

USE OF NONLINEAR ELEMENTS FOR THE CONTROL  
OF A SECOND ORDER LINEAR SYSTEM

by  
Clifton D. <sup>David</sup>Cullum, Jr.

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## I. INTRODUCTION

The use of nonlinear elements in the control of feedback systems is a relatively young area of investigation, especially with respect to its state of development. Strictly linear control systems have been thoroughly investigated and powerful tools developed. In comparison, the use of nonlinear elements in control has only a few tools to draw upon, and even these are useful for the most part in analysis only.

This thesis attempts to present a design procedure for use in attacking one portion of the total problem. Specifically, a method is presented for designing a nonlinear controller to achieve a desired step response in a second order linear system. In a sense, the controller designed herein achieves optimum control of the system in that any reasonable, desired response to an input step can be obtained with good accuracy.

The main body of this thesis consists of the development of the design theory along with its application to several examples. All theoretical calculations for the examples are correlated experimentally with the analog computer. For the most part theoretical analysis is done in the phase plane.

It is presumed that the reader is already familiar with the analysis and design of linear control systems. The tools available for handling such systems will be called on from time to time throughout the thesis. Any introductory book such as Savant<sup>(12)</sup> or Chestnutt and Mayer<sup>(3)</sup> will provide adequate background in linear theory.

If the reader is not familiar with the phase plane method of analysis, it is suggested that he refer to a book such as that by Ku<sup>(8)</sup>. The phase plane will be used extensively throughout the thesis.

All the circuits used in connection with the analog computer are of the type found in books on analog computers such as those by Jackson<sup>(6)</sup> and Johnson<sup>(7)</sup>. The symbols used in the analog computer diagrams are also standard and may be found in either of the above-mentioned books.

## II. REVIEW OF THE LITERATURE

Two of the first papers of significance in the area of using nonlinear elements to aid in control were those by Hopkin<sup>(5)</sup> and MacDonald<sup>(11)</sup>. Hopkin's paper discussed the relay or contactor servomechanism. In this system a relay is used to switch the polarity of the actuating signal while its magnitude remains constant. This is an extreme form of nonlinearity. MacDonald investigated the effects of several different nonlinearities. He was primarily concerned with nonlinearities which produced variable damping effects. This was achieved by making the actuating signal equal to some function involving the sum, product, and quotient of the error and its derivative. Both authors approached the problem through phase plane analysis. Lathrop<sup>(9)</sup> later repeated and extended MacDonald's work with the aid of the analog computer.

J. B. Lewis<sup>(10)</sup> devised a nonlinearity for use with a linear system which can be treated by the method discussed in this thesis. Like MacDonald, Lewis used a combination of the error and its derivative for an actuating signal. The Lewis servomechanism is used as one of the examples later in the thesis. This

system was studied quite extensively on the analog computer by Caldwell and Rideout<sup>(2)</sup>.

In England, Douce, Naylor, and West<sup>(4)</sup> performed an excellent study of the use of nonlinear elements in second order systems with saturation limitations. Their work borders quite closely on the area investigated in this thesis. Specifically, they interpreted the effect of the nonlinear elements on the transient response by observing the shape of the phase plane trajectory corresponding to zero actuating signal. This curve, referred to later as the Zero Force Curve, is a basic part of the design procedure presented in this thesis. Their use of the Zero Force Curve was limited in nature.

A paper with real significance with respect to this thesis was published in 1960 by Athanassiades and Smith<sup>(1)</sup>. In this paper the concept of the Zero Force Curve as a design tool was brought out. Examples of its use were given for second order systems with saturation limitations. The authors used the Zero Force Curve merely as a guide to the response so that the full potentialities of this method were not explored. Their main contribution came in recognizing that the nonlinearity could be designed from the Zero Force Curve, while this thesis develops the correlation between the Zero Force Curve and the system response.

### III. THE ZERO FORCE CURVE METHOD OF DESIGN

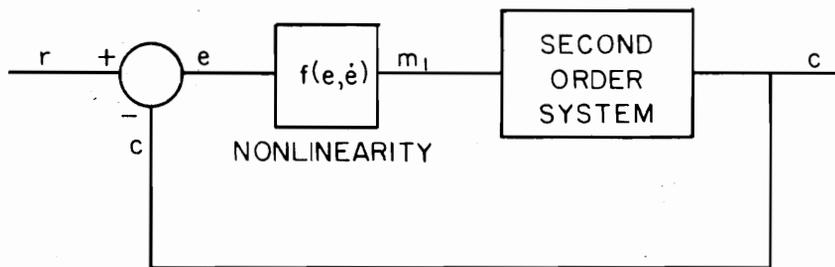
#### Design Theory

The design of nonlinearities for otherwise linear systems will be based on a function referred to as the Zero Force Curve<sup>(1)</sup>. The Zero Force Curve, or briefly ZFC, is defined as the phase plane trajectory a system must follow in response to some disturbance in order to keep the actuating signal zero at all times. Once it has been shown that it is possible to make the actual system trajectory correspond closely to the ZFC, then any desired trajectory may be achieved by designing a nonlinear computer to produce the proper Zero Force Curve.

The basic block diagram of such a system is shown in Figure 1. The nonlinear computer has as an output the function  $f(e, \dot{e})$ . The Zero Force Curve is then,

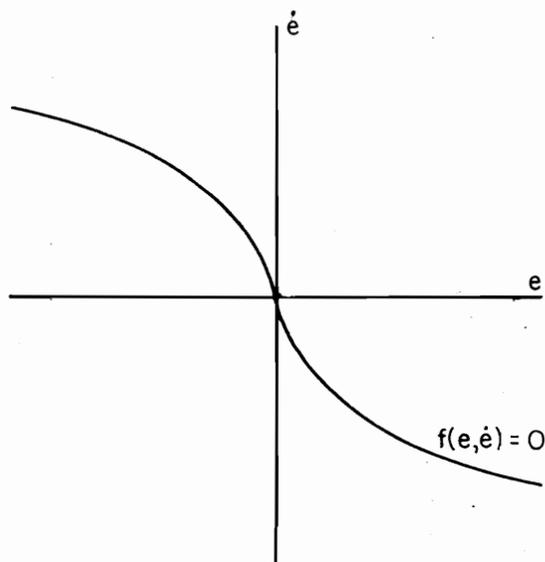
$$f(e, \dot{e}) = 0$$

This equation specifies some curve in the phase plane as shown in Figure 2. For any desired trajectory, a nonlinear computer may be designed so that the ZFC coincides with this trajectory. If the system response in turn coincides with the Zero Force Curve, then the



$r$  = REFERENCE INPUT  
 $c$  = OUTPUT  
 $e = r - c$  = ERROR  
 $\dot{e}$  = TIME DERIVATIVE OF THE ERROR  
 $m_1$  = ACTUATING SIGNAL

**FIGURE 1. BLOCK DIAGRAM OF PROPOSED  
NONLINEAR CONTROL SYSTEM**



**FIGURE 2. TYPICAL ZERO FORCE CURVE**

system response will conform to the desired response. This is the basis of the Zero Force Curve method of design.

It does not seem necessary to comment at length on the design of the nonlinear computer. Curve fitting and function generation are the two areas of primary concern here. These areas are more than adequately covered in the literature.

The key to this entire approach is the premise that the actual system response may be made to correspond very closely to the Zero Force Curve. In this thesis, this premise will be shown to be true for step inputs only, and also for only certain classes of linear second order systems. Therefore, conclusions may be drawn only for systems which fit these restrictions. However, it is not the author's intent to imply that the ZFC approach is valid for these systems alone. Extension to other systems will be commented on at the end of the thesis.

The four types of second order transfer functions for which the above-mentioned premise will be shown to be true are listed below:

CASE I.  $G = K_1/p^2$

CASE II.  $G = K_1/p(p + \frac{1}{T_1})$

CASE III.  $G = K_1/(p + \frac{1}{T_1})(p + \frac{1}{T_2})$

CASE IV.  $G = K_1/(p^2 + 2\zeta \frac{1}{T_1} p + \frac{1}{T_1^2}) \quad \zeta < 1$

where:

$K_1$  = gain constant

$p$  = derivative with respect to time

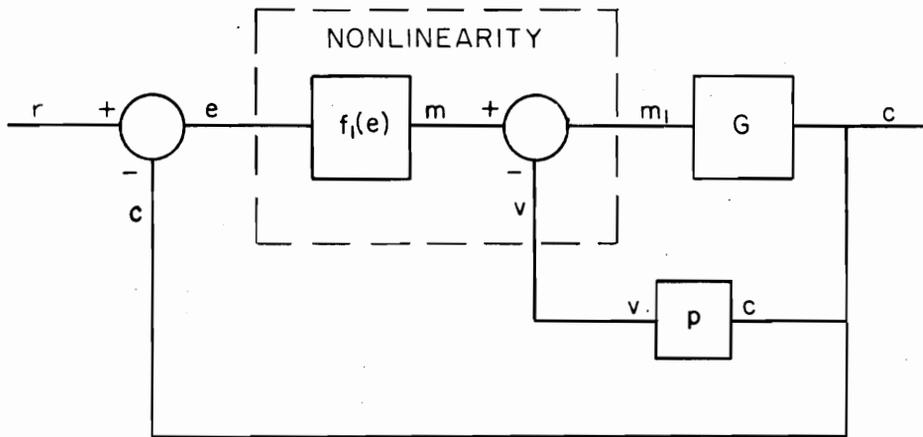
$T_1, T_2$  = time constants, real and positive

$\zeta$  = damping ratio.

It is logical to assume that after a period of time the actuating signal in all four systems will reduce to a very small value. This is a basic property of a stable system with negative feedback, and it will be shown that all of the four systems are stable. If the length of time it takes the actuating signal to reduce to a small value is small enough so that the error will have not changed appreciably over this period, the system trajectory will very nearly be given by the Zero Force Curve. Thus, the problem may be regarded as an investigation of the transient response of the system trajectory about the ZFC.

The Zero Force Curve specifies a particular value of error rate (or output velocity in the case of a step input) for every value of error. The actuating signal and the nonlinear computer may be viewed in the manner shown schematically in Figure 3. The equation of the Zero Force Curve is solved for the output velocity as a function of  $e$ . The nonlinear computer is designed to solve this equation, generating a signal,  $m$ , which corresponds to the value of output velocity required to make the system trajectory correspond to the ZFC for the error existing at that instant. The actual output velocity,  $v$ , is subtracted from  $m$  to give the actuating signal. When the actuating signal is zero, the system trajectory corresponds to the Zero Force Curve as required.

The system response about the ZFC can now be analyzed in terms of this inner feedback loop. If the time required for the system response to correspond closely to the ZFC is small enough so that there is little change in the error over this period, then  $m$ , which is a function of the error, may be considered constant. The inner feedback loop then consists of a step of velocity applied to the linear system element



$$f(e, \dot{e}) = f_1(e) - v = m - v = m_1$$

$m$  = VELOCITY REQUIRED FOR  
ZERO FORCE CURVE

$v$  = ACTUAL OUTPUT VELOCITY

$m_1$  = ACTUATING SIGNAL

$G$  = SYSTEM TRANSFER FUNCTION

**FIGURE 3. BLOCK DIAGRAM OF CONTROL  
SYSTEM SHOWING INNER  
FEEDBACK LOOP**

with the output velocity as the controlled variable.

The closed loop response is given by:

$$\frac{v}{m} = \frac{pG}{1+GH} = \frac{pG}{1+pG} .$$

Now the four cases in question may be considered.

Case I.       $G = \frac{K_1}{p^2}$

Substituting for G,

$$\frac{v}{m} = \frac{K_1}{p+K_1}$$

$$(p+K_1)v = K_1 m .$$

Solving,

$$v = m[1-\exp(-K_1 t)] .$$

This is the response of a first order system. The output velocity becomes exactly equal to m in steady state, and the time constant of the transient is  $1/K_1$ . Therefore, the duration of the transient may be made negligibly small by increasing the system gain,  $K_1$ . The initial assumptions will then be correct, and the system response will agree with the ZFC in a negligibly

small time. The design procedure is therefore valid for system transfer functions of this type.

Case II. 
$$G = \frac{K}{p(p + \frac{1}{T_1})}$$

Substituting,

$$\frac{v}{m} = \frac{K}{p + (K_1 + \frac{1}{T_1})}$$

$$[p + (K_1 + \frac{1}{T_1})]v = K_1 m .$$

Solving,

$$v = m \left\{ 1 - \exp \left[ - \left( K_1 + \frac{1}{T_1} \right) t \right] \right\} .$$

The solution in this case differs from that obtained in Case I only in the time constant. Actually, the added term in the time constant makes the speed of response greater so that the transient dies out faster.

Therefore, systems with transfer functions of this type also follow the ZFC in a satisfactory manner.

Case III.  $G = \frac{K_1}{\left(p + \frac{1}{T_1}\right)\left(p + \frac{1}{T_2}\right)}$

where  $T_1$  and  $T_2$  are real. Substituting,

$$\frac{v}{m} = \frac{pK_1}{p^2 + \left(\frac{1}{T_1} + \frac{1}{T_2} + K_1\right)p + \frac{1}{T_1 T_2}}$$

$$\left[p^2 + \left(\frac{1}{T_1} + \frac{1}{T_2} + K_1\right)p + \frac{1}{T_1 T_2}\right]v = pK_1 m .$$

Under the assumption that  $m$  is constant,

$$pm = 0 .$$

Therefore,

$$[p^2 + C_1 p + C_2]v = 0$$

where

$$C_1 = \left(\frac{1}{T_1} + \frac{1}{T_2} + K_1\right)$$

$$C_2 = \frac{1}{T_1 T_2} .$$

The roots of the characteristic equation are,

$$p_1, p_2 = -\frac{C_1}{2} \pm \sqrt{\frac{C_1^2}{4} - C_2} .$$

The discriminant is then,

$$\frac{C_1^2}{4} - C_2 = \frac{1}{4} \left( \frac{1}{T_1^2} + \frac{1}{T_2^2} + K_1^2 + \frac{2}{T_1 T_2} + 2 \left[ \frac{1}{T_1} + \frac{1}{T_2} \right] K_1 \right) - \frac{1}{T_1 T_2}$$

$$\frac{C_1^2}{4} - C_2 = \frac{1}{4} \left( \frac{1}{T_1} - \frac{1}{T_2} \right)^2 + \frac{K_1^2}{4} + \frac{1}{2} \left( \frac{1}{T_1} + \frac{1}{T_2} \right) K_1 .$$

Examination of the above expression shows that the discriminant is always positive. Therefore the system response is never oscillatory. However, the steady state velocity is seen to be zero. There is an apparent deviation from the type of response encountered in Cases I and II. This must be investigated further.

Initially, an assumption will be made with regard to the parameter  $C_2$ . If  $C_2$  were zero, the response would be the same as for the first two cases. Therefore,  $C_2$  will be assumed to be only slightly different from zero; small enough to allow the inclusion of only the first two terms of the binominal expansion of the radical. In this case,

$$\left[ \frac{C_1^2}{4} - C_2 \right]^{1/2} = \frac{C_1}{2} - \frac{1}{2} \frac{1}{C_1^{1/2}} C_2 = \frac{C_1}{2} - \frac{C_2}{C_1} .$$

Therefore,

$$p_1 = -\frac{C_1}{2} + \frac{C_1}{2} - \frac{C_2}{C_1} = -\frac{C_2}{C_1}$$

$$p_2 = -\frac{C_1}{2} - \frac{C_1}{2} + \frac{C_2}{C_1} = -C_1 + \frac{C_2}{C_1} .$$

The quantity  $C_2/C_1$  has been restricted to very small values in order to use the binominal expansion.

Therefore,

$$p_2 \approx C_1$$

If  $K_1$  is again made large, then  $C_1$  will be large. The exponential term involving the  $p_2$  coefficient will go to zero quickly as is desired.

The other root,  $p_1$ , will be very small. The exponential term corresponding to  $p_1$  will decay very slowly. If the total time of response of the system (not just the time to reach the ZFC condition) is small compared to  $C_1/C_2$ , then this second exponential term may be neglected. The response is again essentially first order. The time constant  $C_1/C_2$  is given in terms of the system parameters below.

$$\frac{C_1}{C_2} = \frac{\frac{1}{T_1} + \frac{1}{T_2} + K_1}{\frac{1}{T_1 T_2}} = T_2 + T_1 + K_1 T_1 T_2 .$$

It can be seen again that the requirements for a large  $C_1/C_2$  can be met by making  $K_1$  sufficiently large. The exact size of  $K_1$  will depend on  $T_1$ ,  $T_2$ , and the total time of response as determined by the ZFC. Notice should be taken of the fact that the requirement that  $C_1/C_2$  be large validates the use of the abbreviated binominal expansion.

$$\text{Case IV. } G = \frac{K_1}{p^2 + 2\zeta \frac{1}{T_1} p + \frac{1}{T_1^2}} \quad \zeta < 1$$

Substituting

$$\frac{v}{m} = \frac{pK_1}{p^2 + (2\zeta \frac{1}{T_1} + K_1)p + \frac{1}{T_1^2}}$$

$$[p^2 + (2\zeta \frac{1}{T_1} + K_1)p + \frac{1}{T_1^2}]v = K_1 pm .$$

If  $m$  is constant,

$$pm = 0 .$$

Therefore,

$$[p^2 + (2\zeta \frac{1}{T_1} + K_1)p + \frac{1}{T_1^2}]v = 0 .$$

This of the form,

$$[p^2 + C_1 p + C_2]v = 0$$

where

$$C_1 = 2\zeta \frac{1}{T_1} + K_1$$

$$C_2 = \frac{1}{T_1^2} .$$

It has already been shown that satisfactory response may be obtained with this form of system equation if

$$C_1 \gg 1$$

$$\frac{C_1}{C_2} \gg 1$$

Substituting for  $C_1$  and  $C_2$ ,

$$(2\zeta \frac{1}{T_1} + K_1) \gg 1$$

$$(2\zeta T_1 + K_1 T_1^2) \gg 1$$

It is evident that both of these requirements may be satisfied by making  $K_1$  large.

Two things have now been shown. First, for the four types of second order transfer functions

considered the actual system trajectory can be made very close to that of the Zero Force Curve. Second, this close agreement between the actual trajectory and the ZFC is achieved primarily by increasing the gain of the system. This does not lead to instability or large oscillations because the response in all four cases is primarily first order. Certain facets of this problem which have not been thoroughly explored will be discussed in the results.

#### Design of an Example System

The design of the nonlinear computer can best be illustrated by taking an example problem. Suppose that it is required to design a nonlinearity for the system of Figure 4 to meet the following step response requirements:

The system is required to follow a phase plane trajectory into the origin which is that of a linear second order system with  $\zeta = 1.1, \omega_0 = 10$  in response to a step input of 10 units. However, the output velocity is to be limited at all times to the maximum velocity obtained by the above mentioned linear system in response to the 10 unit step.

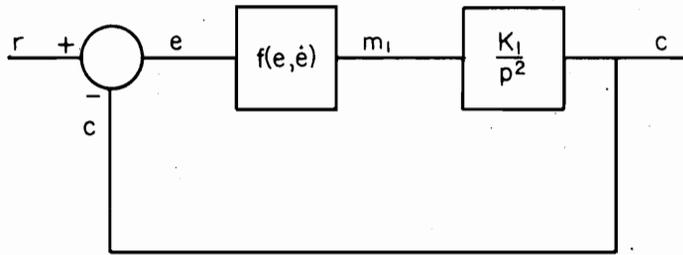


FIGURE 4. BLOCK DIAGRAM OF  
EXAMPLE SYSTEM

The equation of such a trajectory is derived in Appendix I, and the resultant phase plane plot is shown in Figure 5. Only the fourth quadrant is shown since the second quadrant trajectory will be the same. Notice that for step inputs the output velocity and the error velocity are the negative of one another. Only the solid portion of the linear response is to be used. It can be seen that the limiting velocity occurs at an error magnitude of seven units. For any error greater than seven units the error velocity should be -34.4 units per second. This desired trajectory constitutes the required Zero Force Curve.

The design of the nonlinear computer is accomplished by obtaining a function of the error and the error velocity to fit the Zero Force Curve. There are obviously many different forms for functions which may be tried. The form used below was chosen because it was convenient to generate on the analog computer. The trial function is

$$e = -K_1 \dot{e} - K_2 \dot{e} |\dot{e}| .$$

Figure 6 shows a comparison of this curve with the desired curve for  $K_1 = 0.1$ ,  $K_2 = 0.003$

$$e = -0.1\dot{e} - 0.003\dot{e} |\dot{e}| .$$

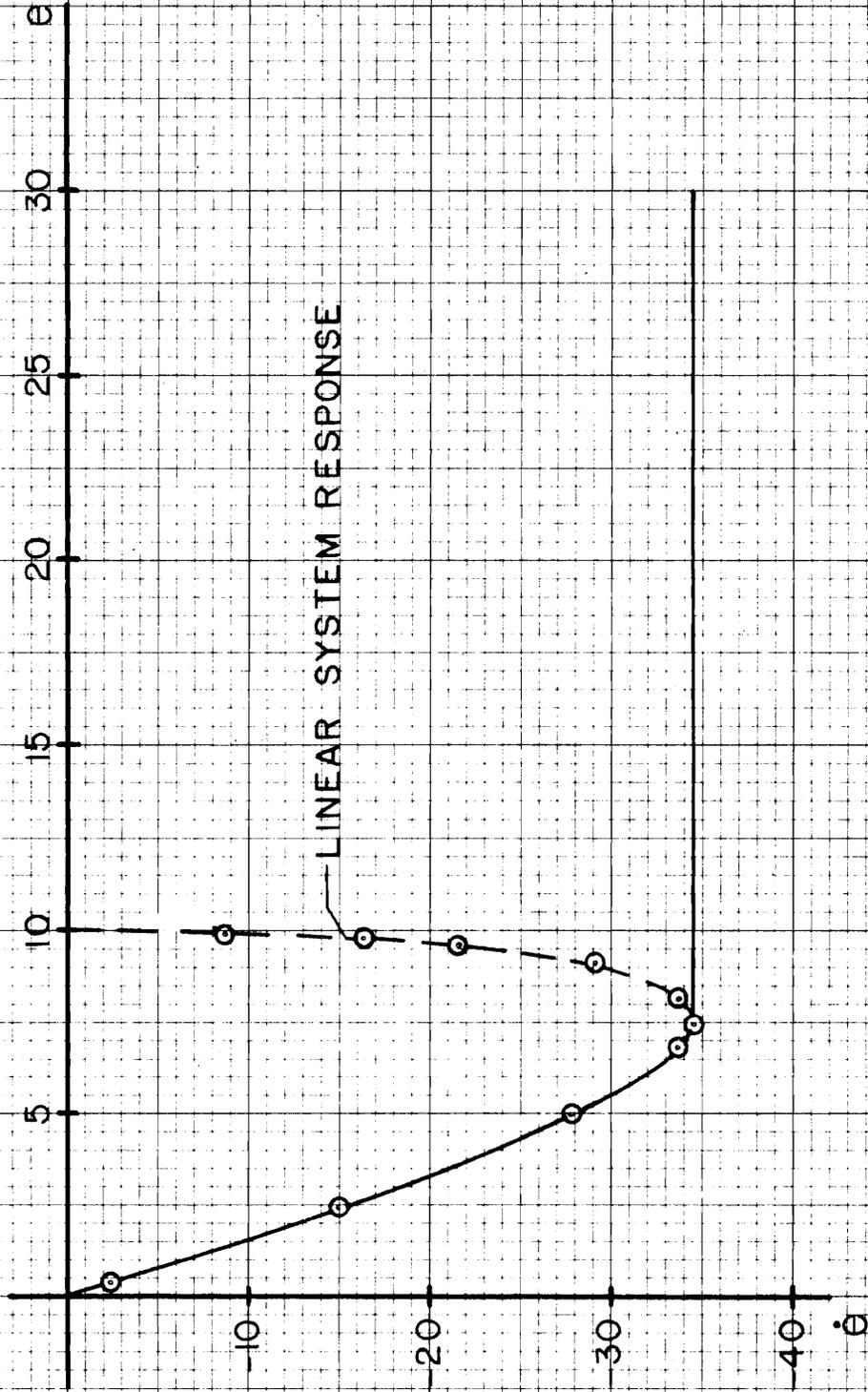


FIGURE 5. DESIRED TRAJECTORY

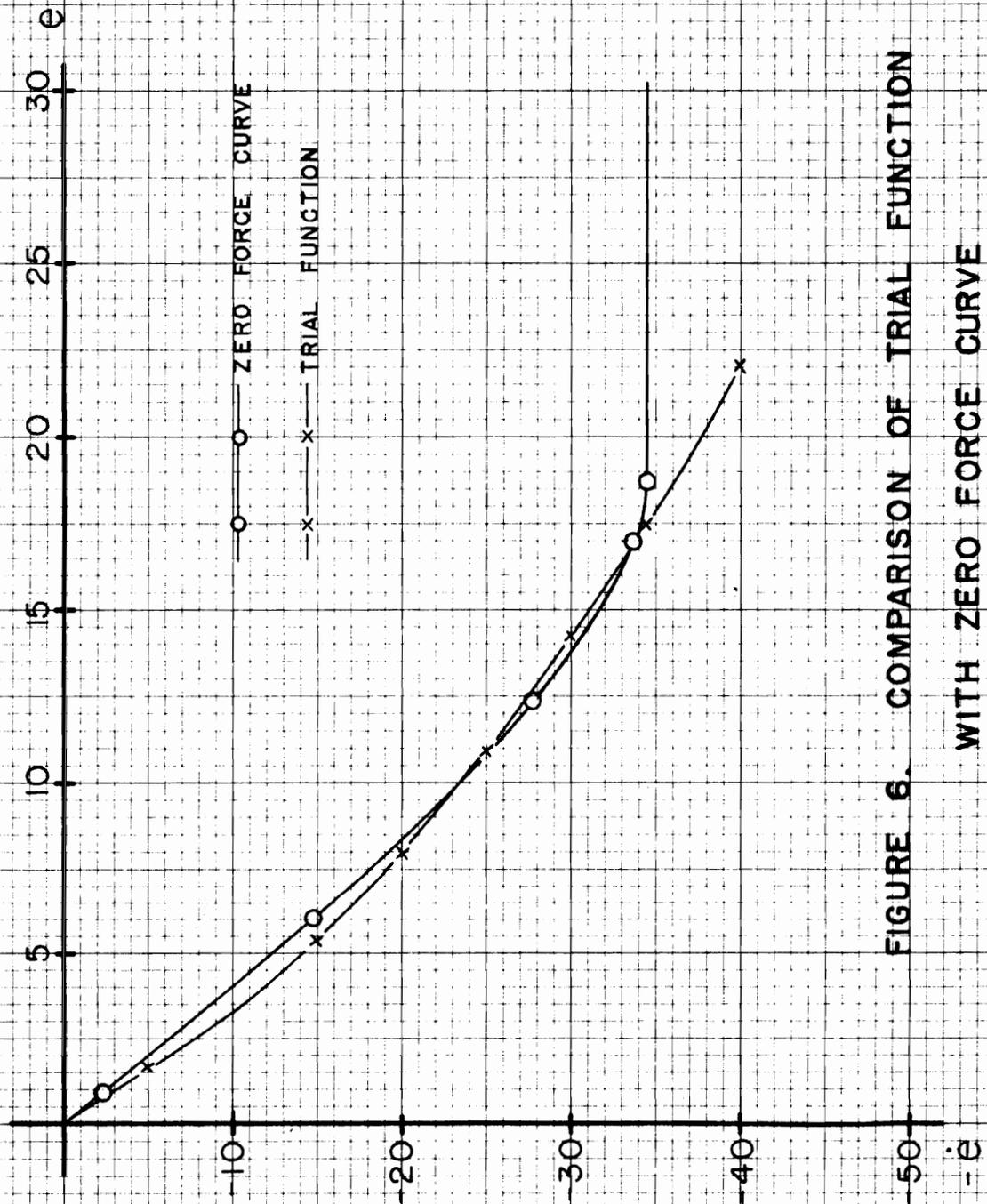


FIGURE 6. COMPARISON OF TRIAL FUNCTION WITH ZERO FORCE CURVE

This is seen to be an excellent approximation to the first part of the curve. The velocity limited portion may be obtained by switching to a different function, a horizontal straight line. The above equation may be rewritten as

$$\dot{e} + \frac{e}{0.1 + 0.003 |\dot{e}|} = 0 .$$

The complete Zero Force Curve is given by

$$f(e, \dot{e}) = 0 = \begin{cases} \dot{e} + \frac{e}{0.1 + 0.003 |\dot{e}|} & |e| \leq 7 \\ \dot{e} + 34.4(\text{sgn } e) & |e| \geq 7 . \end{cases}$$

For convenience in programming this on the analog computer the substitution  $e = -\dot{c}$  will be made. This yields

$$f(e, \dot{c}) = 0 = \begin{cases} -\dot{c} + \frac{e}{0.1 + 0.003 |\dot{c}|} & |e| \leq 7 \\ -\dot{c} + 34.4(\text{sgn } e) & |e| \geq 7 . \end{cases}$$

A scaled analog computer diagram for the entire system is shown in Figure 7. A system gain of 200 is used through most of the work. For a Case I system such as this, the time constant of the transient is 0.005 second.

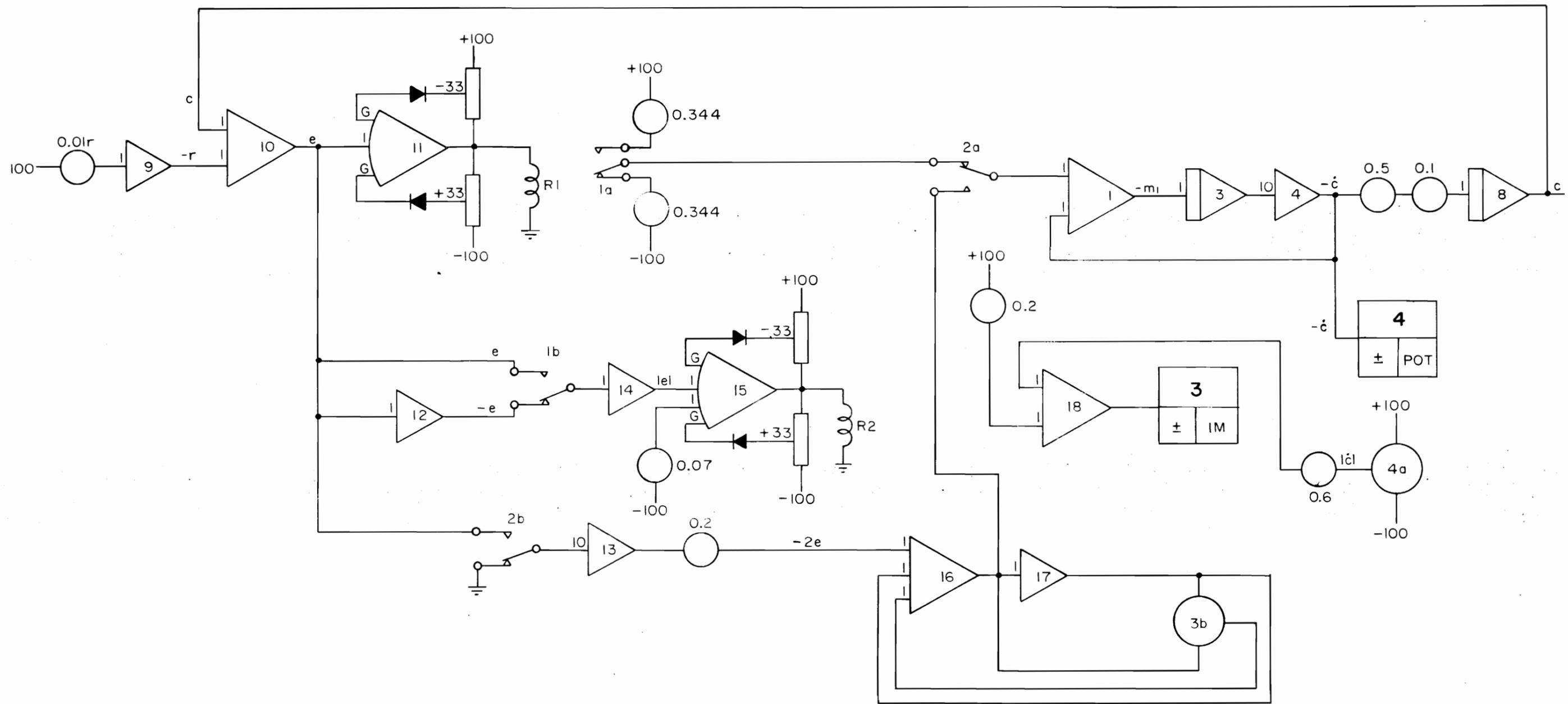


FIGURE 7. ANALOG COMPUTER PROGRAM FOR SIMULATION OF EXAMPLE SYSTEM

If the settling time is regarded as three time constants, then the system trajectory is within five per cent of the ZFC in 0.015 second. The equation for output velocity during this period is

$$\dot{c} = 34.4 \left[ 1 - \exp\left(-\frac{t}{0.005}\right) \right] .$$

Integrating to find the output as a function of time,

$$c = 34.4t + 0.172 \left[ \exp\left(-\frac{t}{0.005}\right) \right] + K .$$

If  $c(0) = 0$ ,

$$K = -0.172$$

$$c = 34.4t - 0.172 \left[ 1 - \exp\left(-\frac{t}{0.005}\right) \right] .$$

At  $t = 0.015$  second,

$$c = 0.352 .$$

Thus the output has changed by just 0.352 unit during the duration of the transient. Since this analysis would apply for step inputs of seven or greater only, this change is only a small portion of the total response time. For step inputs less than seven, the change in output during the duration of the transient would be smaller

because the final velocity would be less than 34.4 units per second.

Computer Solution of Example System. The system shown in Figure 7 was programmed on an Electronics Associates 16-31R Analog Computer located at the Virginia Polytechnic Institute, Blacksburg, Virginia. The computer is a twenty amplifier model with various nonlinear equipment available. The multipliers in particular are the servo-mechanical type, Electronics Associates Model 16-7N-1. The problem was run at one-twentieth of the actual speed to facilitate the recording of results. All results in this section were recorded on a Houston Instruments Corporation, Model HR-92, X-Y Recorder.

Figures 8 through 14 constitute the results of the computer simulation of this system. The large majority of these curves were taken for a step input of 30 units with a system gain of 200. Deviations from this are indicated on the pertinent figures.

There are two significant occurrences on several of the curves which require explanation. One is the initial overshoot of the ZFC present on all of the system phase plane plots. Since previous work specifically precluded such an occurrence, it is

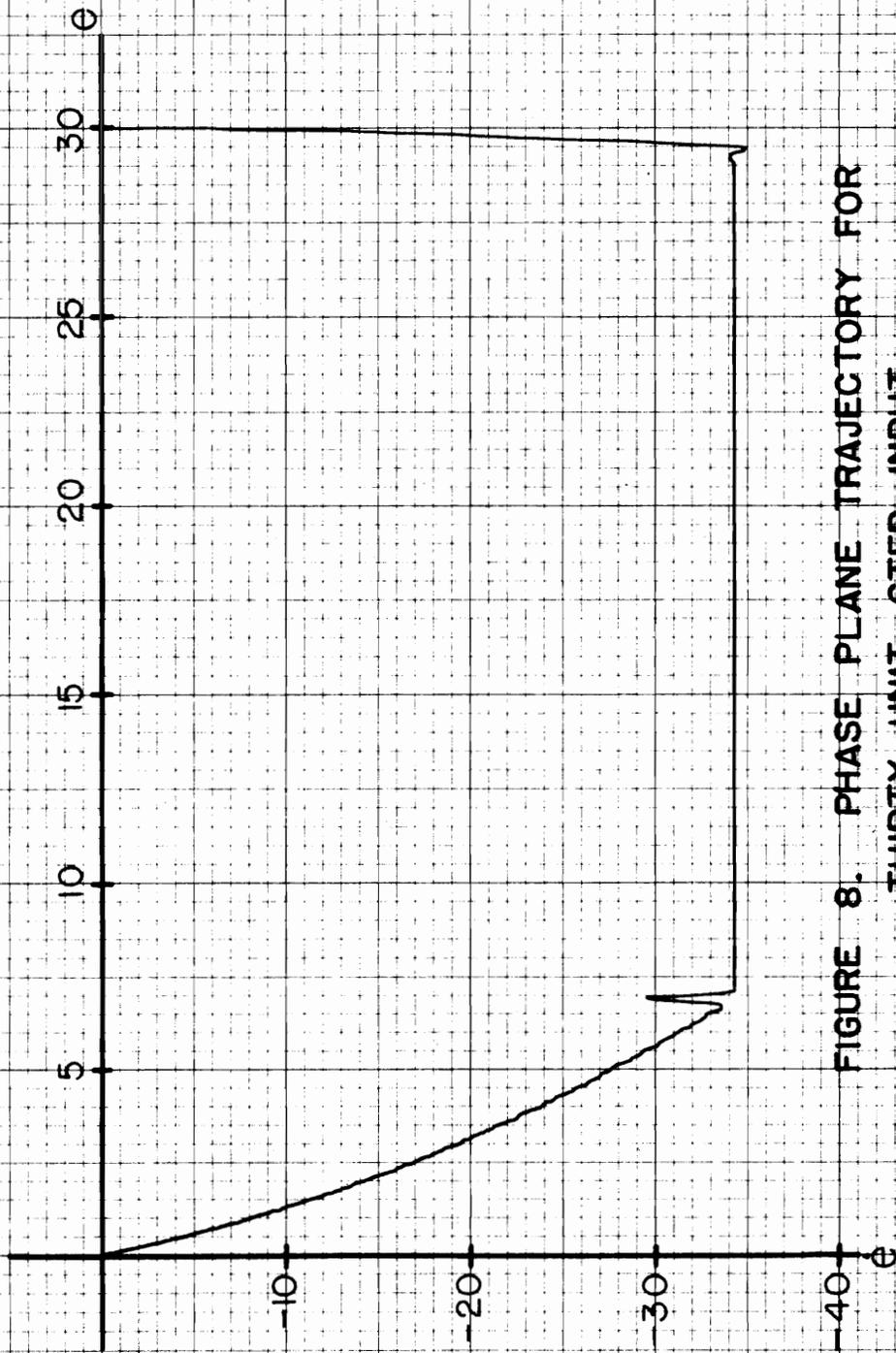


FIGURE 8. PHASE PLANE TRAJECTORY FOR THIRTY UNIT STEP INPUT

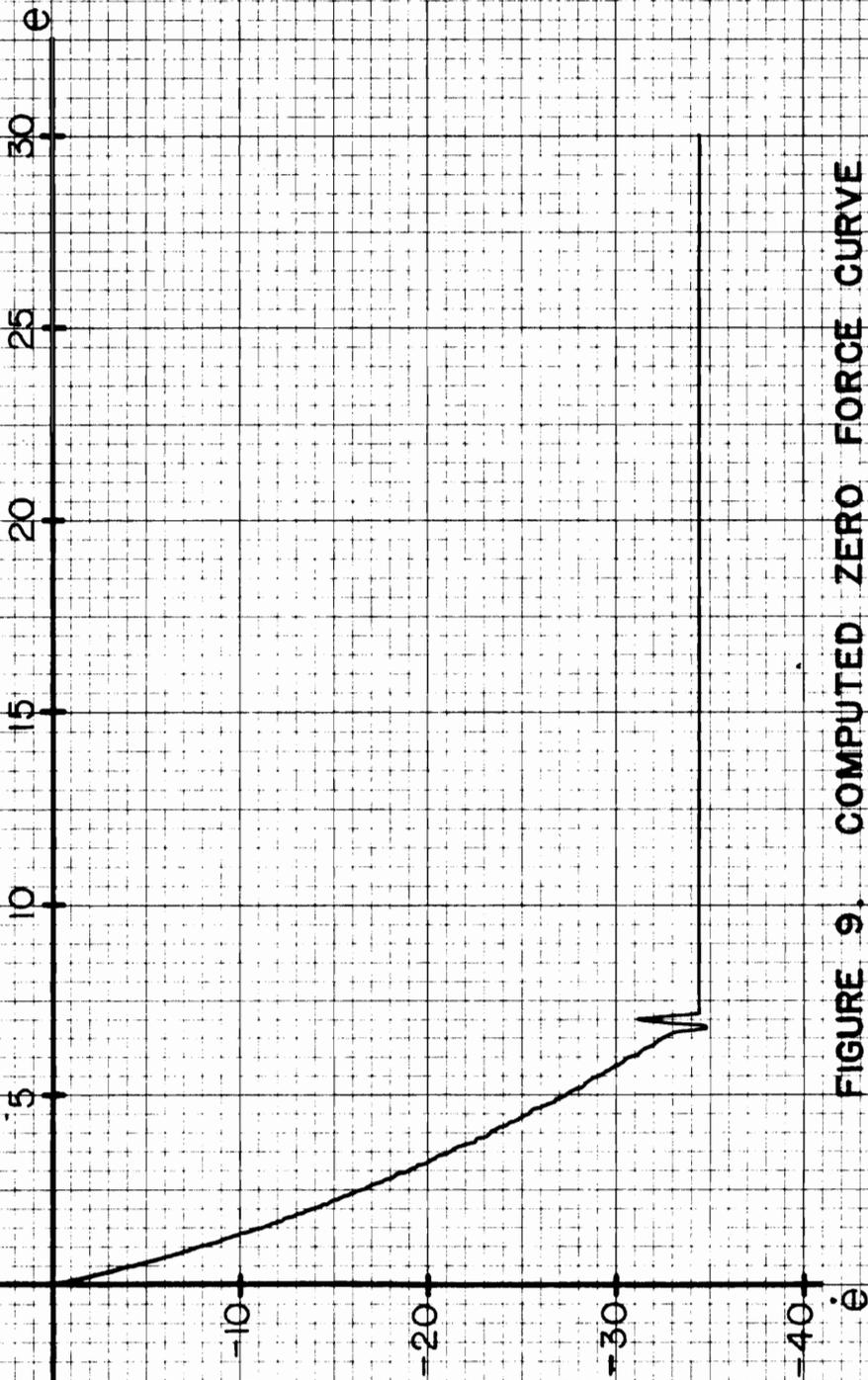


FIGURE 9. COMPUTED ZERO FORCE CURVE

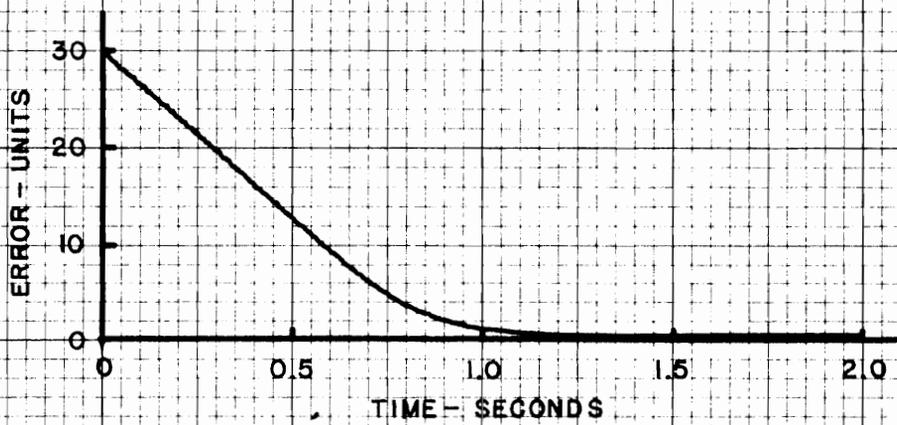
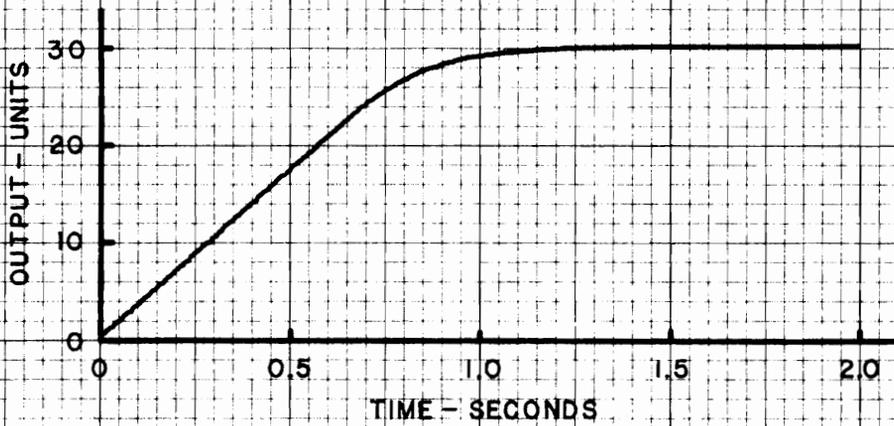


FIGURE 10. OUTPUT AND ERROR VERSUS TIME FOR THIRTY UNIT STEP INPUT

NOTE: SCALES ARE REMOVED FROM THE COORDINATE AXES SO AS NOT TO OBSCURE THE RESPONSE.

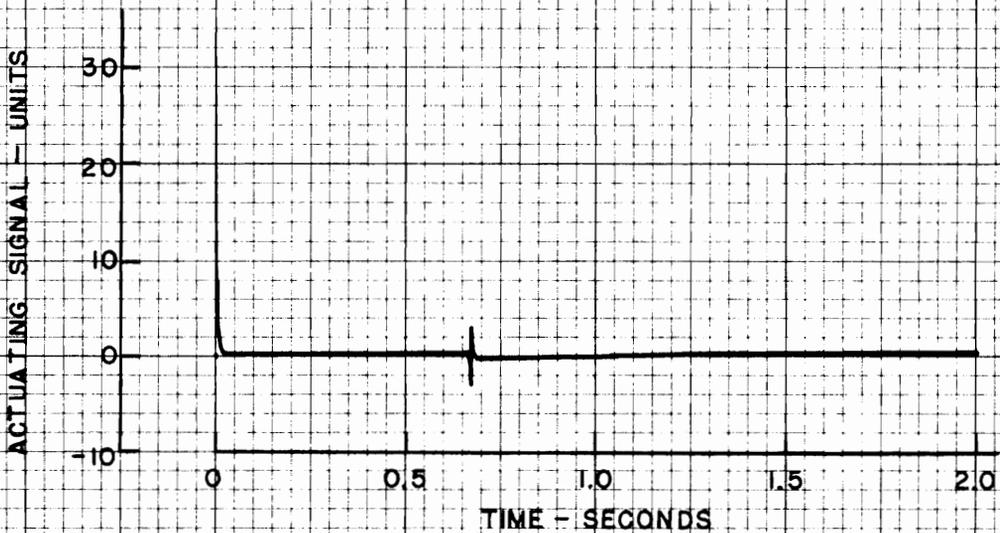
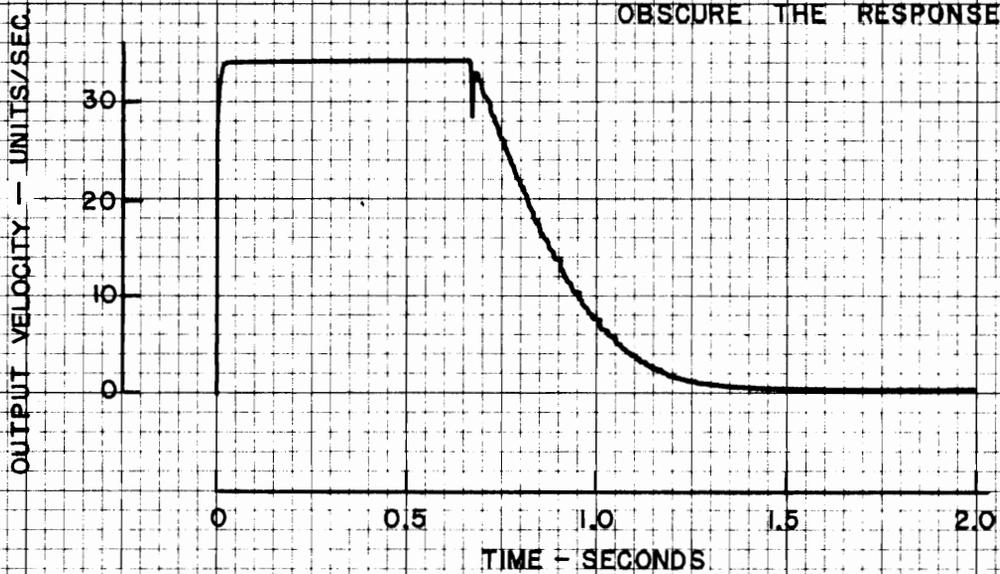


FIGURE II. OUTPUT VELOCITY AND ACTUATING SIGNAL VERSUS TIME FOR THIRTY UNIT STEP INPUT

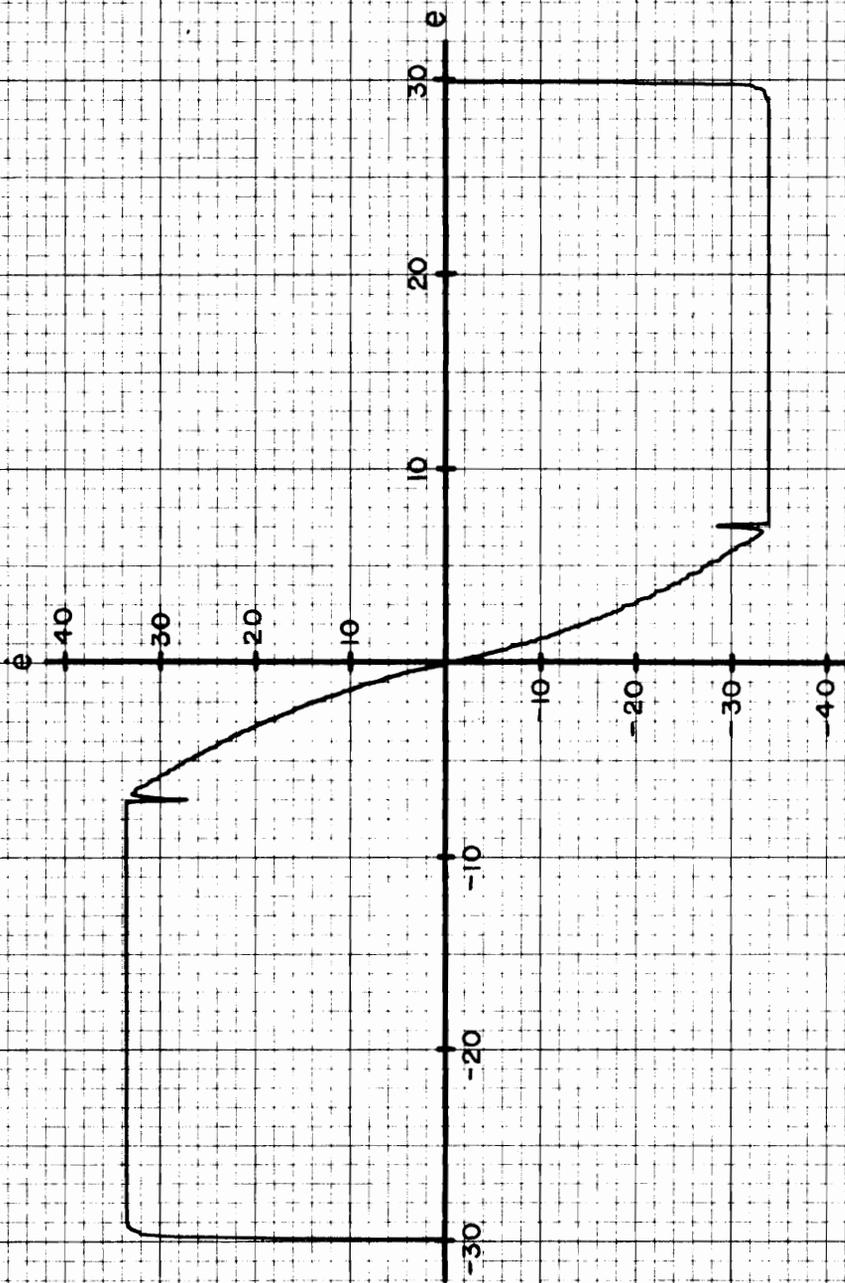


FIGURE 12. SYSTEM TRAJECTORIES FOR OPPOSITE POLARITY STEP INPUTS

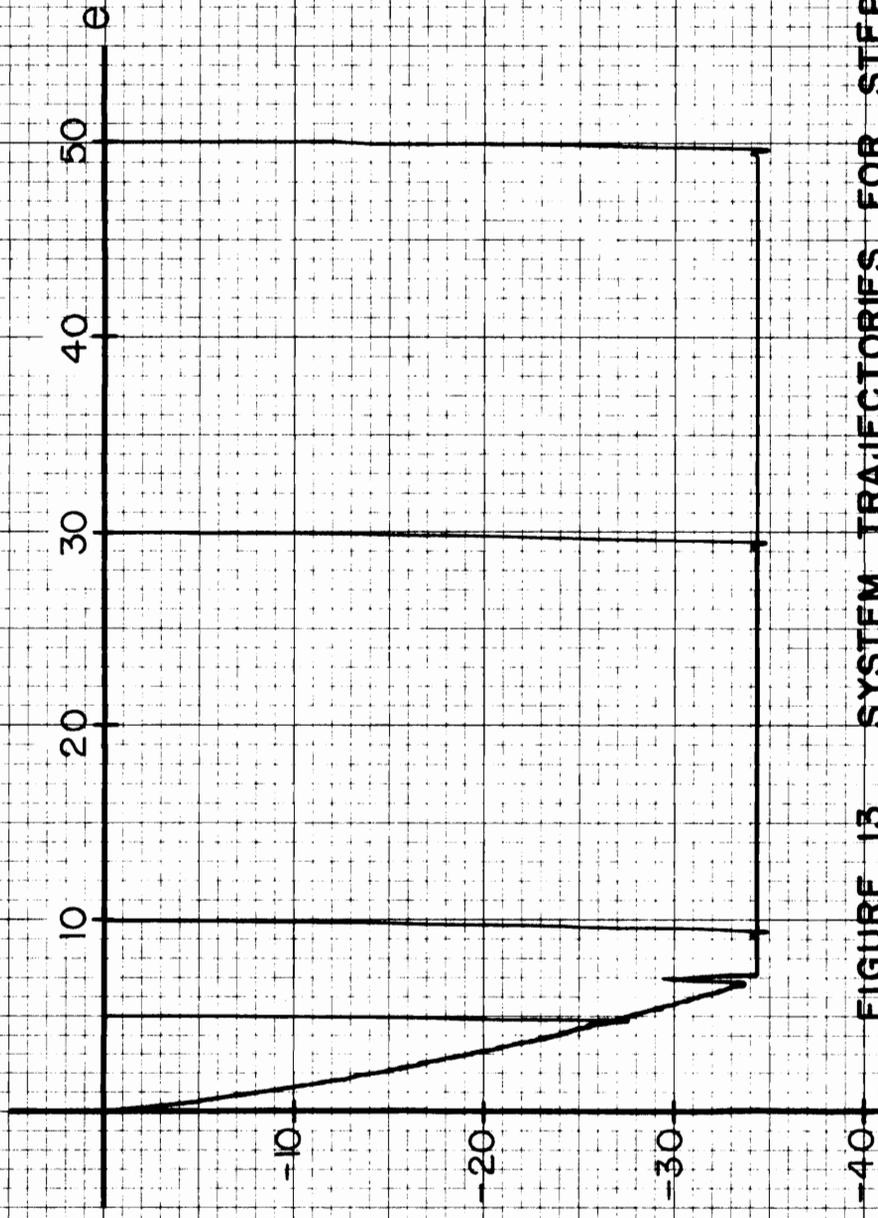


FIGURE 13. SYSTEM TRAJECTORIES FOR STEP INPUTS OF 50, 30, 20, 10, AND 5 UNITS

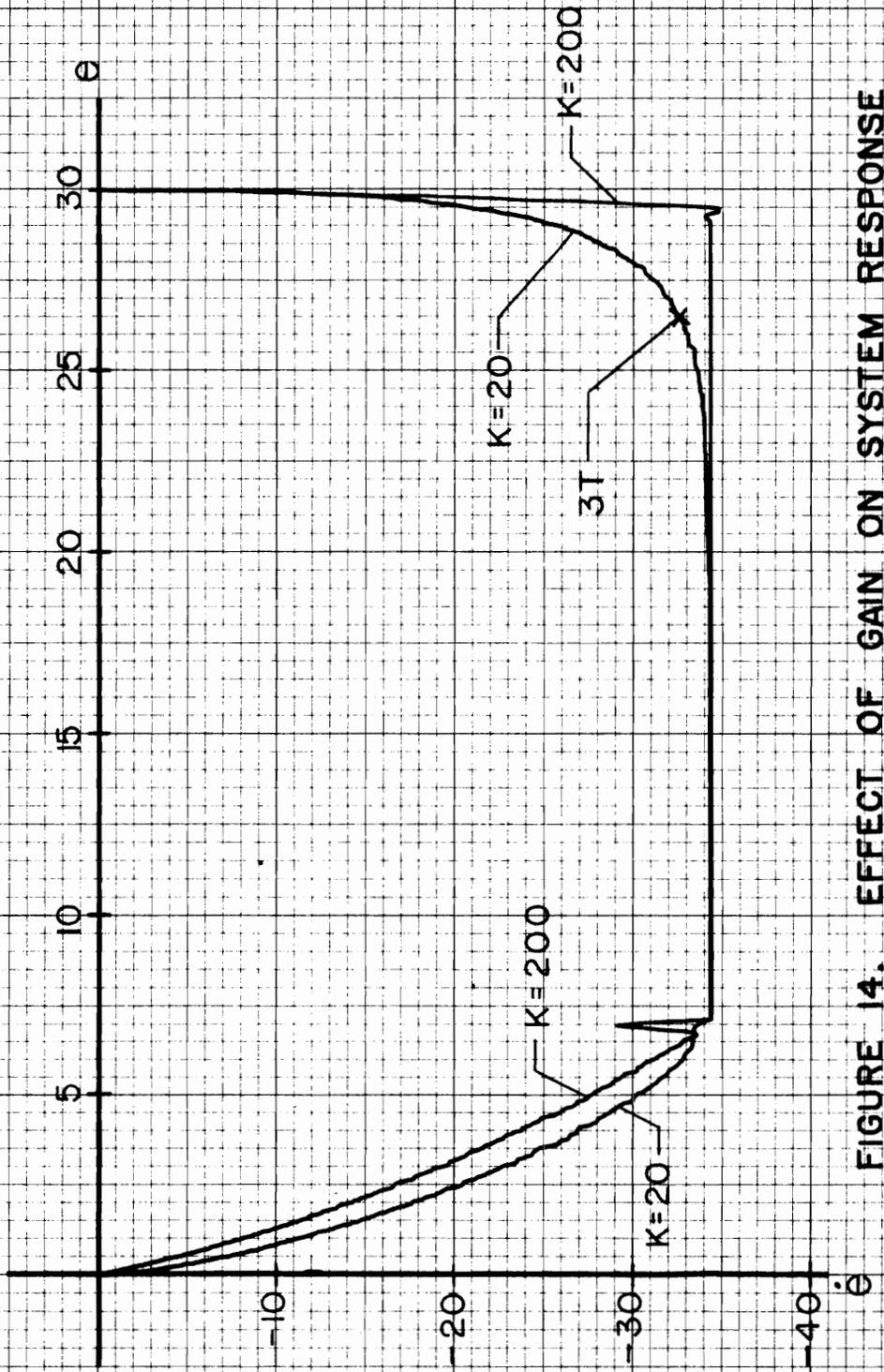


FIGURE 14. EFFECT OF GAIN ON SYSTEM RESPONSE

important to point out that this is recorder overshoot and is not present in the computer output. Figure 11a clearly shows that there is no oscillation in the output velocity.

The second point of importance is the large disturbance occurring at the switching instant on all of the curves. This is the fault of the computer simulation rather than of the actual system. In order to keep from saturating amplifier 13 in Figure 7 it was necessary to limit its input for large errors. All of the diodes on the computer were in use, so the input was simply disconnected by means of relay 2. This relay is used to switch ZFC functions at  $|e| = 7$ . As a result, the input to amplifier 13 was not applied until the very moment it was required. It is probable that the resultant transient condition caused the temporary lack of signal at the output of the computer. The actual switching time of the relay may also have contributed somewhat. In any case, the disturbance is simply a result of poor mechanization of the system. One way to eliminate it would be through the use of diode limiters on amplifier 13.

If these two disturbances are ignored in examining the various curves, a much clearer picture of the essential system response will be obtained.

Figure 8 is the phase plane trajectory of the system response to a 30 unit step input. Comparison of the response with the desired trajectory of Figure 5 shows no discernible difference between the actual and desired responses. The time required for the system trajectory to reach the ZFC is seen to be negligibly small. Actually, the recorder appeared to be velocity limited during the initial period and probably masked the actual transient. There are other curves which show the transient better.

Figure 9 is a plot of the output of the nonlinear computer versus error. As such, it is a plot of the actual Zero Force Curve of the system. Again comparison of Figure 8 with this curve shows no discernible difference. From this it is evident that the actual trajectory is indeed following the Zero Force Curve.

Figures 10 and 11 give the curve of various system variables as functions of time. Figures 10a and 10b, output and error, are presented for the record without comment. Figure 11a contains some significant information. It is a plot of the output velocity

as a function of time. Since the travel of the recorder pen is much less in this diagram than in Figure 8, it would be suspected that the initial transient would be reproduced more faithfully in this case. Examination of this curve reveals that the initial transient does indeed appear to be of a simple exponential type, and the settling time appears to be in the neighborhood of 0.02 second. Considering the scale of the diagram, this is in reasonable agreement with the predicted value of 0.015 second. It is also evident from this curve, as was mentioned previously, that the overshoot apparent in the phase plane plots was not present in the actual output velocity.

Figure 11b shows the variation of the actuating signal,  $m_1$ , with time. This is an excellent example of the validity of the Zero Force Curve method. The signal quickly goes to zero from its initial value. The spikes occurring at 0.67 second correspond to the switching transient previously discussed.

Figure 12 consists of two phase plane trajectories for input steps of opposite polarity. It will be noted that these trajectories are symmetrical as would be expected.

The system design is validated by the curves of Figure 13. The phase plane trajectories of the responses to four different step inputs are superimposed on the same axes. It can be seen that every trajectory quickly assumes the Zero Force Curve. The trajectory is dependent on the input magnitude only to the extent of determining the portion of the Zero Force Curve traced.

Figure 14 shows the effect of the system gain on the phase plane trajectory. The response of the system to a 30 units step is plotted for two different values of gain. The  $K_1 = 200$  plot is the same trajectory observed in the previous figures. The second trajectory is plotted for  $K_1 = 20$ . The exponential approach to the ZFC is clearly visible in this case. The equations derived for Case I are exact for this system because the signal corresponding to the desired velocity is constant as assumed. Analyzing the system for  $K_1 = 20$  shows that after three time constants the error should have changed by 3.5 units. The velocity after three time constants should be 95 per cent of the final velocity or 32.7 units per second. These two values represent a point in the phase plane through which the system trajectory should pass. This point is marked in Figure 14. It can be seen that the trajectory does pass through this point.

Another interesting effect which has not previously been discussed is apparent in Figure 14. Remembering that the  $K_1 = 200$  trajectory is essentially the same as the ZFC, it can be seen that the  $K_1 = 20$  system response lags behind the ZFC on the approach to the origin. This is entirely compatible with work previously done in connection with this system. The lagging response is that which would be expected from a first order linear system with a swiftly changing input. The response about the ZFC has already been shown to be that of a first order system. A lag such as that shown in Figure 14 is exactly the type of response to be expected from this system for low gain.

The results of the computer simulation of this system have been presented here along with that discussion deemed essential to the interpretation of the curves. It should be evident from this discussion that the Zero Force Curve approach worked satisfactorily for this system. Further comment on this system will be reserved for the conclusions.

ZFC Concept Applied to the Lewis Servomechanism

An interesting application of the ZFC concept is in the analysis of the Lewis<sup>(10) (2)</sup> servomechanism for high system gain. A block diagram of this system is shown in Figure 15. It can be seen that this type of nonlinear control fits into the classification for which the ZFC is applicable. There are, however, significant differences between this system and that considered previously.

One major difference is in the form of the equation for the ZFC. This equation is seen to be

$$m_1 = e - A_2 |e| \frac{dc}{dt} = 0 .$$

In those systems considered in Cases I through IV, all had an inner feedback loop in which the output velocity was compared with a signal from the computer to obtain the actuating signal. This system cannot be put in that form, and the previous analyses do not apply. It is possible, however, to obtain essentially the same results. If it is again assumed that the error stays essentially constant over the period of the initial transient, the factor multiplying the derivative of

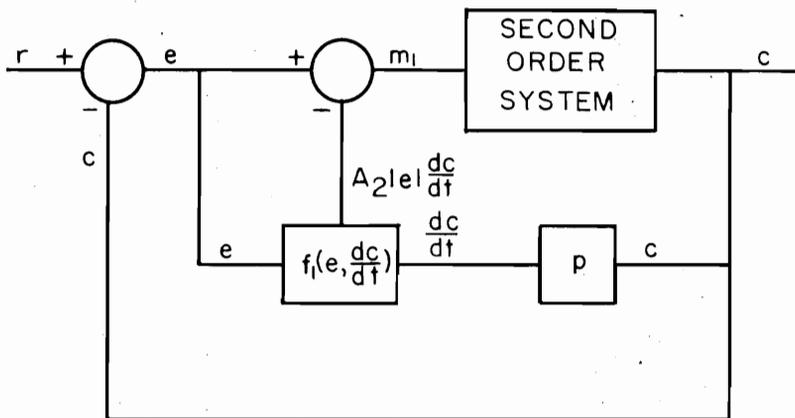


FIGURE 15. BLOCK DIAGRAM OF LEWIS SERVOMECHANISM

the output can be considered to be constant.

Substituting  $\frac{dc}{dt} = v$  and considering the error to be constant yields,

$$m_1 = e - Hv = 0$$

where:

$$H = A_2 |e| = \text{gain constant of the feedback loop.}$$

Figure 16 shows this equation symbolized in block diagram form along with the rest of the system. Now the inner feedback loop is apparent.

There is one important difference between this feedback loop and those considered previously. The gain constant in the feedback loop is now error dependent. In essence, the response time about the ZFC depends on the magnitude of the input step. The response for a large step will be faster than for a small step.

For the sake of simplicity, this response will be analyzed for Case I only. The analyses for the other cases proceed in a similar manner. From Figure 16,

$$\frac{v}{m} = \frac{pG}{1 + GHp} = \frac{p \frac{K_1}{P_2}}{1 + \frac{K_1 H}{p^2} \cdot p} = \frac{K_1}{p + K_1 H}$$

$$(p + K_1 H)v = K_1 m .$$

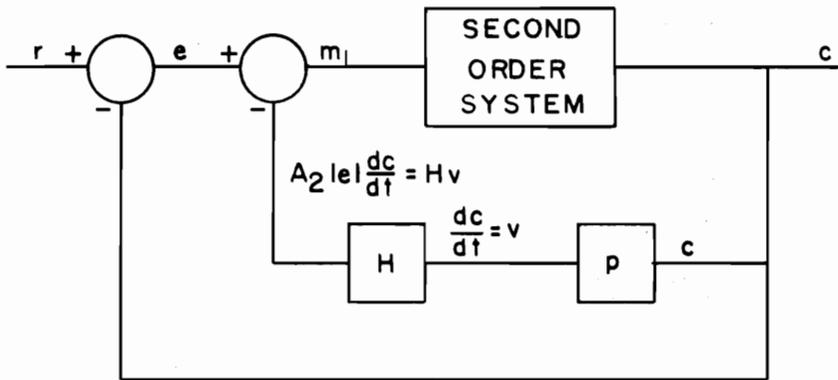


FIGURE 16. BLOCK DIAGRAM OF LEWIS  
SERVOMECHANISM WITH VARIABLE GAIN  
FEEDBACK PATH

Solving,

$$v = \frac{m}{H} [1 - \exp(-K_1 H t)] = \frac{1}{A_2} \frac{e}{|e|} [1 - \exp(-K_1 H t)]$$

$$v = \frac{1}{A_2} (\text{sgn } e) [1 - \exp(-K_1 H t)] .$$

In examining the results of this analysis it is at once apparent that the response time is inversely proportional to the magnitude of the error. The time constant is

$$\tau = \frac{1}{K_1 H} = \frac{1}{K_1 A_2 |e|} .$$

This will have a significant effect on the trajectories for different magnitude inputs.

It is also noted that the final value of the output velocity is independent of the input. The reason for this will become apparent upon examination of the Zero Force Curve for this system.

The Zero Force Curve for this system is plotted in Figure 17. This ZFC is unique in several ways. The velocity limited states on either side of the vertical axis indicate that this system should behave like a constant velocity servomechanism. The abrupt transition from one state to another at zero error should lead one to suspect that the response should

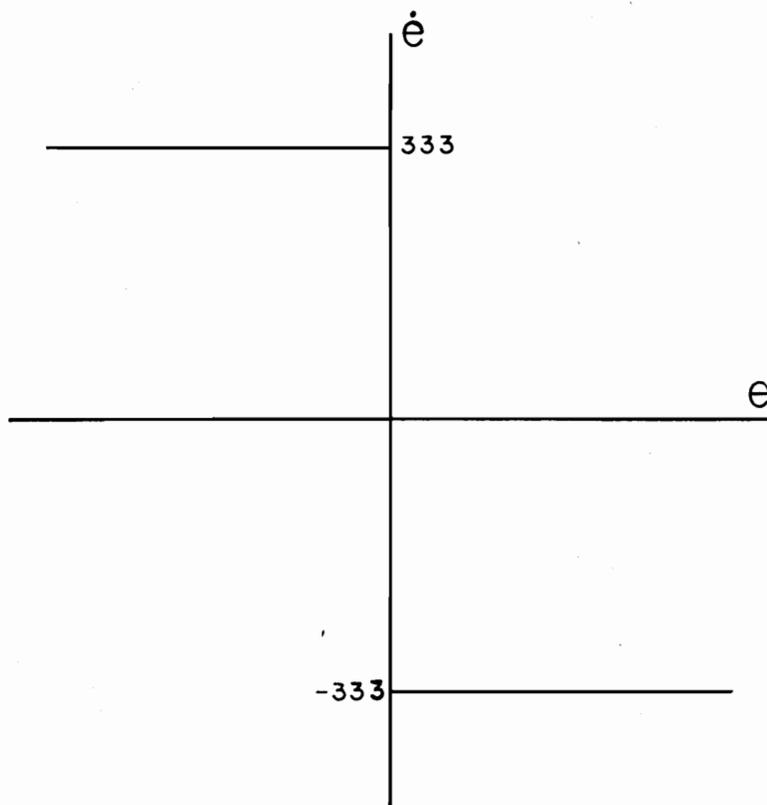


FIGURE 17. ZERO FORCE CURVE  
FOR LEWIS SERVOMECHANISM

be highly oscillatory as is characteristic of systems with this sort of driving force<sup>(11)</sup>. One might even expect to see a limit cycle condition such as is experienced with simple relay servomechanisms<sup>(11)</sup>. This, however, should not be the case since the ZFC curve does not represent a true driving force but rather is a sort of steady state portion of the transient response. The system should be in a state of continuous transient operation as the system tries to move from one portion of the ZFC to the other as the error repeatedly changes sign. The resultant oscillations should damp out, however; since for oscillations near the origin the nonlinear term should have less effect, and the system response should be more or less linear.

Computer Simulation of the System. The Lewis servomechanism with the Case I transfer function was simulated on the analog computer described previously. The following values of the significant parameters were used:

$$A_2 = 0.003$$

$$K_1 = 10,000$$

These values were chosen so that for a 50 unit step input the time constant of the transient would be

0.67 millisecond. This proved to be a sufficiently short time constant for the 50 unit step input. The analog computer program is given in Figure 18. The problem was slowed in time by a factor of one hundred to give satisfactory recorder response.

Figures 19 through 24 show various aspects of the system response as recorded by the X-Y recorder. These curves are the same sets of curves displayed for the previous system. Examination of the phase plane plots shows the presence of the same recorder overshoot discussed for the previous system.

Figure 19 is the phase plane trajectory for a 50 unit step input. The response is in agreement with that predicted except for one portion. The computer shows the output velocity dropping off as the error approaches zero. This was quite puzzling at first. It was only after some experimentation with the computer set up that a satisfactory explanation was obtained. It was found that the shape of the trajectory in the vicinity of the vertical axis was extremely sensitive to changes in the computing cup of the multiplier used in the simulation. Since the input signal to the servo-multiplier was very small in the region in question, the accuracy of the multiplication was quite poor. Tests

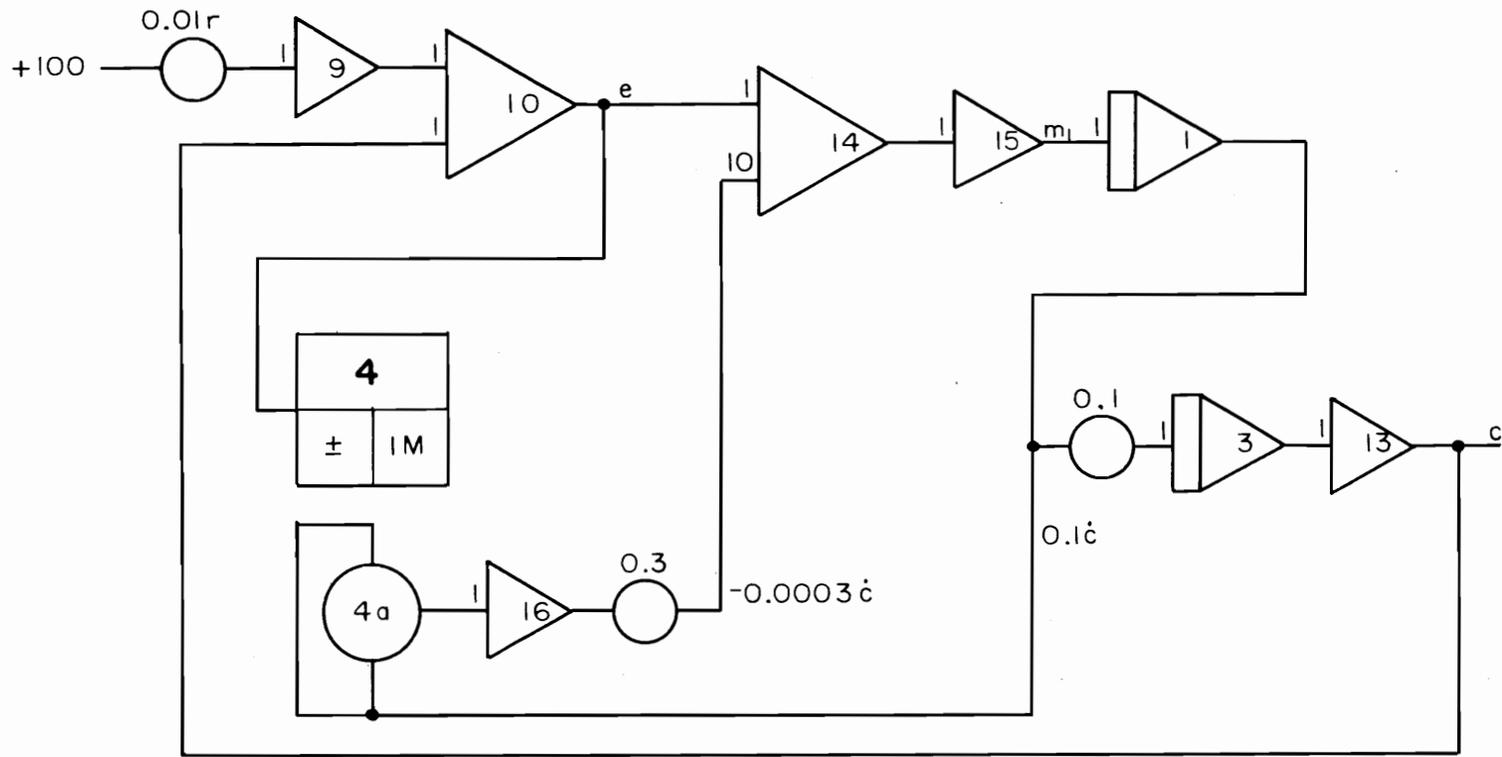
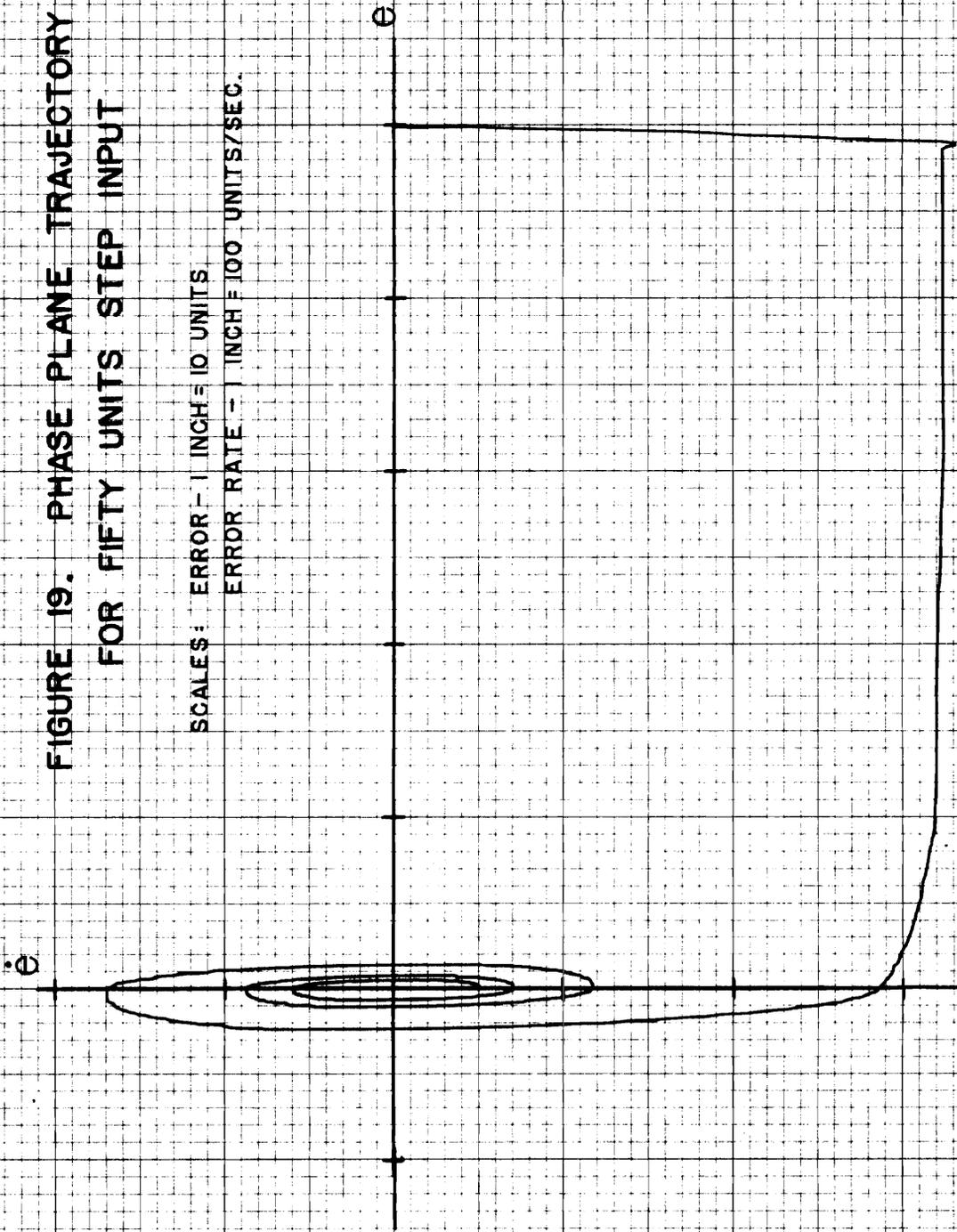


FIGURE 18. ANALOG COMPUTER PROGRAM FOR SIMULATION OF LEWIS SERVOMECHANISM

FIGURE 19. PHASE PLANE TRAJECTORY  
FOR FIFTY UNITS STEP INPUT

SCALES: ERROR - 1 INCH = 10 UNITS  
ERROR RATE - 1 INCH = 100 UNITS/SEC.



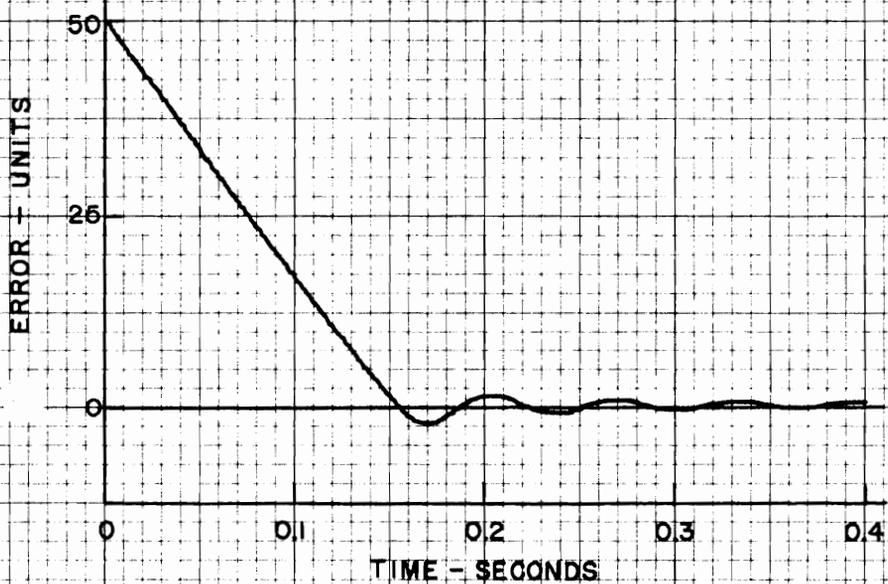
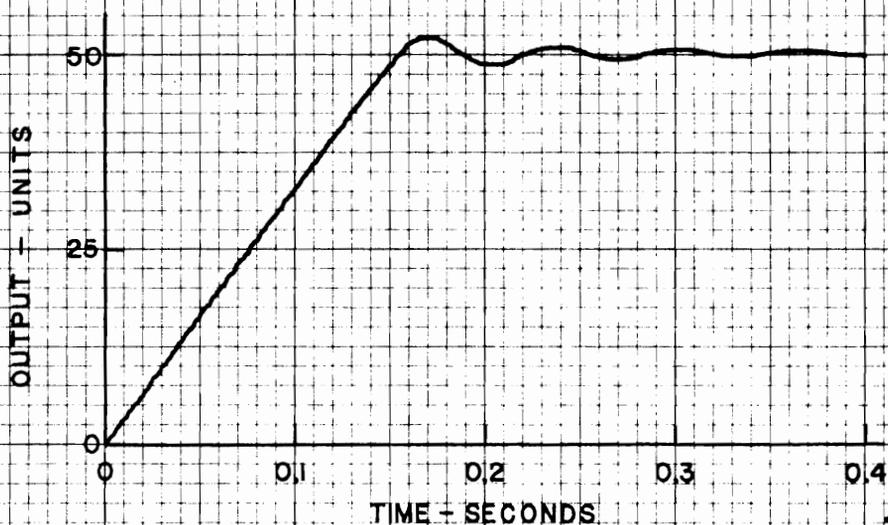


FIGURE 20. OUTPUT AND ERROR VERSUS TIME FOR FIFTY UNIT STEP INPUT

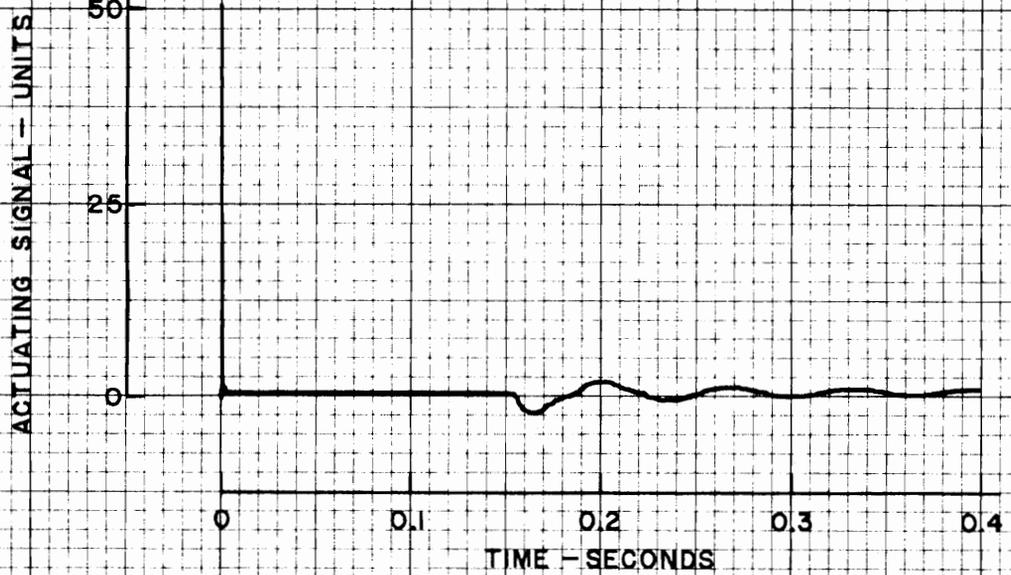
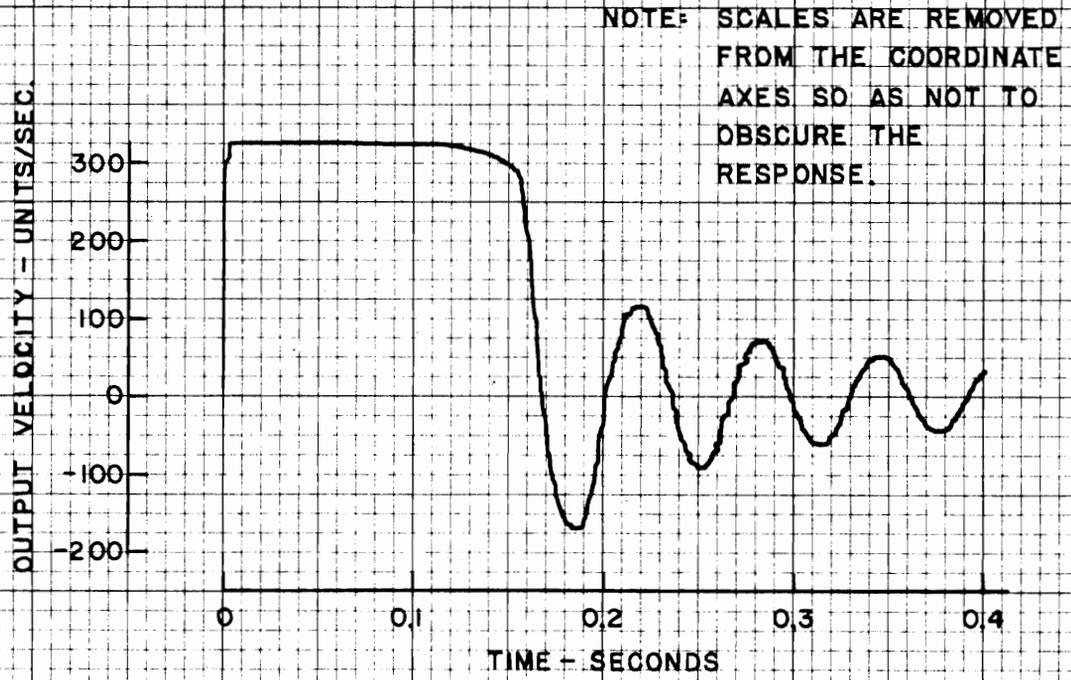


FIGURE 21. OUTPUT VELOCITY AND ACTUATING SIGNAL VERSUS TIME FOR FIFTY UNIT STEP INPUT

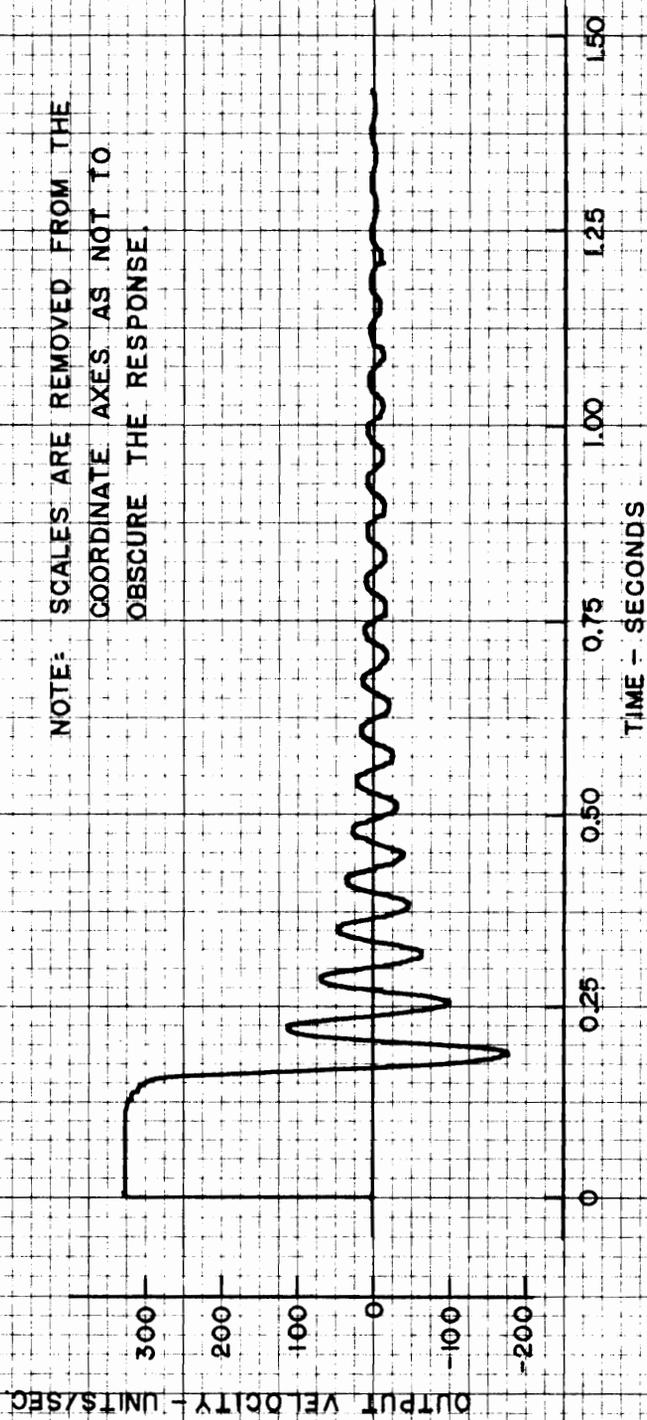


FIGURE 22. OUTPUT VELOCITY VERSUS TIME FOR FIFTY UNIT STEP INPUT

FIGURE 23. SYSTEM TRAJECTORIES  
FOR STEP INPUTS OF 50, 30,  
20, 10, AND 5 UNITS

SCALES: ERROR - 1 INCH = 10 UNITS  
ERROR RATE - 1 INCH = 100 UNITS/SEC.

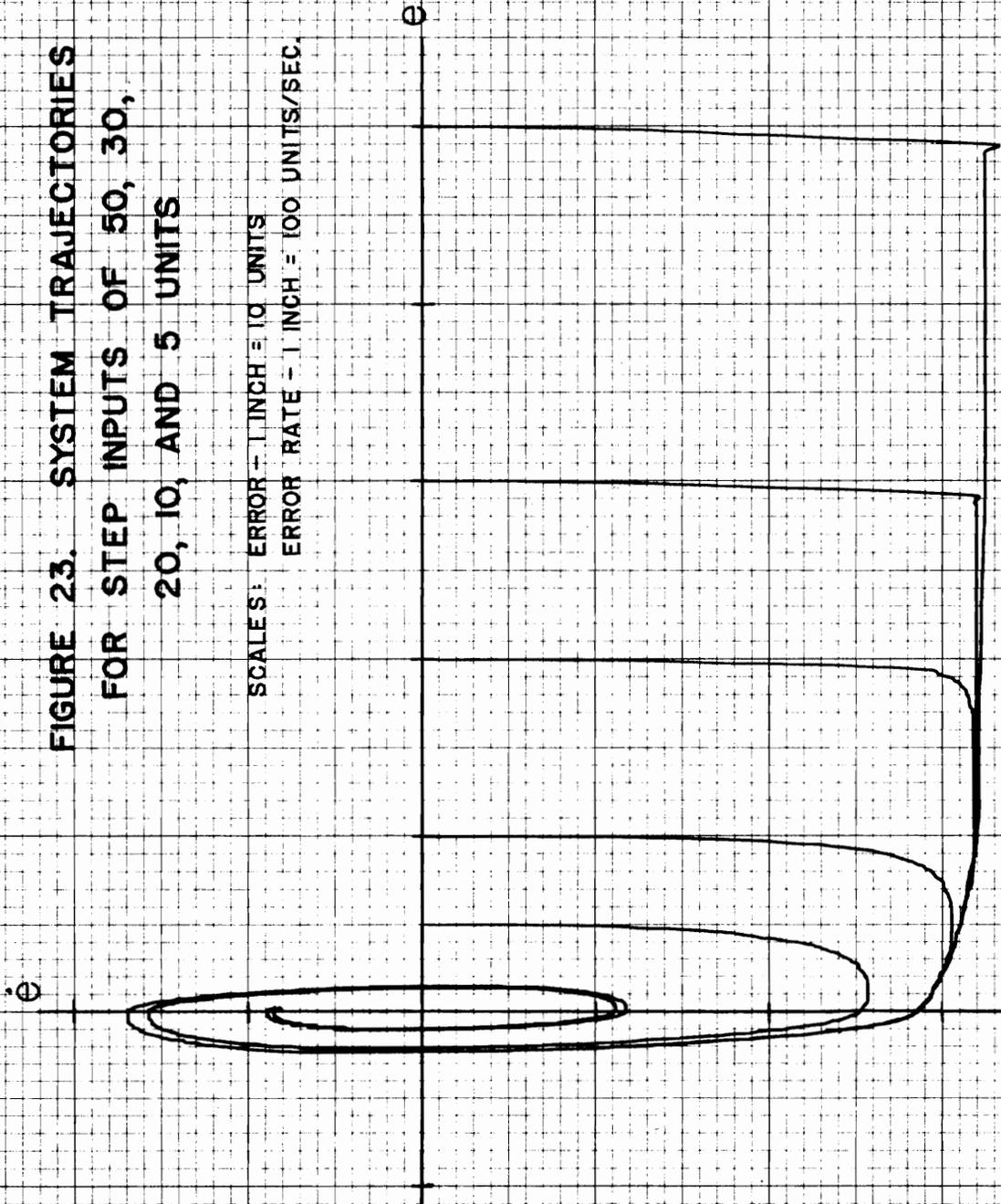
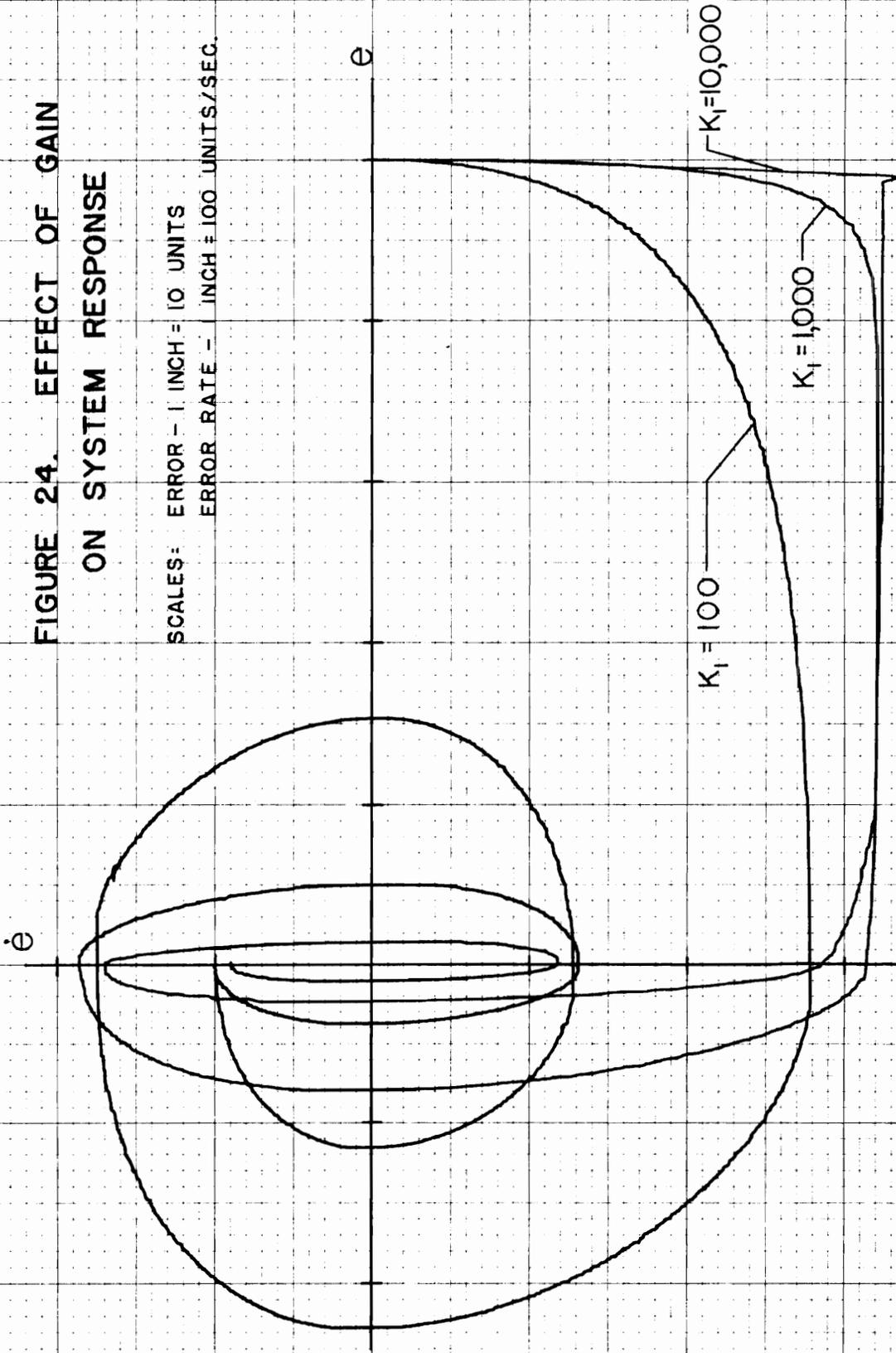


FIGURE 24. EFFECT OF GAIN  
ON SYSTEM RESPONSE

SCALES: ERROR - 1 INCH = 10 UNITS  
ERROR RATE - 1 INCH = 100 UNITS/SEC.



on the computing cup used showed that for small inputs there was about 0.1 volt difference between the input and output when the multiplier was connected for multiplication by unity. The circuit used to perform this test is shown in Figure 25. The difference for small errors is plotted in Figure 26. This inaccuracy is evidently the cause of the unusual behavior for small errors. Except for this, the trajectory is quite satisfactory.

Figure 20 shows the time response of the output and error for a 50 unit step input. It will be noted that the oscillations constitute only about five per cent of the total step.

Figures 21 and 22 show the time response of the output velocity and the actuating signal,  $m_1$ . The highly oscillatory response of the output velocity after the initial velocity limited portion is plainly visible in Figure 21a. It will be noted from Figure 21b that the actuating signal is very definitely not zero during this oscillatory period. In Figure 22 the output velocity has been plotted over a longer period of time so that the exponential decay is plainly visible.

In Figure 23 the change in time constant with error magnitude is plainly visible. The trajectories are

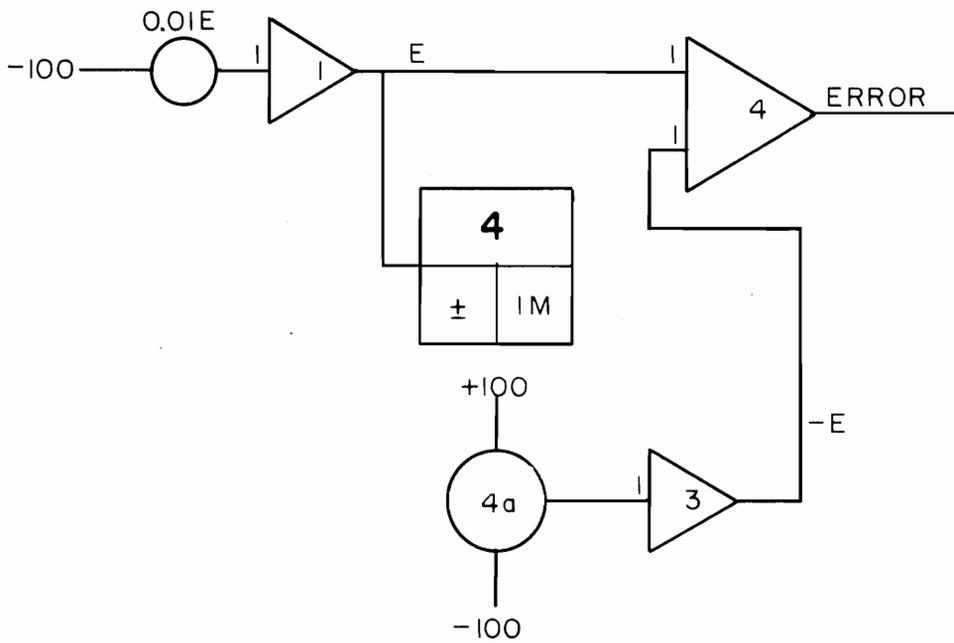


FIGURE 25. CIRCUIT USED TO MEASURE  
SERVO-MULTIPLIER ERROR

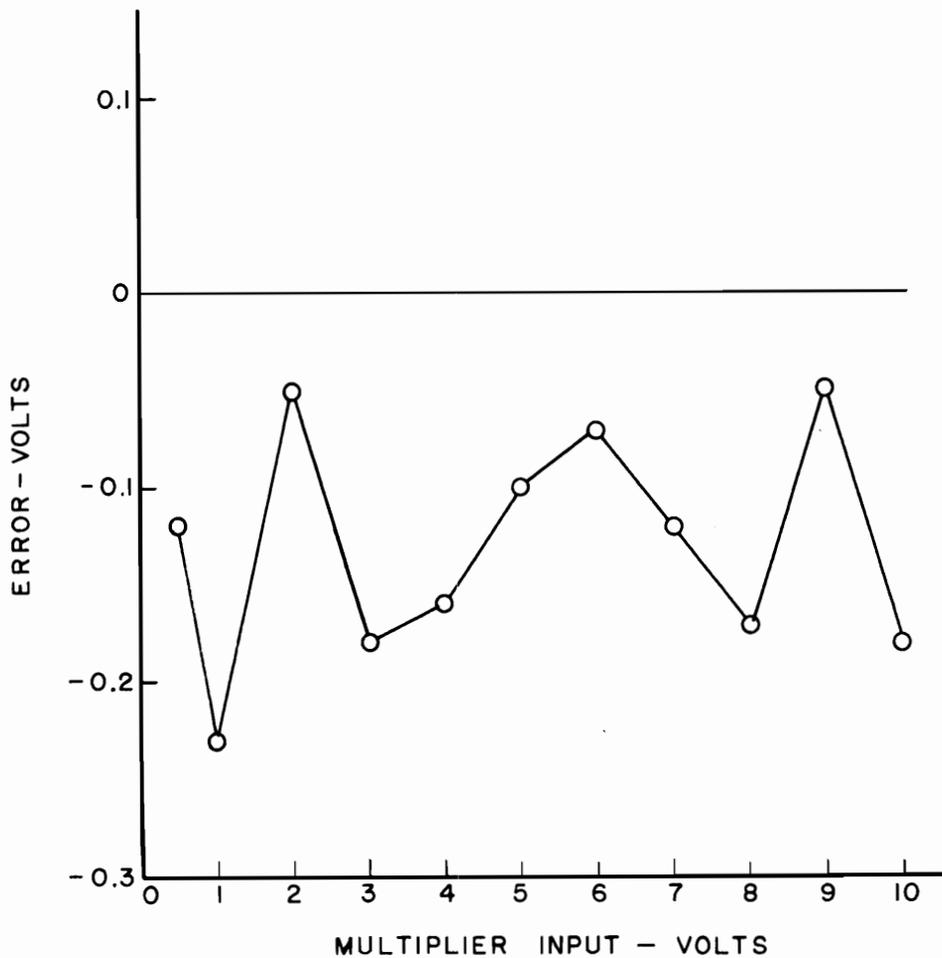


FIGURE 26. ERROR IN SERVO-MULTIPLIER  
16-7N-1, POSITION 4, COMPUTING CUP A,  
ELECTRONICS ASSOCIATES 16-3IR  
ANALOG COMPUTER AT VIRGINIA  
POLYTECHNIC INSTITUTE

plotted for step inputs of 50, 30, 20, 10, and 5 units. The overshoot for steps of five units or greater is seen to be essentially the same. The overshoot for the five unit step is seen to be almost fifty per cent of the step.

Figure 24 shows the effect of gain on the system response. Not only is the system slower in reaching the Zero Force Curve (it does not reach it for  $K_1 = 100$ ), but the overshoot increases greatly for decreased gain as well. It is interesting to note that the  $K_1 = 1000$  trajectory does a better job of maintaining its velocity at small errors than the  $K_1 = 10,000$  trajectory. For lower gains the multiplier inaccuracy is not felt so quickly at the output.

Lewis Servomechanism for a Case III System. The experiment performed above for a Case I system was repeated for the Case III system of Figure 27.  $A_2$  was maintained at 0.003, and  $K_1$  was made equal to 10,000. This made the quantity  $(\frac{1}{T_1} + \frac{1}{T_2} + K_1H)$  approximately equal to  $10,000H$ . This last quantity is the pertinent term in the time constant, and the response should be the same as for the Case I system. The analog computer diagram is given in Figure 28. The solution was again slowed down in time by a factor of 100.

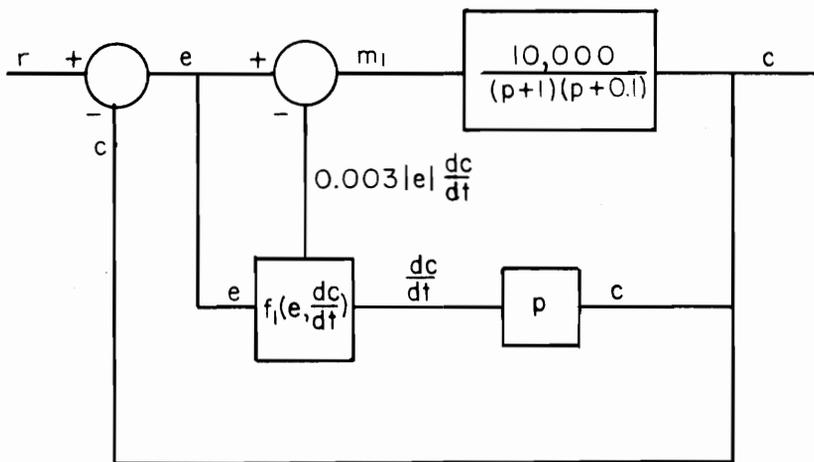


FIGURE 27. BLOCK DIAGRAM OF LEWIS  
SERVOMECHANISM FOR CASE III  
SYSTEM

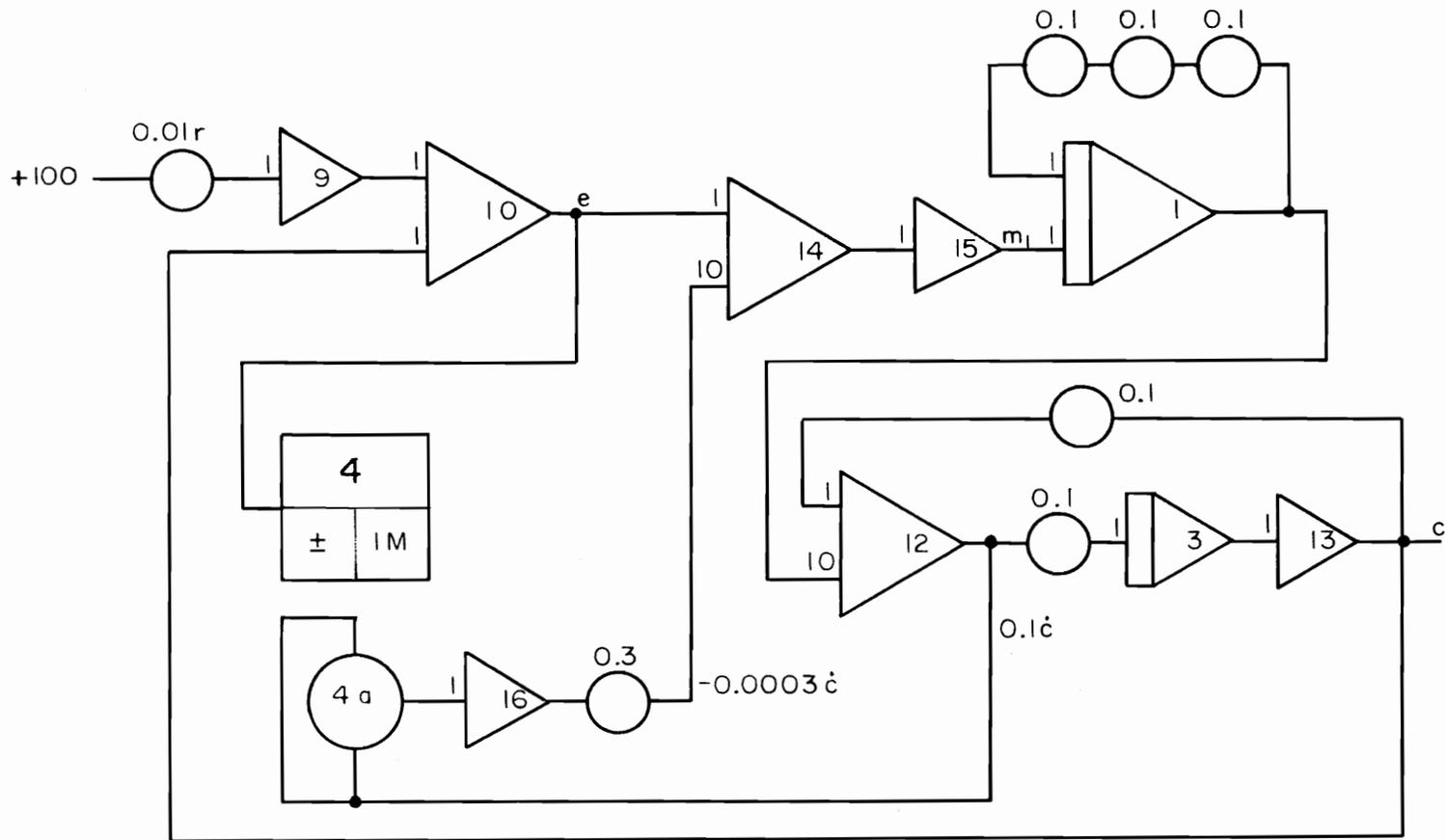


FIGURE 28. ANALOG COMPUTER PROGRAM FOR SIMULATION OF LEWIS SERVOMECHANISM FOR CASE III SYSTEM

Figures 29 through 33 repeat the data taken for Case I. Comparison of the corresponding figures in the two cases shows that the different transfer functions have very little effect on the system response. The one significant difference is in Figure 33. Comparison with Figure 24 shows that the overshoot does not increase as much with decreased gain in the Case III system. There is no other difference worth noting.

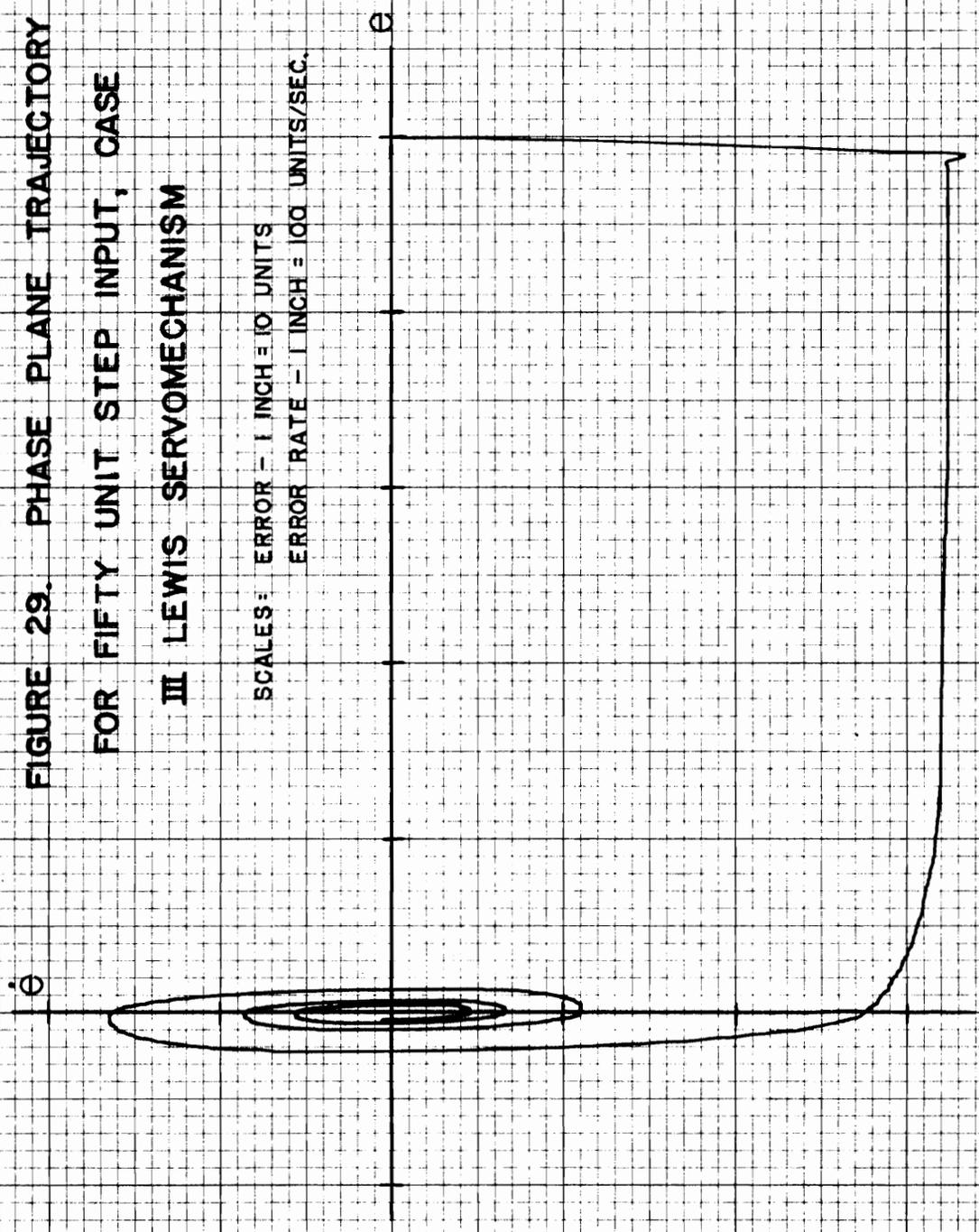
Closed Loop Frequency Response  
of the Lewis System

The information regarding the Lewis servomechanism obtained from the Zero Force Curve analysis makes it possible to determine the closed loop frequency response of this system to a surprising degree of accuracy. It has already been seen that the system of Figure 27 responds to a step input as if it were a constant velocity system. The ZFC sets this velocity at 333 units per second for the chosen system parameters. This approximation gives rise to a very simple sinusoidal response function. The output follows the input precisely as long as the maximum rate of change of the input is less than 333 units per second. Once the input

FIGURE 29. PHASE PLANE TRAJECTORY  
FOR FIFTY UNIT STEP INPUT, CASE

III LEWIS SERVOMECHANISM

SCALES: ERROR - 1 INCH = 10 UNITS  
ERROR RATE - 1 INCH = 100 UNITS/SEC.



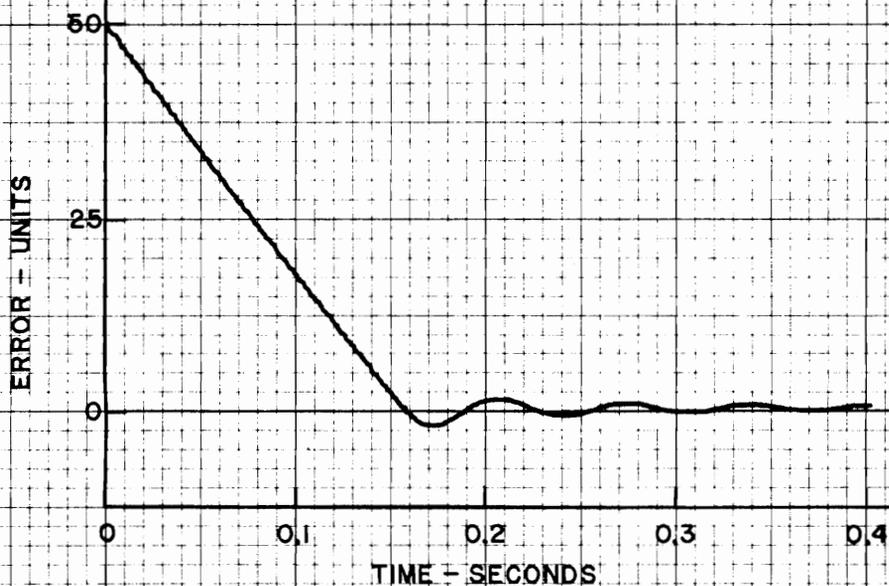
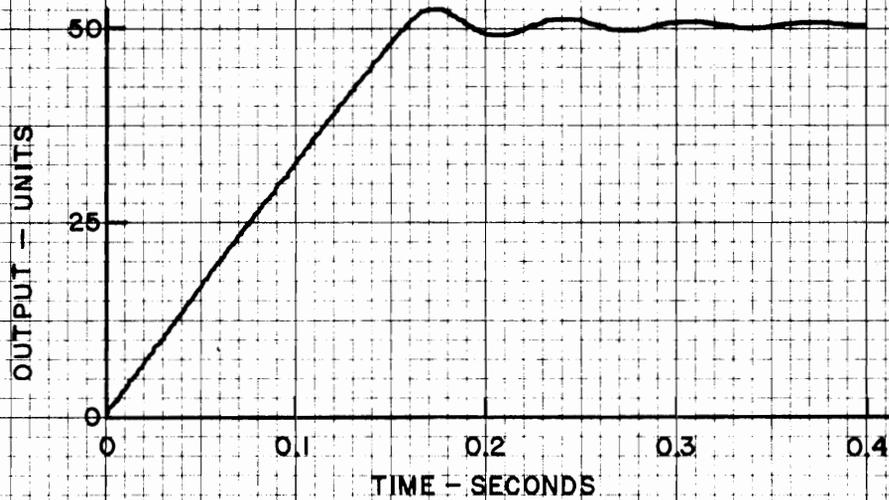


FIGURE 30. OUTPUT AND ERROR VERSUS TIME FOR FIFTY UNIT STEP INPUT, CASE III LEWIS SERVOMECHANISM

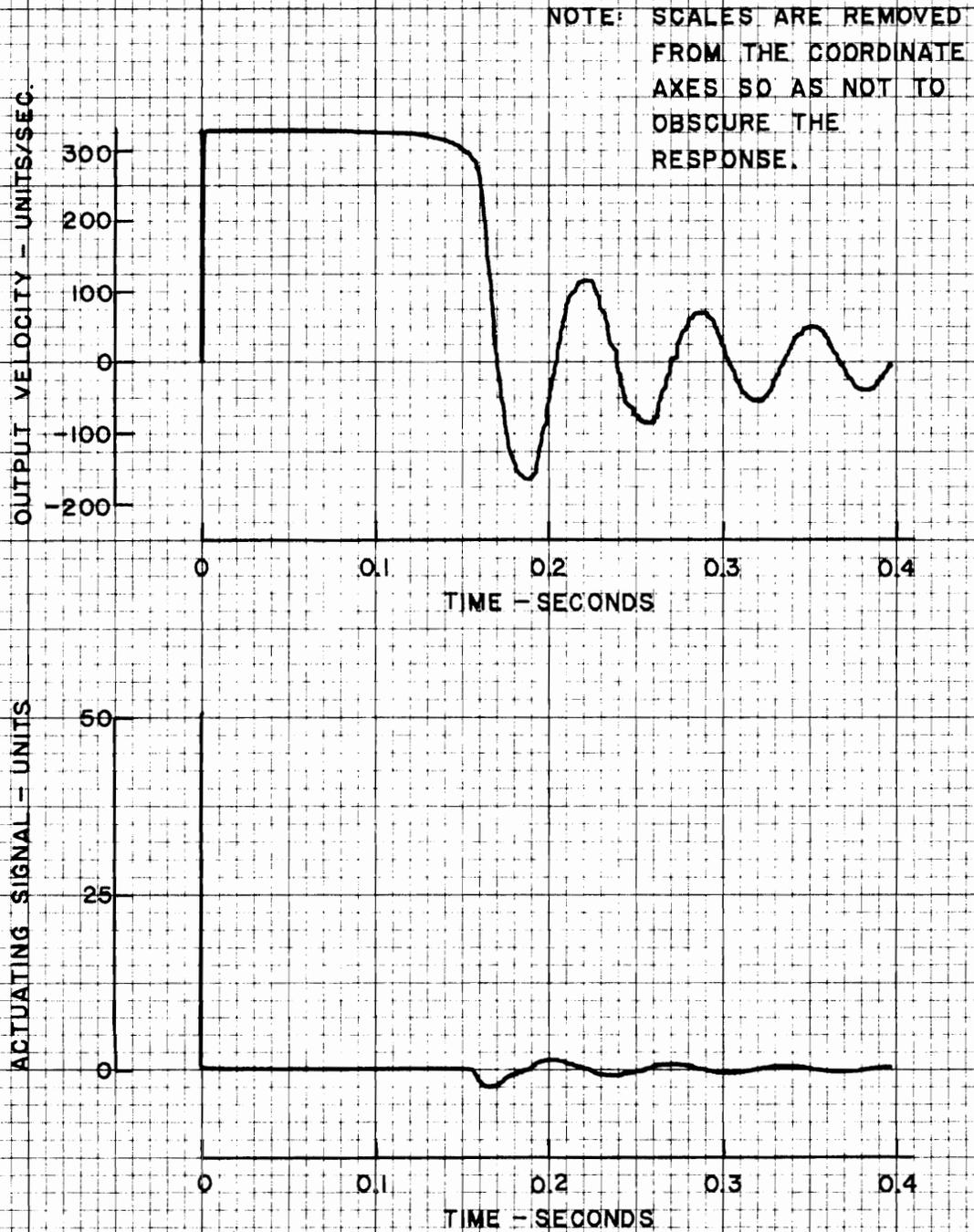


FIGURE 31. OUTPUT VELOCITY AND ACTUATING SIGNAL VERSUS TIME FOR FIFTY UNIT STEP INPUT, CASE III LEWIS SERVOMECHANISM

FIGURE 32. SYSTEM TRAJECTORIES  
 FOR STEP INPUTS OF 50, 30, 20,  
 10, AND 5 UNITS, CASE III  
 LEWIS SERVOMECHANISM

SCALES: ERROR - 1 INCH = 10 UNITS  
 ERROR RATE - 1 INCH = 100 UNITS/SEC.

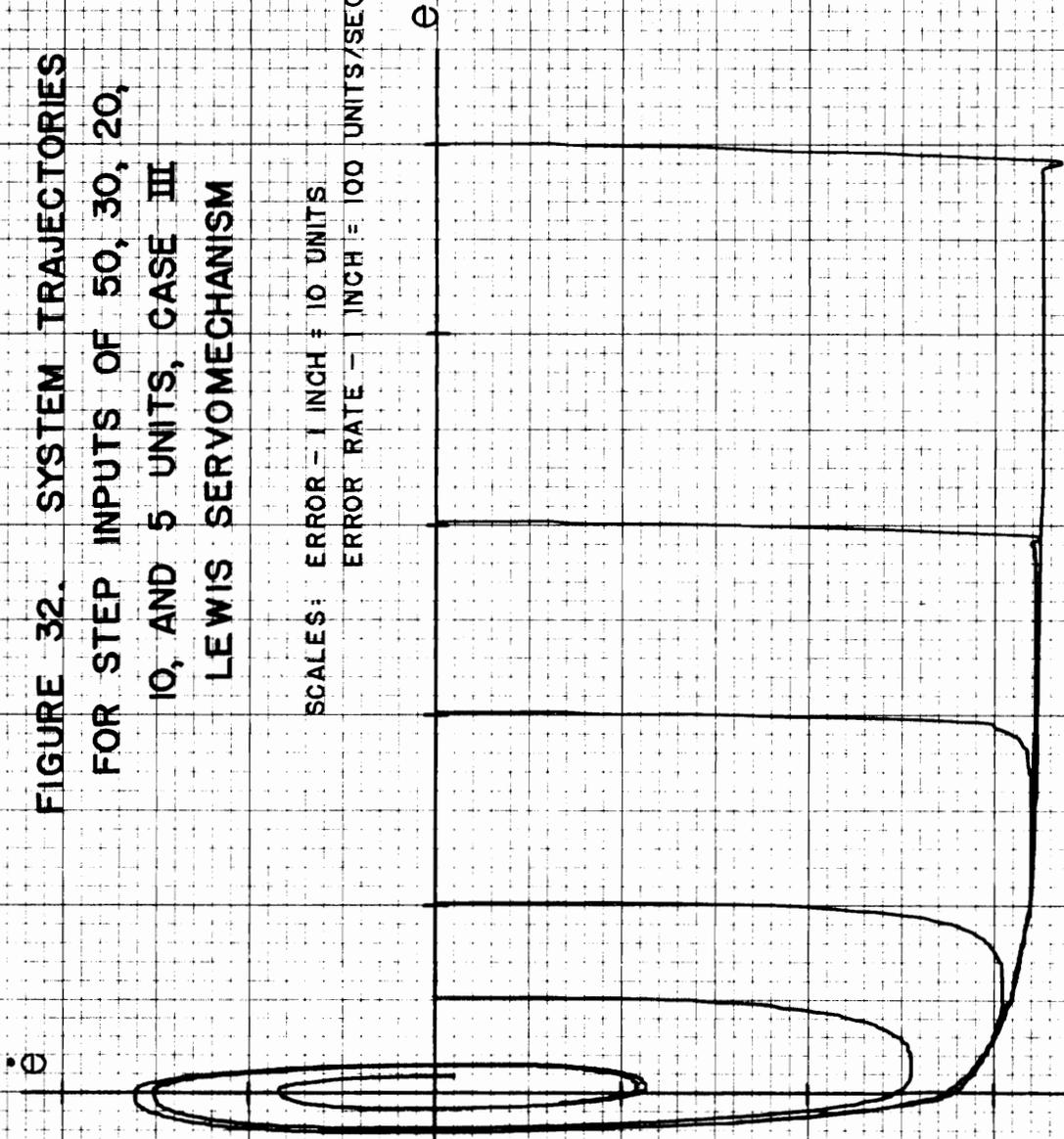
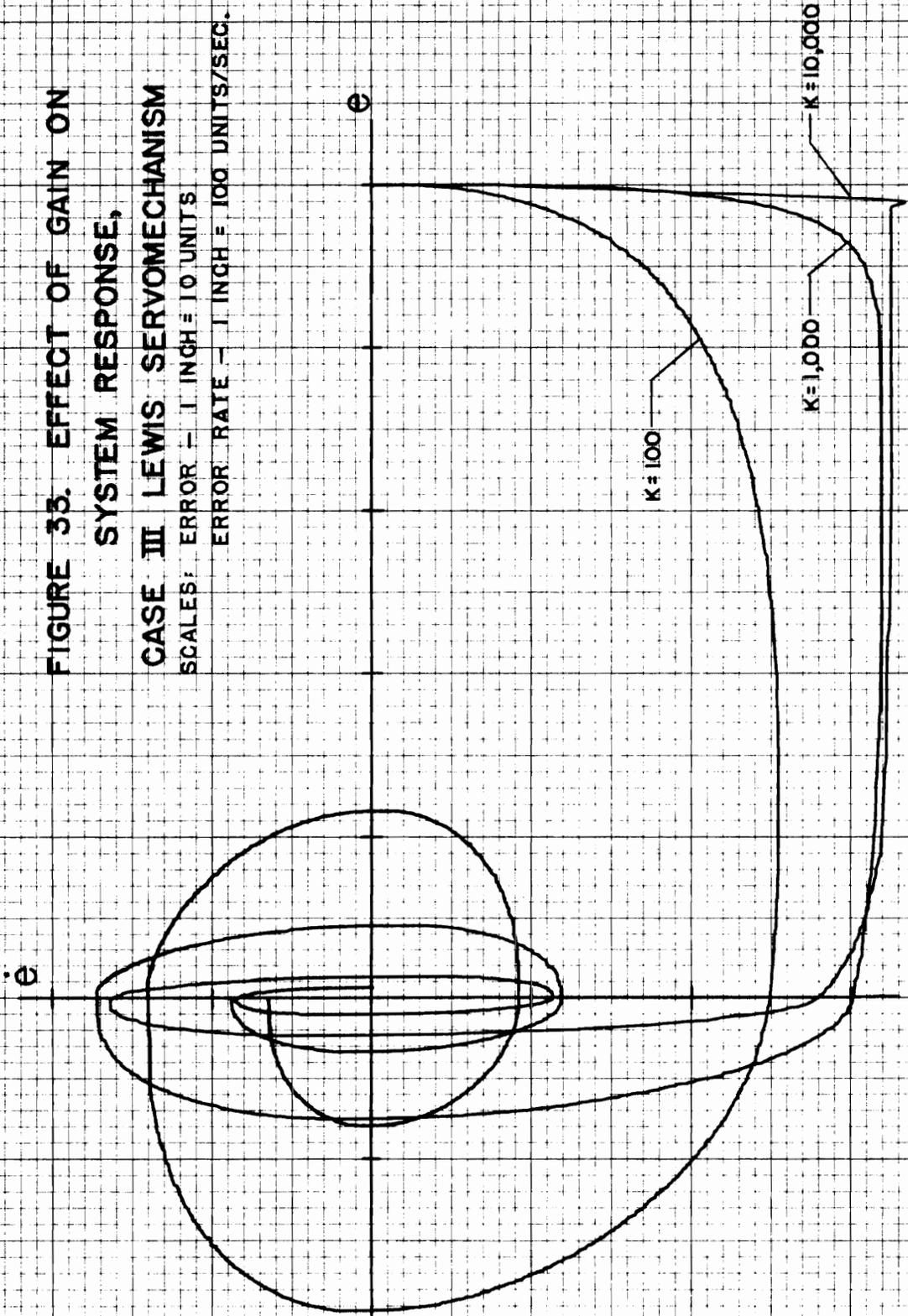


FIGURE 33. EFFECT OF GAIN ON  
SYSTEM RESPONSE,

CASE III LEWIS SERVOMECHANISM

SCALES: ERROR - 1 INCH = 10 UNITS

ERROR RATE - 1 INCH = 100 UNITS/SEC.



changes at a rate faster than this limit, the output rate becomes limited, and the output sine wave becomes distorted in much the same way as occurs with an ordinary diode and capacitor a.m. demodulator used in radio work<sup>(13)</sup>. The output increases or decreases at this maximum rate until the output again equals the input. The output then follows the input until the rate of change of the input again becomes greater than the limit of the output rate. This theory may be used to quantitatively predict the frequency at which distortion occurs, the frequency at which the peak value of the output begins to drop off, and in fact the entire frequency response curve.

The output magnitude versus frequency curve for this system has been calculated in Appendix II for an input sine wave magnitude of 50 units. The resulting curve is shown in Figure 34. The analog computer program of Figure 28 was used to obtain this response experimentally. A Hewlett-Packard 202A Low Frequency Function Generator was used to provide the sinusoidal input. The output was recorded on the six channel strip chart recorder included in the 16-31R computer system. The recorder was manufactured by the Brush Instrument Company. The resultant data is plotted on Figure 34

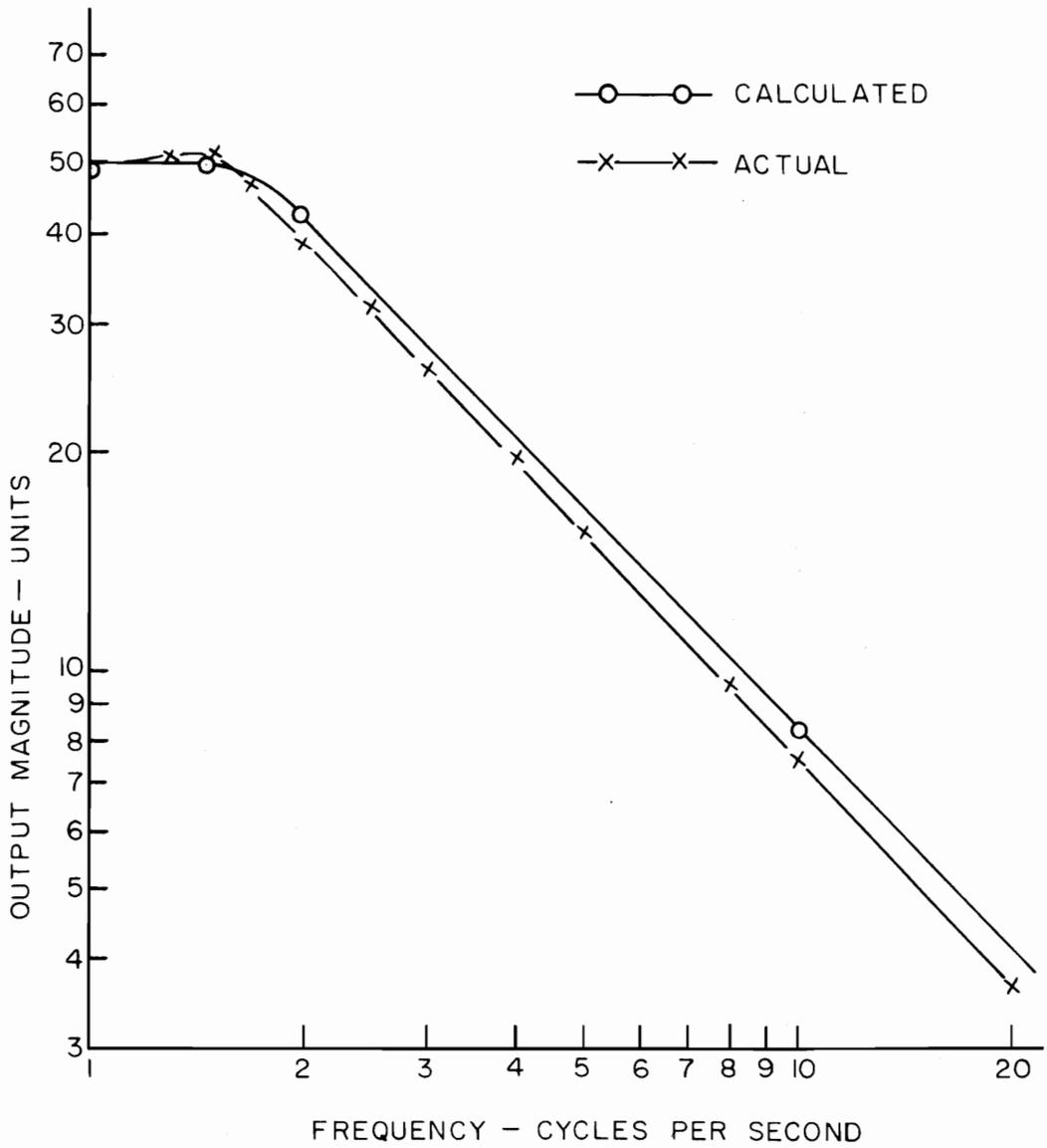


FIGURE 34. CLOSED LOOP FREQUENCY RESPONSE OF LEWIS SERVOMECHANISM

also for comparison with the calculated curve. The degree of accuracy obtained with this method is apparent from the curves. There does appear to be some divergence of the curves at high frequencies, but this may be due to poor calibration of the recorder for small inputs.

It should be pointed out that the frequency response of the Case I system should be exactly the same as that obtained above. This system is unique in that the frequency response appears to be independent of the transfer function.

#### IV. DISCUSSION

In this section an attempt will be made to discuss areas of this problem which indicate the possibility of further work and to point out limitations on the proposed system.

##### Areas of Further Investigation

This method should be investigated for other types of inputs. Examination of the derivation of the response for step inputs shows that there are several important modifications which must be made when the input is a function of time. It is no longer valid to substitute  $\dot{e} = -\dot{c}$ , so that the inner feedback loop is more complicated.

The response to step inputs does not give any indication of instability. However, the study of stability was not a primary goal of this investigation, and further study of step response with emphasis on stability should be carried out. The effect of repeated steps of varying polarities would be an interesting area of study. Caldwell and Rideout<sup>(2)</sup> reported instability in the Lewis servomechanism under such condition with relatively low system gains. If instability is not

present in the system present in this thesis, the reason for the difference should be determined.

The stability of the system for other types of inputs should be examined. There is almost an unlimited amount of work that could be done in this area. As an example, the Lewis servomechanism was seen to exhibit damped oscillations for step inputs. If a sine wave of the same frequency as the oscillations was superimposed on the step with the proper phasing, would it be possible to cause these oscillations to build up? Questions such as this could be investigated for particular systems and also for a general second order system.

Since the transient response has a known form for any magnitude step input, it may be possible to calculate the sinusoidal frequency response of the closed loop system. This was demonstrated in this thesis for the very simple form of step response of the Lewis servomechanism. The possibility of doing the same thing in the general case should be investigated.

Athanassiades and Smith<sup>(1)</sup> have already done some work in the application of this method to systems with saturation limitations. It should be noted, however, that the ZFC method appears to be much more powerful for such systems than they indicate. If the system

limitations are known, they can be taken into account in the design of the Zero Force Curve. As an example, consider a system with torque limitation. Trajectories corresponding to maximum acceleration or deceleration have been shown to be parabolas<sup>(11)</sup>. If a series of these parabolas are drawn on the phase plane they constitute a grid which defines the system limitations at any point in the plane. If at any point on the phase plane the slope of the Zero Force Curve exceeds the slope of the grid at that point, then the system will be required to exceed its acceleration limits. Therefore, for satisfactory ZFC response the Zero Force Curve should never have a slope greater than the slope of the grid at this point. It should be possible to include other types of system limitations in a similar manner.

There is a weakness in the original derivation which indicates the need for further study of this Zero Force Curve method. The assumption that the computed signal,  $m$ , in Figure 3 is a constant requires that it be a function of the error only. Therefore, the Zero Force Curve equation must always be solved for  $\dot{e}$  in terms of  $e$ . It should be possible to examine the response without imposing this restriction.

If an attempt is made to extend this method to higher order systems, it becomes necessary to analyze the problem in a phase space. An immediate consequence is that it is now necessary to speak of a Zero Force Surface in this space. The response of the system with respect to this surface is necessarily different than for the second order system, and requires investigation.

#### Practical Limitations of High Gain Systems

It is very easy to point out practical limitations on the system proposed in this thesis. In the first place, any system is going to have some saturation limitations. A high gain system is obviously going to encounter these limitations for all but the smallest actuating signals. In essence, this system will act like a relay system for large inputs, and the Zero Force Curve will be the switching curve of the system. However, for small actuating signals the system behaves as described in this thesis. As discussed in the previous section, the design of a practical system would have to take this into account in choosing the Zero Force Curve.

Another difficulty with any practical high gain system is the problem of noise in the system. This is

a problem that would have to be handled in the same manner as for any other high gain system, and might be very difficult to surmount.

## V. CONCLUSIONS

A method of obtaining the desired response of a second order linear system for step function inputs has been demonstrated. The method has been shown valid for four classes of linear second order transfer functions if certain conditions are met. The most important restriction was placed on the gain of the system. In general, the gain was required to be large. The exact size of the gain was shown to be a function of the desired response and the system transfer function. Formulae for calculating the required gain were derived for the cases considered. It was shown that the stability of the system was not adversely affected by this high gain.

The theory was applied to two example systems, and the results were verified with an analog computer. No significant discrepancies were noted. In particular, a system was designed for a given response and then simulated on the analog computer. From the recorded results it can be concluded that the system design was completely successful. The computer was also used to

verify the effect of gain on the system. Here again the theory was quantitatively verified.

Finally, the use of nonlinear elements to control second order linear systems was shown to be a useful and versatile form of control.

VI. APPENDIX I

It has been stated in the example design problem that the desired system should follow the phase plane trajectory of a second order linear system with  $\zeta = 1.1$  and  $\omega_0 = 10$  subjected to an input step of ten units. This trajectory will be used for small errors only. For large errors the system will be velocity limited at the maximum velocity of the system described above.

The differential equation of a second order linear system with the parameters given above subjected to a ten unit step input is given below as,

$$[p^2 + 2(1.1)(10)p + (10)^2]e = 0$$

with initial conditions,  $e(0) = 10$ ,  $\dot{e}(0) = 0$ .

The characteristic equation is

$$p^2 + 22p + 100 = 0 .$$

Solving this equation,

$$p = -15.58, -6.42 .$$

Therefore, the solution is given by

$$e = C_1 \exp(-15.58t) + C_2 \exp(-6.42t)$$

where  $C_1$  and  $C_2$  are arbitrary constant.

Differentiating,

$$\dot{e} = -15.58 C_1 \exp(-15.58t) - 6.42 C_2 \exp(-6.42t) .$$

Substitution of the initial conditions into the expressions for the error and its derivative yields the required values for the arbitrary constant. The final expressions for the error and its derivative are,

$$e = 17 \exp(-6.42t) - 7 \exp(-15.58t)$$

$$\dot{e} = -109 \exp(-6.42t) + 109 \exp(-15.58t) .$$

These two equations constitute a set of parametric equations from which the phase plane trajectory may be determined. This trajectory is plotted in Figure 5.

## VII. APPENDIX II

If it is assumed that the Lewis servomechanism operates in a velocity limited mode, the closed loop sinusoidal frequency response may be predicted quite easily. The output will follow the input as long as the input velocity does not exceed the maximum output velocity. For input velocities greater than the limit, the output will be velocity limited. A geometrical examination of the output waveform will yield the output magnitude for any input sine wave. This type of analysis is carried out below.

If the input is a sine wave, it may be assumed to be given by

$$r = R \sin \omega t .$$

Then,

$$\frac{dr}{dt} = \omega R \cos \omega t .$$

The maximum rate of change of the input is

$$\left( \frac{dr}{dt} \right)_{\max} = \omega R .$$

The first appreciable distortion should appear in the output at the point where

$$\left(\frac{dr}{dt}\right)_{\max} = \left(\frac{dc}{dt}\right)_{\text{limit}}$$

$$\omega r = \frac{1}{A_2}$$

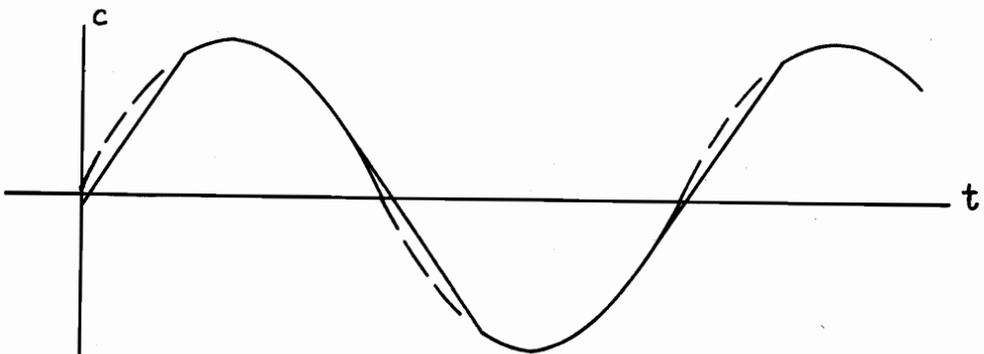
$$\omega = \frac{1}{A_2 R}$$

$$f = \frac{1}{2\pi A_2 R} .$$

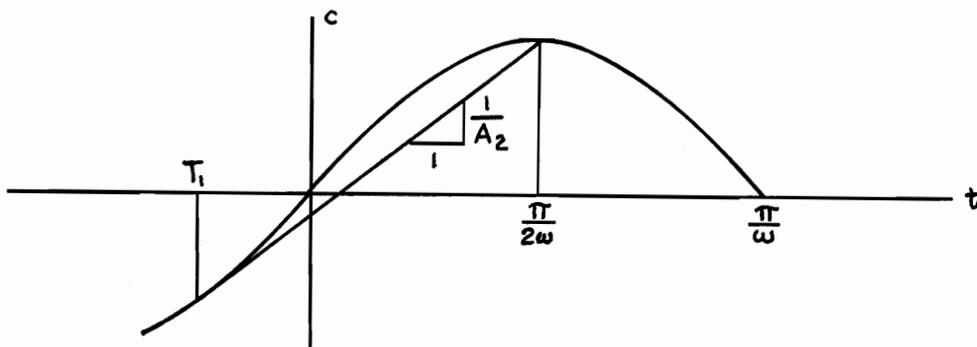
For  $A_2 = 0.003$ ,  $R = 50$ , the frequency at which appreciable distortion occurs is

$$f = 1.06 \text{ cps} .$$

Although the output signal becomes noticeably distorted at this frequency, the maximum value of the output does not begin to drop as yet. This is shown in the figure below.



The location of the frequency at which the output begins to decrease may be done geometrically. The figure below shows the geometry of the problem.



Equating the slope of the sine wave at  $T_1$  to the slope of the straight line gives

$$\omega R \cos \omega T_1 = \frac{1}{A_2} .$$

A second equation may be obtained by equating the ordinates of the two curves at  $t=T_1$ . The equation of the straight line is

$$c = \frac{1}{A_2} \left( t - \frac{\pi}{2\omega} \right) + R .$$

Equating the two curves at  $t=T_1$ ,

$$R \sin \omega T_1 = \frac{1}{A_2} \left( T_1 - \frac{\pi}{2\omega} \right) + R .$$

Thus  $\omega$  and  $T_1$  are defined by the equations

$$\omega R \cos \omega T_1 = \frac{1}{A_2} \quad (1)$$

$$\omega R \sin \omega T_1 = \frac{1}{A_2} \omega T_1 - \frac{\pi}{2A_2} + \omega R . \quad (2)$$

Substitution of (1) into (2) yields the following equation after rearrangement,

$$\tan \omega T_1 = \omega T_1 - \frac{\pi}{2} + \sec \omega T_1 .$$

Solution of this transcendental equation for  $\omega T_1$  gives,

$$\omega T_1 = -0.759 .$$

Substituting this value of  $\omega T_1$  back into equation (1) and solving for  $\omega$  yields.

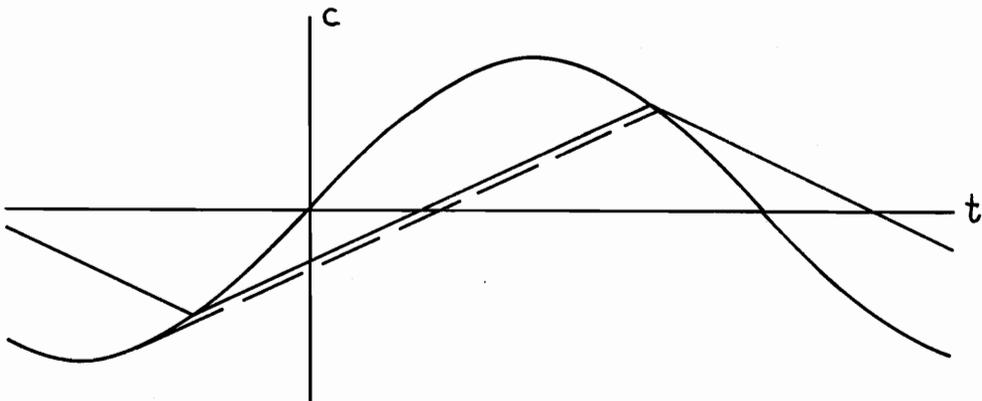
$$\omega = \frac{1}{0.725A_2R}$$

$$f = \frac{1}{4.55A_2R} .$$

For  $A_2 = 0.003$ ,  $R = 50$ , the frequency at which the output amplitude begins to decrease is

$$f = 1.47 \text{ cps} .$$

The method used above can be applied in the case where the frequency is known, and the amplitude of the output is desired. This problem becomes especially easy once the input frequency is above a certain value. The sketch below shows the sine wave and the actual output at high frequencies. The dotted line represents the output based on taking the tangent to the curve at the point where the slope is  $1/A_2$ . However, it will be observed that this straight line intersects the sine wave at a point below the corresponding point where the line left the sine wave. In steady state operation the output will actually look like the solid curve shown. The intersection points with the sine wave are symmetric with respect to the horizontal axis, and the output is a triangular wave.



The maximum value of this triangular wave is given by

$$c_{\max} = \frac{1}{A_2} \left( \frac{T}{4} \right) = \frac{1}{4A_2 f}$$

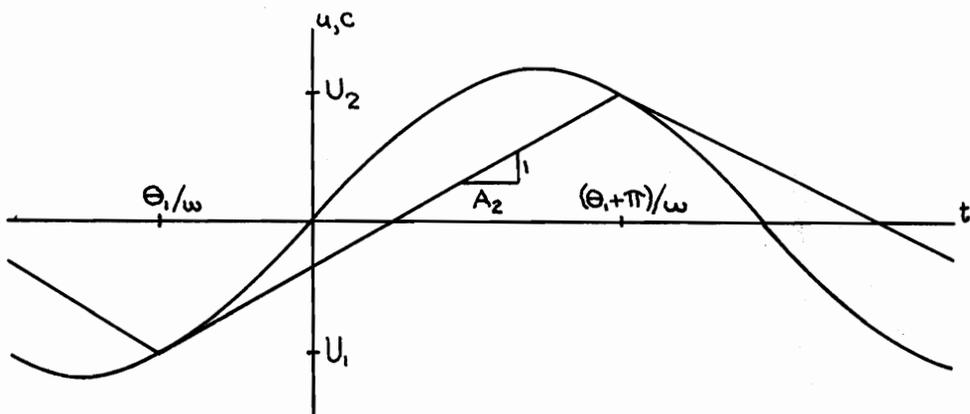
where:

T = period of the sine wave.

For  $A_2 = 0.003$ , this maximum amplitude is given by

$$c_{\max} = \frac{83.2}{f} .$$

There is obviously a low frequency limit for this equation. This limit is the lowest frequency for which the output is a pure triangular wave. At this frequency, the triangle intersects the sine wave at the point where the slope of the sine wave equals the slope of the triangle. This is shown in the figure below.



The sine wave is given by

$$U = R \sin \omega t .$$

Equating the slopes at  $\omega t = \theta_1$

$$\omega R \cos \theta_1 = \frac{1}{A_2} .$$

Recognizing that the ordinates  $U_1$  and  $U_2$  are separated by  $\pi$  radians,  $U_2$  may be obtained from the sine wave and from the triangular wave as,

$$U_2 = R \sin(\theta_1 + \pi) = U_1 + \frac{1}{A_2} \left( \frac{\pi}{\omega} \right) = R \sin \theta_1 + \frac{1}{A_2} \left( \frac{\pi}{\omega} \right) .$$

Solution of this equation for  $\sin \theta$  yields,

$$\sin \theta_1 = \frac{\pi}{2RA_2\omega} .$$

From the equation relating the derivatives of the two curves,

$$\cos \theta_1 = \frac{1}{RA_2\omega} .$$

Using the trigonometric identity,

$$\sin^2 \theta_1 + \cos^2 \theta_1 = 1$$

the following equation is obtained,

$$\frac{\pi^2}{4R^2 A_2^2 \omega^2} + \frac{1}{R^2 A_2^2 \omega^2} = 1 .$$

Solution of this equation for  $\omega$  yields,

$$\omega = \frac{1.86}{RA_2}$$
$$f = \frac{0.296}{RA_2} .$$

Substitution of this value of frequency into the equation for  $c_{\max}$  derived in the previous paragraph yields the following equation for  $c_{\max}$ :

$$c_{\max} = 0.845R .$$

For  $A_2 = 0.003$ ,  $R = 50$ , these equations give,

$$f = 1.97 \text{ cps}$$

$$c_{\max} = 42.2 \text{ units.}$$

Between this frequency and the cutoff frequency defined earlier is a gap which is not covered by any of the equations. Values of output amplitude in this gap can best be obtained by a graphical analysis at each

frequency of interest. The point of tangency of the straight line to the sine wave can be determined, and then the line may be extended to its intersection with the sine wave. The point of intersection will give the maximum value of the output at this frequency.

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X. VITA

Clifton David Cullum, Jr. was born in Baltimore, Maryland, September 10, 1937. He attended Baltimore Polytechnic Institute in Baltimore, graduating in February, 1955.

In September of 1955 he entered Virginia Polytechnic Institute, majoring in Electrical Engineering. During the period of his undergraduate enrollment he participated in the Cooperative Engineering Program with the Martin Co., Middle River, Maryland. Upon receiving his Bachelor of Science Degree in Electrical Engineering in June, 1959, the author joined the Faculty of the Electrical Engineering Department at V. P. I. where he is presently employed.

During the summers of 1960 and 1961 the author worked as a Control Systems Engineer at the Baytown, Texas and Linden, New Jersey refineries of the Humble Oil and Refining Company.

He is a member of the Institute of Radio Engineers, the American Institute of Electrical Engineers, Tau Beta Pi, and Eta Kappa Nu.

In December of 1959 he was married to the former Jane Grace Kehoe of Norfolk, Virginia.

A handwritten signature in cursive script, reading "Clifton D. Cullum, Jr.", written in black ink. The signature is fluid and somewhat stylized, with a horizontal line drawn underneath the name.

## Abstract

This thesis deals with the deliberate insertion of nonlinear elements in second order linear control systems for the purpose of improving their transient response. The main body consists of a method of obtaining a desired step response by placing a nonlinear computer in the forward loop. This computer fixes the system trajectory in the phase plane by determining the required output velocity for the error present at any time. An inner control loop adjusts the output velocity to agree with the computed signal in an extremely short time, thus giving very close agreement between actual and desired responses.

Several examples are presented to show the application of this method, and experimental verification is obtained with an analog computer. Areas of future study and practical limitations are discussed in the final sections of the thesis.