'Real' vs 'Imaginary' Noise in Diffusion-Limited Reactions

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Abstract

Reaction-diffusion systems which include processes of the form $A+A \to A$ or $A+A \to \emptyset$ are characterised by the appearance of 'imaginary' multiplicative noise terms in an effective Langevin-type description. However, if 'real' as well as 'imaginary' noise is present, then competition between the two could potentially lead to novel behaviour. We thus investigate the asymptotic properties of the following two 'mixed noise' reaction-diffusion systems. The first is a combination of the annihilation and scattering processes $2A \to \emptyset$, $2A \to 2B$, $2B \to 2A$, and $2B \to \emptyset$. We demonstrate (to all orders in perturbation theory) that this system belongs to the same universality class as the single species annihilation reaction $2A \to \emptyset$. Our second system consists of competing annihilation and fission processes, $2A \to \emptyset$ and $2A \to (n+2)A$, a model which exhibits a transition between active and absorbing phases.

However, this transition and the active phase are not accessible to perturbative methods, as the field theory describing these reactions is shown to be non-renormalisable. This corresponds to the fact that there is no stationary state in the active phase, where the particle density diverges at finite times. We discuss the implications of our analysis for a recent study of another active / absorbing transition in a system with multiplicative noise.

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1 Introduction

Recently the effects of fluctuations in reaction-diffusion systems have attracted considerable attention (for reviews, see Refs. [1, 2]). In sufficiently low spatial dimensions the presence of microscopic particle density fluctuations causes traditional approaches, such as mean-field rate equations, to break down. This has led to the introduction of field-theoretic methods, based on 'Hamiltonian' representations of the associated classical master equation [3, 4, 5]. These methods allow fluctuations to be handled in a systematic manner. The first system to be analysed in this way was the single species annihilation reaction $A + A \to \emptyset$, [6, 7] where it was shown with renormalisation-group (RG) methods that for dimensions $d < 2$, the average density $\overline{n(t)}$ decays to zero at large times according to the power law

$$\overline{n(t)} \sim E_d \, t^{-d/2} \ , \tag{1}$$

with E_d denoting a universal amplitude (for uncorrelated initial conditions), while $\overline{n(t)} \sim E_2 \, t^{-1} \ln t$ in $d = 2$. Furthermore Peliti has demonstrated that the coagulation reaction $A + A \to A$ belongs to the same universality class as the pure annihilation process $A + A \to \emptyset$ [6]. Physically the anomalously slow decay of eq. (1) results from the *anti*correlation of particles in low dimensions. Due to the 'reentrancy' property of random walks for $d \leq 2$, once two particles are in close proximity they will then tend to react rather quickly. Hence at large times the remaining (unreacted) particles are likely to be situated far from their nearest neighbours (i.e. the particles become anticorrelated).

Markedly more complex behaviour may arise once particle production processes are also permitted. For example, the 'Branching and Annihilating Random Walk' (BARW) system defined by the reactions $2A \to \emptyset$ and $A \to (m + 1)A$, displays a dynamic phase transition between an 'active' ($\overline{n(t)} \to n_s > 0$ for $t \to \infty$) and an 'inactive and absorbing' state ($\overline{n(t)} \to 0$ for $t \to \infty$), with a remarkable difference between the cases of odd and even number of offspring m [8, 9]. For odd m and $d \leq 2$ the transition is basically characterised by the critical exponents of directed percolation (DP) [10, 11, 12, 13], whereas for even m and $d < d_c' \approx 4/3$ the phase transition is described by a new universality class, with the density in the entire absorbing phase decaying according to the power law in eq. (1) [8, 9, 14].

A powerful method for the analysis of such systems is provided by the RG improved perturbation expansion [7, 15, 16]. However, once a field-theoretic action for the system has been derived (from a microscopic master equation), it is also possible to write down effective Langevin-type equations, where the form of the noise can now be specified precisely, without any recourse to assumptions and approximations [2]. The nature of the noise can look somewhat peculiar in this representation, for example in the $A + A \to \emptyset$ reaction we have the exact equation for the field $a(x,t)$:

$$\partial_t a(x,t) = D\,\nabla^2 a(x,t) - 2\lambda\, a(x,t)^2 + a(x,t)\,\eta(x,t)\ , \tag{2}$$

where D is the diffusion constant, λ the reaction rate, and

$$\langle \eta(x,t) \rangle = 0\ , \quad \langle \eta(x,t)\eta(x',t') \rangle = -2\lambda\, \delta^d(x-x')\delta(t-t')\ . \tag{3}$$

Hence the noise η is imaginary, a rather counterintuitive result. Recently Grinstein, Muñoz and Tu [17] have studied an equation superficially similar to that given above, with the aim to model active / absorbing transitions in autocatalytic chemical processes. In the special case of a scalar field with a quadratic nonlinearity, their model is defined by the equation

$$\partial_t a(x,t) = D\,\nabla^2 a(x,t) - r\, a(x,t) - u\, a(x,t)^2 + a(x,t)\,\eta(x,t)\ , \tag{4}$$

where

$$\langle \eta(x,t) \rangle = 0 \; , \quad \langle \eta(x,t)\eta(x',t') \rangle = 2\nu \, \delta^d(x-x')\delta(t-t') \; . \tag{5}$$

Notice that the noise in eq. (5) has the opposite sign to that considered previously (i.e. the noise of Ref. [17] is *real*). However, this is an important point since a positive sign in the noise correlator leads to divergences in the renormalised parameters of the theory:

$$\begin{array}{l} \nu_R = Z\nu \\ u_R = Zu \end{array} \quad \text{with} \quad Z = \frac{1}{1-\nu I_d(r)} \quad \text{and} \quad I_d(r) = \int \frac{d^dk}{(2\pi)^d} \frac{1}{r+Dk^2} \; . \tag{6}$$

Hence new singularities emerge when the denominator of Z vanishes. These divergences have not been present in earlier field-theoretic studies of reaction-diffusion systems. Certainly if the noise in the model of Ref. [17] were resulting from the reaction $A + A \to \emptyset$, its correlator should have a *negative* sign, as described above (and hence *no* extra divergences would appear, rendering much of the interesting behaviour in Ref. [17] obsolete). It is therefore not clear to us how the real multiplicative noise of eqs. (4),(5) could be the *only* type generated — we believe that internal, imaginary reaction noise should generically be present as well. Consequently the physical mechanism behind the noise analysed in Ref. [17] remains somewhat obscure.

One of the objects of this letter is to see if equations similar to that analysed by Grinstein, Muñoz and Tu (with real noise, and hence potentially novel behaviour) can be derived consistently for certain reaction-diffusion systems using field-theoretic methods. Potentially at least, in an emerging competition between 'real' and 'imaginary' noise contributions, the 'real' component might prevail in certain circumstances, conceivably leading to the scenario discussed in Ref. [17]. However, our main finding here is that although we have analysed two systems where the noise correlator has both positive and negative components, we have been unable to recover the new features discussed in Ref. [17]. In fact, in the first of our model systems, a combination of two-particle annihilation and scattering processes for the species A and B, the ('imaginary') reaction noise dominates the long-time behaviour, which is described by the asymptotic power law (1). Neither can the 'imaginary' noise terms be neglected in our second reaction-diffusion system, namely combined annihilation and fission processes of a single species A. In this system a perturbative RG analysis breaks down in the active phase and at the dynamical phase transition separating it from the inactive state. Therefore although we can only address the inactive phase, which is again governed by the pure annihilation model, we believe this system cannot reproduce any of the features in Ref. [17].

In the following Sec. 2, we present a more thorough discussion of how 'imaginary' noise terms emerge in processes dominated by two-particle annihilation reactions. On the other hand, problems belonging to the directed-percolation (DP) universality class, as described by Reggeon field theory [10, 11, 12, 13], can be faithfully represented by a simple Langevin equation for the local particle density with 'real' noise. In Sec. 3, we next present our scattering / annihilation model, which we first analyse to one-loop order, and then to all orders in perturbation theory by solving the coupled Bethe-Salpeter equations for the vertices. In Sec. 4, we proceed to discuss the annihilation / fission reaction system, and show that while its properties in the inactive state may be analysed using field-theoretic methods, this is not the case in the active phase or at the dynamic transition itself. Finally, we summarise and discuss our results in the light of the recently proposed transition scenario for 'real' multiplicative noise problems [17].

2 ‘Real’ vs ‘imaginary’ noise in reaction-diffusion systems

Before turning to our investigation of models with competing ‘real’ and ‘imaginary’ noise terms, we briefly outline and review the general issue of how to systematically include fluctuation effects in reaction-diffusion systems. Above the upper critical dimension, a qualitatively correct analysis may be obtained from the associated mean-field rate equations for the average particle densities. Below this dimension, where fluctuations become important, it is tempting to apply a Langevin equation approach, motivated by the success of this technique in equilibrium critical dynamics. However, one has to be aware that these typically irreversible reactions constitute a dynamical system far away from thermal equilibrium. Thus there is no fluctuation-dissipation theorem available which could serve as a guide to the appropriate form of the Langevin noise correlations. One could of course just try the simplest ansatz, namely some form of white noise multiplicatively coupled to a certain power of the particle densities, in order to ensure that all fluctuations vanish when there are no particles left (i.e., in the absorbing state). But, as we shall see shortly, at least for processes dominated by two-particle annihilation reactions, this generically leads to an incorrect analysis.

Thus, in order to systematically include the effects of microscopic density fluctuations in low dimensions, one can instead start with the corresponding classical master equation, then represent this stochastic process by the action of second-quantised bosonic operators, and finally use a coherent-state path integral representation to map this system onto a field theory. This mapping itself is a standard procedure, and is described in detail in Refs. [2, 4, 5, 7], for instance. Apart from the continuum limit that is usually taken, this procedure provides an *exact* mapping of the initial master equation, and involves no assumptions whatsoever regarding the form of the noise, the relevance or irrelevance of certain terms etc. Note that the resulting bosonic theory applies only to systems where there is *no restriction* on the particle occupation number in the microscopic model. For the description of exclusion processes where the site occupation numbers are restricted to 0 or 1, obviously a fermionic representation is more useful.

For example, for the simple two-particle annihilation reaction $A + A \to \emptyset$ the ensuing field-theoretic action reads

$$S = \int d^d x \int dt \left[\hat{a}(\partial_t - D\nabla^2)a - \lambda(1 - \hat{a}^2)a^2 \right] \ , \tag{7}$$

where we omit boundary terms relating to the initial conditions and the projection state. Here D denotes the diffusion constant, λ the annihilation rate, and $\hat{a}(x,t)$ and $a(x,t)$ are bosonic fields. The stationarity conditions ('classical field equations') $\delta S/\delta a = 0$ and $\delta S/\delta \hat{a} = 0$ yield, respectively, $\hat{a} = 1$ and the mean-field rate equation $\partial_t a = D\nabla^2 a - 2\lambda a^2$. Thus, within the mean-field approximation we can identify $a(x,t)$ with the local 'coarse-grained' particle density. However, fluctuations in $a(x,t)$ may *not* be simply related to density variations, as can be seen by performing the shift $\hat{a} = 1 + \bar{a}$,

$$S = \int d^d x \int dt \left[\bar{a}(\partial_t - D\nabla^2)a + 2\lambda \bar{a} a^2 + \lambda \bar{a}^2 a^2 \right] \ . \tag{8}$$

Integrating out the 'response' field $\bar{a}$ in the functional integral $\int \mathcal{D}a\mathcal{D}\bar{a} \exp(-S)$ then leads precisely to eq. (2) with the *negative* noise correlator (3). Physically this counterintuitive result corresponds to the *anti*correlation of particles in low dimensions. Furthermore, power counting reveals that this noise, which originates from the quartic term in the above action, becomes relevant below a critical dimension $d_c = 2$. Because of this pure imaginary noise, $a(x,t)$ clearly cannot represent a physical density variable. Although $\langle a(x,t)\rangle$ *is* equal to the mean density $\overline{n(x,t)}$, similar relations do not hold for higher correlators, see Ref. [2]. Moreover, since $a(x,t)$ is *not* the density field, this means that the noise must also take a non-standard from — in fact η in eq. (3) represents only the contribution to the overall noise from the reaction process. In reality, of course, diffusive and reaction noise can never be disentangled from one another, and consequently there is no particular reason why the noise in eq. (3) should be 'real', i.e., described by a strictly positive correlator.

However, it is possible to give descriptions where the diffusive noise does appear explicitly. One approach begins with a Langevin equation including both real reaction noise *and* diffusive noise. In that case the field $a(x,t)$ represents a coarse-grained local density. This is the approach taken, for example, by Janssen in Ref. [12]. A second possibility is to begin with the representation (7) and then obtain an *equivalent* description in terms of 'density' variables by considering the canonical transformation $a = \exp(-\tilde{\rho})\rho$, $\hat{a} = \exp(\tilde{\rho})$, $\hat{a}a = \rho$ (see Ref. [13]). The resulting effective action in terms of the new 'density' fields ρ and $\tilde{\rho}$ then includes a term $\propto -\tilde{\rho}^2\rho^2$ which corresponds to pure real noise. However, the ensuing field theory contains an extra 'diffusion noise' contribution.

Hence we see that the 'naive' Langevin equation (i.e., eq. (2), but with a positive noise correlator and no diffusion noise) does not provide an appropriate effective description of the above system. On the other hand, for the standard Gribov process $A \to \emptyset$, $A \to A + A$, $A + A \to A$ [12, 13], the action in terms of the shifted fields reads

$$S = \int d^d x \int dt \left[\bar{a}[\partial_t + D(r - \nabla^2)]a - \sigma \bar{a}^2 a + \lambda \bar{a} a^2 + \lambda \bar{a}^2 a^2 \right] \ . \tag{9}$$

Here D and λ represent the diffusion constant and annihilation rate as before, σ is the branching rate, and $r = (\mu - \sigma)/D$ where μ denotes the spontaneous decay rate. Integrating out the response fields now yields

$$\partial_t a(x,t) = D\,(\nabla^2 - r)a(x,t) - \lambda\, a(x,t)^2 + \eta(x,t) \ , \tag{10}$$

with

$$\langle \eta(x,t) \rangle = 0 \, , \ \langle \eta(x,t) \eta(x',t') \rangle = 2[\sigma\, a(x,t) - \lambda\, a(x,t)^2]\, \delta^d(x-x') \delta(t-t') \ . \tag{11}$$

It should be noted, however, that the effective coupling entering the perturbation expansion is actually $\propto \sigma\lambda$, which leads to an upper critical dimension $d_c = 4$. Consequently, the term $\lambda \bar{a}^2 a^2$ becomes irrelevant, and we are left with a pure positive definite noise correlator $\propto \sigma a(x,t)$ (or 'square-root' multiplicative noise, if we replace η with $\sqrt{a}\eta'$). After a simple rescaling, the action is readily mapped onto Reggeon field theory for directed percolation [10, 11, 12, 13]. Thus, here we encounter the generic case where $a(x,t)$ *can* be identified with a coarse-grained density field. Physically, the above reactions lead to particle clustering, and local densities indeed constitute a natural choice for the order parameter field. We also remark that the signs and magnitudes of the prefactors (which may even be chosen to be imaginary) of the cubic nonlinearities in the action (9) do not matter *provided* their product $-\sigma\lambda$ remains real and negative.

In the above analysis, we have brought out the contrasting features of two categories of reaction-diffusion systems — containing either imaginary (anticorrelating) or real (clustering) noise. However, reaction-diffusion systems with a multiplicative noise term $\propto a\eta$ are characteristically governed by pair reaction processes and thus have *'imaginary'* noise. This is in contrast to the ansatz in Ref. [17]. Of course, one might argue that if there are *both* 'real' and 'imaginary' noise contributions present, then the 'real' parts might prevail and lead to the scenario discussed in [17]. This possibility motivates the following two case studies of combined scattering / annihilation and annihilation / fission reactions to which we now turn our attention.

3 The scattering and annihilation process

The first reaction-diffusion system we want to consider consists of the four reaction processes

$$A + A \to \emptyset \ , \quad A + A \to B + B \ , \quad B + B \to A + A \ , \quad \text{and} \quad B + B \to \emptyset \ , \tag{12}$$

which occur at rates λ_{AA}, λ_{AB}, λ_{BA}, and λ_{BB}, respectively, and with diffusion constants D_A and D_B for the A and B particles. We choose uncorrelated initial conditions where the A and B particles are distributed randomly. Physically the above reaction scheme might occur if the A particles could undergo a scattering process turning into B particles, and vice versa, in addition to the presence of the annihilation reactions. In order to systematically include the effects of microscopic density fluctuations in low dimensions, we represent the corresponding master equation by a coherent-state path integral (see Sec. 2). In terms of the continuous fields a, $\bar{a}$, b, $\bar{b}$, the diffusivities $D_i \neq 0$, continuum reaction rates $\{\lambda_{ij}\}$, and the initial homogeneous densities n_i (where $i, j = A, B$), the action reads (for

$t \geq 0$):

$$S = \int d^d x \int dt \Big[\bar{a}(\partial_t - D_A \nabla^2) a + \bar{b}(\partial_t - D_B \nabla^2) b + 2\lambda_{AA} \bar{a} a^2 + \lambda_{AA} \bar{a}^2 a^2 +$$
$$+ 2\lambda_{BB} \bar{b} b^2 + \lambda_{BB} \bar{b}^2 b^2 + 2\lambda_{AB} \bar{a} a^2 + \lambda_{AB} \bar{a}^2 a^2 - 2\lambda_{AB} \bar{b} a^2 - \lambda_{AB} \bar{b}^2 a^2 +$$
$$+ 2\lambda_{BA} \bar{b} b^2 + \lambda_{BA} \bar{b}^2 b^2 - 2\lambda_{BA} \bar{a} b^2 - \lambda_{BA} \bar{a}^2 b^2 - n_A \bar{a} \delta(t) - n_B \bar{b}\, \delta(t)\Big] \ . \quad (13)$$

If we now integrate out the response fields $\bar{a}$ and $\bar{b}$ from the functional integral $\int \mathcal{D}a\mathcal{D}\bar{a}\mathcal{D}b\mathcal{D}\bar{b} \exp(-S)$, we find that the above reaction-diffusion system can be described *exactly* by a pair of Langevin-type equations

$$\partial_t a(x,t) = D_A \nabla^2 a(x,t) - 2(\lambda_{AA} + \lambda_{AB})\, a(x,t)^2 + 2\lambda_{BA}\, b(x,t)^2 + \eta_A(x,t) \ ,$$
$$\partial_t b(x,t) = D_B \nabla^2 b(x,t) - 2(\lambda_{BB} + \lambda_{BA})\, b(x,t)^2 + 2\lambda_{AB}\, a(x,t)^2 + \eta_B(x,t) \ , \quad (14)$$

with noise correlations

$$\langle \eta_A(x,t) \rangle = \langle \eta_B(x,t) \rangle = 0 \ , \quad (15)$$
$$\langle \eta_A(x,t) \eta_A(x',t') \rangle = [\lambda_{BA}\, b(x,t)^2 - (\lambda_{AA} + \lambda_{AB})\, a(x,t)^2]\, \delta^d(x-x') \delta(t-t') \ ,$$
$$\langle \eta_B(x,t) \eta_B(x',t') \rangle = [\lambda_{AB}\, a(x,t)^2 - (\lambda_{BB} + \lambda_{BA})\, b(x,t)^2]\, \delta^d(x-x') \delta(t-t') \ .$$

Hence, as desired, we have constructed a system where in a Langevin-type formalism we have terms of *both* signs present in the correlator. Thus we can now attempt to answer the question of whether this ‘competition’ alters the structure of the theory in low dimensions where fluctuations are of vital importance.

Power counting on the action (13) reveals that all the reaction rates $\{\lambda_{ij}\}$ have dimension $\sim \mu^{2-d}$, where μ denotes a momentum scale. Hence we expect to find a critical dimension $d_c = 2$, below which fluctuations change the mean-field behaviour qualitatively and the theory must be renormalised. As in the pure annihilation model (8), this renormalisation is simple since the diagrammatic structure of

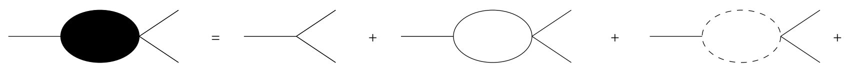

Figure 1: The temporally extended vertex function $\tilde{\lambda}_{AA}(k,s)$ to one-loop order.

the theory does not permit any dressing of the propagators. Hence the only renormalisation required is that for the reaction rates. We now define $\tilde{\lambda}_{AA} = \lambda_{AA} + \lambda_{AB}$, $\tilde{\lambda}_{AB} = \lambda_{AB}$, $\tilde{\lambda}_{BA} = \lambda_{BA}$ and $\tilde{\lambda}_{BB} = \lambda_{BB} + \lambda_{BA}$. The temporally extended vertex function for $\tilde{\lambda}_{AA}(k,s)$ to one-loop order is given by the sum of diagrams shown in Fig. 1 (here s is the Laplace transformed time variable; time runs from right to left). The diagrams for the other vertex functions look quite similar. Evaluation of these one-loop diagrams yields the following form of the renormalised reaction rates: $g_{ij} = C_d \tilde{\lambda}_{ij}(k,s)|_{k^2/4=\mu^2, s=0}/D_i \mu^\epsilon$, where $\epsilon = 2-d$, and $C_d = \Gamma(2-d/2)/2^{d-1}\pi^{d/2}$ is a geometric factor. This leads in a straightforward manner to the following one-loop RG beta functions $\beta_{ij} = \mu\, \partial g_{ij}/\partial \mu$:

$$\beta_{AA} = g_{AA}(-\epsilon + g_{AA}) + g_{AB}\, g_{BA} \;, \tag{16}$$

$$\beta_{AB} = g_{AB}(-\epsilon + g_{AA} + g_{BB}) \;, \tag{17}$$

$$\beta_{BA} = g_{BA}(-\epsilon + g_{AA} + g_{BB}) \;, \tag{18}$$

$$\beta_{BB} = g_{BB}(-\epsilon + g_{BB}) + g_{AB}\, g_{BA} \;. \tag{19}$$

In fact it can be shown that the above one-loop beta functions are actually *exact* to all orders in perturbation theory. This is readily accomplished by writing down the full coupled Bethe-Salpeter equations for the vertices. Diagrammatically, this corresponds to replacing *either* the right-hand *or* the left-hand bare vertices in all the one-loop contributions (see Fig. 1) by their fully renormalised counterparts.

This freedom of choice immediately implies the relation

$$\tilde{\lambda}_{AB}(k,s)/\tilde{\lambda}_{AB} = \tilde{\lambda}_{BA}(k,s)/\tilde{\lambda}_{BA} = N(k,s) \ . \tag{20}$$

After absorbing the diffusivities D_i into the bare couplings $\tilde{\lambda}_{ij}$ and the full vertex functions $\tilde{\lambda}_{ij}(k,s)$, respectively, and introducing the abbreviation $I_i(k,s) = (2\pi)^{-d} \int d^dp \ [p^2 + (k^2/4) + (s/2D_i)]^{-1}$, the coupled *exact* Bethe-Salpeter equations can be explicitly written as

$$\tilde{\lambda}_{AA}(k,s) \, [1 + \tilde{\lambda}_{AA} I_A(k,s)] + \tilde{\lambda}_{AB}(k,s) \, \tilde{\lambda}_{BA} \, I_B(k,s) = \tilde{\lambda}_{AA} \ , \tag{21}$$

$$\tilde{\lambda}_{AA}(k,s) \, \tilde{\lambda}_{AB} \, I_A(k,s) + \tilde{\lambda}_{AB}(k,s) \, [1 + \tilde{\lambda}_{BB} \, I_B(k,s)] = \tilde{\lambda}_{AB} \ , \tag{22}$$

together with a second pair of equations which follow by interchanging $A \leftrightarrow B$ in eqs. (21) and (22). These coupled linear equations (20)–(22) for $\tilde{\lambda}_{ij}(k,s)$ are solved by

$$N(k,s)^{-1} = [1 + \tilde{\lambda}_{AA} I_A(k,s)] \, [1 + \tilde{\lambda}_{BB} \, I_B(k,s)] - \tilde{\lambda}_{AB} \, \tilde{\lambda}_{BA} \, I_A(k,s) \, I_B(k,s) \ , \tag{23}$$

and

$$[\tilde{\lambda}_{AA}(k,s)/\tilde{\lambda}_{AA} \, N(k,s)] = 1 + [\tilde{\lambda}_{BB}(1 - \tilde{\lambda}_{AB}\tilde{\lambda}_{BA}/\tilde{\lambda}_{AA}\tilde{\lambda}_{BB})] \, I_B(k,s) \ , \tag{24}$$

$$[\tilde{\lambda}_{BB}(k,s)/\tilde{\lambda}_{BB} \, N(k,s)] = 1 + [\tilde{\lambda}_{AA}(1 - \tilde{\lambda}_{AB}\tilde{\lambda}_{BA}/\tilde{\lambda}_{AA}\tilde{\lambda}_{BB})] \, I_A(k,s) \ . \tag{25}$$

At the normalisation point one has $I_i(2\mu, 0) = C_d \mu^{-\epsilon}/\epsilon$, and after some tedious but straightforward algebra eqs. (23)–(25) yield again the beta functions (16)–(19).

We can now examine the above eqs. (16)–(19), which we have just demonstrated to hold to *all* orders in perturbation theory, for fixed point solutions g_{ij}^* defined by $\beta_{ij}(\{g_{ij}^*\}) = 0$. For $d > 2$ we find, as expected, merely the trivial Gaussian fixed point where all $g_{ij}^* = 0$. However, for $d < 2$, the only *stable* fixed points are those describing *uncoupled* annihilation processes, i.e.,

$$g^*_{AB} = g^*_{BA} = 0 \ , \qquad g^*_{AA} = g^*_{BB} = \epsilon \ . \tag{26}$$

Furthermore, there are also other solutions, for example the fixed line

$$\begin{aligned} &0 < c = g^*_{AB}\, g^*_{BA} \leq \epsilon^2/4 \quad \text{fixed but arbitrary, and} \\ &2g^*_{AA} = \epsilon \pm \sqrt{\epsilon^2 - 4c} \ , \quad 2g^*_{BB} = \epsilon \mp \sqrt{\epsilon^2 - 4c} \ ; \end{aligned} \tag{27}$$

but these, like the Gaussian fixed point, turn out to be *unstable* for $d < 2$.

Hence the above annihilation / scattering model (12) asymptotically becomes rather simple, and in fact lies in the same universality class as single-species annihilation (with respect to both the decay exponent *and* amplitude). Hence each species of particle decays according to eq. (1) as $t \to \infty$ for $d < 2$. Physically this is a result of the 'reentrancy' property of random walks — as soon as two particles are in close proximity, they will rapidly annihilate, even in the presence of scattering processes. Therefore we conclude that, for this system, the presence of 'real' as well as 'imaginary' noise has not introduced any novel behaviour. We finally remark that the above results also apply in the extreme asymmetric situation where, say, $\tilde{\lambda}_{BA} = 0$ but $\tilde{\lambda}_{AB} > 0$ originally, i.e., when there is spontaneous *unidirectional* transformation of pairs of A particles into pairs of B particles, but not vice versa. At least in this special case, our result that this pairwise transmutation is irrelevant in the long-time limit, appears nontrivial. For example, in the related case of DP processes for A, B particles with coinciding critical points, which are coupled via the reaction $A \to B$, the usual DP critical exponent β is replaced by a much smaller density exponent as a consequence of an ensuing *multicritical* point [18].

4 The annihilation and fission process

Our second reaction-diffusion system consists of the processes

$$A + A \rightarrow \emptyset \quad \text{and} \quad A + A \rightarrow (n+2)A \ , \tag{28}$$

to which we assign the annihilation rate λ and 'fission' rate σ_n. Note that these processes differ from the 'Branching and Annihilating Random Walks' [8, 9, 14] mentioned earlier in that offspring particles can only be produced upon collision of two A particles. The corresponding action derived from the master equation describing the reactions (28) reads in terms of the *unshifted* continuous fields $\hat{a}(x,t)$ and $a(x,t)$

$$S = \int d^dx \int dt \left[\hat{a}(\partial_t - D\nabla^2)a - \lambda(1-\hat{a}^2)a^2 + \sigma_n(1-\hat{a}^n)\hat{a}^2 a^2\right] \tag{29}$$

(the terms depending on the homogeneous, uncorrelated initial density distribution and on the projection state have been omitted here). Once again we point out that this theory is valid only for unrestricted particle occupation numbers in the microscopic model. It is quite possible that altering the microscopic rules for site occupancy (for example by allowing only 0 or 1 particles at a site) may change some of our later conclusions [19]. If we now proceed by performing the shift $\hat{a} = 1 + \bar{a}$, the effective action becomes

$$S = \int d^dx \int dt \Bigg[\bar{a}(\partial_t - D\nabla^2)a + (2\lambda - n\sigma_n)\,\bar{a}a^2 + \\ + \left(\lambda - \frac{n(n+3)}{2}\sigma_n\right)\bar{a}^2a^2 - \sigma_n \sum_{l=3}^{n+2} \binom{n+2}{l} \bar{a}^l a^2\Bigg] \ . \tag{30}$$

If all vertices $\bar{a}^l a^2$ for $l \geq 3$ are neglected, this field theory becomes equivalent to a nonlinear Langevin equation

$$\partial_t a(x,t) = D\,\nabla^2 a(x,t) + (n\sigma_n - 2\lambda)\,a(x,t)^2 + a(x,t)\,\eta(x,t) \ , \tag{31}$$

$$\langle \eta(x,t) \rangle = 0 \ , \quad \langle \eta(x,t)\eta(x',t') \rangle = [n(n+3)\sigma_n - 2\lambda] \ \delta^d(x-x')\delta(t-t') \ , \qquad (32)$$

which again describes competition between 'real' noise (associated with σ_n) and 'imaginary' noise (associated with λ). Upon comparing with the model of Ref. [17], eqs. (4), (5), we see that the annihilation / fission process apparently corresponds to their parameters $r = 0$, $u = 2\lambda - n\sigma_n$, and $\nu = n(n+3)\sigma_n/2 - \lambda$.

Notice that it is potentially dangerous to perform the shift $\hat{a} = 1 + \bar{a}$ and then to arbitrarily omit certain nonlinearities [2, 14], due to the discrete symmetry $\hat{a} \to -\hat{a}$, $a \to -a$, under which the action (29) is invariant (for n even). This symmetry corresponds to local particle number conservation modulo 2, which is lost in the Langevin description based on (30). Furthermore, the neglected terms have the same scaling dimension as those retained. We therefore proceed with the analysis of the *unshifted* theory (29). We find that both the annihilation and fission rate have identical scaling dimension $\sim \mu^{2-d}$ and thus the upper critical dimension is again expected to be $d_c = 2$. For $d > 2$, a description given by the mean-field equation (i.e. eq. (31) without noise) should become qualitatively correct. For $n\sigma_n < 2\lambda$ this leads asymptotically to a density decay $\bar{n}(t) \sim t^{-1}$ (with reduced annihilation rate $\lambda_R = \lambda - n\sigma_n/2$); for $n\sigma_n > 2\lambda$, on the other hand, the density grows rapidly and diverges at $t_c = 1/(n\sigma - 2\lambda)n_0$, where n_0 is the initial density. Thus, there is no stationary state in the active phase.

We now consider a one-loop analysis of the action (29) for $d \leq 2$. To this order all the couplings associated with the interaction vertices in (29) are renormalised. For example the coupling σ_n is renormalised by the diagrams shown in Fig. 2a. However, in addition to this, other processes are also generated at this order: $2A \to nA$ (by a combination of fission and annihilation, see Fig. 2b) and $2A \to 2(n+1)A$ (by two successive fission reactions, see Fig. 2c). Furthermore, the mechanisms

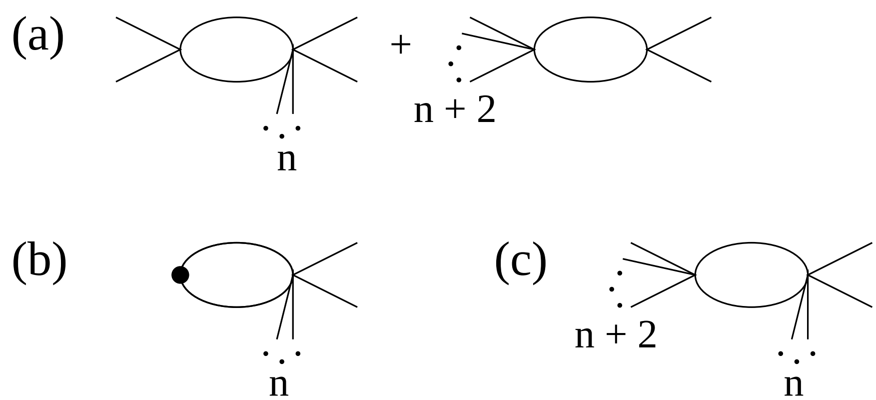

Figure 2: One-loop diagrams for (a) the renormalisation of the σ_n coupling, (b) the generation of the process $2A \to nA$ from a combination of $2A \to (n+2)A$ and $2A \to \emptyset$, and (c) the generation of the process $2A \to 2(n+1)A$ by two successive fission reactions.

producing these processes (with increasingly large n) are not simply restricted to the one-loop level — new particle creation vertices are effectively generated at each successive order in perturbation theory. Therefore the number of (relevant) higher-order couplings, each of which requires renormalisation, increases without bound as higher and higher orders of perturbation theory are considered. Hence we must conclude that the field theory is non-renormalisable: an infinite number of renormalisations would be needed to render the theory free from divergences.

Nevertheless some further progress is possible by considering the original master equation for the annihilation-fission process [20]. Omitting the diffusive terms, we have

$$\frac{\partial P(\{m_i\};t)}{\partial t} = \lambda^{\mathrm{lat}} \sum_i [(m_i+2)(m_i+1)P(\{\ldots,m_i+2,\ldots\};t) - m_i(m_i-1)P(\{m_i\};t)]$$
$$+\sigma_n^{\mathrm{lat}} \sum_i [(m_i-n)(m_i-n-1)P(\{\ldots,m_i-n,\ldots\};t) - m_i(m_i-1)P(\{m_i\};t)], \quad (33)$$

where $P(\{m_i\};t)$ is the configuration probability for finding occupation numbers $\{m_i\}$ at time t, and where λ^{lat} and σ_n^{lat} are the lattice annihilation and fission rates, respectively. Using the relation $\overline{m(t)} = \sum_{\{m_i\}} m_i P(\{m_i\};t)$, eq. (33) implies that

$$\frac{d\overline{m(t)}}{dt} = (n\sigma_n^{\text{lat}} - 2\lambda^{\text{lat}})\,\overline{m(m-1)}. \tag{34}$$

Since $m_i = 0, 1, 2, \ldots$, we see that $\overline{m(m-1)}$ is non-negative, and hence that $n\sigma_n^{\text{lat}} = 2\lambda^{\text{lat}}$ marks the transition point between the active and inactive phases. Note that actually at the transition ($n\sigma_n^{\text{lat}} = 2\lambda^{\text{lat}}$), the average density will remain constant, whereas in the active phase it will diverge. These conclusions can be confirmed by studying the shifted action (30). If we have the equality $2\lambda = n\sigma_n$ for the *bare* field-theoretic parameters, then the bare cubic coupling vanishes. However, the structure of the higher-order vertices ensures that a cubic coupling cannot then be regenerated *at any order* in perturbation theory. Hence we can again conclude that $2\lambda = n\sigma_n$ is the transition point between the active and inactive phases. Furthermore in the inactive phase, where the annihilation mechanism dominates, and the successive generation of an infinite series of fission processes is probably suppressed, we might expect the density to decay as $t^{-d/2}$ (for $d < 2$) due to the strong particle *anti*correlations which emerge as a result of the annihilation process in low dimensions.

However, the non-renormalisability of the field theory means that we are unable to fully address the properties of either the active phase or the active / inactive transition. This failure may be associated with the fact that the active phase is not in a stationary state (at least in mean-field theory, and with unrestricted site occupancy $m_i = 0, 1, \ldots, \infty$, the density in this phase diverges in finite time). This behaviour, taken together with the massless nature of our field theory (i.e., there is no term proportional to the field $a(x,t)$ in eq. (31)), implies that the transition from the absorbing to the active phase cannot be in the DP universality class. Rather the transition is closer to being 'first order', as suggested by an exact evaluation of the correlation function at criticality [20]. As field-theoretic methods clearly cannot shed any further light on this problem, we hope that our analysis will stimulate further work using, for example, exact one-dimensional methods. In addition, numerical simulations presently in progress seem to indicate that this annihilation / fission system may display remarkably rich behaviour [19].

5 Summary

In this paper we have studied the effects of various types of noise in diffusion limited reactions. In section 2 we emphasised that 'naive' Langevin equations (with positive noise correlators) fail to accurately describe systems controlled by pair reaction processes, where the noise is in fact 'imaginary'. Physically this failure is associated with the anticorrelation of particles in low dimensions. On the other hand such a naive approach does indeed work for the Gribov process, where the noise turns out to be 'real' (related to particle clustering).

We then studied two diffusion-limited reaction systems with both real and imaginary noise components: the annihilation / scattering processes (12) and the annihilation / fission processes (28). We have shown (to all orders in perturbation theory) that the first of these belongs to the same universality class as the pure annihilation model in dimensions $d \leq 2$, while for $d > 2$ the mean-field rate equations apply. However, the second system displays a transition between an active and absorbing state, which is not accessible to perturbative analysis. In both cases, despite the competition between 'real' and 'imaginary' noise, we have been unable to recover any of the interesting behaviour discussed in Ref. [17]. In fact, considering that the processes discussed here, along with BARW, are amongst the *simplest* reactions leading to both 'real' and 'imaginary' multiplicative noise, it is rather unclear which *physical* system might be described by the Langevin equation (4) with purely 'real' multiplicative noise (5), and thus display the nontrivial effects of Ref. [17].

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