Supply Chain Revenue Management Considering Components' Quality and Reliability

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(ABSTRACT)

The reliability and quality of suppliers' components are inevitably two factors that impact the performance of the supply chain. Stochastic reliability affects the final production quantity and hence makes it more difficult to predict the manufacturer's best ordering quantity as opposed to the simpler traditional news vendor model. In addition, the quality of suppliers' products directly influence the potential demand in the market. Hence every firm in the supply chain system faces the needs to invest time, money and effort to improve the product quality even though it may bring a higher production and investment cost.

Thus our dissertation is divided into two parts. In the first part, we build a model for a two echelon supply chain system in which a single manufacturer sells his product to a market with stochastic demand. A group of suppliers provide essential components for the manufacturer. They may be: 1) homogeneous component suppliers, 2) complementary component suppliers or 3) divided into subgroups, suppliers in the same subgroup provide the same component while the components from different subgroups are assembled in the final product. The fraction of effective component ordered from each supplier is a random variable. We first analyze the manufacturer's optimal ordering quantity decision. We identify several important properties of the optimal decision. Then based on those properties, we devise optimal solution procedures and heuristic methods for the above three systems. Finally, in the case of Bernoulli reliability, we investigate the suppliers' price competition by non-cooperative game theory.

In the second part, we model a two echelon assembly system which faces deterministic demand affected by the market price and quality of the product. Therefore, the decisions of the firms are divided into two stages: in the first stage, they decide on how much effort to invest in the quality of the components or the final product to stimulate the market. They may make decisions simultaneously or sequentially. Then after the efforts are invested, in the second stage, the component suppliers first decide on their components' wholesale price and then the manufacture decides on the market price given the wholesale price. We identify the existence of Nash equilibrium in each stage through potential functions. Moreover, in the first stage decision, we find that the competition with a leader can always benefit the whole system compared with simultaneous competition.

Dedication

To my mother Youying and father Shuanglin.

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Chapter 1

Introduction

In supply chain management, inventory control and product pricing are two major decisions. Building from the very basic news vendor model only considering the volatile market demand, researchers included more and more factors into it: lead time, risk pooling, multiple periods, pricing. Up until the late of 1980s, people started to notice that the reliability of the components could play an important role in manufacturer's ordering quantity decision. From then on, the topic of multiple sourcing with unreliable suppliers attracts many researchers' attention in supply chain management [see Federgruen [11] & [12], Dada [9], Tomlin [42], Gurnani [20], [21] & [23] etc.]. They published a series of papers starting from dual sourcing to multiple sourcing in multiple periods with multiple constraints. But until now, as far as we know, no one can give very detailed answers to the sensitivity analysis on the manufacturer's optimal ordering quantity and whether there exists a pure strategy Nash equilibrium in the suppliers pricing game followed by the manufacturer's optimal decision, even in special cases. The first topic of our research thus contributes to the academic literature by offering some clue to the above two question. We are also the first ones who combine the sourcing and assembly system under stochastic reliability and suggest an effective solution procedure.

In our first topic, we consider a situation in which a manufacturer procures components from multiple suppliers to produce the final product in an uncertain market. If there is only one supplier and the yield rate is constant, the problem can be easily handled by the traditional news vendor model, in which the optimal ordering quantity will make sure the service level is the ratio of marginal underage cost over the summation of underage and overage cost. However, if the yield rate (fraction of effective components) is a random variable, even for suppliers providing homogeneous components with the same wholesale price, the manufacturer may have the incentive to order from both suppliers to reduce the risk of upstream uncertainty. In our research, we explore the manufacturer's supplier selection and order allocation decisions under both downstream and upstream uncertainty in a single period in two different production systems: the sourcing system and the assembly system. In the sourcing system, the final product is produced from a single component which can be provided by multiple suppliers. On the other hand, in the assembly system, several complementary components are used to produce the final product, each provided by a single supplier. Furthermore, for suppliers with Bernoulli reliability, we analyze the suppliers' pricing game given the manufacturer's optimal decision. We find that, in a sourcing system with two suppliers and uniform market demand, there exists a case with no pure strategy Nash equilibrium. If each supplier's best response is an increasing function in the other supplier's wholesale price, then a Nash equilibrium exits in suppliers' pricing game and the manufacturer will always prefer the suppliers' competition compared with suppliers' coordination. This happens for uniform demand or exponential demand under some restriction on parameters. In the assembly model, the equilibrium always exists under any continuous demand. The Nash equilibrium with nonzero ordering quantity is unique for the demand distribution with an increasing generalized failure rate under mild restriction on the distribution support. Based on this, we further analyze the supply chain efficiency and separation of profit. We also analyze the suppliers investment decision on reliability. Finally, we consider a more complicated combined sourcing and assembly system in which there are multiple complementary components with a group of suppliers for each component. We suggest an effective algorithm based on an iterative procedure of expansion in the set of suppliers.

Another trend in recent supply chain management is to include product innovation investment in supply chain design. This trend started from the economics and marketing literature which emphasize the impact of product quality on the market demand function. In this way, for substitutable products, the majority of research is concentrated in two main areas: product pricing and quality differentiation which analyzes the initial product introduction and process of product development which studies the product improvement by innovation. To our knowledge, there is only one paper that studies the case with a single product composed of a single component [Gurnani et al. [23]]. In their model, both the final product manufacturer and the component supplier can make efforts on the quality innovation. They studies the impact of pricing and innovation investment decision sequence. In this topic, our contribution to the academic literature is to extend and enrich their results to the case with single product composed of multiple components under general quality and cost function.

In our second topic, we consider the impact of the decision sequence in an assembly system: in first stage, multiple component producers invest in technology independently to improve their own quality and the manufacturer makes its own effort in improving its sales or production system. After the effort dependent demand is realized, component producers provide their components based on the wholesale price contracts. In the end, after assembly, the manufacturer sells the final product to the market at his optimal market price. We assume that the component producers play a simultaneous pricing competition game given the manufacturer's optimal sale price. We compare three scenarios in the effort investment stage: simultaneous competition, one or more firms as a group in the lead followed by simultaneous competition among the other firms and system wide coordination of effort to maximize the supply chain's profit. We found, for most general demand functions and for continuous cost functions, the Nash equilibrium exists in the above effort and price games. Furthermore, in the case of a linear demand function with an additive effort effect on demand or exponential demand function with a multiplicative effort effect on demand, it turns out that every player, system and customer prefer the scenario with one or more firms as a group in the lead to simultaneous competition. Furthermore, if the first group's effort decision can be increased by enlarging the group size, then everyone will benefit this enlargement. Finally if the equilibrium is unique, the system and the customers prefer system coordination to simultaneous competition.

1.1. Literature Review

1.1.1 Suppliers' Reliability Literature

Previous research incorporating suppliers' reliability in supply chain management focuses on the manufacturer's determination of best ordering policy. This research can be divided into two group: a single component with multiple suppliers referred as the sourcing system in our paper and multiple components each provided by a single supplier and referred as the assembly system in our work.

For the sourcing system with the suppliers' reliability problem, The first work deals with single sourcing problem analyzed by Bassok and Akella [6]. In their model, to minimize the expected cost constrained by capacity. The decisions to be made are: the ordering quantity of the component from a single supplier and the production level for multiple products that share the component. Later, a great deal of work was done on a single supplier with reliability problems. This area of research is surveyed well by Yano and Lee [48] and Grosfeld-Nir and Gerchak [19]. For the benefit of dual sourcing problem in the presence of supply uncertainty, Gerchak and Parlar [16], Yano [47] and Parlar and Wang [37] initialize the model from an EOQ setting. Moreover, Swaminathan and Shanthikumar [41] extends their result to the case with discrete demand. Anupindi and Akella [3], in particular, analyze the dual sourcing system in multiple periods in which the optimal policy is similar to the base stock policy but to order from single vs. two suppliers depends on the initial inventory level in that period. The multiple sourcing problem with the suppliers' reliability issue is first discussed in Agrawal, N. and S. Nahmias, [1], where demand is assumed to be deterministic. Two papers by J.Burke et al. [7] and Dada et al.[9] analyze the optimal ordering property in models similar to ours. J.Burke et al.'s work focuses on the uniform demand function. Dada et al.'s work adopts a similar model to ours, which assumes general demand distribution. However, there may be capacity constraints on the suppliers' production quantity in their model. Compared with these works, we provide more properties for optimal decisions, especially in sensitivity analysis. For the computational algorithm, Yang and Yang [46] provide an algorithm to get optimal ordering quantities based on Active Set Method combined with Newton search for general demand functions and linear wholesale prices. For a fixed plus linear wholesale price under the manufacturer's service level constraint, the multiple sourcing problem becomes NP complete even for special cases and Normal distributed functions. In [11] and [12], Federgruen and Yang provide procedures for optimal selection of suppliers and ordering allocation based on CLT-based approximation and Large Deviation Technique.

Another trend in research related to our paper with respect to the assembly system in literature is on the multidimensional news vendor problem. Readers can refer to Harrison and Van Mieghem [24], Van Mieghem [30], and Rudi and Zheng [39] for details. Papers considering random yield rate come much later than the sourcing system. Gerchak et al.[17] approaches the problem by assuming deterministic demand and hence simplifying the analysis of "Minimum" supply. The model with two complementary components each provided by single supplier in multiple periods problem is analyzed by Gurnani et al. [21] and similar results as the dual sourcing system in multiple period are provided. They approach the problem by objective function approximation. The model by Van Mieghem and Rudi [31] handles a multiple complementary components assembly system problem in single period with random yield rate.

Another line of research on the multiple sourcing problem concerns the impact of the suppliers' lead time. Fukuda [15], Lau and Zhao [27] and Feng et al.[13] study the single components dual sourcing in multiple periods with different lead time problem. In general base-stock policy is not optimal but is still widely used in practice. Gurnani et al. [20] study the manufacturer's optimal decision in an assembly system with two complementary critical components with stochastic lead time in multiple periods.

The current research tends to consider the suppliers decision based on the manufacturer's optimal ordering quantity. Gurnani et al.[23] study a two suppliers' production quantity game in an assembly system with deterministic demand random yield rate. Furthermore they show that an additional penalty contract can coordinate the system. In contrast to the suppliers' production game, we analyze suppliers' pricing game by taking suppliers' reliability into consideration, hence preventing the suppliers from being trapped in the "*Bertrand pardox*" (all suppliers set wholesale prices at their marginal cost and make zero profit). Based on our analysis, in assembly system, the Bernoulli reliability of the component suppliers will make the system degenerate to a deterministic selling to the news vendor problem with multiple suppliers. Lariviere and Porteus analyze this problem with single supplier under a price-only contract in [26]. They investigate the division of system profit and supply chain efficiency for demand with increasing generalized failure rate. Therefore, our pricing game analysis in assembly system is an extension of their work to the case with multiple suppliers. Albeniz [2] consider two homogeneous suppliers' pricing game based on different stochastic supply lead time. However, in their work, the manufacturer's decision is not the optimal one but a sub-optimal one with a base-stock policy. Recently, Babich et al. [5] starts to analyze the suppliers' pricing game given manufacturer's optimal ordering decision in sourcing system. They show the existence of a pure strategy Nash equilibrium in the case of the suppliers' Bernouli default probability under constant market demand. While for the stochastic market demand, no conditions for the existence of Nash equilibrium are given since the supplier's profit function is neither supermodular nor quasi-concave. We continue their work for stochastic demand.

1.1.2 Product Effort Investment Literature

For the product effort investment decision, most literature focuses on differential products, i.e. there are multiple products in the market and then model the demand for differentiated products in different approaches. Those approaches including 1): Representative consumer model (e.g., aggregate linear demand model), discrete choice model (e.g., the multinomial logit model) and the location model (e.g., Hotelling's model). For each of the above demand models, there is a representative paper which studies two firms' optimal quality innovation decision game combined with their price competition. In the model studied by Matsubayashi [28], he uses the aggregate linear demand model with quadratic initial investment costs and linear production costs with respect to the product quality for two identical firms to study the simultaneous quality and price decision game. Nash equilibrium is proved to exist only if the both firms are relatively differentiated. Multinomial logit demand model is applied by Moorthy [33]. Again the two firms are identical with the same quadratic unit production cost and non initial investment cost. He analyzes the firms quality decision game followed by price competition. The paper shows that the pure Nash equilibrium always exists and the company who chooses higher quality will also select a higher marginal profit. In [35] by Osborne and Pitchik, they study Hotelling's two-stage model of spatial competition, in which two firms first choose the location simultaneously and then choose their price. Under uniform distributed customers and linear cost to distance, they show Nash equilibrium exists in price competition, but they do not show that a pure Nash equilibrium exists in the location game. Recently, there is also some other literature based on the above demand models. Choudhary et al. [8] extend Moorthy's model by allowing personalized pricing, i.e. firms charge different prices to different consumers based on their willingness to pay. Klastorin and Tsai [25] also try to extend Moorthy's model. They consider the issue of entry timing as well as a finite product life cycle. Melumad and Ziv [29] analyze production quantity competition constrained by production capacity between two manufacturers when the market price is a linear function of average quality and production quantity. Actually this model is a variation of the aggregate linear demand model. They study the quantity equilibrium then do sensitive analysis with respect to quantity differentiation.

In the literature on the process of product development, the main approach is to model the problem as an optimal dynamic control (in monopoly case) and a differential game(in duopoly case). For a single firm's decision, Ouardighi and Tapiero [36] solve the optimal control of product quality diffusion in a continuous time domain assuming that price acts a signal of quality and demand is more sensitive to quality than to price. Voros [45] further combines the price and quality control in a continuous time model. Gjerde et al. [18] approaches the product innovation with multiple features constrained by technology in a discrete time domain by dynamic programming. For the multiple firm's differential game, Nair and Narasimhan [34] study the model when demand is affected by goodwill which can be controlled by product quality and advertisement.

As far as we know, only the paper [22] by Gurnani et al. is closely related to our work. They study the model with a single product composed by a single component, both the component supplier and the final product manufacturer can make investments on the product quality. The market demand is linear in price and quality. The production cost is linear and the initial investment cost is quadratic in quality decision. After quality investment decision is made, supplier decide on wholesale price of the component, and based on that, manufacturer decide on market price. The paper shows that Nash equilibrium exists in simultaneous quality investment decision under concave pay-off condition. In addition, the manufacturer is worse off if he or she follows component supplier's quality investment decision compared to their making investment decisions simultaneously. Our study indicates that everyone should be benefiting from having a leader in the investment decision, which is different to Gurnani's finding.

1.2. Overview and Outline

Compared with the relevant work, for the suppliers' reliability model, we focus on providing more properties of the optimal ordering strategy for the manufacturer. Based on those properties, we devise several effective solution procedure. In addition , suppliers' price decisions are also considered to analyze the performance of the supply chain under both supplier's competition and coordination. And for the product investment and pricing model, we generalize the demand function and investment cost function and prove the existence of pure strategy equilibrium for the case with multiple component suppliers. Further more, we show a system with a leader or a group of leaders will perform better than if they compete with each other.

The remainder of this dissertation is organized as follows. We first introduce the manufacturer's ordering strategy under stochastic demand and suppliers' reliability for both sourcing and assembly system. Mathematical model and properties of the optimal solution are also analyzed. We suggest optimal solution procedures and heuristic approaches, numerical experimental results are also provided. For both of the two systems, we further analyze the pricing game among suppliers with Bernouli reliability and analyze the system performance compared with system coordination. Finally a general model that combine the structure of sourcing and assembly system is investigated. In Section 3, we analyze the competition of effort and price decision sequence in assemble production system. Then we compare the system performance under the simultaneous effort decision with the sequential effort decision and the system coordination. Summary of the study and guidance for future research is provided in section 4. To improve the presentation and readability, we delegate most of the proofs to the Appendix.

Chapter 2

Manufacturer's Sourcing Strategy and Supplier's Pricing Game under Uncertain Reliability

2.1. The Notation and the Model

Consider a single period two-echelon supply chain with n suppliers, each providing a component to a unique manufacturer that produces a final product which sells for p per unit. The demand D of the final product is a stochastic variable with probability density function $f(\cdot)$ and cumulative distribution function $F(\cdot)$. The per unit salvage value is s for unsold stock and the underage cost is u for unsatisfied demand. R_i is the proportion of effective components provided by supplier i. They are independent random variables with density function $g_i(\cdot)$ and mean $\bar{R}_i > 0$ and standard deviation $\sigma_i > 0$. Based on the per unit cost c_i, supplier i offers an average per unit wholesale price w_i for the components ordered by the manufacturer. The manufacturer decides on the ordering quantities $q_i, i = 1, \ldots, n$ so as to maximize its own expected profit. Although in our model, the payment of purchasing is only proportional to the ordering quantity. In the case that the payment for the order from supplier i depends on the price for each unit ordered, w_i^o, and the price for each arrived effective unit, w_i^e, the total expected payment for supplier i is $w_i^o q_i + w_i^e R_i q_i$. Correspondingly, to model it, we can replace $w_i^o + w_i^e \bar{R}_i$ with our unit wholesale price w_i without affecting the solution. In the coordinated model, the suppliers decide on the component wholesale prices together so as to maximize their total expected profit. In the competition model, component prices are decided by the suppliers individually so as to maximize their own expected profit.

In the sourcing system, only one component, which can be provided by any one of the suppliers, is required for producing the final product. In the assembly system, a different component provided by each supplier is required to produce the final product. All the parameters are normalized to reflect the costs and prices associated with the components only.

We assume full information and $p+u > s \geq 0$ (per unit revenue is greater than salvage value to ensure that the manufacturer will satisfy any demand with available stocking).

We analyze the manufacturer's optimal decision in section two and the suppliers' pricing game in the sourcing system under Bernoulli reliability in section three. Then in section four, we investigate the property of the manufacturer's decision in the assembly system and provide efficient solution procedures. In section five, we study the suppliers' pricing game in the assembly system under Bernoulli reliability. Finally, we model the system which combines the structure of sourcing and assembly system and solve the problem numerically in section six. A summery of component reliability model is concluded in section seven.

Table 2.1: Summary of Notation

n = Number of suppliers.

q_i = Decision variable denoting ordering quantity from supplier i.

$\mathbf{q} = \{q_1, \ldots, q_n\}$ = Order quantity vector.

$q_i^*; \mathbf{q}^*$ = Optimal value of q_i and corresponding vector of optimal decision.

D = Nonnegative demand random variable.

$F(\cdot); f(\cdot)$ = Cdf and pdf,respectively characterizing demand D

p = Constant per unit market price for final product.

u = Constant per unit penalty cost for unsatisfied demand.

s = Constant per unit salvage value for left over inventory.

w_i = Average per unit component wholesale price set by supplier i.

$\mathbf{w}$ = Wholesale price vector.

c_i = Constant per unit production cost for supplier i.

R_i = Nonnegative and independent random variable representing reliability of supplier i.

P_i = The probability of supplier i to successfully deliver q_i when R_i is Bernoulli random variable.

r_i = A realization of R_i.

$\bar{R}_i = E[R_i]$ = Expectation of random variable R_i.

$G_i(\cdot); g_i(\cdot)$ = Cdf and pdf,respectively characterizing reliability R_i.

$Q = \sum_i^n R_i q_i$ = A random variable representing the total delivered effective components.

$q^r = \sum_i^n r_i q_i$ = A realization of total effective delivery

π = Expected profit of manufacturer.

π_{si} = Expected profit of component supplier i.

2.2. Manufacture's Optimal Decision in Sourcing system

We first analyze the manufacturer's optimal decision in the sourcing system and characterize the property of the optimal ordering quantity. Here we assume demand and reliability are continuous random variables.

2.2.1 Manufacturer's optimal decision in sourcing system

The manufacturer's object is to maximize his expected profit:

$$\max_{\mathbf{q}\geq 0} \pi = pE[D-(D-Q)^+] + sE[(Q-D)^+] - uE[(D-Q)^+] - \sum_{i=1}^{N} w_i q_i \tag{2.1}$$

where $Q = \sum_{i=1}^{N} R_i q_i$. Here $pE[D-(D-Q)^+]$ is the expected revenue, $sE[(Q-D)^+]$ is the salvage profit from left over inventory and $uE[(D-Q)^+]$ is the total penalty cost for unsatisfied demand. And each expectation is taken over both stochastic demand and random reliability. To make sure the manufacture won't make profit by sale the components directly to the salvage market we assume $w_i \geq c_i > \bar{R}_i s$ for all i.

Lemma 2.1. *The manufacturer's expected profit function is jointly concave in the ordering quantities* q_i, $i = 1, \ldots, n$.

One easy way to check the concavity of the objective function is based on the fact that the expectation of convex (concave) function is convex (concave) and the operator $()^+$ is convex over the linear operator. Hence the concavity will still hold even if the reliability of each suppliers is mutually dependent or non-continuous random variables. A detailed proof for lemma 2.1 by checking the hessian is provided in the appendix. In addition, since the density distribution function of demand is continuous, the expected profit function is twice differentiable. Because the objective function is concave, the **KKT** necessary and sufficient conditions imply the following corollary.

Corollary 2.1. *q_i^* $(i = 1, \dots, n)$ with $Q^* = \sum_{i=1}^n R_i q_i^*$ is the optimal ordering quantity for the manufacturer if and only if:*
For each $i \in \{1, \dots, n\}$, $q_i^* = 0$, when $w_i \geq \bar{R}_i(p+u) - (p+u-s)E[R_i F(Q^*)]$, otherwise

$$w_i = \bar{R}_i(p+u) - (p+u-s)E[R_i F(Q^*)] \tag{2.2}$$

Based on corollary 2.1, [46] applies the Active Set Method combined with the Newton search procedure to solve the problem. Note that the calculation of first order derivatives will involve the evaluation of $E[R_i F(Q)]$. In [46], they use the Monte Carlo sampling methods for problems.
Corollary 2.1 implies that if the component price $w_i \geq \bar{R}_i(p+u)$, then the manufacturer will not obtain any profit by ordering from this supplier. As a result, supplier i will select component prices with $w_i \leq \bar{R}_i(p+u)$. Furthermore, the vector of component prices has the following property.

Lemma 2.2. *The set $\{w_i, i = 1, \dots, n | q_i \geq 0$ with $w_i = \bar{R}_i(p+u) - (p+u-s)E[R_i F(Q)]$ and $0 \leq w_i \leq \bar{R}_i(p+u), i = 1, \dots, n\}$ is either empty or a convex set.*

The following lemma identifies conditions for positive ordering quantities.

Lemma 2.3. *If there exits two suppliers i and j such that $w_i/\bar{R}_i > w_j/\bar{R}_j$ with $q_i^* > 0$, then $q_j^* > 0$.*

If we define $w_i/\bar{R}_i$ as the effective wholesale price of supplier i, which indicates the expected cost for one unit of effective component. Actually this value reflects the real cost to order from each supplier. By lemma 2.3, if there is any supplier who gets a positive ordering quantity, then those suppliers with lower effective wholesale prices should also get a positive ordering quantity. Based on that, we have:

Lemma 2.4. *If there is a supplier i with $w_i \leq (p+u)\bar{R}_i$, then the manufacturer must make some positive order from some supplier.*

Corollary 2.2. *Supplier i will get a positive ordering quantity if $w_i < (p+u)$ and $w_i/\bar{R}_i < w_j/\bar{R}_i$ for all $j \neq i$.*

In the following lemma, we provide bounds on the ordering quantities.

Lemma 2.5. *The total optimal quantity $\sum_j q_j^*$ ordered by the manufacturer is no less than the optimal ordering quantity under full reliability from a single supplier with $w = \min w_j/\bar{R}_j$.*

Lemma 2.5 suggests a lower bound of total ordering quantity.

Although Yang et al. [46] suggests an Active Set method combined with a Newton search procedure to solve the problem, they do not apply the useful property of lemma 2.3 to 2.5, and the active set (suppliers with positive order) may shrink or enlarge during the procedure. This cause the performance of the algorithm to deteriorate when only small portion of the suppliers get positive optimal order. Here we devise an Iterative Expansion algorithm based on our findings:

Algorithm 2.1 Iterative Expansion Procedure

1. (*Initialization*): Order the suppliers by $w_i/\bar{R}_i$ bin ascending order. Starting with $m = 1$, let $\mathbf{q}_m$ be the ordering quantities from suppliers $1, \ldots, m$, initialize $\mathbf{q}_1 = \{F^{-1}((p+u-w_1/\bar{R}_1)/(p+u-s)\}$.

2. (*Optimality Test*): Evaluate $\mathbf{g} = \frac{\partial \pi}{\partial \mathbf{q}_m}$, if $\sum_{i=1}^{m} |g_i| < \epsilon$ for a predetermined positive tolerance level ϵ, then check if $m = n$ or $g_{m+1} = \frac{\partial \pi}{\partial q_{m+1}} < 0$, then stop and report $\mathbf{q}_m$ as optimal solution. Otherwise $m = m + 1$.

3. (*Computation of direction*): Evaluate hessian $\mathbf{H}_m = \{\frac{\partial^2 \pi}{\partial q_i \partial q_j} : i, j = 1, \ldots, m\}$. Try to calculate $\mathbf{d}_m = -\mathbf{H}_m^{-1}\mathbf{g}_m$ using the principal-element Gauss-Jordan method. If in the process, one of the principal element is smaller than a pre-determined positive level ϵ', then let $\mathbf{d}_m = \mathbf{g}_m$ (steepest ascent direction) instead.

4. (*Computation of step size*): Let $\alpha = \min\{q_j/(-d_j)|j = 1, \ldots, m \text{ and } d_j < 0\}$. If α exits then $\mathbf{q}_m = \mathbf{q}_m + \alpha \mathbf{d}_m$ otherwise $\mathbf{q}_m = \mathbf{q}_m + \mathbf{d}_m$ and go back to step 2.

Note that the hessian here can be calculated by the following formula:

$$\frac{\partial^2 \pi}{\partial q_i \partial q_j} = -(p+u-s)E[R_i R_j f(Q)] \qquad \forall i,j = 1,\ldots,n \tag{2.3}$$

The fundamental reason why this algorithm ensures optimality is lemma 2.3. Hence, in every step 2-4, we only concentrate on the suppliers with positive ordering quantity. The algorithm stops if we find the optimal solution for m suppliers with the least effective wholesale price and adding on the $m+1$ supplier will not benefit the system. Our numerical study shows that the iterative expansion procedure works more efficiently than the active set method in [46] for cases that no more than half of the suppliers get positive order in optimal solution.

Lemma 2.6. *If the support of the demand is an interval:* $[0,b)$ *(*$f(x) > 0$*,* $\forall 0 < x < b$ *and* $f(x) = 0$*, if* $x \geq b$*, the manufacturer's expected profit is strictly concave in* $\mathbf{q}$ *at optimal point* $\mathbf{q}^* \neq 0$ *and hence the optimal solution is unique.*

Lemma 2.7. *If the manufacturer's expected profit is strictly concave in* $\mathbf{q}$ *at the optimal point , the optimal ordering quantity* $\mathbf{q}^*$ *is a continuous function in the price* $\mathbf{w}$*.*

We can find that even for general uniform demand, the optimal solution may be not unique. However, the above two lemmas examine the condition for unique optimal solution. Therefore, the uniqueness and continuous of the manufacturer's best response gives us a starting point to further analyze the supplier's price decision in the following chapter.

For all the following sensitivity analysis and comparative results, we assume that the expected profit function is strictly concave at optimal point and hence the optimal solution is unique. The effects of system parameters on ordering quantities are summarized below.

Theorem 2.1. *The optimal ordering quantity* q_i^* *of a component received by supplier* i *is non-increasing with respect to its wholesale prices* w_i*. In addition, if there are only two suppliers, then the optimal ordering quantity* q_i^* *of a component received by supplier* i *is non-decreasing with respect to the other supplier's prices* w_j *,* $j \neq i$*.*

Observation 2.1. *If* $n \geq 3$*, the optimal ordering quantity from supplier may decrease with the increasing of the other suppliers' price. The example is as follows:*

Example 2.1. *Case with $n = 3$: Suppose market demand is uniformly distributed $D \sim U[100, 170]$, product price is \$8 each with salvage value \$3 each. The three component suppliers' average unit price is $w_1 = \$2.64$, $w_2 = \$2.64$ and $w_3 = \$6.09$. The reliability density function for each supplier is as:*

$$
\begin{aligned}
R_1 &= \begin{cases} X, & \textit{with probability 0.5} \\ Y, & \textit{with probability 0.5} \end{cases} \\
R_2 &= \begin{cases} X, & \textit{with probability 0.5} \\ Y, & \textit{with probability 0.5} \end{cases} \\
R_3 &= Y
\end{aligned}
$$

Where $X \sim U[0, 0.1]$ and $Y \sim U[0.9, 1]$. In this case, $q_1^* = q_2^* = 100$, while $q_3^* = 10$. While at the optimal point:

$$
\begin{aligned}
\frac{\partial q_1^*}{\partial w_2} &= -10815 \\
\frac{\partial q_2^*}{\partial w_2} &= -10884 \\
\frac{\partial q_3^*}{\partial w_2} &= 11411
\end{aligned}
$$

Which means with the increasing of wholesale price of the second supplier, the manufacturer's optimal ordering quantity from both first supplier and second supplier will decrease while the ordering quantity from the third supplier will increase.

By theorem 2.1 the supplier always gets less ordering quantity if he increase his own wholesale price. In case of two suppliers, this action will shift the ordering quantity to the other supplier. While if there are more than two suppliers, there always exists some re-mix effect: if the less reliable supplier increases his wholesale price, the decreasing of ordering quantity from him actually leads to the need for more reliable supply to cover the rest part of the demand. Hence the mixture of ordering from the other suppliers will lean to the more reliable supplier. Thus the optimal ordering quantity from the risky supplier will decrease.

Theorem 2.2. *If $n = 2$, and demand $D = \delta + X$, where X is a random variable with log-concave density distribution f and $\delta \geq 0$, then the optimal ordering quantity for both supplier q_i^*, $i = 1, 2$ is increasing with δ.*

The above theorem shows that if demand is shifted up, both supplier will get benefit. A function $f(\cdot)$ is said to be log-concave if its natural log, $\ln(f(x))$ is a concave function; that is, assuming f is differentiable, $f''(x)/f(x) - f'(x)^2 \leq 0$. Since *log* is a strictly concave function, any concave function is also log-concave.

A random variable is said to be log-concave if its density function is log-concave. A wide range of distributions happen to be log-concave. These include uniform, normal, beta, exponential, extreme value distributions, the two-parameter Weibull, Gamma, Pareto, lognormal, and linear failure rate distributions. If pdf $f(\cdot)$ is log-concave, then so is its cdf $F(\cdot)$ and $1 - F(\cdot)$. The truncated version of a log-concave function is also log-concave. In practice the intuitive meaning of the assumption that a distribution is log-concave is that: (a) it doesn't have multiple separate maxima (although it could be flat on top), and (b) the tails of the density function are not "too thick".

If the demand is not log-concave, then increasing the demand may lead to the decreasing of optimal ordering quantity from one of the suppliers as shown in example 2.2.

Example 2.2. *Suppose density function of market demand is*

$$D = \begin{cases} D_1, & \textit{with probability 0.5} \\ D_2, & \textit{with probability 0.5} \end{cases}$$

Where $D_1 \sim U[10, 20]$ and $D_2 \sim U[100, 110]$ are two uniform distributed random variables. Product price is \$8 each with salvage value \$3 each. The two component supplier's average unit price is $w_1 = \$2.69$, $w_2 = \$5.30$. The reliability of supplier 1 is $R_1 \sim U[0, 1]$ and $R_2 \sim U[0.95, 1]$ for supplier 2. In this case $q_1^* = 90$ and $q_2^* = 20$. If we increase the market *demand by* 1 *unit such that the demand function is increased by* 1 *unit:*

$$D' = \begin{cases} D'_1 \sim U[11, 21], & \textit{with probability 0.5} \\ D'_2 \sim U[101, 111], & \textit{with probability 0.5} \end{cases}$$

The optimal ordering quantity is changed to $q_1^* = 89.996$ *and* $q_2^* = 21.02$. *So here, the ordering quantity from first supplier is decreased.*

Example 2.2 provides some insight into why the manufacturer will decrease the ordering quantity from some supplier when demand is increasing. In the case of bimodal demand density, if there are two suppliers, one supplier is more reliable (less variance) with higher cost and the other is less reliable (larger variance) with lower cost. In optimal solution, the ordering quantity from the risky supplier is mainly to satisfy the stochastic part (in the support of the distribution) of the demand while the ordering quantity from the more reliable supplier is mostly used to satisfy the deterministic part of the demand. Note that even the more reliable supplier is still subject to variability and so increasing the ordering quantity from him may cover some part of the stochastic demand. Therefore, increasing the demand by some deterministic constant, may cause the shifting of the ordering quantity from the less reliable supplier to more reliable supplier.

Lemma 2.8. *In case* $2f(x) \geq$ *(or* $\leq$*)* $- xf'(x)$ *for* $x \geq 0$, *total optimal ordering quantities from two identical suppliers does not increase (or decrease) when they are replaced by a single one.*

By lemma 2.8, we can find with risk pooling, the manufacturer may not necessarily order more. It depends on wether $rF(rq)$ is concave or convex in r. If $rF(rq)$ is convex, then the manufacturer will order less. If $rF(rq)$ is concave, he will order more.

Although lemma 2.3 provides some insight into when a supplier receives a positive order compared with the other supplier given first order information of reliability distribution. However, with more information on the reliability distributions, we can compare the ordering quantities between two suppliers:

Theorem 2.3. *In case $w_i > w_j$ and $R_i \leq_{rh} R_j$ (R_i is smaller than R_j in reverse hazard rate order)*[1]*, then $q_i^* \leq q_j^*$.*

Since $R_i \leq_{rh} R_j$ implies $R_i \leq_{st} R_j$ [2], thus $E[R_i^n] \leq E[R_j^n]$ for all $n > 0$. Note here, the supplier with same effective wholesale price and lower variance can guarantee a higher ordering quantity.

2.3. Suppliers' pricing game in sourcing system

For the pricing game among the suppliers, consider the case with two suppliers $i \in \{1, 2\}$, under Bernoulli reliability, i.e. with probability P_i to successfully deliver the whole ordering quantity and $1 - P_i$, ($P_i \in (0, 1)$) to default. Denote index $-i$ as the supplier i's opponent. Without loss of generality, we let $p + u = 1$, which means we adopt the price unit as $p + u$ and this won't affect the analysis of the pricing game. We also assume $s = 0$. By previous result we have:

$$\begin{aligned} q_i > 0: &\qquad w_i = P_i(p+u) - (p+u)P_i\big(P_{-i}F(q_i + q_{-i}) + (1 - P_{-i})F(q_i)\big) \\ q_i = 0: &\qquad w_i \geq P_i(p+u) - (p+u)P_iF(q_{-i}) \end{aligned}$$

In [5], Babich et al. give equilibrium equations for two suppliers with Bernouli reliability, i.e. with probability P_i to successfully deliver the whole ordering quantity and $1-P_i$ ($P_i \in (0, 1]$). Assuming the existence of equilibrium, their implicit equilibrium formula is derived from the

[1] Let X and Y be two random variables with absolutely continuous distribution functions and with reversed hazard rate function $\widetilde{r}$ and $\widetilde{q}$, respectively such tat $\widetilde{r}(t) \leq \widetilde{q}(t)$, $\forall t \in \mathbb{R}$. Then X is said to be smaller than Y in the reversed hazard rate order (denoted as $X \leq_{rh} Y$).
In fact, the absolute continuity is not really need. And $X \leq_{rh} Y$ if and only if $G(t)/F(t)$ increases in $t \in (\min(l_X, l_Y), \infty)$, here l_X, l_Y is the left end point of support of X and Y.

[2] Let X and Y be two random variables such that $P\{X > x\} \leq P\{Y > x\}$ for all $x \in (-\infty, \infty)$. Then X is said to be smaller than Y in the usual stochastic order (denoted by $X \leq_{st} Y$).

first order condition for each supplier's optimal profit target:

$$
\begin{aligned}
\max_{q_1,q_2} \quad & \pi_{si}(q_1,q_2) = ((p+u)P_i - (p+u)P_i(P_{-i}F(q_1+q_2) + (1-P_{-i})F(q_i)) - c_i)q_i \\
subject \quad & w_{-i} = (p+u)P_{-i} - (p+u)P_{-i}\big(P_iF(q_1+q_2) + (1-P_i)F(q_{-i})\big) \\
& q_1 \geq 0, q_2 \geq 0
\end{aligned}
$$

Clearly, this result is incomplete and neglects the case when $q_i = 0$. We also find that even for a uniform demand distribution, there may not exist any equilibrium in the suppliers' price competition.

One big issue that comes up with Bernouli reliability is that since the reliability density function is not continuous, our lemma 2.7 can't be applied here and the optimal ordering quantity (suppliers' profit) is no longer a function of their wholesale price (for a given whole sale price there may be multiple optimal ordering quantity). For example, if $w_1 = 5/9$ and $w_2 = 1/54$ in example 2.3, there are multiple optimal ordering quantity points: $q_1^* = 5/2$ and $q_2^* \in [1/2, 1]$. Even if we assume that the manufacturer will always select the largest best ordering quantity, Nash equilibrium may not exist as shown in following example:

Example 2.3. *Consider the case with demand uniformly distributed on* $[1,3]$*. Let marketing price be* $p = 1$*, salvage value* s *and penalty cost* u *be* 0*. Production cost* $c_1 = 1/2$*,* $c_2 = 0$ *and Reliability* $P_1 = 5/6$ *and* $P_2 = 1/9$*. There is no equilibrium point in price competition. The best response trajectories on price 2.2 and quantity 2.1 are shown in figure 2.1 and 2.2.*

2.3.1 Suppliers' price competition under uniform demand distribution

If the demand is uniformly distributed on $[0, b]$, in this case, the optimal ordering quantities is unique for given suppliers wholesale price, and hence the suppliers' profit is a function of their wholesale price. Clearly, $w_i \in W_i = [0, P_i]$ and the optimal ordering quantity $0 \leq q_i \leq b$. Let $-i$ be the index of supplier i's opponent, $i = 1, 2$. By the **KKT** condition at optimal

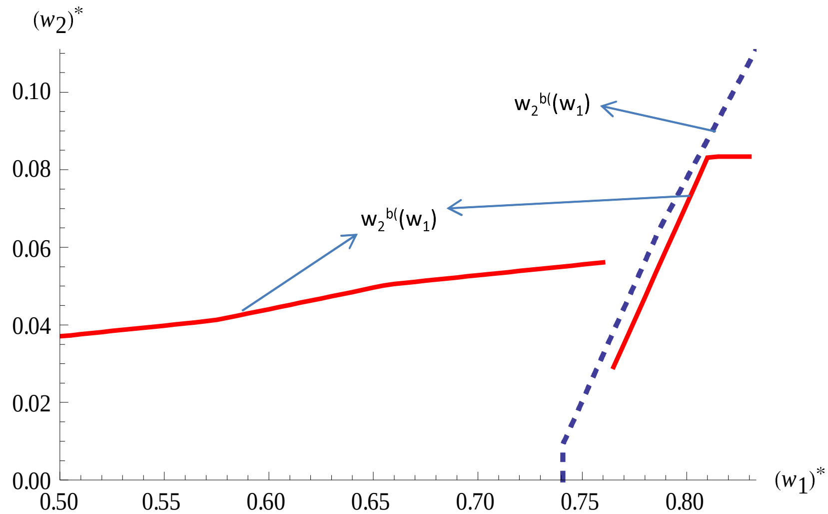

Figure 2.1: Suppliers' Best Response Wholesale Price

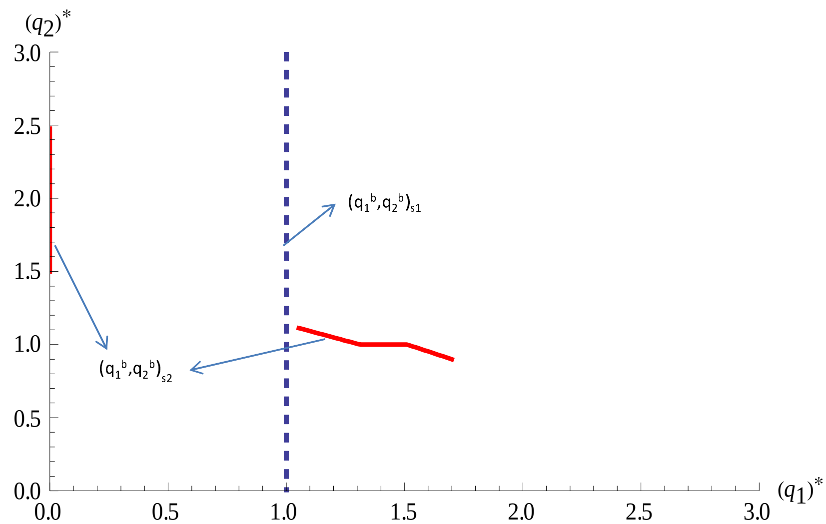

Figure 2.2: Suppliers' Best Response Production Quantity

points stated in Lemma 2.1, we have:

In case $w_{-i} \in \mathcal{W}^l_{-i} = [0, P_{-i}(1-P_i)]$

$$\pi_{si}(w_i|w_{-i}) = \begin{cases} \pi^1_{si}(w_i|w_{-i}) = \frac{b(P_i(1-P_{-i})-w_i)(w_i-c_i)}{P_i(1-P_{-i})}, & w_i \in \mathcal{W}^1_i = [0, P_i(1-P_{-i}) - w_{-i}\frac{P_i(1-P_{-i})}{P_{-i}(1-P_i)}] \\ \pi^2_{si}(w_i|w_{-i}) = \frac{b(P_i(1-P_{-i}+w_{-i})-w_i)^+(w_i-c_i)}{P_i(1-P_iP_{-i})}, & w_i \in \mathcal{W}^{2l}_i = [P_i(1-P_{-i}) - w_{-i}\frac{P_i(1-P_{-i})}{P_{-i}(1-P_i)}, P_i] \end{cases}$$

Otherwise if $w_{-i} \in \mathcal{W}_{-i}^u = [P_{-i}(1-P_i), P_{-i}]$

$$\pi_{si}(w_i|w_{-i}) = \begin{cases} \pi_{si}^2(w_i|w_{-i}) = \frac{b(P_i(1-P_{-i}+w_{-i})-w_i)^+(w_i-c_i)}{P_i(1-P_iP_{-i})}, w_i \in \mathcal{W}_i^{2u} = [\frac{w_{-i}}{P_{-i}} - (1-P_i), P_i] \\ \pi_{si}^3(w_i|w_{-i}) = \frac{b(P_i-w_i)(w_i-c_i)}{P_i}, \quad w_i \in \mathcal{W}_i^3 = [0, \frac{w_{-i}}{P_{-i}} - (1-P_i)] \end{cases}$$

The maximizer for $\pi_{si}(w_i|w_{-i})$ may be not unique, especially if w_i' is the maximizer such that the corresponding ordering quantity $q_i^*(w_i', w_{-i}') = 0$. In this is the case, then supplier i's profit in optimal decision is zero and for all $w_i \geq w_i'$ are also optimal decisions which lead to zero ordering quantity and profit (because of the $()^+$ operator in the profit function). In this case we reduce the best response set so that the supplier i will only choose the minim w_i that leads to 0 ordering quantity. By the reduction of the domain, we can get rid of the $()^+$ operator in the profit function. Clearly if there exists a fixed point in the reduced best response correspondence, then there exits an equilibrium in the suppliers' pricing game. After the restriction on the best response, we can rewrite the profit function as follows:
In case $w_{-i} \in \mathcal{W}_{-i}^l = [0, P_{-i}(1-P_i)]$

$$\pi_{si}(w_i|w_{-i}) = \begin{cases} \pi_{si}^1(w_i|w_{-i}) = \frac{b(P_i(1-P_{-i})-w_i)(w_i-c_i)}{P_i(1-P_{-i})}, \quad w_i \in \mathcal{W}_i^1 = [0, P_i(1-P_{-i}) - w_{-i}\frac{P_i(1-P_{-i})}{P_{-i}(1-P_i)}] \\ \pi_{si}^2(w_i|w_{-i}) = \frac{b(P_i(1-P_{-i}+w_{-i})-w_i)(w_i-c_i)}{P_i(1-P_iP_{-i})}, w_i \in \mathcal{W}_i^{2l} \end{cases}$$

Here $\mathcal{W}_i^{2l} = [P_i(1-P_{-i}) - w_{-i}\frac{P_i(1-P_{-i})}{P_{-i}(1-P_i)}, P_i(1-P_{-i}+w_{-i})]$.
In case $w_{-i} \in \mathcal{W}_{-i}^u = [P_{-i}(1-P_i), P_{-i}]$

$$\pi_{si}(w_i|w_{-i}) = \begin{cases} \pi_{si}^2(w_i|w_{-i}) = \frac{b(P_i(1-P_{-i}+w_{-i})-w_i)(w_i-c_i)}{P_i(1-P_iP_{-i})}, w_i \in \mathcal{W}_i^{2u} = [\frac{w_{-i}}{P_{-i}} - (1-P_i), P_i(1-P_{-i}+w_{-i})] \\ \pi_{si}^3(w_i|w_{-i}) = \frac{b(P_i-w_i)(w_i-c_i)}{P_i}, \quad w_i \in \mathcal{W}_i^3 = [0, \frac{w_{-i}}{P_{-i}} - (1-P_i)] \end{cases}$$

The trajectory of suppliers' sales quantity with respect to $\mathbf{w}$ is shown in figure 2.3 and the plot of supplier's profit as a function of its own wholesale price is shown in figure 2.4.

Let $w_i^b(w_{-i}) = argmax_{w_i \in \mathcal{W}_i} \ \pi_{si}(w_i|w_{-i})$ be the best response map of supplier i with respect to $w_{-i} \in \mathcal{W}_{-i}$. Even after the reduction of the domain, $w_i^b(w_{-i})$ may be singleton, since both of the local maximal could be global optimal solution. We have:

Lemma 2.9. *$w_i^b(w_{-i})$ is an increasing correspondence from $\mathcal{W}_{-i}$ to $\mathcal{W}_i$.*

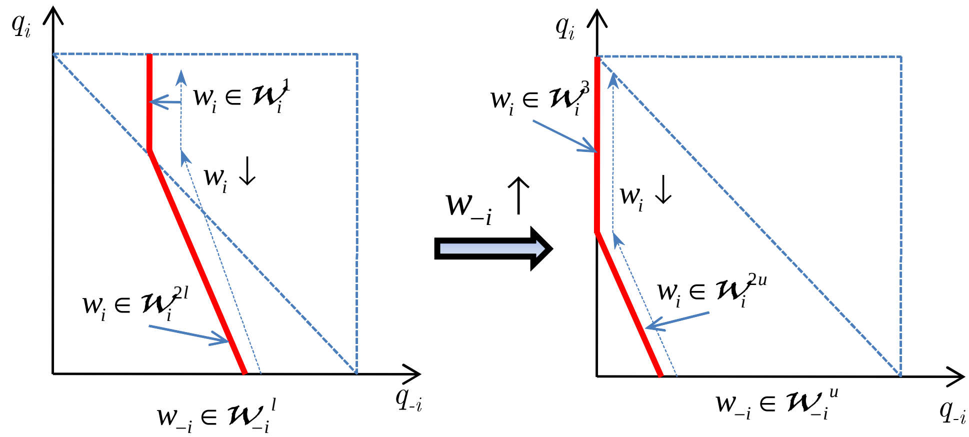

Figure 2.3: The trajectory of suppliers' sales quantity q_i, q_{-i} changes with w_i, w_{-i}

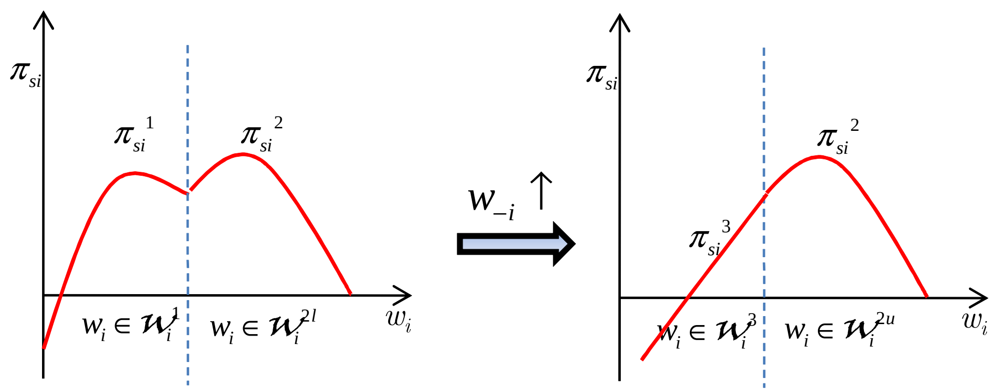

Figure 2.4: Supplier i's profit as a function of w_i changes with w_{-i}

The following theorem is proved by Topkis in [43] as theorem 2.4.1, which gives the sufficient condition for the existence of a fixed point.

Theorem 2.4. *Suppose that* $\mathbf{X}$ *is a nonempty complete lattice,* $\mathbf{Y}(\mathbf{x})$ *is an increasing correspondence from* $\mathbf{X}$ *into* $\mathcal{L}(\mathbf{X})$ *with* $\mathcal{L}(\mathbf{X})$ *having the induced set ordering* $\sqsubseteq$ *and* $\mathbf{Y}(\mathbf{x})$ *is subcomplete for each* $\mathbf{x}$ *in* $\mathbf{X}$*.*
(a) The set of fixed points of $\mathbf{Y}(\mathbf{x})$ *in* $\mathbf{X}$ *is nonempty.*
(b) The set of fixed points of $\mathbf{Y}(\mathbf{x})$ *in* $\mathbf{X}$ *is a nonempty complete lattice.*

Note here if $\mathbf{Y}(\mathbf{x})$ is singleton for every given $\mathbf{x}$ then $\mathbf{Y}(\mathbf{x})$ is an increasing function and we have the the following lemma as a simplified version of the theorem 2.4. This lemma is proved in our appendix without using lattice theory.

Lemma 2.10. *Let $\mathcal{X} = \times_{i=1}^{n}[l_i, u_i] \subseteq \mathcal{R}^n$ which is a hypercube defined in an n dimensional space and f be an increasing function on $\mathcal{X}$ to $\mathcal{X}$ such that $\forall \mathbf{x}', \mathbf{x}'' \in \mathcal{X}, x' \geq x'' \Rightarrow f(\mathbf{x}') \geq f(\mathbf{x}'')$. The set of all the fixed points of f is nonempty.*

To apply the above theorem in our case, define a best joint response function as follows

$$\begin{aligned} \pi_{\sum}(x_1, x_2, w_1, w_2) &= \pi_{s1}(x_1, w_2) + \pi_{s2}(x_2, w_1) \\ \mathbf{w}^b(\mathbf{w}) &= argmax_{x_1 \in \mathcal{W}_1, x_2 \in \mathcal{W}_2} \pi_{\sum}(x_1, x_2, w_1, w_2) \end{aligned}$$

Clearly $\mathbf{w}^b(\mathbf{w}) = w_1^b(w_2) \times w_2^b(w_1)$, by lemma 2.9, $\mathbf{w}^b(\mathbf{w})$ is is an increasing complete sub-complete sublattice. Hence $\mathbf{w}^b(\mathbf{w})$ has a fixed point. And by lemma 4.2.1 in [43], the fixed points of $\mathbf{w}^b(\mathbf{w})$ are equilibrium points, hence

Theorem 2.5. *For the two supplier's pricing game with Bernoulli unreliability, if demand is uniformly distributed on $[0, b]$ and $s = 0$, there exists a pure strategy Nash equilibrium. Furthermore, the set of Nash equilibria $\mathcal{W}^*$ forms a lattice and exists a least equilibrium $\mathbf{w}_l \leq \mathbf{w}^*$ and a greatest equilibrium $\mathbf{w}_u \geq \mathbf{w}^*$ for all $\mathbf{w}^* \in \mathcal{W}^*$.*

Since π_{si} is increasing with w_{-i}, both of the suppliers will get more profit in the largest equilibrium compared with the other equilibria.

2.3.2 Suppliers' price competition under exponential demand distribution

Suppose that the demand is exponentially distributed: $f(x) = \lambda e^{-\lambda x}$, $x \geq 0$. Using a similar approach as uniform demand function, let $p + u = 1$ and assume salvage value $s = 0$. We further assume that the production cost c_i for each supplier is not so small such that $c_i/P_i \geq e^{-1}$, which means the marginal profit in the supply chain is no more than 1.7 of the production cost. This is a reasonable assumption for today's manufacturer system in a fierce competition environment. Let $x_i = e^{-\lambda q_i}$, thus, at manufacturer's optimal decision we have:

$$\begin{aligned} \text{if } q_i^* > 0 \Rightarrow 0 < x_i < 1 \quad &: \qquad w_i = P_i x_i (1 - P_{-i}(1 - x_{-i})) \\ \text{otherwise } q_i^* = 0 \Rightarrow x_i = 1 \quad &: \qquad w_i \geq P_i (1 - P_{-i}(1 - x_{-i})) \end{aligned}$$

Here x_i is a one to one decreasing map of q_i from $[0, +\infty)$ to $(0, 1]$. Since for any given supplier $-i$'s wholesale price, the best response wholesale price for supplier i: $w_i \geq c_i$, combine with equation 2.4, we have:

$$w_i = P_i x_i (1 - P_{-i}(1 - x_{-i})) \leq P_i x_i \Rightarrow x_i \geq \frac{w_i}{P_i} \geq \frac{c_i}{P_i} \geq e^{-1}$$

which means supplier i will only select $x_i \in [e^{-1}, 1]$, $i = 1, 2$.

In case $x_{-i} < 1$, solve :

$$\left.\begin{array}{l} w_i = P_i x_i (1 - P_{-i}(1 - x_{-i})) \\ w_{-i} = P_{-i} x_{-i} (1 - P_i (1 - x_i)) \end{array}\right\} \Rightarrow \left\{\begin{array}{l} w_i = P_i x_i (1 - P_{-i}(1 - \frac{w_{-i}}{P_{-i}(1 - P_i(1 - x_i))})) \\ x_i \geq 1 - \frac{P_{-i} - w_{-i}}{P_{-i} P_i} \end{array}\right.$$

$$\Rightarrow \quad \pi_{si}^1(x_i | w_{-i}) = -\frac{1}{\lambda}(P_i x_i (1 - P_{-i}(1 - \frac{w_{-i}}{P_{-i}(1 - P_i(1 - x_i))})) - c_i) \log x_i$$

In case $x_{-i} = 1$, solve:

$$\left.\begin{array}{l} w_i = P_i x_i (1 - P_{-i}(1 - x_{-i})) \\ w_{-i} \geq P_{-i} (1 - P_i (1 - x_i)) \end{array}\right\} \Rightarrow \left\{\begin{array}{l} w_i = P_i x_i \\ x_i \leq 1 - \frac{P_{-i} - w_{-i}}{P_{-i} P_i} \end{array}\right.$$

$$\Rightarrow \quad \pi_{si}^2(x_i | w_{-i}) = -\frac{1}{\lambda}(P_i x_i - c_i) \log x_i$$

Then for any given w_{-i}, supplier i's profit as a function of x_i (also a function of q_i) is:

$$\pi_{si}(x_i | w_{-i}) = \begin{cases} \pi_{si}^1(x_i | w_{-i}) : \text{where } x_i \in \mathcal{X}_i^u = [\max\{e^{-1}, 1 - \frac{P_{-i} - w_{-i}}{P_{-i} P_i}\}, 1] \\ \pi_{si}^2(x_i | w_{-i}) : \text{where } x_i \in \mathcal{X}_i^l = [e^{-1}, 1 - \frac{P_{-i} - w_{-i}}{P_{-i} P_i}] \end{cases}$$

We will first analyze the best ordering quantity $-\frac{\log x_i}{\lambda}$ for supplier i's optimal decision when w_{-i} is given, then study the wholesale price corresponding to each optimal ordering quantity and lastly get the property of the best response wholesale price. Let $x_i^{b1} = argmax_{x_i \in \mathcal{X}_i^u} \pi_{si}^1(x_i | w_{-i})$, $x_i^{b2} = argmax_{x_i \in \mathcal{X}_i^l} \pi_{si}^2(x_i | w_{-i})$ and $x_i^b = argmax_{x_i \in [e^{-1}, 1]} \pi_{si}(x_i | w_{-i})$.

Lemma 2.11. *For any given w_{-i}, $\pi^1_{si}(x_i|w_{-i})$ is strictly quasi-concave in x_i. In addition, if $x_i^{1o}(w_{-i})$ is the point that satisfies the first order condition for function $\pi^1_{si}(x_i|w_{-i})$ in its feasible region, then $x_i^{1o}(w_{-i})$ is decreasing with respect to w_{-i}. Also, the corresponding $w_i^{1o} = P_i x_i^{1o}(w_{-i})(1 - P_{-i}(1 - \frac{w_{-i}}{P_{-i}(1-P_i(1-x_i^{1o}(w_{-i})))}))$ is increasing in w_{-i}*

Lemma 2.12. *$\pi^2_{si}(x_i|w_{-i})$ is strictly concave in x_i. Additionally when $x_i^{1o}(w_{-i}) = 1 - \frac{P_{-i}-w_{-i}}{P_{-i}P_i}$, we have:*

$$\frac{\partial \pi^2_{si}(x_i|w_{-i})}{\partial x_i}\Big|_{x_i = 1-\frac{P_{-i}-w_{-i}}{P_{-i}P_i}} \geq \frac{\partial \pi^1_{si}(x_i|w_{-i})}{\partial x_i}\Big|_{x_i = 1-\frac{P_{-i}-w_{-i}}{P_{-i}P_i}} = 0$$

If $x_i^{b2}(w_{-i}) \in x_i^b(w_{-i})$, then $P_i x_i^{b2} \in w_i^b(w_{-i})$ is the best response wholesale price of supplier i. In the case $x_i^{b1}(w_{-i}) \in x_i^b(w_{-i})$ and $x_i^{b1}(w_{-i}) < 1$, $P_i x_i^{b1}(w_{-i})(1 - P_{-i}(1 - x_{-i})) \in w_i^b(w_{-i})$ where $x_{-i} = \frac{w_{-i}}{P_{-i}(1-P_i(1-x_i^{b1}(w_{-i})))}$, is the best response wholesale price of supplier i. In the case that $x_i^b(w_{-i}) = x_i^{b1} = 1$ (corresponding optimal sale quantity is 0), $w_i^b(w_{-i}) \geq P_i(1-P_{-i}+w_{-i})$ is the best response wholesale price for supplier i. To simplify the analysis, we restrict the best response to be $w_i^b(w_{-i}) = P_i(1 - P_{-i} + w_{-i})$ if $x_i^b(w_{-i}) = 1$. Then to show the existence of Nash equilibrium, we can check if there exists an intersection point in the best response map $w_i^b(w_{-i}) = P_i x_i^b(w_{-i})(1 - P_{-i}(1 - x_{-i}))$ for $i = 1, 2$.

Lemma 2.13. *$w_i^b(w_{-i})$ is an increasing correspondence from $\mathbf{W}_{-i}$ to $\mathbf{W}_i$*

Again, the best response w_{-i}, $w_i^b(w_{-i})$ may not be a convex set. Hence we can't apply Brouwer's Fixed point theorem. However theorem 2.4 can be applied here, and hence we have

Theorem 2.6. *For two supplier's pricing game with Bernoulli unreliability, if demand is exponentially distributed $c_i/P_i \geq e^{-1}$, there exists a pure strategy Nash equilibrium. Further more, the set of Nash equilibria $\mathcal{W}^*$ forms a lattice and exists a least equilibrium $\mathbf{w}_l \leq \mathbf{w}^*$ and a greatest equilibrium $\mathbf{w}_u \geq \mathbf{w}^*$ for all $\mathbf{w}^* \in \mathcal{W}^*$.*

Since π_{si} is increasing with w_{-i}, both of the suppliers will get more profit in the largest equilibrium compared with the other equilibria.

2.3.3 Suppliers coordination under uniform demand distribution or exponential demand distribution

By theorem 2.5 and 2.6, we know that an equilibrium exists in the two suppliers's pricing game when demand is exponentially distributed and $c_i/P_i \geq e^{-1}$, $i = 1, 2$ or uniformly distributed on $[0, b]$. Also, the best response whole sale price $w_i^b(w_{-i})$ is increasing with the opponent's wholesale price w_{-i}. Then let $\mathcal{W}^*$ be the set of equilibrium and let $\mathcal{W}'$ be set of points that achieve system coordination: $\mathcal{W}' = argmax_{w_i, w_{-1}} \pi_{si}(\mathbf{w}) + \pi_{s-i}(\mathbf{w})$, then we have

Theorem 2.7. *If there are two suppliers and $w_i^b(w_{-i})$ is increasing with w_{-i}, for $i = 1, 2$. Then $\forall \mathbf{w}^* \in \mathcal{W}^*$, if $\mathbf{w}^* \notin \mathcal{W}'$, then $\mathbf{w}^* \leq \mathbf{w}'$ for all $\mathbf{w}' \in \mathcal{W}'$*

Therefore the suppliers will set a lower wholesale price in competition which is preferred by the manufacturer.

2.4. Manufacturer's Optimal Decision in Assembly System

Similar to the sourcing system, we first analyze the manufacturer's optimal decision, and based on his optimal ordering quantity we discuss the suppliers' pricing game.

The manufacturer tries to maximize the expected profit in this assembly system:

$$\begin{aligned} \max_{\mathbf{q} \geq 0} \pi &= pE[D - (D - minR_i q_i)^+] + sE[(minR_i q_i - D)^+] \\ &\quad -uE[(D - minR_i q_i)^+] - \sum_{i=1}^{n} w_i q_i \end{aligned} \tag{2.4}$$

Here $pE[D - (D - minR_i q_i)^+]$ is the expected revenue, $sE[(minR_i q_i - D)^+]$ is the salvage profit from left over inventory and $uE[(D - minR_i q_i)^+]$ is the total penalty cost for unsatisfied demand. After simplification, we have:

$$\begin{aligned}\pi &= -\sum_{i=1}^{n} w_i q_i + pE[D] - sE[D] + sE[min(R_i q_i)] \\ &- (p+u-s)E[(D - min(R_i q_i))^+]\end{aligned}$$

Lemma 2.14. *The manufacturer's expected profit function is concave in ordering quantity q_i, $i = 1, \ldots, n$.*

Corollary 2.3. *If $q_i^* = 0$ for some i, then $q_j^* = 0$ for all $j \in \{1, \ldots, n\}$.*

Given q_i, to calculate the expected profit π, the information of density function $g_m(x)$ of $\min_i R_i q_i$ can save us a lot of work and it can be calculated by the following formula:

$$g_m(x) = \sum_{i=1}^{n} g_i(x/q_i)/q_i \prod_{j \neq i} (1 - G_j(x/q_j)) \tag{2.5}$$

Similar to the sourcing system, the manufacturer's expected profit is a concave function. Hence, the optimal ordering quantities are defined by the KKT conditions.

Define the operator $E[\chi(\mathbf{R})]_{s_i}$ on function of random variable $\chi(\mathbf{R})$ as follows

$$E[\chi(\mathbf{R})]_{s_i} = E[\chi(\mathbf{R})|R_i q_i = min_j\{R_j q_j\}]Pr(R_i q_i = min_j\{R_j q_j\})$$

Hence $E[\chi(\mathbf{R})]_{s_i}$ is the integration of $\chi(\mathbf{R})$ with density $\mathbf{R}$ in the region where component i's realized yield $R_i q_i$ bounds the production quantity. In other words, the integration is over the region $R_i \leq \frac{1}{q_i} \min_{j \neq i} R_j q_j$. By formula 2.4, the density function $g_{mi}(x)$ for random variable $min_{j \neq i} R_j q_j$ can be formulated as:

$$g_{mi}(x) = \sum_{j \neq i} g_j(x/q_j)/q_j \prod_{k \neq i,j} (1 - G_k(x/q_k))$$

Therefore, with $E[x]_{s_i}$ can be written in the following double integration:

$$\begin{aligned}E[\chi(\mathbf{R})]_{s_i} &= E[\chi(\mathbf{R})|R_i q_i = min_j\{R_j q_j\}]Pr(R_i q_i = min_j\{R_j q_j\}) \\ &= E[\chi(\mathbf{r})|R_i q_i \leq \min_{j \neq i} R_j q_j]Pr(R_i q_i \leq \min_{j \neq i} R_j q_j) \\ &= \int_0^{+\infty} \int_0^{y/q_i} \chi(\mathbf{r}) g_i(r_i) g_{mi}(y) dr_i dy \\ &= \int_0^1 \chi(\mathbf{r}) g_i(r_i)(1 - G_{mi}(r_i q_i) dr_i\end{aligned}$$

Lemma 2.15. *If optimal ordering quantities $q_i^* > 0$ $(i = 1, \dots, N)$, then they are defined by*

$$w_i = (p+u)E[R_i]_{s_i} - (p+u-s)E[R_i F(R_i q_i^*)]_{s_i} \qquad \textit{for } q_i^* > 0$$

The following two lemmas compare the optimal ordering quantities among the suppliers.

Lemma 2.16. *If R_i and R_j are independently and identically distributed and $w_i > w_j$, then $q_i^* \leq q_j^*$.*

Note that in deterministic reliability case, all the suppliers will receive the same ordering quantity even if their wholesale prices are different. In our stochastic reliability model, manufacturer tries to lower his over stocking cost for higher wholesale price components by ordering less.

Lemma 2.17. *If $R_i \geq R_j$ almost surely, then $q_i^* \leq q_j^*$.*

Hence by lemma 2.17, a less reliable supplier will get more ordering quantity. This is opposite to sourcing system.

The case in which all the optimal ordering quantities are zero is identified in the following lemma.

Lemma 2.18. *Not ordering is an optimal decision for the manufacturer if and only if $\sum_{i=1}^{n} w_i \xi_i \geq (p+u)E[\min w_i \xi_i]$, $\forall \xi_i \geq 0$, $i = 2, \dots, n$ and $\xi_1 = 1$*

The following theorem characterizes the effect of the component prices on the optimal ordering quantities:

Theorem 2.8. *The expected profit function of the manufacturer is supermodular in $\{\mathbf{q}, -\mathbf{w}\}$. Hence the optimal ordering quantity q_i^* for all $i = 1, \dots, n$ is monotone decreasing in w_j for all $j = 1, \dots, n$.*

2.4.1 Optimal solution procedure based on supermodularity

Compared with with the sourcing system in which the calculation of expected profit will involve n dimensional integration which is very time consuming, the calculation of optimal ordering quantity in assembly system only involves double integration. Also, since the expected profit function of the manufacturer is both supermodular and concave in q_i, [44] suggests a so-called tatonnement or round-robin scheme which accelerate the computation of optimal decision:

Algorithm 2.2 Round-Robin Scheme

1. Starting with an arbitrary quantity vector $\mathbf{q^0} = (q_1^{max}, q_2^{max}, \ldots, q_n^{max})$ where q_i^{max} is the upper bound for q_i^* which can be estimated by the upper bound of demand divided by the lower bound of the reliability. Let $k = 1$.

2. In k^{th} iteration, $\mathbf{q^k}$ is obtained from $\mathbf{q^{k-1}}$ by determining:

$$q_i^k = argmax_{q_i} E[\pi](\ldots, q_{i-1}^k, q_i, q_{i+1}^{k-1}, \ldots) \qquad \forall i = 1, \ldots, n$$

3. If $\|\frac{\partial E[\pi]}{\partial q}\| \leq \varepsilon$ stops, otherwise increase k by 1 and go back to step 2.

Since the function is concave, by supermodularity, $\mathbf{q^k}$ converges to $\mathbf{q}^*$.

2.4.2 Heuristic solution procedure

Even with the property of supermodularity and applying the Round-Robin scheme, the double integral involved in our algorithm is time consuming. Our computational experiments suggest that for three suppliers, the computation time is more than 20 mins. One alternative approach is Monte Carlo sampling method, which draw M samples independently through the joint distribution of the reliability. Here we suggest a heuristic which approximates $\min_i R_i q_i$ in the first order condition with some deterministic target production quantity q^t. The q^t is estimated by solving the news vendor problem with constant reliability $\bar{R}_i$.

Algorithm 2.3 Target Production Quantity Heuristic

1. Let $q^t = F^{-1}((p + u - \sum_i w_i/\bar{R}_i)/(p + u - s))$

2. $\forall i \in 1 \ldots n$, our heuristic solution q_i^o is the root for

$$\begin{aligned} w_i &= \psi(q_i) \\ &= (p+u)E[R_i|R_i \leq q^t/q_i]Pr(R_i q_i \leq q^t) - (p+u-s)E[R_i F(R_i q_i)|R_i \leq q^t/q_i]Pr(R_i q_i \leq q^t) \end{aligned}$$

Note that as $q_i \to +\infty$, $\psi(q_i) \to 0$ and as $q_i \to 0$ $\psi(q_i) \to p + u$. Besides,

$$\frac{\partial \psi(q_i)}{\partial q_i} = -\frac{{q^t}^2 g_i(q^t/q_i)}{q_i^3}((p+u) - (p+u-s)F(q^t)) - (p+u-s)\int_0^{q^t/q_i} x^2 g_i(x) f(q_i x)dx \leq 0$$

Hence there must exists a solution for $w_i = \psi(q_i)$ in step 2. In [21], Gurnani et al. apply a similar approach to their problem, they also set the same target production quantity as ours. But instead of fully utilizing of the reliability distribution function, they only use the first order information of R_i and set $q_i^o = q_t/\bar{R}_i$. We will compare these two heuristic in our numerical examples.

2.4.3 Numerical examples

To evaluate the performance of our heuristic, we applied it to 3 component problems. By [49], most yield reliability density functions tend to be unimodal with finite support. Using their same approach, we assume that the density function has a triangular distribution and test the problem on different distribution parameters.

For all the test problems, we assume demand is also triangularly distributed with mean at 200 but with different variances depending on different experiment scenarios.

In all of the following experiments, the market price is fixed to be 70, the lost sale penalty is assumed to be 5 and the salvage value is assumed to be 6. Also the wholesale price for component one and two is fixed to be 6 and 5 respectively .

We compare the optimal solution by Round-Robin Scheme with the solutions by target

Table 2.2: Yield reliability shape parameters

Shape	Min(a)	Mode(b)	Max(c)	Mean	Variance	Skewness
NS1:	0.4000	0.8000	0.6000	0.6000	0.0067	0.0000
NSS1:	0.5000	0.7000	0.6000	0.6000	0.0017	0.0000
NSS2:	0.4586	0.7414	0.6000	0.6000	0.0033	0.0000
NSF1:	0.3172	0.8828	0.6000	0.6000	0.0133	0.0000
NSF2:	0.2000	1.0000	0.6000	0.6000	0.0267	0.0000
LS1:	0.4104	0.8090	0.5806	0.6000	0.0067	0.1412
LS2:	0.4231	0.8170	0.5599	0.6000	0.0067	0.2828
LS3:	0.4401	0.8243	0.5356	0.6000	0.0067	0.4241
LS4:	0.4845	0.8309	0.4845	0.6000	0.0067	0.5657
RS1:	0.3910	0.7896	0.6194	0.6000	0.0067	-0.1412
RS2:	0.3830	0.7770	0.6401	0.6000	0.0067	-0.2828
RS3:	0.3757	0.7599	0.6644	0.6000	0.0067	-0.4241
RS4:	0.3691	0.7155	0.7155	0.6000	0.0067	-0.5657

Table 2.3: Demand Distribution parameters

Shape	Min(a)	Mode(b)	Max(c)	Mean	Variance	Skewness
DNS:	180	200	220	200	66.67	0
DNSS:	133.33	200	266.67	200	740.74	0
DNS1	166.67	200	233.33	200	185.18	0
DNS2	152.87	200	247.13	200	370.26	0
DNF1	105.72	200	294.28	200	1481.48	0
DNF2	66.67	200	333.33	200	2962.96	0

production quantity heuristic proposed in this paper (TPQ Heuristic) and the heuristic applied by [21] (Gurnani Heuristic). In Round-Robin Scheme, we use the absolute norm of partial derivative as $\|\frac{\partial E[\pi]}{\partial q}\|$ and set optimal tolerance $\varepsilon = 0.001$. Since the comparison of the expected profit for different algorithms may be affected by the constant part $pE[D]$, here we only compare the manufacturer's expected cost which is $\pi + pE[D]$. The detailed characteristics of twenty nine test problems are displayed in Table A.4 and the computational result is attached in Table A.4 in appendix.

In general, our result shows that for Round-Robin scheme we applied, even though our solution procedure converges very quickly to the optimal solution (within 10 iterations), the calculation of double integral consumes most of the time and the total computation time is more than twenty minutes for each single case. Monte Carlo sampling method may be applied here to accelerate the integration speed.Our results also suggest that our TPQ heuristic performs very well (within 0.05% to the optimal solution) while the average optimality gap of Gurnani heuristic is around 10%. With negligible computation time, our TPQ heuristic performs more than 1000 times better than the Gurnani heuristic and can be used to replace the time consuming Round-Robin scheme in most cases.

We design several scenario groups to check the change of system performance with respect to different changes in input parameters.

2.4.3.1 Scenario group one: Change of wholesale price

Problem (1,2,3,4), see table A.4 in appendix.
In the first scenario group shown in figure 2.6, the only change we make is to increase the wholesale price for component three from 3 to 6 each, and the computed results reflect our theorem 2.8: the optimal ordering quantity for each component is decreasing and the optimal cost is increasing. We also find that the decreasing rate of q_3^* is higher than the other two. This is because only component three's wholesale price is increasing, whereas the corresponding change of optimal ordering quantity has larger impact on supplier 3 himself. Also for the same density of reliability, by lemma 2.16 the component supplier with higher w_i will get less ordering quantity.

2.4.3.2 Scenario group two: Change of reliability variance for one Component

Including Problem (1,5,6,7,8), see table A.4 in appendix.
In the second scenario group shown in figure 2.6, the only change we make is to increase the

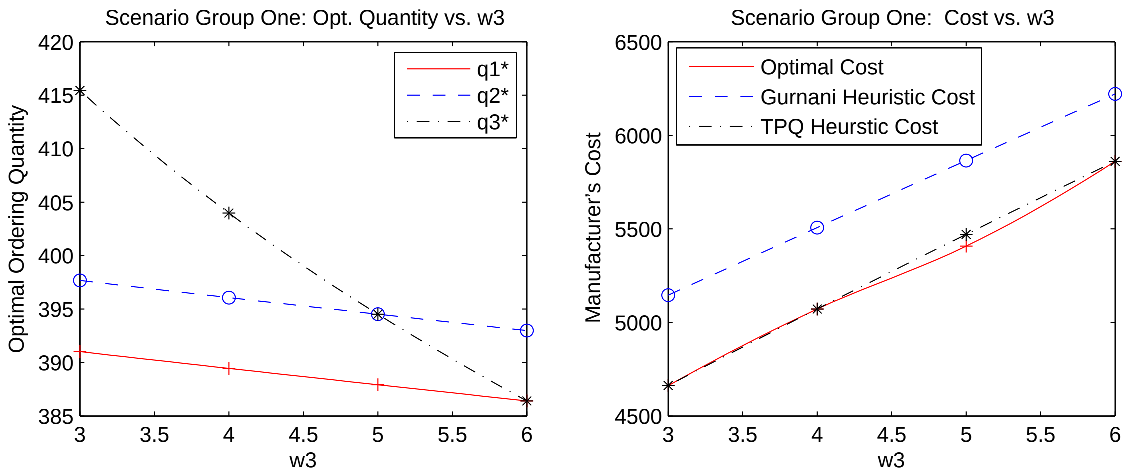

Figure 2.5: The system performance under w_3

variance of the reliability for component three (keep mean at 0.6). The result suggests an increase in the optimal cost due to the manufacturer facing a higher risk in upper flow. Also we find that the optimal ordering quantity for component three increases quickly. The reason for this can be relatively easy to find by checking our TPQ heuristic (close in numerical result): since the mean of R_3 doesn't change, the target production quantity remain the same, then we have to increase q_i to make sure $\psi(q_i)$ remain the same when the support of g_i is wider. This reason also explains why the optimal ordering quantity for the other two product is stable.

2.4.3.3 Scenario group three: Change of reliability skewness for one component

Including Problem (1,9-16), see table A.4 in appendix.

Scenario three (figure 2.7) studies the system performance with increasing skewness of component three (keeping mean and variance same). In this case, the numerical results show both optimal cost and ordering quantity for component three decreases (elongated tail at right). To explain this, again we can look at our TPQ heuristic. Since the target production

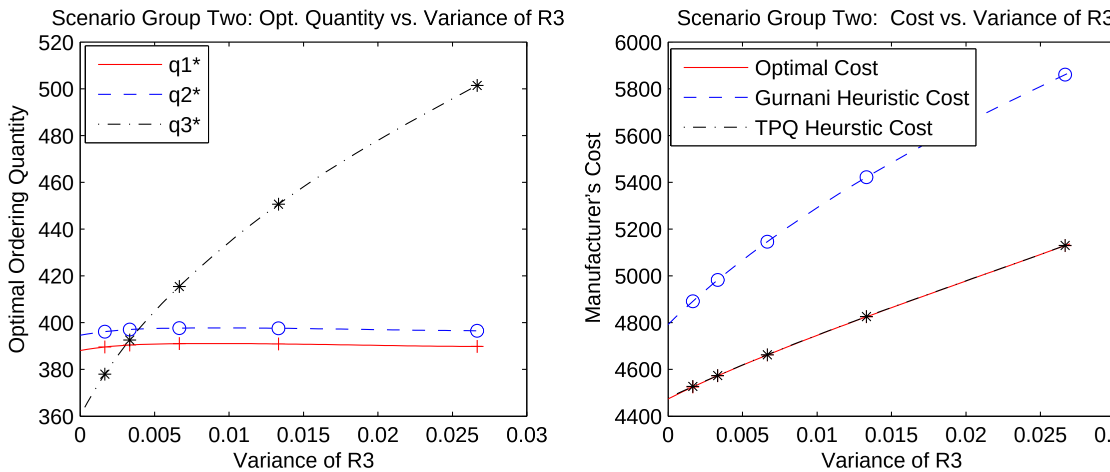

Figure 2.6: The system performance under $\sigma^2_{R_3}$

quantity remains the same, $\psi(q_i)$ is decreasing with q_i. For more negative skewness, the density g_i is larger in the integration region compared with positive skewness, and $\psi(q_i)$ will be larger with same q_i, to counteract this effect, we have to decrease q_i. Also note that under-stocking component three will create more cost than over-stocking component 3. Negative skewness with an elongated tail at left is more likely to cause component 3 to be under-stocked. It turns out thus giving the reason for the lower optimal cost. Again, the change in skewness of component three has a very small impact on optimal ordering quantity for the other components.

2.4.3.4 Scenario group four: Change of demand variance

Including Problem (1,17-21), see table A.4 in appendix.

Only demand variance is increasing in scenario four of the numeric study (shown in figure 2.8). This increase leads to an increase in the optimal cost and optimal ordering quantity for each component. Empirically, increasing the demand variance causes a higher underage and overage risk, which leads directly an increase in cost. The TPQ heuristic shows when demand

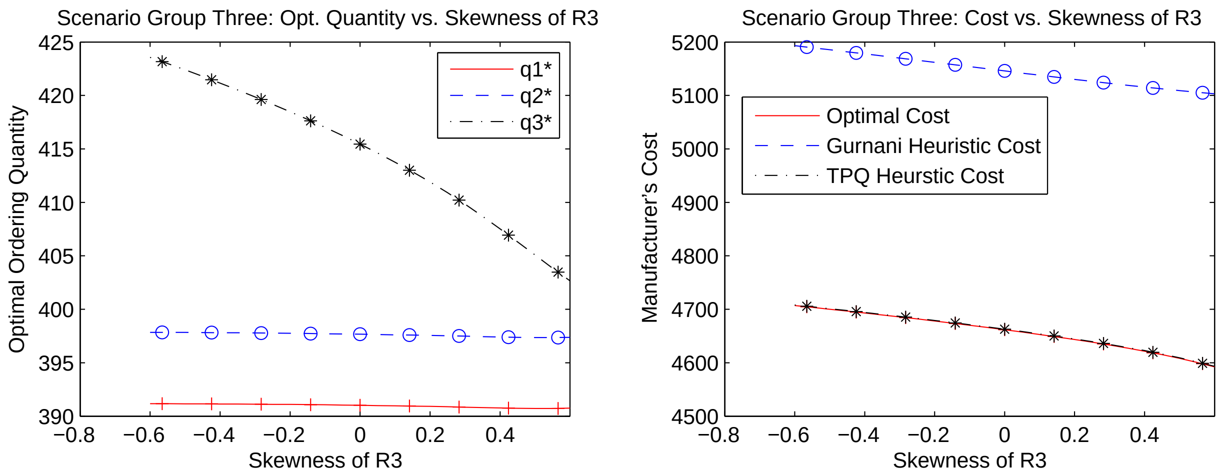

Figure 2.7: The system performance under skewness of R_3

variance is enlarged, the corresponding target production quantity for high service level (> 0.5) will increase. This directly leads to increase in ordering quantity for each components. Another interesting observation is that with the increase of the demand variance, the Gurnani Heuristic performs better. This is because that Gurnani heuristic views the reliability of component as a deterministic input and only cares about the risk of demand, then when demand risk increases dominating the upper flow risk (unreliability of the components), the Gurnani heuristic approaches the main part of the cost and performs better.

2.4.3.5 Scenario group five: Change of reliability variance for all components

Including Problem (1,22-25), see table A.4 in appendix.

Compared with scenario two, we increase the variation for the reliability of all three components (see figure 2.9). In this case, we find that the increasing rate of the optimal ordering quantity is higher for lower price components. The reason for this is that for the different wholesale price components we can normalize them into the same price with different units ($3\$/unit- > 6\$/2unit$), and hence, the manufacturer will order more quantities for lower

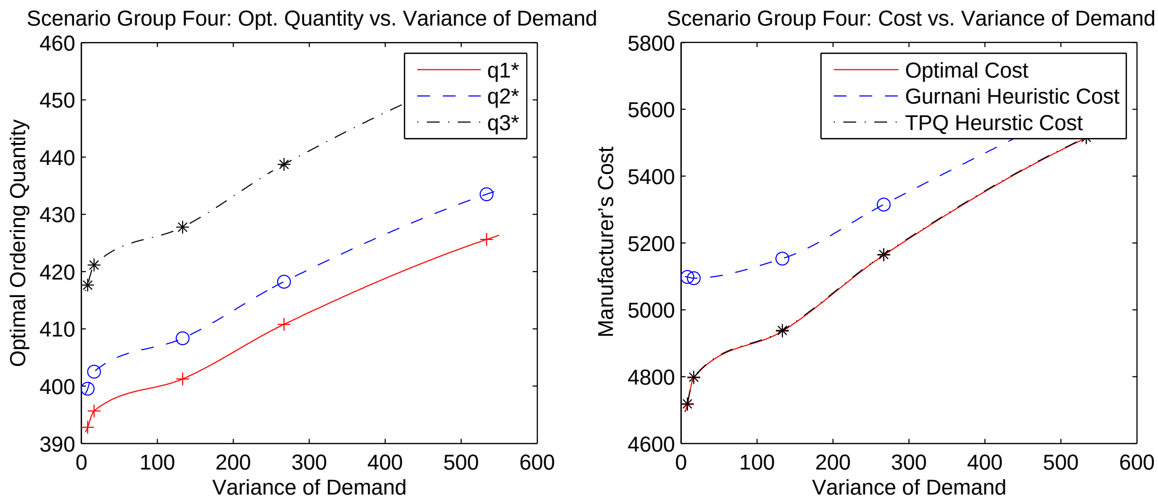

Figure 2.8: The system performance under demand variance σ_D^2

wholesale price components. The optimal cost is also increasing with the reliability of variance. An interesting issue is which of the following two factors has a larger impact on the optimal cost: increasing demand variance or increasing component reliability variance. (see figure 2.10). To unify the unit, we plot the change in optimal cost versus the coefficient of variation (ρ). The result suggests a more noticeable impact by the increase in variation of the component reliability.

2.4.3.6 Scenario group six: Change of reliability skewness in different direction for two components

Including Problem (1,26-29), see table A.4 in appendix.

In the last scenario group (see figure 2.11, we test on positive skewness of component one and negative skewness of component three (same absolute skewness). The numeric study shows that optimal ordering quantity for component three is more sensitive to increasing skewness while the quantity for component one decreases a little when absolute skewness is increased. For the TPQ heuristic's mechanism, with same mean, the target production

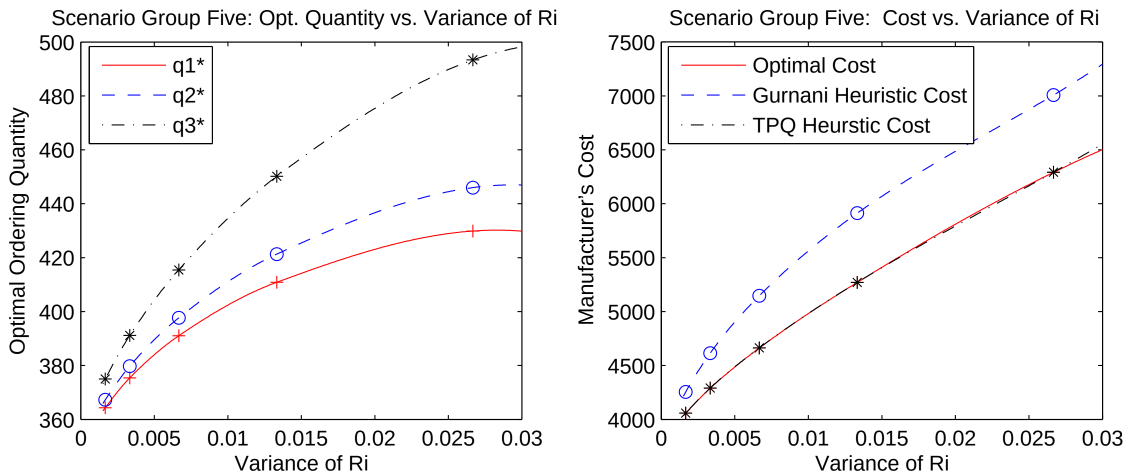

Figure 2.9: The system performance under $\sigma^2_{R_1}, \sigma^2_{R_2}, \sigma^2_{R_3}$

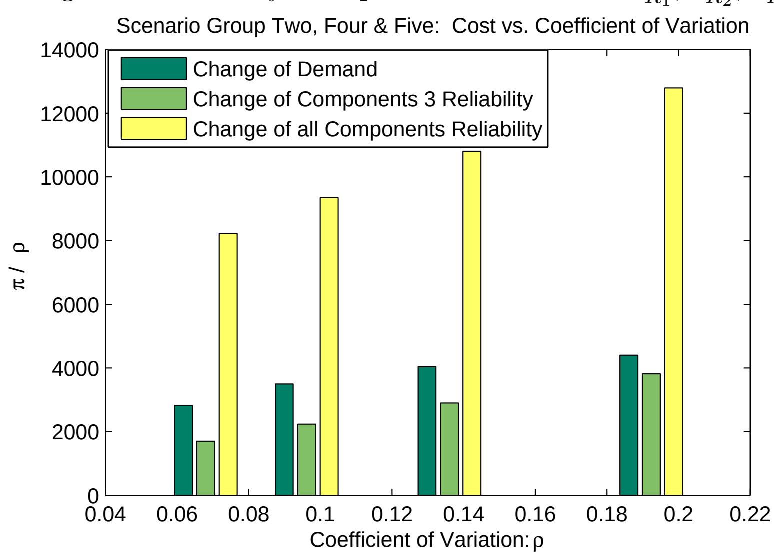

Figure 2.10: Sensitivity analysis of optimal profit under variation in demand and in components

quantity will not change, and the change of R_i has very little effect on the ordering quantity for the other components. Having similar reasoning to scenario five, the ordering quantity for the component with lower wholesale price is more sensitive to the absolute skewness. Also we find that the optimal cost is decreasing with absolute skewness, meaning that the change in distribution skewness of component with a lower wholesale price has a dominant impact on the optimal cost.

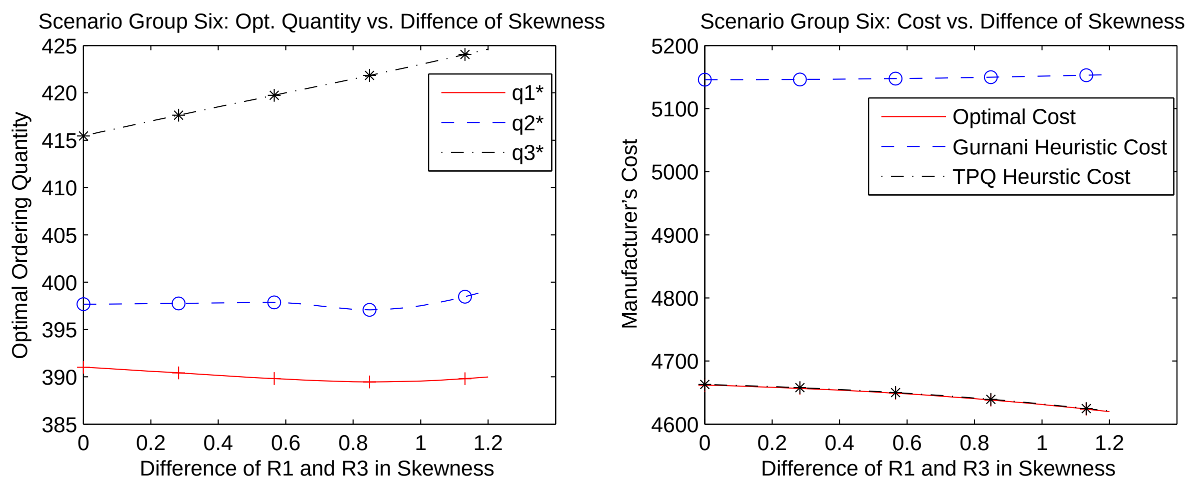

Figure 2.11: The system performance under non symmetric reliability skewness

2.5. Suppliers' Pricing Game in Assembly System

For the pricing game among the suppliers, consider the case with n suppliers $i \in \{1, \ldots\}$, with Bernoulli reliability, i.e. with probability P_i to successfully deliver the whole ordering quantity and $1 - P_i$, ($P_i \in (0, 1)$).

2.5.1 Manufacturer's problem and optimal decision

The manufacturer tries to maximize the expected profit in this assembly system as stated in equation 2.4. In this case, the manufacture can only sell his product when all components arrived successfully with probability $\prod_{i=1}^{n} P_i$ and loss all the demand when any of the component order is default with probability $1 - \prod_{i=1}^{n} P_i$. So we have

$$\begin{aligned}\max_{\mathbf{q}\geq 0} \pi &= \prod_{i=1}^{n} P_i\big(pE[D-(D-\min_i q_i)^+] + sE[(\min_i q_i - D)^+] \\ &\quad -uE[(D-\min_i q_i)^+]\big) - (1-\prod_{i=1}^{n} P_i)uE[D] - \sum_{i=1}^{n} w_i q_i\end{aligned}$$

By lemma 2.14 the manufacturer's profit is concave in $\mathbf{q}$.

Proposition 2.1. *In manufacturer's optimal decision, $q_i^* = q^*$ for all $i = 1, \ldots, n$.*

To make sure there is no trivial solution in which the manufacturer can always make positive profit from salvage market, we assume $w_i \geq c_i > s/\prod_{i=1}^{n} P_i$. Thus the manufacturer will face a simple news vendor problem, in which he only has to decide on optimal ordering quantity q:

$$\begin{aligned}\max_{q\geq 0} \pi &= \prod_{i=1}^{n} P_i(p+u)E[D] - uE[D] - q\sum_{i=1}^{n} w_i \\ &\quad -\prod_{i=1}^{n} P_i\big((p+u)E[(D-q)^+] + sE[(q-D)^+]\big)\end{aligned}$$

Therefore optimal ordering quantity can be calculated by replacing the underage cost with $(p+u)\prod_{i=1}^{n} P_i - \sum_i w_i$ and the overage cost with $\sum_i w_i - s\prod_{i=1}^{n} P_i$. When demand is continuous distributed, the manufacturer will order $F^{-1}\big(\frac{(p+u)\prod_{i=1}^{n} P_i - \sum_i^n w_i}{(p+u-s)\prod_{i=1}^{n} P_i}\big)$. Since it is possible that F^{-1} may not be unique, we assume that the manufacturer will always choose the maximum indifferent optimal ordering quantity: $\max_x : F(x) = \frac{(p+u)\prod_{i=1}^{n} P_i - \sum_i^n w_i}{(p+u-s)\prod_{i=1}^{n} P_i}$. Also for $\sum_i w_i > (p+u)\prod_{i=1}^{n} P_i$, the optimal ordering quantity is 0. Hence given $\mathbf{w}$, the optimal ordering quantity is:

$$q^*(\sum_i w_i) = \begin{cases} \max \quad x : F(x) = \frac{(p+u)\prod_{i=1}^{n} P_i - \sum_i w_i}{(p+u-s)\prod_{i=1}^{n} P_i} & \text{if } \sum_i^n w_i \leq (p+u)\prod_{i=1}^{n} P_i \\ 0 \quad \text{otherwise} \end{cases}$$

2.5.2 Suppliers' simultaneous pricing game

For each supplier i, he wants to maximize its own individual profit:

$$\max_{w_i \geq c_i} \pi_{si}(w_i) = (w_i - c_i)q^*(\mathbf{w})$$

Define an Nash equilibrium $\mathbf{w}^*$ to be trivial if $\pi_{si}(\mathbf{w}^*) = 0$ for all $i = 1, \ldots, n$.

Proposition 2.2. *There always exits a trivial Nash equilibrium $\mathbf{w}^*$ in the the noncooperative pricing game among suppliers $\{\{1 \ldots, n\}, \{w_i \geq c_i\}, \{\pi_{s_i} : i = 1 \ldots, n\}\}$.*

Theorem 2.9. *If the demand distribution F is continuous on $[0, +\infty)$ with $F(0) = 0$, there exists an positive optimal solution for the following potential function:*

$$\max_{\mathbf{w} \geq \mathbf{c}} \psi(\mathbf{w}) = \prod_{i=1}^{n} (w_i - c_i) q^*(\sum_i w_i)$$

Also, the maximizers are non trivial equilibrium points for the suppliers's noncooperative pricing game $\{\{1 \ldots, n\}, \{w_i \geq c_i\}, \{\pi_{s_i} : i = 1 \ldots, n\}\}$.

Hence for most common demand distributions, there always exists equilibrium points with positive profits for every one. Clearly such equilibrium strongly dominates the trivial Nash equilibrium in which no one get any positive profit. So in the following analysis we concentrate on the non-trivial equilibrium points.

Now suppose demand is a continuous random variable, and $f(x)$ has positive support on $[a, b)$ where $a \geq 0$. Therefore $F(x)$ is strictly increasing on $[a, b)$, and $q^*(\sum_i w_i) = F^{-1}(\frac{(p+u)\prod_{i=1}^n P_i - \sum_i w_i}{(p+u-s)\prod_{i=1}^n P_i})$ for $\sum_i w_i < (p+u)\prod_{i=1}^n P_i$, $q^*(\sum_i w_i) = 0$ for $\sum_i w_i > (p+u)\prod_{i=1}^n P_i$ and $q^*(\sum_i w_i) = a$ for $\sum_i w_i = (p+u)\prod_{i=1}^n P_i$ thus $q^*(\sum_i w_i)$ is differentiable in $[\sum_i c_i, (p+u)\prod_{i=1}^n P_i)$ with $\frac{\partial q^*(\sum_i w_i)}{\partial w_i} = -\frac{1}{(p+u-s)\prod_{i=1}^n P_i f(q^*(\sum_i w_i^*))}$ with positive profit for every one.

Assumption 2.1. *Demand is a continuous random variable and has a probability density function with positive support on $[a, b)$*

Lemma 2.19. *Under assumption 2.1*

a) If $a = 0$ then any non-trivial equilibrium point $\mathbf{w}^*$ is an interior point with $\sum_i w_i^* < (p+u)\prod_i P_i$.

b) For any interior non-trivial equilibrium point $\mathbf{w}^*$ such that $\sum_i w_i^* < (p+u)\prod_i P_i$, every supplier have the same profit:

$$\begin{aligned} \pi_{si}(\mathbf{w}^*) &= (p+u-s)\prod_{i=1}^{n} P_i F^{-1}\Big(\frac{(p+u)\prod_{i=1}^{n} P_i - \sum_i w_i^*}{(p+u-s)\prod_{i=1}^{n} P_i}\Big)^2 f\Big(F^{-1}\Big(\frac{(p+u)\prod_{i=1}^{n} P_i - \sum_i w_i^*}{(p+u-s)\prod_{i=1}^{n} P_i}\Big)\Big) \\ w_i^* - c_i &= (p+u-s)\prod_{i=1}^{n} P_i F^{-1}\Big(\frac{(p+u)\prod_{i=1}^{n} P_i - \sum_i w_i^*}{(p+u-s)\prod_{i=1}^{n} P_i}\Big) f\Big(F^{-1}\Big(\frac{(p+u)\prod_{i=1}^{n} P_i - \sum_i w_i^*}{(p+u-s)\prod_{i=1}^{n} P_i}\Big)\Big) \end{aligned}$$

Furthermore, if $\mathbf{w}^{*1}$ and $\mathbf{w}^{*2}$ are two equilibrium points, then we have either $\mathbf{w}^{*1} > \mathbf{w}^{*2}$ or $\mathbf{w}^{*1} < \mathbf{w}^{*2}$. Additionally, without loss of generality, if $\mathbf{w}^{*1} > \mathbf{w}^{*2}$ then $\pi_{si}(\mathbf{w}^{*2}) > \pi_{si}(\mathbf{w}^{*1})$ for all $i = 1, \dots, n$.

c) If $a > 0$, then any boundary non-trivial equilibrium point $\mathbf{w}^*$ satisfy $\sum_i w_i^* = (p+u)\prod_i P_i$ and $q^*(\sum_i w_i^*) = a$. In addition, if there exists any equilibrium such that $\sum_i w_i^* = (p+u)\prod_i P_i$, then $\lim_{x\to a^+} xf(x) \geq \frac{(p+u)\prod_{i=1}^{n} P_i - \sum_{i=1}^{n} c_i}{m(p+u-s)\prod_{i=1}^{n} P_i}$, where m is the number of suppliers with $w_i^* > c_i$.

To further analyze non-trivial equilibrium and the division of system profits, we assume there is no salvage value and no penalty cost for lost sale: $u = s = 0$, and we let $\bar{p} = p\prod_{i=1}^{n} P_i$, $\bar{c} = \sum_i c_i$. Thus the manufacturer's expected profit is:

$$\pi = \bar{p}E[D] - \bar{p}E[(D-q)^+] - q\sum_i w_i$$

At any equilibrium point $\mathbf{w}^*$, the optimal ordering quantity satisfies: $\bar{p} - \sum_i w_i^* = \bar{p}F(q^*(\sum_i w_i^*))$.

Let $q^* = q^*(\mathbf{w}^*)$ for some non-trivial equilibrium point $\mathbf{w}^*$ and π_{si}^* and π^* be the corresponding supplier i and manufacturer's profit in equilibrium respectively. Most of the following results in this chapter are extensions of [26] to the multiple supplier case.

Lemma 2.20. *Under assumption 2.1, suppose that the demand distribution has increasing* *generalized failure rate (IGFR)*[3] *and a finite mean. Assume* $a = 0$ *or* $\lim_{x\to a^+} xf(x) < \frac{\bar{p}-\bar{c}}{n\bar{p}}$. *The suppliers' simultaneous pricing game has a unique non-trivial Nash equilibrium with the corresponding* q^* *which is the unique solution for following equation:*

$$1 - F(q) - nqf(q) = \frac{\bar{c}}{\bar{p}} \tag{2.6}$$

Lemma 2.20 shows that unique non-trivial equilibrium exists when $a > 0$ and $\lim_{x\to a^+} xf(x) < \frac{\bar{p}-\bar{c}}{n\bar{p}}$ for IGFR demand distribution. The IGFR assumption is not very restrictive, since most common distributions are within this category such as normal, uniform, gamma and Weibull for all parameters.

$$\text{gamma density :} \qquad f(x) = e^{-\frac{x}{\beta}} x^{\alpha-1}\beta^{-\alpha}/\Gamma(\alpha) \tag{2.7}$$

$$\text{Weibull density :} \qquad f(x) = \theta^{-1}kx^{k-1}e^{-x^k/\theta} \tag{2.8}$$

In case $\lim_{x\to a^+} xf(x) \geq \frac{\bar{p}-\bar{c}}{n\bar{p}}$, the non-trivial equilibrium may be not unique, and we have the following lemma:

Lemma 2.21. *Under assumption 2.1 and* $a > 0$, *suppose demand distribution has IGFR and a finite mean. If for some* $m \in \{1, \ldots, n\}$, *we have* $\frac{\bar{p}-\bar{c}}{m\bar{p}} \leq \lim_{x\to a^+} xf(x) < \frac{\bar{p}-\bar{c}}{(m-1)\bar{p}}$ *then the non-trivial equilibrium in suppliers' simultaneous pricing game is defined by the set:* $\mathcal{W}^* = \{\mathbf{w}^* : \sum_{i=1}^n w_i^* = \bar{p}-\bar{c}$ *and* $\lim_{x\to a^+} xf(x) \leq \frac{w_i^*-c_i}{\bar{p}}$ *for all* $w_i^* - c_i > 0\}$. *Thus there are no more than* $n-m$ *suppliers get* 0 *profit in the equilibrium. Especially when* $\lim_{x\to a^+} xf(x) = \frac{\bar{p}-\bar{c}}{n\bar{p}}$, *the equilibrium is unique:* $\mathbf{w}^* = \mathbf{c}^* + \frac{\bar{p}-\bar{c}}{n\bar{p}}$.

Through lemma 2.21, we can find that if the demand contains a larger deterministic portion (a is larger), the possible status at equilibrium is complicated. However, all those different equilibria just alter the share of profit among suppliers, they still lead to a common ordering quantity. From the customer (manufacture)'s point of view, they are all the same.

[3] A distribution has an increasing generalized failure rate if $\frac{xf(x)}{1-F(x)}$ is weakly increasing for all x such that $F(x) < 1$.(here f is the density function and F is the cumulative distribution function)

2.5.3 The division of system profit

To analyze the division of system profit, we assume that either $a = 0$ or $\lim_{x\to a^+} xf(x) \leq \frac{\bar{p}-\bar{c}}{n\bar{p}}$ and hence the non-trivial equilibrium points must be interior points and every supplier gets the same profit. Then by lemma 2.19, for each retailer i we have: $w_i^* - c_i = \bar{p}q^*(\sum_i w_i^*)f(q^*(\sum_i w_i^*))$. Define $\tau = q^*(\sum_i w_i^*)f(q^*(\sum_i w_i^*))/F(q^*(\sum_i w_i^*))$. Thus we have:

$$\frac{w_i^* - c_i}{\bar{p} - \sum_i w_i^*} = \tau$$

Since $(\bar{p} - \sum_i w_i^*)q^* \geq \pi^*$ and $(w_i^* - c_i)q^* = \pi_{si}^*$ for all $i = 1, \ldots, n$, we have $\pi_{si}^*/\pi^* \geq \tau$ Additionally, we take the service level in the case all components are successfully delivered as the realized service level of the manufacturer which is defined by $\frac{\bar{p}-\sum_i w_i^*}{\bar{p}} = \frac{1}{n\tau+1}\frac{\bar{p}-\bar{c}}{\bar{p}}$. Clearly the true service level is the realized service level times the probability that all components are successfully delivered: $\prod_{i=1}^n P_i$.

Lemma 2.22. *For any non-trivial equilibrium point* $\mathbf{w}^*$*, if* F *is convex in* $(0, q^*)$*, the realized service level is less than* $\frac{\bar{p}-\bar{c}}{(n+1)\bar{p}}$ *and each individual supplier gets more than the manufacturer. If* F *is concave in* $(0, q^*)$*, the realized service level lies in the interval* $[\frac{\bar{p}-\bar{c}}{(n+1)\bar{p}}, \frac{\bar{p}-\bar{c}}{\bar{p}}]$

In case F is concave $(0, q^*)$, then $\tau \leq 1$. The manufacturer's profit may be more or less than each supplier's profit.

Example 2.4. *Consider Weibull demand with* $\theta = 1$ *. Suppose* $n = 2$*,* $\bar{p} = 1$ *and* $c_i = 0$ *for* $i = 1, 2$*. In equilibrium, manufacturer order* $q^* = (\frac{1}{nk})^{1/k}$*. To see that the manufacturer may capture more than the supplier, set* $k = 0.2$*. The numerical solution shows that the manufacturer get* 125% *of each supplier's profit with a realized service level of* 0.92 *which is between* 0.5 *and* 1*. While if* $k = 0.6$*,* F *is still concave for* $x \in (0, q^*)$*, however in this case the manufacturer gets* 80% *of each supplier's profit with an realized service level of* 0.57*.*

Corollary 2.4. *The equilibrium* q^* *falls in the convex portion of* F *if:*
a) F *is normal distribution with mean* μ*, standard deviation* σ*, and* $\sqrt{2/\pi}\sigma/\mu < n\bar{p}/(\bar{p}-2\bar{c})$*.*
b) F *is a gamma distribution with density defined by 2.7 and* $\alpha > 3.83$

c) F is a Weibull distribution with density defined by 2.8 and $k \geq 1 + 1/n$

2.5.4 Supply chain efficiency

For the supply chain efficiency, we consider the profit of the three systems: the decentralized system profit π^D which is the total profit for all the companies in the system with the suppliers' price competition, the the centralized system profit π^D which is the optimal profit for the whole system and the supplier coordination profit π^S which is the system profit when all the suppliers coordinate to maximize their total profit and sell components to the manufacture. The supplier coordination system is the same as a single supplier with component price $\bar{c}$, therefore, by [26], when demand distribution is IGFR with finite mean and support on $[a, b)$, if $a > 0$ or $\lim_{x \to a^+} xf(x) \geq \frac{\bar{p}-\bar{c}}{\bar{p}}$, the corresponding best sell quantity for supplier is $q' = a$ otherwise it is the unique solution for $1 - F(q) - qf(q) = \bar{c}/\bar{p}$. Theorem 5 in [26] presents the ratio of π^S/π^I:

Theorem 2.10 (Theorem 5 in [26])**.** *If demand is given by $D = \delta + \lambda X$ where X is an IGFR random variable with cumulative distribution F, support $[0, b)$, mean μ and standard deviation σ, $\delta \geq 0$ and $\lambda > 0$ are constants.*
a) If $\delta = 0$, π^S/π^I is independent of λ.
b) For any λ and $\rho > 0$, let $\delta(\lambda) = \lambda(\sigma/\rho - \mu)$. Thus ρ is the coefficient of variation of the demand. If $f(0) > 0$, then there exits a $\rho' > 0$ such that π^S/π^I is decreasing in ρ for $\rho < \rho'$ and goes to one as ρ goes to zero. There also exists a ρ'' such that π^S_M/π^I is decreasing in ρ for all $\rho < \rho''$ (here π^S_M is the manufacture's under supplier coordination).

For the decentralized system, note that if $\lim_{x \to a^+} xf(x) \geq \frac{\bar{p}-\bar{c}}{n\bar{p}}$, there may be multiple non-trivial equilibrium points while all of them lead to the same $q^* = a$ and the same system profit. Hence it still makes sense for us to compare π^D/π^I for general demand shifting effect.

Theorem 2.11. *If demand is given by $D = \delta + \lambda X$ where X is an IGFR random variable* *with cumulative distribution F, support $[0, b)$, mean μ and standard deviation σ, $\delta \geq 0$ and $\lambda > 0$ are constants.*

a) If $\delta = 0$, π^D/λ and π^I/λ are constant. Hence π^D/π^I is independent of λ.

b) For any λ and $\rho > 0$, let $\delta(\lambda) = \lambda(\sigma/\rho - \mu)$. If $f(0) > 0$, then there exits a $\rho' > 0$ such that π^D/π^I is decreasing in ρ for $\rho < \rho'$ and goes to one as ρ goes to zero.

Theorem 2.12. *If demand is given by $D = \delta + \lambda X$ where X is an IGFR random variable with cumulative distribution F, support $[0, b)$, mean μ and standard deviation σ, $\delta \geq 0$ and $\lambda > 0$ are constants.*

a) Let q^* be the equilibrium ordering quantity for π^D and q^S be the suppliers coordination ordering quantity for π^S. Then we have $q^* \leq qS$ and $\pi^D \leq \pi^S$. In case if F_D is convex in $[0, q^S]$, $q^* \geq nq^S$ and $\pi^D/\pi^S \geq 1/(2n)$

b) If $\delta = 0$, π^D/π^S is independent of λ.

c) For any λ and $\rho > 0$, let $\delta(\lambda) = \lambda(\sigma/\rho - \mu)$. If $f(0) > 0$, then there exits a $\rho' > 0$ such that π^D/π^S is decreasing in ρ for $\rho \leq \rho'$ where $\rho' \geq n\bar{p}f(0)\sigma/(n\bar{p}\mu f(0) + \bar{p} - \bar{c})$ and equals to one for $\rho \leq \bar{p}f(0)\sigma/(\bar{p}\mu f(0) + \bar{p} - \bar{c})$.

In case F_D is concave in $[0, q^S)$, q^S maybe greater or less than nq^*:

Example 2.5. *Consider Gamma demand with $\alpha = 0.3$ and $\beta = 10$, in this case F is concave in $[0, +\infty)$. Suppose $n = 10$, $\bar{p} = 1$ and $\bar{c} = 1/3$ for $i = 1, 2$. Then $q^S = 0.84$ and $q^* = 0.018$ with $\pi^D/\pi^S = 0.043 \leq 1/20$. If $\alpha = 1$ and $\beta = 10$, F is still concave, however in this case $q^S = 4.68$ and $q^* = 0.64$ with $\pi^D/\pi^S = 0.19 \geq 1/20$.*

These results show that when the coefficient of variance ρ is reduced, suppliers will increase their wholesale price, which leads to a lower service level. However, the reduction of ρ may lower the absolute difference between the fractiles of the demand distribution. The gaps of the ordering quantity between the competition system, suppliers' coordination system and centralized system may decrease, which results in the increasing in the efficiency ratio. In the extreme case of pure deterministic demand, all profit goes to the suppliers in all the three systems, thus the competition system is more efficient under a larger coefficient of variance.

2.5.5 Reliability investment decision

Suppose before the component suppliers set their whole sale price, they can improve the reliability probability $P_i \in \mathcal{P}_i = [a_i, b_i] \subseteq (0,1)$ with some fixed investment cost $k_i(P_i)$ and unit production cost $c_i(P_i)$ so that $p\prod_{i=1}^n P_i > \sum_{i=1}^n c_i(P_i)$. Let $\mathbf{P} = (P_i)_{i=1}^n$ and $\mathcal{P} = \times_{i=1}^n \mathcal{P}_i$. Suppose in this case demand is IGFR with finite mean on support $[a,b)$ and either $a = 0$ or $\lim_{x\to a^+} xf(x) \le \frac{p\prod_{i=1}^n P_i - \sum_{i=1}^n c_i(P_i)}{np\prod_{i=1}^n P_i}$ for all $\mathbf{P} \in \mathcal{P}$. Therefore, for any give $\mathbf{P}$, by lemma 2.20, there exists a unique nontrivial Nash equilibrium $\mathbf{w}^*$ with the corresponding q^* satisfy

$$1 - F(q) - nqf(q) = \frac{\sum_{i=1}^n c_i(P_i)}{p\prod_{i=1}^n P_i} \tag{2.9}$$

Let $q(\mathbf{P})$ be the unique solution for above equation, then the supplier i's profit (excluding the investment cost) is $p\prod_i P_i q(\mathbf{P})^2 f(q(\mathbf{P}))$. Now suppose each supplier i will take the unique non-trivial equilibrium in the next pricing competition stage as response and maximize his own profit including the investment cost by control P_i:

$$\max_{P_i\in\mathcal{P}_i} \pi_{si}(P_i, \mathbf{P}_{-i}) = p\prod_{i=1}^n P_i q(\mathbf{P})^2 f(q(\mathbf{P})) - k_i(P_i)$$

Theorem 2.13. *If $c_i(P_i)$ and $k_i(P_i)$ are continuous in P_i for all $i = 1,\dots,n$, there exists an optimal solution for the following potential function*

$$\max_{\mathbf{P}\in\mathcal{P}} \psi(\mathbf{P}) = p\prod_{i=1}^n P_i q(\mathbf{P})^2 f(q(\mathbf{P})) - \sum_{i=1}^n k_i(P_i)$$

Besides, the maximizers of this potential function are equilibrium points for the suppliers's noncooperative reliability investment game $\{\{1\dots,n\}, \mathbf{P}\in\mathcal{P}, \{\pi_{s_i} : i = 1\dots,n\}\}$.

Lemma 2.23. *If c_i is constant then π_{si} is increasing in P_j, $j \neq i$.*

2.6. Manufacturer's Optimal decision in assembly-sourcing system

Now we consider the production system such that the final product is composed of one of each component in the set $\mathcal{C}$. For each component $i \in \mathcal{C}$, there is a set of different suppliers $\mathcal{T}_i$ who can provide. The component i provided by supplier $j \in \mathcal{T}_i$ has wholesale price at w_{ij} with independent stochastic reliability R_{ij}. Then the manufacture should try to maximize his expected profit by selecting the ordering quantity from each supplier:

$$\begin{aligned}
\max_{q_{ij}\geq 0, i\in\mathcal{C}, j\in\mathcal{T}_i} \pi &= -\sum_{i\in\mathcal{C}}\sum_{j\in\mathcal{T}_i} w_{ij}q_{ij} + pE[D-(D-\min_{i\in\mathcal{C}}(\sum_{j\in\mathcal{T}_i} R_{ij}q_{ij}))^+] \\
&\quad +sE[(\min_{i\in\mathcal{C}}(\sum_{j\in\mathcal{T}_i} R_{ij}q_{ij})-D)^+ - uE[(D-\min_{i\in mathcalC}(\sum_{j\in\mathcal{T}_i} R_{ij}q_ij))^+] \\
&= (p-s)E[D] + sE[\min_{i\in\mathcal{C}}(\sum_{j\in\mathcal{T}_i} R_{ij}q_{ij})] - \sum_{i\in\mathcal{C}}\sum_{j\in\mathcal{T}_i} w_{ij}q_{ij} \\
&\quad -(p+u-s)E[(D-\min_{i\in\mathcal{C}}(\sum_{j\in\mathcal{T}_i} R_{ij}q_{ij}))^+]
\end{aligned}$$

Recall that $p+u>s$, then we have:

Theorem 2.14. *The manufacturer's expected profit function is jointly concave in the ordering quantities q_{ij}, $i \in \mathcal{C}, j \in \mathcal{T}_i$.*

Let $\mathbf{q}_i \in \mathcal{R}^{|\mathcal{T}_i|}$, $i \in \mathcal{C}$ be the ordering quantity vector for component i from the suppliers in set $\mathcal{T}_i$.

Corollary 2.5. *If $\mathbf{q}_i^* = 0$ for some $i \in \mathcal{C}$, then $\mathbf{q}_j^* = 0$ for all $j \in \mathcal{C}$.*

Let $\mathbf{R}$ be the vector of reliability of all the suppliers. For any ordering quantity $\mathbf{q}$, let:

$$\begin{aligned}
\mathcal{S}_i(\mathbf{q}) &= \{r_{jk}, j \in \mathcal{C}, k \in \mathcal{T}_j | \sum_{k\in\mathcal{T}_i} r_{ik}q_{ik} = \min_{j\in\mathcal{C}} \sum_{k\in\mathcal{T}_j} r_{jk}q_{jk}\} \\
E[\chi(\mathbf{R})]_{\mathcal{S}_i(\mathbf{q})} &= E[\chi(\mathbf{R})|\mathbf{R} \in \mathcal{S}_i(\mathbf{q})]Pr(\mathbf{R} \in \mathcal{S}_i(\mathbf{q})) \\
&= \int \dots \int_{\mathbf{r}\in\mathcal{S}_i(\mathbf{q})} \chi(\mathbf{r})dG(\mathbf{r}) \\
&\quad \text{here } G(\mathbf{r}) = \prod_{i\in\mathcal{C}} \prod_{j\in\mathcal{T}_i} g_{ij}(r_{ij})
\end{aligned}$$

In case $\mathbf{q} \neq 0$, let $\mathbf{q}_{-i}$ be the ordering quantity vector for components $\forall j \neq i$ from suppliers in $\{k|k \in \mathcal{T}_j\}$. Define $G_i(x|\mathbf{q}_i)$ ($g_i(x|\mathbf{q}_i)$) as the cumulative (density) distribution for random variable $\sum_{j\in\mathcal{T}_i} R_{ij}q_{ij}$ which is the effective component ordered. Also define $G_{-i}(x|\mathbf{q}_{-i})$ ($g_{-i}(x|\mathbf{q}_{-i})$) as the cumulative (density) distribution for random variable $\min_{k\neq i}(\sum_{j\in\mathcal{T}_k} R_{kj}q_{kj})$. Thus the density function $\varphi(x|\mathbf{q})$ for random variable $\min_{i\in\mathcal{C}}(\sum_{j\in\mathcal{T}_i} R_{ij}q_{ij})$ is:

$$\varphi(x|\mathbf{q}) = (1 - G_{-i}(x|\mathbf{q}_{-i}))g_i(x|\mathbf{q}_i) + (1 - G_i(x|\mathbf{q}_i))g_{-i}(x|\mathbf{q}_{-i})$$

In addition,

$$\begin{aligned}
E[\chi(\mathbf{r})]_{\mathcal{S}_i(\mathbf{q})} &= \int_0^1 \dots \int_0^1 \int_{\sum_{k\in\mathcal{T}_i} r_{ik}q_{ik}}^{+\infty} \chi(\mathbf{r})g_{-i}(x|\mathbf{q}_{-i})dx\big(\prod_{k\in\mathcal{T}_i} g_{ik}(r_{ik})dr_{ik}\big) \\
&= \int_0^1 \dots \int_0^1 \chi(\mathbf{r})(1 - G_{-i}(\sum_{k\in\mathcal{T}_i} r_{ik}q_{ik}|\mathbf{q}_{-i}))\big(\prod_{k\in\mathcal{T}_i} g_{ik}(r_{ik})dr_{ik}\big)
\end{aligned}$$

Lemma 2.24. *If the optimal ordering quantity* $\mathbf{q}^* \neq 0$ *then* $\forall i \in \mathcal{C}$ *and* $j \in \mathcal{T}_i$*,*

$$\begin{aligned}
&\textit{if } q^*_{ij} > 0 \;\Rightarrow\; w_{ij} = (p+u)E[R_{ij}]_{\mathcal{S}_i(\mathbf{q}^*)} - (p+u-s)E[R_{ij}F(\sum_{j\in\mathcal{T}_i} R_{ij}q^*_{ij})]_{\mathcal{S}_i(\mathbf{q}^*)} \\
&\textit{if } q^*_{ij} = 0 \;\Rightarrow\; w_{ij} \geq (p+u)E[R_{ij}]_{\mathcal{S}_i(\mathbf{q}^*)} - (p+u-s)E[R_{ij}F(\sum_{j\in\mathcal{T}_i} R_{ij}q^*_{ij})]_{\mathcal{S}_i(\mathbf{q}^*)}
\end{aligned}$$

Here $E[\chi(\mathbf{r})]_{\mathcal{S}_i(\mathbf{q})}$ can be evaluated through Monte Carlo sampling method, however the analytical solution is complicated for general cases.

Lemma 2.25. *Consider the following two scenarios: in scenario one the market demand is* D' *with optimal profit* π'^* *and in scenario two the market demand is* D'' *with optimal profit* π''^**, if* $E[D'] = E[D'']$ *and* $D' \leq_{disp} D''$*,* [4] *then* $\pi'^* \geq \pi''^*$*.*

For most distribution families (like Normal, Uniform), the above condition is satisfied if D' and D'' have the same mean but D'' has a larger variance.

Lemma 2.26. *Consider the following two scenarios: in scenario one the reliability vector is* R' *with optimal profit* π'^* *and in scenario two the reliability vector is* R'' *with optimal profit* π''^**, if* $E[R'_{ij}] \geq E[R''_{ij}]$ *and* $R'_{ij} \leq_{disp} R''_{ij}$*,* $\forall i \in \mathcal{C}, j \in \mathcal{T}_i$*, then* $\pi'^* \geq \pi''^*$*.*

For most distribution families (like Normal, Uniform), the above condition is satisfied if R'_{ij} has a larger mean but R''_{ij} has a larger variance.

Lemma 2.26 and 2.25 are illustrated by our numerical examples in assembly model (see figure 2.6 and 2.8). Comparing the two results, the one with lower disperse order (hence lower variation) and higher expected reliability will always lead to a higher profit. However this is not true for demand. This is because if the penalty for underage cost u is larger, higher expected demand may cause higher negative profit.

Lemma 2.27. *If there exits two suppliers* j *and* k *who provide the same component* i *such that* $w_{ij}/\bar{R}_{ij} > w_{ik}/\bar{R}_{ik}$ *with* $q^*_{ij} > 0$*, then* $q^*_{ik} > 0$*.*

Lemma 2.27 is an extension of lemma 2.3 which is is a special case when $|\mathcal{C}| = 1$. Thus for every component $i \in \mathcal{C}$, we order the suppliers $j \in \mathcal{T}_i$ in ascending order of $w_{ij}/\bar{R}_{ij}$. Then by the above lemma, we know that the manufacturer makes production if and only if he makes

[4] X and Y be two random variables with distribution function F and G respectively. Let F^{-1} an G^{-1} be the right continuous inverse of F and G respectively and assume that

$$F^{-1}(\beta) - F^{-1}(\alpha) \leq G^{-1}(\beta) - G^{-1}(\alpha) \qquad \text{whenever } 0 < \alpha \leq \beta < 1 \tag{2.10}$$

Then X is said to be smaller than Y in dispersive order denote $X \leq_{disp} Y$.

positive ordering quantity in an assembly system with only suppliers $(i, 1)$ for all $i \in \mathcal{C}$. Thus by lemma 2.18, we have :

Corollary 2.6. *Not ordering is an optimal decision for the manufacturer if and only if* $\sum_{i\in\mathcal{C}}^{n} w_{i1}\xi_i \geq (p+u)E[\min w_{i1}\xi_i]$, $\forall \xi_i \geq 0$.

Therefore we can extend our Iterative Expansion algorithm in sourcing model to the assembly-sourcing model here:

Algorithm 2.4 Extended Iterative Expansion Procedure

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1. (*Initialization*): For each component $i \in \mathcal{C}$, ascending order the suppliers $j \in \mathcal{T}_i$ in $w_{ij}/\bar{R}_{ij}$. Starting with $\mathcal{M} = \{(i,1) : \forall i \in \mathcal{C}\}$, let $\mathbf{q}_{\mathcal{M}}$ be the ordering quantities from suppliers $(i,j) \in \mathcal{M}$, starting with $q_{i1} = F^{(\frac{p+u-\sum_{i\in\mathcal{C}} w_{i1}/\bar{R}_{i1}}{p+u-s})}/\bar{R}_{i1}$ and $q_{ij} = 0$ for $j > 1$. Define the number of suppliers with positive ordering quantity for component i as m_i, hence we begin with $m_i = 1$, $\forall i \in \mathcal{C}$.

2. (*Optimality Test*): Evaluate $\mathbf{g} = \frac{\partial \pi}{\partial \mathbf{q}_{\mathcal{M}}}$, if $\sum_{(i,j)\in\mathcal{M}} |g_{ij}| > \epsilon$ for a predetermined positive tolerance level ϵ then continue to step 3. Otherwise check if $\mathcal{M} = \bigcup_{i\in\mathcal{C}} \mathcal{T}_i$ or $g_{i,m_i+1} = \frac{\partial \pi}{\partial q_{i,m_i+1}} \leq 0$, then stop and report $\mathbf{q}$ as optimal solution. Otherwise $m_i = m_i + 1$ for all i such that $g_{i,m_i+1} > 0$ and update $\mathcal{M} = \{(i, t_i) : \forall i \in \mathcal{C}, 1 \leq t_i \leq m_i\}$.

3. (*Computation of direction*): Compute $\mathbf{d}_{\mathcal{M}}$ by conjugate gradient method, see the detail in step 4 of algorithm 2.5. And $\mathbf{d}_{\mathcal{M}^c} = \mathbf{0}$

4. (*Computation of step size*): Let $\alpha = \min\{q_{ij}/(-d_{ij}) | (i,j) \in \mathcal{M} \text{ and } d_{ij} < 0\}$. If α exits then $\mathbf{q} = \mathbf{q} + \alpha\mathbf{d}$ otherwise $\mathbf{q} = \mathbf{q} + \mathbf{d}$ and go back to step 2.

Compared with the algorithm 2.1, we do not have the hessian information here, hence we apply a simple conjugate gradient method which only requires the information of the first order value and last step to improve direction. The detail in step 3 is the same as step 4 in algorithm 2.5. In the case of only a small portion of suppliers with positive optimal ordering quantity, we can anticipate a better performance by our Extended Iterative Expansion procedure as compared with the general active set method which is applied in our numerical examples.

In case that the reliability random variable R_{ij} for supplier $i \in \mathcal{C}$ and $j \in \mathcal{T}_i$ is Bernoulli distributed with probability P_{ij} to successfully deliver all the ordering quantity and $1 - P_{ij}$ to default. Let $\mathcal{T} = \bigcup_{i \in \mathcal{C}} \mathcal{T}_i$ be the set of all suppliers, then the manufacturer's expected profit can be rewritten as:

$$\begin{aligned}\pi &= (p-s)E[D] - \sum_{i \in \mathcal{C}} \sum_{j \in \mathcal{T}_i} w_{ij} q_{ij} \\ &\quad + \sum_{\mathcal{S} \subseteq \mathcal{T}} \Big(\prod_{(i,j) \in \mathcal{S}} P_{ij} \prod_{(i',j') \in \mathcal{S}^c} (1 - P_{i'j'}) sE[\min_{i \in \mathcal{C}} (\sum_{(i,j) \in \mathcal{S}} q_{ij})] - (p+u-s)E[(D - \min_{i \in \mathcal{C}} (\sum_{(i,j) \in \mathcal{S}} q_{ij}))^+] \Big)\end{aligned}$$

Lemma 2.28. *When reliability are Bernoulli random variables, for any optimal solution* $\mathbf{q}^*$*,* $\forall i \in \mathcal{C}$*,* $j \in \mathcal{T}_i$*, the following is true:*

$$q^*_{ij} \leq \sum_{s \in \mathcal{T}_k} q^*_{ks} \qquad \forall k \in \mathcal{C}, k \neq i$$

Hence the total ordering quantity for any component is always more than the ordering quantity from any single supplier. Note that for general reliability function, this is not true.

2.6.1 Solution Procedure and Numerical Examples

We already suggested an Extended Iterative Expansion algorithm for the assembly-sourcing problem. It performs better in the case that a small portion of suppliers are selected in the final positive ordering set. For general setting of the problems, we devise the following solution procedure based on active set method combined with conjugate gradient procedure:

In the algorithm, we start with a heuristic ordering quantity solution which takes the sum of the average effective wholesale price among suppliers for each component as production cost, calculates an target production quantity, and evenly allocates this quantity to every

Algorithm 2.5 Active Set Method Combined with Conjugate Gradient Procedure

1. (*Initialization*): Let n_i be the number of suppliers who provide component i and m be the number of components. Choose $q_{ij} = F^{-1}(\frac{p+u-\sum_{i\in\mathcal{C}}(\sum_{j\in\mathcal{T}_i} w_{ij})/n_i}{p+u-s})/(n_i E[R_{ij}])$ as a starting point, and let $k = 1$, and the nonactive set N, be set of indices at which the non-negative requirement for the ordering quantity vector $\mathbf{q}$ are non-binding, be $\{\forall(i,j) : i \in \mathcal{C}, j \in \mathcal{T}_i\}$

2. (*Optimality Test*): Evaluate $\mathbf{g}^k = \frac{\partial\pi}{\partial\mathbf{q}}$, if $\sum\limits_{(i,j)\in N} |g^k_{ij}| + \sum\limits_{i\in\mathcal{C},j\in\mathcal{T}_i,(i,j)\in N^c} \max\{0, g^k_{ij}\} < \epsilon$ for a predetermined positive tolerance level ϵ, then stop and report $\mathbf{q}$ as optimal solution. Otherwise continue to Set 3.

3. (*Expansion of Nonactive set*): If $\sum_{(i,j)\in N} g^k_{ij} < \epsilon'$ for another predetermined positive tolerance level ϵ', then expansion N by letting $N = N \cup \{i, j | g^k_{ij} > 0\}$; otherwise do nothing. Continue to Step 4.

4. (*Computation of of direction*): Let $\mathbf{g}^k_N = \{g^k_{ij} : (i,j) \in N\}$ and $\mathbf{d}^k_N = \mathbf{g}^k_N + \alpha_k \mathbf{id}^{k-1}_N$, where $\alpha_1 = 1$ and $\mathbf{id}^0 = \mathbf{g}^1_N$. For $k > 1$, $\alpha_k = \min\{\frac{(\|\mathbf{g}^k_N\|)^2}{(\|\mathbf{g}^{k-1}_N\|)^2}, \frac{(\|\mathbf{g}^{k-1}_N\|)^2}{(\|\mathbf{g}^k_N\|)^2}\}$ and $\mathbf{id}^{k-1}_N = \{id^{k-1}_{ij} : (i,j) \in N\}$. Let $\mathbf{d}^k_{(i,j)\in N^c} = \mathbf{0}$ and go to Step 5

5. (*Shrinkage of nonactive set*): Let $\beta = \min\{q_{ij}/(-d_{ij}) | \forall(i,j) : d_{ij} < 0\}$, if $\beta \leq 1$, $\mathbf{q} = \mathbf{q} + \beta\mathbf{d}^k$, $\mathbf{id}^k = \beta\mathbf{d}$ and $N = \{(i,j) : q_{ij} > 0\}$; otherwise let $\mathbf{q} = \mathbf{q} + \mathbf{d}^k$ and $\mathbf{id}^k = \mathbf{d}^k$ Increase k by 1 goto Step 2.

supplier for each component component. Then in each step, we identify the set of suppliers with their ordering quantity $q_{ij} > 0$ as N. Then check the **KKT** condition for optimality in step 2. In step 3, when the current solution quantity among suppliers in the nonactive set is close to optimal, we check if we can improve the solution by enlarging the nonactive set. In step 4, we calculate the conjugate gradient improving direction among suppliers in N. Note that in the non-constrained conjugate gradient method, $\alpha_k = \frac{(\|\mathbf{g}^k_N\|)^2}{(\|\mathbf{g}^{k-1}_N\|)^2}$, and the direction $\mathbf{d}^k$ is calculated for line search along $\mathbf{q} + \lambda\mathbf{d}^K$. In our constrained case, we do a unit step search in step 5 in stead of line search. To make sure that the procedure will converge to optimal solution we let $\alpha_k = \min\{\frac{(\|\mathbf{g}^k_N\|)^2}{(\|\mathbf{g}^{k-1}_N\|)^2}, \frac{(\|\mathbf{g}^{k-1}_N\|)^2}{(\|\mathbf{g}^k_N\|)^2}\}$. Then in step 5, we increase the ordering quantity along $\mathbf{d}$ by either one unit or until some supplier in nonactive set get 0 order. When the latter occurs, we update the nonactive set. To evaluate $\frac{\partial\pi}{\partial\mathbf{q}}$ needed in the algorithm, we use the Monte Carlo sampling method. We draw M samples independently from the joint distribution of the reliability, and denote sample m by $\{r_{ij}^m | i \in \mathcal{C}, j \in \mathcal{T}_i\}$. Then we use the following approximations:

$$\frac{\partial \pi}{\partial q_{ij}} = -w_{ij} + \sum_{m=1}^{M} [(p+u)r_{ij}^m - (p+u-s)r_{ij}^m F(\sum_{j \in \mathcal{T}_i} r_{ij} q_{ij})] 1(\sum_{j \in \mathcal{T}_j} r_{ij}^m q_{ij} = min_{k \in \mathcal{C}} \sum_{j \in \mathcal{T}_j} r_{ij}^m q_{ij})$$

Here the function 1() returns 1 if the inside condition is satisfied, otherwise it returns 0.

2.6.1.1 Computation Results

In our experiments, we let $p = 60$, $u = 5$ and $s = 2$. We let the overall tolerance level ϵ be 0.01. We also keep the tolerance for nonactive set ϵ' be 0.0001. A sample size of 200000 is selected for the Monte Carlo sampling. Our algorithm is programmed in Matlab and run on a PC with an AMD 64 X2 processor.

2.6.1.1.1 Computation Effort We test our algorithm on problem sets with n component and n suppliers for each component, $n = 2, \ldots, 6$. The demand is uniformly distributed on $[100, 200]$ and the reliability R_{ij} are independent Beta distributed with α_{ij} and β_{ij} ranging from 1/2 to 10. The time consumption and the convergence rate are shown in table 2.6.1.1.2. From the table, we can see that the computation time is doubled if one more component and supplier for each component are added. Hence to solve a problem with 10 components and 10 suppliers each component (which is close to the industrial problem size) will consume about two hours.

Table 2.4: Computation Time and Convergence Iterations v.s. Problem Scale

Problem size: n x n	2	3	4	5	6
Computation Time:(S):	54	125	263	761	1580
Convergence Iterations:(k)	192	171	191	310	423

To make sure that the error caused by the sample size is small enough to do the sensitivity analysis, we evaluate the relative error in our algorithm for different sample sizes for the problem with three components and three suppliers each:

Table 2.5: Relative Error v.s. Sample Szie

Sample size: m	5×10^4	10^5	2×10^5	4×10^5	8×10^5
Relative error in Optimal Profit:	0.000819	0.000159	0.0000954	0.000143	0
Relative error in Optimal Ordering quantity	0.0113	0.00578	0.00702	0.00584	0

Here the relative error in optimal profit is defined by $|\pi_m - \pi_{8\times10^5}|/\pi_{8\times10^5}$ and the relative error in optimal ordering quantity is defined by $\|\mathbf{q}_m - \mathbf{q}_{8\times10^5}\|/\|\mathbf{q}_{8\times10^5}\|$. From the table we can find sample size of 2×10^5 is enough for us to do our sensitivity analysis numerically.

2.6.1.1.2 Sensitivity Analysis We do the sensitivity analysis of ordering quantity and optimal profit based on our basic problem instance with three components and three suppliers for each component:

Table 2.6: Basic Problem Set

Parameters	α_{ij}	β_{ij}	$E[R_{ij}]$	$Var[R_{ij}]$	w_{ij}	$w_{ij}/E[R_{ij}]$
(1,1):	1	2	0.333	0.0556	3.06	9.18
(1,2):	1.5	1.5	0.5	0.0625	3.69	7.38
(1,3):	1	1	0.5	0.0833	4.816	6.88
(2,1):	1.5	1.5	0.5	0.0625	3.93	7.86
(2,2):	2	4	0.333	0.0317	1.74	5.22
(2,3):	2.5	0.5	0.8333	0.0347	2.66	3.192
(3,1):	1	5	0.1667	0.0198	1.5	9.00
(3,2):	2.5	4	0.3845	0.0316	3.23	8.39
(3,3):	3.5	4.5	0.4375	0.0273	2.07	4.7314

In our table for component i, we order the suppliers in descending sequence of effective wholesale price: $w_{ij}/E[R_{ij}]$. In all of our numerical study instances, we find that for some component i, a supplier with a higher effective wholesale price gets a higher optimal ordering quantity. This result is stronger than our lemma 2.28, but it is not always true for the general case.

In our scenario group one, we test the effect of w_{13} on the system performance. As w_{13}

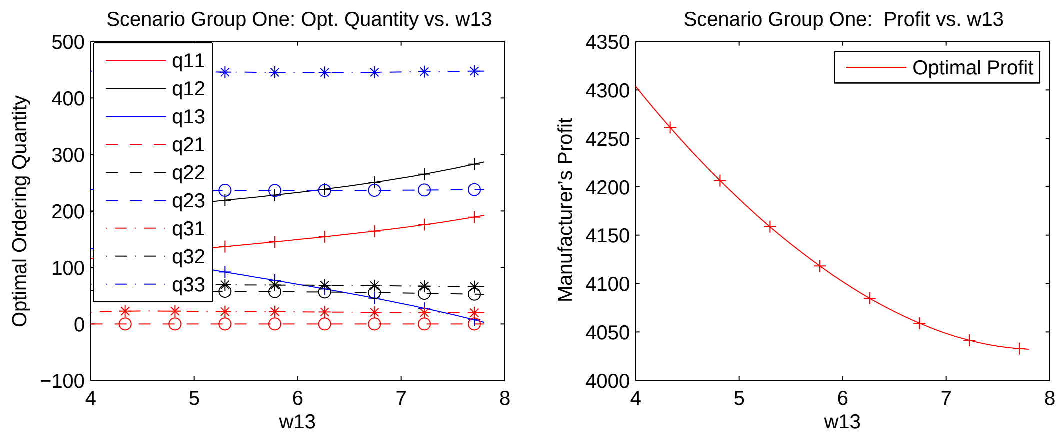

Figure 2.12: The system performance under w_{13}

increases from 4 to 8, the manufacturer's optimal profit is decreasing. The optimal ordering quantity from supplier $(1, 3)$ is also decreasing to 0. While by the substitutable property, the ordering quantity from the other two suppliers for this component increases. It seems that with more than one possible source for each component, the complementary effect shown in section 2.4 becomes weaker. And the corresponding decrease in the optimal ordering quantity from the suppliers out of the group is no more than 15%. We also find some suppliers may even receives lower ordering quantity with increasing of w_{13}. However this reduction is around 0.1% and neglectable hence we categorize it as computation error. One interesting phenomena here is that the suppliers with a higher effective wholesale price are more sensitive (in sense of both absolute and relative values) to the the changes.

In our scenario group two, we test the effect of $Var[R_{13}]$ on the system performance. Note that R_{13} is beta distributed and that during the test we keep the mean unchanged and maintain the skewness to be 0. With the increase of $Var[R_{13}]$ from 0.01 to 0.08, the manufacturer's optimal profit reduces. This observation coincides with our lemma 2.26. The increase of the reliability variance also causes a distinct effect on the reallocation of the optimal ordering quantity from supplier 3 to the other two suppliers for component 1.

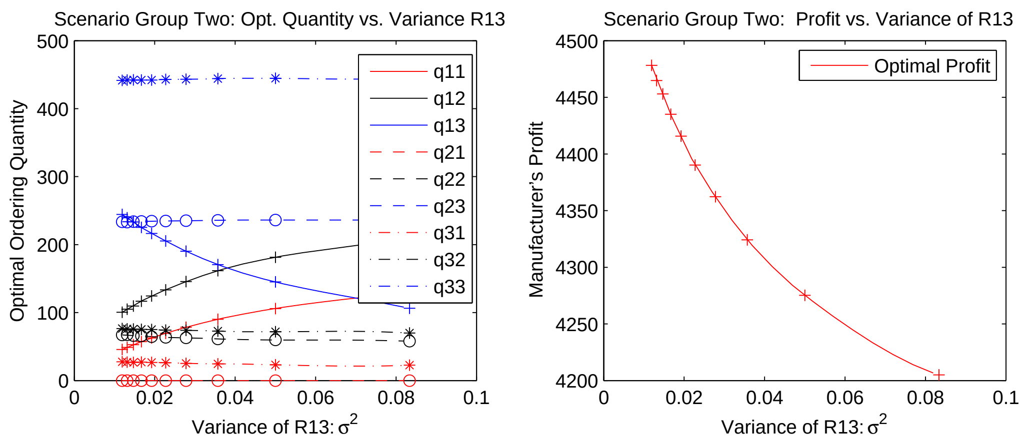

Figure 2.13: The system performance under $Var[R_{13}]$ when demand is uniform in [100,200]

Again, the increasing of reliability variance has a very slight impact (no more than 15%) on the suppliers for the other components. However we still can see a generally decreasing effect in the optimal ordering quantity for them. Some suppliers may receive a negligible increase in orders. Similar as the phenomena in scenario group one, suppliers out of the group 1 with higher effective wholesale price is more sensitive to the changes.

Compared with scenario group two, in scenario group three we test the effect of $Var[R_{13]}$ on the system performance under a demand with density function uniformly distributed on $[145, 155]$ instead of uniform on $[100, 200]$. The figure 2.14 shows a trend of change similar to scenario group two. While from figure 2.15, we can see that the optimal profit is more sensitive to reliability variance under smaller demand variance. This is because that for every ineffective component, lower demand variance has more effect on the lost sale.

Finally we increase the $E[R_{13}]$ in our scenario group four. Since we maintain the same mean for R_{13} which is beta distributed, thus with the increasing of expectation, R_{13} becomes more left-skewness. As the numerical results in the assembly system show, the reduction of skewness alone also causes the increasing of the optimal profit. Therefore the corresponding increase in the optimal profit is the combined effect of the increase in the mean and decrease

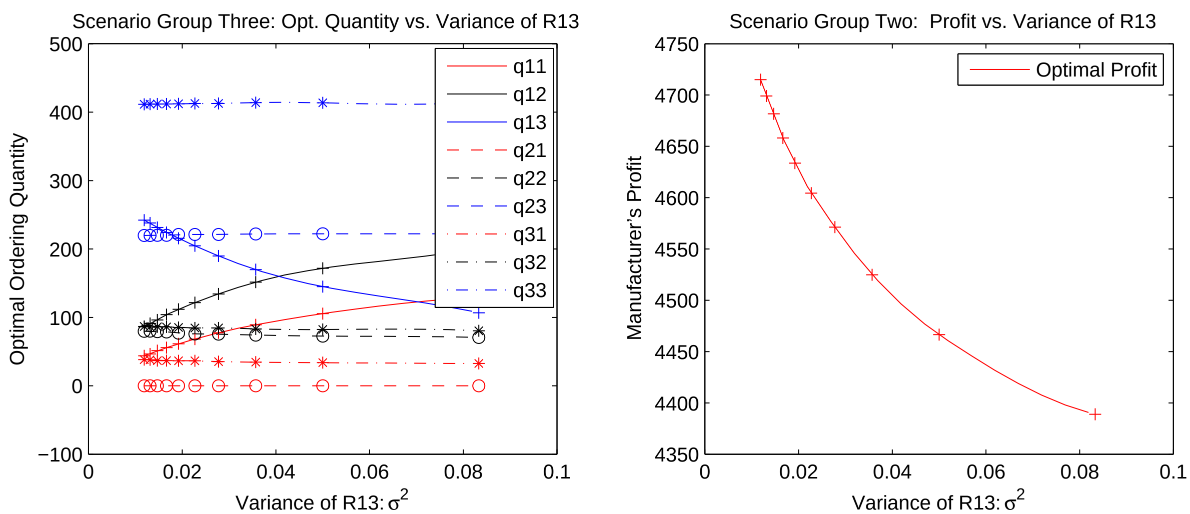

Figure 2.14: The system performance under $Var[R_{13}]$ when demand is uniform in [145,155]

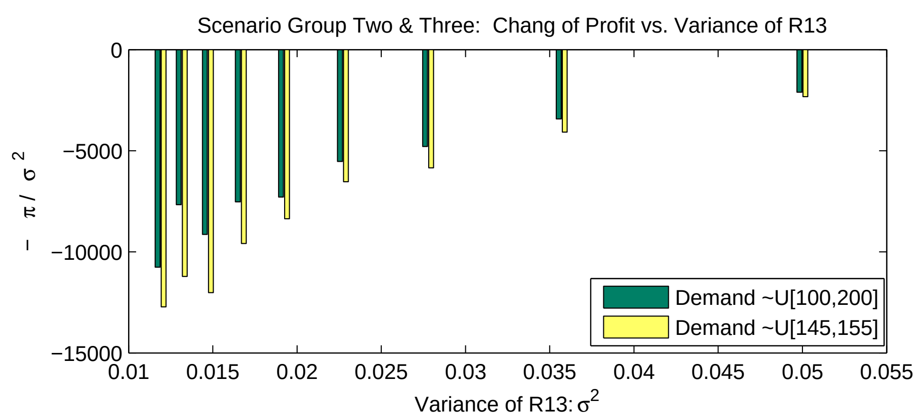

Figure 2.15: The change ratio in optimal profit over reliability variance affected by demand variance

in the skewness. With the increasing of the $E[R_{13}]$, the manufacturer shifts his orders from supplier 1 and 2 to supplier 3. Similar to the previous scenarios, the change in expectation of supplier 3 for component 1 has a slight impact on the suppliers for the other components, especially for those with lower effective wholesale prices. For the suppliers in group 3 and group 2 with higher effective wholesale prices, a $10\% - 20\%$ of increasing in optimal ordering

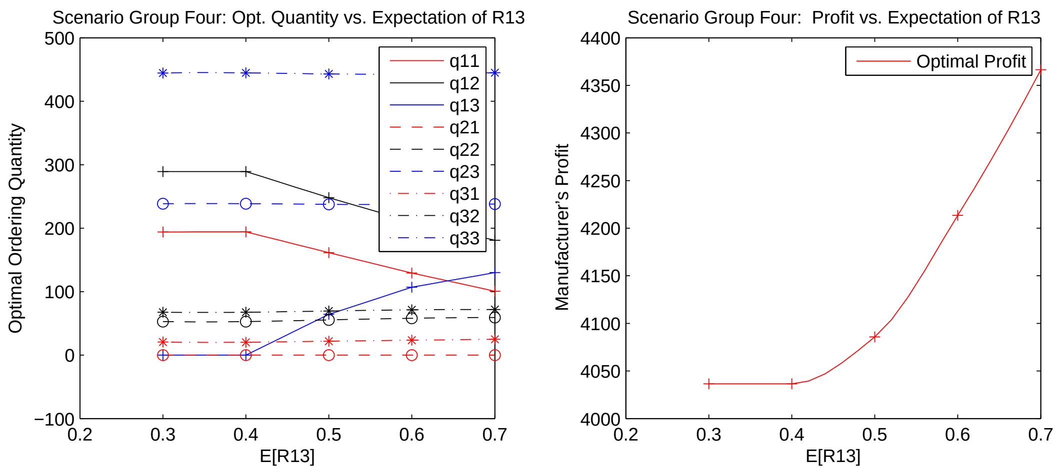

Figure 2.16: The system performance under $E[R_{13}]$

quantity can be observed.

To conclude, in the assembly-sourcing system, the properties that we get in the assembly or the sourcing system seem to be extendable in our case. For example, the optimal profit increasing with lower wholesale price, higher expected reliability and lower variance. However, the change of the parameters of some suppliers have a strong impact on the optimal ordering quantities within the group. In general, the unfavorable changes (higher price, higher reliability variance and lower reliability expectation) cause the transformation of ordering quantity to the other suppliers in the group along with lowering optimal profit. The suppliers for different component will be affected slightly. This is especially true for suppliers with lower effective wholesale price. However we can find a complementary effect from those impacts as we saw in assembly model: the unfavorable changes for one component supplier causes a decrease ordering quantity from the suppliers for the other components.

2.7. Summary

In this chapter, we examine a two echelon supply chain system in which the manufacture's optimal decision is affected by the upper stream of component suppliers' stochastic reliability and the lower stream of stochastic demand. The traditional news vendor model is the extreme case when all reliability is deterministic. There are several variations of the system structure in the model: the sourcing system, when multiple suppliers provide the same component, the assembly system, when multiple complementary components are provided by single supplier each and the assembly-sourcing model, when multiple complementary components are provided by a group of suppliers. In general for all those systems, the manufacturer's expected profit is jointly concave in his ordering quantity from each supplier. Also his optimal profit will be increasing with the increasing of the variation of the demand and the reliability. While for different system, based on the relationship between the suppliers, the optimal solution shows different properties.

In the sourcing system, when the number of suppliers is restricted to two, an increase of the wholesale price by one supplier will lead to an increase of the optimal ordering quantity for the other supplier and a decrease in the optimal ordering quantity for himself. Also an increase in the demand always increases the optimal ordering quantity when the demand is log-concave. However those effects will not always hold for multiple suppliers. We also find that a supplier with lower wholesale price and higher reverse hazard rate order will always get more.

The corresponding suppliers' pricing game analysis for the sourcing system is complicated even for Bernoulli reliability. The supplier's profit function is bimodal in his own wholesale price and hence the best response function may be not a convex set or continuous. We identify the case when there is no Nash equilibrium. However if the distribution of demand is restricted with some parameter setting constraints, equilibrium can be guaranteed by the monotonicity of the best response. In those cases, the manufacturer will prefer the competition of suppliers compared with suppliers coordination because the former case leads to a lower equilibrium wholesale price.

In the assembly system, since the relationship between suppliers is complementary, the manufacturer's expected profit is supermodular in the ordering quantities. Thus the optimal ordering quantity always decreases with the wholesale price. Based on the supermodularity, the Round-Robin scheme is an effective algorithm to find optimal solution for a small number of suppliers if the multi-dimension integration can be calculated efficiently. While in case the number of suppliers is larger, we suggest a target production quantity heuristic based on the intuition that the actual production quantity is relative stable for the same mean of reliability under different variation. This happens when there are a lot of suppliers and the reliability density assumes an unimodal shape.

When reliability are Bernoulli random variables, every supplier will get the same ordering quantity in the assembly system. In this case, the suppliers' pricing game degenerates to a deterministic assembly model where the ordering quantity can be simply solved by the news vendor model. In this case, Nash equilibrium always exists. We concentrate ourselves on the non-trivial equilibrium where not every one get zero profit. Under the mild restriction of demand distribution, we get the uniqueness of non-trivial equilibrium. Then we further analyze the division of profit and supply chain efficiency. We find the suppliers will get more than the manufacturer in case of a convex-concave distribution which is true for several common distribution families. Also if the coefficient of demand variation is small enough, the competition of suppliers leads to an efficient system with system profit close to the centralized system. In case that suppliers can make investments on the reliability by themselves at the very beginning stage to maximize their own individual profit, there is also an equilibrium.

Finally we extend our results to an assembly-sourcing system, and devise an Extended Iterative Expansion solution procedure based on the property in the optimal solution that the positive optimal ordering quantity on the supplier with higher effective wholesale price ensures a positive ordering quantity on the suppliers with lower effective wholesale price in the same group. Our numerical experiment shows the assembly-sourcing system inherits most in-group effects inherited from sourcing system and out-group effects from assembly system. However, the change of suppliers for some component has a prominent impact on the suppliers within this group and a slight impact on the suppliers for the other components. Moreover, we also find that suppliers with higher effective wholesale price out of the group are more sensitive to the changes.

Chapter 3

Impact of Investment Decision Sequence in Assembly Manufacture System

3.1. The Notation and the Model

Consider a two echelon supply chain with a single manufacturer who sells the final product that consists of n different components, each provided by a single component producer in a single selling season. Demand $d(p, \mathbf{e})$ is a function of the market price p set by the manufacturer and effort vector $\mathbf{e} = [e_0, e_1, \ldots, e_n]$ by the manufacturer (e_0) and component supplier i (e_i), $i = 1, \ldots, n$. We assume $\lim_{p \to \infty} d(p, \mathbf{e})p = 0$. Now consider the following two stage decisions for them:

Stage one: Each party selects an effort on the product, this effort will cause a fixed investment cost $k_i(e_i)$ and a unit production cost $c_i(e_i)$ for each player to produce one unit of the final product/componet.

Stage two: After the efforts are observed by all the players, the component producers simultaneously select the wholesale price w_i, $i = 1. \dots, n$ for their components so as to maximize their individual profit. Based on the wholesale price of the components and production cost, the manufacturer select the market price p so as to maximize his own profit.

Keeping price competition in the second stage, we consider three different scenarios for the first stage decision based on the order of the decisions and the objective of the decision makers:

Case 1: Manufacturer and component producers make their effort investment decision simultaneously to maximize their own individual profit.

Case 2: Any one of the firms announce his effort decision first and the other firms make their effort investment decision simultaneously.

Case 3: The players decide on their joint effort strategy together to maximize the total system profit.

3.2. Second Stage Price Decision

For the second stage decision, after all the efforts $\mathbf{e}$ are realized, the manufacturer maximizes his profit by controlling the market price given $\mathbf{w}$ is known:

$$\max_{p \geq \sum_{i=1}^{n} w_i + c_0(e_0)} \pi_0(p) = d(p, \mathbf{e})(p - \sum_{i=1}^{n} w_i - c_0(e_0)) - k_0(e_0)$$

Assumption 3.1. *The component providers select their wholesale price in a compact set:* $\mathbf{w} \in \prod_{i=1}^{n}[\underline{w}_i, \bar{w}_i] = \mathcal{W}$ *such that*

$$\underline{w}_i - c_i(e_i) \geq 0 \qquad i = 1, .., n$$

and under manufacturer's optimal price decision $p^*(\mathbf{w})$*, demand is a nonnegative continuous function of* $\sum w_i$*:*

$$d(p*_{(\mathbf{w})}, \mathbf{e}) = d^*(\sum w_i, \mathbf{e}) \geq 0$$

Table 3.1: Summary of Notation

$n =$ Number of component suppliers.
$N = \{0, 1, \ldots, n\}$ Set of players including the manufacturer indexed by 0 and component supplier $i = 1, \ldots, n$.
$p =$ Per unit market price for final product set by manufacturer.
$w_i =$ Per unit wholesale price for component i set by component supplier i, $i = 1, \ldots, n$.
$\mathbf{w} =$ Wholesale price vector. $e_i =$ Effort invested by firm i, $i = 0, \ldots, n$, while e_0 is the effort invested by manufacturer.
$\mathbf{e} =$ Effort vector.
$c_i(e_i) =$ Per unit production cost for firm i, $i = 0, \ldots, n$, while $c_0(e_0)$ is the per unit production cost for manufacturer.
$k_i(e_i) =$ Fixed effort investment cost for firm i, $i = 0, \ldots, n$, while $k_0(e_0)$ is the fixed effort investment cost for manufacturer .
$\pi_i =$ Profit for firm i, $i = 0, \ldots, n$, while π_0 is the profit for manufacturer.
$x_i =$ Effective effort invested by firm i, $i = 0, \ldots, n$
$\widehat{k}_i x_i =$ Fixed investment cost for firm i as a function of effective effort, $i = 0, \ldots, n$.
$\widehat{\pi}_i =$ Profit for firm i as a function of effective effort.

Assumption 3.1 is true for most commonly considered demand functions like

$$\begin{aligned}
\textbf{Linear}: &\quad d(p) = a(\mathbf{e}) - p \\
\textbf{Truncate Linear}: &\quad d(p) = [a(\mathbf{e}) - p]^+ \\
\textbf{CES}: &\quad d(p) = a(\mathbf{e})p^{-b} \qquad b > 1 \\
\textbf{Exponential}: &\quad d(p) = a(\mathbf{e})e^{-\lambda p}
\end{aligned}$$

Here $a(\mathbf{e})$ is the market size after the efforts are realized. Then the component producer i's profit function given his effort investment and full information about manufacturer's best market price response $d(p^*_{(\sum w_i)}, \mathbf{e})$ is:

$$\pi_i(\mathbf{w}) = d(p^*_{(\sum w_i)}, \mathbf{e})(w_i - c_i(e_i)) - k_i(e_i)$$

And define the gross profit for each player i by:

$$\psi_i(\mathbf{w}) = d(p^*_{(\sum w_i)}, \mathbf{e})(w_i - c_i(e_i)) \qquad \forall i = 1, \ldots, n$$
$$\psi_0(\mathbf{w}) = d(p^*_{(\sum w_i)}, \mathbf{e})(p^*_{(\sum w_i)} - \sum w_i - c_0(e_0))$$

Lemma 3.1. *Under assumption 3.1, a Nash Equilibrium exists in the component producers' pricing game.*

Following lemma shows that every component supplier get the same gross profit.

Lemma 3.2. *Under assumption 3.1:*

a) At any Nash Equilibrium point, component producers whose equilibrium strategies are interior points, have the same gross profit.

b) If $d^*(\sum w_i, \mathbf{e})$ is twice differentiable in $\sum w_i$, and both the Equilibrium point $\mathbf{w}^*$ and manufacturer's best response $p^*(\mathbf{w}^*)$ are interior points, then in addition to the result stated in part a),

$$\psi_0(p^*(\mathbf{w}^*)) = \xi(\mathbf{w}^*)\psi_i(\mathbf{w}^*) \qquad \forall i = 1, \ldots, n$$
$$\textit{where: } \xi(\mathbf{w}^*) = \frac{\partial p^*(\sum w_i)}{\partial \sum w_i}|_{\mathbf{w}=\mathbf{w}^*}$$

Part a of lemma 3.1 follows from the fact that at interior equilibrium, the first order condition should be satisfied, hence every supplier has the same marginal unit profit. Because in the assembly system every supplier get the same ordering quantity, they should have the same gross profit. This result is true when there is no alternative supplier for the same components (no competition in homogenous component). Also we do not consider the suppliers reliability here. By the result of chapter 2, if the effective delivered portion of the ordering quantity received by each supplier is not deterministic, the manufacturer may order differently from each supplier, which means that even every one has the same marginal profit, they may get different gross profits. Note that in the real life system, system situation is complicated. Homogenous competition, non-perfect information and reliability are all decision factors which may lead to different gross profits for each supplier. However we capture the key structure of the assembly model, and the results do hold in a lot of situations.

Assumption 3.2. *Demand is Linear, Truncate linear or Exponential with* $p \in \Re$ *and* $\mathbf{w} \in \Re^n$. *For linear demand , we assume that* $a - \sum_{i=0}^{n} c_0(e_0) \geq 0$.

Lemma 3.3. *Under assumption 3.2, the net profit under Nash Equilibrium is unique. For linear demand function:*

$$\begin{aligned} \pi_i^*(\mathbf{e}) &= \frac{(a - \sum_{i=0}^{n} c_i(e_i))^2}{2(n+1)^2} - k_i(e_i) \qquad \forall i = 1, \ldots, n \\ \pi_0^*(\mathbf{e}) &= \frac{(a - \sum_{i=0}^{n} c_i(e_i))^2}{4(n+1)^2} - k_0(e_0) \end{aligned}$$

for exponential demand function:

$$\pi_i^*(\mathbf{e}) = \frac{a}{\lambda} \exp(-2 - \lambda \sum_{i=0}^{n} c_i(e_i)) - k_i(e_i) \qquad \forall i = 0, \ldots, n \qquad \forall i = 1, \ldots, n$$

for truncate linear demand function:

$$\begin{aligned} \pi_i^*(\mathbf{e}) &= \frac{(a - \sum_{i=0}^{n} c_i(e_i))^{+2}}{2(n+1)^2} - k_i(e_i) \qquad \forall i = 1, \ldots, n \\ \pi_0^*(\mathbf{e}) &= \frac{(a - \sum_{i=0}^{n} c_i(e_i))^{+2}}{4(n+1)^2} - k_0(e_0) \end{aligned}$$

The above lemma suggests that in the linear and truncate linear demand function case, the manufacture gets half of each component supplier' gross profit, while in an exponential demand case, manufacture get the same gross profit as each component supplier.

Although for the second stage decision, we assume simultaneous price competition between suppliers followed by the manufacturer's price decision in the following of the chapter, here we give a brief comparison with the system coordination in the second stage. If the system is coordinated, the system profit is found by maximizing the objective in 3.1 with

$w_i = c_i(e_i)$, $\forall i = 1 \dots n$. The optimal system gross profit is $\frac{(a-\sum_{i=0}^{n} c_i(e_i))^2}{2}$ for linear demand, $\frac{(a-\sum_{i=0}^{n} c_i(e_i))^{+2}}{2}$ for truncate linear demand and $\frac{a}{\lambda}\exp(-1-\lambda\sum_{i=0}^{n} c_i(e_i))$ for exponential demand. Hence by competition, the system gross profit is less than $1/(n+1)$ of the coordination system in linear and truncated linear demand, while it is e^{-1} in the coordination system for exponential demand.

3.3. First Stage Effort Decision

3.3.1 First stage simultaneous effort decision

Assumption 3.3. *Given any first stage manufacturer and component producers' investment effort* $\mathbf{e}$ *chosen from a compact space* $\mathcal{E}$*, the component producers' pricing game results in a unique common gross profit* $\psi_i(\mathbf{e}) = \psi(\mathbf{e})$*,* $\forall i = 1 \dots n$*. Also, the manufacturer's gross profit* $\psi_0(\mathbf{e}) = \xi\psi(\mathbf{e})$ *where* ξ *is a constant for any given* $\mathbf{e}$*. Furthermore,* $\psi(\mathbf{e})$ *is a continuous function for* $\mathbf{e} \in \mathcal{E}$*,*

Clearly, assumption 3.3 is satisfied if assumption 3.2 is valid.

Lemma 3.4. *Under assumption 3.3, for any* $\mathcal{I} \subseteq \{0, \dots, n\}$ *with* e_i *given for* $i \in \mathcal{I}^c$*, and the strategy space* $\mathcal{E}_{\mathcal{I}}$ *for* $i \in \mathcal{I}$ *is compact, the simultaneous investment effort game among players in* $\mathcal{I}$ *has a Nash Equilibrium.*

Although we know the equilibrium exists from lemma 3.4, we do not have any further information about the system behavior under the equilibrium. To gain some insight into the property of the equilibrium point, for the remainder of this chapter, we assume $p \in \Re^+$ and $w_i \in \Re^+$ in the seconde stage price decision and $\mathbf{e} \in \prod_{i=0}^{n}[\underline{e}_i, \bar{e}_i] = \mathcal{E}$ for $i \in \{0, \dots, n\}$ in the first stage effort investment decision. Furthermore, we consider the special case of the market size $a(\mathbf{e})$ as a function of investment effort. For linear demand function or truncate linear demand function, we consider additive effect of the investment effort on the final demand

$$\text{Linear Demand :} \qquad d(p,\mathbf{e}) = a(\mathbf{e}) - p = \alpha + \sum_{i=0}^{n} \theta_i(e_i) - p$$

or

$$\text{Truncate Linear Demand :} \qquad d(p,\mathbf{e}) = [a(\mathbf{e}) - p]^+ = [\alpha + \sum_{i=0}^{n} \theta_i(e_i) - p]^+$$

where $a + \sum_{i=0}^{n} \theta_i(e_i) - \sum_{i=0}^{n} c_i(e_i) \geq 0$ for $\forall \mathbf{e} \in \mathcal{E}$.

For exponential demand function, we consider the case that each player has an multiplicative effect of the investment effort on the final demand:

$$d(p,\mathbf{e}) = a(\mathbf{e})e^{-\lambda p} = \alpha \prod_{i=0}^{n} \theta_i(e_i) e^{-\lambda p}$$

where $\theta_i(e_i) \geq 0$ for $\forall \mathbf{e} \in \mathcal{E}$ and $\forall i \in \{0, \ldots, n\}$.

To further analyze the property of effort equilibrium behavior, we first transform the strategy space:

For Linear demand function (Truncate linear demand function), $\forall i \in \{0, \ldots, n\}$, let:

$$\begin{aligned}
x_i &= \theta_i(e_i) - c_i(e_i) \\
\widehat{k}_i(x_i) &= \min_{e_i \in [\underline{e}_i, \bar{e}_i]} \{k_i(e_i) : \qquad \theta_i(e_i) - c_i(e_i) = x_i\} \\
\underline{x}_i &= \min_{e_i \in [\underline{e}_i, \bar{e}_i]} \theta_i(e_i) - c_i(e_i) \\
\bar{x}_i &= \max_{e_i \in [\underline{e}_i, \bar{e}_i]} \theta_i(e_i) - c_i(e_i)
\end{aligned}$$

For Exponential demand function, $\forall i \in \{0, \ldots, n\}$, let:

$$\begin{aligned}
x_i &= \theta_i(e_i) \exp(-\lambda c_i(e_i)) \\
\widehat{k}_i(x_i) &= \min_{e_i \in [\underline{e}_i, \bar{e}_i]} \{k_i(e_i) : \qquad \theta_i(e_i) \exp(-\lambda c_i(e_i)) = x_i\} \\
\underline{x}_i &= \min_{e_i \in [\underline{e}_i, \bar{e}_i]} \theta_i(e_i) \exp(-\lambda c_i(e_i)) \\
\bar{x}_i &= \max_{e_i \in [\underline{e}_i, \bar{e}_i]} \theta_i(e_i) \exp(-\lambda c_i(e_i))
\end{aligned}$$

The above $\underline{x}_i$ and $\bar{x}_i$ are well defined since it is a min (max) of a continuous function in a compact interval. Also $\widehat{k}_i(x_i)$ is well defined, since the left side of the equality in the constraint is continuous, then the inverse set of the point x_i in the range must be closed in a bounded set and non-empty, so the minimizer of the continuous function in a compact interval exits.

After the transformation, we actually delete all the points in $\mathcal{E}$ with same gross profit but higher fixed investment cost for each of the player. Then for each player i we consider the net profit function $\pi_i(\mathbf{x})$ in new strategy space $\mathbf{x} \in \prod_{i=0}^{n}[\underline{x}_i, \bar{x}_i] = \mathcal{X}$.
For Linear demand function:

$$\begin{aligned}\widehat{\pi}_i(\mathbf{x}) &= \frac{(a+\sum x_i)^2}{2(n+1)^2} - \widehat{k}_i(x_i) \qquad \forall i \in \{1,\ldots,n\} \\ \widehat{\pi}_0(\mathbf{x}) &= \frac{(a+\sum x_i)^2}{4(n+1)^2} - k_0'(x_0)\end{aligned}$$

For Truncate linear demand function:

$$\begin{aligned}\widehat{\pi}_i(\mathbf{x}) &= \frac{(a+\sum x_i)^{+2}}{2(n+1)^2} - \widehat{k}_i(x_i) \qquad \forall i \in \{1,\ldots,n\} \\ \widehat{\pi}_0(\mathbf{x}) &= \frac{(a+\sum x_i)^{+2}}{4(n+1)^2} - k_0'(x_0)\end{aligned}$$

For Exponential demand function:

$$\widehat{\pi}_i(\mathbf{x}) = \frac{\alpha e^{-2}}{\lambda}\prod_{i=0}^{n} x_i - \widehat{k}_i(x_i) \qquad \forall i \in \{0,\ldots,n\}$$

Let $\widehat{\psi}_i(\mathbf{x}) = \widehat{\pi}_i(\mathbf{x}) - \widehat{k}_i(x_i)$ for all $i = 1,\ldots,n$.

Notation: $\forall \mathcal{I} \subseteq \{0,\ldots,n\}$, let $\mathcal{X}_{\mathcal{I}} = \prod_{i\in\mathcal{I}}[\underline{x}_i, \bar{x}_i]$.

Proposition 3.1. *Let $\mathcal{E}^*$ be the set of all equilibrium points for the noncooperative game $(\mathcal{N}, \mathcal{E}, \{\pi_i : i \in \mathcal{N}\})$ and $\mathcal{X}^*$ be the set of all equilibrium points for the noncooperative game after transformation $(\mathcal{N}, \mathcal{X}, \{\widehat{\pi}_i : i \in \mathcal{N}\})$. Then the function defined by $\{x_i(\mathbf{e}) = \theta_i(e_i) - c_i(e_i)\}$, $i \in \mathcal{N}$ for linear and truncated linear demand with additive effort effect or $\{x_i(\mathbf{e}) = \theta_i(e_i)exp(-\lambda c_i(e_i))\}$, $i \in \mathcal{N}$ for exponential demand with multiplicative effort effect is a function map from $\mathcal{E}^*$ onto $\mathcal{X}^*$ (Surjection). In addition, $\forall \mathbf{e} \in \mathcal{E}^*$, $\pi_i(\mathbf{e}) = \widehat{\pi}_i(\mathbf{x}(\mathbf{e}))$, $\forall i \in \mathcal{N}$*

The above proposition suggests that for every Nash Equilibrium point $\mathbf{e}^*$ in $\mathcal{E}$, there is a unique corresponding Nash Equilibrium $\mathbf{x}^*$ in $\mathcal{X}$ with same profit for each player: $\widehat{\pi}_i(\mathbf{x}(\mathbf{e}^*)) = \pi_i(\mathbf{e}^*)$. And for every Nash Equilibrium point $\mathbf{x}^*$ in $\mathcal{X}$, there exists at least one corresponding Nash Equilibrium point $\mathbf{e}^*$ in $\mathcal{E}$ with the same profit for each player: $\pi_i(\mathbf{e}^*) = \widehat{\pi}_i(\mathbf{x}^*)$

By the above proposition, the equilibrium profit results in the transformed game are identical to the original investment effort game. Therefore, to analyze the behavior of the players in the original ivestment effort space $\mathcal{E}$, it's enough for us to investigate their behavior in the new space $\mathcal{X}$ and we call $\mathbf{x}$ effective effort.

Lemma 3.5. *After the transformation, for any subset $\mathcal{S} \subseteq \{0, \ldots, n\}$, function*

$$f(\mathbf{x}) = \sum_{i \in \mathcal{S}} \widehat{\pi}_i(\mathbf{x})$$

is supermodular in $\mathbf{x}$ and has increasing difference in (x_i, α), $\forall i \in \mathcal{S}$, $\alpha \in \mathcal{A} = [\underline{\alpha}, \overline{\alpha}]$. Especially, if demand is exponential with multiplicative effort effect (when $\underline{x}_i > 0, \forall i = 0, \ldots, n$ or linear with additive effort effect, then the supermodularity is strict.

Lemma 3.6. *After the transformation,$\forall i \in \{0, \ldots, n\}$, $\widehat{\pi}_i(\mathbf{x})$ is nondecreasing in x_j, for $\forall j \neq i$*

Lemma 3.7. *For any $\mathcal{I} \subseteq \{0, \ldots, n\}$. with given $\mathbf{x}_{\mathcal{I}^c} \in \mathcal{I}^c$ such that the players in $\mathcal{I}$ makes a simultaneous decision on x_i. We have:*
a) The Nash Equilibrium in $\mathcal{X}_{\mathcal{I}}$ forms a compact sublattice: if $\mathbf{x}_{\mathcal{I}}^{\mathbf{1}}$ and $\mathbf{x}_{\mathcal{I}}^{\mathbf{2}}$ are equilibrium points, $\mathbf{x}_{\mathcal{I}}^{\mathbf{1}} \wedge \mathbf{x}_{\mathcal{I}}^{\mathbf{2}}$ and $\mathbf{x}_{\mathcal{I}}^{\mathbf{1}} \vee \mathbf{x}_{\mathcal{I}}^{\mathbf{2}}$ are also Nash equilibrium points.
b) There exists a smallest and largest Nash equilibrium points $\mathbf{x}_{\mathcal{I}}^{\mathbf{l}}$ and $\mathbf{x}_{\mathcal{I}}^{\mathbf{u}}$, such that $x_i^l \leq x_i^u$ for all $i \in \mathcal{I}$.
c) Among all the Nash equilibriums, every player prefer the largest Nash equilibrium $\mathbf{x}_{\mathcal{I}}^{\mathbf{u}}$.

Lemma 3.8 is a direct result from lemma 3.5 and 3.7 by Theorem 4.2.2. in Topkis [44]

Lemma 3.8. *With increasing market size α, the equilibrium effective investment effort and net profit is non decreasing.*

3.3.2 First stage non simultaneous effort decision

For any given set $\mathcal{S} \subset \{0, \ldots, n\}$ with given x_i, $\forall i \in \mathcal{S}^c$, for the remainder of the chapter, we assume the outcome of the firms' simultaneous investment game among firms in $\mathcal{S}$ is the largest Nash equilibrium, which is preferred by every player.

Let $\mathcal{I} \subset \{0, \ldots, N\}$, and $\mathcal{S} \subseteq \mathcal{I}^c$. Let $Y_{\mathcal{I}}(\mathbf{x}_s | \mathbf{x}_{\mathcal{I}\mathcal{S}^c})$ be the largest Nash equilibrium in the simultaneous investment game among players in $\mathcal{I}$ as a function of $\mathbf{x}_s$ while keeping the effective effort of all the other players in $\mathcal{I}\mathcal{S}^c = \mathcal{I}^c \bigcap \mathcal{S}^c$ fixed.

The following two lemmas suggest that if one firm makes an investment effort decision first followed by the other firms' simultaneous investment decision game, every firm can benefit, and the resulting equilibrium effective effort is higher.

Lemma 3.9. *Let $\mathcal{I} \subset \{0, \ldots, n\}$ and $m = |\mathcal{I}| < n$. For any any given $\mathbf{x}_{\mathcal{I}} \in \mathcal{X}_{\mathcal{I}}$, $\forall k \in \mathcal{I}^c$, denote $\mathcal{I}^{ck} = \mathcal{I}^c - \{k\}$. Let $x'_k > x''_k$ and $\mathbf{x}'_{\mathcal{I}^{ck}} = Y_{\mathcal{I}^{ck}}(x'_k | \mathbf{x}_I)$, $\mathbf{x}''_{\mathcal{I}^{ck}} = Y_{\mathcal{I}^{ck}}(x''_k | \mathbf{x}_I)$, we have $\mathbf{x}'_{\mathcal{I}^{ck}} \succeq \mathbf{x}''_{\mathcal{I}^{ck}}$ which means $x'_i \geq x''_i$ for $\forall i \in \mathcal{I}^{ck}$. Besides $\widehat{\pi}_i(x'_k, \mathbf{x}'_{\mathcal{I}^{ck}} | \mathbf{x}_{\mathcal{I}}) \geq \widehat{\pi}_i(x''_k, \mathbf{x}''_{\mathcal{I}^{ck}} | \mathbf{x}_{\mathcal{I}})$ for $\forall i \in \mathcal{I}^{ck}$.*

Lemma 3.10. *Let $\mathcal{I} \subset \{0, \ldots, n\}$ and $m = |\mathcal{I}| < n$. For any any given $\mathbf{x}_{\mathcal{I}} \in \mathcal{X}_{\mathcal{I}}$, denote the largest equilibrium among in the simultaneous investment effort game among players in $\mathcal{I}^c$ be $\mathbf{x}_{\mathcal{I}^c}{}'$. Now player $\forall k \in \mathcal{I}^c$ deviate from x'_k followed by simultaneous effort game among players in $\mathcal{I}^{ck} = \mathcal{I}^c - \{k\}$. Let $x^*_k = \max \operatorname{argmax}_{\mathrm{x_k}} \widehat{\pi}_{\mathrm{k}}(\mathrm{x_k}, \mathrm{Y}_{\mathcal{I}^{\mathrm{ck}}}(\mathrm{x_k} | \mathbf{x}_{\mathcal{I}}) | \mathbf{x}_{\mathcal{I}})$ and $\mathbf{x}^*_{\mathcal{I}^{ck}} = Y_{\mathcal{I}^{ck}}(x_k | \mathbf{x}_{\mathcal{I}})$. We have $x'_i \leq x^*_i$ for $\forall i \in \mathcal{I}^c$. Besides $\widehat{\pi}_i(\mathbf{x}^* | \mathbf{x}_{\mathcal{I}}) \geq \widehat{\pi}_i(\mathbf{x}' | \mathbf{x}_{\mathcal{I}})$ for $\forall i = 0, \ldots, n$*

In lemma 3.11, the leader is a single player, then the question is whether the results can be extended to the case we have a group of plays as leader. To examine that, we should first make sure that the simultaneous effort decision among players as leader should hold an equilibrium if they take the followers' largest effort equilibrium as response:

Lemma 3.11. *Let $\mathcal{I} \subset \{0, \ldots, n\}$ and $m = |\mathcal{I}| < n$. For any given $\mathbf{x}_{\mathcal{I}} \in \mathcal{X}_{\mathcal{I}}$, denote the largest equilibrium among in the simultaneous invest effort game among players in $\mathcal{I}^c$ be*

$\mathbf{x}_{\mathcal{I}^c}{}'$. *Now suppose players* $\forall k \in \mathcal{S} \subset \mathcal{I}^c$ *deviate from* x'_k *and make their own decision early to maximize each individual profit taking the largest equilibrium among players in* $\mathcal{T} = \mathcal{I}^c \backslash \mathcal{S}$ *as response:* $\max_{x_i} \widehat{\pi}_i(x_i, Y_{x_{\mathcal{T}}}(\mathbf{x}_{\mathcal{S}}|\mathbf{x}_{\mathcal{I}})|\mathbf{x}_{\mathcal{I}})$, $\forall i \in \mathcal{S}$. *Then the equilibrium exists in the game among players in* $\mathcal{S}$ *if there exists a maximum for the potential function:* $\Psi_{\mathcal{S}}(\mathbf{x}_{\mathcal{S}}|\mathbf{x}_{\mathcal{I}}) = \widehat{\psi}(\mathbf{x}_{\mathcal{S}}, Y_{\mathcal{T}}(\mathbf{x}_{\mathcal{S}}|\mathbf{x}_{\mathcal{I}}), \mathbf{x}_{\mathcal{I}}) - \sum_{i \in \mathcal{S}} \widehat{k}_i(x_i)/\gamma_i$ *where* $\gamma_i = 1$ *if* $i = 1 \ldots n$ *and* $\gamma_0 = 1/\xi$.

Lemma 3.11 is a direct result from the property of potential function. Thus if the equilibrium among the players as leader uniquely exists, we may take the group of suppliers as a single player and lemma 3.12 follows from lemma 3.10.

Lemma 3.12. *Suppose* $\widehat{k}_i$ *is strictly convex in* x_i *for additive effort effect and non-decreasing and strictly convex in* x_i *for exponential demand with multiplicative effort effect. Let* $\mathcal{I} \subset \{0, \ldots, n\}$ *and* $m = |\mathcal{I}| < n$. *For any given* $\mathbf{x}_{\mathcal{I}} \in \mathcal{X}_{\mathcal{I}}$, *denote the largest equilibrium among in the simultaneous investment effort game among players in* $\mathcal{I}^c$ *be* $\mathbf{x}_{\mathcal{I}^c}{}'$. *Now suppose for all the players* $k \in \mathcal{S} \subset \mathcal{I}^c$ *deviate from* x'_k *and make their own simultaneous decision early to maximize its own profit taking the largest equilibrium among players in* $\mathcal{T} = \mathcal{I}^c \backslash \mathcal{S}$ *as response:* $\max_{x_i} \widehat{\pi}_i(x_i, Y_{\mathcal{T}}(\mathbf{x}_{\mathcal{S}}|\mathbf{x}_{\mathcal{I}}))$, $\forall i \in \mathcal{S}$. *If there exists a unique Nash equilibrium* $\mathbf{x}_{\mathcal{S}}{}^*$ *in the game among players in* $\mathcal{S}$, *let* $\mathbf{x}_{\mathcal{T}}{}^* = Y_{\mathcal{T}}(\mathbf{x}_{\mathcal{S}}{}^*|\mathbf{x}_{\mathcal{I}})$. *Suppose for every given* $\mathbf{x}_{\mathcal{T}}, \mathbf{x}_{\mathcal{I}}$, *there exists a unique Nash equilibrium in the simultaneous decision game among players in* $\mathcal{S}$, *and the corresponding potential function* $\Psi_{\mathcal{S}}(\mathbf{x}_{\mathcal{S}})$ *has a maxima. Then* $x'_i \leq x^*_i$ *and* $\widehat{\pi}_i(\mathbf{x}_{\mathcal{S}}{}', \mathbf{x}_{\mathcal{T}}{}', |\mathbf{x}_{\mathcal{I}}) \leq \widehat{\pi}_i(\mathbf{x}_{\mathcal{S}}{}^*, \mathbf{x}_{\mathcal{T}}{}^*|\mathbf{x}_{\mathcal{I}})$ *for any* $i \in \{0, \ldots, n\}$.

Lemma 3.12 is an extension of lemma 3.10 for the case with simultaneous competition among group of players in both stages of the Stackelburg game. Here convexity in the effort investment cost is important to make sure that everyone in the group has the same behavior as a single one.

As a special case, let us consider the case for exponential demand function with multiplicative effort effect and $\widehat{k}_i(x_i) = \lambda_i x_i^{n_i}$. Suppose $x_i \in [l_i, u_i]$ for all $i = 0, \ldots, n$

Lemma 3.13. *If* $n_i > n + 1$, $0 < l_i = l < (\frac{\alpha}{\lambda_i n_i})^{\frac{1}{n_i - n - 1}}$ *and* $u_i = u > (\frac{\alpha}{\lambda_i n_i})^{\frac{1}{n_i - n - 1}}$ *for all*

$i = 0, \ldots, n$, then there exists a uniqe Nash equilibrium $\mathbf{x}'$ in the simultaneous effort decision game among all the suppliers and manufacturer with $\prod_{i=0}^{n} x_i' = (\frac{\alpha^{1-\eta}}{\gamma})^{1/\eta}$, $x_i' = (\frac{\alpha \prod_{i=0}^{n} x_i'}{n_i \lambda_i})^{1/n_i}$ and $\widehat{\pi}_i' = \alpha(1 - \frac{1}{n_i}) \prod_{i=0}^{n} x_i'$ where $\eta = 1 - \sum_{i=0}^{n} 1/n_i > 0$ and $\gamma = \prod_{i=0}^{n} (n_i \lambda_i)^{1/n_i}$.

Lemma 3.13 suggests that every player with same n_i will share the same net profit regardless with cost scaler λ_i. Besides the ratio of profit between two supplier j and k is $\frac{1-1/n_j}{1-1/n_k}$, hence the player with higher n_i gets more profit. Now suppose players in $\mathcal{S} \subset \{0, \ldots, n\}$ make decison first followed by simultaneous decison among players in $\mathcal{T} = \mathcal{S}^c$. Let $s =\mid \mathcal{S} \mid$ and $t =\mid \mathcal{T} \mid$. Define $\gamma_S = \prod_{i \in \mathcal{S}} (n_i \lambda_i)^{1/n_i}$, $\gamma_T = \prod_{i \in \mathcal{T}} (n_i \lambda_i)^{1/n_i}$ and $\eta_S = 1 - \sum_{i \in \mathcal{S}} 1/n_i$ and $\eta_T = 1 - \sum_{i \in \mathcal{T}} 1/n_i$.

Lemma 3.14. *Suppose $n_i > n + 1$ and $0 < l_i = l < \underline{l}$, $u_i = u > \overline{u}$ for all $i = 0, \ldots, n$, where $\underline{l}$, $\overline{u}$ are constant defined as follows:*

$$\begin{aligned} \underline{l} &= \min\{\min_{i \in \mathcal{T}}\{(\frac{\alpha}{\lambda_i n_i})^{\frac{1}{n_i - n - 1}}\}, \min_{i \in \mathcal{S}}\{\big(\frac{1}{\eta_T n_i \lambda_i}(\frac{\alpha}{\gamma_T})^{\frac{1}{\eta_T}}\big)^{\frac{1}{n_i - s/\eta}}\}\} \\ \overline{u} &= \max\{\max_{i \in \mathcal{T}}\{(\frac{\alpha}{\lambda_i n_i})^{\frac{1}{n_i - n - 1}}\}, \max_{i \in \mathcal{S}}\{\big(\frac{1}{\eta_T n_i \lambda_i}(\frac{\alpha}{\gamma_T})^{\frac{1}{\eta_T}}\big)^{\frac{1}{n_i - s/\eta}}\}\} \end{aligned}$$

Then there exists a unique equilibrium in the above Stackelburg game with $\prod_{i \in \mathcal{S}} x_i^* = (\frac{\alpha^{1-\eta_S}}{\gamma_S^{\eta_T} \gamma_T^{1-\eta_S} \eta_T^{(1-\eta_S)\eta_T}})^{1/\eta}$, $x_i^* = (\frac{\alpha^{1/\eta}}{n_i \lambda_i \eta_T^{\eta_T/\eta} \gamma_S^{1/\eta} \gamma_T^{1/\eta}})^{1/n_i}$, $\widehat{\pi}_i^* = (1 - \frac{1}{n_i \eta_T}) \frac{\alpha^{1/\eta}}{\eta_T^{1-\eta_S/\eta} \gamma_S^{1/\eta} \gamma_T^{1/\eta}}$ for $i \in \mathcal{S}$ and $\prod_{i \in \mathcal{T}} x_i^* = (\frac{\alpha^{1-\eta_T}}{\gamma_S^{1-\eta_T} \gamma_T^{\eta_S} \eta_T^{(1-\eta_S)(1-\eta_T)}})^{1/\eta}$, $x_i^* = (\frac{\alpha^{(2-\eta)/\eta}}{n_i \lambda_i \gamma_S^{1/\eta} \gamma_T^{1/\eta} \eta_T^{(1-\eta)/\eta}})^{1/n_i}$, $\widehat{\pi}_i^* = (1 - \frac{1}{n_i}) \frac{\alpha^{(2-\eta)/\eta}}{\gamma_S^{1/\eta} \gamma_T^{1/\eta} \eta_T^{(1-\eta)/\eta}}$ for $i \in \mathcal{T}$

Here we have several observations from lemma 3.14 and 3.13. Firstly, since $0 < \eta_T < 1$ and $2/\eta - 1 > 1/\eta$, thus compare the simultaneous competition equilibrium $\widehat{\pi}_i'$ with Stackelburg game equilibrium $\widehat{\pi}_i^*$, we have $\widehat{\pi}_i^* > \widehat{\pi}_i'$ for all $i = 0, \ldots, n$. Secondly, note that $\prod_i x_i^* = (\frac{\alpha^{1-\eta}}{\gamma \eta_T^{1-\eta^S}})^{1/\eta}$. Since η_T is decreasing with the sum of $1/n_i$ for members in $\mathcal{T}$. Thus the product of effective effort (also proportional to demand and gross profit) is increasing with the set of followers $\mathcal{T}$. This observation suggests that among all the scenarios of two groups

Stackelburg game, the scenario with $\mathcal{S} = \{i : \max_i n_i\}$ leads to best social welfare and most profit for the suppliers other than i. Therefore in a production system, to let the company with "heaviest" fixed investment cost burden make the effort decision will be the best choice. Finally, the follower player j in $\mathcal{T}$'s profit is decreasing with η_T. Thus for the same players' profile, compare two scenarios: players $\mathcal{S}_1$ as leader followed by players in $\mathcal{T}_1$ and players in $\mathcal{S}_2$ as leader followed by players in $\mathcal{T}_2$. If $\mathcal{T}_1 \subset \mathcal{T}_2$, then all players in $\mathcal{T}_1$ get more profit in scenario two than what they get in scenarios one.

Here is the example:

Example 3.1. *Let $n = 3$, $\widehat{k}_0(x_0) = x_0^5$, $\widehat{k}_1(x_1) = x_1^6$, $\widehat{k}_2(x_2) = x_2^5$, $\widehat{k}_3(x_3) = x_3^6$ and $\alpha = 1$. If $l_i = 10^{-10}$ and $u_i = 10^{10}$ for all i. Both the equilbrium of simultaneous decision among all players and Stackelburg game defined by player $0, 1$ first followed by $2, 3$ is unique and the result is in table 3.1*

Table 3.2: Equilibrium Results

Player	x_i'	x_i^*	$\widehat{k}_i'$	$\widehat{k}_i^*$
0	0.286	0.355	0.00762	0.0122
1	0.342	0.409	0.00794	0.0132
2	0.286	0.324	0.00762	0.0143
3	0.342	0.379	0.00794	0.0167

Lemma 3.15. *If demand is exponential with multiplicative effort effect ($\bar{x}_i > 0, \forall i = 0, \ldots, n$) or (truncated) linear with additive effort effect and if $n = 2$, consider the Stackelburg game in the following two scenarios: player i and j make simultaneous decisions first followed by player k and player i makes decision first followed by player j and k's simultaneous decisions. Let $\mathbf{x}'$ be the largest equilibrium in the first scenario and $\mathbf{x}^*$ be the largest equilibrium for the second scenario, then if $x_i' > x_i^*$ then $x_j' \geq x_j^*$ and $x_k' \geq x_k^*$, besides, $\widehat{\pi}_i' > \widehat{\pi}_i^*$, $\widehat{\pi}_j' \geq \widehat{\pi}_j^*$ and $\widehat{\pi}_k' \geq \widehat{\pi}_k^*$.*

Lemma 3.15 shows that if there are three player (two supplier and one manufacturer), every one can benefit from enlarging leader group if the enlarged leaders set a higher effective effort. In the case that the original player as the leader does not set a higher effective effort after some other players make effort decisions earlier with him, the system could have various outcomes which are shown in the following examples when $n = 2$ and $\psi(\mathbf{x}) = x_0 x_1 x_2$:

Example 3.2. *If* $\mathcal{X}_i = \{1, 2\}$ *and* $\widehat{k}_0(1) = 0$*,* $\widehat{k}_0(2) = 6$*,* $\widehat{k}_1(1) = 0$*,* $\widehat{k}_1(2) = 6$*,* $\widehat{k}_2(1) = 0$*,* $\widehat{k}_2(2) = 1/2$

Scenario 1: player 0 and 1 make simultaneous decisions first followed by player 2

In equilibrium, $x'_0 = 1$*,* $\widehat{\pi}'_0 = 1$*,* $x'_1 = 1$*,* $\widehat{\pi}'_1 = 1$ *and* $x'_2 = 1$*,* $\widehat{\pi}'_2 = 1$

Scenario 2: player 0 makes decision first followed by player 1 and 2's simultaneous decision

In equilibrium, $x^*_0 = 2$*,* $\widehat{\pi}^*_0 = 2$*,* $x^*_1 = 2$*,* $\widehat{\pi}^*_1 = 2$ *and* $x^*_2 = 2$*,* $\widehat{\pi}^*_2 = 15/2$

This example shows that to let a player make decision earlier in the Stackelberg game will make everyone better.

Example 3.3. *If* $\mathcal{X}_0 = \{5/4, 1\}$*,* $\mathcal{X}_1 = \{2, 1\}$ *and* $\mathcal{X}_2 = \{33/16, 2, 1, 0\}$*. Suppose* $\widehat{k}_0(1) = 0$*,*$\widehat{k}_0(5/4) = 6/5$*;* $\widehat{k}_1(1) = 0$*,* $\widehat{k}_1(2) = 2$*;* $\widehat{k}_2(0) = 0$*,* $\widehat{k}_2(1) = 9/8$*,* $\widehat{k}_2(2) = 11/4$*,*$\widehat{k}_2(33/16) = 185/64$

Scenario 1: player 0 and 1 make simultaneous decisions first followed by player 2

In equilibrium, $x'_0 = 1$*,* $x'_1 = 2$ *and* $x'_2 = 2$ *with* $\widehat{\pi}'_0 = 4$*,* $\widehat{\pi}'_1 = 2$ *and* $\widehat{\pi'_2} = 5/4$

Scenario 2: player 0 makes decision first followed by player 1 and 2's simultaneous decisions

In equilibrium, $x^*_0 = 5/4$*,* $x^*_2 = 1$ *and* $x^*_3 = 1$ *with* $\widehat{\pi}^*_0 = 1/20$*,* $\widehat{\pi}^*_1 = 5/4$ *and* $\widehat{\pi}^*_3 = 1/8$

This example shows that to letting a player make decision earlier in the Stackelberg game makes everyone better off but someone will set a lower effort.

Example 3.4. *If* $\mathcal{X}_0 = \{5/4, 1\}$*,* $\mathcal{X}_1 = \{2, 1\}$ *and* $\mathcal{X}_2 = \{49/32, 3/2, 1, 0\}$*. Suppose* $\widehat{k}_0(1) = 0$*,*$\widehat{k}_0(5/4) = 1.83$*;* $\widehat{k}_1(1) = 0$*,* $\widehat{k}_1(2) = 1.9$*;* $\widehat{k}_2(0) = 0$*,* $\widehat{k}_2(1) = 9/8$*,* $\widehat{k}_2(3/2) = 31/16$*,* $\widehat{k}_2(49/32) = 257/128$

Scenario 1: player 0 and 1 make simultaneous decisions first followed by player 2

In equilibrium, $x_0' = 1$*,* $x_1' = 2$ *and* $x_2' = 1$ *with* $\widehat{\pi}_0' = 2$*,* $\widehat{\pi}_1' = 0.1$ *and* $\widehat{\pi_2'} = 0.875$

Scenario 2: player 0 make decision first followed by player 1 and 2's simultaneous decision

In equilibrium, $x_0^* = 5/4$*,* $x_2^* = 1$ *and* $x_3^* = 3/2$ *with* $\widehat{\pi}_0^* = 0.045$*,* $\widehat{\pi}_1^* = 1.875$ *and* $\widehat{\pi}_3^* = -0.0625$

This example shows that to let a player make decision earlier in the Stackelberg game has mixed effects on different players.

Example 3.5. *If* $\mathcal{X}_0 = \{0.56, 0.57, 0.91\}$*,* $\mathcal{X}_1 = \{0.47, 0.55, 0.77\}$ *and* $\mathcal{X}_2 = \{0.05, 0.14, 0.75\}$*. Suppose* $\widehat{k}_0(0.56) = 0.8$*,* $\widehat{k}_0(0.57) = 0.83$*,* $\widehat{k}_0(0.91) = 0.88$*;* $\widehat{k}_1(0.47) = 0.04$*,* $\widehat{k}_1(0.55) = 0.29$*,* $\widehat{k}_1(0.77) = 0.34$*;* $\widehat{k}_2(0.05) = 0.54$*,* $\widehat{k}_2(0.14) = 0.65$*,* $\widehat{k}_2(0.74) = 0.77$*,*

Scenario 1: player 0, 1 and 2 make simultaneous decisions

In equilibrium, $x_0' = 0.91$*,* $x_1' = 0.47$ *and* $x_2' = 0.75$ *with* $\widehat{\pi}_0' = -0.5635$*,* $\widehat{\pi}_1' = 0.2765$ *and* $\widehat{\pi_2'} = -0.4535$

Scenario 2: player 0 make decision first followed by player 1 and then at last player 2 make decision

In equilibrium, $x_0^* = 0.56$*,* $x_2^* = 0.77$ *and* $x_3^* = 0.75$ *with* $\widehat{\pi}_0^* = -0.4809$*,* $\widehat{\pi}_1^* = -0.0209$ *and* $\widehat{\pi}_3^* = -0.4509$

This example shows that compared with simultaneous competition, totaly sequential decision makes someone better and someone worse. Note that the one who makes first decision can always do better in in the three players' case.

Lemma 3.16. *If* $\widehat{k}_i$ *is strictly convex in* x_i $\forall i = 0, \ldots, n$ *for (truncated) linear demand with additive effort effect and strictly increasing convex for exponential demand with multiplicative effort effect (assuming* $\underline{x}_i > 0$*,* $\forall i = 0, \ldots, n$*. In addition, for any given* $\mathcal{S} \subset \{0, \ldots, n\}$*, there exists a unique Nash equilibrium in the Stackelburg game defined as players in* $\mathcal{S}$ *make simultaneous decision first followed by players in* $\mathcal{S}^c$*'s simultaneous decision and there exists a maxima in the corresponding potential function* $\Psi_{\mathcal{S}}(\mathbf{x}_{\mathcal{S}})$*. Let* $\mathcal{S}_1$*,* $\mathcal{S}_2$ *and* $\mathcal{S}_3$ *be nonempty multi-exclusive subsets of* $\{0, \ldots, n\}$ *and* $\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 = \{0, \ldots, n\}$*. Consider the Stackelburg game in the following two scenarios: players in* $\mathcal{S}_1 \cup \mathcal{S}_2$ *make simultaneous decision first* *followed by plays in $\mathcal{S}_3$'s simultaneous decision and players in $\mathcal{S}_1$ make simultaneous decisions first followed by plays in $\mathcal{S}_2 \cup \mathcal{S}_3$'s simultaneous decision.* $\mathbf{x}'$ *be the equilibrium in the first scenario and* $\mathbf{x}^*$ *be the equilibrium for the second scenario, then if* $\forall j \in \mathcal{S}_1$ *such that* $x'_j \geq x^*_j$*, then* $x'_i \geq x^*_i$ *and* $\widehat{\pi}_i(\mathbf{x}') \geq \widehat{\pi}_i(\mathbf{x}^*)$ *for all* $i = 0, \ldots, n$.

Lemma 3.16 is an extension of lemma 3.15 for the case of individual players being replaced by a group of players.

3.3.3 First stage cooperative effort decision

Now suppose the players cooperate together in the first stage effort decision while keeping the second stage in price competition, we have the following results:

Lemma 3.17. *For exponential demand with a multiplicative effect of investment effort or (truncated) linear demand with additive effect of investment effort, the function* $f(t, \mathbf{x}) = t\widehat{\psi}(\mathbf{x}) - \sum_{i=0}^{n} \widehat{k}_i(x_i)$ *has increasing difference in* $(t, \mathbf{x})$ *when* $t > 0$.

The following lemma shows that if the demand is exponential with a multiplicative effect of investment effort and the firms coordinate together in investment effort to maximize the system profit, every firm will set a higher effective effort and get higher gross profit (with larger demand) compared with competition system.

Lemma 3.18. *If the demand is exponential with a multiplicative effect of investment effort, let* $\mathbf{x}^* = \max \operatorname{argmax}_{\mathbf{x}} \sum_{\mathrm{i=0}}^{\mathrm{n}} \widehat{\pi}_{\mathrm{i}}(\mathbf{x})$ *and* $\mathcal{X}^b$ *be the set of all the Nash Equilibria of investment effort game among all the firms. Then* $\exists \mathbf{x}^b \in \mathcal{X}^b$ *such that* $\mathbf{x}^* \succeq \mathbf{x}^b$ *and* $\widehat{\psi}_i(\mathbf{x}^*) \geq \widehat{\psi}_i(\mathbf{x}^b)$ *for all* $i = 0, \ldots, n$.

Lemma 3.19. *If* $\widehat{k}_0$ *is increasing in* x_0*, then the function* $f(s, \mathbf{x}) = \widehat{\psi}(\mathbf{x}) - \sum_{i=1}^{n} \widehat{k}_i(x_i) - s\widehat{k}_0(x_0)$ *has increasing difference in* $(-s, \mathbf{x})$ *when* $s > 0$.

Thus we can extend the results of lemma 3.18 to linear or truncated linear demand with an additive effort effect when the manufacturer's effective effort cost is increasing in his effort.

Lemma 3.20. *If the demand is linear or truncated linear with an additive effect of investment effort and $\widehat{k}_0$ is increasing in x_0, let $\mathbf{x}^* = \max \operatorname{argmax}_{\mathbf{x}} \sum_{i=0}^{n} \widehat{\pi}_i(\mathbf{x})$ and $\mathcal{X}^b$ be the set of all the Nash Equilibria of investment effort game among all the firms. Then $\exists \mathbf{x}^b \in \mathcal{X}^b$ such that $\mathbf{x}^* \succeq \mathbf{x}^b$ and $\widehat{\psi}_i(\mathbf{x}^*) \geq \widehat{\psi}_i(\mathbf{x}^b)$ for all $i = 0, \ldots, n$.*

Hence if the equilibrium is unique, we are sure that the coordination system will set a higher pure effective effort than the competition system.

3.4. Summary

In this chapter, a two echelon assembly system involving pricing and investment decisions is investigated. Compared with the assembly model in chapter two, demand in this model is deterministic as a function of price and product quality and vendors' effort like promotion or advertisement. The suppliers sell the components to the manufacturer with a price only contract taking into consideration of the manufacturer's best price decision. We find that for demand function with a smooth manufacturer's best price response, equilibrium exists in the suppliers' competition, while for several common demand functions, the unique Nash equilibrium leads to identical gross profit for all suppliers which is a constant proportion of the manufacturer's gross profit. In this stage, because of the competition, the supply chain profit is no more than $1/(n+1)$ of the coordination system.

Taking the unique price competition as a response, we analyze the effort investment decision in the beginning stage. We find that in the effort decision stage, Nash equilibrium always exists for simultaneous competition. After transforming the effort variables to effective effort variables, the players' profit is supermodular in effective effort and increasing with the other players' effective effort. Thus we know that the equilibrium set consists of a largest and a smallest equilibrium such that in the largest equilibrium every one get most of the profit. We also find that if some one makes an effort decision earlier followed by the others' simultaneous decision, every one can benefit. Since we can replace a group of players by an artificial single player in the case the investment cost function is increasing convex and the Nash equilibrium is unique, the previous results hold in “group” followed by ”group” Stackelburg decision. If the enlargement of the leading group’s size will induce in a higher group effective effort decision, everyone can benefit. Coordination in effort investment will lead to a higher effective effort and higher profit for every player. Finally, for more than two suppliers’ case, making decisions in a totaly sequential order (one by one) may be better or worse than simultaneous competition.

Chapter 4

Summary and Future Research

The objective of this dissertation is to analyze the behavior of a two echelon supply chain system considering the components' reliability and quantity.

For the stochastic reliability model, the manufacture's expected profit is always concave in his ordering quantity. Furthermore, when component suppliers are complementary, the expected profit is also supermodular in the ordering quantities. Thus an effective optimal scheme solution and a heuristic algorithm are suggested. The target production quantity heuristic provides an effective methodology when multiple variable integration is hard to calculate. Although this heuristic performs well for unimodal reliability density distributions in our numerical experiments, its performance under non-unimodal reliability density distribution and worst case is not known, so it could be interesting for future research. Our numerical study suggests that variation in reliability has more impact on optimal profit than the variation of demand. This phenomena needs to be further analyzed. In the assembly system, when each supplier's reliability is a Bernoulli random variable, the problem of the analysis degenerate to become deterministic, and we can easily go further to analyze the pricing game among the suppliers. Under a mild restriction on the demand, the non-trivial equilibrium is unique, and based on that, we further investigate the effort decision to improve the successful delivery probability. Nash equilibrium is found to exists in the above games.

The supply chain seems to be more efficient when the coefficient of demand variance is small in which case the demand degenerates to become deterministic and suppliers get all the system profits. For the general reliability distribution, the behavior of the pricing game among suppliers, even the existence of equilibrium is still unknown and would be worthy of investigation in the future. A little insight here is that based on the property of complementary and non-functional difference between the components that affects demand, suppliers should get similar profit in equilibrium. For the sourcing model where the relation between the suppliers are substitutable, unlike the assembly model, even the sensitive analysis results are restrictive to the number of suppliers and the demand distribution function. In case there are more than three suppliers, they may get higher ordering quantities when the other supplier raises his wholesale price. This non-monotone response will cause further difficulty in the pricing game analysis among suppliers. We find that for two suppliers' case with Bernoulli reliability, under restricted conditions on the demand distribution and cost parameters, the existence of Nash equilibrium could be guaranteed by the increasing best response property. For the general demand or reliability distribution, future research may be conducted in this direction. For the model of the assembly-sourcing system, we can extend some basic results from the sourcing system and assembly system. However, more in depth decision analysis is very complicated and requires further investigation. Our numerical study through active set method suggests a strong in-group effect. For those suppliers out of the group, the one with the higher wholesale price is more sensitive to the changes. The mechanism for this phenomena is of further research interest.

For the quality investment model in the deterministic demand, based on the approach by construction of potential function, Nash equilibrium is shown to exist. After some transformation, we can identify the supermodularity of the players profit function in their effective effort decision. We show that the competition with a leader can benefit every one. This nice property does not apply when we try to extend it to a totally sequential decision. The condition for the successful extension of this property is an interesting question. Our results can be easily extended to the case with stochastic demand if the stochastic factor is realized before price decision. However, if the manufacturer faces a news vendor problem with a price decision, although the existence of Nash equilibrium still exists, the property of system behavior is then hard to analyze. This would be an interesting open question for future research.

In our dissertation, for substitutable suppliers, we differentiate them with their reliability and price. There are some other approaches in the suppliers' differentiation like lead time, location, quality and negotiate power. The combination of those factors on the manufacturer's ordering decisions and suppliers competition is attracting more and more research interests. Our work just opens a window for that direction.

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Appendix A

Proofs and Tables

A.1. Proofs for Preliminary Results

Lemma A.1. *Given X and Y are two independent random variables supported on the bounded positive intervals and $f(x,y)$ is a positive integrable function. If $E[Xf(X,Y)]E[Y[f(X,Y)] \geq E[XYf(X,Y)]E[f(X,Y)]$ then*

$$E[X^2 f(X,Y)]E[Yf(X,Y)] \geq E[XYf(X,Y)]E[Xf(X,Y)] \tag{A.1}$$

Proof of Lemma A.1. First, if $E[f(X,Y)] = 0$, since X and Y are supported on bounded interval, we have $E[X^2 f(X,Y)] = E[Yf(X,Y)] = E[XYf(X,Y)] = E[Xf(X,Y)] = 0$, and the inequality A.1 hold.

Otherwise $E[f(X,Y)] > 0$, since $f(x,y)$ is positive integrable, by **Cauchy-Schwarz Inequality** we have:

$$E[f(X,Y)]E[X^2 f(X,Y)] \geq E^2[Xf(X,Y)].$$

Since y and $f(x,y)$ are always positive, we have

$$\begin{aligned} E[f(X,Y)]E[X^2 f(X,Y)]E[Yf(X,Y)] &\geq E^2[Xf(X,Y)]E[Yf(X,Y)] \\ &\geq E[Xf(X,Y]E[XYf(X,Y]E[f(X,Y)] \end{aligned}$$

Divide both sides of the inequality by $E[f(X,Y)]$ and the result follows.

Q.E.D.

Lemma A.2. *Suppose X and Y are two independent random variable supported on bounded positive intervals and $f(\bullet)$ is a log-concave density function, then*

$$E[Xf(X+Y)]E[Yf(X+Y)] \geq E[XYf(X+Y)]E[f(X+Y)]. \tag{A.2}$$

Proof of Lemma A.2. Without loss of generality, let $g_1(x)$ be the density function of the random variable X supported on $[l_1, u_1]$ and $g_2(y)$ be the density function of the random variable Y supported on $[l_2, u_2]$. Define the following function:

$$\begin{aligned}
F_1(x) &= x \\
F_2(x) &= \int_{l_2}^{u_2} f(x+y)g_2(y)dy \\
G_1(x) &= 1 \\
G_2(x) &= \int_{l_2}^{u_2} yf(x+y)g_2(y)dy
\end{aligned}$$

From this, we have:

$$\begin{aligned}
& E[Xf(X+Y]E[Yf(X+Y)] - E[XYf(X+Y)]E[f(X+Y)] \\
= & \int_{l_1}^{u_1} xg_1(x) \int_{l_2}^{u_2} f(x+y)g_2(y)dydx \int_{l_1}^{u_1} g_1(x) \int_{l_2}^{u_2} yf(x+y)g_2(y)dydx \\
& - \int_{l_1}^{u_1} g_1(x) \int_{l_2}^{u_2} f(x+y)g_2(y)dydx \int_{l_1}^{u_1} xg_1(x) \int_{l_2}^{u_2} yf(x+y)g_2(y)dydx \\
= & \int_{l_1}^{u_1} F_1(x)F_2(x)g_1(x)dx \int_{l_1}^{u_1} G_1(x)G_2(x)g_1(x)dx \\
& - \int_{l_1}^{u_1} F_1(x)G_2(x)g_1(x)dx \int_{l_1}^{u_1} F_2(x)G_1(x)g_1(x)dx \\
= & \frac{1}{2}\int_{l_1}^{u_1}\int_{l_1}^{u_1} (F_1(x_1)G_1(x_2) - F_1(x_2)G_1(x_1))(F_2(x_1)G_2(x_2) - F_2(x_2)G_2(x_1))g_1(x_1)g_1(x_2)dx_1dx_2
\end{aligned}$$

Hence to prove the inequality (A.2), it is enough to show that for any $x_1, x_2 \in [l_1, u_1]$, the expression $F_1(x_1)G_1(x_2) - F_1(x_2)G_1(x_1)$ and $F_2(x_1)G_2(x_2) - F_2(x_2)G_2(x_1)$ have the same sign.

By our definition of the functions, it is equivalent to show that for any $u_1 \geq x_2 \geq x_1 \geq l_1$, the following inequality holds:

$$\int_{l_2}^{u_2} f(x_1+y)g_2(y)dy \int_{l_2}^{u_2} yf(x_2+y)g_2(y)dy - \int_{l_2}^{u_2} f(x_2+y)g_2(y)dy \int_{l_2}^{u_2} yf(x_1+y)g_2(y)dy \geq 0$$

Similar as before, we define:

$$\begin{aligned}\bar{F}_1(y) &= 1\\ \bar{F}_2(y) &= f(x_1+y)\\ \bar{G}_1(y) &= y\\ \bar{G}_2(y) &= f(x_2+y)\end{aligned}$$

From this

$$\begin{aligned}&\int_{l_2}^{u_2} f(x_1+y)g_2(y)dy \int_{l_2}^{u_2} yf(x_2+y)g_2(y)dy - \int_{l_2}^{u_2} f(x_2+y)g_2(y)dy \int_{l_2}^{u_2} yf(x_1+y)g_2(y)dy\\ =\ & \frac{1}{2}\int_{l_2}^{u_2}\int_{l_2}^{u_2} (\bar{F}_1(y_1)\bar{G}_1(y_2) - \bar{F}_1(y_2)\bar{G}_1(y_1))(\bar{F}_2(y_1)\bar{G}_2(y_2) - \bar{F}_2(y_2)\bar{G}_2(y_1))g_2(y_1)g_2(y_2)dy_1dy_2\end{aligned}$$

Hence, if for any y_1 and y_2, $\bar{F}_1(y_1)\bar{G}_1(y_2) - \bar{F}_1(y_2)\bar{G}_1(y_1)$ and $\bar{F}_2(y_1)\bar{G}_2(y_2) - \bar{F}_2(y_2)\bar{G}_2(y_1)$ have the same sign, then the above integration is nonnegative. Based on our definition of the functions, this condition holds if for any $u_2 \geq y_2 \geq y_1 \geq l_2$:

$$f(x_2+y_2)f(x_1+y_1) \leq f(x_2+y_1)f(x_1+y_2). \tag{A.3}$$

By the definition of log-concave density function, $f(\bullet)$ satisfies:

$$f(\alpha x_1 + (1-\alpha)x_2) \geq f(x_1)^{\alpha} f(x_2)^{(1-\alpha)}.$$

for all x_1 and x_2 in the support of $\mathbf{x}$ and $\alpha \in [0,1]$. Now if $x_2 = x_1$ and $y_2 = y_1$, then (A.3) holds as equality. Otherwise, let $\alpha = \frac{x_2-x_1}{x_2-x_1+y_2-y_1}$ and $(1-\alpha) = \frac{y_2-y_1}{x_2-x_1+y_2-y_1}$. We have:

$$\begin{aligned}f(x_2+y_1) &= f(\alpha(x_2+y_2) + (1-\alpha)(x_1+y_1)) \geq f(x_2+y_2)^{\alpha} f(x_1+y_1)^{(1-\alpha)}\\ f(x_1+y_2) &= f((1-\alpha)(x_2+y_2) + \alpha(x_1+y_1)) \geq f(x_2+y_2)^{(1-\alpha)} f((x_1+y_1)^{\alpha}\\ &\Rightarrow f(x_2+y_1)f(x_1+y_2) \geq f(x_2+y_2)f(x_1+y_1)\end{aligned}$$

As a result, (A.2) holds.

Q.E.D.

Lemma A.3. *If $f_i(x_i)$ is strictly convex in x_i, for $i \in \mathcal{S}$, define*

$$\begin{aligned} \mathbf{x}_{\mathcal{S}}(t_s) \quad &= \quad argmin \sum_{i\in\mathcal{S}} f_i(x_i) \\ subject\ to \quad & \sum_{i\in\mathcal{S}} x_i = t_s \\ & x_i \in [l_i, u_i], \forall i \in \mathcal{S}, \qquad (l_i \le u_i) \end{aligned}$$

then $\mathbf{x}_{\mathcal{S}}(t_s)$ is nondecreasing with respect to t_s. If $f_i(x_i)$ is non-decreasing, then the result also holds for

$$\begin{aligned} \mathbf{x}_{\mathcal{S}}(t_s) \quad &= \quad argmin \sum_{i\in\mathcal{S}} f_i(x_i) \\ subject\ to \quad & \prod_{i\in\mathcal{S}} x_i = t_s \\ & x_i \in [l_i, u_i], \forall i \in \mathcal{S}, \qquad (0 \le l_i \le u_i) \end{aligned}$$

Proof of Lemma A.3. First we show that it is true for the first case. Suppose not, then for some $t''_s > t'_s$ and in the optimal solution $\mathbf{x}''$ for t''_s and $\mathbf{x}'$ for t'_s, there exist i and j such that $x''_i > x'_i$ and $x''_j < x'_j$.
1) If $x''_i - x'_i > x'_j - x''_j$, since f_k is strictly convex for all $k \in \mathcal{S}$, $\sum_{i\in\mathcal{S}} f_i(x_i)$ is strictly jointly convex in $\mathbf{x}_{\mathcal{S}}$, then $\mathbf{x}'$ is the unique optimal solution for t'_s. Thus

$$f_i(x'_i)+f_j(x'_j) < f_i(x'_i+x'_j-x''_j)+f_j(x'_j-(x'_j-x''_j)) \Rightarrow f_j(x'_j)-f_j(x''_j) < f_i(x'_i+x'_j-x''_j)-f_i(x'_i)$$

Now construct a feasible solution $\mathbf{x}^*$ for t''_s such that $x^*_k = x''_k$ for $k \neq i, j$, $x^*_j = x''_j+(x'_j-x''_j) = x'_j$ and $x^*_i = x''_i - (x'_j - x''_j)$. Hence,

$$\sum_{k\in\mathcal{S}} f_k(x^*_k) - \sum_{k\in\mathcal{S}} f_k(x''_k) = f_i(x''_i - (x'_j - x''_j)) - f_i(x''_i) + (f_j(x'_j) - f_j(x''_j))$$

and by the strictly convex property of f_i, we have:

$$\begin{aligned} & f_i(x_i'') - f_i(x_i'' - (x_j' - x_j'')) > f_i(x_i' + x_j' - x_j'') - f_i(x_i') \\ \Rightarrow\ & f_i(x_i'' - (x_j' - x_j'')) - f_i(x_i'') + f_i(x_i' + x_j' - x_j'') - f_i(x_i') < 0 \\ \Rightarrow\ & f_i(x_i'' - (x_j' - x_j'')) - f_i(x_i'') + f_j(x_j') - f_j(x_j'') < 0 \\ \Rightarrow\ & \sum_{k\in\mathcal{S}} f_k(x_k^*) < \sum_{k\in\mathcal{S}} f_k(x_k'') \end{aligned}$$

Which leads to a contradiction.

2) $x_i'' - x_i' \leq x_j' - x_j''$, f_k is strictly convex for all $k \in \mathcal{S}$, then $\mathbf{x}'$ is the unique optimal soltuion for t_s'. Thus, $f_i(x_i'') - f_i(x_i') > f_j(x_j') - f_j(x_j' - (x_i'' - x_i'))$. Now construct a feasible solution $\mathbf{x}^*$ for t_s'' such that $x_k^* = x_k''$ for $k \neq i, j$, $x_j^* = x_j'' + (x_i'' - x_i')$ and $x_i^* = x_i'$. Hence,

$$\sum_{k\in\mathcal{S}} f_k(x_k^*) - \sum_{k\in\mathcal{S}} f_k(x_k'') = f_j(x_j'' + (x_i'' - x_i')) - f_j(x_j'') + (f_i(x_i') - f_i(x_i''))$$

and by the strictly convex property of f_i, we have:

$$\begin{aligned} & f_j(x_j') - f_j(x_j' - (x_i'' - x_i') > f_j(x_j'' + (x_i'' - x_i')) - f_i(x_j'') \\ \Rightarrow\ & f_j(x_j'' + (x_i'' - x_i')) - f_i(x_j'') + f_j(x_j'' + (x_i'' - x_i')) - f_i(x_j'') < 0 \\ \Rightarrow\ & f_j(x_j'' + (x_i'' - x_i')) - f_i(x_j'') + f_i(x_i') - f_i(x_i'') < 0 \\ \Rightarrow\ & \sum_{k\in\mathcal{S}} f_k(x_k^*) < \sum_{k\in\mathcal{S}} f_k(x_k'') \end{aligned}$$

Which leads to a contradiction.

For the second part of the lemma with the multiplicative constraint, in case $t_s = 0$, since $f_i(x_i)$ is non-decreasing and strictly convex, it must be strictly increasing. Hence the unique optimal solution is $x_i(0) = \max\{0, l_i\}$, for $i \in \mathcal{S}$, and for any $t_s' > 0$, the corresponding optimal solution $\mathbf{x}_{\mathcal{S}}(t_s') \succeq \mathbf{x}_{\mathcal{S}}(0)$.

In cast $t_s > 0$, let $z_i = \log x_i$, then the original problem is equivalent to:

$$\begin{aligned} \mathbf{z}_{\mathcal{S}}(t_s) \quad = \quad & argmin \sum_{i\in\mathcal{S}} f_i(e^{z_i}) \\ \text{subject to} \quad & \sum_{i\in\mathcal{S}} z_i = \log t_s \\ & z_i \in [\log l_i, \log u_i], \forall i \in \mathcal{S} \end{aligned}$$

Note that since f_i is monotone increasing and strictly convex and e^{z_i} is convex in z_i, $f_i(e^{z_i})$ is strictly convex in z_i. By the previous result, $\mathbf{z}_{\mathcal{S}}(t_s)$ is monotone non-decreasing in $\log t_s$ and non-decreasing in t_s. Since $x_i = e_i^z$ is monotone increasing map, $\mathbf{x}_{\mathcal{S}}(t_s)$ is monotone non-decreasing in t_s.

Q.E.D.

A.2. Proof for Chapter Two

Proof of Lemma 2.1.

$$\begin{aligned}
\frac{\partial \pi}{\partial q_i} &= -w_i + \int_0^1 g_1(r_1)\dots\int_0^1 g_n(r_n)\Big(\int_0^{\sum q_i r_i} s r_i f(x)dx \\
&\quad + \int_{\sum q_i r_i}^{\infty} (p+u) r_i f(x)dx\Big) dr_n \dots r_1 \\
&= -w_i + sE[R_i F(Q)] + (p+u)E[R_i(1-F(Q)] \\
&= -w_i + \bar{R}_i(p+u) - (p+u-s)E[R_i F(Q)] \\
\frac{\partial^2 \pi}{\partial q_i^2} &= -(p+u-s)E[R_i^2 f(Q)] \\
\frac{\partial^2 \pi}{\partial q_i \partial q_j} &= -(p+u-s)E[R_i R_j f(Q)]
\end{aligned}$$

$\forall \mathbf{x} \in \mathbf{R}^n$:

$$\begin{aligned}
\mathbf{x}H\mathbf{x}' &= -(p+u-s)\sum_{i=1}^{n}\sum_{j=1}^{n} E[R_i x_i R_j x_j f(Q)] \\
&= -(p+u-s)E[(\sum_{i=1}^{n} x_i R_i)^2 f(Q)] \leq 0 \qquad \text{(A.4)}
\end{aligned}$$

for $p+u<s$. Hence the expected profit function is jointly concave in q_i. **Q.E.D.**

Proof of Theorem 2.2. Define $rge(f)$ as the range of the map f. Define $T'(\mathbf{q}) : \mathbb{R}^{+n} \rightrightarrows \mathbb{R}^n = \{\bar{R}_i(p+u)-(p+u-s)E[R_iF(Q)] : i = 1,\dots n\}$ for $\mathbf{q} \geq \mathbf{0}$. Then the set $\{w_i, i = 1,\dots,n | q_i \geq 0$ with $w_i = \bar{R}_i(p+u)-(p+u-s)E[R_iF(Q)]$ and $0 \leq w_i \leq \bar{R}_i(p+u), i = 1,\dots\}$ is actually $rge(T') \cap \times_{i=1}^{n}[0, \bar{R}_i(p+u)]$. Since $\times_{i=1}^{n}[0, \bar{R}_i(p+u)]$ is convex, it is enough for us to show that $rge(T')$ is also convex. Now define the following function:

$$\psi(\mathbf{q}) = \begin{cases} -\pi + \sum_{i=1}^{n} w_i q_i = -(pE[D-(D-Q)^+] + sE[(Q-D)^+] - uE[(D-Q)^+]), & \mathbf{q} \geq \mathbf{0} \\ \infty, \quad otherwise \end{cases}$$

Since π is concave and differentiable in $\mathbf{q}$ and the set $\mathbb{R}^{+n}$ is convex, $\psi(\mathbf{q})$ is a proper convex function[1] on $\mathbb{R}^{+n}$. In addition, we have:

$$T(\mathbf{q}) = \partial\psi = \begin{cases} \{\bar{R}_i(p+u) - (p+u-s)E[R_iF(Q)] : i = 1,\ldots,n\} & \mathbf{q} > \mathbf{0} \\ \emptyset, \quad otherwise \end{cases}.$$

Clearly, the effective domain of $T(\mathbf{q})$ is the same as the domain $T'(\mathbf{q})$. Since T' is continuous, $rge(T')$ is the closure of $rge(T)$. By Theorem 12.17 in [38], for any proper convex function $\psi : \mathbb{R}^n \rightrightarrows \overline{\mathbb{R}}$, the mapping $\partial\psi$ is monotone [2]. Furthermore, since T is single valued mapping, it is maximal monotone[3] by Corollary 12.27 in [38]. In addition, by Proposition 6.4.1 in [4], the closure of $rge(T)$ is convex. Therefore, $rge(T')$ is a convex set and $\{w_i, i = 1,\ldots,n | q_i \geq 0$ with $w_i = \bar{R}_i(p+u) - (p+u-s)E[R_iF(Q)]$ and $0 \leq w_i \leq \bar{R}_i(p+u), i = 1,\ldots\}$ is convex.

Q.E.D.

Proof of Lemma 2.3. We prove the result by contradiction. Suppose not, then there exists i and j such that $w_i/\bar{R}_i > w_j/\bar{R}_j$, $q_i^* > 0$ with $q_j^* = 0$. By lemma 2.1, we have:

$$w_i = \bar{R}_i(p+u) - (p+u-s)E[R_iF(Q^*)] \quad (A.5)$$

$$w_j \geq \bar{R}_j(p+u) - (p+u-s)E[R_jF(Q^*)] \quad (A.6)$$

Since $q_j^* = 0$, R_j and $F(Q^*)$ are independent random variables, we have $E[R_jF(Q^*)] = \bar{R}_jE[F(Q^*)]$. Since $\bar{R}_j > 0$, thus

$$w_j/\bar{R}_j \geq (p+u) - (p+u-s)E[F(Q^*)]. \quad (A.7)$$

[1] A function is called proper if $f(x) < \infty$ for at least one $x \in \mathbb{R}^n$

[2] A map $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is called monotone if its graph is monotone, namely:

$$\langle u - v, x - y\rangle \geq 0, \forall(x,u) \in \text{gph}T, \forall(y,v) \in \text{gph}T,$$

[3] A monotone map $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is maximal if

$$\langle u - v, x - y\rangle \geq 0, \forall v \in T(y), y \in \text{dom}T \Rightarrow u \in T(x)$$

Also since $q_i^* > 0$, the function $F(\sum_i q_i^* r_i)$ is monotone increasing in r_i, which implies that there is a strictly positive correlation between R_i and $F(Q^*)$. We have $E[R_i F(Q^*)] \geq \bar{R}_i E[F(Q^*)]$. Therefore should hold:

$$\begin{aligned} w_i &= \bar{R}_i(p+u) - (p+u-s)E[R_i F(Q^*)] \\ &\leq \bar{R}_i(p+u) - \bar{R}_i(p+u-s)E[F(Q^*)] \\ &\Rightarrow w_i/\bar{R}_i \leq (p+u) - (p+u-s)E[F(Q^*)] \leq w_j/\bar{R}_j \end{aligned}$$

which contradicts with $w_i/\bar{R}_i > w_j/\bar{R}_j$. Hence, $q_j^* > 0$.

Q.E.D.

Proof of Lemma 2.4. We prove the result by contradiction. Suppose in the optimal decision every supplier get zero ordering quantity $q_j^* = 0$ for all $j = 1, \ldots, n$. Since the solution is optimal, **KKT** condition should be satisfied for supplier i, that is:

$$w_i \geq (p+u)\bar{R}_i - (p+u-s)E[\bar{R}_i F(Q^*)] = (p+u)\bar{R}_i$$

which contradicts with $w_i < (p+u)\bar{R}_i$. Hence, there must exists some supplier j with $q_i^* > 0$.

Q.E.D.

Proof Corollary 2.2. By lemma 2.4, there must be some supplier who receives positive ordering quantity. Suppose supplier i get 0 ordering quantity while supplier j,$j \neq i$ get a positive ordering quantity, since $w_i/\bar{R}_i < w_j/\bar{R}_j$, then by Lemma 2.3, q_i should be positive which leads to a contradiction. Hence, supplier i must get a positive ordering quantity.

Q.E.D.

Proof of Lemma 2.5. First, since $R_j \leq 1$,for all $j = 1, \ldots, n$, we have

$$\frac{E[R_j F(Q)]}{\bar{R}_j} \leq \frac{E[R_j F(\sum_i q_i)]}{\bar{R}_j} = F(\sum_i q_i). \qquad \text{(A.8)}$$

By lemma 2.1, in optimal decision we have

$$w_j \geq \bar{R}_j(p+u) - (p+u-s)E[R_jF(Q^*)]$$

$$\begin{aligned}
\Rightarrow \quad & F(\sum_i q_i^*) \geq \frac{\bar{R}_j(p+u) - w_j}{\bar{R}_j(p+u-s)} \qquad \text{for all } j \\
\Rightarrow \quad & \sum_i q_i^* \geq F^{-1}(\frac{p+u-w_j/\bar{R}_j}{p+u-s}) \qquad \text{for all } j \\
\Rightarrow \quad & \sum_i q_i^* \geq F^{-1}(\frac{p+u-w}{p+u-s})
\end{aligned}$$

where $F^{-1}(\frac{p+u-w}{p+u-s})$ is the optimal ordering quantity for the manufacturer ordering from a single supplier with full reliability $R = 1$ and wholesale price $w = min_{j=1}^n w_j/R_j$.

Q.E.D.

Proof of Lemma 2.6 We prove the strictly concavity at the optimal point by showing that the hessian at optimal point is positive definite .

By the assumption, $0 < F(x) < 1$ for any $x \in (0, b)$. If $w_i \geq \bar{R}_i(p+u)$ for all i, suppose there exists some j with $q_j^* > 0$, then $E[R_jF(Q^*)] > 0$. However, check the **KKT** condition in corollary 2.1,

$$w_j = \bar{R}_j(p+u) - (p+u-s)E[R_jF(Q^*)] < \bar{R}_j(p+u)$$

which leads to a contradiction. Thus, $q_i^* = 0$ for all i is unique optimal solution.

Otherwise, suppose there exits some i with $w_i < \bar{R}_i(p+u)$, then by lemma 2.4, in the manufacturer's optimal decision, there must exits a supplier j with positive ordering quantity, from corollary 2.1:

$$w_j = \bar{R}_j(p+u) - (p+u-s)E[R_jF(Q^*)].$$

Since $s\bar{R}_j < w_j$, $E[R_jF(Q^*)] < \bar{R}_j$. In addition, $0 < E[R_jF(Q^*)]$ because $w_j < \bar{R}_j(p+u)$. Note that R_i, $\forall i = 1, \ldots, n$, is continuously distributed random variable supported on $[0, 1]$, thus $P(Q^* = 0) = 0$ and $P(Q^* \in (0, b)) > 0$. Hence, there exists a set $\mathcal{S} \subset \mathcal{R}^n$ such that

$P(\mathbf{R} \in \mathcal{S}) > 0$ and for all $\mathbf{r} \in S$, $f(\sum_i q_i^* r_i) > 0$. Consequently, for any $\mathbf{x} \neq 0$ with the hessian H at the optimal point:

$$\mathbf{xHx}' = -(p+u-s)E[(\sum_{i=1}^{n} x_i R_i)^2 f(Q)].$$

Note that $(\sum_i x_i r_i)^2 = 0$ only if $\sum_i x_i r_i = 0$, which is defined by a hyperplane $\mathcal{T}$ in an n dimensional space. Furthermore, since R_i are independent random variables with a continuous distribution, $P(\mathbf{R} \in \mathcal{T}) = 0$ and $P(\mathbf{R} \in \mathcal{S} \setminus \mathcal{T}) > 0$. Since $f(\sum_i q_i^* r_i) > 0$ and $(\sum_i x_i r_i)^2 > 0$ in the set $\mathcal{S} \setminus \mathcal{T}$,

$$\mathbf{xHx}' = -(p+u-s)E[(\sum_{i=1}^{n} x_i R_i)^2 f(Q)] < -(p+u-s)\int_{\mathbf{r}\in\mathcal{S}\setminus\mathcal{T}} (\sum_i x_i R_i)^2 f(\sum_i q_i^* R_i) dG(\mathbf{r}) < 0.$$

Q.E.D.

Proof of Lemma 2.7. Since the expected profit is strictly joint concave at the maximal point, the optimal ordering quantity is unique and can be represented by a multi-value function from $\mathbf{w}$. Also by theorem 2.2.13 in Fiacco 1983 [14], since the expected profit function is jointly continuous in $(\mathbf{q}, \mathbf{w})$, linear independence of boundary on q_i, strict complementary slackness and the second-order sufficiency conditions hold at any optimal point $\mathbf{q}^*$, $\mathbf{q}^*$ is continuous in $\mathbf{w}$.

Q.E.D.

Proof of Theorem 2.1. Since the the profit function is strictly concave at the optimal point, the optimal ordering quantity is continuous in the wholesale price. We only have to show that for both of the cases $q_i^* = 0$ and $q_i^* > 0$, our result holds. Therefore, by continuity, the monotonicity result holds for all $q_i^* \geq 0$.

First, for any supplier i who receives optimal ordering quantity $q_i^* = 0$, in increasing w_i, the **KKT** optimal condition still holds under current $\mathbf{q}^*$. This means that the optimal ordering quantity does not change. Otherwise $q_i^* > 0$, then equality 2.2 holds for all the supplers in the set $\mathcal{S} = \{j \in \{1..n\} | q_j^* > 0\}$. By implicity function theorem, we have:

$$\frac{\partial q_i^*}{\partial w_i} = [\mathbf{H}_{\mathcal{S}}^{-1}]_{ii}$$

where $\mathbf{H}_{\mathcal{S}}$ is the hessian of the expected profit function involved only suppliers in the set $\mathcal{S}$ at the optimal point. Since $E[\pi]$ is concave, $[\mathbf{H}_{\mathcal{S}}^{-1}]_{ii}$ is non positive. Therefore, q_i^* is non increasing with w_i.

When $n = 2$, without loss of generality, let $i = 1$. If $q_1^* = 0$, the **KKT** optimal condition still holds when w_1 is increased from the original optimal solution, hence q_2^* is nondecreasing with w_1. In case $q_1^* > 0$, since $q_2^* \geq 0$ for any w_1, we only have to consider the case $q_2^* > 0$. In this case, equality 2.2 holds for both suppliers, and by the implicity function theorem, we have:

$$\frac{\partial q_2^*}{\partial w_1} = \frac{(p + u - s)E[R_1 R_2 f(Q)]}{\det \mathbf{H}} > 0.$$

Q.E.D.

Proof of Theorem 2.2. In case $q_i^* = 0$, our result always holds. We only have to consider the case $q_i^* > 0$. Note that the density distribution of demand at δ is $f_\delta(d) = f(d - \delta)$, since f is log-concave, $f_\delta(d)$ is also log-concave. To show that the optimal ordering quantity is increasing with δ, it is equivalent to show $\frac{\partial q_i^*}{\partial \delta}|_{c=0} \geq 0$. Without loss of generality, let $i = 1$, then

$$\begin{aligned}\frac{\partial q_1^*}{\partial \delta} &= -\frac{E[R_2^2 f(Q^* - \delta)]E[R_1 f(Q^* - \delta)] - E[R_1 R_2 f(Q^* - \delta)]E[R_2 f(Q^* - \delta)]}{\det \mathbf{H}} \\ &= -\frac{1}{q_2^{*2} q_1^* \det \mathbf{H}}(E[(R_2 q_2^*)^2 f_\delta(Q^*)]E[R_1 q_1^* f_\delta(Q^*)] - E[R_1 q_1^* R_2 q_2^* f_\delta(Q^*)]E[R_2 q_2^* f_\delta(Q^*)]).\end{aligned}$$

By lemma A.1, since $f_\delta(d)$ is log-concave, we have

$$E[(R_2 q_2^*)^2 f_\delta(Q^*)]E[R_1 q_1^* f_\delta(Q^*)] - E[R_1 q_1^* R_2 q_2^* f_\delta(Q^*)]E[R_2 q_2^* f_\delta(Q^*) \geq 0.$$

In addition, $\det \mathbf{H} \leq 0$ because the expected profit function is concave. Combine the above results together, we have $\frac{\partial q_i^*}{\partial \delta} \geq 0$.

Q.E.D.

Proof of Lemma 2.8. Since supplier 1 and 2 are identical, $q_1^* = q_2^* = q^*$ in optimal decision. In case $q_i^* = 0$, our result always holds. We only consider the case $q_i^* > 0$ for $i = 1, 2$. By **KKT** condition, we have:

$$w_1 + w_2 = (p+u)(\bar{R}_1 + \bar{R}_2) - (p+u-s)E[(R_1+R_2)F(q^*(R_1+R_2))].$$

Let $\frac{R_1+R_2}{2} = R$, then

$$w_1 = (p+u)\bar{R} - (p+u-s)E[RF(2q^*R)].$$

In addition, since

$$\frac{\partial^2 rF(qr)}{\partial r^2} = q(2f(qr) + qrf'(qr)),$$

if $2f(x) \geq (\leq) - xf'(x)$, then $rF(qr)$ is convex (concave).

Note that R_1 and R_2 are identical independent random variables, therefore, $R_1 \geq_{cx} \frac{R_1+R_2}{2} = R$ (R_1 is larger than R in convex order). Now consider the scenario that only supplier 1 remains in the system and the optimal ordering quantity is q_1'.

First, consider the case that $2f(x) \geq -xf'(x)$ and suppose $q_1' > 2q^*$:

$$w_1 = (p+u)\bar{R}_1 - (p+u-s)E[R_1F(q_1'R_1)] = (p+u)\bar{R} - (p+u-s)E[RF(2q^*R)]$$

Therefore $E[R_1F(q_1'R_1)] = E[RF(2q^*R)]$. But by the monotonicity of $rF(rq)$, we have

$$E[R_1F(q_1'R_1)] > E[R_1F(2q^*R_1)].$$

In this case $rf(rq)$ is convex. By the convexity order of R and R_1,

$$E[R_1F(2q^*R_1)] \geq E[RF(2q^*R)].$$

Thus, $E[R_1F(q_1'R_1)] > E[RF(2q^*R)]$, which leads to a contradiction. Therefore $q_1' \leq 2q^*$.

Second, consider the case that $2f(x) \leq -xf'(x)$ and suppose $q_1' < 2q^*$:

$$w_1 \geq (p+u)\bar{R}_1 - (p+u-s)E[R_1F(q_1'R_1)].$$

Therefore $E[R_1F(q_1'R_1)] \geq E[RF(2q^*R)]$. By the monotonicity of $rF(rq)$,

$$E[R_1F(q_1'R_1)] < E[R_1F(2q^*R_1)]$$

In this case $rf(rq)$ is concave. By the convexity order of R and R_1,

$$E[R_1F(2q^*R_1)] \leq E[RF(2q^*R)].$$

Therefore $E[R_1F(q_1'R_1)] < E[RF(2q^*R)]$, which leads to a contradiction. Hence, $q_1' \geq 2q^*$.

Q.E.D.

Proof of Theorem 2.3. Consider the following two case:

Case 1: $q_j^* = 0$. $R_i \leq_{rh} R_j \Rightarrow \bar{R}_i \leq \bar{R}_j$. $q_i^* = 0 \leq q_j^*$ because of lemma 2.3 .

Case 2: $q_j^* > 0$ and $q_i^* > 0$. Since $w_i > w_j$ and $\bar{R}_i \leq \bar{R}_j$, by the result of lemma 2.1 and 2.3, the following inequality holds:

$$\begin{aligned} w_i = \bar{R}_i(p+u) - (p+u-s)E[R_iF(Q^*)] &> w_j = \bar{R}_j(p+u) - (p+u-s)E[R_jF(Q^*)] \\ &\Rightarrow \quad E[R_iF(Q^*)] < E[R_jF(Q^*)] \end{aligned} \tag{A.9}$$

Now suppose $q_i^* > q_j^*$, define $Q_{ij^c}^* = \sum_{k\neq i,j} q_k^* R_k$. Consider the following two functions:

$$\begin{aligned} \phi_1(r_i, r_j) &= r_j E_{R_k, k\neq i,j}[F(Q_{ij^c}^* + q_i^* r_i + q_j^* r_j)] \\ \phi_2(r_i, r_j) &= r_i E_{R_k, k\neq i,j}[F(Q_{ij^c}^* + q_i^* r_i + q_j^* r_j)] \\ \Delta\phi_{21}(r_i, r_j) &= (r_i - r_j) E_{R_k, k\neq i,j}[F(Q_{ij^c}^* + q_i^* r_i + q_j^* r_j)]. \end{aligned}$$

The following inequalities hold when $r_i \leq r_j$:

$$\begin{aligned} \frac{\partial \Delta\phi_{21}(r_i, r_j)}{\partial r_j} &= q_j^*(r_i - r_j) E_{R_k, k\neq i,j}[f(Q_{ij^c}^* + q_i^* r_i + q_j^* r_j)] - E_{R_k, k\neq i,j}[F(Q_{ij^c}^* + q_i^* r_i + q_j^* r_j)] \leq 0 \\ \Delta\phi_{21}(r_i, r_j) &= (r_i - r_j) E_{R_k, k\neq i,j}[F(Q_{ij^c}^* + q_i^* r_i + q_j^* r_j)] \\ &\leq -(r_j - r_i) E_{R_k, k\neq i,j}[F(Q_{ij^c}^* + q_i^* r_j + q_j^* r_i)] = -\Delta\phi_{21}(r_j, r_i). \end{aligned}$$

By Theorem 1.B.48 [40], $E_{R_i,R_j}[\phi_1(R_i, R_j)] \leq E_{R_i,R_j}[\phi_2(R_i, R_j)]$ holds when R_i and R_j are two independent random variables and $R_i \leq_{rh} R_j$. Therefore,

$$E[R_iF(Q^*)] = E_{R_i,R_j}[\phi_2(R_i, R_j)] \geq E_{R_i,R_j}[\phi_1(R_i, R_j)] = E[R_jF(Q^*)].$$

which leads to a contradiction with inequality A.9. Hence, $q_i^* \leq q_j^*$.

Q.E.D.

Proof of Theorem 2.9. Since $\mathcal{W}_{-i}^l$ and $\mathcal{W}_{-i}^u$ are both closed intervals and at $w_{-i} = P_{-i}(1-P_i)$, $\mathcal{W}_i^3 = \mathcal{W}_i^1 = 0$ and $\mathcal{W}_i^{2l} = \mathcal{W}_i^{2u}$, it is enough for us to show $w_i^b(w_{-i})$ is monotone nondecreasing in $\mathcal{W}_{-i}^l$ and $\mathcal{W}_{-i}^u$.

In case $w_{-i} \in \mathcal{W}_{-i}^u$, since both $\pi_{si}^2(w_i|w_{-i})$ and $\pi_{si}^3(w_i|w_{-i})$ are strictly concave and continuous in w_{-i}, in addition, the lower bound and upper bounds for $\mathcal{W}_i^3$ and $\mathcal{W}_i^{2u}$ are linear functions of w_{-i}, the local maximizer of $\pi_{si}^2(w_i|w_{-i})$ in $\mathcal{W}_i^{2u}$ and $\pi_{si}^3(w_i|w_{-i})$ in $\mathcal{W}_i^3$ are unique and continuous in w_{-i}.

For $w_i \in \mathcal{W}_i^{2u}$, the lower bound $\frac{w_{-i}}{P_{-i}} - (1-P_i)$ and the upper bound $P_i(1-P_{-i}+w_{-i})$ are both increasing in w_{-i}. Furthermore,

$$\frac{\partial^2 \pi_{si}^2(w_i|w_{-i})}{\partial w_i \partial w_{-i}} = \frac{b(w_i - c_i)}{P_i(1-P_iP_{-i})}. \tag{A.10}$$

Note that if the local maximal point for $\pi_{si}^2(w_i|w_{-i})$ in $\mathcal{W}_i^{2u}$ is an interior point w_i^o, then

$$\begin{aligned}
& \pi_{si}^2(w_i^o|w_{-i}) = \frac{b(P_i(1-P_{-i}+w_{-i}) - w_i^o)(w_i^o - c_i)}{P_i(1-P_iP_{-i})} \\
\geq \;& \pi_{si}^2(P_i(1-P_{-i}+w_{-i})|w_{-i}) = 0 \\
\Rightarrow \;& w_i^o - c_i \geq 0 \\
\Rightarrow \;& \frac{\partial w_i^o}{\partial w_{-i}} = -\frac{\frac{\partial^2 \pi_{si}^2(w_i|w_{-i})}{\partial w_i \partial w_{-i}}}{\frac{\partial^2 \pi_{si}^2(w_i|w_{-i})}{\partial w_i^2}}\Big|_{w_i = w_i^o} = w_i^o - c_i \geq 0.
\end{aligned} \tag{A.11}$$

Therefore, the local maximizer of $\pi_{si}^2(w_i|w_{-i})$ in $\mathcal{W}_i^{2u}$ is nondecreasing in w_{-i}.

Let $q_{-i}^o(w_i^o(w_{-i}), w_{-i})$ be the ordering quantity for supplier $-i$ when the local maximizer for $\pi_{si}^2(w_i|w_{-i})$ is an interior point, then

$$q_{-i}^o(w_i^o(w_{-i}), w_{-i}) = b\frac{c_i - P_i(1-P_{-i}) - P_iw_{-i}}{2P_i(1-P_iP_{-i})}.$$

Clearly, $q_{-i}^o\big(w_i^o(w_{-i}), w_{-i}\big)$ is monotone decreasing with w_{-i}. Therefore, if $w_{-i} = P_{-i}(1-P_i)$, the local maximizer of $\pi_{si}^2(w_i|w_{-i})$ leads to 0 ordering quantity from supplier $-i$ (w_i set at its lower bound in $\mathcal{W}_i^{2u}$). As a result, with the increasing of w_{-i}, the local maximizer is always $w_i = \frac{w_{-i}}{P_{-i}} - (1-P_i)$. Hence, the best response is the local maximizer of $\pi_{si}^3(w_i|w_{-i})$ in $\mathcal{W}_i^3$. Otherwise, since $q_{-i}^o\big(w_i^o(w_{-i}), w_{-i}\big)$ is strictly positive and linearly decrease with w_{-i} at $w_{-i} = P_{-i}(1-P_i)$, thus $q_{-i}^o\big(w_i^o(w_{-i}), w_{-i}\big) = 0$ at $w'_{-i} = \frac{c_i - P_i(1-P_{-i})}{2(1-P_iP_{-i})}$. For $w_{-i} \geq w'_{-i}$, the local maximizer of $\pi_{si}^2(w_i|w_{-i})$ in $\mathcal{W}_i^{2u}$ is its lower bound and hence the local maximizer of $\pi_{si}^3(w_i|w_{-i})$ in $\mathcal{W}_i^3$ is the best response. For $w_{-i} < w'_{-i}$, we know:

$$\begin{aligned}
&\frac{\partial \pi_{si}^2(w_i|w_{-i})}{\partial w_i}\Big|_{w_i=\frac{w_{-i}}{P_{-i}}-(1-P_i)} = b\Big(\frac{c_i+P_i-2w_i}{P_i(1-P_{-i})}\Big|_{w_i=\frac{w_{-i}}{P_{-i}}-(1-P_i)} - \frac{P_{-i}-w_{-i}}{1-P_{-i}}\Big) \leq 0 \\
\Rightarrow\ &\frac{\partial \pi_{si}^2(w_i|w_{-i})}{\partial w_i}\Big|_{w_i=\frac{w_{-i}}{P_{-i}}-(1-P_i)} = b\frac{c_i+P_i-2w_i}{P_i}\Big|_{w_i=\frac{w_{-i}}{P_{-i}}-(1-P_i)} \\
&= \Big(\frac{\partial \pi_{si}^2(w_i|w_{-i})}{\partial w_i}\Big|_{w_i=\frac{w_{-i}}{P_{-i}}-(1-P_i)} + b\frac{P_{-i}-w_{-i}}{1-P_{-i}}\Big)(1-P_{-i}) \geq 0
\end{aligned}$$

which means the local maximizer of $\pi_{si}^3(w_i|w_{-i})$ in $\mathcal{W}_i^3$ is always at its upper bound and therefore, the best response $w_i^b(w_{-i})$ is the local maximizer of $\pi_{si}^2(w_i|w_{-i})$ in $\mathcal{W}_i^{2u}$. Consequently, by the monotone increasing property of the local maximizer of $\pi_{si}^2(w_i|w_{-i})$ in $\mathcal{W}_i^{2u}$ and $\pi_{si}^3(w_i|w_{-i})$ in $\mathcal{W}_i^3$, the best response is monotone increasing.

In case $w_{-i} \in \mathcal{W}_{-i}^l$, we consider the following two cases:

case a): $c_i \geq P_i(1-P_{-i})$

For $w_i \in \mathcal{W}_i^1$, $w_i \leq P_i - P_iP_{-i}$, thus

$$\frac{\partial \pi_{si}^1(w_i|w_{-i})}{\partial w_i} = b\frac{c_i+P_i-P_iP_{-i}-2w_i}{P_i-P_iP_{-i}} \geq 0.$$

The best response $w_i^b(w_{-i})$ is the local maximizer of $\pi_{si}^2(w_i|w_{-i})$ in $\mathcal{W}_i^{2l}$. Also note that for $w_i \in \mathcal{W}_i^{2l}$, $w_i \leq P_i - P_iP_{-i}$, hence,

$$\frac{\partial \pi_{si}^2(w_i|w_{-i})}{\partial w_i} = b\frac{c_i+P_i-P_iP_{-i}-2w_i+P_iw_{-i}}{P_i(1-P_iP_{-i})} \geq 0.$$

Therefore, the local maximizer for $\pi_{si}^2(w_i|w_{-i})$ in $\mathcal{W}_i^{2l}$ is either the interior point which is increasing with w_{-i} by inequality A.11 or the upper bound of $\mathcal{W}_i^{2l}$ which is also increasing in w_{-i}. Thus, by the continuous property of the local maximizer of $\pi_{si}^2(w_i|w_{-i})$ in $\mathcal{W}_i^{2l}$, the best response $w_i^b(w_{-i})$ is monotone nondecreasing.

case b): $c_i < P_i(1 - P_{-i})$

In this case

$$\frac{\partial \pi_{si}^2(w_i|w_{-i})}{\partial w_i}|_{w_i=P_i(1-P_{-i}+w_{-i})} = b\frac{c_i - P_i(1-P_{-i}) - P_i w_{-i})}{P_i(1-P_iP_{-i})} < 0$$

which means the local optimal point of $\pi_{si}^2(w_i|w_{-i})$ is either at $P_i(1-P_{-i}) - w_{-i}\frac{P_i(1-P_{-i})}{P_{-i}(1-P_i)}$, the lower bound of $\mathcal{W}_i^{2l}$ or $w_i^o(w_{-i})$, the interior point satisfying the first ordering condition for π_{si}^2. Also note that

$$\begin{aligned}
\frac{\partial \pi_{si}^2(w_i|w_{-i})}{\partial w_i} &= b\frac{c_i - 2w_i + P_i(1-P_{-i}+w_{-i})}{P_i(1-P_iP_{-i})} = 0 \\
\Rightarrow \quad w_i^o &= \frac{1}{2}\big(c_i + P_i(1-P_{-i}+w_{-i})\big) \\
\Rightarrow \quad w_i^o|_{w_{-i}=P_{-i}(1-P_i)} &= \frac{1}{2}(c_i + P_i - P_i^2 P_{-i}).
\end{aligned}$$

Clearly $w_i^o|_{w_{-i}=P_{-i}(1-P_i)} \geq 0$. Since $c_i < P_i(1-P_{-i})$, we have:

$$w_i^o|_{w_{-i}=P_{-i}(1-P_i)} = \frac{1}{2}(c_i + P_i - P_i^2P_{-i}) < P_i - P_i^2 P_{-i} = P_i(1-P_{-i}+w_{-i})|_{w_{-i}=P_{-i}(1-P_i)}$$

Thus $w_i^o|_{w_{-i}=P_{-i}(1-P_i)} \in \mathcal{W}_i^{2l}$. In addition, because $w_i^o(w_{-i})$ is increasing with w_{-i} and the lower bound of $\mathcal{W}_i^{2l}$ is decreasing with w_{-i}, there exists $\widehat{w}_{-i}$ such that for $w_{-i} \leq \widehat{w}_{-i}$, the local maximal of $\pi_{si}^2(w_i|w_{-i})$ is the lower bound of $\mathcal{W}_i^{2l}$, and for $w_{-i} \geq \widehat{w}_{-i}$, the local maximal of $\pi_{si}^2(w_i|w_{-i})$ is $w_i^o(w_{-i})$. Especially, at $w_{-i} = \widehat{w}_{-i}$, the lower bound of $\mathcal{W}_i^{2l}$ satisfies the first order condition. Also check:

$$\begin{aligned}
\frac{\partial \pi_{si}^1(w_i|w_{-i})}{\partial w_i} &= b\frac{c_i + P_i(1-P_{-i}) - 2w_i}{P_i(1-P_{-i})} = \frac{\frac{\partial \pi_{si}^2(w_i|w_{-i})}{\partial w_i}(1-P_iP_{-i}) - P_i w_{-i}}{1-P_i} \\
\Rightarrow \frac{\partial \pi_{si}^1(w_i|w_{-i})}{\partial w_i}|_{w_i=0} &= b\frac{c_i + P_i(1-P_{-i})}{P_i(1-P_{-i})} \geq 0 \qquad \text{And} \\
\frac{\partial \pi_{si}^1(w_i|w_{-i})}{\partial w_i}|_{w_i=w_i^o(\widehat{w}_{-i}), w_{-i}=\widehat{w}_{-i}} &= \frac{-P_i \widehat{w}_{-i}}{1-P_i} < 0
\end{aligned}$$

Combining with the fact that $\pi_{si}^1(w_i|w_{-i})$ is a strictly concave function only in w_i and the upper bound of its domain $\mathcal{W}_i^1$ is monotone increasing with w_{-i}, we know the following facts: i) For $w_{-i} \leq \widehat{w}_{-i}$, the best response $w_i^b(w_{-i})$ is actually the local maximizer of $\pi_{si}^1(w_i|w_{-i})$, which is the interior point of $\mathcal{W}_i^1$ and is a constant with respect to w_{-i}.

ii) For $w_{-i} > \widehat{w}_{-i}$, the local maximal of $\pi^1_{si}(w_i|w_{-i})$ is monotone non-increasing with w_{-i}. Furthermore, at $w_{-i} = P_{-i}(1-P_i)$, the local maximal of $\pi^1_{si}(w_i|w_{-i})$ is strictly less than the local maximal of $\pi^2_{si}(w_i|w_{-i})$ in $\mathcal{W}^{2l}_i$.

Therefore, let us check:

$$\frac{\partial \pi^2_{si}(w_i|w_{-i})|_{w_i=w^o_i}}{\partial w_{-i}} = -b\frac{c_i - P_i(1-P_{-i}) - P_i w_{-i}}{2(1-P_iP_{-i})} > 0$$

which means the local maximal of $\pi^2_{si}(w_i|w_{-i})$ in $\mathcal{W}^{2l}_i$ is strictly increasing in w_{-i} when $w_{-i} \in (\widehat{w}_{-i}, P_{-i}(1-P_i))$. Thus there must exist a $\widetilde{w}_{-i} \in (\widehat{w}_{-i}, P_{-i}(1-P_i))$, such that for $w_{-i} \in [\widetilde{w}_{-i}, P_{-i}(1-P_i)]$, the global optimal is the local maximal of $\pi^2_{si}(w_i|w_{-i})$ in $\mathcal{W}^{2l}_i$, and for $w_{-i} \in (\widehat{w}_{-i}, \widetilde{w}_{-i}]$, the global optimal is the local maximal of $\pi^1_{si}(w_i|w_{-i})$ in $\mathcal{W}^1_i$. In addtion, in this region:

$$\frac{\partial \pi^2_{si}(w_i|w_{-i})}{\partial w_i}\Big|_{w_i=P_i(1-P_{-i})-w_{-i}\frac{P_i(1-P_{-i})}{P_{-i}(1-P_i)}} > 0.$$

Since the global maximal is in $\mathcal{W}^1_i$, we must have

$$\frac{\partial \pi^1_{si}(w_i|w_{-i})}{\partial w_i}\Big|_{w_i=P_i(1-P_{-i})-w_{-i}\frac{P_i(1-P_{-i})}{P_{-i}(1-P_i)}} < 0.$$

Therefore, for all $w_{-i} \in [0, \widetilde{w}_{-i}]$, the best response $w^b_i(w_{-i})$ is the local maximizer of $\pi^1_{si}(w_i|w_{-i})$, which is the interior point in $\mathcal{W}^1_i$ and a constant with respect to w_{-i}. Also for all $w_{-i} \in [\widetilde{w}_{-i}, P_{-i}(1-P_i)]$, the best response is the local maximizer of $\pi^2_{si}(w_i|w_{-i})$ which is an interior point in $\mathcal{W}^{2l}_i$ and increasing with w_{-i}. Finally, at $w_{-i} = \widetilde{w}_{-i}$, the best response is a set of two points: the smaller one is the local maximizer of $\pi^1_{si}(w_i|w_{-i})$ and the larger one is the local maximizer of $\pi^2_{si}(w_i|w_{-i})$. Therefore, in the domain $w_{-i} \in \mathcal{W}^l_{-i}$, $w^b_i(w_{-i})$ is monotone non decreasing.

Combine the results of case a and case b, we know $w^b_i(w_{-i})$ is an increasing correspondence from $\mathcal{W}_{-i}$ to $\mathcal{W}_i$.

Q.E.D.

Proof of Lemma 2.10. Let $\mathcal{S} = \{\mathbf{x} : \mathbf{x} \in \mathcal{X}, f(\mathbf{x}) \geq \mathbf{x}\}$. Since $(l_1, \ldots, l_n) \in \mathcal{X}$ and $f(l_1, \ldots, l_n) \geq (l_1, \ldots, l_n)$, $\mathcal{S} \neq \emptyset$. Let $\mathbf{x^u} = \sup_{\mathcal{X}} \mathcal{S}$, which is the least upper bound of $\mathcal{S}$ in

$\mathcal{X}$. Hence $\forall \mathbf{x} \in \mathcal{S}$, by the increasing property of $f(\mathbf{x})$, we have $\mathbf{x} \leq \mathbf{x^u}$ and $\mathbf{x} \leq f(\mathbf{x}) \leq f(\mathbf{x^u})$. Therefore $f(\mathbf{x^u})$ is an upper bound of $\mathcal{S}$. Note that $\mathbf{x^u}$ is least upper bound of $\mathcal{S}$, we conclude that:

$$\mathbf{x^u} \leq f(\mathbf{x^u})$$

Thus by the nondecreasing property of $f(\mathbf{x})$:

$$f(\mathbf{x^u}) \leq f(f(\mathbf{x^u}))$$

Therefore, $f(\mathbf{x^u}) \in \mathcal{S}$. Since $\mathbf{x^u}$ is the least upper bound of $\mathcal{S}$, $f(\mathbf{x^u}) \leq \mathbf{x^u}$. Combine with A.12, we have $f(\mathbf{x^u}) = \mathbf{x^u}$ and consequently, the set of the fixed points of f is nonempty.

Q.E.D.

Proof Lemma 2.11. For any given w_{-i}, since $x_{-i} = \frac{w_{-i}}{P_{-i}(1-P_i(1-x_i))}$ for $x_i \in \mathcal{X}_i^u$, we can rewrite $w_i(x_i)$ as:

$$w_i(x_i) = w_i^1(x_i) = P_i(1 - P_{-i} + \frac{w_{-i}}{1 - P_i(1 - x_i)})x_i.$$

Note that:

$$\begin{aligned}
\frac{\partial w_i(x_i)}{\partial x_i} &= P_i(1 - P_{-i} + w_{-i}\frac{1 - P_i}{(1 - P_i(1 - x_i))^2}) > 0 \\
\frac{\partial^2 w_i(x_i)}{\partial x_i^2} &= -\frac{2(1 - P_i)P_i^2 w_{-i}}{(1 - P_i(1 - x_i))^3} \leq 0
\end{aligned}$$

Therefore π_{si}^1 is either monotone in its feasible interval (hence the function is quasi-concave) or there exists some point x_i^{1o} such that the first order condition is satisfied:

$$\frac{\partial \pi_{si}^1}{\partial x_i}|_{x_i = x_i^{1o}} = -\lambda(\frac{w_i(x_i^{1o}) - c_i}{x_i^{1o}} + \frac{\partial w_i(x_i)}{\partial x_i}|_{x_i = x_i^{1o}} \log x_i^{1o}) = 0$$

At $x_i = x_i^{1o}$, check the hessian of $\pi_{si}^2(x_i)$ while the equation A.12 is satisfied :

$$\begin{aligned}
\frac{\partial^2 \pi_{si}^1}{\partial x_i^2}|_{x_i = x_i^{1o}} &= -\lambda(-\frac{w_i(x_i^o) - c_i}{x_i^{1o2}} + 2\frac{w_i(x_i^{1o})}{x_i^{1o}} + \frac{\partial^2 w_i(x_i)}{\partial x_i^2}|_{x_i = x_i^{1o}} \log x_i^{1o}) \\
&= -\lambda(\frac{\partial w_i(x_i)}{\partial x_i}|_{x_i = x_i^{1o}} \frac{2 + \log x_i^{1o}}{x_i^{1o}} + \frac{\partial^2 w_i(x_i)}{\partial x_i^2}|_{x_i = x_i^{1o}} \log x_i^{1o}).
\end{aligned}$$

Since $\frac{\partial^2 w_i(x_i)}{\partial x_i^2} \leq 0$, $x_i^{1o} \geq e^{-1} > e^{-2}$, we have $\frac{\partial^2 \pi_{si}^1}{\partial x_i^2}|_{x_i=x_i^{1o}} < 0$. Therefore, π_{si}^1 is strictly quasi-concave in x_i. When the first order condition is satisfied, let:

$$h(x_i^{1o}, w_{-i}) = \frac{w_i(x_i^{1o}) - c_i}{x_i^{1o}} + \frac{\partial w_i(x_i)}{\partial x_i}|_{x_i=x_i^{1o}} \log x_i^{1o}$$

Hence

$$\frac{\partial x_i^{1o}}{\partial w_{-i}} = -\frac{\frac{\partial h(x_i^{1o}, w_{-i})}{\partial w_{-i}}}{\frac{\partial h(x_i^{1o}, w_{-i})}{\partial x_i^{1o}}}$$

where

$$\begin{aligned}
\frac{\partial h(x_i^{1o}, w_{-i})}{\partial w_{-i}} &= \frac{P_i((1-P_i)(1+\log x_i^{1o}) + P_i x_i^{1o})}{(1-P_i(1-x_i^{1o}))^2} \geq 0 \\
\frac{\partial h(x_i^{1o}, w_{-i})}{\partial x_i^{1o}} &= -\frac{1}{\lambda}\frac{\partial^2 \pi_{si}^1}{\partial x_i^2}|_{x_i=x_i^{1o}} > 0
\end{aligned}$$

Therefore $\frac{\partial x_i^{1o}}{\partial w_{-i}} \leq 0$.

Since $w_i^{1o} = P_i(1 - P_{-i} + \frac{w_{-i}}{1-P_i(1-x_i)})x_i$, thus:

$$\frac{\partial w_i^{1o}}{\partial w_{-i}} = \frac{P_i x_i^{1o}(w_{-i})}{1-P_i(1-x_i^{1o}(w_{-i}))} + \frac{\partial w_i}{\partial x_i}|_{x_i=x_i^{1o}(w_{-i})} \frac{\partial x_i^{1o}(w_{-i})}{\partial w_{-i}}.$$

Note that

$$\begin{aligned}
-\frac{1}{\lambda}\frac{\partial^2 \pi_{si}^1}{\partial x_i^2}|_{x_i=x_i^{1o}} &= \frac{\partial w_i(x_i)}{\partial x_i}|_{x_i=x_i^{1o}} \frac{2+\log x_i^{1o}}{x_i^{1o}} + \underbrace{\frac{\partial^2 w_i(x_i)}{\partial x_i^2}|_{x_i=x_i^{1o}} \log x_i^{1o}}_{\geq 0} \\
&\geq \frac{\partial w_i(x_i)}{\partial x_i}|_{x_i=x_i^{1o}} \frac{2+\log x_i^{1o}}{x_i^{1o}}
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{\partial w_i}{\partial x_i}|_{x_i=x_i^{1o}(w_{-i})} \frac{\partial x_i^{1o}(w_{-i})}{\partial w_{-i}} &= -\frac{\partial w_i}{\partial x_i}|_{x_i=x_i^{1o}(w_{-i})} \frac{\frac{\partial h(x_i^{1o}, w_{-i})}{\partial w_{-i}}}{\frac{\partial h(x_i^{1o}, w_{-i})}{\partial x_i^{1o}}} \\
&= -\frac{\partial w_i}{\partial x_i}|_{x_i=x_i^{1o}(w_{-i})} \frac{\frac{P_i((1-P_i)(1+\log x_i^{1o})+P_i x_i^{1o})}{(1-P_i(1-x_i^{1o}))^2}}{-\frac{1}{\lambda}\frac{\partial^2 \pi_{si}^1}{\partial x_i^2}|_{x_i=x_i^{1o}}} \\
&\geq -\frac{\partial w_i}{\partial x_i}|_{x_i=x_i^{1o}(w_{-i})} \frac{\frac{P_i((1-P_i)(1+\log x_i^{1o})+P_i x_i^{1o})}{(1-P_i(1-x_i^{1o}))^2}}{\frac{\partial w_i(x_i)}{\partial x_i}|_{x_i=x_i^{1o}}} \frac{x_i^{1o}}{2+\log x_i^{1o}} \\
&= -\frac{P_i((1-P_i)(1+\log x_i^{1o}) + P_i x_i^{1o})\frac{x_i^{1o}}{2+\log x_i^{1o}}}{(1-P_i(1-x_i^{1o}))^2}
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{\partial w_i^{1o}}{\partial w_{-i}}|_{x_i=x_i^{1o}} &\geq \frac{P_i x_i^{1o}}{1-P_i(1-x_i^{1o})} - \frac{P_i((1-P_i)(1+\log x_i^{1o})+P_i x_i^{1o})\frac{x_i^{1o}}{2+\log x_i^{1o}}}{(1-P_i(1-x_i^{1o}))^2} \\
&= P_i \frac{x_i^{1o}(1-P_i(1-x_i^{1o})) - ((1-P_i)(1+\log x_i^{1o})+P_i x_i^{1o})\frac{x_i^{1o}}{2+\log x_i^{1o}}}{(1-P_i(1-x_i^{1o}))^2}
\end{aligned}$$

Note that

$$\begin{aligned}
&\frac{\partial x_i^{1o}(1-P_i(1-x_i^{1o})) - ((1-P_i)(1+\log x_i^{1o})+P_i x_i^{1o})\frac{x_i^{1o}}{2+\log x_i^{1o}}}{\partial x_i^{1o}} \\
= &\frac{1+P_i(5x_i^{1o}-1)+\log x_i^{1o}(1+P_i(6x_i^{1o}-1))+2P_i x_i^{1o}(\log x_i^{1o})^2}{(2+\log x_i^{1o})^2} \\
= &\frac{\overbrace{(1+\log x_i^{1o})}^{\geq 0}+P_i x_i^{1o}+\overbrace{(1+\log x_i^{1o})}^{\geq 0}P_i(4x_i^{1o}-1)+2P_i x_i^{1o}\overbrace{\log x_i^{1o}(\log x_i^{1o}+1)}^{\geq -1/4}}{(2+\log x_i^{1o})^2} \\
\geq &\frac{P_i x_i^{1o}/2}{(2+\log x_i^{1o})^2} \geq 0
\end{aligned}$$

Hence, for $x_i^{1o} \in [e^{-1}, 1]$, $x_i^{1o}(1-P_i(1-x_i^{1o})) - ((1-P_i)(1+\log x_i^{1o})+P_i x_i^{1o})\frac{x_i^{1o}}{2+\log x_i^{1o}}$ achieves its minimum $\frac{1-P_i}{e}$ at $x_i^{1o} = e^{-1}$. Therefore

$$\frac{\partial w_i^{1o}}{\partial w_{-i}} \geq 0.$$

Q.E.D.

Proof of Lemma 2.12. By the definition of π_{si}^2,

$$\begin{aligned}
\pi_{si}^2(x_i) &= -\frac{1}{\lambda}(P_i x_i - c_i)\log x_i \\
\frac{\partial \pi_{si}^2}{\partial x_i} &= -\frac{1}{\lambda}(P_i - \frac{c_i}{x_i} + P_i \log x_i) \\
\frac{\partial^2 \pi_{si}^2}{\partial x_i^2} &= -\frac{1}{\lambda x_i}(\frac{c_i}{x_i} + P_i) < 0.
\end{aligned}$$

Hence, π_{si}^2 is strictly concave in x_i. Furthermore, define $w_i^2(x_i) = P_i x_i$ and $w_i^1(x_i) = P_i x_i(1-$

$P_{-i}\big(1-\frac{w_{-i}}{P_{-i}(1-P_i(1-x_i))})\big)$, then

$$\begin{aligned}\frac{\partial \pi^1_{si}}{\partial x_i} &= -\frac{(w^1_i(x_i)-c_i)/x_i+\frac{\partial w^1_i(x_i)}{x_i}\log x_i}{\lambda}\\ \frac{\partial \pi^2_{si}}{\partial x_i} &= -\frac{(w^2_i(x_i)-c_i)/x_i+\frac{\partial w^2_i(x_i)}{x_i}\log x_i}{\lambda}\end{aligned}$$

$$\begin{aligned}\frac{\partial w^2_i}{\partial x_i} &= P_i\\ \frac{\partial w^1_i}{\partial x_i} &= P_i(1-P_{-i}+w_{-i}\frac{1-P_i}{(1-P_i(1-x_i))^2})\\ &= \frac{\partial w^2_i}{\partial x_i}-P_i(P_{-i}-\frac{w_{-i}(1-P_i)}{(1-P_i(1-x_i))^2}.\end{aligned}$$

While at $x_i=1-\frac{P_{-i}-w_{-i}}{P_{-i}P_i}$, we have $w_{-i}=P_{-i}(1-P_i)+x_iP_{-i}P_i\geq P_{-i}(1-P_i)$. Hence,

$$\frac{\partial w^1_i}{\partial x_i}=\frac{\partial w^2_i}{\partial x_i}-\frac{P_iP_{-i}(w_{-i}-(1-P_i)P_{-i})}{w_{-i}}\leq\frac{\partial w^2_i}{\partial x_i}.$$

Since $w^1_i(1-\frac{P_{-i}-w_{-i}}{P_{-i}P_i})=w^2_i(1-\frac{P_{-i}-w_{-i}}{P_{-i}P_i})$, thus

$$\frac{\partial \pi^1_{si}(x_i|w_{-i})}{\partial x_i}\Big|_{x_i=1-\frac{P_{-i}-w_{-i}}{P_{-i}P_i}}\leq\frac{\partial \pi^2_{si}(x_i|w_{-i})}{\partial x_i}\Big|_{x_i=1-\frac{P_{-i}-w_{-i}}{P_{-i}P_i}}.$$

Q.E.D.

Proof of Lemma 2.13. We first show that there exists at most one point w^c_{-i} such that if $w_{-i}\leq w^c_{-i}$ then $x^b_i(w_{-i})=x^{b1}_i(w_{-i})$ and $x^b_i(w_{-i})=x^{b2}_i(w_{-i})$ for $w_{-i}\geq w^c_{-i}$.
By lemma 2.11, $x^{1o}_i(w_{-i})$ is decreasing with w_{-i}. Hence there exists a unique solution w^c_i to the following equation:

$$x^{1o}_i(w^c_{-i})=1-\frac{P_{-i}-w^c_{-i}}{P_{-i}P_i}\Rightarrow\frac{\partial \pi^1_{si}}{\partial x_i}\Big|_{x_i=1-\frac{P_{-i}-w^c_{-i}}{P_{-i}P_i}}=0.$$

Then for $w_{-i}\geq w^c_{-i}$,

$$\frac{\partial \pi^1_{si}}{\partial x_i}\leq 0\qquad\forall x_i\in[\max\{e^{-1},1-\frac{P_{-i}-w_{-i}}{P_{-i}P_i}\},1].$$

Therefore, $x_i^b(w_{-i}) = x_i^{b2}(w_{-i})$. While in the region $w_i \leq w_{-i}^c$,

$$\frac{\partial \pi_{si}^2}{\partial x_i} \geq 0 \qquad \forall x_i \in [e^{-1}, 1 - \frac{P_{-i} - w_{-i}}{P_{-i}P_i}].$$

Hence, $x_i^b(w_{-i}) = x_i^{b1}(w_{-i})$ and $x_i^b = x_i^{b1} = x_i^{b2}$ at $w_i \geq w_{-i}^c$. Combine them together,

$$w_i^b(w_{-i}) = w_i^1(x_i^{b1}(w_{-i})) = P_i(1 - P_{-i} + \frac{w_{-i}}{1 - P_i(1 - x_i^{b1}(w_{-i}))})x_i^{b1}(w_{-i}) \qquad w_{-i} \leq w_{-i}^c$$
$$w_i^b(w_{-i}) = w_i^2(x_i^{b2}(w_{-i})) = P_i x_i^{b2}(w_{-i}) \qquad w_{-i} \geq w_{-i}^c.$$

Note that for $w_{-i} \geq w_{-i}^c$, since π_{si}^2 is a not a function in w_{-i} and only the upper bound of the feasible interval: $1 - \frac{P_{-i} - w_{-i}}{P_i P_{-i}}$ is increasing with w_{-i}, thus, x_i^{b2} and $w_i^b(w_{-i}) = P_i x_i^{b2}(w_{-i})$ are increasing with w_{-i}. When $w_{-i} \leq w_{-i}^c$, by lemma 2.11, the FOC point x_i^{1o} is decreasing with w_{-i} and the lower bound of the feasible interval is increasing with w_{-i} while the upper bound remains as a constant. Therefore, with the increasing of w_{-i}, the optimal x_i^{b1} starts with upper bound, then decreases as an interior optimal point until the interior optimal point touches the lower bound. Hence the the corresponding x_i^{b1} is continuous. Note that by lemma 2.11, the w_i corresponding to x_i^{1o} is increasing with w_{-i}, which is also true for the w_i corresponding to the upper bound and lower bound. Thus for all $w_i \leq w_{-i}^c$, w_i^b is increasing in w_{-i}. At $w_i = w_i^c$, since $x_i^{b1} \geq x_i^{b2}$,

$$\begin{aligned} w_i^1(x_i^{b1}(w_{-i})) &= P_i x_i^{b1}(1 - P_{-i}(1 - x_{-i})) \\ &\geq P_i x_i^{b1} \geq P_i x_i^{b2} = w_i^2(x_i^{b2}(w_{-i})). \end{aligned}$$

Therefore, for all $w_{-i} \geq c_i$, $w_i^b(w_{-i})$ is increasing.

Q.E.D.

Proof of Theorem 2.7. By the increasing best response property, the equilibrium set $\mathcal{W}^*$ is always non-empty. Suppose that the conclusion does not hold, then $\exists \mathbf{w}^* \in \mathcal{W}^*$ and $\exists \mathbf{w}' \in \mathcal{W}'$ such that $\sum_{i=1}^2 \pi_{si}(\mathbf{w}^*) < \sum_{i=1}^2 \pi_{si}(\mathbf{w}')$ and $w_i^* > w_i'$ for $i = 1$ or 2. Let $w_i^b(w_{-i})$ be the best response wholesale price of supplier i. Consider the following two cases,

case one: $w_{-i}^* \leq w_{-i}'$. By the increasing best response property, we have

$$w_i^b(w_{-i}') \geq w_i^b(w_{-i}^*) \ni w_i^* > w_i'.$$

Thus,

$$\begin{aligned}\pi_{si}(w'_i, w'_{-i}) &\leq \pi_{si}(w^b_i(w'_{-i}), w'_{-i}) \\ \pi_{s-i}(w'_i, w'_{-i}) &\leq \pi_{s-i}(w^b_i(w'_{-i}), w'_{-i}).\end{aligned}$$

The first inequality is because that $w^b_i(w'_{-i})$ is the best response of supplier i. The second inequality holds because for given w_{-i}, q_{-i} and π_{s-i} increase with w_i. Note that the equality in the first set holds only if $w'_i \in w^b_i(w^b_{-i})$, in which case $w'_i \geq w^*_i$ and this leads to a contradiction. Therefore $\pi_{si}(\mathbf{w}') + \pi_{s-i}(\mathbf{w}') < \pi_{si}(w^b_i(w'_{-i}), w'_{-i}) + \pi_{s-i}(w^b_i(w'_{-i}), w'_{-i})$, which contradicts with $\mathbf{w}' \in \mathcal{W}'$.

case two: $w^*_{-i} \geq w'_{-i}$. By the increasing best response property, we have:

$$\begin{aligned} w^*_i \in w^b_i(w^*_{-i}) \geq w^b_i(w'_{-i}) \\ w^*_{-i} \in w^b_{-i}(w^*_i) \geq w^b_{-i}(w'_i).\end{aligned}$$

In addition,

$$\begin{aligned}\pi_{si}(w'_i, w'_{-i}) \leq \pi_{si}(w^b_i(w'_{-i}), w'_{-i}) \leq \pi_{si}(w^b_i(w'_{-i}), w^*_{-i}) \leq \pi_{si}(w^*_i, w^*_{-i}) \\ \pi_{s-i}(w'_i, w'_{-i}) \leq \pi_{s-i}(w'_i, w^b_{-i}(w'_i) \leq \pi_{s-i}(w^*_i, w^b_{-i}(w'_i)) \leq \pi_{s-i}(w^*_i, w^*_{-i}).\end{aligned}$$

In above two inequality sets, the first and the third inequality in each set are because of the best response property, the second inequality in each set because the π_{si} or π_{s-i} is increasing in the opponent's wholesale price given his wholesale price fixed. Thus, $\pi_{si}(\mathbf{w}') + \pi_{s-i}(\mathbf{w}') \leq \pi_{si}(\mathbf{w}^*) + \pi_{s-i}(\mathbf{w}^*)$. However, $\mathbf{w}^* \notin \mathcal{W}'$, and this leads to contradiction. Therefore $\mathbf{w}^* \leq \mathbf{w}'$.
Q.E.D.

Proof of Lemma 2.14. We show the concavity of the expected profit by showing that the expected profit function is a composition of concave function by concave operator. First Let's show the function $\min_i r_i q_i$ is jointly concave in $\mathbf{q}$. Let $\mathbf{q}^1$ and $\mathbf{q}^2$ be two vector. Hence, for any $\lambda \in (0, 1)$, we have:

$$\begin{aligned}\min_i\{\lambda r_i q^1_i + (1-\lambda) r_i q^2_i\} &= \lambda r_j q^1_j + (1-\lambda) r_j q^2_j \qquad \text{for some } j \in 1, \ldots, n \\ &\geq \lambda \min_i\{r_i q^1_i\} + (1-\lambda)\min_i\{r_i q^2_i\}.\end{aligned}$$

Thus $d - \min_i\{r_i q_i\}$ is jointly convex in $\mathbf{q}$. Since the operator $()^+$ is monotone increasing and convex, $(d - \min_i\{r_i q_i\})^+$ is convex in $\mathbf{q}$. Also note that the convexity (concavity) preserved under expectation, we get that $sE[\min_i R_i q_i] - (p+u-s)E[(D - \min_i\{R_i q_i\})^+]$ is jointly concave in $\mathbf{q}$ given that $s \geq 0$ and $p+u-s > 0$. Therefore, the whole function is concave in $\mathbf{q}$.

Q.E.D.

Proof of Corollary 2.3. If there exists some supplier j with $q_j^* = 0$, then $E[\min_i(R_i q_i^*)] = 0$ and $E[(D - \min_i R_i q_i^*)^+] = E[D]$. Therefore, the manufacturer's profit:

$$\pi = -\sum_{i=1}^{n} w_i q_i - uE[D].$$

Since $w_i > 0$, the manufacturer's profit is strictly decreasing with q_i for $i \neq j$. Hence in optimal solution, $q_i^* = 0$ for all i.

Q.E.D.

Proof of Lemma 2.15. Since π is concave, for optimal solution with $q_i > 0$, the first order condition should be satisfied:

$$\pi = pE[D] - \sum_{i=1}^{N} w_i q_i - (p+u)E[(D - minR_i q_i)^+] + sE[(minR_i q_i - D)^+]$$

$$\Rightarrow \quad \frac{\partial \pi}{\partial q_i} = -w_i - (p+u)\frac{\partial E[(D - minR_i q_i)^+]}{\partial q_i} + s\frac{\partial sE[(minR_i q_i - D)^+]}{\partial q_i}, \forall i = 1, \ldots, n$$

Since $g_m(x) = g_{mi}(x)(1 - G_i(\frac{x}{q_i})) + \frac{1}{q_i} g_i(\frac{x}{q_i})(1 - G_{mi}(x))$, where $G_{mi}(x)$ is the cumulative distribution function for the random variable $\min_{j \neq i} R_j q_j$. We can write π in integration form:

$$\begin{aligned} E[(D - minR_i q_i)^+] &= \int_0^{+\infty} \int_0^v (v-x) f(v) \Big(g_{mi}(x)(1 - G_i(\frac{x}{q_i})) + \frac{1}{q_i} g_i(\frac{x}{q_i})(1 - G_{mi}(x)) \Big) dx dv \\ \frac{\partial E[(D - minR_i q_i)^+]}{\partial q_i} &= \int_0^{+\infty} f(v) \Big(\int_0^v (v-x) \big(\frac{x g_i(x/q_i) g_{mi}(x)}{q_i^2} \\ &\quad - \frac{g_i(x/q_i)(1 - G_{mi}(x))}{q_i^2} - \frac{x g_i'(x/q_i)(1 - G_{mi}(x))}{q_i^3} \big) dx \Big) dv. \qquad \text{(A.12)} \end{aligned}$$

Since

$$\begin{aligned}
&\int_0^v (v-x)x\frac{g_i'(x/q_i)(1-G_{mi}(x))}{q_i^3}dx = \int_0^v \frac{(v-x)x(1-G_{mi}(x))}{q_i^2}dg_i(x/q_i) \\
= \;& \frac{(v-x)x(1-G_{mi}(x))}{q_i^2}g_i(x/q_i)|_0^v - \int_0^v g_i(x/q_i)d\frac{(v-x)x(1-G_{mi}(x))}{q_i^2}dx \\
= \;& -\int_0^v \frac{g_i(x/q_i)}{q_i^2}\Big((v-2x)(1-G_{mi}(x)) - (v-x)xg_{mi}(x)\Big)dx, \qquad \text{(A.13)}
\end{aligned}$$

replace A.13 back to A.12 and we get:

$$\begin{aligned}
\frac{\partial E[(D-minR_iq_i)^+]}{\partial q_i} &= -\int_0^{+\infty} f(v)\int_0^v \frac{xg_i(x/q_i)(1-G_{mi}(x))}{q_i^2}dxdv \\
&= -\int_0^{+\infty} \frac{xg_i(x/q_i)(1-G_{mi}(x))}{q_i^2}\int_x^{+\infty} f(v)dvdx \\
&= -\int_0^{+\infty} \frac{x(1-F(x))g_i(x/q_i)(1-G_{mi}(x))}{q_i^2}dx \\
&= -\int_0^1 r_i(1-F(r_iq_i))g_i(r_i)(1-G_{mi}(r_iq_i))dr_i \\
&= -E[r_i(1-F(r_iq_i)]_{s_i}.
\end{aligned}$$

Note that we assume the differentiability of g_i here for equation A.12. However, if we do the integration transformation first, we can get the same result without the assumption of differentiability of g_i:

$$\begin{aligned}
E[(D-minR_iq_i)^+] &= \int_0^{+\infty}\int_0^v (v-x)f(v)\big(g_{mi}(x)(1-G_i(\frac{x}{q_i})) + \frac{1}{q_i}g_i(\frac{x}{q_i})(1-G_{mi}(x))\big)dxdv \\
&= \int_0^{+\infty} f(v)\Big(\int_0^v (v-x)g_{mi}(x)(1-G_i(\frac{x}{q_i}))dx \\
&\quad +\int_0^v (v-x)(1-G_{mi}(x))dG_i(x/q_i)\Big)dv \\
&= \int_0^{+\infty} f(v)\Big(\int_0^v (v-x)g_{mi}(x)(1-G_i(\frac{x}{q_i}))dx \\
&\quad +(v-x)(1-G_{mi}(x))G_i(x/q_i)|_0^v \\
&\quad +\int_0^v G_i(x/q_i)\big((v-x)g_{mi}(x) + (1-G_{mi}(x))\big)dx\Big)dv \\
&= \int_0^{+\infty}\int_0^v (v-x)f(v)\big(g_{mi}(x) + G_i(x/q_i)(1-G_{mi}(x))\big)dxdv \\
\frac{\partial E[(D-minR_iq_i)^+]}{\partial q_i} &= -\int_0^{+\infty} \frac{xg_i(x/q_i)(1-G_{mi}(x))}{q_i^2}\int_x^{+\infty} f(v)dvdx.
\end{aligned}$$

Similarly, we have:

$$
\begin{aligned}
E[(minR_iq_i - D)^+] &= \int_0^{+\infty}\int_v^{+\infty}(x-v)f(v)\big(g_{mi}(x)(1-G_i(\frac{x}{q_i})) + \frac{1}{q_i}g_i(\frac{x}{q_i})(1-G_{mi}(x))\big)dxdv \\
\frac{\partial E[(minR_iq_i - D)^+]}{\partial q_i} &= \int_0^{+\infty} f(v)\Big(\int_v^{+\infty}(x-v)\big(\frac{xg_i(x/q_i)g_{mi}(x)}{q_i^2} \\
&\quad -\frac{g_i(x/q_i)(1-G_{mi}(x))}{q_i^2} - \frac{xg_i'(x/q_i)(1-G_{mi}(x))}{q_i^3}\big)dx\Big)dv \\
&= -\int_0^{+\infty} f(v)\int_v^{+\infty}\frac{xg_i(x/q_i)(1-G_{mi}(x))}{q_i^2}dxdv \\
&= -\int_0^{1}\frac{xF(x)g_i(x/q_i)(1-G_{mi}(x))}{q_i^2} \\
&= -E[r_iF(r_iq_i)]_{s_i}
\end{aligned}
$$

Hence,

$$
\frac{\partial \pi}{\partial q_i} = -w_i + (p+u)E[r_i(1-F(r_iq_i)]_{s_i} - sE[r_iF(r_iq_i)]_{s_i}.
$$

Therefore at optimal point we have:

$$
w_i = (p+u)E[r_i(1-F(r_iq_i)]_{s_i} - (p+u-s)E[F(r_iq_i)]_{s_i} \qquad \forall i = 1,\dots,n
$$

Q.E.D.

Proof of Lemma 2.16. First, if $q_j^* = 0$, then by Corollary 2.3, we have $q_i^* = 0$ and the result holds. Now consider the case $q_i^* > 0$ and suppose $q_i^* > q_j^* > 0$, we can show the result by contradiction. Let $s_i(x) = \{r_i | r_ix \le \min_{j\ne i} r_jq_j^*\}$. Clearly $s_i(q_i^*) \subseteq s_i(q_j^*)$. By **KKT** condition we have

$$\begin{aligned}
w_j - w_i &= (p+u)(E[R_j]_{s_j} - E[R_i]_{s_j}) - (p+u-s)(E[R_jF(R_jq_j)]_{s_j} - E[R_iF(R_iq_i)]_{s_i}) \\
&= (p+u-s)\big(E[R_j(1-F(R_jq_j))]_{s_j(q_j)} - E[R_i(1-F(R_iq_i))]_{s_i(q_i)}\big) \\
&\quad +s(E[R_j]_{s_j} - E[R_i]_{s_i}) \\
&\geq (p+u-s)\big(E[R_j(1-F(R_jq_j))]_{s_j(q_j)} - E[R_i(1-F(R_iq_i))]_{s_i(q_i)}\big) \\
&= (p+u-s)\big(E[R_j(1-F(R_jq_j))]_{s_j(q_j)} - E[R_i(1-F(R_iq_i))]_{s_i(q_i)}\big) \\
&\quad +(p+u)E[R_j(1-R_jq_j)]_{s_j(q_j)\backslash s_j(q_i)} \\
&\geq (p+u)E[R_j(1-R_jq_j)]_{s_j(q_j)\backslash s_j(q_i)} \geq 0,
\end{aligned}$$

which contradicts with $w_i > w_j$.

Q.E.D.

Proof of Lemma 2.17. We prove the result by contradiction. Suppose $q_i^* > q_j^*$, since $R_i \geq_{as} R_j$, we have $Pr(R_iq_i = \min_j R_jq_j) = 0$. Hence $E[R_i]_{s_i} = 0$ and $E[R_iF(R_iq_i)]_{s_i} = 0$. Therefore, by lemma 2.15, $q_i^* = 0$ which contradicts with $q_i^* > q_j^* \geq 0$.

Q.E.D.

Proof of Lemma 2.18. By corollary 2.3, if $q_i^* = 0$ for some i, then $q_j^* = 0$ for all $j \in \{1, \ldots, N\}$. Therefore, for any optimal solution $\mathbf{q}^*$, there exist a vector $(\xi_i)_{i=1,\ldots,n} \geq 0$ such that $q_i^* = \xi_i q_1^*$, $\forall i = 1, \ldots, n$ ($\xi_1 \equiv 1$). Hence,

$$\max_{\mathbf{q}\geq 0} \pi = pE[D] + sE[(\min_i q_iR_i - D)^+] - (p+u)E[(D - \min_i q_iR_i)^+] - \sum_{i=1}^N w_iq_i$$

$$= \max_{q_1\geq 0, \xi_i \geq 0, \xi_1 = 1} \pi = \{pE[D] + sE[(q_1 \min_i \xi_iR_i - D)^+] - (p+u)E[(D - q_1 \min_i \xi_iR_i)^+] - q_1 \sum_{i=1}^N w_i\xi_i$$

Thus the manufacturer does not make any order if and only if for all $\xi_i \geq 0, \forall i = 2, \ldots, n$ and $\xi_1 = 1$, there is no positive optimal solution q_1^*.

Let $Rm(\xi) = \min_i \xi_iR_i$. Since π is concave in q_1, it's equivalent to show that $\forall \gamma_i \geq 0, \forall i =$

$2, \ldots, n$ and $\xi_1 = 1$ the following inequality holds:

$$\begin{aligned}
& \frac{\partial \pi(q)}{\partial q}|_{q=0} \leq 0 \\
\Leftrightarrow \quad & -\sum_{i=1}^{N} w_i\xi_i + (p+u)E[Rm(\xi)] - (p+u)E[Rm(\xi)F(Rm(\xi q)]|_{q=0} \leq 0 \\
\Leftrightarrow \quad & \sum_{i=1}^{N} w_i\xi_i \geq (p+u)E[Rm(\xi)]
\end{aligned}$$

Q.E.D.

Proof of lemma 2.8. First, let us show the function $s(\min_i r_i q_i - d)^+ - (p+u)(d - \min_i r_i q_i)^+$ is supermodular in $\mathbf{q}$. $\forall \mathbf{q}^1, \mathbf{q}^2 \in \mathbb{R}^{+n}$,

$$\begin{aligned}
& \min_i r_i(q_i^1 \wedge q_i^2) = \min_i(\min\{r_i q_i^1, r_i q_i^2\}) = \min\{\min_i r_i q_i^1, \min_i r_i q_i^2\} \\
\Rightarrow \quad & s(\min_i r_i(q_i^1 \wedge q_i^2) - d)^+ - (p+u)(d - \min_i r_i(q_i^1 \wedge q_i^2))^+ \\
& = s(\min\{\min_i r_i q_i^1, \min_i r_i q_i^2\} - d)^+ - (p+u)(d - \min\{\min_i r_i q_i^1, \min_i r_i q_i^2\})^+
\end{aligned}$$

Let $\max\{\min_i r_i q_i^1, \min_i r_i q_i^2\} = r_j q_j^t$ and $\min_i r_i(q_i^1 \vee q_i^2) = r_k q_k^{t'}$.

$$\begin{aligned}
\text{if } t = t' \quad &\Rightarrow \quad r_k q_k^{t'} \geq \min_i r_i q_i^t = r_j q_j^t \\
\text{if } t \neq t' \quad &\Rightarrow \quad r_k q_k^{t'} \geq r_k q_k^t \geq \min_i r_i q_i^t = r_j q_j^t
\end{aligned}$$

Therefore $\min_i r_i(q_i^1 \vee q_i^2) \geq \max\{\min_i r_i q_i^1, \min_i r_i q_i^2\}$. Note that function $f(x) = s(x-d)^+ - (p+u)(d-x)$ is increasing in x because $p+u \geq 0$ and $s \geq 0$. Thus

$$\begin{aligned}
& s(\min_i r_i(q_i^1 \vee q_i^2) - d)^+ - (p+u)(d - \min_i r_i(q_i^1 \vee q_i^2))^+ \\
& \geq s(\max\{\min_i r_i q_i^1, \min_i r_i q_i^2\} - d)^+ - (p+u)(d - \max\{\min_i r_i q_i^1, \min_i r_i q_i^2\})^+
\end{aligned}$$

Combine together we have

$$\begin{aligned}
& s(\min_i r_i(q_i^1 \wedge q_i^2) - d)^+ - (p+u)(d - \min_i r_i(q_i^1 \wedge q_i^2))^+ \\
& + s(\min_i r_i(q_i^1 \vee q_i^2) - d)^+ - (p+u)(d - \min_i r_i(q_i^1 \vee q_i^2))^+ \\
\geq \;& s(\min\{\min_i r_i q_i^1, \min_i r_i q_i^2\} - d)^+ - (p+u)(d - \min\{\min_i r_i q_i^1, \min_i r_i q_i^2\})^+ \\
& + s(\max\{\min_i r_i q_i^1, \min_i r_i q_i^2\} - d)^+ - (p+u)(d - \max\{\min_i r_i q_i^1, \min_i r_i q_i^2\})^+ \\
= \;& s(\min_i r_i q_i^1 - d)^+ - (p+u)(d - \min_i r_i q_i^1)^+ + s(\min_i r_i q_i^2 - d)^+ - (p+u)(d - \min_i r_i q_i^2)^+.
\end{aligned}$$

Therefore $s(\min_i r_i q_i - d)^+ - (p+u)(d - \min_i r_i q_i)^+$ is supermodular in $\mathbf{q}$. Furthermore,

$$\begin{aligned}
\frac{\partial^2(-\sum_{i=1}^n w_i q_i)}{\partial(-w_j)\partial q_k} &= \begin{cases} -1 & \forall j = k \\ 0 & \forall j \neq k \end{cases} \\
\frac{\partial^2(-\sum_{i=1}^n w_i q_i)}{\partial(-w_j)\partial - w_i} &= 0 \qquad \forall i \neq j.
\end{aligned}$$

Therefore $-\sum_{i=1}^n w_i q_i$ is supermodular in $(-\mathbf{w}, \mathbf{q})$. Hence $-\sum_{i=1}^n w_i q_i + s(\min_i r_i q_i - d)^+ - (p+u)(d - \min_i r_i q_i)^+$ is supermodular in $(-\mathbf{w}, \mathbf{q})$. Since supermodularity is preserved under expectation by corollary 2.6.2 in [44], we have

$$\pi = pE[D] - (p+u)E[(D - minR_i q_i)^+] + sE[(minR_i q_i - D)^+] - \sum_{i=1}^N w_i q_i$$

is supermodular in $(-\mathbf{w}, \mathbf{q})$. By the result of Milgrom and Shannon 1994 [32], if a function is supermodular in $(-\mathbf{w}, \mathbf{q})$ then the optimal ordering quantity q_i^* is monotone non-increasing in w_j for all $i, j = 1, \ldots, n$.

Q.E.D.

Proof of proposition 2.1. We prove the result by contradiction. Suppose not and there exists an optimal ordering quantity $\mathbf{q}^*$ that such that $q_i^* \neq q_j^*$ for some i, j. Now construct another ordering strategy $\mathbf{q}'$ such that $q_i' = \min_{j=1}^n q_j^*$. Thus $E[(D - \min_i q_i^*)^+] = E[(D - \min_i q_i')^+]$ and $E[(\min_i q_i^* - D)^+] = E[(\min_i q_i' - D)^+]$. Therefore

$$\pi(\mathbf{q}^*) - \pi(\mathbf{q}') = -\sum w_i (q_i^* - q_i') > 0,$$

which leads to a contradiction.

Q.E.D.

Proof of proposition 2.2. Let $w_i^* = p + u$ for all $i = 1, \dots, n$. Therefore, $q^*(\sum_i w_i^*) = 0$. Consequently, $\forall i = 1, \dots, n$, $\forall w_i \geq c_i$, we have:

$$\pi_{si}(w_i, \mathbf{w}_{-i}^*) = (w_i - c_i)q^*(\sum_{j \neq i} w_j^* + w_i) = 0 = \pi_{si}(\mathbf{w}^*).$$

Hence, $\mathbf{w}^*$ is a trivial equilibrium point such that every supplier i get 0 profit.

Q.E.D.

Proof of theorem 2.9. We first show that $\psi(\mathbf{w})$ has a positive maximum in the compact domain: $\{w_i \geq c_i, \sum_i w_i \leq (p+u)\prod_{i=1}^n P_i : i = 1, \dots, n\}$. Since $q^*(\mathbf{w}) = 0$ and $\psi(\mathbf{w}) = 0$ for $\sum_i w_i > (p+u)\prod_{i=1}^n P_i$, thus the maximum is global in $\mathbf{w} \geq \mathbf{c}$.

By Bolzano-Weierstrass theorem, to show $\psi(\mathbf{w})$ has a maximum in the compact domain, it is enough for us to show $\psi(\mathbf{w})$ is upper semicontinuous in $\mathbf{w}$. Since $\prod_{i=1}^n (w_i - c_i) \geq 0$, the upper semicontinuous property preserves if $q^*(\sum_i w_i)$ is upper semicontinuous in $\sum_i w_i$ for

$$\sum_i w_i \in \mathcal{S} = [\sum_i c_i, (p+u)\prod_{i=1}^n P_i]$$

Now $\forall \sum_i w_i \in \mathcal{S}$, by the increasing property of $F(x)$ we have :

$$\begin{aligned} q^*(\sum_i w_i) &= \max \quad x : F(x) = \frac{(p+u)\prod_{i=1}^n P_i - \sum_i w_i}{(p+u-s)\prod_{i=1}^n P_i} \\ &= \max \quad x : F(x) \leq \frac{(p+u)\prod_{i=1}^n P_i - \sum_i w_i}{(p+u-s)\prod_{i=1}^n P_i} \\ &= \max_{x \geq 0} \quad x : (p+u-s)\prod_{i=1}^n P_i F(x) - (p+u)\prod_{i=1}^n P_i \leq -\sum_i w_i. \end{aligned}$$

Note that the function $(p+u-s)\prod_{i=1}^n P_i F(x) - (p+u)\prod_{i=1}^n P_i$ is uniformly continuous in x for $x \in [0, \max F^{-1}(\frac{(p+u)\prod_{i=1}^n P_i - \sum_i c_i}{(p+u-s)\prod_{i=1}^n P_i})]$. Hence $\forall \sum_i w_i \in [\sum_i c_i, (p+u)\prod_{i=1}^n P_i]$, the set valued map defined by $T_{-\sum_i w_i} = \{x : (p+u-s)\prod_{i=1}^n P_i F(x) - (p+u)\prod_{i=1}^n P_i \leq -\sum_i, x \geq 0\}$

is non-empty and compact. By Theorem 1 in [10], the map T is upper semicontinuous in $\sum_i w_i$ in $\mathcal{S}$.[4] Since x is continuous, by theorem 3 in [10], q^* is upper semicontinuous in $\sum_i w_i$ and the maximizer exists.

Since $(p+u)\prod_i P_i > \sum_i c_i$, then there exists a $\mathbf{w}$ such that $w_i > c_i$ and $\sum_i w_i < (p+u)\prod_i P_i$. Also given that demand distribution F is continuous on $[0, +\infty)$ with $F(0) = 0$, there exists a $x > 0$ such that $F(x) < \frac{(p+u)\prod_i P_i - \sum_i w_i}{(p+u-s)\prod_i P_i}$. Hence $q^*(\sum_i w_i) \geq x > 0$ with corresponding $\psi(\mathbf{w}) > 0$. Therefore the optimal ψ is positive. Since the maximizer $\mathbf{w}^*$ satisfies $\psi(\mathbf{w}^*) > 0$, thus $w_i^* > c_i$ for all $i = 1, \dots, n$. In addition, $\forall i = 1 \dots, n$, $\forall w_i' \geq c_i$ we have:

$$\begin{aligned} \psi(\mathbf{w}^*) = \prod_{j=1}^{n}(w_j^* - c_j) q^*(\sum_j w_j^*) &\geq (w_i' - c_i)\prod_{j\neq i}(w_j^* - c_j) q^*(\sum_{j\neq i} w_j^* + w_i') \\ \Rightarrow \quad (w_i^* - c_i) q^*(\sum_j w_j^*) &\geq (w_i' - c_i) q^*(\sum_{j\neq i} w_j^* + w_i') \\ \Rightarrow \quad \pi_{si}(w_i^*, \mathbf{w}_{-i}^*) &\geq \pi_{si}(w_i', \mathbf{w}_{-i}^*). \end{aligned}$$

Therefore, $\mathbf{w}^*$ is an equilibrium point in the non-cooperative game.

Q.E.D.

Proof of lemma 2.19. To show part a), we prove by contradiction. Let $\mathbf{w}^*$ be a non-trivial equilibrium point, then there exists a supplier i such that $\pi_{si}(\mathbf{w}^*) > 0$. Therefore, $q^*(\sum_i w_i^*) > 0$ and $\sum_i w_i < (p+u)\prod_i P_i$. Suppose $\exists i$ such that $w_i^* = c_i$, thus $\pi_{si}(\mathbf{w}^*) = 0$. Also by the continuity of q^*, $\exists w_i' > w_i^* = c_i$ such that $q^*(\sum_{j\neq i} w_j^* + w_i') > 0$, and thus $\pi_{si}(w_i', \mathbf{w}_{-i}^*) > 0 = \pi_{si}(w_i^*, \mathbf{w}_{-i}^*)$ which leads to a contradiction. Therefore the equilibrium point must be an interior point.

For part b). For any interior non-trivial equilibrium $\mathbf{w}^*$ such that $\sum_i w_i^* < (p+u)\prod_i P_i$. Therefore, $q^* \in [a, b)$. Since $q^*(\sum_i w_i)$ is differentiable, the first order condition must be

[4] Definition: Suppose $S_{\bar{b}}$ is compact. The mapping S is said to be upper semicontinuous (u.s.c.) at $\bar{b}$ if $\forall \varepsilon > 0, \exists \delta > 0, \ni \mid b - \bar{b} \mid < \delta \Rightarrow S_b \subseteq \eta_\varepsilon(S_{\bar{b}})$, here $\eta_\varepsilon(S_{\bar{b}})$ is an ε-neighborhood of S_b, defined by

$$\eta_\varepsilon(S_{\bar{b}}) = \{x : \rho(x, S_b) < \varepsilon\} = \cup_{x\in S_b} N_\varepsilon(x)$$

satisfied :

$$\begin{aligned}\frac{\partial \pi_{si}(\mathbf{w})}{\partial w_i} &= q^*(\sum_i w_i) + (w_i - c_i)\frac{\partial q^*(\sum_i w_i)}{\partial w_i} \\ &= F^{-1}\big(\frac{(p+u)\prod_{i=1}^n P_i - \sum_i w_i}{(p+u-s)\prod_{i=1}^n P_i}\big) - \frac{(w_i - c_i)}{(p+u-s)\prod_{i=1}^n P_i f\Big(F^{-1}\big(\frac{(p+u)\prod_{i=1}^n P_i - \sum_i w_i}{(p+u-s)\prod_{i=1}^n P_i}\big)\Big)} = 0\end{aligned}$$

Thus the unit profit is the same for every supplier. Since the ordering quantity for each supplier is the same, every supplier have the same profit in equilibrium point $\mathbf{w}^*$:

$$\begin{aligned}\pi_{si}(\mathbf{w}^*) &= (p+u-s)\prod_{i=1}^n P_i q^*(\sum_i w_i^*)^2 f\Big(F^{-1}\big(\frac{(p+u)\prod_{i=1}^n P_i - \sum_i w_i^*}{(p+u-s)\prod_{i=1}^n P_i}\big)\Big) \\ &= (p+u-s)\prod_{i=1}^n P_i F^{-1}\big(\frac{(p+u)\prod_{i=1}^n P_i - \sum_i w_i^*}{(p+u-s)\prod_{i=1}^n P_i}\big)^2 f\Big(F^{-1}\big(\frac{(p+u)\prod_{i=1}^n P_i - \sum_i w_i^*}{(p+u-s)\prod_{i=1}^n P_i}\big)\Big)\end{aligned}$$

Therefore $w_i^* - c_i = \xi$ for all $i = 1, \ldots, n$. Thus, for any two equilibrium $\mathbf{w}^1, \mathbf{w}^2$, let $\xi^1 = w_i^{*1} - c_i$ and $\xi^2 = w_i^{*2} - c_i$. Since $\mathbf{w}^{*1} \neq \mathbf{w}^{*2}$, we have either $\xi^1 > \xi^2 \Rightarrow \mathbf{w}^{*1} > \mathbf{w}^{*2}$ or $\xi^1 < \xi^2 \Rightarrow \mathbf{w}^{*1} < \mathbf{w}^{*2}$. Therefore, for all $i = 1, \ldots, n$:

$$\begin{aligned}\pi_{si}(\mathbf{w}^{*2}) &= (w_i^{*2} - c_i)q^*(\sum_j w_j^{*2}) \\ &\geq (w_i^{*1} + \sum_j (w_j^{*2} - w_j^{*1}) - c_i)q^*(\sum_j w_j^{*2} + \sum_j (w_j^{*1} - w_j^{*2})) \\ &> (w_i^{*2} - c_i)q^*(\sum_j w_j^{*1}).\end{aligned}$$

For part c), since $\mathbf{w}^*$ is a boundary non-trivial equilibrium, we have $\sum_i w_i^* \leq (p+u)\prod_i P_i$. Suppose $\sum_i w_i^* < (p+u)\prod_{i=1}^n P_i$, then there musts exists a supplier i such that $w_i^* = c_i$ with $\pi_{si}(\mathbf{w}^*) = 0$. Since $q^*(\sum_i w_i^*) > 0$ and $q^*(\sum_i w_i)$ is continuous on $[\sum_i c_i, (p+u)\prod_i P_i]$, there must exists a $w_i' > c_i$ such that and $\sum_{j\neq i} w_i^* + w_i' < (p+u)\prod_i P_i$:

$$\pi_{si}(w_i', \mathbf{w}_{-i}^*) = (w_i' - c_i')q^*(\sum_{j\neq i} w_i^* + w_i') > 0$$

which contradict with the fact that $\mathbf{w}^*$ is an equilibrium point. Therefore $\sum_{i=1}^n w_i^* = (p+u)\prod_{i=1}^n P_i$ with $q^*(\mathbf{w}^*) = F^{-1}(0) = a$.

For any non-trivial equilibrium point such that $\sum_{i=1}^n w_i^* = (p+u)\prod_i P_i$, $\forall i \in \mathcal{I}^+ = \{j = 1,\ldots,n | w_j^* > c_j^*\}$ and let $m = |\mathcal{I}^+|$, we must have

$$\begin{aligned}
& \lim_{w_i \to w_i^{*-}} \frac{\partial \pi_{si}}{\partial w_i} \geq 0 \\
\Rightarrow \quad & \lim_{w_i \to w_i^{*-}} \Big\{ F^{-1}\big(\frac{(p+u)\prod_{i=1}^n P_i - \sum_{j\neq i} w_j^* - w_i}{(p+u-s)\prod_{i=1}^n P_i}\big) \\
& - \frac{(w_i - c_i)}{(p+u-s)\prod_{i=1}^n P_i f\Big(F^{-1}\big(\frac{(p+u)\prod_{i=1}^n P_i - \sum_{j\neq i} w_j^* - w_i}{(p+u-s)\prod_{i=1}^n P_i}\big)\Big)} \Big\} \geq 0. \\
& \text{Let q=}F^{-1}\big(\frac{(p+u)\prod_{i=1}^n P_i - \sum_{j\neq i} w_j^* - w_i}{(p+u-s)\prod_{i=1}^n P_i}\big) \ . \\
& \text{Since } q \text{ is continuously decreasing in } w_i, \\
& \text{and } q \to a^+ \text{ as } w_i \to w_i^{*-}, \text{ then we have} \\
& \lim_{x \to a^+} x - \frac{(w_i^* - c_i)}{(p+u-s)\prod_{i=1}^n P_i f(x)} \geq 0. \\
\text{Sum all } i \in \mathcal{I}^+ \text{ up} \quad \Rightarrow \quad & m \lim_{x \to a^+} x \geq \frac{\sum_{i\in\mathcal{I}^+}(w_i^* - c_i)}{(p+u)\prod_{i=1}^n P_i \lim_{x\to a^+} f(x)} \\
\Rightarrow \quad & \lim_{x \to a^+} x f(x) \geq \frac{(p+u)\prod_{i=1}^n P_i - \sum_{i=1}^n c_i}{m(p+u-s)\prod_{i=1}^n P_i}.
\end{aligned}$$

Q.E.D.

Proof of lemma 2.20. Since $a = 0$ or $\lim_{x\to a^+} x f(x) < \frac{\bar{p}-\bar{c}}{n\bar{p}}$, by lemma 2.19, the non-trivial equilibrium points must be interior points and satisfy the following equation:

$$w_i^* - c_i = \bar{p} q^* f(q^*) \qquad \forall i = 1,\ldots,n$$

In addition, for the manufacturer's optimal decision, we have:

$$\begin{aligned}
& \bar{p} - \sum_i w_i^* = \bar{p} F(q^*) \\
\Rightarrow \quad & \sum_i w_i^* - \bar{c} = \bar{p}(1 - F(q^*)) - \bar{c} \\
\Rightarrow \quad & w_i^* - c_i = \frac{\bar{p} - \bar{c} - \bar{p}F(q^*)}{n} \qquad \forall i = 1,\ldots,n.
\end{aligned}$$

Thus, if we can show that there exists a unique q for the following equation, we get the uniqueness of the equilibrium $\mathbf{w}$. Note that

$$\begin{aligned} \frac{\bar{p} - \bar{c} - \bar{p}F(q)}{n} &= \bar{p}qf(q) \\ \Rightarrow \quad 1 - F(q) - nqf(q) &= \frac{\bar{c}}{\bar{p}}. \end{aligned} \tag{A.14}$$

Let $v(q) = 1/\frac{qf(q)}{1-F(q)}$, then

$$\begin{aligned} \frac{\partial[1 - F(q) - nqf(q)]}{\partial q} &= -f(q) - nf(q) - nqf'(q) \\ &= -f(q)(1 - n\frac{qf(q)}{1-F(q)}) \\ &\quad + n\frac{-f(q)^2 q - (1 - F(q))(f(q) + qf'(q))}{(qf(q))^2}(1 - F(q))\frac{(qf(q))^2}{(1 - F(q))^2} \\ &= -f(q)(1 - \frac{n}{v(q)}) + nv'(q)(1 - F(q))/v(q)^2. \end{aligned}$$

Define $\bar{q}$ to be $\sup_q\{v(q) \geq n\}$. Since $\lim_{q \to 0} v(q) = +\infty$, by lemma 1 in [26], the $\bar{q}$ exists and $\bar{q} < \infty$. By the IGFR property, $v'(q) \leq 0$ for $q \geq 0$. Hence, for $q < \bar{q}$, $1 - F(q) - nqf(q)$ is strictly decreasing and for $q \geq \bar{q}$, $1 - F(q) - nqf(q) < 0$. Consequently, the left side of the equation A.14 is either strictly monotone decreasing or negative while the right side of the equation is a positive constant. Therefore, there exists a unique q^* satisfies the equation. By lemma 2.19, there always exists an equilibrium $\mathbf{w}^*$ with corresponding $q^*(\mathbf{w}^*)$ satisfy the equation. Hence, there exists a unique equilibrium q^* with corresponding unique $\mathbf{w}^*$.

Q.E.D.

Proof of lemma 2.21 First, we show that $\sum_i w_i^* = \bar{p}$ for any non-trivial equilibrium point $\mathbf{w}^*$ by contradiction. Suppose not, then $\sum_i w_i^* < \bar{p}$ and $q^* > a$. By lemma 2.19, $w_i^* - c_i = \bar{p}q^*f(q^*)$ for all i and q^* must be the solution for

$$1 - F(q) - nqf(q) = \frac{\bar{c}}{\bar{p}}.$$

In addition,

$$\begin{aligned} &\frac{\bar{p}-\bar{c}}{n\bar{p}} \leq \lim_{q\to a^+} qf(q) \\ \Rightarrow \quad &\lim_{q\to a^+} 1 - F(q) - nqf(q) \leq \bar{c}/\bar{p}. \end{aligned}$$

By lemma 2.20, $1 - F(q^*) - nq^* f(q^*)$ is strictly decreasing and then negative for $q \geq a$. Therefore, when $q^* > a$, we have $1 - F(q^*) - nq^* f(q^*) < \bar{c}/\bar{p}$ which leads to a contradiction. Hence, $\sum_i w_i^* = \bar{p}$ for all the non-trivial equilibrium $\mathbf{w}^*$. In addition, for all $w_i^* > c_i$, $\lim_{q\to a^+} qf(q) \geq \frac{w_i^*-c_i}{\bar{p}}$ by lemma 2.19.
Also note that since $\frac{\bar{p}-\bar{c}}{n\bar{p}} \leq \lim_{q\to a^+} qf(q)$, the set of of points defined as $\mathcal{W}^* = \{\mathbf{w}^* : \sum_i w_i^* = \bar{p}$ and $\lim_{x\to a^+} xf(x) \leq \frac{w_i^*-c_i}{\bar{p}}$ for all $w_i^* - c_i > 0\}$ is non-empty and contains the point $w_i^* = c_i + (\bar{p}-\bar{c})/n$, $\forall i = 1, \ldots, n$.
Let $\mathbf{w}^* \in \mathcal{W}^*$. For all i such that $w_i^* = c_i$, we have $\pi_{si}(w_i, \mathbf{w}^*_{-i}) = 0 = \pi_{si}(\mathbf{w}^*)$ for all $w_i \geq c_i$. Furthermore, for all i such that $w_i^* > c_i$, $\pi_{si}(w_i, \mathbf{w}^*_{-i}) = 0$ because $\forall w_i > w_i^*$. Thus to show $\mathbf{w}^*$ is an equilibrium, it's enough for us to show that

$$\begin{aligned} &\frac{\partial \pi_{si}(w_i, \mathbf{w}^*_{-i})}{\partial w_i} \\ = \quad & q^*(\sum_{j\neq i} w_j^* + w_i) - \frac{(w_i - c_i)}{\bar{p} f(q^*(\sum_{j\neq i} w_j^* + w_i))} > 0, \qquad \forall w_i < w_i^*. \end{aligned}$$

For $w_i < w_i^*$, $q^*(\sum_{j\neq i} w_j^* + w_i) > a$ and $w_i = (1 - F(q^*(\sum_{j\neq i} w_j^* + w_i)))\bar{p} - \sum_{j\neq i} w_j^*$. Let $q = q^*(\sum_{j\neq i} w_j^* + w_i)$ and replace w_i by $(1-F(q))\bar{p} - \sum_{j\neq i} w_j^*$. It's equivalent to show that

$$q - \frac{(1-F(q))\bar{p} - \sum_{j\neq i} w_j^* - c_i}{\bar{p} f(q)} > 0 \qquad \forall q > a$$

$$\text{it's equivalent to:} \quad 1 - F(q) - qf(q) < \frac{\sum_{j\neq i} w_j^* + c_i}{\bar{p}} = 1 - \frac{w_i^* - c_i}{\bar{p}} \quad \forall q > a$$

Note that, by lemma 1 in [26], for $q \geq a$, $1 - F(q) - qf(q)$ is strictly continuously decreasing and then have a negative value, so for all i such that $w_i^* - c_i > 0$ we have:

$$\begin{aligned} \lim_{q\to a^+} qf(q) \geq \frac{w_i^* - c_i}{\bar{p}} &\Rightarrow \lim_{q\to a^+} 1 - F(q) - qf(q) \leq 1 - \frac{w_i^* - c_i}{\bar{p}} \\ &\Rightarrow 1 - F(q) - qf(q) < 1 - \frac{\bar{w}_i^* - c_i}{\bar{p}} \quad \forall q > a \end{aligned}$$

Therefore, w_i^* maximize $\pi_{si}(w_i|\mathbf{w}_{-i}^*)$ and hence $\mathbf{w}^*$ is an equilibrium point.

Note that, when $\frac{\bar{p}-\bar{c}}{n\bar{p}} = \lim_{x\to a^+} xf(x)$, the only point satisfy $\sum_{i=1}^n w_i^* = \bar{p}-\bar{c}$ and $\lim_{x\to a^+} xf(x) \le \frac{w_i^*-c_i}{\bar{p}}$ for all $w_i^* - c_i > 0$ is $\mathbf{w}^* = \mathbf{c}^* + \frac{\bar{p}-\bar{c}}{n\bar{p}}$.

Q.E.D.

Proof of lemma 2.22. Since $F(q^*) = \int_0^{q^*} f(x)dx = qf(q) - \int_0^{q^*} qdf(q)$, if F is convex (concave), then $q^*f(q^*) \ge [\le] F(q^*)$. Hence in case that F is convex, $\tau = q^*f(q^*)/F(q^*) \ge 1$. Note that $\pi_{si}(q^*)/\pi(q^*) \ge \tau \ge 1$, which means the each individual supplier get more profit. In addition, the realized service level is $\frac{1}{n\tau+1}\frac{\bar{p}-\bar{c}}{\bar{p}} \le \frac{\bar{p}-\bar{c}}{(n+1)\bar{p}}$. In case that F is concave, then $0 \le \tau \le 1$ and the realized service level is $\frac{\bar{p}-\bar{c}}{\bar{p}} \ge \frac{1}{n\tau+1}\frac{\bar{p}-\bar{c}}{\bar{p}} \ge \frac{\bar{p}-\bar{c}}{(n+1)\bar{p}}$. **Q.E.D.**

Proof of corollary 2.4 For all the three distributions which are unimodal, F is convex in the area when q^* is less than the mode of the F. Hence for Normal distribution with mode is μ, we solve:

$$\begin{aligned} & \big(1 - F(q) - nqf(q)\big)|_{q=\mu} \le \bar{c}\bar{p} \\ \Rightarrow\ & \frac{1}{2} - \frac{n\mu}{\sqrt{2\pi}\sigma} \le \frac{\bar{c}}{\bar{p}} \\ \Rightarrow\ & \frac{\sigma}{\mu} \le \sqrt{2/\pi}\,n\bar{p}/(\bar{p}-2\bar{c}) \end{aligned}$$

Similarly, for gamma distribution with mode $(\alpha-1)\beta$, we know that:

$$(1 - F(q) - qf(q)|_{q=(\alpha-1)\beta} \le \frac{\bar{c}}{\bar{p}} \Rightarrow (1 - F(q) - nqf(q)|_{q=(\alpha-1)\beta} \le \frac{\bar{c}}{\bar{p}}$$

And relax $\frac{\bar{c}}{\bar{p}} = 0$, by the numerical result of corollary 1 in [26], we get $\alpha > 3.83$.

For weibull distribution with mode $(\frac{(k-1)\theta}{k})^{1/k}$, we have

$$\begin{aligned} & (1 - F(q) - nqf(q)|_{q=(\frac{(k-1)\theta}{k})^{1/k}} \le \frac{\bar{c}}{\bar{p}} \\ \Rightarrow\ & e^{-\frac{k-1}{\theta^{k-1}k}}(1 - n(k-1)) \le \frac{\bar{c}}{\bar{p}} \end{aligned}$$

For $k \ge 1 + 1/n$, the above results always holds.

Q.E.D.

Proof of theorem 2.11 First, if $\delta = 0$, then the cumulative distribution of demand $F_D(x) = F(x/\lambda)$ and the density of demand $f_D(x) = f(x/\lambda)/\lambda$. For any $\lambda_1, \lambda_2 > 0$, the equilibrium ordering quantity $q^*_{\lambda_1}$ for λ_1 and $q^*_{\lambda_2}$ for λ_2 should satisfy the equation 2.6:

$$\underbrace{1 - F(q/\lambda) - nq/\lambda f(q/\lambda) = \frac{\bar{c}}{\bar{p}}}_{\text{unique solution for } q/\lambda} \Rightarrow q^*_{\lambda_1}/\lambda_1 = q^*_{\lambda_2}/\lambda_2.$$

Hence

$$\lambda_2 \pi^*_{si\lambda_1} = \bar{p}\lambda_2 q^{*2}_{\lambda_1} f(q^*_{\lambda_1}/\lambda_1)/\lambda_1 = \bar{p}\lambda_1 q^{*2}_{\lambda_2} f(q^*_{\lambda_2}/\lambda_2)/\lambda_2 = \lambda_1 \pi^*_{si\lambda_2}.$$

Also since $w^*_i - c_i = \bar{p}q^*/\lambda f(q^*/\lambda)$, which means $\sum_i w^*_i$ is a constant for different λ, thus:

$$\begin{aligned}
\lambda_2 \pi^*_{\lambda_1} &= \lambda_2(-\sum_i w_i q^*_{\lambda_1} + \bar{p}\lambda_1\mu - \bar{p}E[(\lambda_1 X - q^*_{\lambda_1})^+]) \\
&= \lambda_1(-\sum_i w_i q^*_{\lambda_2} + \bar{p}\lambda_2\mu - \bar{p}E[(\lambda_2 X - q^*_{\lambda_2})^+]) \\
&= \lambda_1 \pi^*_{\lambda_2}
\end{aligned}$$

Similarly, the optimal ordering quantity q^I_λ satisfy $F(q^I_\lambda/\lambda) = 1 - \bar{c}/\bar{p}$ and $\lambda_1 q^I_{\lambda_2} = \lambda_2 q^I_{\lambda_1}$. Thus:

$$\begin{aligned}
\lambda_2 \pi^I_{\lambda_1} &= \lambda_2(-\sum_i w_i q^I_{\lambda_1} + \bar{p}\lambda_1\mu - \bar{p}E[(\lambda_1 X - q^I_{\lambda_1})^+]) \\
&= \lambda_1(-\sum_i w_i q^I_{\lambda_2} + \bar{p}\lambda_2\mu - \bar{p}E[(\lambda_2 X - q^I_{\lambda_2})^+]) \\
&= \lambda_1 \pi^I_{\lambda_2}
\end{aligned}$$

Hence efficiency and split of profits are independent of λ.

For part b), without loss of generality, let $\lambda = 1$ and $\delta > 0$. Thus the demand distribution $F_\delta(x) = F(x - \delta)$ and $\pi^I_\delta = \pi^I_0 + \delta(\bar{p} - \bar{c})$. Since $f(0) > 0$, then $\bar{\delta} = \frac{1-\bar{c}/\bar{p}}{nf(0)}$ is well defined. In addition, at $\delta = \bar{\delta}$, the solution for equation 2.6 is $q = \bar{\delta}$. For $\delta > \bar{\delta}$, by lemma 2.21, although there are multiple non-trivial equilibria, they all lead to the same $\sum_i w^*_i = \bar{p}$ and $q^* = \delta$ with

$\pi^D_\delta = (\bar{p}-\bar{c})\delta$. Note that $\pi^I_\delta = (\bar{p}-\bar{c})\delta + \pi^I_\delta$, hence $\pi^D_\delta/\pi^I_\delta = 1-(\bar{p}-\bar{c})\delta/((\bar{p}-\bar{c})\delta+\pi^I_0)$ for $\delta \geq \bar{\delta}$ which is increasing in δ. Therefore, there exists $\delta' \leq \bar{\delta}$ such that $\pi^D_\delta/\pi^I_\delta$ is increasing for $\delta \geq \delta'$. To convert from values of δ to values of the coefficient of variance, note that $D = \delta(\lambda) + \lambda X = \lambda(\widehat{\delta} + X)$ where $\widehat{\delta} = \sigma/\rho - \mu$. An increasing in the coefficient of variance ρ is thus equivalent to a decrease in δ.

Q.E.D.

Proof of theorem 2.12. Since q^S is the suppliers coordination ordering quantity, then in case that $\delta > 0$ and $\lim_{x\to\delta} xf(x) \geq \frac{\bar{p}-\bar{c}}{\bar{p}}$, we have $q^S = q^* = \delta$. Otherwise, q^S satisfies the first order condition and we have:

$$\frac{\bar{c}}{\bar{p}} = 1 - F(q^S) - q^S f(q^S) \geq 1 - F(q^S) - nq^S f(q^S)$$

Since $1 - F(q) - nqf(q)$ is strictly decreasing then negative, therefore there will not exist any $q^* > q^S$ such that $1 - F(q) - nqf(q) = \bar{c}/\bar{p}$ can hold. Therefore $q^* \leq q^S$.

Also note that the system wise coordination ordering quantity $q^I \geq q^S \geq q^*$, then by the concavity of the system profit in q, $\pi^S \geq \pi^D$.

When F is convex in $[0, q^S]$ (f is increasing), we have:

$$1 - F(q^S) - q^S f(q^S) = 1 - F(nq^S/n) - nq^S/nf(nq^S/n) \leq 1 - F(q^S/n) - nq^S/nf(q^S/n).$$

Hence $q^* \geq q^S/n$. By lemma 2.22, since F is convex in $[0, q^S]$, $\pi^S = \pi^S_s + \pi^S_M \leq 2\pi^S_s = 2(\sum w_i - \bar{c})q^S = 2q^{S2}f(q^S)$. Also $\pi^D \geq n(w_i - c_i)q^* = nq^{*2}f(q^*)$. Therefore:

$$\frac{\pi^D}{\pi^S} \geq \frac{nq^*f(q^*)}{2\underbrace{q^{S2}}_{\leq n^2q^{*2}}\underbrace{f(q^S)}_{\leq f(q^*)}} \geq \frac{nq^*f(q^*)}{2nq^{*2}f(q^*)} = \frac{1}{2n}.$$

Part b) follows from theorem 2.10 and 2.11. For part c, let $\lambda = 1$ and $\delta > 0$, then $\bar{\delta} = \frac{1-\bar{c}/\bar{p}}{f(0)}$ is well defined. For $\delta \geq \bar{\delta}$, we have $q^* = \delta$ and $q^S = \delta$, with $\sum w^*_i = \bar{p}$ and $\sum w^S_i = \bar{p}$ ($\mathbf{w}^S$ is the wholesale price under suppliers coordination). Therefore $\pi^D = \pi^S$ for $\delta \geq \bar{\delta}$ [corresponding $\bar{\rho} = \bar{p}f(0)\sigma/(\bar{p}\mu f(0) + \bar{p} - \bar{c})$]. Also define $\widetilde{\delta} = \frac{1-\bar{c}/\bar{p}}{nf(0)}$. Thus for $\bar{\delta} \geq \delta \geq \widetilde{\delta}$, we have $q^* = \delta$,

$\pi_\delta^* = \delta(\bar{p} - \bar{c})$ and q_δ^S satisfy:

$$1 - F(\delta - q_\delta^S) - q_\delta^S f(\delta - q_\delta^S) = \bar{c}/\bar{p}.$$

Thus for any $\triangle\delta > 0$, we have:

$$\begin{aligned} & 1 - F(\delta + \triangle\delta - (q_\delta^S + \triangle\delta)) - (q_\delta^S + \triangle\delta) f(\delta + \triangle\delta - (q_\delta^S + \triangle\delta)) \\ \leq\ & 1 - F(\delta - q_\delta^S) - q_\delta^S f(\delta - q_\delta^S) = \bar{c}/\bar{p}. \end{aligned}$$

Therefore, the corresponding $q_{\delta+\triangle\delta}^S$ is less than $q_\delta^S + \triangle\delta$. Hence $\frac{\partial q_\delta^S}{\partial \delta} \leq 1$ and $(D - q_\delta^S)^+$ is increasing in δ. Since

$$\begin{aligned} & \pi_\delta^S = (\bar{p} - \bar{c}) q_\delta^S - \bar{p} E[(D - q_\delta^S)^+] \\ \Rightarrow\ & \frac{\partial \pi_\delta^S}{\partial \delta} = (\bar{p} - \bar{c}) \underbrace{\frac{\partial q_\delta^S}{\partial \delta}}_{\leq 1} - \bar{p} \underbrace{\frac{\partial E[(D - q_\delta^S)^+]}{\partial \delta}}_{\geq 0} \\ \Rightarrow\ & \frac{\partial \pi_\delta^S}{\partial \delta} \leq \bar{p} - \bar{c}, \end{aligned}$$

we have:

$$\frac{\partial \pi_\delta^D / \pi_\delta^S}{\partial \delta} = \frac{\frac{\pi_\delta^D}{\partial \delta} \pi_\delta^S - \frac{\pi_\delta^S}{\partial \delta} \pi_\delta^D}{\pi_\delta^{S2}} \geq \frac{(\bar{p} - \bar{c}) \pi_\delta^S - (\bar{p} - \bar{c}) \pi_\delta^D}{\pi_\delta^{S2}} \geq 0.$$

Hence there exists $\delta' \leq \widetilde{\delta}$ such that for all $\delta \geq \delta'$, $\pi_\delta^D / \pi_\delta^S$ is increasing.

Q.E.D.

Proof of theorem 2.13. Let $\nu(x) = 1 - F(x) - nf(x)$. Since F is IGFR, by lemma 2.20, $\nu(x)$ is strictly decreasing then negative and continuous for $x \in [a, b)$. Thus ν^{-1} is strictly decreasing and continuous in $(0, 1)$. And since $\sum_i c_i P_i$ (also $\frac{\sum_{i=1}^n c_i(P_i)}{p \prod_{i=1}^n P_i}$) is continuous, $q(\mathbf{P}) = \nu^{-1}(\frac{\sum_{i=1}^n c_i(P_i)}{p \prod_{i=1}^n P_i})$ is continuous in $\mathbf{P}$. Hence $\psi(\mathbf{P}) = p \prod_{i=1}^n P_i q(\mathbf{P})^2 f(q(\mathbf{P})) - \sum_{i=1}^n k_i(P_i)$ continuous in $\mathbf{P}$. The maximizer exists because $\mathcal{P}$ is compact. Let P^* be one of the maximizers, then $\forall i = 1, \ldots, n$, $P_i \in \mathcal{P}_i$, we have :

$$\begin{aligned}
& \psi(P_i^*, \mathbf{P}_{-i}^*) - \psi(P_i, \mathbf{P}_{-i}^*) \geq 0 \\
\Rightarrow \quad & p \prod_{j=1}^{n} P_j^* q(\mathbf{P}^*)^2 f(q(\mathbf{P}^*)) - \sum_{i=j}^{n} k_j(P_j^*) \\
& \geq p \prod_{j \neq i}^{n} P_j^* P_i' q(P_i, \mathbf{P}_{-i}^*)^2 f(q(P_i, \mathbf{P}_{-i}^*)) - \sum_{j \neq i} k_j(P_j^*) - k_i(P_i) \\
\Rightarrow \quad & p \prod_{j=1}^{n} P_j^* q(\mathbf{P}^*)^2 f(q(\mathbf{P}^*)) - k_i(P_i^*) - \Big(p \prod_{j \neq i}^{n} P_j^* P_i' q(P_i, \mathbf{P}_{-i}^*)^2 f(q(P_i, \mathbf{P}_{-i}^*)) - k_i(P_i) \Big) \geq 0 \\
\Rightarrow \quad & \pi(P_i^*, \mathbf{P}_{-i}^*) \geq \pi(P_i, \mathbf{P}_{-i}^*).
\end{aligned}$$

Therefore, $\mathbf{P}^*$ is the Nash equilibrium for the reliability investment game.

Q.E.D.

Proof of Theorem 2.23: By equation 2.9, we can rewrite supplier i's profit as:

$$\pi_{si}(\mathbf{P}) = p \prod_i P_i q(\mathbf{P})^2 f(q(\mathbf{P})) - k_i(P_i) = \frac{1}{n} q(\mathbf{P})(\bar{p} - \bar{p}F(q(\mathbf{P})) - \bar{c}) - k_i(P_i).$$

Thus, for $i \neq j$,

$$\frac{\partial \pi_{si}(\mathbf{P})}{\partial P_j} = \frac{1}{n}\Big\{ \underbrace{\big(p \prod_k P_k (1 - F(q(\mathbf{P})) - qf(q(\mathbf{P}))) - \bar{c}\big)}_{\geq 0 \text{ since } 1 - F(q(\mathbf{P})) - nqf(q(\mathbf{P})) \geq \bar{c}/(p \prod_k P_k)} \frac{\partial q(\mathbf{P})}{\partial P_j} + \underbrace{(1 - F(q(\mathbf{P})))q(\mathbf{P}) p \prod_{k \neq j} P_k}_{\geq 0} \Big\}.$$

Note that $q(\mathbf{P})$ is the solution for $1 - F(q) - nqf(q) = \bar{c}/(p \sum_i P_i)$, where the left hand side of equation is decreasing with q and the right hand side is decreasing with P_j. Therefore, with increasing of P_j, $q(\mathbf{P})$ is increasing. Hence, $\frac{\partial \pi_{si}(\mathbf{P})}{\partial P_j} \geq 0$.

Q.E.D.

Proof of Lemma 2.14. Similar to the approach used to prove lemma 2.14, we demonstrate the concavity of the expected profit by showing that it is a composition of concave function. First, let us show that the function $\min_{i \in \mathcal{C}}(\sum_{j \in \mathcal{T}_i} r_{ij} q_{ij})$ is jointly concave in $\mathbf{q}$. Let$\mathbf{q}^1$ and

$\mathbf{q}^2$ be two vector then for any $\lambda \in (0,1)$, we have

$$\begin{aligned}
& \min_{i\in\mathcal{C}} \sum_{j\in\mathcal{T}_i} \{\lambda r_{ij} q_{ij}^1 + (1-\lambda) r_{ij} q_{ij}^2\} \\
= & \sum_{j\in\mathcal{T}_i} \lambda (r_{ij} q_{ij}^1 + (1-\lambda) r_{ij} q_{ij}^2) \qquad \text{for some } i \in \mathcal{C} \\
= & \lambda \sum_{j\in\mathcal{T}_i} r_{ij} q_{ij}^1 + (1-\lambda) \sum_{j\in\mathcal{T}_i} r_{ij} q_{ij}^2 \qquad \text{for some } i \in \mathcal{C} \\
\geq & \lambda \min_{i\in\mathcal{C}} (\sum_{j\in\mathcal{T}_i} r_{ij} q_{ij}^1) + (1-\lambda) \min_{i\in\mathcal{C}} (\sum_{j\in\mathcal{T}_i} r_{ij} q_{ij}^2).
\end{aligned}$$

Hence, $d - \min_{i\in\mathcal{C}}(\sum_{j\in\mathcal{T}_i} r_{ij} q_{ij})$ is jointly convex in $\mathbf{q}$. Since the operator $()^+$ is monotone increasing and convex, $(d - \min_{i\in\mathcal{C}}(\sum_{j\in\mathcal{T}_i} r_{ij} q_{ij}))^+$ is convex in $\mathbf{q}$. Also, since the convexity (concavity) is preserved under expectation, $sE[\min_{i\in\mathcal{C}}(\sum_{j\in\mathcal{T}_i} R_{ij} q_{ij})] - (p+u-s)E[(D - \min_{i\in\mathcal{C}}(\sum_{j\in\mathcal{T}_i} R_{ij} q_{ij}))^+]$ is jointly concave in $\mathbf{q}$ when $s \geq 0$ and $p+u-s > 0$. Therefore, the whole function is concave in $\mathbf{q}$.

Q.E.D.

Proof of corollary 2.5: If there exists a component i such that in optimal solution $\mathbf{q}_i^* = 0$, then $E[\min_{i\in\mathcal{C}}(\sum_{j\in\mathcal{T}_i} R_{ij} q_{ij})] = 0$ and $E[(D - \min_{i\in\mathcal{C}}(\sum_{j\in\mathcal{T}_i} R_{ij} q_{ij}))^+] = E[D]$. Therefore, the manufacturer's profit is

$$\pi = -\sum_{k\in\mathcal{C}, k\neq i} \sum_{j\in\mathcal{T}_k} w_{ij} q_{ij} - uE[D].$$

Since $w_{ij} > 0$, the manufacturer's profit is strictly decreasing with q_{kj} for $k \neq i$, $j \in \mathcal{T}_k$. Hence, in an optimal solution, $\mathbf{q}_i^* = 0$ for all $i \in \mathcal{C}$.

Q.E.D.

Proof of lemma 2.24: We can rewrite the expected function as :

$$\begin{aligned}
\pi = & pE[D] - \sum_{i\in\mathcal{C}} \sum_{j\in\mathcal{T}_i} w_{ij} q_{ij} \\
& -(p+u)E[(D - \min_{i\in\mathcal{C}}(\sum_{j\in\mathcal{T}_i} R_{ij} q_{ij}))^+] + s[(\min_{i\in\mathcal{C}}(\sum_{j\in\mathcal{T}_i} R_{ij} q_{ij}) - D)^+]
\end{aligned}$$

$$E[(D-\min_{i\in\mathcal{C}}(\sum_{j\in\mathcal{T}_i}R_{ij}q_{ij}))^+] = \int_0^{+\infty}\int_0^v (v-x)f(v)\Big((1-G_{-i}(x|\mathbf{q}_{-i}))g_i(x|\mathbf{q}_i) + (1-G_i(x|\mathbf{q}_i))g_{-i}(x|\mathbf{q}_{-i})\Big)dxdv.$$

Note that

$$\begin{aligned}
&\int_0^v (v-x)f(v)(1-G_{-i}(x|\mathbf{q}_{-i}))g_i(x|\mathbf{q}_i)dx \\
= &\int_0^v (v-x)f(v)(1-G_{-i}(x|\mathbf{q}_{-i}))dG_i(x|\mathbf{q}_i) \\
= &(v-x)f(v)(1-G_{-i}(x|\mathbf{q}_{-i}))G_i(x|\mathbf{q}_i)|_0^v - \int_0^v f(v)G_i(x|\mathbf{q}_i)d(v-x)(1-G_{-i}(x|\mathbf{q}_{-i})) \\
= &\int_0^v f(v)G_i(x|\mathbf{q}_i)\Big((v-x)g_{-i}(x|\mathbf{q}_{-i}) + (1-G_{-i}(x|\mathbf{q}_{-i}))\Big)dx
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial E[(D-\min_{i\in\mathcal{C}}(\sum_{j\in\mathcal{T}_i}R_{ij}q_{ij}))^+]}{\partial q_{ij}} \\
= &\frac{\partial\int_0^{+\infty}\int_0^v f(v)G_i(x|\mathbf{q}_i)(1-G_{-i}(x|\mathbf{q}_{-i}))dxdv}{\partial q_{ij}} + \frac{\partial\int_0^{+\infty}\int_0^v (v-x)f(v)g_{-i}(x|\mathbf{q}_{-i})dxdv}{\partial q_{ij}} \\
= &\frac{\partial\int_0^{+\infty}(1-F(x))(1-G_{-i}(x|\mathbf{q}_{-i}))\int_0^1\dots\int_0^1\int_0^{\sum_{k\neq j,k\in\mathcal{T}_i}r_{ik}q_{ik}}\prod_{k\in\mathcal{T}_i}g_{ik}(r_{ik})\prod_{k\in\mathcal{T}_i}dr_{ik}dx}{\partial q_{ij}} + 0 \\
= &-\int_0^{+\infty}\int_0^1\dots\int_0^1(1-F(x))(1-G_{-i}(x|\mathbf{q}_{-i}))\frac{x-\sum_{k\neq j,k\in\mathcal{T}_i}r_{ik}q_{ik}}{q_{ij}^2}\prod_{k\neq j,k\in\mathcal{T}_i}g_{ik}(r_{ik})\prod_{k\neq j,k\in\mathcal{T}_i}dr_{ik}dx \\
&\text{let } \frac{x-\sum_{k\neq j,k\in\mathcal{T}_i}r_{ik}q_{ik}}{q_{ij}} = r_{ij}\text{, we have} \\
= &-\int_0^1\dots\int_0^1 r_{ij}(1-F(\sum_{k\in\mathcal{T}_i}r_{ik}q_{ik}))(1-G_{-i}(\sum_{k\in\mathcal{T}_i}r_{ik}q_{ik}|\mathbf{q}_{-i}))\prod_{k\in\mathcal{T}_i}g_{ik}(r_{ik})\prod_{k\in\mathcal{T}_i}dr_{ik} \\
= &-E[R_{ij}(1-F(\sum_{k\in\mathcal{T}_i}r_{ik}q_{ik}))]_{\mathcal{S}_i(\mathbf{q})}.
\end{aligned}$$

Similarly,

$$
\begin{aligned}
& E[(\min_{i\in\mathcal{C}}(\sum_{j\in\mathcal{T}_i} R_{ij}q_{ij}) - D)^+] \\
= \ & \int_0^{+\infty}\int_v^{+\infty}(x-v)f(v)\Big((1-G_{-i}(x|\mathbf{q}_{-i}))g_i(x|\mathbf{q}_i) + (1-G_i(x|\mathbf{q}_i))g_{-i}(x|\mathbf{q}_{-i})\Big)dxdv \\
& \frac{\partial E[(D-\min_{i\in\mathcal{C}}(\sum_{j\in\mathcal{T}_i} R_{ij}q_{ij}))^+]}{\partial q_{ij}} \\
= \ & -\frac{\partial \int_0^{+\infty} F(x)(1-G_{-i}(x|\mathbf{q}_{-i}))\int_0^1\dots\int_0^1\int_0^{\sum_{k\neq j,k\in\mathcal{T}_i} r_{ik}q_{ik}}\prod_{k\in\mathcal{T}_i} g_{ik}(r_{ik})\prod_{k\in\mathcal{T}_i} dr_{ik}dx}{\partial q_{ij}} \\
= \ & \int_0^1\dots\int_0^1 r_{ij}F(\sum_{k\in\mathcal{T}_i} r_{ik}q_{ik})(1-G_{-i}(\sum_{k\in\mathcal{T}_i} r_{ik}q_{ik}|\mathbf{q}_{-i}))\prod_{k\in\mathcal{T}_i} g_{ik}(r_{ik})\prod_{k\in\mathcal{T}_i} dr_{ik} \\
= \ & E[R_{ij}F(\sum_{k\in\mathcal{T}_i} r_{ik}q_{ik})]_{\mathcal{S}_i(\mathbf{q})}.
\end{aligned}
$$

Therefore, by the concavity of π, the **KKT** condition is necessary and sufficient. Thus $\forall i\in\mathcal{C},\ j\in\mathcal{T}_i$, we have

$$
\begin{aligned}
q_{ij}^* > 0 \ \Rightarrow \ & \frac{\partial\pi}{q_{ij}} = -w_{ij} + (p+u)E[R_{ij}(1-F(\sum_{k\in\mathcal{T}_i} R_{ik}q_{ik}^*))]_{\mathcal{S}_i(\mathbf{q}^*)} + E[R_{ij}F(\sum_{k\in\mathcal{T}_i} R_{ik}q_{ik}^*)]_{\mathcal{S}_i(\mathbf{q}^*)} = 0 \\
\Rightarrow \ & w_{ij} = (p+u)E[R_{ij}]_{\mathcal{S}_i(\mathbf{q}^*)} - (p+u-s)E[R_{ij}(1-F(\sum_{k\in\mathcal{T}_i} R_{ik}q_{ik})^*)]_{\mathcal{S}_i(\mathbf{q}^*)} \\
q_{ij}^* = 0 \ \Rightarrow \ & \frac{\partial\pi}{q_{ij}} \le 0 \\
\Rightarrow \ & w_{ij} \ge (p+u)E[R_{ij}]_{\mathcal{S}_i(\mathbf{q}^*)} - (p+u-s)E[R_{ij}(1-F(\sum_{k\in\mathcal{T}_i} R_{ik}q_{ik})^*)]_{\mathcal{S}_i(\mathbf{q}^*)}
\end{aligned}
$$

Q.E.D.

Proof of lemma 2.25. Let $\mathbf{q}''^*$ be the optimal solution under demand D''. Consider the function:

$$\nu(d,\mathbf{r}|\mathbf{q}''^*) = pd - (p+u-s)(d - \min_{i\in\mathcal{C}}(\sum_{j\in\mathcal{T}_i} r_{ij}q_{ij}''^*))^+ \tag{A.15}$$

Clearly, $\nu(d,\mathbf{r}|\mathbf{q}''^*)$ is increasing and concave in d. Since $E[D'] = E[D'']$ and $D' \le_{disp} D''$, by

Theorem 3.B.2 in [40], we have $E[\nu(D', \mathbf{r}|\mathbf{q}''^*)] \geq E[\nu(D'', \mathbf{r}|\mathbf{q}''^*)]$ for any given $\mathbf{r}$. Therefore

$$\begin{aligned}
\pi''(\mathbf{q}''^*) &= -sE[D''] + sE[\min_{i\in\mathcal{C}}(\sum_{j\in\mathcal{T}_i} R_{ij}q''^*_{ij})] - \sum_{i\in\mathcal{C}}\sum_{j\in\mathcal{T}_i} w_{ij}q''^*_{ij} + E[\nu(D'', \mathbf{R}|\mathbf{q}''^*)] \\
&\leq -sE[D'] + sE[\min_{i\in\mathcal{C}}(\sum_{j\in\mathcal{T}_i} R_{ij}q''^*_{ij})] - \sum_{i\in\mathcal{C}}\sum_{j\in\mathcal{T}_i} w_{ij}q''^*_{ij} + E[\nu(D', \mathbf{R}|\mathbf{q}''^*)] \\
&= \pi'(\mathbf{q}''^*) \leq \pi'^*
\end{aligned}$$

Thus, we have $\pi'^* \geq \pi''^*$.

Q.E.D.

Proof of lemma 2.26. Let $\mathbf{q}''^*$ be the optimal solution under demand D''. Consider the function:

$$\nu(\mathbf{r}, d|\mathbf{q}''^*) = s(\min_{i\in\mathcal{C}}(\sum_{j\in\mathcal{T}_i} r_{ij}q''^*_{ij}) - d)^+ - (p+u)(d - \min_{i\in\mathcal{C}}(\sum_{j\in\mathcal{T}_i} r_{ij}q''^*_{ij}))^+.$$

Since $p+u > s$, clearly, $\nu(\mathbf{r}, d|\mathbf{q}''^*)$ is increasing and concave in $\mathbf{r}$ for give $\mathbf{q}^*$. Since $E[R'_{ij}] \geq E[R''_{ij}]$ and $R'_{ij} \leq_{disp} R''_{ij}$ for all $i \in \mathcal{C}$ and $j \in \mathcal{T}_i$, by Theorem 3.B.2 in [40], we have $E[\nu(\mathbf{R}', d|\mathbf{q}''^*)] \geq E[\nu(\mathbf{R}'', d|\mathbf{q}''^*)]$ for any given d. Therefore,

$$\begin{aligned}
\pi''(\mathbf{q}''^*) &= pE[D] - \sum_{i\in\mathcal{C}}\sum_{j\in\mathcal{T}_i} w_{ij}q''^*_{ij} + E[\nu(\mathbf{R}'', D|\mathbf{q}''^*)] \\
&\leq pE[D] - \sum_{i\in\mathcal{C}}\sum_{j\in\mathcal{T}_i} w_{ij}q''^*_{ij} + E[\nu(\mathbf{R}', D|\mathbf{q}''^*)] \\
&= \pi'(\mathbf{q}''^*) \leq \pi'^*
\end{aligned}$$

Thus, we have $\pi'^* \geq \pi''^*$.

Q.E.D.

Proof of Lemma 2.27. We prove the result by contradiction. Suppose there exist $i \in \mathcal{C}$ and

$j, k \in \mathcal{T}_i$ such that $w_{ij}/\bar{R}_{ij} > w_{ik}/\bar{R}_{ik}$, $q^*_{ij} > 0$ with $q^*_{ik} = 0$. By lemma 2.24 we have:

$$\begin{aligned} w_{ij} &= (p+u)E[R_{ij}]_{\mathcal{S}_i(\mathbf{q}^*)} - (p+u-s)E[R_{ij}F(\sum_{m\in\mathcal{T}_i} R_{im}q^*_{im})]_{\mathcal{S}_i(\mathbf{q}^*)} \\ &= (p+u-s)E[R_{ij}(1-F(\sum_{m\in\mathcal{T}_i} R_{im}q^*_{im}))]_{\mathcal{S}_i(\mathbf{q}^*)} + sE[R_{ij}]_{\mathcal{S}_i(\mathbf{q}^*)} \\ w_{ik} &\geq (p+u)E[R_{ik}]_{\mathcal{S}_i(\mathbf{q}^*)} - (p+u-s)E[R_{ik}F(\sum_{m\in\mathcal{T}_i} R_{im}q^*_{im})]_{\mathcal{S}_i(\mathbf{q}^*)} \\ &= (p+u-s)E[R_{ik}(1-F(\sum_{m\in\mathcal{T}_i} R_{im}q^*_{im}))]_{\mathcal{S}_i(\mathbf{q}^*)} + sE[R_{ik}]_{\mathcal{S}_i(\mathbf{q}^*)}. \end{aligned}$$

Since $q^*_{ik} = 0$, we have

$$\begin{aligned} E[R_{ik}]_{\mathcal{S}_i(\mathbf{q}^*)} &= \int_0^1 \dots \int_0^1 r_{ik}(1 - G_{-i}(\sum_{m\in\mathcal{T}_i} r_{im}q^*_{im}|\mathbf{q}^*_{-i}))\big(\prod_{m\in\mathcal{T}_i} g^*_{im}(r_{im})dr_{im}\big) \\ &= E[R_{ik}]E[(1-G_{-i}(\sum_{m\in\mathcal{T}_i} R_{im}q^*_{im}|\mathbf{q}^*_{-i})] \\ E[R_{ik}F(\sum_{m\in\mathcal{T}_i} R_{im}q^*_{im})]_{\mathcal{S}_i(\mathbf{q}^*)} &= \int_0^1 \dots \int_0^1 r_{ik}F(\sum_{m\in\mathcal{T}_i} r_{im}q^*_{im}) \\ &\quad (1-G_{-i}(\sum_{m\in\mathcal{T}_i} r_{im}q^*_{im}|\mathbf{q}^*_{-i}))\big(\prod_{m\in\mathcal{T}_i} g_{im}(r_{im})dr_{im}\big) \\ &= E[R_{ik}]E[(1-G_{-i}(\sum_{m\in\mathcal{T}_i} R_{im}q^*_{im}|\mathbf{q}^*_{-i}))]. \end{aligned}$$

Also note that

$$G_{-i}(\sum_{m\in\mathcal{T}_i} r_{im}q^*_{im}|\mathbf{q}^*_{-i}) = Pr(\sum_{m\in\mathcal{T}_i} r_{im}q^*_{im} \geq \min_{i'\neq i} \sum_{m'\in\mathcal{T}_{i'}} R_{i'm'}q^*_{i'm'}).$$

Hence, $G_{-i}(\sum_{m\in\mathcal{T}_i} r_{im}q^*_{im}|\mathbf{q}^*_{-i}))$ is increasing with r_{ij}.

Therefore, $(1-F(\sum_{m\in\mathcal{T}_i} R_{im}q^*_{im}))(1-G_{-i}(\sum_{m\in\mathcal{T}_i} R_{im}q^*_{im}|\mathbf{q}^*_{-i}))$ and R_{ij} are negatively correlated. Thus

$$\begin{aligned}
& E[R_{ij}(1-F(\sum_{m\in\mathcal{T}_i} R_{im}q^*_{im}))]_{\mathcal{S}_i(\mathbf{q}^*)} \\
= & \int_0^1 \dots \int_0^1 r_{ij}(1-F(\sum_{m\in\mathcal{T}_i} r_{im}q^*_{im})) \\
& (1-G_{-i}(\sum_{m\in\mathcal{T}_i} r_{im}q^*_{im}|\mathbf{q}^*_{-i}))\big(\prod_{m\in\mathcal{T}_i} g_{im}(r_{im})dr_{im}\big) \\
\leq & \int_0^1 \dots \int_0^1 r_{ij}\big(\prod_{m\in\mathcal{T}_i} g_{im}(r_{im})dr_{im}\big) * \\
& \int_0^1 \dots \int_0^1 (1-F(\sum_{m\in\mathcal{T}_i} r_{im}q^*_{im}))(1-G_{-i}(\sum_{m\in\mathcal{T}_i} r_{im}q^*_{im}|\mathbf{q}^*_{-i}))\big(\prod_{m\in\mathcal{T}_i} g_{im}(r_{im})dr_{im}\big) \\
= & E[R_{ij}]E[(1-F(\sum_{m\in\mathcal{T}_i} R_{im}q^*_{im}))G_{-i}(\sum_{m\in\mathcal{T}_i} R_{im}q^*_{im}|\mathbf{q}^*_{-i})].
\end{aligned}$$

In addition, $(1-G_{-i}(\sum_{m\in\mathcal{T}_i} R_{im}q^*_{im}|\mathbf{q}^*_{-i}))$ and R_{ij} are negatively correlated, which implies

$$E[R_{ij}]_{\mathcal{S}_i(\mathbf{q}^*)} \leq E[R_{ij}]E[G_{-i}(\sum_{m\in\mathcal{T}_i} R_{im}q^*_{im}|\mathbf{q}^*_{-i})]$$

and hence,

$$\begin{aligned}
w_{ik} & \geq (p+u-s)E[R_{ik}]E[(1-F(\sum_{m\in\mathcal{T}_i} R_{im}q^*_{im}))G_{-i}(\sum_{m\in\mathcal{T}_i} R_{im}q^*_{im}|\mathbf{q}^*_{-i})] \\
& \quad +sE[R_{ik}]E[G_{-i}(\sum_{m\in\mathcal{T}_i} R_{im}q^*_{im}|\mathbf{q}^*_{-i})] \\
w_{ij} & \leq (p+u-s)E[R_{ik}]E[(1-F(\sum_{m\in\mathcal{T}_i} R_{im}q^*_{im}))G_{-i}(\sum_{m\in\mathcal{T}_i} R_{im}q^*_{im}|\mathbf{q}^*_{-i})] \\
& \quad +sE[R_{ij}]E[G_{-i}(\sum_{m\in\mathcal{T}_i} R_{im}q^*_{im}|\mathbf{q}^*_{-i})] \\
\Rightarrow & \quad w_{ik}/E[R_{ik}] \geq w_{ij}/E[R_{ij}],
\end{aligned}$$

which contradicts with $w_{ik}/E[R_{ik}] < w_{ij}/E[R_{ij}]$. Hence $q^*_{ik} > 0$.

Q.E.D.

Proof of lemma 2.28. Suppose not and there exists an optimal ordering quantity $\mathbf{q}^*$ such that $q^*_{ij} > \sum_{s\in\mathcal{T}_k} q^*_{ks}$ for some $i,k \in \mathcal{C}$ and $j \in \mathcal{T}_i$. Now construct another ordering strategy

$\mathbf{q}'$ such that $q'_{i'j'} = q^*_{i'j'}$ for all $(i', j') \neq (i, j)$ and $q_{ij} = \sum_{s \in \mathcal{T}_k} q^*_{ks}$. Thus, $\forall \mathcal{S} \subseteq \mathcal{T}$,

$$\min_{k \in \mathcal{C}} \Big(\sum_{\substack{(k,s) \in \mathcal{S} \\ s \in \mathcal{T}_k}} q^*_{ks} \Big) = \min_{k \in \mathcal{C}} \Big(\sum_{\substack{(k,s) \in \mathcal{S} \\ s \in \mathcal{T}_k}} q'_{ks} \Big)$$

Therefore,

$$\pi(\mathbf{q}^*) - \pi(\mathbf{q}') = -w_i (q^*_{ij} - q'_{ij}) > 0$$

which leads to a contradiction.

Q.E.D.

A.3. Proof for Chapter Three

Proof of lemma 3.1. Suppose $d(p^*_{(\mathbf{w})}, \mathbf{e}) = 0$ for all $\mathbf{w} \in \mathcal{W}$, then $\pi_i(\mathbf{w}) \equiv 0$ for all $i \in \{1, \ldots, n\}$ and $\mathbf{w} \in \mathcal{W}$. Hence, any points in $\mathcal{W}$ is an Equilibrium points.

Otherwise, there exists $\mathbf{w} \in \mathcal{W}$ such that $d(p^*_{(\mathbf{w})}, \mathbf{e}) > 0$. Let $\mathcal{I}= \{i|\underline{w}_i < \bar{w}_i, \forall i = 1, \ldots, n\}$, then $w_i^* = \underline{w}_i$ is always the best response for $i \in \mathcal{I}$. By assumption 3.1, $d(p^*_{(\mathbf{w})}, \mathbf{e})$ is continuous in w_i and $w_i \geq c_i(e_i)$ for all $i \in \{1, \ldots, n\}$ and there exists a $\mathbf{w}^0 \in \mathcal{W}$ such that $d(p^*_{(\mathbf{w}^0)}, \mathbf{e}) > 0$ with $w_i^0 > c_i(e_i)$, $\forall i \in \mathcal{I}$. Now consider the potential function:

$$\Psi(\mathbf{w}) = d(p^*_{(\sum w_i)}, \mathbf{e}) \prod_{i \in \mathcal{I}} (w_i - c_i(e_i))$$

Clearly, $\Psi(\mathbf{w})$ is continuous in $\mathbf{w}$ when $\mathbf{w} \in \mathcal{W}$, which is a compact set. Hence, the maximizer of $\Psi(\mathbf{w})$ exists and let us denote it by $\mathbf{w}^*$. Therefore, $\Psi(\mathbf{w}^*) \geq \Psi(\mathbf{w}^0) > 0$. For any $i \in \mathcal{I}$, given the other component providers' strategy is fixed at $\mathbf{w}^*_{-i}$, $\forall w_i' \neq w_i^*$ and $(w_i', \mathbf{w}^*_{-i}) \in \mathcal{W}$, we have

$$\begin{aligned}
& \Psi(w_i^*|\mathbf{w}^*_{-i}) \geq \Psi(w_i'|\mathbf{w}^*_{-i}) \\
\Rightarrow \quad & (d(w_i^*, \mathbf{w}^*_{-i}, \mathbf{e})(w_i^* - c_i(e_i)) - d(w_i', \mathbf{w}^*_{-i}, \mathbf{e})(w_i' - c_i(e_i))) \prod_{j \neq i, j \in \mathcal{I}} (w_j^* - c_j(e_j)) \geq 0 \\
\Rightarrow \quad & (\pi_i(w_i^*, \mathbf{w}^*_{-i}) - \pi_i(w_i', \mathbf{w}^*_{-i})) \prod_{j \neq i, j \in \mathcal{I}} (w_j^* - c_j(e_j)) \geq 0.
\end{aligned}$$

Since $\Psi(\mathbf{w}^*) > 0$, we know that $\prod_{j \neq i, j \in \mathcal{I}} (w_j^* - c_j(e_j)) > 0$. Therefore, $\pi_i(w_i^*, \mathbf{w}^*_{-i}) \geq \pi_i(w_i', \mathbf{w}^*_{-i})$. Hence, $w_i = w_i^*$ for $i \in \mathcal{I}$ and $w_i = \underline{w}_i$ for $i \in \mathcal{I}^c$ is an Equilibrium point for the component producers' pricing game.

Q.E.D.

Proof of lemma 3.2. At the equilibrium $\mathbf{w}$, if the component producer i's equilibrium strategy

w_i^* is an interior point, then the first order condition must be satisfied:

$$\begin{aligned}
\frac{\partial \psi_i(\mathbf{w})}{\partial w_i}|_{\mathbf{w}=\mathbf{w}^*} &= d(p^*_{(\sum w_i^*)}) + (w_i^* - c_i)\frac{\partial d(p^*_{(\sum w_i)})}{\partial w_i}|_{\mathbf{w}=\mathbf{w}^*} \\
\Rightarrow \quad (w_i - c_i) &= -\frac{d(p^*_{(\sum w_i^*)})}{\frac{\partial d(p^*_{(\sum w_i)})}{\partial w_i}|_{\mathbf{w}=\mathbf{w}^*}} \\
\Rightarrow \quad \psi_i(\mathbf{w}^*) &= -\frac{d(p^*_{(\sum w_i^*)})^2}{\frac{\partial d(p^*_{(\sum w_i)})}{\partial w_i}|_{\mathbf{w}=\mathbf{w}^*}}.
\end{aligned}$$

Since $\frac{\partial d(p^*_{(\sum w_i^*)})}{\partial w_i}$ is same for all $i = 1, \ldots, n$, $\psi_i(\mathbf{w}^*)$ is same for all the producers using the interior equilibrium strategy.

For part b), since $p^*_{(\sum w_i^*)}$ is an interior point, first order condition is satisfied by the manufacturer's gross profit,

$$\begin{aligned}
\frac{\partial \psi_0(p)}{\partial p}|_{p=p^*_{(\mathbf{w}^*)}} &= d(p^*_{(\sum w_i^*)}) + (p^*_{(\sum w_i^*)} - \sum w_i^* - c_0(e_0))\frac{\partial d(p, \mathbf{e})}{\partial p}|_{p=p^*_{(\sum w_i^*)}} \\
\Rightarrow \quad \psi_0(p^*_{(\sum w_i^*)}) &= -\frac{d(p^*_{(\sum w_i^*)})^2}{\frac{\partial d(p,\mathbf{e})}{\partial p}|_{p=p^*_{(\sum w_i^*)}}}.
\end{aligned}$$

Note that

$$\begin{aligned}
\frac{\partial d(p^*_{(\sum w_i^*)})}{\partial w_i}|_{\mathbf{w}=\mathbf{w}^*} &= \frac{\partial d(p, \mathbf{e})}{\partial p}|_{p=p^*_{(\sum w_i^*)}} \frac{\partial p^*_{(\sum w_i)}}{\partial \sum w_i}|_{\mathbf{w}=\mathbf{w}^*} \\
\Rightarrow \quad \psi_0(p^*_{(\mathbf{w}^*)}) &= \frac{\partial p^*_{(\sum w_i)}}{\partial \sum w_i}|_{\mathbf{w}=\mathbf{w}^*} \psi_i(\mathbf{w}^*).
\end{aligned}$$

Q.E.D.

Proof of lemma 3.3. For linear demand function, the manufacturer's decision for any given $\mathbf{w}$ is provided by:

$$\max_p \pi_0(p) = (a - p)(p - \sum_{i=1}^{n} w_i - c_0(e_0)) - k_i(e_i).$$

First order derivative of $\pi_0(p)$ is:

$$\frac{\partial \pi_0(p)}{\partial p} = a - 2p + \sum_{i=1}^{n} w_i + c_0(e_0).$$

Clearly $\pi_0(p)$ is strictly concave and the unique optimal solution is

$$\begin{aligned} p^*(\mathbf{w}|\mathbf{e}) &= \frac{a + \sum_{i=1}^n w_i + c_0(e_0)}{2} \\ d^*(\mathbf{w}|\mathbf{e}) &= \frac{a - \sum_{i=1}^n w_i - c_0(e_0)}{2} \end{aligned}$$

The objective for component producer i is to:

$$\max_{w_i} \pi_i(\mathbf{w}) = \frac{a - \sum_{i=1}^n w_i - c_0(e_0)}{2}(w_i - c_i(e_i)) - k_i(e_i).$$

The first order derivative of $\pi_i(\mathbf{w})$ with respect to w_i is:

$$\frac{\partial \pi_i(\mathbf{w})}{\partial w_i} = 1/2(a - \sum_{i=1}^n w_i - c_0(e_0) - w_i + c_i(e_i)),$$

which means that $\pi_i(w_i)$ is strictly concave, and the best response for each component producer i is

$$w_i = a - \sum_{i=1}^n w_i - c_0(e_0) + c_i(e_i) \qquad \forall i = 1, \ldots, n. \tag{A.16}$$

By solving this system of linear equations, we get the unique best solution in the strategy domain:

$$\begin{aligned} \sum_{i=1}^n w_i^* &= \frac{n(a - c_0(e_0)) + \sum_{i=1}^n c_i(e_i))}{n+1} \\ w_i^* &= \frac{a - c_0(e_0) - \sum_{j \neq i} c_j(e_j) + nc_i(e_i)}{n+1}. \end{aligned}$$

For this unique Nash equilibrium, we have:

$$\begin{aligned} \pi_i^*(\mathbf{e}) &= \frac{(a - \sum_{i=0}^n c_i(e_i))^2}{2(n+1)^2} - k_i(e_i) \\ \pi_0^*(\mathbf{e}) &= \frac{(a - \sum_{i=0}^n c_i(e_i))^2}{4(n+1)^2} - k_0(e_0). \end{aligned}$$

For truncated linear demand function, if $a - \sum_{i=0}^n c_i(e_i) \geq 0$, then the result and steps are actually the same as linear demand function. Otherwise, we can show that $d(\mathbf{w}^*) = 0$ and $\pi_i = -k_i(e_i)$ for all $i = 0, \ldots, n$ is the unique Nash Equilibrium result.

Clearly, for the given $\mathbf{w}$, the best price decision for manufacturer is:

$$\begin{aligned} p^*(\mathbf{w}|\mathbf{e}) &= \frac{a + \sum_{i=1}^n w_i + c_0(e_0)}{2} \\ d^*(\mathbf{w}|\mathbf{e}) &= \frac{[a - \sum_{i=1}^n w_i - c_0(e_0)]^+}{2}. \end{aligned}$$

First, we show that $w_i^* = c_i(e_i)$ and $d(\mathbf{w}^*)$ is an equilibrium point.
For any $i \in \{1, \dots, n\}$, if firm i deviate from this point, there are two choices: increasing w_i or decreasing w_i. For the first choice, based on the best response of manufacturer, $d^*(\mathbf{w})$ is still 0 and there is no change in final profit. Otherwise, $w_i - c_i(e_i) < 0$ and $d^*(\mathbf{w})$ is nonnegative, still he can not benefit from the deviation. Hence, $w_i = c_i(e_i)$ is an equilibrium point.

Now, suppose there exists a Nash equilibrium $\mathbf{w}^*$ with $d(\mathbf{w}^*) > 0$. Clearly, based on the best response of the manufacturer, $a - \sum_{i=1}^n w_i^* - c_0(e_0) > 0$. In case $a - \sum_{i=0}^n c_i(e_i) < 0$, there must exists $i \in \{1, \dots, n\}$ with $w_i^* < c_i(e_i)$ and $\pi_i(\mathbf{w}^*) < -k_i(e_i)$. Let $w_i' = a - \sum_{j \neq i} w_j^* - c_0(e_0)$, then

$$\pi_i(w_i', \mathbf{w}_{-i}^*) = -k_i(e_i) > \pi_i(w_i^*, \mathbf{w}_{-i}^*).$$

Hence, $\pi_i(\mathbf{w}^*) = -k_i(e_i)$ is the unique Nash equilibrium outcome if $a - \sum_{i=0}^n c_i(e_i) < 0$.

For exponential demand function, the manufacturer's decision problem for any given $\mathbf{w}$ is

$$\max_p \pi_0(p) = a \exp(-\lambda p)\big(p - \sum_{i=1}^n w_i - c_0(e_0)\big) - k_i(e_i).$$

The first order derivative of $\pi_0(p)$ is:

$$\frac{\partial \pi_0(p)}{\partial p} = a \exp(-\lambda p)\Big(1 - \lambda\big(p - \sum_{i=1}^n w_i - c_0(e_0)\big)\Big)$$

Clearly, $\pi_0(p)$ is unimodal and the unique optimal solution is

$$\begin{aligned} p^*(\mathbf{w}|\mathbf{e}) &= 1/\lambda + \sum_{i=1}^n w_i + c_0(e_0) \\ d^*(\mathbf{w}|\mathbf{e}) &= a \exp(-1 - \lambda(\sum_{i=1}^n w_i + c_0(e_0))). \end{aligned}$$

The objective for each component producer i' is to:

$$\max_{w_i} \pi_i(\mathbf{w}) = a(w_i - c_i) \exp\Big(-1 - \lambda\big(\sum_{i=1}^{n} w_i + c_0(e_0)\big)\Big) - k_i(e_i).$$

The first order derivative of $\pi_i(\mathbf{w})$ with respect to w_i is

$$\frac{\partial \pi_i(\mathbf{w})}{\partial w_i} = a \exp(-1 - \lambda(\sum_{i=1}^{n} w_i + c_0(e_0)))\big(1 - \lambda(w_i - c_i(e_i)\big)$$

which means that $\pi_i(w_i)$ is unimodal, and the best response for each component producer i is uniquely given by

$$w_i^* = 1/\lambda + c_i(e_i) \qquad \forall i = 1, \ldots, n.$$

For this unique Nash equilibrium, we have:

$$\pi_i^*(\mathbf{e}) = \frac{a}{\lambda} \exp(-2 - \lambda \sum_{i=0}^{n} c_i(e_i)) - k_i(e_i) \qquad \forall i = 0, \ldots, n.$$

Q.E.D.

Proof of lemma 3.4. The utility function for any component producer $i \in \mathcal{I}$ is given by:

$$\begin{aligned} \pi_i(\mathbf{e}_{\mathcal{I}}|\mathbf{e}_{\mathcal{I}^c}) &= \psi(\mathbf{e}_{\mathcal{I}}|\mathbf{e}_{\mathcal{I}^c}) - k_i(e_i) \qquad \forall i = 1 \ldots n \\ \pi_0(\mathbf{e}_{\mathcal{I}}|\mathbf{e}_{\mathcal{I}^c}) &= \xi\psi(\mathbf{e}_{\mathcal{I}}|\mathbf{e}_{\mathcal{I}^c}) - k_0(e_0) \end{aligned}$$

Let $\gamma_i = 1$ for $i = 1 \ldots n$ and $\gamma_0 = 1/\xi$. Consider the potential function:

$$\Psi(\mathbf{e}_{\mathcal{I}}|\mathbf{e}_{\mathcal{I}^c}) = \psi(\mathbf{e}_{\mathcal{I}}|\mathbf{e}_{\mathcal{I}^c}) - \sum_{i \in \mathcal{I}} k_i(e_i)/\gamma_i.$$

For any component producer $i \in \mathcal{I}$, let the other component producers' strategy be given $\mathbf{e}_{-i}$. For all $(e_i', \mathbf{e}_{-i})$ and $(e_i'', \mathbf{e}_{-i}) \in \mathcal{E}_{\mathcal{I}}$ such that $e_i' \neq e_i''$, we have:

$$\begin{aligned} & \Psi(e_i'|(\mathbf{e}_{-i}, \mathbf{e}_{\mathcal{I}^c})) \geq \Psi(e_i''|(\mathbf{e}_{-i}, \mathbf{e}_{\mathcal{I}^c})) \\ \Leftrightarrow \quad & \psi(e_i', \mathbf{e}_{-i}, \mathbf{e}_{\mathcal{I}^c}) - \psi(e_i'', \mathbf{e}_{-i}, \mathbf{e}_{\mathcal{I}^c}) - (k_i(e_i')/\gamma_i - k_i(e_i'')/\gamma_i) \geq 0 \\ \Leftrightarrow \quad & \pi_i(e_i'|(\mathbf{e}_{-i}, \mathbf{e}_{\mathcal{I}^c})) \geq \pi_i(e_i''|(\mathbf{e}_{-i}, \mathbf{e}_{\mathcal{I}^c})) \end{aligned} \tag{A.17}$$

Therefore, the maximizer of function $\Psi(\mathbf{e}_\mathcal{I})$ is a Nash Equilibrium for the simultaneous effort game among the players in $\mathcal{I}$. The maximizer exists since $\Psi(\mathbf{e}_\mathcal{I})$ is continuous and the strategy space $\mathcal{E}_\mathcal{I}$ is compact.

Q.E.D.

Proof of proposition 3.1 We first show that $\forall \mathbf{e}' \in \mathcal{E}^*$, $\pi_i(\mathbf{e}') = \widehat{\pi}_i(\mathbf{x}(\mathbf{e}'))$, $\forall i \in \mathcal{N}$. In the case of exponential demand function:

$$\begin{aligned}
\pi_i(\mathbf{e}') &= \frac{\alpha e^{-2}}{\lambda}\theta_i(e'_i)\exp(-\lambda c_i(e'_i))\prod_{k\neq i}\theta_k(e'_k)\exp(-\lambda c_k(e'_k)) - k_i(e'_i) \\
&= \frac{\alpha e^{-2}}{\lambda}\theta_i(e'_i)\exp(-\lambda c_i(e'_i))\prod_{k\neq i}x_k(e'_k) - k_i(e'_i) \\
&= \frac{\alpha e^{-2}}{\lambda}x_i(e'_i)\prod_{k\neq i}x_k(e'_k) - k_i(e'_i) - \min_{\theta_i(e_i)\exp(-\lambda c_i(e_i))=x_i(e'_i)} k_i(e_i) \\
&= \frac{\alpha e^{-2}}{\lambda}x_i(e'_i)\prod_{k\neq i}x_k(e'_k) - \widehat{k}_i(x_i(e'_i)) = \widehat{\pi}_i(\mathbf{x}(\mathbf{e}')).
\end{aligned}$$

In the case of linear demand function and $i \in \{1, \ldots, n\}$:

$$\begin{aligned}
\pi_i(\mathbf{e}') &= \frac{(\alpha + \sum_{j\in\mathcal{N}}(\theta_j(e'_j) - c_j(e'_j)))^2}{2(n+1)^2} - k_i(e'_i) \\
&= \frac{(\alpha + \sum_{j\in\mathcal{N}} x_j(e'_j))^2}{2(n+1)^2} - \min_{\theta_i(e_i)-c_i(e_i)=x_i(e'_i)} k_i(e_i) \\
&= \frac{(\alpha + \sum_{j\in\mathcal{N}} x_j(e'_j))^2}{2(n+1)^2} - \widehat{k}_i(x_i(e'_i)) = \widehat{\pi}_i(\mathbf{x}(\mathbf{e}')).
\end{aligned}$$

Note if $i = 0$, the only difference is changing the denominator from $2(n+1)^2$ to $4(n+1)^2$. In the case of truncated linear demand function and $i \in \{1, \ldots, n\}$:

$$\begin{aligned}
\pi_i(\mathbf{e}') &= \frac{(\alpha + \sum_{j\in\mathcal{N}}(\theta_j(e'_j) - c_j(e'_j)))^{+2}}{2(n+1)^2} - k_i(e'_i) \\
&= \frac{(\alpha + \sum_{j\in\mathcal{N}} x_j(e'_j))^{+2}}{2(n+1)^2} - \min_{\theta_i(e_i)-c_i(e_i)=x_i(e'_i)} k_i(e_i) \\
&= \frac{(\alpha + \sum_{j\in\mathcal{N}} x_j(e'_j))^{+2}}{2(n+1)^2} - \widehat{k}_i(x_i(e'_i)) = \widehat{\pi}_i(\mathbf{x}(\mathbf{e}'))
\end{aligned}$$

Note if $i = 0$, the only difference is changing the denominator from $2(n+1)^2$ to $4(n+1)^2$.

To show that $\{x_i(\mathbf{e})\}$ for $i \in \mathcal{N}$ is a Surjection function from $\mathcal{E}^*$ to $\mathcal{X}^*$, we have to show the following two facts:

i) If $\mathbf{e}' \in \mathcal{E}^*$, then $\mathbf{x}' = \{x_i(\mathbf{e}') : i \in \mathcal{N}\} \in \mathcal{X}^*$.

ii) $\forall \mathbf{x}' \in \mathcal{X}^*$, $\exists \mathbf{e}' \in \mathcal{E}^*$ such that $\mathbf{x}' = \{x_i(\mathbf{e}') : i \in \mathcal{N}\}$

For part i) Suppose not and there exits $\mathbf{x}' \notin \mathcal{X}^*$. Thus $\exists x''_j \in \mathcal{X}_j$ for some $j \in \mathcal{N}$ such that

$$\widehat{\pi}_j(x''_j, \mathbf{x}'_{-j}) > \widehat{\pi}_j(x'_j, \mathbf{x}'_{-j}).$$

Let

$$\begin{aligned} e''_j &\in argmin_{e_j \in [\underline{e}_j, \bar{e}_j]}\{k_j(e_j) : \quad \theta_j(e_j) - c_j(e_j)) = x''_j\} \quad \text{for linear or truncated linear demand} \\ e''_j &\in argmin_{e_j \in [\underline{e}_j, \bar{e}_j]}\{k_j(e_j) : \quad \theta_j(e_j) \exp(-\lambda c_j(e_j)) = x''_j\} \quad \text{for exponential demand} \end{aligned}$$

The minimizer of the continuous fixed investment cost function k_i, e''_j exists because the left side of the equality in the constraint is continuous and the inverse set of the point x''_j in the range is closed in a bounded and non-empty set. Therefore, in the case of exponential demand function,

$$\begin{aligned} \pi_j(e''_j, \mathbf{e}'_{-j}) &= \frac{\alpha e^{-2}}{\lambda} \theta_j(e''_j) \exp(-\lambda c_j(e''_j)) \prod_{k \neq j} \theta_j(e'_k) \exp(-\lambda c_k(e'_k)) - k_j(e''_j) \\ &= \frac{\alpha e^{-2}}{\lambda} x''_j \prod_{k \neq j} x'_k - \widehat{k}_j(x''_j) \\ &= \widehat{\pi}_j(x''_j, \mathbf{x}'_{-j}) > \widehat{\pi}_j(x'_j, \mathbf{x}'_{-j}) = \pi_j(\mathbf{e}'). \end{aligned}$$

In the case of linear demand function, for $j \in \{1, \dots, n\}$,

$$\begin{aligned} \pi_j(e''_j, \mathbf{e}'_{-j}) &= \frac{\left(\alpha + \sum_{k \neq j}(\theta_k(e'_k) - c_k(e'_k)) + \theta_j(e''_j) - c_j(e''_j)\right)^2}{2(n+1)^2} - k_j(e''_j) \\ &= \frac{(\alpha + \sum_{k \neq j} x'_k + x''_j)^2}{2(n+1)^2} - \widehat{k}_j(x''_j) \\ &= \widehat{\pi}_j(x''_j, \mathbf{x}'_{-j}) > \widehat{\pi}_j(x'_j, \mathbf{x}'_{-j}) = \pi_j(\mathbf{e}'). \end{aligned}$$

In the case of truncated linear demand function, for $j \in \{1, \ldots, n\}$,

$$\begin{aligned}
\pi_j(e''_j, \mathbf{e}'_{-j}) &= \frac{\left(\alpha + \sum_{k\neq j}(\theta_k(e'_k) - c_k(e'_k)) + \theta_j(e''_j) - c_j(e''_j)\right)^2}{2(n+1)^{+2}} - k_j(e''_j) \\
&= \frac{(\alpha + \sum_{k\neq j} x'_k + x''_j)^{+2}}{2(n+1)^2} - \widehat{k}_j(x''_j) \\
&= \widehat{\pi}_j(x''_j, \mathbf{x}'_{-j}) > \widehat{\pi}_j(x'_j, \mathbf{x}'_{-j}) = \pi_j(\mathbf{e}').
\end{aligned}$$

For linear or truncated linear demand function, if $j = 0$, just replace the denominator $2(n+1)^2$ by $4(n+1)^2$. This result contradicts with the fact that $\mathbf{e}'$ is the equilibrium point and hence $\pi_j(\mathbf{e}') \geq \pi_j(e''_j, \mathbf{e}'_{-j})$.

For part ii), given that $\mathbf{x}' \in \mathcal{X}^*$, for all $j \in \mathcal{N}$, let

$$\begin{aligned}
e'_j &\in argmin_{e_j \in [\underline{e}_j, \bar{e}_j]}\{k_j(e_j) : \quad \theta_j(e_j) - c_j(e_j)) = x'_j\} \quad \text{for linear or truncated linear demand} \\
e'_j &\in argmin_{e_j \in [\underline{e}_j, \bar{e}_j]}\{k_j(e_j) : \quad \theta_j(e_j)\exp(-\lambda c_j(e_j)) = x'_j\} \quad \text{for exponential demand}
\end{aligned}$$

By the argument stated in part i), e'_j exists. Next, we show that $\{e'_j : j \in \mathcal{N}\} \in \mathcal{E}^*$. It is enough for us to show that $\pi_i(e''_i, \mathbf{e}'_{-i}) \leq \pi_i(e'_i, \mathbf{e}'_{-i})$ for any $i \in \mathcal{N}$ and $e''_i \in \mathcal{E}_i$ such that $e''_i \neq e'_i$. In the case of exponential demand function:

$$\begin{aligned}
\pi_j(e''_i, \mathbf{e}'_{-i}) &= \frac{\alpha e^{-2}}{\lambda}\theta_i(e''_i)\exp(-\lambda c_i(e''_i))\prod_{k\neq i}\theta_k(e'_k)\exp(-\lambda c_k(e'_k)) - k_i(e''_i) \\
&= \frac{\alpha e^{-2}}{\lambda}x_i(e''_i)\prod_{k\neq i}x_k(e'_k) - k_i(e''_i) \\
&\leq \frac{\alpha e^{-2}}{\lambda}x_i(e''_i)\prod_{k\neq i}x_k(e'_k) - \min_{\theta_i(e_i)\exp(-\lambda c_i(e_i)) = x_i(e''_i)} k_i(e_i) \\
&= \widehat{\pi}_j(x''_j, \mathbf{x}'_{-j}) \leq \widehat{\pi}_i(x'_i, \mathbf{x}'_{-i}) = \pi_i(\mathbf{e}').
\end{aligned}$$

In the case of linear demand function, for $i \in \{1, \ldots, n\}$:

$$\begin{aligned}
\pi_j(e''_i, \mathbf{e}'_{-i}) &= \frac{\left(\alpha + \sum_{k\neq i}(\theta_k(e'_k) - c_k(e'_k)) + \theta_i(e''_i) - c_i(e''_i)\right)^2}{2(n+1)^2} - k_i(e''_i) \\
&= \frac{(\alpha + \sum_{k\neq i} x'_k + x''_i)^2}{2(n+1)^2} - k_i(e''_i) \\
&\leq \frac{(\alpha + \sum_{k\neq i} x'_k + x''_i)^2}{2(n+1)^2} - \min_{\theta_i(e_i) - c_i(e_i)) = x_i(e''_i)} k_i(e_i) \\
&= \widehat{\pi}_j(x''_j, \mathbf{x}'_{-j}) \leq \widehat{\pi}_i(x'_i, \mathbf{x}'_{-i}) = \pi_i(\mathbf{e}').
\end{aligned}$$

In the case of truncated linear demand function, for $i \in \{1, \ldots, n\}$:

$$\begin{aligned}
\pi_j(e''_i, \mathbf{e}'_{-i}) &= \frac{\left(\alpha + \sum_{k\neq i}(\theta_k(e'_k) - c_k(e'_k)) + \theta_i(e''_i) - c_i(e''_i)\right)^2}{2(n+1)^{+2}} - k_i(e''_i) \\
&= \frac{(\alpha + \sum_{k\neq i} x'_k + x''_i)^{+2}}{2(n+1)^2} - k_i(e''_i) \\
&\leq \frac{(\alpha + \sum_{k\neq i} x'_k + x''_i)^{+2}}{2(n+1)^2} - \min_{\theta_i(e_i) - c_i(e_i)) = x_i(e''_i)} k_i(e_i) \\
&= \widehat{\pi}_j(x''_j, \mathbf{x}'_{-j}) \leq \widehat{\pi}_i(x'_i, \mathbf{x}'_{-i}) = \pi_i(\mathbf{e}').
\end{aligned}$$

For linear and truncated linear demand function if $i = 0$, just replace the denominator $2(n+1)^2$ by $4(n+1)^2$. Together with part i), we have $\{x_i(e_i) : i \in \mathcal{N}\}$ is a Surjection from $\mathcal{E}^*$ to $\mathcal{X}^*$.

Q.E.D.

Proof of lemma 3.5. Since for every given α, the strategy space $\mathcal{X}$ do not change, by Theorem 2.6.1 in [44], to show that f is supermodular in $\mathbf{x}$ and has increasing difference in (x_i, α), it is enough to show that f is supermodular in $(\mathbf{x}, \alpha)$. Let $\tau = |\mathcal{S}| - 1 + \beta$ if $\{0\} \in \mathcal{S}$ otherwise $\tau = |\mathcal{S}|$, where $\beta = 1/2$ for linear and truncate linear demand and $\beta = 1$ for exponential demand function. For all $(\alpha, \mathbf{x})^1$, $(\alpha, \mathbf{x})^2 \in \mathcal{A} \times \mathcal{X}$. By definition, to show the supermodularity of $f(\alpha, \mathbf{x})$, we have to check that:

$$\begin{aligned}
& f((\alpha,\mathbf{x})^1 \wedge (\alpha,\mathbf{x})^2) + f((\alpha,\mathbf{x})^1 \vee (\alpha,\mathbf{x})^2) - f((\alpha,\mathbf{x})^1) - f((\alpha,\mathbf{x})^2) \\
= \;& \tau\widehat{\psi}((\alpha,\mathbf{x})^1 \wedge (\alpha,\mathbf{x})^2) + \tau\widehat{\psi}((\alpha,\mathbf{x})^1 \vee (\alpha,\mathbf{x})^2) - \tau\widehat{\psi}((\alpha,\mathbf{x})^1) - \tau\widehat{\psi}((\alpha,\mathbf{x})^2) \geq 0
\end{aligned}$$

For linear demand function:

$$\frac{\partial^2 \widehat{\psi}(\mathbf{x})}{\partial x_i \partial x_j} = \frac{1}{(n+1)^2} > 0 \qquad \forall i, j \in \{0, \ldots, n\} \text{ and } i \neq j$$

$$\frac{\partial^2 \widehat{\psi}(\mathbf{x})}{\partial x_i \partial \alpha} = \frac{1}{(n+1)^2} > 0 \qquad \forall i \in \{0, \ldots, n\}.$$

For exponential demand function:

$$\frac{\partial^2 \widehat{\psi}(\mathbf{x})}{\partial x_i \partial x_j} = \frac{\alpha e^{-2}}{\lambda} \prod_{k \neq i,j} x_k \geq 0 \qquad \forall i,j \in \{0,\ldots,n\} \text{ and } i \neq j$$

$$\frac{\partial^2 \widehat{\psi}(\mathbf{x})}{\partial x_i \partial \alpha} = \frac{e^{-2}}{\lambda} \prod_{k \neq i} x_k \geq 0 \qquad \forall i \in \{0,\ldots,n\}.$$

The above two inequality is strict when $x_i \geq \underline{x}_i > 0$, $\forall i = 0,\ldots,n$.

For truncated linear demand function, since the operator $()^+$ is a convex function, by lemma 2.6.1 in [44], $(\alpha + \sum x_i)^+$ is supermodular in $(\alpha, \mathbf{x})$. Since the operator $()^2$ is an increasing and convex operator in a non-negative domain, by Lemma 2.6.4 in [44], $(\alpha + \sum x_i)^{+2}$ is supermodular in $(\alpha, \mathbf{x})$. Therefore, $\widehat{\psi}(\mathbf{x})$ is supermodular in $\mathbf{x}$ for linear, truncated linear and exponential demand functions. Since

$$\tau\widehat{\psi}(\mathbf{x^1} \wedge \mathbf{x^2}) + \tau\widehat{\psi}(\mathbf{x^1} \vee \mathbf{x^2}) - \tau\widehat{\psi}(\mathbf{x^1}) - \tau\widehat{\psi}(\mathbf{x^2}) \geq 0,$$

$f(\mathbf{x})$ is supermodular in $\mathbf{x}$ and has increasing difference in (x_i, α). In case $\frac{\partial^2 \widehat{\psi}(\mathbf{x})}{\partial x_i \partial x_j} > 0$ when demand is exponential with multiplicative effort effect with $\bar{x}_i > 0, \forall i = 0,\ldots,n$ or linear with additive effort effect, the above inequality is strict and hence $f(\mathbf{x})$ is strict supermodular in $\mathbf{x}$.

Q.E.D.

Proof of lemma 3.6. For linear demand function and exponential demand function, the result comes directly from $\frac{\partial \widehat{\psi}_i(\mathbf{x})}{\partial x_j} \geq 0$. Therefore, for any $\mathbf{x}'$, $\mathbf{x}''$ such that $x'_k = x''_k$, $k \neq j$ and $x'_j \geq x''_j$, $\widehat{\pi}_i(\mathbf{x}') - \widehat{\pi}_i(\mathbf{x}'') = \widehat{\psi}_i(\mathbf{x}') - \widehat{\psi}_i(\mathbf{x}'') \geq 0$.

For truncate linear demand function, given $\mathbf{x}_{-j}$, if $x_j \geq -(a + \sum_{k \neq j} x_k)$, then for any $x'_j > x_j$

$$\begin{aligned} a + \sum_{k \neq j} x_k + x'_j > a + \sum_{k \neq j} x_k + x_j \quad &\Rightarrow \quad (a + \sum_{k \neq j} x_k + x'_j)^2 > (a + \sum_{k \neq j} x_k + x_j)^2 \\ &\Rightarrow \quad \widehat{\pi}_i(x'_j, \mathbf{x}_{-j}) > \widehat{\pi}_i(x_j, \mathbf{x}_{-j}), \end{aligned}$$

otherwise $x_j < -a - \sum_{k \neq j} x_k$, then for any $x'_j > x_j$ we have

$$\widehat{\pi}_i(x'_j, \mathbf{x}_{-j}) \geq -\widehat{k}_i(x_i) = \widehat{\pi}_i(x_j, \mathbf{x}_{-j}).$$

Hence, $\pi_i(\mathbf{x})$ is nondecreasing in x_j, for $\forall j \neq i$.

Q.E.D.

Proof of lemma 3.7. a) and b) follows directly from the fact that $\pi_i(\mathbf{x}_\mathcal{I})$ is a supermodular function on $\mathbf{x}_\mathcal{I}$ and $\mathcal{X}_\mathcal{I}$ is a compact sublattice and lemma 4.2.2 in [44]. To prove part c), let $\mathbf{x}^*_\mathcal{I}$ be an equilibrium point, for $\forall i \in \mathcal{I}$, we have

$$\widehat{\pi}_i(x^*_i, \mathbf{x}^*_{-i}|\mathbf{x}_{\mathcal{I}^\mathbf{c}}) \leq \widehat{\pi}_i(x^*_i, \mathbf{x}^u_{-i}|\mathbf{x}_{\mathcal{I}^\mathbf{c}}) \leq \widehat{\pi}_i(x^u_i, \mathbf{x}^u_{-i}|\mathbf{x}_{\mathcal{I}^\mathbf{c}}) = \widehat{\pi}_i(\mathbf{x}^\mathbf{u}_\mathcal{I}|\mathbf{x}_{\mathcal{I}^\mathbf{c}})$$

Since x^u_i is the best response given $\mathbf{x}^u_{-i}$, the second inequality holds. On the other hand, the first inequality follows from the fact that π_i is increasing in j for all $j \neq i$ which also implies that $\forall i \in \mathcal{I}^c$:

$$\widehat{\pi}_i(\mathbf{x}_{\mathcal{I}^\mathbf{c}}, \mathbf{x}^\mathbf{u}_\mathcal{I}) \geq \widehat{\pi}_i(\mathbf{x}_{\mathcal{I}^\mathbf{c}}, \mathbf{x}^*_\mathcal{I}).$$

Note that in the original theorem, they assume that the objective function $f_i(x_i, \mathbf{x}_{-i})$ is upper semicontinuous in x_i (*Lemma 4.2.2. in Topkis [44]*). The only purpose of this assumption is to make sure that $\mathrm{argmax}_{\mathrm{x_i} \in [\underline{\mathrm{x}}_\mathrm{i}, \bar{\mathrm{x}}_\mathrm{i}]} \mathrm{f_i}(\mathrm{x_i}, \mathbf{x}_{-\mathrm{i}})$ is a nonempty compact set. Actually we achieve this precondition by our selection of $\mathbf{x}$.

Q.E.D.

Proof of lemma 3.9. Since the strategy space $\mathcal{X}$ do not change and $\widehat{\pi}_i$ is supermodular in $\mathbf{x}$, $\widehat{\pi}_i$ has increasing difference in $(x_k, \mathbf{x}^{ck}_\mathcal{I})$ for every given $\mathbf{x}_\mathcal{I}$ for all $i \in \mathcal{I}^{ck}$. By lemma 4.2.2 in [44], the largest best response of players in $\mathcal{I}^{ck}$ is increasing with x_k. Furthermore,

$$\widehat{\pi}_i(x'_k, \mathbf{x}'_{\mathcal{I}^{ck}}|\mathbf{x}_\mathcal{I}) = \widehat{\pi}_i(x'_k, x'_i, \mathbf{x}'_{\mathcal{I}^{ck}\setminus\{i\}}|\mathbf{x}_\mathcal{I}) \geq \widehat{\pi}_i(x'_k, x''_i, \mathbf{x}'_{\mathcal{I}^{ck}\setminus\{i\}}|\mathbf{x}_\mathcal{I}) \geq \widehat{\pi}_i(x''_k, \mathbf{x}''_{\mathcal{I}^{ck}}|\mathbf{x}_\mathcal{I}).$$

The first inequality follows from the fact that $\mathbf{x}'_{\mathcal{I}^{ck}}$ is the equilibrium for x'_k, and the second inequality follows from the fact that $\widehat{\pi}_i$ is nondecreasing with respect to x_j for $j \neq i$.

Q.E.D.

Proof of lemma 3.10 Clearly, if $x_k^* \geq x_k'$, then by lemma 3.9, $x_i^* \geq x_i'$ for $i \in \mathcal{I}^{ck}$. Otherwise, suppose $x_k^* < x_k'$, then $x_i^* \leq x_i'$ for $\forall i \in \mathcal{I}^{ck}$. Since x_k' is also a feasible solution to $\max_{x_k} \widehat{\pi}_k(x_k, Y_{\mathcal{I}^{ck}}(x_k)|\mathbf{x}_{\mathcal{I}})$, we have

$$\widehat{\pi}_k(\mathbf{x}'_{\mathcal{I}^{\mathbf{c}}}|\mathbf{x}_{\mathcal{I}}) < \widehat{\pi}_k(\mathbf{x}_{\mathcal{I}^{\mathbf{c}}}{}^*|\mathbf{x}_{\mathcal{I}}).$$

However, note that

$$\begin{aligned}
& \widehat{\pi}_k(\mathbf{x}_{\mathcal{I}^{\mathbf{c}}}{}^*|\mathbf{x}_{\mathcal{I}}) = \widehat{\pi}_k(x_k^*, \mathbf{x}^*_{\mathcal{I}^{ck}}|\mathbf{x}_{\mathcal{I}}) \\
\leq \quad & \widehat{\pi}_k(Y_k(\mathbf{x}^*_{\mathcal{I}^{ck}}|\mathbf{x}_{\mathcal{I}}), \mathbf{x}^*_{\mathcal{I}^{ck}}|\mathbf{x}_{\mathcal{I}}) \qquad \text{since } Y_k() \text{ is the best response operator} \\
\leq \quad & \widehat{\pi}_k(Y_k(\mathbf{x}^*_{\mathcal{I}^{ck}}|\mathbf{x}_{\mathcal{I}}), \mathbf{x}'_{\mathcal{I}^{ck}}|\mathbf{x}_{\mathcal{I}}) \qquad \text{since } \pi_k \text{ is nondecreasing in } x_j, j \neq k \\
\leq \quad & \widehat{\pi}_k(Y_k(\mathbf{x}'_{\mathcal{I}^{ck}}|\mathbf{x}_{\mathcal{I}}), \mathbf{x}'_{\mathcal{I}^{ck}}|\mathbf{x}_{\mathcal{I}}) \qquad \text{since } Y_k() \text{ is the best response operator} \\
= \quad & \widehat{\pi}_k(\mathbf{x}'_{\mathcal{I}^{\mathbf{c}}}|\mathbf{x}_{\mathcal{I}})
\end{aligned}$$

which leads to a contradiction with A.18. Therefore, $x_k^* \geq x_k'$, and by lemma 3.9, $x_i^* \geq x_i'$ for $\forall i \in \mathcal{I}^{ck}$.

$\widehat{\pi}_i(\mathbf{x}^*|\mathbf{x}_{\mathcal{I}}) \geq \widehat{\pi}_i(\mathbf{x}'|\mathbf{x}_{\mathcal{I}})$ for $\forall i \in \mathcal{I}^{ck}$ follows from lemma 3.9

$\widehat{\pi}_i(\mathbf{x}^*|\mathbf{x}_{\mathcal{I}}) \geq \widehat{\pi}_i(\mathbf{x}'|\mathbf{x}_{\mathcal{I}})$ for $\forall i \in \mathcal{I}$ follows from the fact that π_i is a nondecreasing function of x_j, $\forall j \neq i$ and $\mathbf{x}_{\mathcal{I}^{\mathbf{c}}}{}^* \succeq \mathbf{x}_{\mathcal{I}^{\mathbf{c}}}{}'$.

$\widehat{\pi}_k(\mathbf{x}^*|\mathbf{x}_{\mathcal{I}}) \geq \widehat{\pi}_k(\mathbf{x}'|\mathbf{x}_{\mathcal{I}})$ because that x_k' is also a feasible solution to

$$\max_{x_k \in \mathcal{X}_k} \widehat{\pi}_k(x_k, Y_{\mathcal{I}^{ck}}(x_k)|\mathbf{x}_{\mathcal{I}}).$$

Q.E.D.

Proof of lemma 3.12. By lemma 3.11, since the equilibrium is unique in sequential decisions, it must be the maximizer of the potential function:

$$\mathbf{x}_{\mathcal{S}}{}^* = argmax_{\mathbf{x}_{\mathcal{S}} \in \mathcal{X}_{\mathcal{S}}} \qquad \Psi_{\mathbf{x}_{\mathcal{S}}}(\mathbf{x}_{\mathcal{S}}|\mathbf{x}_{\mathcal{I}}) = \widehat{\psi}(\mathbf{x}_{\mathcal{S}}, Y_{\mathcal{T}}(\mathbf{x}_{\mathcal{S}}|\mathbf{x}_{\mathcal{I}}), \mathbf{x}_{\mathcal{I}}) - \sum_{i \in \mathcal{S}} \widehat{k}_i(x_i)/\gamma_i$$

Since $\widehat{\psi}$ and $Y_{x_{\mathcal{T}}}(\mathbf{x}_{\mathcal{S}}|\mathbf{x}_{\mathcal{I}})$ is only a function of $\sum_{i \in \mathcal{S}} x_i$ for additive demand effort (or $\prod_{i \in \mathcal{S}} x_i$

for multiplicative demand effort), so

$$\Psi_{\mathbf{x}_\mathcal{S}}(\mathbf{x}_\mathcal{S}|\mathbf{x}_\mathcal{I}) = \widehat{\psi}(t(\mathbf{x}_\mathcal{S}), Y_\mathcal{T}(t(\mathbf{x}_\mathcal{S})|\mathbf{x}_\mathcal{I}), \mathbf{x}_\mathcal{I}) - \sum_{i\in\mathcal{S}} \widehat{k}_i(x_i)/\gamma_i,$$

where $t(\mathbf{x}_\mathcal{S}) = \sum_{i\in\mathcal{S}} x_i$ for additive effort effect and $t(\mathbf{x}_\mathcal{S}) = \prod_{i\in\mathcal{S}} x_i$ for multiplicative effort effect. Therefore, in case demand is additive in effort effect, the equilibrium is the solution for:

$$\begin{aligned} t_s^* &= argmax_{\sum_{i\in\mathcal{S}} \underline{x}_i \le t_s \le \sum_{i\in\mathcal{S}} \overline{x}_i} \qquad \psi(t_s, Y_\mathcal{T}(t_s|\mathbf{x}_\mathcal{I}), \mathbf{x}_\mathcal{I}) - \widehat{k}_s(t_s) \\ \text{where:} \quad \widehat{k}_s(t_s) &= \min_{\mathbf{x}_\mathcal{S}\in\mathcal{X}_\mathcal{S}:\sum_{i\in\mathcal{S}} x_i = t_s} \sum_{i\in\mathcal{S}} \widehat{k}_i(x_i)/\gamma_i. \end{aligned}$$

In case the demand is multiplicative in effort effect, the equilibrium is the solution for:

$$\begin{aligned} t_s^* &= max_{\prod_{i\in\mathcal{S}} \underline{x}_i \le t_s \le \prod_{i\in\mathcal{S}} \overline{x}_i} \qquad \psi(t_s, Y_\mathcal{T}(t_s|\mathbf{x}_\mathcal{I}), \mathbf{x}_\mathcal{I}) - \widehat{k}_s(t_s) \\ \text{where:} \quad \widehat{k}_s(t_s) &= \min_{\mathbf{x}_\mathcal{S}\mathcal{X}_\mathcal{S}:\prod_{i\in\mathcal{S}} x_i = t_s} \sum_{i\in\mathcal{S}} \widehat{k}_i(x_i) \end{aligned}$$

Compare with the simultaneous decision case:

$$\mathbf{x}'_\mathcal{T} = Y_\mathcal{T}(\mathbf{x}'_\mathcal{S}|\mathbf{x}_\mathcal{I}) = Y_\mathcal{T}(t'_s|\mathbf{x}_\mathcal{I}).$$

Since the equilibrium is unique for given $\mathbf{x}_\mathcal{T}$, $\mathbf{x}_\mathcal{I}$, we have

$$\mathbf{x}'_\mathcal{S} = argmax_{\mathbf{x}_\mathcal{S}\in\mathcal{X}_\mathcal{S}} \qquad \Psi_{\mathbf{x}_\mathcal{S}}(\mathbf{x}_\mathcal{S}, \mathbf{x}'_\mathcal{T}|\mathbf{x}_\mathcal{I}) = \widehat{\psi}(\mathbf{x}_\mathcal{S}, \mathbf{x}'_\mathcal{T}|\mathbf{x}_\mathcal{I}), \mathbf{x}_\mathcal{I}) - \sum_{i\in\mathcal{S}} \widehat{k}_i(x_i)/\gamma_i.$$

Same as before, we can rewrite

$$\Psi_{\mathbf{x}_\mathcal{S}}(\mathbf{x}_\mathcal{S}, \mathbf{x}'_\mathcal{T}|\mathbf{x}_\mathcal{I}) = \widehat{\psi}(t(\mathbf{x}_\mathcal{S}), \mathbf{x}'_\mathcal{T}|\mathbf{x}_\mathcal{I}), \mathbf{x}_\mathcal{I}) - \sum_{i\in\mathcal{S}} \widehat{k}_i(x_i)/\gamma_i.$$

In case of exponential demand with multiplicative effort effect, this maximizer is the solution for:

$$\begin{aligned} t'_s &= argmax_{\prod_{i\in\mathcal{S}} \underline{x}_i \le t_s \le \prod_{i\in\mathcal{S}} \overline{x}_i} \qquad \widehat{\psi}(t_s, \mathbf{x}'_\mathcal{T}, \mathbf{x}_\mathcal{I}) - \widehat{k}_s(t_s) \\ \text{where:} \quad \widehat{k}_s(t_s) &= \min_{\mathbf{x}_\mathcal{S}\mathcal{X}_\mathcal{S}:\prod_{i\in\mathcal{S}} x_i = t_s} \sum_{i\in\mathcal{S}} \widehat{k}_i(x_i). \end{aligned}$$

In case the demand is additive in effort effect, the equilibrium is the solution for:

$$t'_s = argmax_{\sum_{i\in\mathcal{S}}\underline{x}_i \le t_s \le \sum_{i\in\mathcal{S}}\overline{x}_i} \quad \widehat{\psi}(t_s, \mathbf{x}'_{\mathcal{T}}, \mathbf{x}_{\mathcal{I}}) - \widehat{k}_s(t_s)$$
$$\text{where:} \quad \widehat{k}_s(t_s) = \min_{\mathbf{x}_{\mathcal{S}}\in\mathcal{X}_{\mathcal{S}}:\sum_{i\in\mathcal{S}} x_i = t_s} \sum_{i\in\mathcal{S}} \widehat{k}_i(x_i)/\gamma_i.$$

Hence, for comparison, we replace the group of player $\mathcal{S}$ by a single player s with $\widehat{k}(t_s)$ and $t_s \in [\sum_{i\in\mathcal{S}}\underline{x}_i, \sum_{i\in\mathcal{S}}\overline{x}_i]$ for additive demand effect or $t_s \in [\prod_{i\in\mathcal{S}}\underline{x}_i, \prod_{i\in\mathcal{S}}\overline{x}_i]$ for multiplicative demand effect. By the result of lemma 3.10, we have $t'_s \le t^*_s$ and $x'_i \le x^*_i$, $\widehat{\pi}_i(t'_s, \mathbf{x}'_{\mathcal{T}}|\mathbf{x}_{\mathcal{I}}) \le \widehat{\pi}_i(t^*_s, \mathbf{x}^*_{\mathcal{T}}|\mathbf{x}_{\mathcal{I}})$ for all $i \in \mathcal{S}^c$. In addition, by lemma A.3, the minimizer of

$$\widehat{k}_s(t_s) = \min_{\mathbf{x}_{\mathcal{S}}\in\mathcal{X}_{\mathcal{S}}:\sum_{i\in\mathcal{S}} x_i = t_s} \sum_{i\in\mathcal{S}} \widehat{k}_i(x_i)/\gamma_i \qquad \text{for linear demand with additive effort effect}$$
$$\widehat{k}_s(t_s) = \min_{\mathbf{x}_{\mathcal{S}}\mathcal{X}_{\mathcal{S}}:\prod_{i\in\mathcal{S}} x_i = t_s} \sum_{i\in\mathcal{S}} \widehat{k}_i(x_i) \qquad \text{for exponential demand with multiplicative effort effect}$$

is nondecreasing with x_s, and hence $x'_i \le x^*_i$ for $i \in \mathcal{S}$. Also, since $Y_{x_{\mathcal{T}}}(\mathbf{x}_{\mathcal{S}}|\mathbf{x}_{\mathcal{I}})$ in non decreasing in $\mathbf{x}_{\mathcal{S}}$, for any $i \in \mathcal{S}$,

$$Y_{\mathcal{T}}(x^*_i, \mathbf{x}^*_{\mathcal{S}\setminus i}|\mathbf{x}_{\mathcal{I}}) \ge Y_{\mathcal{T}}(x'_i, \mathbf{x}^*_{\mathcal{S}\setminus i}|\mathbf{x}_{\mathcal{I}}) \ge Y_{\mathcal{T}}(x'_i, \mathbf{x}'_{\mathcal{S}\setminus i}|\mathbf{x}_{\mathcal{I}}).$$

Therefore,

$$\begin{aligned}\widehat{\pi}_i(\mathbf{x}^*) &\ge \widehat{\pi}_i(x'_i, \mathbf{x}^*_{\mathcal{S}\setminus i}, Y_{\mathcal{T}}(x'_i, \mathbf{x}^*_{\mathcal{S}\setminus i}|\mathbf{x}_{\mathcal{I}}), \mathbf{x}_{\mathcal{I}}) \\ &\ge \widehat{\pi}_i(\mathbf{x}'_{\mathcal{S}}, Y_{\mathcal{T}}(\mathbf{x}'_{\mathcal{S}}|\mathbf{x}_{\mathcal{I}}), \mathbf{x}_{\mathcal{I}}) = \widehat{\pi}_i(\mathbf{x}').\end{aligned}$$

The first inequality follows from the fact that $\mathbf{x}^*_{\mathcal{S}}$ is the equilibrium for sequential decisions, and the second inequality follows from the fact that $\widehat{\pi}_i$ is nondecreasing with x_j for $j \ne i$.

Q.E.D.

Proof of lemma 3.15. If $x'_i > x^*_i$ and suppose $x'_j \le x^*_j$, then by the result of lemma 3.8, the largest best response $y_k(x_i, x_j)$ is nondecreasing with $x_i x_j$ for multiplicative effort effect(or $x_i + x_j$ for additive effort effect), thus $y_k(x'_i, x^*_j) \ge \max\{x'_k, x^*_k\}$. Since $\mathbf{x}'$ is the equilibrium for the first scenario,

$$\widehat{\pi}_j(x_i', x_j', x_k') \geq \widehat{\pi}_j(x_i', x_j^*, y_k(x_i', x_j^*)) \geq \widehat{\pi}_j(x_i', x_j^*, \max\{x_k', x_k^*\}). \tag{A.18}$$

Since $\mathbf{x}^*$ is the largest equilibrium for the second scenario,

$$\widehat{\pi}_j(x_i^*, x_j^*, x_k^*) \geq \widehat{\pi}_j(x_i^*, x_j', x_k^*) \geq \widehat{\pi}_j(x_i^*, x_j', \min\{x_k^*, x_k'\}).$$

Therefore,

$$\widehat{\pi}_j(x_i', x_j', x_k') + \widehat{\pi}_j(x_i^*, x_j^*, x_k^*) \geq \widehat{\pi}_j(x_i', x_j^*, \max\{x_k', x_k^*\}) + \widehat{\pi}_j(x_i^*, x_j', \min\{x_k^*, x_k'\})$$

which contradicts with the fact that π_j is strictly supermodular when demand is exponential with multiplicative effort effect in case $\bar{x}_i > 0, \forall i = 0, \ldots, n$ or linear with additive effort effect. Hence if $x_i' > x_i^*$ then $x_j' \geq x_j^*$. Also by the nondecreasing property of $y_k(x_i, x_j)$, $x_k' \geq x_k^*$. In addition,

$$\begin{aligned}
\widehat{\pi}_k(x_i', x_j', x_k') &\geq \widehat{\pi}_k(x_i', x_j', y_k(x_i^*, x_j^*)) \geq \widehat{\pi}_k(x_i^*, x_j^*, y_k(x_i^*, x_j^*)) = \widehat{\pi}_k(x_i^*, x_j^*, x_k^*) \\
\widehat{\pi}_i(x_i', x_j', x_k') &> \widehat{\pi}_i(x_i^*, x_j', y_k(x_i', x_j')) \geq \widehat{\pi}_i(x_i^*, x_j^*, y_k(x_i^*, x_j^*)) = \widehat{\pi}_i(x_i^*, x_j^*, x_k^*)) \\
\widehat{\pi}_j(x_i', x_j', x_k') &\geq \widehat{\pi}_j(x_i', x_j^*, y_k(x_i', x_j^*)) \geq \widehat{\pi}_j(x_i^*, x_j^*, y_k(x_i', x_j^*)) = \widehat{\pi}_i(x_i^*, x_j^*, x_k^*)
\end{aligned}$$

The first inequality in above inequalities is because $\mathbf{x}'$ is the equilibrium in scenario, and the second inequality is because that $\widehat{\pi}_i$ ($\widehat{\pi}_j$ and $\widehat{\pi}_k$) in increasing with the other players' effective effort.

Q.E.D.

Proof of lemma 3.13. To show that the Nash equilibrium is unqiue, we first prove that for any equilibrium point $\mathbf{x}'$, we have $u > x_i' > l$ for all $i = 0, \ldots, n$ and hence the Nash equiibrium must be an interior point. In the next step, we show that the interior point defined by first order condition sets is unique.

For $i = 0, \ldots, n$, clearly we have:

$$\frac{\partial \widehat{\pi}_i}{\partial x_i}|_{x_i=u} = \alpha \prod_{j \neq i} x_j - \lambda_i n_i x_i^{n_i - 1} = \alpha \prod_{j \neq i} x_j - \lambda_i n_i u^{n_i - 1}.$$

Since $x_i \le u$, $u > (\frac{\alpha}{\lambda_i n_i})^{\frac{1}{n_i-n-1}}$ and $n_i > n+1$ for all $i = 0,\dots,n$, we have:

$$\frac{\alpha\prod_{j\neq i} x_j}{n_i\lambda_i u^{n_i-1}} \le \frac{\alpha u^n}{\lambda_i n_i u^{n_i-1}} = \frac{\alpha}{\lambda_i n_i u^{n_i-n-1}} < 1.$$

Therefore,

$$\frac{\partial\widehat{\pi}_i}{\partial x_i}|_{x_i=u} < 0.$$

Similarly, $\forall i = 0,\dots,n$, clearly we have:

$$\frac{\partial\widehat{\pi}_i}{\partial x_i}|_{x_i=l} = \alpha\prod_{j\neq i} x_j - \lambda_i n_i x_i^{n_i-1} = \alpha\prod_{j\neq i} x_j - \lambda_i n_i l^{n_i-1}. \quad (A.19)$$

Since $x_i \ge l$, $0 < l < (\frac{\alpha}{\lambda_i n_i})^{\frac{1}{n_i-n-1}}$ and $n_i < n+1$ for all $i = 0,\dots,n$, we have:

$$\frac{\alpha\prod_{j\neq i} x_j}{n_i\lambda_i l^{n_i-1}} \ge \frac{\alpha l^n}{\lambda_i n_i l^{n_i-1}} = \frac{\alpha}{\lambda_i n_i l^{n_i-n-1}} > 1.$$

Therefore,

$$\frac{\partial\widehat{\pi}_i}{\partial x_i}|_{x_i=l} > 0$$

Since an equilibrium always exists, and it can not be at the boundary, it must be an interior point and satsify the first order condition:

$$\begin{aligned}
&\frac{\partial\widehat{\pi}_i}{\partial x_i} = \alpha\prod_{j\neq i} x_j - \lambda_i n_i x_i^{n_i-1} = 0 \qquad \forall i,\dots,n \\
\Rightarrow\ & \alpha\prod_{j=0}^{n} x'_j = \lambda_i n_i x_i'^{n_i} \qquad \forall i = 0,\dots,n \qquad (A.20)\\
\Rightarrow\ & x'_i = \Big(\frac{\alpha}{\lambda_i n_i}\prod_{j=0}^{n} x'_j\Big)^{1/n_i} \qquad \forall i = 0,\dots,n \qquad (A.21)\\
\text{Multiply equation A.21 for all i}\ \Rightarrow\ & \prod_{j=0}^{n} x'_j = \frac{\alpha^{\sum_{j=0}^{n} 1/n_j}}{(\prod_{j=0}^{n}\lambda_j n_j)^{1/n_j}}\Big(\prod_{j=0}^{n} x'_j\Big)^{\sum_{j=0}^{n} 1/n_j}
\end{aligned}$$

Let $\eta = 1 - \sum_{i=0}^{n} 1/n_i > 0$, the solution is unique as follows:

$$\begin{aligned}
\prod_{i=0}^{n} x'_i &= \Big(\frac{\alpha^{1-\eta}}{\prod_{i=0}^{n}(n_i\lambda_i)^{1/n_i}}\Big)^{1/\eta} = \Big(\frac{\alpha^{1-\eta}}{\gamma}\Big)^{1/\eta},\\
\text{By equation A.21:}\quad x'_i &= \Big(\frac{\alpha\prod_{i=0}^{n} x'_i}{n_i\lambda_i}\Big)^{1/n_i} = \Big(\frac{(\frac{\alpha}{\gamma})^{1/\eta}}{n_i\lambda_i}\Big)^{1/n_i} \quad \forall i = 0,\dots,n,\\
\text{From equation A.20:}\quad \widehat{\pi}'_i &= \alpha\prod_{i=0}^{n} x'_i - \lambda_i x_i'^{n_i} = \alpha\Big(1 - \frac{1}{n_i}\Big)\prod_{i=0}^{n} x'_i = \Big(1 - \frac{1}{n_i}\Big)\Big(\frac{\alpha}{\gamma}\Big)^{1/\eta}.
\end{aligned}$$

Q.E.D.

Proof of lemma 3.14. For any given $\mathbf{x}_{\mathcal{S}}$, since $n_i > n+1$, $0 < l < (\frac{\alpha}{\lambda_i n_i})^{\frac{1}{n_i-n-1}}$ and $u > (\frac{\alpha}{\lambda_i n_i})^{\frac{1}{n_i-n-1}}$, for $\forall i \in \mathcal{T}$, we have :

$$\frac{\partial \widehat{\pi}_i(\mathbf{x}_{\mathcal{S}})}{\partial x_i}|_{x_i=u} = \alpha \prod_{j \neq i} x_j - \lambda_i n_i x_i^{n_i-1} = \alpha \prod_{j \neq i} x_j - \lambda_i n_i u^{n_i-1} \geq \alpha u^n - \lambda n_i u^{n_i-1} > 0$$
$$\frac{\partial \widehat{\pi}_i(\mathbf{x}_{\mathcal{S}})}{\partial x_i}|_{x_i=l} = \alpha \prod_{j \neq i} x_j - \lambda_i n_i x_i^{n_i-1} = \alpha \prod_{j \neq i} x_j - \lambda_i n_i l^{n_i-1} \leq \alpha l^n - \lambda n_i u^{n_i-1} < 0$$

Therefore, the best response $\mathbf{Y}_{\mathcal{T}}(\mathbf{x}_{\mathcal{S}})$ of players in $\mathcal{T}$ must be interior point. The equilibrium is the same as the equilibrium in the system with t players in $\mathcal{T}$ and the market capacity α replaced by $\alpha \prod_{j \in \mathcal{S}} x_j$. By the result of lemma 3.13, after replacing η by $\eta_{\mathcal{T}} = 1 - \sum_{i \in \mathcal{T}} 1/n_i > 0$ and α by $\alpha \prod_{j \in \mathcal{S}} x_j$, the best response is unique and satisfies:

$$\prod_{i \in \mathcal{T}} x_i = (\frac{(\alpha \prod_{j \in \mathcal{S}} x_j)^{1-\eta_T}}{\prod_{i \in \mathcal{T}} (n_i \lambda_i)^{1/n_i}})^{1/\eta_T} = (\frac{(\alpha \prod_{j \in \mathcal{S}} x_j)^{1-\eta_T}}{\gamma_T})^{1/\eta_T}$$

where $\gamma_T = \prod_{i \in \mathcal{T}} (n_i \lambda_i)^{1/n_i}$.

With a similar approach as we do in proof of lemma 3.13, to show the Nash equilibrium among players in $\mathcal{S}$ is unique, we first prove that for any equilibrium point $\mathbf{x}_{\mathcal{S}}{}^*$, we have $u > x_i^* > l$ for all $i \in \mathcal{S}$ and hence the Nash equilibrium must be an interior point. In the next step, we show that the interior point defined the by first order equation sets is unique as we give.

$\forall i \in \mathcal{S}$, we have

$$\widehat{\pi}_i(\mathbf{x}_{\mathcal{S}}) = (\frac{\alpha}{\gamma_T})^{1/\eta_T} \prod_{i \in \mathcal{S}} x_i^{1/\eta_T} - \lambda_i x_i^{n_i} \qquad \forall i \in \mathcal{S}$$

$$\frac{\partial \widehat{\pi}_i}{\partial x_i}|_{x_i=u} = \frac{1}{\eta_T}(\frac{\alpha}{\gamma_T})^{\frac{1}{\eta_T}} u^{\frac{1-\eta_T}{\eta_T}} \prod_{j \in \mathcal{S}, j \neq i} x_j^{\frac{1}{\eta_T}} - n_i \lambda_i u^{n_i-1} \leq \frac{1}{\eta_T}(\frac{\alpha}{\gamma_T})^{\frac{1}{\eta_T}} u^{\frac{1-\eta_T}{\eta_T}} u^{\frac{s-1}{\eta_T}} - n_i \lambda_i u^{n_i-1}$$

$$\text{While:} \qquad \frac{\frac{1}{\eta_T}(\frac{\alpha}{\gamma_T})^{\frac{1}{\eta_T}} u^{\frac{1-\eta_T}{\eta_T}} u^{\frac{s-1}{\eta_T}}}{n_i \lambda_i u^{n_i-1}} = \frac{1}{\eta_T n_i \lambda_i}(\frac{\alpha}{\gamma_T})^{\frac{1}{\eta_T}} / u^{n_i - \frac{s}{\eta_T}}$$

Since $\eta_{\mathcal{T}} = 1 - \sum_{j\in\mathcal{T}} \frac{1}{n_j} \geq 1 - \frac{t}{n+1} = \frac{s}{n+1}$, $n_i - \frac{s}{\eta_{\mathcal{T}}} \geq n_i - (n+1) > 0$. Furthermore, since $u > \big(\frac{1}{\eta_T n_i \lambda_i}(\frac{\alpha}{\gamma_T})^{\frac{1}{\eta_T}}\big)^{\frac{1}{n_i - s/\eta}}$, we have:

$$\frac{\partial \widehat{\pi}_i}{\partial x_i}|_{x_i=u} \leq \frac{1}{\eta_T}(\frac{\alpha}{\gamma_T})^{\frac{1}{\eta_T}} u^{\frac{1-\eta_T}{\eta_T}} u^{\frac{s-1}{\eta_T}} - n_i\lambda_i u^{n_i-1} < 0.$$

Similarly,

$$\frac{\partial \widehat{\pi}_i}{\partial x_i}|_{x_i=l} = \frac{1}{\eta_T}(\frac{\alpha}{\gamma_T})^{\frac{1}{\eta_T}} l^{\frac{1-\eta_T}{\eta_T}} \prod_{j\in\mathcal{S}, j\neq i} x_j^{\frac{1}{\eta_T}} - n_i\lambda_i l^{n_i-1} \geq \frac{1}{\eta_T}(\frac{\alpha}{\gamma_T})^{\frac{1}{\eta_T}} l^{\frac{1-\eta_T}{\eta_T}} u^{\frac{s-1}{\eta_T}} - n_i\lambda_i l^{n_i-1}.$$

Since $l < \big(\frac{1}{\eta_T n_i \lambda_i}(\frac{\alpha}{\gamma_T})^{\frac{1}{\eta_T}}\big)^{\frac{1}{n_i - s/\eta}}$, we have

$$\frac{\partial \widehat{\pi}_i}{\partial x_i}|_{x_i=u} \geq \frac{1}{\eta_T}(\frac{\alpha}{\gamma_T})^{\frac{1}{\eta_T}} l^{\frac{1-\eta_T}{\eta_T}} l^{\frac{s-1}{\eta_T}} - n_i\lambda_i l^{n_i-1} > 0.$$

Therefore, the equilibrium among players in $\mathcal{S}$ must be an interior point and satisfies the first order condition. For players in $\mathcal{S}$, now check:

$$\begin{aligned} \widehat{\pi}_i(\mathbf{x}_{\mathcal{S}}) &= (\frac{\alpha}{\gamma_T})^{\frac{1}{\eta_T}} (\prod_{i\in\mathcal{S}} x_i)^{1/\eta_{\mathcal{T}}} - \lambda_i x_i^{n_i} \\ \frac{\partial \widehat{\pi}_i(\mathbf{x}_{\mathcal{S}})}{\partial x_i} &= \frac{1}{\eta_{\mathcal{T}}}(\frac{\alpha}{\gamma_T})^{\frac{1}{\eta_T}} x_i^{1/\eta_{\mathcal{T}}-1} (\prod_{j\in\mathcal{S}, j\neq i} x_j)^{1/\eta_{\mathcal{T}}} - n_i\lambda_i x_i^{n_i-1} \end{aligned}$$

We solve the equation set at $\frac{\partial \widehat{\pi}_i(\mathbf{x}_{\mathcal{S}})}{\partial x_i} = 0$. For $x_i > 0$, the solution is unique as:

$$\frac{1}{\eta_{\mathcal{T}}}(\frac{\alpha}{\gamma_T})^{\frac{1}{\eta_T}} (\prod_{i\in\mathcal{S}} x_i^*)^{1/\eta_{\mathcal{T}}} = n_i\lambda_i x_i^{*n_i} \tag{A.22}$$

$$\Rightarrow \quad \frac{\alpha^{1/\eta_T}}{n_i\lambda_i\eta_T\gamma_T^{1/\eta_T}} (\prod_{i\in\mathcal{S}} x_i)^{1/\eta_T} = x_i^{n_i} \quad \forall i \in \mathcal{S} \tag{A.23}$$

Multiply equation A.23 for all $i \in \mathcal{S}$

$$\Rightarrow \quad \prod_{i\in\mathcal{S}} x_i^* = (\frac{\alpha^{1-\eta_S}}{\gamma_S^{\eta_T}\gamma_T^{1-\eta_S}\eta_T^{(1-\eta_S)\eta_T}})^{1/\eta}$$

By equation A.23

$$x_i^* = (\frac{\alpha^{1/\eta_T}}{n_i\lambda_i\eta_T\gamma_T^{\frac{1}{\eta_T}}} (\prod_{i\in\mathcal{S}} x_i^*)^{1/\eta})^{1/n_i} = (\frac{\alpha^{1/\eta}}{n_i\lambda_i\eta_T^{\eta_T/\eta}\gamma_S^{1/\eta}\gamma_T^{1/\eta}})^{1/n_i} \quad \forall i \in \mathcal{S}$$

By equation A.22

$$\widehat{\pi}_i^* = \frac{\alpha^{1/\eta_T}}{\gamma_T^{1/\eta_T}}(1 - \frac{1}{n_i\eta_T})(\prod_{i\in\mathcal{S}} x_i)^{1/\eta_T} = (1 - \frac{1}{n_i\eta_T})\frac{\alpha^{1/\eta}}{\eta_T^{1-\eta_S/\eta}\gamma_S^{1/\eta}\gamma_T^{1/\eta}} \quad \forall i \in \mathcal{S}$$

Substituting the solution into the equation, we have :

$$\begin{aligned}
\prod_{i\in\mathcal{T}} x_i^* &= \Big(\frac{(\alpha\prod_{i\in\mathcal{S}} x_i^*)^{1-\eta_T}}{\gamma}\Big)^{1/\eta_T} = \Big(\frac{\alpha^{1-\eta_T}}{\gamma_S^{1-\eta_T}\gamma_T^{\eta_S}\eta_T^{(1-\eta_S)(1-\eta_T)}}\Big)^{1/\eta} \\
x_i^* &= \Big(\frac{\alpha\prod_{i\in\mathcal{S}} x_i^*\prod_{i\in\mathcal{T}} x_i^*}{n_i\lambda_i}\Big)^{1/n_i} = \Big(\frac{\alpha^{(2-\eta)/\eta}}{n_i\lambda_i\gamma_S^{1/\eta}\gamma_T^{1/\eta}\eta_T^{(1-\eta)/\eta}}\Big)^{\frac{1}{n_i}} \quad \forall i\in\mathcal{T} \\
\widehat{\pi}_i^* &= \alpha\prod_{i\in\mathcal{S}} x_i^*\prod_{i\in\mathcal{T}} x_i^*\Big(1-\frac{1}{n_i}\Big) = \Big(1-\frac{1}{n_i}\Big)\frac{\alpha^{(2-\eta)/\eta}}{\gamma_S^{1/\eta}\gamma_T^{1/\eta}\eta_T^{(1-\eta)/\eta}} \quad \forall i\in\mathcal{T}
\end{aligned}$$

Q.E.D.

Proof of lemma 3.16. By lemma 3.11, since the equilibrium is unique in sequential decisions, it must be the maximizer of the potential function, therefore in scenario 1, we have:

$$\begin{aligned}
\mathbf{x}'_{\mathcal{S}_1\cup\mathcal{S}_2} &= argmax_{\mathbf{x}_{\mathcal{S}_1\cup\mathcal{S}_2}\in\mathcal{X}_{\mathcal{S}_1\cup\mathcal{S}_2}} \quad \Psi_{\mathbf{x}_{\mathcal{S}_1\cup\mathcal{S}_2}}(\mathbf{x}_{\mathcal{S}_1\cup\mathcal{S}_2}) = \widehat{\psi}(\mathbf{x}_{\mathcal{S}_1\cup\mathcal{S}_2}, Y_{\mathcal{S}_3}(\mathbf{x}_{\mathcal{S}_1\cup\mathcal{S}_2})) - \sum_{i\in\mathcal{S}_1\cup\mathcal{S}_2}\widehat{k}_i(x_i)/\gamma_i \\
\mathbf{x}'_{\mathcal{S}_3} &= argmax_{\mathbf{x}_{\mathcal{S}_3}\in\mathcal{X}_{\mathcal{S}_3}} \quad \Psi_{\mathbf{x}_{\mathcal{S}_3}}(\mathbf{x}_{\mathcal{S}_3}) = \widehat{\psi}(\mathbf{x}_{\mathcal{S}_3}|\mathbf{x}'_{\mathcal{S}_1\cup\mathcal{S}_2}) - \sum_{i\in\mathcal{S}_3}\widehat{k}_i(x_i)/\gamma_i.
\end{aligned}$$

For scenario 2, we have:

$$\begin{aligned}
\mathbf{x}^*_{\mathcal{S}_1} &= argmax_{\mathbf{x}_{\mathcal{S}_1}\in\mathcal{X}_{\mathcal{S}_1}} \quad \Psi_{\mathbf{x}_{\mathcal{S}_1}}(\mathbf{x}_{\mathcal{S}_1}) = \widehat{\psi}(\mathbf{x}_{\mathcal{S}_1}, Y_{\mathcal{S}_2\cup\mathcal{S}_3}(\mathbf{x}_{\mathcal{S}_1})) - \sum_{i\in\mathcal{S}_1}\widehat{k}_i(x_i)/\gamma_i \\
\mathbf{x}^*_{\mathcal{S}_2\cup\mathcal{S}_3} &= argmax_{\mathbf{x}_{\mathbf{x}_{\mathcal{S}_2\cup\mathcal{S}_3}}\in\mathcal{X}_{\mathcal{S}_2\cup\mathcal{S}_3}} \quad \Psi_{\mathbf{x}_{\mathcal{S}_2\cup\mathcal{S}_3}}(\mathbf{x}_{\mathcal{S}_2\cup\mathcal{S}_3}) = \widehat{\psi}(\mathbf{x}_{\mathcal{S}_2\cup\mathcal{S}_3}|\mathbf{x}^*_{\mathcal{S}_1}) - \sum_{\mathbf{x}_{\mathcal{S}_2\cup\mathcal{S}_3}}\widehat{k}_i(x_i)/\gamma_i.
\end{aligned}$$

Since $\widehat{\psi}$ and $Y_{\mathcal{S}}(\mathbf{x}_{\mathcal{S}})$ is only a function of $\sum_{i\in\mathcal{S}} x_i$ for additive demand effort (or $\prod_{i\in\mathcal{S}} x_i$ for multiplicative demand effort), so we can replace the game with three groups by the game with three artificial players s_1, s_2 and s_3, such that:

$$\begin{aligned}
\text{for linear additive demand:} \quad \widehat{k}_{s_i}(t_{s_i}) &= \min_{\mathbf{x}_{\mathcal{S}}\in\mathcal{X}_{\mathcal{S}}:\sum_{j\in\mathcal{S}} x_j = t_{s_i}} \sum_{i\in\mathcal{S}}\widehat{k}_j(x_j)/\gamma_j \quad \forall i = 1,2,3 \\
\text{for exponential multiplicative demand:} \quad \widehat{k}_{s_i}(t_{s_i}) &= \min_{\mathbf{x}_{\mathcal{S}}\in\mathcal{X}_{\mathcal{S}}:\prod_{j\in\mathcal{S}} x_j = t_{s_i}} \sum_{i\in\mathcal{S}}\widehat{k}_j(x_j)/\gamma_j \quad \forall i = 1,2,3
\end{aligned}$$

Therefore, by the potential function as we defined:

$$\begin{aligned}
(t_{s_1}, t_{s_2})' &= argmax_{t_{s_1}\in\mathcal{T}_{s_1}, t_{s_2}\in\mathcal{T}_{s_2}} \widehat{\psi}(t_{s_1}, t_{s_2}, Y_{s_3}(t_{s_1}, t_{s_2})) - \widehat{k}_{s_1}(t_{s_1}) - \widehat{k}_{s_2}(t_{s_2}) \\
t'_{s_3} &= argmax_{t_{s_3}\in\mathcal{T}_{s_3}} \widehat{\psi}(t_{s_3}|(t_{s_1}, t_{s_2})') - \widehat{k}_{s_3}(t_{s_3}) \\
t^*_{s_1} &= argmax_{t_{s_1}\in\mathcal{T}_{s_1}} \widehat{\psi}(t_{s_1}, Y_{s_2,s_3}(t_{s_1})) - \widehat{k}_{s_1}(t_{s_1}) \\
(t_{s_2}, t_{s_3})^* &= argmax_{t_{s_2}\in\mathcal{T}_{s_2}, t_{s_3}\in\mathcal{T}_{s_3}} \widehat{\psi}(t_{s_2}, t_{s_3}|t^*_{s_1}) - \widehat{k}_{s_2}(t_{s_2}) - \widehat{k}_{s_3}(t_{s_3})
\end{aligned}$$

where $\mathcal{T}_{s_i} = [\sum_{j\in\mathcal{S}_i} \underline{x}_j, \sum_{j\in\mathcal{S}_i} \overline{x}_j]$ for linear additive demand and $\mathcal{T}_{s_i} = [\prod_{j\in\mathcal{S}_i} \underline{x}_j, \prod_{j\in\mathcal{S}_i} \overline{x}_j]$ for exponential multiplicative demand. In addition, $\mathbf{x}'_{\mathcal{S}_i}$ (or $\mathbf{x}^*_{\mathcal{S}_i}$) is the minimizer for $\widehat{k}_{s_i}(t'_{s_i})$ (or $\widehat{k}_{s_i}(t^*_{s_i})$) for $i = 1, 2, 3$.

If $\forall i \in \mathcal{S}_1$, $x'_i \geq x^*_i$, then $t'_{s_1} = \sum_{i\in\mathcal{S}_1} x'_i \geq \sum_{i\in\mathcal{S}_1} x^*_i = t^*_{s_1}$. Thus, by the lemma 3.15, $t'_{s_2} > t^*_{s_2}$, $t_{s'_3} > t^*_{s_3}$. Moreover, by lemma A.3, $x'_i \geq x^*_i$ for all $i = 0, \ldots, n$. Hence,

$$\forall i \in \mathcal{S}_3 : \widehat{\pi}_i(\mathbf{x}'_{\mathcal{S}_1}, \mathbf{x}'_{\mathcal{S}_2}, \mathbf{x}'_{\mathcal{S}_3}) \geq \widehat{\pi}_i(\mathbf{x}'_{\mathcal{S}_1}, \mathbf{x}'_{\mathcal{S}_2}, x^*_i, \mathbf{x}'_{\mathcal{S}_3\backslash i}) \geq \widehat{\pi}_i(\mathbf{x}^*_{\mathcal{S}_1}, \mathbf{x}^*_{\mathcal{S}_2}, \mathbf{x}^*_{\mathcal{S}_3})$$

$$\begin{aligned}
\forall i \in \mathcal{S}_1 : \quad & \widehat{\pi}_i(\mathbf{x}'_{\mathcal{S}_1}, \mathbf{x}'_{\mathcal{S}_2}, \mathbf{x}'_{\mathcal{S}_3}) \geq \widehat{\pi}_i(x^*_i, \mathbf{x}'_{\mathcal{S}_1\backslash i}, \mathbf{x}'_{\mathcal{S}_2}, Y_{\mathcal{S}_3}(x^*_i, \mathbf{x}'_{\mathcal{S}_1\backslash i}, \mathbf{x}'_{\mathcal{S}_2})) \\
\geq \; & \widehat{\pi}_i(\mathbf{x}^*_{\mathcal{S}_1}, \mathbf{x}^*_{\mathcal{S}_2}, Y_{\mathcal{S}_3}(\mathbf{x}^*_{\mathcal{S}_1}, \mathbf{x}^*_{\mathcal{S}_2})) = \pi_i(\mathbf{x}^*_{\mathcal{S}_1}, \mathbf{x}^*_{\mathcal{S}_2}, \mathbf{x}^*_{\mathcal{S}_3})
\end{aligned}$$

$$\begin{aligned}
\forall i \in \mathcal{S}_2 : \quad & \widehat{\pi}_i(\mathbf{x}'_{\mathcal{S}_1}, \mathbf{x}'_{\mathcal{S}_2}, \mathbf{x}'_{\mathcal{S}_3}) \geq \widehat{\pi}_i(\mathbf{x}'_{\mathcal{S}_1}, x^*_i, \mathbf{x}'_{\mathcal{S}_2\backslash i}, Y_{\mathcal{S}_3}(\mathbf{x}'_{\mathcal{S}_1}, x^*_i, \mathbf{x}'_{\mathcal{S}_2\backslash i})) \\
\geq \; & \widehat{\pi}_i(\mathbf{x}^*_{\mathcal{S}_1}, \mathbf{x}^*_{\mathcal{S}_2}, Y_{\mathcal{S}_3}(\mathbf{x}^*_{\mathcal{S}_1}, \mathbf{x}^*_{\mathcal{S}_2})) = \pi_i(\mathbf{x}^*_{\mathcal{S}_1}, \mathbf{x}^*_{\mathcal{S}_2}, \mathbf{x}^*_{\mathcal{S}_3})
\end{aligned}$$

For each case, the first inequality is because $(\mathbf{x}'_{\mathcal{S}_1}, \mathbf{x}'_{\mathcal{S}_2}, \mathbf{x}'_{\mathcal{S}_3})$ is the equilibrium strategy, while the second one is because $\widehat{\pi}_i$ is non-decreasing in j, $\forall j \neq i$.

Q.E.D.

Proof of lemma 3.17. Since for every given t, the strategy space $\mathcal{X}$ do not change, by Theorem 2.6.1 in [44], to show that f has increasing difference in $(t, \S)$, it is enough to show that f is supermodular in $(t, \mathbf{x})$. For all $(t, \mathbf{x})^1$, $(t, \mathbf{x})^2 \in (0, +\infty) \times \mathcal{X}$. By definition, to show the supermodularity of $f(\alpha, \mathbf{x})$, we have to check that:

$$\begin{aligned}
& f((t, \mathbf{x})^1 \wedge (t, \mathbf{x})^2) + f((t, \mathbf{x})^1 \vee (t, \mathbf{x})^2) - f((t, \mathbf{x})^1) - f((t, \mathbf{x})^2) \\
= \; & (t^1 \wedge t^2)\widehat{\psi}(\mathbf{x}^1 \wedge \mathbf{x}^2) + (t^1 \vee t^2)\widehat{\psi}(\mathbf{x}^1 \vee \mathbf{x}^2) - t_1\widehat{\psi}(\mathbf{x}^1) - t_2\widehat{\psi}(\mathbf{x}^2) \geq 0.
\end{aligned}$$

For exponential demand function with multiplicative effort effect:

$$\frac{\partial^2 t\widehat{\psi}(\mathbf{x})}{\partial x_i \partial x_j} = t\frac{\alpha e^{-2}}{\lambda}\prod_{k\neq i,j} x_k \geq 0 \qquad \forall i,j \in \{0,\ldots,n\} \text{ and } i \neq j$$

$$\frac{\partial^2 t\widehat{\psi}(\mathbf{x})}{\partial x_i \partial t} = \alpha\frac{e^{-2}}{\lambda}\prod_{k\neq i} x_k \geq 0 \qquad \forall i \in \{0,\ldots,n\}.$$

Thus, $(t^1 \wedge t^2)\widehat{\psi}(\mathbf{x}^1 \wedge \mathbf{x}^2) + (t^1 \vee t^2)\widehat{\psi}(\mathbf{x}^1 \vee \mathbf{x}^2) - t_1\widehat{\psi}(\mathbf{x}^1) - t_2\widehat{\psi}(\mathbf{x}^2) \geq 0$.

Q.E.D.

Proof of lemma 3.18. Let $\mathbf{x}^t = \max \operatorname{argmax}_{\mathbf{x}\in\mathcal{X}} \mathrm{t}\widehat{\psi}(\mathbf{x}) - \sum_{\mathrm{i}} \widehat{\mathrm{k}}_{\mathrm{i}}(\mathrm{x}_{\mathrm{i}})$. As explained in the proof of lemma 3.4,if $t = 1$, $\widehat{\psi}(\mathbf{x}) - \sum_i \widehat{k}_i(x_i)$ is a potential function of the transformed investment effort game, and $\mathbf{x}^t \in \mathcal{X}^b$. Also note that the function $t\widehat{\psi}(\mathbf{x}) - \sum_i \widehat{k}_i(x_i)$ is a supermodular function in $\mathbf{x}$ and has increasing difference in $(t, \mathbf{x})$, hence $\mathbf{x}^t$ is increasing with t. On the other hand, in system coordination, $t = n$ and $\mathbf{x}^n = \mathbf{x}^* = \max \operatorname{argmax}_{\mathbf{x}\in\mathcal{X}} \mathrm{n}\widehat{\psi}_{\mathrm{i}}(\mathbf{x}) - \sum_{\mathrm{i}} \widehat{\mathrm{k}}_{\mathrm{i}}(\mathrm{x}_{\mathrm{i}})$. Hence $\mathbf{x}^* \succeq \mathbf{x}^1$ and $\widehat{\psi}(\mathbf{x}^*) \geq \widehat{\psi}(\mathbf{x}^1)$ while $\mathbf{x}^1 \in \mathcal{X}^b$.

Q.E.D.

Proof lemma 3.19. Since $t > 0$, $\frac{\partial f(s,x)}{\partial -s} = \widehat{k}_0(x_0)$. Note that $\widehat{k}_0$ is only a function of x_0 and increasing in x_0, this implies $f(s, x)$ has increasing difference in $(-s, \mathbf{x})$.

Q.E.D.

Proof of lemma 3.20. Let $\mathbf{x}^{t,s} = \max \operatorname{argmax}_{\mathbf{x}\in\mathcal{X}} \mathrm{t}\widehat{\psi}(\mathbf{x}) - \sum_{\mathrm{i=1}}^{\mathrm{n}} \widehat{\mathrm{k}}_{\mathrm{i}}(\mathrm{x}_{\mathrm{i}}) - \mathrm{s}\widehat{\mathrm{k}}_0(\mathrm{x}_0)$. By lemma 3.17, since the function $t\widehat{\psi}(\mathbf{x}) - \sum_{i=0}^n \widehat{k}_i(x_i)$ is a supermodular function in $\mathbf{x}$ and has increasing difference in $(t, \mathbf{x})$, $\mathbf{x}^{t,1}$ is increasing with t. Therefore, $\mathbf{x}^{n,1} \succeq \mathbf{x}^{1,1}$ and $\widehat{\psi}(\mathbf{x}^{n,1}) \geq \widehat{\psi}(\mathbf{x}^{1,1})$. Also by lemma 3.19, the function $\widehat{\psi}(\mathbf{x}) - \sum_{i=1}^n \widehat{k}_i(x_i) - s\widehat{k}_0(x_0)$ is supermodular in $\mathbf{x}$ and has increasing difference in $(-s, \mathbf{x})$. Hence, $\mathbf{x}^{1,s}$ is decreasing with s. Therefore, $\mathbf{x}^{1,1} \succeq \mathbf{x}^{1,2}$ and $\widehat{\psi}(\mathbf{x}^{1,1}) \geq \widehat{\psi}(\mathbf{x}^{1,2})$. Note that in system coordination, $t = n$ an $s = 1$ so $\mathbf{x}^{n,1} = \mathbf{x}^*$. While in system competition, $t\widehat{\psi}(\mathbf{x}) - \sum_{i=1}^n \widehat{k}_i(x_i) - s\widehat{k}_0(x_0)$ is the potential function of the transformed effort game when $t = 1$ and $s = 2$. Hence $\mathbf{x}^{1,2} \in \mathbf{x}^b$. Therefore $\exists \mathbf{x}^b \in \mathcal{X}^b$ such that $\mathbf{x}^* \succeq \mathbf{x}^b$ and $\widehat{\psi}_i(\mathbf{x}^*) \geq \widehat{\psi}_i(\mathbf{x}^b)$ for all $i = 0, \ldots, n$.

Q.E.D.

A.4. Tables for Numerical Examples

Table A.1: Problem Characteristics

Problem	$g_1(x)$	$g_2(x)$	$g_3(x)$	$f(x)$	w_1	w_2	w_3
1	NS1	NS1	NS1	DNS	6	5	3
2	NS1	NS1	NS1	DNS	6	5	4
3	NS1	NS1	NS1	DNS	6	5	5
4	NS1	NS1	NS1	DNS	6	5	6
5	NS1	NS1	NSS1	DNS	6	5	3
6	NS1	NS1	NSS2	DNS	6	5	3
7	NS1	NS1	NSF1	DNS	6	5	3
8	NS1	NS1	NSF2	DNS	6	5	3
9	NS1	NS1	LS1	DNS	6	5	3
10	NS1	NS1	LS2	DNS	6	5	3
11	NS1	NS1	LS3	DNS	6	5	3
12	NS1	NS1	LS4	DNS	6	5	3
13	NS1	NS1	RS1	DNS	6	5	3
14	NS1	NS1	RS2	DNS	6	5	3
15	NS1	NS1	RS3	DNS	6	5	3
16	NS1	NS1	RS4	DNS	6	5	3
17	NS1	NS1	NS1	DNS1	6	5	3
18	NS1	NS1	NS1	DNS2	6	5	3
19	NS1	NS1	NS1	DNSS	6	5	3
20	NS1	NS1	NS1	DNF1	6	5	3
21	NS1	NS1	NS1	DNF2	6	5	3
22	NSS1	NSS1	NSS1	DNS	6	5	3
23	NSS2	NSS2	NSS2	DNS	6	5	3
24	NSF1	NSF1	NSF1	DNS	6	5	3
25	NSF2	NSF2	NSF2	DNS	6	5	3
26	LS1	NS1	RS1	DNS	6	5	3
27	LS2	NS1	RS2	DNS	6	5	3
28	LS3	NS1	RS3	DNS	6	5	3
29	LS4	NS1	RS4	DNS	6	5	3

Table A.2: Computational Results

No.	Optimal Cost	q_1*	q_2*	q_3*	Time (s)	Heuristic Gurnani	Optim Gap (%)	Heuristic TPQ	Optim Gap (%)
1	4661.81	391.02	397.67	415.45	1744	5145.81	10.38	4662.84	0.020
2	5071.31	389.45	396.07	403.98	1913	5506.72	8.59	5071.76	0.008
3	5407.43	387.92	394.52	394.52	1949	5865.03	7.21	5470.54	0.002
4	5860.8	386.41	392.99	386.40	1761	6220.92	6.14	5860.82	0.0003
5	4524.96	389.58	396.17	377.97	1649	4891.46	8.10	4526.94	0.043
6	4572.81	390.44	397.06	392.57	1724	4982.71	8.96	4573.98	0.026
7	4825.12	390.91	397.60	450.66	1677	5422.08	12.37	4826.07	0.019
8	5129.36	389.82	396.55	501.42	1607	5860.16	14.25	5129.85	0.009
9	4649.12	390.95	397.59	413.01	1578	5134.62	10.44	4650.14	0.022
10	4635.03	390.86	397.50	410.21	1579	5123.97	10.55	4636.03	0.022
11	4618.66	390.75	397.39	406.93	1542	5114.12	10.73	4619.68	0.022
12	4598.18	390.73	397.36	403.47	1621	5105.18	11.03	4599.08	0.019
13	4673.5	391.08	397.73	417.63	1526	5157.20	10.35	4674.55	0.022
14	4684.44	391.12	397.78	419.62	1516	5168.55	10.33	4685.50	0.023
15	4694.76	391.15	397.82	421.46	1507	5179.68	10.33	4695.84	0.023
16	4704.59	391.17	397.84	423.17	1517	5190.53	10.33	4705.68	0.023
17	4716.98	392.81	399.53	417.64	1420	5098.46	8.09	4718.36	0.029
18	4796.62	395.67	402.52	421.15	1735	5094.74	6.22	4798.32	0.035
19	4936.09	401.26	408.35	427.77	1688	5153.42	4.40	4938.11	0.041
20	5163.57	410.79	418.24	438.72	2020	5314.96	2.93	5165.56	0.038
21	5514.46	425.61	433.51	455.33	2591	5617.29	1.86	5515.9	0.026
22	4057.75	364.33	367.28	375.01	1534	4255.21	4.87	4059.34	0.039
23	4289.40	375.41	379.76	391.17	1799	4614.70	7.58	4290.82	0.033
24	5270.62	410.88	421.30	450.17	1951	5912.65	12.18	5270.97	0.006
25	6290.77	429.88	445.98	493.34	1907	7007.67	11.40	6292.43	0.027
26	4656.66	390.41	397.75	417.64	2041	5146.23	10.51	4657.71	0.022
27	4649.00	389.81	397.86	419.74	2053	5147.54	10.72	4650.03	0.022
28	4638.27	389.46	397.07	421.83	2033	5149.78	11.03	4639.24	0.021
29	4623.85	389.80	398.47	424.05	1632	5152.91	11.44	4624.68	0.018

VITA

Chengbin Zhu was born in Shanghai in 1977. He graduated with a Bachelor of Science degree in Electrical Engineering in 1999 from Hefei University of Technology. After his graduation, he worked as a telecom network consultant in Shanghai Post&Telecom Design Institute for four years.

In 2003, Chengbin moved to the United States to pursue a Ph.D. degree in the Grado Department of Industrial and System Engineering at Virginia Polytechnic Institute and State University (Virginia Tech) under the guidance of Dr. L.M. Ann Chan. In 2005, Chengbin received a Master's degree in Industrial and System Engineering. During his Ph.D. studies at Virginia Tech. During his study in Virginia Tech, he worked as research assistant and teaching assistant. Chengbin Zhu was awarded the Grado Fellowship during 2006 to 2008 due to his performance as a student and his area of research. He gave several presentations in INFORMS national meeting and MSOM conference. In August 2008, he successfully defend his Ph.D. dissertation. His research interests include inventory control, revenue management and game theory.