converges to strategy profile (T, L). The sequence of play cannot follow any cycles.

Proposition 4.5 shows that with perturbations, fictitious play cannot converge cyclically to a mixed Nash equilibrium in general. Therefore, it is very difficult for players to learn mixed strategy equilibria under fictitious play with perturbations. More assumptions of behavior rules and assessment rules should be introduced to achieve better properties of convergence.

4.5.2 The perturbation term is updated

The convergence criterion that I discussed in the previous section captures the idea of updating beliefs including perturbations. It does work when we consider convergence to a pure strategy profile. However, it may fail to help us draw nice conclusions when it comes to the convergence to a mixed strategy profile. To begin with, let us reconsider Example 4.1.

In Table 4.4, we can see that both players keep switching between the two pure strategies when ε is large. If perturbations are updated in accordance with the ε-updating rule, it is easy to check that the perturbation term will not shrink to a very small value. The empirical distribution converges to a mixed strategy Nash equilibrium.

Observation 4.1. *For 2×2 matrix games with two pure strategy equilibria and one mixed strategy equilibrium $(\frac{1}{2}, \frac{1}{2})$. If behavior rules and assessment rules are as in the model of fictitious play with perturbations, and perturbation term ε^i, which is very close to 1 for all $i = 1, 2$, is updated in accordance with the ε-updating rule, then the empirical distribution converges to the unique mixed strategy Nash equilibrium for initial weight $|\eta_0^i(s_1^i) - \eta_0^i(s_2^i)| < 2\varepsilon^i - 1$ for $i = 1, 2$.*

That is if players' priors about their rivals' strategies are close, then no pure strategy Nash equilibria can be reached in the model of fictitious play with perturbations. It seems that the perturbation term can lead players to learn the mixed strategy Nash equilibrium. Then the question is, can we extend the above conclusion to more general strategic-form games?

Unfortunately, the answer is no. We can find that for many games, perturbations can get players to learn a mixed strategy profile which is not a mixed strategy Nash equilibrium.

Observation 4.2. *Take the game in Example 4.1. But now let* $U^1(B, A) = U^2(A, B) = 2$ *while all other payoffs remain the same. If behavior rules and assessment rules are as in the model of fictitious play with perturbations, and the perturbation term* ε^i *is updated in accordance with the* ε*-updating rule, initial weight* $\eta_0^i \in \mathfrak{R}^+$ *of two players are very close, then the empirical distribution converges together with assessments to the mixed strategy profile* $(\frac{1}{2}, A; \frac{1}{2}, B)$.

Observation 4.2 shows that for some games, even though empirical distributions converge together with assessments to a mixed strategy profile, that strategy profile is not a mixed strategy Nash equilibrium.

Let $\#_p(\zeta_t)(s) : S \to \mathfrak{R}^+$ assigns the number of times strategy $s \in S$ was played from time 1 to $t-1$. (That is, $\#_p(\zeta_t)(s) + \#_{np}(\zeta_t)(s) = t-1$.) And let $\bar{\sigma}(\zeta_t)(s) = \frac{\#_p(\zeta_t)(s)}{t-1}$ be the proportion of strategy s was played from time 1 to $t-1$. So we have $\bar{\sigma}^i(\zeta_t)$ which is the marginal empirical distribution of player i. Then we have the following definition.

Definition 4.5. The assessment rule μ^i is asymptotically empirical if for every $\zeta \in \mathcal{Z}$,

$$\lim_{t\to\infty} \left\| \mu_t^i(\zeta_t) - \bar{\sigma}^{-i}(\zeta_t) \right\| = 0,$$

where the ζ_t are the sub-histories of the fixed ζ.

Proposition 4.6. *Let* $s_t^i \in \phi_t^i(\zeta_t)$ $\forall t = 1, 2, \ldots$ *and* $\forall i = 1, 2,$ *such that for some* $\sigma_* \in \Sigma$, $\lim_{t\to\infty} \bar{\sigma}^i(\zeta_t) = \sigma_*^i$. *If behavior rules* ϕ^i *are asymptotically myopic relative to assessment rules, assessment rules are asymptotically empirical, and perturbations of assessment rules are updated in accordance with the* ε*-updating rule, then* σ_* *is not guaranteed to be a mixed strategy Nash equilibrium of the repeated game.*

Proposition 4.6 shows that when the assessment rule is not so stationary (The perturbation term does not have to be very close to 1), the sequence of play does not converge to a mixed strategy Nash equilibrium for sure. Perturbations can lead players to learn the mixed strategy Nash equilibrium only for some specific games.

4.6 Conclusion

I propose a variation of traditional fictitious play where each player has a perturbation term that measures to what extent his rival will stick to the rules when the game is repeated. The model provides two main extensions, first, with perturbations, players are not so naive as those in traditional fictitious play. Players still assume their rivals choose strategies based on past play while they also assume their rivals may deviate from their past play. Second, with perturbations, assessments are no longer stationary, players do not update their beliefs based on a specific distribution.

Some studies show that if the assessment is not stationary, then the sequence of play cannot converge to a pure strategy Nash equilibrium (Camerer and Hua Ho (1999)). Proposition 4.3 shows that as long as the perturbation term is bounded, the empirical distribution of fictitious play with perturbations can converge to a pure strategy Nash equilibrium. Furthermore, if we allow players to update their perturbations based on the past play, the rate of convergence will be faster. When it comes to the convergence to a mixed strategy profile, Proposition 4.6 shows that perturbations can help players to learn the mixed strategy Nash equilibrium, but only for some specific games. However, Proposition 4.6 indicates that although the empirical distribution converges to a mixed strategy profile that is not a mixed strategy Nash equilibrium, nevertheless the assessment converges together with the empirical distribution and players learn perturbations.

In this paper, I propose a very specific learning model based on traditional fictitious play to discuss how players do learn and what players can learn when no true environment is assumed. Many questions need to be considered. For instance, how do players learn if players assume that two possible rules, such as fictitious play and reinforcement learning, can be chosen? Furthermore, what if not just the rules, strategies that can be used, are not common knowledge for players? A new framework should be considered to discuss these questions.

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Appendices

Appendix A

Appendix for Chapter 2

A.1 Proofs of Theorem 2.1

For the necessity part, it is straightforward to show that a complexity averse preference $\succsim$ satisfies Axiom 2.1-2.6. The proof of *Only if* part proceeds as a sequence of Lemmas.

Lemma 1. *For any $f \in \mathcal{F}$, if there exists $x \in \overline{\mathcal{F}}$ with $f \succsim x$, then there exist a $x_f \in \overline{\mathcal{F}}$ such that $x_f \sim f$.*

Proof. Let x' be a best outcome and x'' be a worst outcome that some acts induce. By Axiom 2.4, we have $x' \succsim f \succsim x''$. Then, by Axiom 2.2, the following two sets are closed

$$A_1 = \{\alpha \in [0,1] : \alpha x' + (1-\alpha)x'' \succsim f\} \quad \text{and} \quad A_2 = \{\alpha \in [0,1] : f \succsim \alpha x' + (1-\alpha)x''\}$$

Since $A_1 \cup A_2$ is connected, we must have $\alpha \in A_1 \cap A_2$ such that $x_f = \alpha x' + (1-\alpha)x'' \sim f$. $\square$

Lemma 2. *There exist an affine utility function $u : X \to \mathbb{R}$ with unbounded range and a prior probability measure μ over Ω such that*

$$U(f) = \int_{\Omega} u(f(\omega))\mu(d\omega).$$

Proof. For $\succsim$ on $\mathcal{F}$, by Axiom 2.3, if $x \sim y$, we have $\alpha x + (1-\alpha)y \sim \alpha y + (1-\alpha)y$. Then by Axiom 2.5, $\alpha x + (1-\alpha)z \succsim \alpha y + (1-\alpha)z$. Using Axiom 2.3 again, we have $\alpha y + (1-\alpha)y \sim \alpha x + (1-\alpha)y$. By Axiom 2.5, $\alpha y + (1-\alpha)z \succsim \alpha x + (1-\alpha)z$. So we can conclude that if $x \sim y$, we have $\alpha x + (1-\alpha)z \sim \alpha y + (1-\alpha)z$.

Moreover, by Lemma 1, we can find $x_f \sim f$ for all $f \in \{f \in \mathcal{F} : \exists x \in \overline{\mathcal{F}} \text{ s.t } f \succsim x\}$. Then by Axiom 2.6, we have the unboundedness of affine utility function. Then let

$$U(f) = \begin{cases} \int_{\Omega} u(x_f(\omega))\mu(d\omega) & \text{if } f \in \{f \in \mathcal{F} : \exists x \in \overline{\mathcal{F}} \text{ s.t } f \succsim x\}, \\ -\infty & \text{if otherwise.} \end{cases}$$

Just consider $g \in \{f \in \mathcal{F} : \exists x \in \overline{\mathcal{F}} \text{ s.t } f \succsim x\}$. $f \succsim g \Leftrightarrow x_f \succsim x_g$. By Lemma 1, we have the result. Thus, together with Axiom 2.1 and Axiom 2.2, by Theorem 8 (Herstein and Milnor (1953)), a measurable utility can be defined on $\mathcal{F}$. And combining Monotonicity, the DM is a standard expected utility maximizer (Anscombe and Aumann (1963)). □

Lemma 3. *If $f \in \{f \in \mathcal{F} : \exists x \in \overline{\mathcal{F}}$ s.t $f \succsim x\}$, and $\sigma(f) = \sigma(g)$, then $g \in \{f \in \mathcal{F} : \exists x \in \overline{\mathcal{F}}$ s.t $f \succsim x\}$.*

Proof. Construct $f' \in \mathcal{F}$ and $x' \in \overline{\mathcal{F}}$ such that $u(f'(\omega)) = 2u(f(\omega))$ and $u(x'(\omega)) = 0$ for all $\omega \in \Omega$. Then, we have $\frac{1}{2}u(f'(\omega)) + \frac{1}{2}u(x'(\omega)) = u(f(\omega))$, by Lemma 2, it can be rewrite as $u(\frac{1}{2}f'(\omega)+\frac{1}{2}x(\omega))=u(f(\omega))$. Again, by Lemma 2, we have $\frac{1}{2}f'+\frac{1}{2}x'\sim f$. Using order and continuity, we can find $x''\in\overline{\mathcal{F}}$ such that $\frac{1}{2}x''+\frac{1}{2}x'\sim\frac{1}{2}f'+\frac{1}{2}x'$. Then by Axiom 2.3, for any $f\in\mathcal{F}$, we have $\frac{1}{2}x''+\frac{1}{2}f\sim\frac{1}{2}f'+\frac{1}{2}f$. Therefore, we can choose $f\in\mathcal{F}$, such that $u(g(\omega))=u(\frac{1}{2}f'(\omega)+\frac{1}{2}f(\omega))$. By Lemma 2, $g\sim\frac{1}{2}f'+\frac{1}{2}f\sim\frac{1}{2}x''+\frac{1}{2}f$. By Compete and Lemma 1, it is easy to find $g\in\mathcal{F}$ and $z\in\overline{\mathcal{F}}$, such that $\frac{1}{2}x''+\frac{1}{2}f\sim g\succsim\frac{1}{2}x''+\frac{1}{2}z$, where $f\succsim z$ with some $z\in\overline{\mathcal{F}}$. And by order and continuity, there exists $y\in\overline{\mathcal{F}}$, such that $\frac{1}{2}x''+\frac{1}{2}z\sim y$. Finally, we find such $y\in\overline{\mathcal{F}}$, such that $g\succeq y$. Thus, $g\in\{f\in\mathcal{F}:\exists x\in\overline{\mathcal{F}}\text{ s.t } f\succsim x\}$. □

Lemma 4. *Given any $S\in\mathbb{P}$, there exists $f\in\{f\in\mathcal{F}:\exists x\in\overline{\mathcal{F}}\text{ s.t } f\succsim x\}$ such that $\sigma(f)=\sigma(S)$.*

Proof. By contradiction, suppose there exists such $S\in\mathbb{P}$ such that all f that satisfies $\sigma(f)=\sigma(S)$ is not in $\{f\in\mathcal{F}:\exists x\in\overline{\mathcal{F}}\text{ s.t } f\succsim x\}$. That is

$$\sigma(f)\neq\sigma(S)\text{ for all } f\in\{f\in\mathcal{F}:\exists x\in\overline{\mathcal{F}}\text{ s.t } f\succsim x\}.$$

Consider another $S'\in\mathbb{P}$, there exists $f\in\{f\in\mathcal{F}:\exists x\in\overline{\mathcal{F}}\text{ s.t } f\succsim x\}$ such that $\sigma(f)=\sigma(S')$. That is we can find an act g such that $g\succsim x$. Therefore, it is easy to construct an act f' that is $f\sim g$ but with corresponding partition S. □

Lemma 5. *There exists a cost function $\mathcal{C}:\{\sigma(S):S\in\mathbb{P}\}\to\overline{\mathbb{R}}_+$ such that*

$$V(f)=\int_\Omega u(f(\omega))\mu(d\omega)-\mathcal{C}(\sigma(f))$$

where $\sigma(S)=\cap\{\mathcal{A}\subset\Sigma:S\subset\mathcal{A}\text{ and }\mathcal{A}\text{ is a }\sigma\text{-algbra}\}$, $\mathcal{A}$ is a σ-algebra of subsets of Ω; and $\overline{\mathbb{R}}_+\equiv[0,\infty]$.

Proof. Consider a information partition $S \in \mathbb{P}$, by Lemma 4, there exists $f \in \{f \in \mathcal{F} : \exists x \in \overline{\mathcal{F}} \text{ s.t } f \succsim x\}$ with $\sigma(S) = \sigma(f)$. Construct an act $g_S \in \mathcal{F}$ such that $\sigma(g_S) = \sigma(S)$. For example, suppose $S = \{s_1, \cdots, s_n\}$:

$$g_S(\omega) = \begin{cases} x_1 & \text{if } \omega \in s_1, \\ \vdots & \\ x_i & \text{if } \omega \in s_i, \\ \vdots & \\ x_n & \text{if } \omega \in s_n. \end{cases}$$

where $x_i \neq x_j$ for all $i, j \in \{1, \cdots, n\}$. So we have $\sigma(g_S) = \sigma(f)$. By Lemma 3, $g_S \in \{f \in \mathcal{F} : \exists x \in \overline{\mathcal{F}} \text{ s.t } f \succsim x\}$. Then, by Lemma 1, we can find $x_S \in \overline{\mathcal{F}}$ be such that $\frac{1}{2}x_S + \frac{1}{2}x' \sim \frac{1}{2}g_S + \frac{1}{2}x'$ (x' is the same as what we defined in the proof of Lemma 3.). Then, we can pick $f' \in \mathcal{F}$ be such that $u(f'(\omega)) = 2u(f(\omega))$ for any $f \in \{f \in \mathcal{F} : \exists x \in \overline{\mathcal{F}} \text{ s.t } f \succsim x\}$ and any $\omega \in \Omega$. According to this construction, we have $\sigma(f) = \sigma(f')$. Then by Lemma 2, we have

$$f \sim \frac{1}{2}g_S + \frac{1}{2}f' \sim \frac{1}{2}x_S + \frac{1}{2}f'.$$

Thus, we have $U(f) = U(\frac{1}{2}x_S + \frac{1}{2}f') = \frac{1}{2}U(x_S) + \frac{1}{2}U(f') = \frac{1}{2}U(x_S) + \frac{1}{2}U(f')$, that is

$$U(f) = \frac{1}{2}U(x_S) + \frac{1}{2}\int_\Omega u(2f(\omega)(d\omega)) = \frac{1}{2}u(x_S) + \int_\Omega u(f(\omega)(d\omega)).$$

We can define $\mathcal{C}(\sigma(S)) = -\frac{1}{2}U(x_S)$, then $U(P) = V(P)$. $\square$

Lemma 6. *If $\succsim$ satisfies Axiom 2.5, then $\sigma(S) \subset \sigma(S')$ implies $\mathcal{C}(\sigma(S)) \leq \mathcal{C}(\sigma(S'))$.*

Proof. Consider $\alpha x + (1-\alpha)f \sim \beta x + (1-\beta)f$, by Lemma 2, we have

$$\begin{aligned}
&\int_\Omega u(x(\omega))\mu(d\omega) = \int_\Omega u(f(\omega))\mu(d\omega)\\
&\Longrightarrow \lambda \int_\Omega u(x(\omega))\mu(d\omega) = \lambda \int_\Omega u(f(\omega))\mu(d\omega)\\
&\Longrightarrow \lambda \int_\Omega u(x(\omega))\mu(d\omega) + (1-\lambda)\int_\Omega u(g(\omega))\mu(d\omega)\\
&= \lambda \int_\Omega u(f(\omega))\mu(d\omega) + (1-\lambda)\int_\Omega u(g(\omega))\mu(d\omega)
\end{aligned}$$

Since we have $\lambda x + (1-\lambda)g \succsim \lambda f + (1-\lambda)g$, that is $\sigma(g) \subset \sigma(\lambda f + (1-\lambda)g)$ implies

$$\mathcal{C}(\sigma(g)) \leq \mathcal{C}(\sigma(\lambda f + (1-\lambda)g))$$

by the fact that $\mathcal{C}(\sigma(g)) = \mathcal{C}(\sigma(\lambda x + (1-\lambda)g)$. ☐

This completes the proof of sufficiency.

A.2 Proofs of Corollary 2.1

Suppose $\succsim$ is a complexity aversion preference represented by $\langle u, \mu, \mathcal{C}\rangle$. Then by the proof of Theorem 2.1, $\int_\Omega u(f(\omega)(d\omega)) - \mathcal{C}\big(\sigma(f)\big)$ represents $\succsim$. Suppose for contradiction that there exists another complexity cost function $\mathcal{C}'$ where $\int_\Omega u(f(\omega)(d\omega)) - \mathcal{C}'\big(\sigma(f)\big)$ represents $\succsim$. Suppose $\mathcal{C}'\big(\sigma(f)\big) > \mathcal{C}\big(\sigma(f))\big)$ for all $f \in \mathcal{F}^{Sf}$ and $\mathcal{C}'\big(\sigma(g)\big) = \mathcal{C}\big(\sigma(g)\big)$ for all other acts $g \notin \mathcal{F}^{Sf}$.

For an act $f^* \in \mathcal{F}^{Sf}$, we construct a new act $f' \in \mathcal{F}$ such that $f'(\omega) = f^*(\omega) - \epsilon$ for all $\omega \in \Omega$ and a very small but positive $u(\epsilon)$. Suppose $f' \sim g$ where $g \in \mathcal{F}$ but

$g \notin \mathcal{F}^{Sf}$, by $\sigma(f^*) = \sigma(f')$, we know $f^* \succ g$. That is

$$\int_\Omega u(f^*(\omega)(d\omega)) - \mathcal{C}\big(\sigma(f^*)\big) > \int_\Omega u(g(\omega)(d\omega)) - \mathcal{C}\big(\sigma(g)\big).$$

By $\mathcal{C}'\big(\sigma(g)\big) = \mathcal{C}\big(\sigma(g)\big)$, we have

$$\int_\Omega u(f^*(\omega)(d\omega)) - \mathcal{C}\big(\sigma(f^*)\big) > \int_\Omega u(g(\omega)(d\omega)) - \mathcal{C}'\big(\sigma(g)\big).$$

If $\mathcal{C}'\big(\sigma(f)\big) >> \mathcal{C}\big(\sigma(f^*)\big)$, we may have

$$\int_\Omega u(f^*(\omega)(d\omega)) - \mathcal{C}'\big(\sigma(f^*)\big) < \int_\Omega u(g(\omega)(d\omega)) - \mathcal{C}'\big(\sigma(g)\big).$$

Until now, we still can find a $\mathcal{C}'$ that represents the same preference $f^* \succ g$. However, consider following

$$\begin{aligned}
&\int_\Omega u(f'(\omega)(d\omega)) - \mathcal{C}\big(\sigma(f')\big) = \int_\Omega u(g(\omega)(d\omega)) - \mathcal{C}\big(\sigma(g)\big).\\
\iff &\int_\Omega u(f^*(\omega)(d\omega)) - u(\epsilon) - \mathcal{C}\big(\sigma(f^*)\big) = \int_\Omega u(g(\omega)(d\omega)) - \mathcal{C}\big(\sigma(g)\big).\\
\iff &\mathcal{C}\big(\sigma(f^*)\big) + u(\epsilon) - \mathcal{C}\big(\sigma(g)\big) > \mathcal{C}'\big(\sigma(f^*)\big) - \mathcal{C}'\big(\sigma(g)\big)\\
\iff &\mathcal{C}\big(\sigma(f^*)\big) + u(\epsilon) > \mathcal{C}'\big(\sigma(f^*)\big)
\end{aligned}$$

This holds for any very small $u(\epsilon)$, so we have $\mathcal{C}\big(\sigma(f^*)\big) = \mathcal{C}'\big(\sigma(f^*)\big)$, a contradiction. The same logic to prove for a complexity cost function $\mathcal{C}'$ that gives more different costs of acts compared to $\mathcal{C}$. □

A.3 Proofs of Corollary 2.2

Suppose $\langle u, \mu, \mathcal{C}\rangle$ and $\langle u', \mu', \mathcal{C}'\rangle$ represent the same preferences relations, and $\mathcal{C}$ and $\mathcal{C}'$ are canonical. By Lemma 2, the preference relation has an expected utility representation, so $\mu = \mu'$ and $\exists \beta_1 > 0$ and $\beta_2 \in \mathbb{R}$ such that $u = \beta_1 u' + \beta_2$.[1]

Then we turn to prove $\mathcal{C} = \alpha\mathcal{C}$. Suppose u and u' represent the same preference relations. Consider act $f \in \mathcal{F}$ such that there exists $x \in \overline{\mathcal{F}}$ with $f \succsim x$. By Lemma 1 there exist a $x_f \in \overline{\mathcal{F}}$ such that $x_f \sim f$. So we have

$$\int_\Omega u(x)\mu(d\omega) = \int_\Omega u\big(f(\omega)\big)\mu(d\omega) - \mathcal{C}\big(\sigma(f)\big)$$

and

$$\begin{aligned}
&\int_\Omega u'(x)\mu(d\omega) = \int_\Omega u'\big(f(\omega)\big)\mu(d\omega) - \mathcal{C}'\big(\sigma(f)\big)\\
&\Longrightarrow \alpha \int_\Omega u(x)\mu(d\omega) + \beta = \alpha \int_\Omega u\big(f(\omega)\big)\mu(d\omega) + \beta - \mathcal{C}'\big(\sigma(f)\big)\\
&\Longrightarrow \alpha\Big(\int_\Omega u(x)\mu(d\omega) - \int_\Omega u\big(f(\omega)\big)\mu(d\omega)\Big) = \mathcal{C}'\big(\sigma(f)\big)\\
&\Longrightarrow \alpha\mathcal{C}'\big(\sigma(f)\big) = \mathcal{C}'\big(\sigma(f)\big).
\end{aligned}$$

□

[1]For the proof of the uniqueness of μ and u, we refer to Fishburn(1970).

A.4 Proofs of Theorem 2.2

Only if part. Suppose $\succsim^1$ has higher degree of complexity aversion than $\succsim^2$. If $x \succ^1 f$, then

$$\int_\Omega u^1(x(\omega))\mu^1(d\omega) - \mathcal{C}^1\big(\sigma(x)\big) > \int_\Omega u^1(f)(\omega))\mu^1(d\omega) - \mathcal{C}^1\big(\sigma(f)\big)$$

Since $x \succ^1 f$ implies $x \succ^2 f$, then

$$\int_\Omega u^2(x(\omega))\mu^2(d\omega) - \mathcal{C}^2\big(\sigma(x)\big) > \int_\Omega u^2(f(\omega))\mu^2(d\omega) - \mathcal{C}^2\big(\sigma(f)\big)$$

We know $(u^1, \mu^1) = (u^2, \mu^2)$ and $\mathcal{C}\big(\sigma(x)\big) = 0$ for all $x \in \overline{\mathcal{F}}$, then

$$\int_\Omega u^1(x(\omega))\mu^1(d\omega) > \int_\Omega u^1(f(\omega))\mu^1(d\omega) - \mathcal{C}^1\big(\sigma(f)\big) \geq \int_\Omega u^2(f(\omega))\mu^1(d\omega) - \mathcal{C}^2\big(\sigma(f)\big)$$

Thus, we have $\mathcal{C}^1\big(\sigma(f)\big) \leq \mathcal{C}^2\big(\sigma(f)\big)$.

If part. If $\mathcal{C}^1\big(\sigma(f)\big) \leq \mathcal{C}^2\big(\sigma(f)\big)$, then

$$\int_\Omega u^1(x(\omega))\mu^1(d\omega) > \int_\Omega u^1(f(\omega))\mu^1(d\omega) - \mathcal{C}^1\big(\sigma(f)\big) \geq \int_\Omega u^2(f(\omega))\mu^1(d\omega) - \mathcal{C}^2\big(\sigma(f)\big)$$

□

A.5 Proofs of Theorem 2.3

Only if part. Suppose $\succsim^1$ has higher capacity for more complex acts than $\succsim^2$. If $supp(\mathcal{C}^1) \nsubseteq supp(\mathcal{C}^2)$, then we can find an act $h \in supp(\mathcal{C}^2) \backslash supp(\mathcal{C}^1)$. Since $h \notin supp(\mathcal{C}^1)$, we cannot find an act $h' \in supp(\mathcal{C}^1)$ such that $\sigma(h) \subset \sigma(h')$, which implies $\mathcal{C}^1\big(\sigma(h')\big) \geq \mathcal{C}^2\big(\sigma(h)\big)$.

Since $\alpha f + (1-\alpha)x \succ^1 \alpha f + (1-\alpha)g$ implies $\alpha f + (1-\alpha)x \succ^2 \alpha f + (1-\alpha)g$, we have

$$\mathcal{C}^1\Big(\sigma\big(\alpha f+(1-\alpha)x\big)\Big)-\mathcal{C}^1\Big(\sigma\big(\alpha f+(1-\alpha)g\big)\Big) \geq \mathcal{C}^2\Big(\sigma\big(\alpha f+(1-\alpha)x\big)\Big)-\mathcal{C}^2\Big(\sigma\big(\alpha f+(1-\alpha)g\big)\Big).$$

Adding $\mathcal{C}^1\Big(\sigma\big(\alpha f + (1-\alpha)g\big)\Big)$ to both sides, we get

$$\begin{aligned}
&\mathcal{C}^1\Big(\sigma\big(\alpha f + (1-\alpha)g\big)\Big) + \mathcal{C}^1\Big(\sigma\big(\alpha f + (1-\alpha)x\big)\Big) - \mathcal{C}^1\Big(\sigma\big(\alpha f + (1-\alpha)g\big)\Big)\\
&\quad \geq \mathcal{C}^2\Big(\sigma\big(\alpha f + (1-\alpha)x\big)\Big) + \mathcal{C}^1\Big(\sigma\big(\alpha f + (1-\alpha)g\big)\Big) - \mathcal{C}^2\Big(\sigma\big(\alpha f + (1-\alpha)g\big)\Big).\\
&\Longrightarrow \mathcal{C}^1\Big(\sigma\big(\alpha f + (1-\alpha)g\big)\Big) \geq \mathcal{C}^2\Big(\sigma\big(\alpha f + (1-\alpha)x\big)\Big)\\
&\quad + \mathcal{C}^1\Big(\sigma\big(\alpha f + (1-\alpha)g\big)\Big) - \mathcal{C}^2\Big(\sigma\big(\alpha f + (1-\alpha)g\big)\Big)\\
&\quad + \mathcal{C}^1\Big(\sigma\big(\alpha f + (1-\alpha)g\big)\Big) - \mathcal{C}^1\Big(\sigma\big(\alpha f + (1-\alpha)x\big)\Big),
\end{aligned}$$

If $\sigma\big(\alpha f+(1-\alpha)x\big) \subset \sigma\big(\alpha f+(1-\alpha)g\big)$, we have $\mathcal{C}^1\Big(\sigma\big(\alpha f+(1-\alpha)g\big)\Big) \geq \mathcal{C}^1\Big(\sigma\big(\alpha f+(1-\alpha)x\big)\Big)$. Thus, above inequality can be rewrite as

$$\begin{aligned}
\mathcal{C}^1\Big(\sigma\big(\alpha f + (1-\alpha)g\big)\Big) &\geq \mathcal{C}^2\Big(\sigma\big(\alpha f + (1-\alpha)x\big)\Big)\\
&\quad + \mathcal{C}^1\Big(\sigma\big(\alpha f + (1-\alpha)x\big)\Big) - \mathcal{C}^2\Big(\sigma\big(\alpha f + (1-\alpha)g\big)\Big)\\
&\quad + \mathcal{C}^1\Big(\sigma\big(\alpha f + (1-\alpha)g\big)\Big) - \mathcal{C}^1\Big(\sigma\big(\alpha f + (1-\alpha)x\big)\Big)\\
&= \mathcal{C}^2\Big(\sigma\big(\alpha f + (1-\alpha)x\big)\Big) + \mathcal{C}^1\Big(\sigma\big(\alpha f + (1-\alpha)g\big)\Big)\\
&\quad - \mathcal{C}^2\Big(\sigma\big(\alpha f + (1-\alpha)g\big)\Big),
\end{aligned}$$

By Theorem 2.2, we get $\mathcal{C}^1\Big(\sigma\big(\alpha f + (1-\alpha)x\big)\Big) \leq \mathcal{C}^2\Big(\sigma\big(\alpha f + (1-\alpha)g\big)\Big)$. Thus, we have

$$\mathcal{C}^1\Big(\sigma\big(\alpha f + (1-\alpha)g\big)\Big) \geq \mathcal{C}^2\Big(\sigma\big(\alpha f + (1-\alpha)x\big)\Big).$$

Therefore, we find this act h'. For the *If part*, it is obvious. □

A.6 Proofs of Propositions

Proof of Proposition 2.1. The principal solves the problem with two steps.

Step1: choosing the optimal wage scheme for different complexity level. Consider following three cases.

Case1: $|S^w| = 1$, that is $W(\omega_1) = W(\omega_2) = W(\omega_3) = W$

$\sum_\omega \mu_1(\omega)\mu(\omega)\sqrt{W} = 0 < c$ violates constraint (2).

Case2: $|S^w| = 3$, that is $W(\omega_1) \neq W(\omega_2) \neq W(\omega_3)$

The Lagrangian for this this problem is

$$\begin{aligned}\mathcal{L} = &-\sum_\omega \mu_1(\omega)W(\omega) - 3\delta - \lambda\left[\bar{u} - \sum_\omega \mu_1(\omega)\sqrt{W(\omega)}\right] \\ &- \beta\left[c - \sum_\omega \mu_1(\omega)\mu(\omega)\sqrt{W(\omega)}\right]\end{aligned}$$

The first conditions are

$$\frac{\partial\mathcal{L}}{\partial W(\omega)} = -\mu_1(\omega) + \lambda\mu_1(\omega)\frac{1}{2\sqrt{W(\omega)}} + \beta\mu_1(\omega)\mu(\omega)\frac{1}{2\sqrt{W(\omega)}} = 0, \quad \text{(c1)}$$

$$\frac{\partial\mathcal{L}}{\partial\lambda} = \bar{u} - \sum_\omega \mu_1(\omega)\sqrt{W(\omega)} \leq 0, \quad \text{(c2)}$$

$$\frac{\partial \mathcal{L}}{\partial \beta} = c - \sum_{\omega} \mu_1(\omega)\mu(\omega)\sqrt{W(\omega)} \leq 0. \tag{c3}$$

where (c2) and (c3) hold with equality if $\lambda \neq 0$ and $\beta \neq 0$.

(c1) can be rewritten as

$$\frac{1}{2\sqrt{W(\omega)}} = \frac{1}{\lambda + \beta\mu(\omega)}. \tag{c4}$$

Suppose that $\beta = 0$. Then (c4) implies $W(\omega)$ is a constant. This means (c3) fails. Thus, $\beta \neq 0$.

Suppose that $\lambda = 0$. Then (c4) implies $\sqrt{W(\omega)}$ is negative for at least one ω. Thus, $\lambda \neq 0$.

Hence, (c2) and (c3) hold with equality. From (c1), we have

$$\lambda = 2\bar{u}; \beta = \frac{-2c}{\sum_{\omega} \mu_0(\omega)\mu(\omega)}.$$

So we have

$$W(\omega) = (\bar{u} - c\mu(\omega)(\sum_{\omega} \mu_0(\omega)\mu(\omega))^{-1})^2,$$

and the minimal cost:

$$\bar{u}^2 + \frac{c^2 \sum_{\omega} \mu_1(\omega)\mu^2(\omega)}{\sum_{\omega} \mu_0(\omega)\mu(\omega)} + 3\delta$$

Case3: $|S^w| = 2$, that is $W(\omega_i) = W(\omega_j) \neq W(\omega_k)$ for any $i, j, k \in \{1, 2, 3\}$

The Lagrangian for this this problem is

$$\begin{aligned}\mathcal{L} = &-(1-\mu_1(\omega_k))W - \mu_1(\omega_k)W(\omega_k) - 2\delta \\ &- \lambda\left[\bar{u} - (1-\mu_1(\omega_k))\sqrt{W} - \mu_1(\omega_k)\sqrt{W(\omega_k)}\right] \\ &- \beta\left[c - \mu_1(\omega_k)\mu(\omega_k)(\sqrt{W(\omega_k)} - \sqrt{W})\right]\end{aligned}$$

We have

$$(\lambda, \beta) = (2\bar{u}, \beta = \frac{2c}{\mu^2(\omega_k)/(1-\mu_1(\omega_k))}),$$

So we have

$$W(\omega_i) = W(\omega_j) = (\bar{u} - c\mu_1(\omega_k)/\mu(\omega_k))^2.$$

By $\mu_1(\omega) \neq \mu_0(\omega)$ for at least one ω, there must exist $k \in \{1, 2, 3\}$ such that $\mu(\omega_k) < 0$. Hence, above result must exist. Moreover, we have

$$W(\omega_k) = (\bar{u} + c(1-\mu_1(\omega_k))/\mu(\omega_k))^2.$$

and the minimal cost:

$$\bar{u}^2 + \frac{c^2\mu_1(\omega_k)(1-\mu_1(\omega_k))}{\mu^2(\omega_k)} + 2\delta$$

Step2: comparing the design cost among above three cases. The optimal wage scheme is $|S^W| = 2$ if

$$\begin{aligned}\bar{u}^2 + \frac{c^2\mu_1(\omega_k)(1-\mu_1(\omega_k))}{\mu^2(\omega_k)} + 2\delta &\leq \bar{u}^2 + \frac{c^2\sum_\omega \mu_1(\omega)\mu^2(\omega)}{\sum_\omega \mu_0(\omega)\mu(\omega)} + 3\delta \\ &= \frac{c^2\sum_\omega \mu_1(\omega)\mu^2(\omega)}{\sum_\omega \mu_0(\omega)\mu(\omega)} + c^2 + 3\delta - c^2 \\ &= 3\delta - c^2\end{aligned}$$

That is

$$\frac{\mu_1(\omega_k)(1-\mu_1(\omega_k))}{\mu^2(\omega_k)} \le \delta/c^2 - 1.$$

□

Proof of Proposition 2.2. It is immediately from the proof of Proposition 2.1. □

Appendix B

Appendix for Chapter 3

B.1 Proofs of Theorem 3.1

Only if part.

Step 1. The first part of the proof is the same as the proof of Theorem 2.1.

Step 2. *Minimal complexity cost function.* By Axiom 3.3, given any $E \in \Sigma'$, any $f \in \mathcal{F}$, and $x, z \in \overline{\mathcal{F}}$. If $fEz \succsim_E x$, then we have $fEz' \succsim_E x$ for any $z' \in \overline{\mathcal{F}}$ such that $\sigma(fEz) \subset \sigma(fEz')$. Then by Lemma 5,

$$\begin{aligned}
&\int_\Omega u\big((fEz)(\omega)\big)\mu_E(d\omega) - \mathcal{C}_{E,\mu}\big(\sigma(fEz)\big) \geq \int_\Omega u(x(\omega))\mu_E(d\omega) - \mathcal{C}_{E,\mu}\big(\sigma(x)\big) \\
\implies &\int_\Omega u\big((fEz')(\omega)\big)\mu_E(d\omega) - \mathcal{C}_{E,\mu}\big(\sigma(fEz')\big) \geq \int_\Omega u(x(\omega))\mu_E(d\omega) - \mathcal{C}_{E,\mu}\big(\sigma(x)\big)
\end{aligned}$$

Since $\int_\Omega u\big((fEz)(\omega)\big)\mu_E(d\omega) = \int_\Omega u\big(f(\omega)\big)\mu_E(d\omega)$, we have

$$\mathcal{C}_{E,\mu}\big(\sigma(fEz)\big) \geq \mathcal{C}_{E,\mu}\big(\sigma(fEz')\big).$$

However, by monotonicity of cost function, $\sigma(fEz) \subset \sigma(fEz')$ implies $\mathcal{C}_{E,\mu}\big(\sigma(fEz)\big) \leq \mathcal{C}_{E,\mu}\big(\sigma(fEz')\big)$. Thus, we have $\mathcal{C}_{E,\mu}\big(\sigma(fEz)\big) = \mathcal{C}_{E,\mu}\big(\sigma(fEz')\big)$.

The result holds for any $x, z \in \overline{\mathcal{F}}$ with $\sigma(fEz) \subset \sigma(fEz')$. Therefore, we can find another z'' such that $\sigma(fEz'') \subset \sigma(fEz)$. If $fEz'' \succsim_E x$, then we have

$fEz' \succsim_E x$. That is $\mathcal{C}_{E,\mu}\big(\sigma(fEz'')\big) = \mathcal{C}_{E,\mu}\big(\sigma(fEz')\big)$. To conclude, there exists $fEz^* \in \mathcal{F}^{min}(f,E)$ such that $\mathcal{C}_{E,\mu}\big(\sigma(fEz')\big) = \mathcal{C}_{E,\mu}\big(\sigma(fEz^*)\big)$ for any $z' \in \overline{\mathcal{F}}$ with $\sigma(fEz^*) \subset \sigma(fEz')$. $\square$

Step 3. *Ordinal preference consistency.* Follows from the following lemma.

Lemma 7. *Consider a collection of preferences* $\big(\succsim, \{\succsim_E\}_{E\in\Sigma'}\big)$ *such that* $\succsim$ *satisfies Axiom 3.1 (i) and (iv) and* $\{\succsim_E\}_{E\in\Sigma}$ *satisfies Axiom 3.4. Then for all* $x, y \in \overline{\mathcal{F}}$ *and* $E \in \Sigma'$*, we have*

$$x \succsim y \Longleftrightarrow x \succsim_E y.$$

Proof. By completeness, suppose $x \succsim y$. By Axiom 3.1 (iv), $x \succsim y$ implies $xEy \succsim y$ for any $E \in \Sigma$. Then, by *Dynamic Complexity Aversion*, $xEy \succsim yEy$ implies $x \succsim_E y$. Conversely, suppose $x \succsim_E y$ and $y \succ x$. Again, by Axiom 3.1 (iv), $y \succ x$ implies $yEx \succsim x$ for any $E \in \Sigma$. With Dynamic Complexity Aversion, we can conclude $y \succ_E x$, which contradicts with $x \succsim_E y$. Thus, $x \succsim y$. $\square$

Step 4. *Bayesian Updating.* Suppose for any $E \in \Sigma'$, and any $f, g \in \mathcal{F}$ such that

$x_f = x_g$ where $fEx_f \in \mathcal{F}^{min}(f,E)$ and $gEx_g \in \mathcal{F}^{min}(g,E)$. Then for $x_g \in \overline{\mathcal{F}}$ we have

$$
\begin{aligned}
f \succsim_E g &\iff fEx_g \succsim gEx_g \\
&\iff \int_\Omega u\big((fEx)(\omega)\big)\mu(d\omega) - \mathcal{C}\big(\sigma(fEx_g)\big) \\
&\quad \geq \int_\Omega u\big((gEx)(\omega)\big)\mu(d\omega) - \mathcal{C}\big(\sigma(gEx_g)\big) \\
&\iff \int_E u\big(f(\omega)\big)\mu(d\omega) + \int_{\Omega\setminus E} u\big(x(\omega)\big)\mu(d\omega) - \mathcal{C}\big(\sigma(fEx_g)\big) \\
&\quad \geq \int_E u\big(g(\omega)\big)\mu(d\omega) + \int_{\Omega\setminus E} u\big(x(\omega)\big)\mu(d\omega) - \mathcal{C}\big(\sigma(gEx_g)\big) \\
&\iff \int_E u\big(f(\omega)\big)\mu(d\omega) - \mathcal{C}\big(\sigma(fEx_g)\big) \geq \int_E u\big(g(\omega)\big)\mu(d\omega) - \mathcal{C}\big(\sigma(gEx_g)\big) \\
&\iff \frac{1}{\mu(E)}\int_E u\big(f(\omega)\big)\mu_E(d\omega) - \mathcal{C}_{E,\mu}\big(\sigma(f)\big) \geq \frac{1}{\mu(E)}\int_E u\big(g(\omega)\big)\mu_E(d\omega) - \mathcal{C}_{E,\mu}\big(\sigma(g)\big)
\end{aligned}
$$

The last equation comes from step 2. □

If part.

Axiom 3.1. It is standard practice to show that Axiom 3.1 are immediate from the representation.

Minimal Complexity Updating. With the existence of minimal complexity cost function, we have

$$
\begin{aligned}
\mathcal{C}_{E,\mu}\big(\sigma(fEz)\big) &= \mathcal{C}_{E,\mu}\big(\sigma(fEz^*)\big). \\
&= \mathcal{C}_{E,\mu}\big(\sigma(fEz^{*\prime})\big)
\end{aligned}
$$

for any $z, z' \in \overline{\mathcal{F}}$, $\sigma(fEz) \subset \sigma(fEz')$, and $fEz^* \in \mathcal{F}^{min}(f,E)$. Thus, if $fEz \succsim_E$

fEz', then

$$\begin{aligned}
&\int_\Omega u\big((fEz)(\omega)\big)\mu_E(d\omega) - \mathcal{C}_{E,\mu}\big(\sigma(fEz)\big) \geq \int_\Omega u(x(\omega))\mu_E(d\omega) - \mathcal{C}_{E,\mu}\big(\sigma(x)\big) \\
&\Rightarrow \int_\Omega u\big(f(\omega)\big)\mu_E(d\omega) - \mathcal{C}_{E,\mu}\big(\sigma(fEz^*)\big) \geq \int_\Omega u(x(\omega))\mu_E(d\omega) - \mathcal{C}_{E,\mu}\big(\sigma(x)\big) \\
&\Rightarrow \int_\Omega u\big((fEz')(\omega)\big)\mu_E(d\omega) - \mathcal{C}_{E,\mu}\big(\sigma(fEz^*)\big) \geq \int_\Omega u(x(\omega))\mu_E(d\omega) - \mathcal{C}_{E,\mu}\big(\sigma(x)\big) \\
&\Rightarrow \int_\Omega u\big((fEz')(\omega)\big)\mu_E(d\omega) - \mathcal{C}_{E,\mu}\big(\sigma(fEz')\big) \geq \int_\Omega u(x(\omega))\mu_E(d\omega) - \mathcal{C}_{E,\mu}\big(\sigma(x)\big)
\end{aligned}$$

Hence, we have $fEz' \succsim_E x$.

Dynamic Complexity Aversion. For any $E \in \Sigma'$, and any $f, g \in \mathcal{F}$, we can find $h \in \mathcal{F}^{min}(g, E)$. Then, there exists $x \in g(E)$ such that $h = gEx$. If $fEx \succsim gEx$, then by Lemma 5 and Axiom 3.3 we have

$$\begin{aligned}
&\int_\Omega u\big((fEx)(\omega)\big)\mu(d\omega) - \mathcal{C}\big(\sigma(fEx)\big) \geq \int_\Omega u\big((gEx)(\omega)\big)\mu(d\omega) - \mathcal{C}\big(\sigma(gEx)\big) \\
&\Rightarrow \int_E u\big(f(\omega)\big)\mu(d\omega) + \int_{\Omega\setminus E} u\big(x(\omega)\big)\mu(d\omega) - \mathcal{C}\big(\sigma(fEx)\big) \\
&\quad \geq \int_E u\big(g(\omega)\big)\mu(d\omega) + \int_{\Omega\setminus E} u\big(x(\omega)\big)\mu(d\omega) - \mathcal{C}\big(\sigma(gEx)\big) \\
&\Rightarrow \int_E u\big(f(\omega)\big)\mu(d\omega) - \mathcal{C}\big(\sigma(fEx)\big) \geq \int_E u\big(g(\omega)\big)\mu(d\omega) - \mathcal{C}\big(\sigma(gEx)\big) \\
&\Rightarrow \int_E u\big(f(\omega)\big)\mu_E(d\omega) - \frac{1}{\mu(E)}\mathcal{C}\big(\sigma(fEx)\big) \geq \int_E u\big(g(\omega)\big)\mu_E(d\omega) - \frac{1}{\mu(E)}\mathcal{C}\big(\sigma(gEx)\big) \\
&\Rightarrow \int_E u\big(f(\omega)\big)\mu_E(d\omega) - \int_E u\big(g(\omega)\big)\mu_E(d\omega) \\
&\quad \geq \frac{1}{\mu(E)}\mathcal{C}\big(\sigma(fEx)\big) - \frac{1}{\mu(E)}\mathcal{C}\big(\sigma(gEx)\big) \\
&\quad \geq \mathcal{C}_{E,\mu}\big(\sigma(f)\big) - \mathcal{C}_{E,\mu}\big(\sigma(g)\big) \\
&\Rightarrow \int_E u\big(f(\omega)\big)\mu_E(d\omega) - \mathcal{C}_{E,\mu}\big(\sigma(f)\big) \geq \int_E u\big(g(\omega)\big)\mu_E(d\omega) - \frac{1}{\mu(E)}\mathcal{C}_{E,\mu}\big(\sigma(g)\big)
\end{aligned}$$

The last inequality comes from the two facts: $\mathcal{C}_{E,\mu}\big(\sigma(g)\big) = \frac{1}{\mu(E)}\mathcal{C}\big(\sigma(gEx)\big)$ and

$\mathcal{C}_{E,\mu}\big(\sigma(f)\big) = \frac{1}{\mu(E)}\mathcal{C}\big(\sigma(fEx_f)\big) \geq \frac{1}{\mu(E)}\mathcal{C}\big(\sigma(fEx)\big)$ where $fEx_f \in \mathcal{F}^{min}(f,E)$. Therefore, we can conclude $fEx \succsim gEx \Longrightarrow f \succsim_E g$. □

This completes the proof of necessity of Theorem 3.1.

B.2 Proofs of Corollary 3.1

No act is costless. By the definition of $\mathcal{C}_{E,\mu}$, given $E \in \Sigma'$ and $\mu \in \Delta(\Omega)$, $\mathcal{C}_{E,\mu}(x) = min\{\mathcal{C}(h)/\mu(E) : h \in \mathcal{F}, \text{ and } \sigma(h|E) = \sigma(x|E)\} = 0$. And $\mathcal{C}_{E,\mu}(f) > 0$ for all $f \in \mathcal{F}\backslash\overline{\mathcal{F}}$ and $\sigma(f|E) \neq \{E\}$

Monotonicity. Suppose $\sigma(g|E) \subset \sigma(f|E)$, it is easy to see that $\sigma(gEx) \subset \sigma(fEx)$ for any $x \in \overline{\mathcal{F}}$.By the definition of $\mathcal{C}_{E,\mu}$, we have $\sigma(fEx_f) \subset \sigma(fEx_g)$ where $fEx_f \in \mathcal{F}^{min}(f,E)$. Since $\sigma(gEx_f) \subset \sigma(fEx_f)$ and $\sigma(gEx_g) \subset \sigma(gEx_f)$ where $gEx_g \in \mathcal{F}^{min}(g,E)$, we have $\sigma(gEx_g) \subset \sigma(fEx_f)$, which implies $\mathcal{C}\big(\sigma(gEx_g)\big) \leq \mathcal{C}\big(\sigma(fEx_f)\big)$. Hence, we have $\mathcal{C}_{E,\mu}\big(\sigma(f)\big) \geq \mathcal{C}_{E,\mu}\big(\sigma(g)\big)$.

Uniqueness. Then by the proof of Theorem 3.1, $\int_\Omega u(f(\omega)\mu_E(d\omega)) - \mathcal{C}_{E,\mu}\big(\sigma(f)\big)$ represents $\succsim_E$. Suppose for contradiction that there exists another conditional complexity cost function $\mathcal{C}'_{E,\mu}$ where $\int_\Omega u(f(\omega)\mu_E(d\omega)) - \mathcal{C}'_{E,\mu}\big(\sigma(f)\big)$ represents $\succsim$. Suppose $\mathcal{C}'_{E,\mu}\big(\sigma(f)\big) < \mathcal{C}_{E,\mu}\big(\sigma(f))\big)$ for all $f \in \mathcal{F}^{S^f}$ and $\mathcal{C}'\big(\sigma(g)\big) = \mathcal{C}\big(\sigma(g)\big)$ for all other acts $g \notin \mathcal{F}^{S^f}$.

For an act $f^* \in \mathcal{F}^{S^f}$, we construct a new act $f' \in \mathcal{F}$ such that $f'(\omega) = f^*(\omega) + \epsilon$ for all $\omega \in \Omega$ and a very small but positive $u(\epsilon)$. Suppose $f' \sim_E g$ where $g \in \mathcal{F}$ but

$g \notin \mathcal{F}^{S^f}$, by $\sigma(f^*) = \sigma(f')$, we know $g \succ_E f^*$. That is

$$\begin{aligned}
&\int_\Omega u(g(\omega)\mu_E(d\omega)) - \mathcal{C}_{E,\mu}\big(\sigma(g)\big) = \int_\Omega u(f'(\omega)\mu_E(d\omega)) - \mathcal{C}_{E,\mu}\big(\sigma(f')\big) \\
\iff &\int_\Omega u(g(\omega)\mu_E(d\omega)) - \mathcal{C}_{E,\mu}\big(\sigma(g)\big) = \int_\Omega u(f^*(\omega)\mu_E(d\omega)) + u(\epsilon) - \mathcal{C}_{E,\mu}\big(\sigma(f^*)\big) \\
\iff &\mathcal{C}_{E,\mu}\big(\sigma(f^*)\big) - u(\epsilon) - \mathcal{C}_{E,\mu}\big(\sigma(g)\big) < \mathcal{C}'_{E,\mu}\big(\sigma(f^*)\big) - \mathcal{C}'_{E,\mu}\big(\sigma(g)\big) \\
\iff &\mathcal{C}_{E,\mu}\big(\sigma(f^*)\big) - u(\epsilon) < \mathcal{C}'_{E,\mu}\big(\sigma(f^*)\big)
\end{aligned}$$

This holds for any very small $u(\epsilon)$, so we have $\mathcal{C}_{E,\mu}\big(\sigma(f^*)\big) = \mathcal{C}'_{E,\mu}\big(\sigma(f^*)\big)$, a contradiction. □

B.3 Proofs of Corollary 3.2

Suppose $\langle u, \mu, \mathcal{C}, \{\mathcal{C}_{E,\mu}\}_{E\in\Sigma'}\rangle$ and $\langle u', \mu', \mathcal{C}', \{\mathcal{C}'_{E,\mu}\}_{E\in\Sigma'}\rangle$ represent the same preferences relations, and $\mathcal{C}$ and $\mathcal{C}'$ are defined as in Definition 2.1. By Lemma 2, the preference relation has an expected utility representation, so $\mu = \mu'$ and $\exists \beta_1 > 0$ and $\beta_2 \in \mathbb{R}$ such that $u = \beta_1 u' + \beta_2$.

Then we turn to prove $\mathcal{C}_{E,\mu} = \alpha \mathcal{C}_{E,\mu}$. Suppose u and u' represent the same preference relations. Given any $E \in \Sigma'$, consider act $f \in \mathcal{F}$ such that there exists $x \in \overline{\mathcal{F}}$ with $f \succsim_E x$. By Lemma 1, there exist a $x_f \in \overline{\mathcal{F}}$ such that $x_f \sim_E f$. So we have

$$\int_\Omega u(x)\mu_E(d\omega) = \int_\Omega u\big(f(\omega)\big)\mu_E(d\omega) - \mathcal{C}_{E,\mu}\big(\sigma(f)\big)$$

and

$$\begin{aligned}
&\int_\Omega u'(x)\mu_E(d\omega) = \int_\Omega u'\big(f(\omega)\big)\mu_E(d\omega) - \mathcal{C}'_{E,\mu}\big(\sigma(f)\big) \\
&\Longrightarrow \alpha \int_\Omega u(x)\mu_E(d\omega) + \beta = \alpha \int_\Omega u\big(f(\omega)\big)\mu_E(d\omega) + \beta - \mathcal{C}'_{E,\mu}\big(\sigma(f)\big) \\
&\Longrightarrow \alpha\Big(\int_\Omega u(x)\mu_E(d\omega) - \int_\Omega u\big(f(\omega)\big)\mu_E(d\omega)\Big) = \mathcal{C}'\big(\sigma(f)\big) \\
&\Longrightarrow \alpha\mathcal{C}'_{E,\mu}\big(\sigma(f)\big) = \mathcal{C}'_{E,\mu}\big(\sigma(f)\big).
\end{aligned}$$

□

Appendix C

Appendix for Chapter 4

C.1 Proofs of Proposition 4.1

We show that $s_{t+\tau} \neq s_* = (s_*^i, s_*^{-i})$. From the model, we have:

$$\mu_{t+1}^i(s_*^{-i}) = \frac{\eta_t^i(\zeta_t)(s_*^{-i}) + 1 - \varepsilon^i}{\sum_{s' \in S^{-i}} \eta_t^i(\zeta_t)(s') + 1}, \text{ and}$$
$$\mu_{t+1}^i(s^{-i}) = \frac{\eta_t^i(\zeta_t)(s^{-i}) + \varepsilon_{s^{-i}}^i}{\sum_{s' \in S^{-i}} \eta_t^i(\zeta_t)(s') + 1}, \text{ with } s^{-i} \neq s_*^{-i} \text{ at time } t$$

If s_* is played until $t + \tau - 1$, then for $t + \tau$, we have:

$$\mu_{t+\tau}^i(s_*^{-i}) = \frac{\eta_t^i(\zeta_t)(s_*^{-i}) + \tau - \tau\varepsilon^i}{\sum_{s' \in S^{-i}} \eta_t^i(\zeta_t)(s') + \tau}, \text{ and}$$
$$\mu_{t+\tau}^i(s^{-i}) = \frac{\eta_t^i(\zeta_t)(s^{-i}) + \tau\varepsilon_{s^{-i}}^i}{\sum_{s' \in S^{-i}} \eta_t^i(\zeta_t)(s') + \tau}, \text{ with } s^{-i} \neq s_*^{-i} \text{ at time } t + \tau - 1.$$

Then player i maximizes:

$$u_i(s^i;\mu^i_{t+\tau}) = \frac{\eta^i_t\left(\zeta_t\right)\left(s^{-i}_*\right)+\tau-\tau\varepsilon^i}{\sum_{s'\in S^{-i}}\eta^i_t\left(\zeta_t\right)\left(s'\right)+\tau}u(s^i,s^{-i}_*) + \sum_{\substack{s^{-i}\neq s^{-i}_* \\ s^{-i}\in S^{-i}}}\left[\frac{\eta^i_t\left(\zeta_t\right)\left(s^{-i}\right)+\tau\varepsilon^i_{s^{-i}}}{\sum_{s'\in S^{-i}}\eta^i_t\left(\zeta_t\right)\left(s'\right)+\tau}u(s^i,s^{-i})\right]$$

$$= \frac{(1-\varepsilon^i)\tau}{\sum_{s'\in S^{-i}}\eta^i_t\left(\zeta_t\right)\left(s'\right)+\tau}u(s^i,s^{-i}_*) + \frac{\eta^i_t\left(\zeta_t\right)\left(s^{-i}_*\right)}{\sum_{s'\in S^{-i}}\eta^i_t\left(\zeta_t\right)\left(s'\right)+\tau}u(s^i,s^{-i}_*)$$
$$+ \sum_{\substack{s^{-i}\neq s^{-i}_* \\ s^{-i}\in S^{-i}}}\left[\frac{\eta^i_t\left(\zeta_t\right)\left(s^{-i}\right)}{\sum_{s'\in S^{-i}}\eta^i_t\left(\zeta_t\right)\left(s'\right)+\tau}u(s^i,s^{-i})\right]$$
$$+ \sum_{\substack{s^{-i}\neq s^{-i}_* \\ s^{-i}\in S^{-i}}}\left[\frac{\tau\varepsilon^i_{s^{-i}}}{\sum_{s'\in S^{-i}}\eta^i_t\left(\zeta_t\right)\left(s'\right)+\tau}u(s^i,s^{-i})\right]$$

With $\alpha = \frac{1}{\sum_{s'\in S^{-i}}\eta^i_t(\zeta_t)(s')+\tau}$, we have:

$$u_i(s^i;\mu^i_{t+\tau}) = (1-\alpha)\sum_{s^{-1}\in S^{-1}}u^i\left(s^i,s^{-i}\right)\mu^i_t\left(s^{-i}\right) + \tau\alpha(1-\varepsilon^i)u(s^i,s^{-i}_*)$$
$$+ \tau\alpha\sum_{\substack{s^{-i}\neq s^{-i}_* \\ s^{-i}\in S^{-i}}}[\varepsilon^i_{s^{-i}}u(s^i,s^{-i})].$$

So we have:

$$u_i(s^{i\prime};\mu^i_{t+\tau}) - u_i(s^i_*;\mu^i_{t+\tau}) = (1-\alpha)\sum_{s^{-1}\in S^{-1}}\mu^i_t(s^{-i})[u^i(s^{i\prime},s^{-i}) - u^i(s^i_*,s^{-i})]$$
$$+ \tau\alpha(1-\varepsilon^i)[u^i(s^{i\prime},s^{-i}_*) - u^i(s^i_*,s^{-i}_*)]$$
$$+ \tau\alpha\sum_{\substack{s^{-i}\neq s^{-i}_* \\ s^{-i}\in S^{-i}}}\varepsilon^i_{s^{-i}}[u^i(s^{i\prime},s^{-i}) - u^i(s^i_*,s^{-i})].$$

Since s^i_* maximizes date t expected utility, s^i_* maximizes the first term. Since s_*

is a strict Nash equilibrium, s^i_* maximizes the second term. Thus,

$$u^i(s^{i'}, s^{-i}) - u^i(s^i_*, s^{-i}) \leq 0, \text{ and } u^i(s^{i'}, s^{-i}_*) - u^i(s^i_*, s^{-i}_*) \leq 0 \text{ with } < \text{ if } s^i \neq s^i_*.$$

Since there is no weakly dominant strategy, so we can find $s^{i'}$ and $\varepsilon^i_{s^{-i}}$, such that

$$\sum_{\substack{s^{-i} \neq s^{-i}_* \\ s^{-i} \in S^{-i}}} \varepsilon^i_{s^{-i}} [u^i(s^{i'}, s^{-i}) - u^i(s^i_*, s^{-i})] > 0$$

Let ε^i be close to 1, so the second term is very close to 0. As long as τ is large enough, we can have:

$$(1-\alpha) \sum_{s^{-1} \in S^{-1}} \mu^i_t(s^{-i})[u^i(s^{i'}, s^{-i}) - u^i(s^i_*, s^{-i})] + \tau\alpha \sum_{\substack{s^{-i} \neq s^{-i}_* \\ s^{-i} \in S^{-i}}} \varepsilon^i_{s^{-i}} [u^i(s^{i'}, s^{-i}) - u^i(s^i_*, s^{-i})] > 0$$

Thus, we can find $\varepsilon^i \in (0, 1)$, such that at time $t + \tau$, play i will choose $s^{i'}$ instead of s^i_*. □

C.2 Proofs of Proposition 4.4

Since $s_t = s_* = (s^1_*, s^2_*)$ is played for all $t \geq T$, both players will not switch any more. That is the perturbation term will shrink to 0 as t goes to infinity. Which also means that player 1 will assign very high probability to strategy s^2_*. Here we suppose the probability is $1 - \delta$, that is $\mu^1_t(s^2_*) = 1 - \delta$.

If s_* is not a Nash equilibrium, then player 1 can find another pure strategy $s^1_l \neq s^1_*$, such that $u(s^1_l, s^2_*) > u(s^1_*, s^2_*)$. We can set $u(s^1_l, s^2_*) - u(s^1_*, s^2_*) = \frac{2\delta}{(1-\delta)}$, where

$\lambda > 0$. Then we have

$$u(s_l^1, s_*^2; \mu_t^1(s^2)) - u(s_*^1, s_*^2; \mu_t^1(s^2)) = [u(s_l^1, s_*^2) - u(s_*^1, s_*^2)] \cdot (1-\delta) - \sum_{k=2}^{K} \left[\left(u(s_*^1, s_k^2) - u(s_l^1, s_k^2) \right) \cdot \mu_t^1(s_k^2) \right]$$

We can minimize the difference by normalizing the payoff, that is $\max\left(u(s_*^1, s_k^2) - u(s_l^1, s_k^2)\right) = 1$, so we have $\min\left(u(s_l^1, s_*^2; \mu_t^1(s^2)) - u(s_*^1, s_*^2; \mu_t^1(s^2)\right) \geq \frac{2\delta}{(1-\delta)} \cdot (1-\delta) - \delta = \delta$. Thus we have $u(s_l^1, s_*^2; \mu_t^1(s^2)) - u(s_*^1, s_*^2; \mu_t^1(s^2)) > 0$. Which contradicts the fact that s_*^1 maximizes $u(s_*^1, s_*^2; \mu_t^1(s^2))$. □

C.3 Proofs of Observations

Proof of Observation 4.1. Take the general form of a 2×2 matrix game:

Player 2

		R	L
Player 1	T	(a_1, a_2)	(c_1, c_2)
	B	(b_1, b_2)	(d_1, d_2)

If $(d_1 - c_1) \cdot (a_1 - b_1) > 0$, $(d_2 - b_2) \cdot (a_2 - c_2) > 0$ and $(d_1 - c_1) \cdot (d_2 - b_2) > 0$, then the game has two pure strategy equilibria and one mixed strategy equilibrium. That is $d_1 - c_1 > 0$, $a_1 - b_1 > 0$, $d_2 - b_2 > 0$, and $a_2 - c_2 > 0$ hold. Since the mixed strategy equilibrium is $(\frac{1}{2}, T; \frac{1}{2}, R)$, we have the following form:

Player 2

		R	L
Player 1	T	(a_1, a_2)	$(d_1 - \alpha_1, a_2 - \alpha_2)$
	B	$(a_1 - \alpha_1, d_2 - \alpha_2)$	(d_1, d_2)

where α_1, and $\alpha_2 \in \mathbb{R}^+$.

Let $(\eta_0^1(R), \eta_0^1(L)) = (x_1, y_1)$, and $(\eta_0^2(T), \eta_0^2(B)) = (x_2, y_2)$. We have four cases.

Case 1. $(\frac{x_1}{x_1+y_1} \geq \frac{1}{2}, \frac{x_2}{x_2+y_2} \geq \frac{1}{2})$, that is $(x_1 \geq y_1, x_2 \geq y_2)$.

At $t = 1$, $\eta_0^1 = (x_1, x_1 - a)$, and $\eta_0^2 = (x_2, x_2 - b)$, $a, b \in \mathbb{R}^+$. (T, R) is played.

At $t = 2$, $\eta_1^1 = (x_1 + 1 - \varepsilon^1, x_1 - a + \varepsilon^1)$, and $\eta_1^2 = (x_2 + 1 - \varepsilon^2, x_2 - b + \varepsilon^2)$. Since $a < 2\varepsilon^1 - 1$ and $b < 2\varepsilon^2 - 1$, $a, b \in \mathbb{R}^+$. (B, L) is played.

At $t = 3$, since both players switch at $t = 2$, perturbations for both players are still ε^i. That is $\eta_2^1 = (x_1 + 1, x_1 - a + 1)$, and $\eta_1^2 = (x_2 + 1, x_2 - b + 1)$. (T, R) is played.

Finally, the sequence of play can follow the cycle $((T, R)$,$(B, L))$, and the empirical distribution converges to the mixed strategy equilibrium. The other three cases are: $(x_1 \leq y_1, x_2 \leq y_2)$, $(x_1 \geq y_1, x_2 \leq y_2)$, and $(x_1 \leq y_1, x_2 \geq y_2)$. They are dealt with easily. □

Proof of Observation 4.2. It is very similar to the proof of Observation 1. □