VARIABLE SAMPLING INTERVALS FOR CONTROL CHARTS USING COUNT DATA

by

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(ABSTRACT)

This thesis examines the use of variable sampling intervals as they apply to control charts that use count data. Papers by Reynolds, Arnold, and R. Amin developed properties for charts with an underlying normal distribution. These properties are extended in this thesis to accomodate an underlying Poisson distribution.

ACKNOWLEDGEMENTS

This thesis is dedicated to my parents, and . Without their support and encouragement this thesis could never have been successful.

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1.1 Introduction to Control Charts

A control chart is used to monitor a quality variable to detect changes in the parameter of the distribution of this variable. This is done by taking samples from the process and using the samples to compute values of a control statistic that is either an estimate of the monitored parameter or a function of the estimate. For example, if the variable being monitored is the process mean μ, then the logical control statistic would be the sample mean $\bar{X}$. After the control statistic has been computed, it is plotted on the control chart. A control chart is divided into two regions, say Z and Z'. If the control statistic is plotted within predetermined control limits, then it is in region Z and the decision is that the monitored parameter is maintaining its target value. In this case the process is said to be in control. If the control statistic is plotted outside the predetermined control limits, then it is in region Z' and the decision is that the monitored process is no longer maintaining its target value. In this case the parameter is said to be out of control. When a variable has been declared out of control the process is often stopped and a cause for the change is sought.

Traditionally, control charts use a fixed sampling interval (FSI). That is, a sample is taken every d time units, where d is some constant. However, if the control statistic is near the control limit, one might want to take a sample in less than d time units to see if a process change has occurred. Or, if the control statistic is very near the required value, a time longer than d may be used to decrease the amount of samples taken unnecessarily. This logical extension of FSI control charts is called variable sampling interval (VSI) control charts.

All previous VSI control chart work (see literature review) has been done for quality variables that follow a normal distribution. There are, however, many situations where the assumption of normality is not appropriate. In many industrial processes, the objective is to monitor a discrete count variable. For example, the number of defects found on an electronic component or the number of blemishes seen on a mass produced wood product. The underlying distribution of these count data examples is usually assumed to be Poisson. The objective of this thesis is to take the existing properties of VSI control charts developed for the normal distribution and apply them to the discrete Poisson case. The control charts to be considered are the c-chart and cumulative sum (cusum) chart for Poisson data. Also included in this thesis will be some general guidelines that can be used to construct VSI discrete control charts.

1.2 Literature Review

Process/quality control is the group heading for methods that monitor a product in order to detect a deviation from a predetermined standard. The history of process/quality control goes back as far as 1924 when W.A. Shewhart first conceptualized a control chart. Reasoning that when product quality is high, inspection should be low, and that when product quality is low, inspection should be high, Dodge(1943) introduced a method of acceptance sampling that inspected all or a fraction of the items being produced. His method is referred to as CSP-1 (continuous sampling plan). Wald and Wolfowitz(1945) devised a plan that was more effective when there was a sequence of low quality items. Lieberman and Soloman(1955) developed a design using multiple inspection levels (multilevel plan MLP) where several inspection fractions were used and switching could only be done to adjacent adjacent levels. Dermann, Littauer, and Soloman(1957) proposed three "tightened" variations of the MLP. These plans made it possible to switch back more than one level at a time, producing plans with an increased sensitivity to large process shifts.

While work was being done on continuous sampling plans, E.S. Page(1955) introduced warning lines within the control limits of a Shewhart chart. The idea was that if a significant number of points fell outside of the warning lines but inside of the control limits, an action should be taken. Page(1962) then showed that warning lines were faster at detecting a large process shift than the runs rules indroduced by Moore(1958). Page(1954) also designed the cumulative sum chart (cusum chart). This chart plots a running sum of the monitored count variable and thereby makes use of past sample results to determine if the process is in or out of control. The average run length (ARL) of a discrete one-sided cusum chart was computed using a Markov chain by Brook and Evans(1972). This method also gave good approximations for continuous cusum charts. Lucas(1982) added the control limits of a Shewhart chart to the cusum chart. In this way, small and moderate shifts as well as large shifts could be detected. Lucas and Crosier(1982b) developed a procedure that was more sensitive to outliers than was the Lucas(1982) procedure.

All work on control charts cited thus far has been done under the assumption of taking one sample per time unit, where a time unit is the waiting period being used between samples. However, it seems more reasonable to wait a longer period of time (an example is two time units) if the process is in control or wait a shorter period of time (an example is a fourth of a time unit) if something appears to be going wrong with the process. This possibility was investigated by Arnold(1970). He used a Markovian structure and developed an expected sample size for which he evaluated a number of sampling schemes. Crigler(1973) derived an economically optimal sampling plan based on Arnold's work. Crigler and Arnold(1979,1986) combined to extend these results. Hui and Jensen(1980) further extended this idea to the multivariate case. Their work included time needed for process adjustments, and expected number of inspections and adjustments. Up to this point however, there had been no work done in the area of applying variable sampling interval (VSI) procedures to standard Shewhart and cusum charts.

Then, Reynolds and Arnold(1987) showed that by using VSI procedures, the $\bar{x}$- chart became faster at detecting shifts. Reynolds, Arnold, Amin, and Nachlas(1987) developed formulae for the average time to signal (ATS) and the average number of samples to signal (ANSS) in a VSI $\bar{x}$-chart. They also investigated how many intervals should be used and of what length they should be. Reynolds, Arnold, and Amin (1987) applied VSI procedures to cusum charts to again achieve faster detection of process changes.

Chapter II
Variable Sampling Interval C-Chart

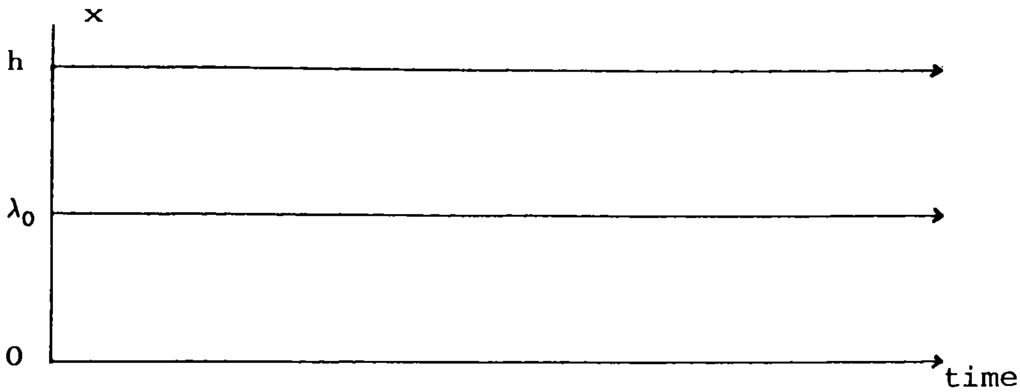
2.1 Introduction to the c-chart

The first control chart to be investigated will be a specific Shewhart chart, the upper one-sided c-chart. In this case the monitored variable is a discrete count variable. The control chart will plot the count per sample in order to detect an upward shift in the mean count per sample. The challenge is to design a VSI control chart based on the Poisson distribution:

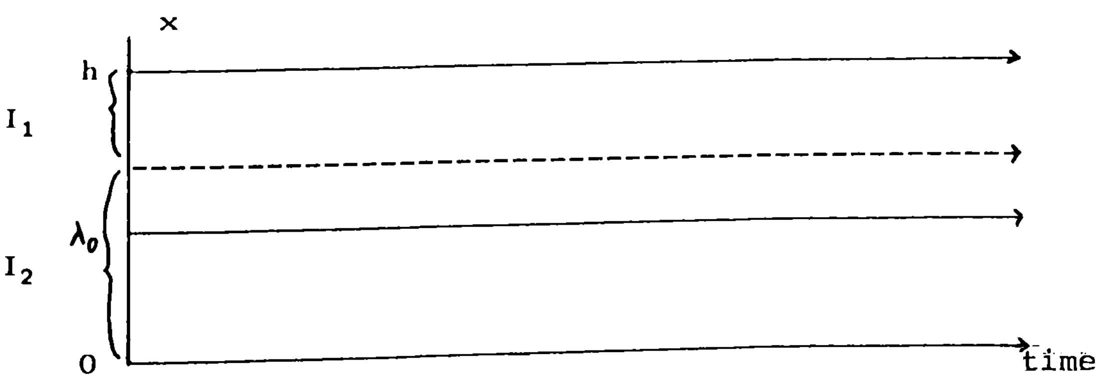
$$P(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

where λ is the target count per sample and x is the count in the current sample. x can be the direct count at a single observation or the sum of counts for a sample of n observations. Either case is acceptable as long as λ is the average count per observation.

The standard fixed sampling interval (FSI) c-chart can be set up as in the following figure:



where h is the control limit signaling a shift and λ_0 is the target count per sample. The procedure for the FSI c-chart is to plot samples taken at a fixed interval and conclude the process is out of control when a sample crosses the control limit, i.e. the count found in the current sample is greater than h. The area greater than h is referred to as the signal region or the out of control area. The area between and inclusive of 0 and h is referred to as the in control area. In a VSI c-chart, the objective is to detect process shifts faster by dividing the in control area of the chart into $I_1, I_2, \ldots I_n$ sampling intervals that will use $d_1, d_2, \ldots d_n$ time intervals depending on which sampling interval the previous sample was plotted. An example of a VSI c-chart with two sampling intervals is:



The intervals I_1 and I_2 would use time intervals of d_1 and d_2 respectively. The probability of using a specific interval i is $P(x \in I_i | \lambda) = P_i(\lambda)$.

2.2 Signaling

The control limit is the line that once crossed indicates a process shift. On continuous variable Shewhart charts this line is often drawn at h equal to the target plus three standard deviations of the control statistic. When the Poisson distribution is used, this may not be feasible since $\sigma^2 = \lambda$ may cause h to be too far from target to be effective. Therefore, an alternate method of choosing a control limit is needed. The method used is to choose a control limit corresponding to the probability of a signal when $\lambda = \lambda_0$. When $\lambda = \lambda_0$ the probability of a signal should be small. Signaling when no shift has occured is called a false alarm. This is written as

$$P(\text{false alarm}) = P(x \geq h \mid \lambda = \lambda_0) = q(\lambda_0).$$

When a c-chart is designed, $q(\lambda_0)$ is used to generate the control limit h. Since the distribution of count data is discrete, there are only a finite number of possible values for $q(\lambda_0)$. For example, for $\lambda_0 = 3.0$, $q(\lambda_0) = 0.002$ may be the desired in-control signal or false alarm probability but this is not possible to achieve exactly so $q = 0.001103$ can be used because it is the closest value less than 0.002. Using $q(\lambda_0) = 0.001103$ gives a control limit of h=10.

When $q(\lambda)$ is the probability of a signal and N is the number of samples until a signal, the average number of samples to signal (ANSS) can be found by finding the expected value of N. N has a geometric distribution with parameter $q(\lambda)$. Therefore,

$$E(N) = \frac{1}{q(\lambda_0)}.$$

This can be interpreted as $1/q(\lambda_0)$ being the average number of samples until a signal. For example, when $q(\lambda_0) = 0.001103$, the ANSS would be approximately 907.

In future equations, $q(\lambda)$ will be written as q unless a specific λ value needs to be indicated.

2.3. Average Time to Signal (ATS)

The average time to signal (ATS) for the general Shewhart chart will be given first. This derivation will then be used to compute examples and tables for the c-chart.

The ATS for both an FSI and VSI Shewhart chart can be found by finding

$$ATS = E(T) = E(\sum_{i=1}^{N} R_i)$$

where T is real time, N is as previously defined, and R_i is the time interval used before the i^{th} sample. Using Wald's Identity, this becomes

$$ATS = E(T) = E(N)E(R_i).$$

$E(R_i)$ is the average sampling interval, and E(N) is as previously defined.

In the FSI Shewhart chart where one sample is taken every d time units

$$E(R_i) = d,$$

and therefore

$$ATS = E(T) = \frac{1}{q}(d) = \frac{d}{q}. \qquad (2.1)$$

In the VSI Shewhart chart where multiple time intervals can be used, we let $d_1, d_2, \ldots d_n$ be the lengths in time units such as hours or minutes for the $I_1, I_2, \ldots I_n$ intervals, respectively. Each sampling interval I_i has a probability of occurrence p_i that depends on λ. Using this notation

$$E(N) = \frac{1}{q} \quad \text{and} \quad E(R_i) = \frac{\Sigma d_j p_j}{1-q},$$

and therefore,

$$ATS = E(T) = \frac{\Sigma d_j p_j}{q(1-q)}.$$

In Reynolds and Arnold(1986) it was suggested that the optimum number of intervals to use is two and that they be spaced far apart. Using only two intervals, the ATS for the VSI Shewhart chart now becomes

$$ATS = \frac{d_1 p_1(\lambda) + d_2 p_2(\lambda)}{q(\lambda)[1-q(\lambda)]}. \qquad (2.2)$$

When $\lambda = \lambda_0$, it is desired to have the ATS for a VSI control chart equal to a specified value for two reasons. One is that in designing VSI charts, a specified time to false alarm or a specified average sampling rate when $\lambda=\lambda_0$ is often needed. The other reason is that when comparisons are made between the FSI and VSI charts, both charts need to have the same false alarm rate and average sampling rate when $\lambda=\lambda_0$. The VSI ATS and FSI ATS will have the same specified value if $d_1, d_2, p_1,$ and p_2 are chosen to satisfy

$$\frac{d_1 p_1 + d_2 p_2}{q(1-q)} = \frac{d}{q}$$

or

$$\frac{d_1 p_1 + d_2 p_2}{(1-q)} = d.$$

Since p_1 and p_2 can only achieve discrete values, it is suggested that they be chosen first. In continuous control charts, Reynolds and Arnold(1987) found that setting p_1 approximately equal to p_2 gives the best properties over a wide range of shifts. In discrete control charts p_1 and p_2 will be chosen as close to equality as possible. Also, d_1 and d_2 are to be spaced far apart. So if all d_i must be $l_1 \leq d_i \leq l_2$ where l_1 is the smallest time interval needed to take a reliable sample and l_2 is the maximum time interval allowed, let $d_1 = l_1$ and compute d_2 after d_1 is chosen. For the examples in this thesis, we let $d_1 = 0.1 * d$ or one tenth of the average sampling interval. Then we find d_2 by solving

$$d_2 = \frac{d(1-q) - d_1 p_1}{p_2}. \tag{2.3}$$

In the $\bar{x}$-chart of Reynolds et al(1987) d_1 and d_2 were symmetric about d. Since the c-chart uses a discrete distribution however, this is seldom possible since the possibilities for p_1 and p_2 are limited.

After d_1 and d_2 have been computed it is a simple matter to calculate the ATS for a shift to λ_1. Compute signal and interval probabilities substituting λ_1 for λ_0, then use equation (2.2) to find the ATS at λ_1. A simple example using the c-chart follows:

A process is monitored using a target of $\lambda_0 = 3.0$ defects per sample. The FSI c-chart is set up using a sampling interval of 2.0 hours. Let the upper control limit be set at h=10. Then $P(x \geq 10 | \lambda_0) = 0.001103$ gives a false alarm rate of one in approximately 907 samples. The VSI c-chart will use the same target $c = \lambda_0 = 3.0$, the same false alarm rate of $P(x \geq 10 | \lambda=\lambda_0) = 0.001103$ and the same average sampling interval of 2 hours. The in control area of the VSI c-chart is divided into an upper interval of $I_1=3 \leq x \leq 9$ with an associated probability $p_1(\lambda_0) = P(3 \leq x \leq 9 | \lambda=\lambda_0) = 0.5757$, and a lower interval $I_2=x \leq 2$ with an associated probability $p_2(\lambda_0) = P(x \leq 2 | \lambda=\lambda_0) = 0.4232$. For I_1, a time interval of $d_1 = 0.1*2.0 = 0.2$ of an hour or 12 minutes is computed. For I_2, a time interval of d_2 is computed as

$$d_2 = \frac{2.0(1-0.001103) - 0.2(0.5757)}{0.4232} = 4.4487 \text{ hours.}$$

These numbers mean that a point in I_2 corresponds to a count of 0,1, or 2. A point in I_1 corresponds to a count of 3,4,5,6,7,8,or 9. When a point falls in I_2, wait $d_2 = 4.4487$ hours until the next sample is taken. In practice 4.4487 may be rounded to 4.4 or 4.5. If a point falls in I_1, wait $d_1 = 0.2$ of a hour (12 minutes) until the next sample is taken. If the number of defects is greater than or equal to 10, the plotted point is above the control limit. This signals a shift in λ.

The ATS at λ_0 is the average number of hours until a false alarm and is computed using equation (2.2). The in control ATS is

$$\text{ATS} = \frac{0.2(0.5757) + 4.4486(0.4232)}{0.001103(1-0.001103)} = 1813.74 \text{ hours.}$$

Note that the in control ATS for the FSI chart $= \frac{d}{q} = \frac{2.0}{0.001103} = 1813.74$ hours. The FSI ATS and VSI ATS are the same when $\lambda=\lambda_0$.

If a process shift of 50% occurs, $\lambda_1 = 4.5$ and the probabilities computed at the new λ are:

$q(\lambda_1) = 0.01709342$
$p_1(\lambda_1) = 0.8093285$
$p_2(\lambda_1) = 0.1735780.$

The new probabilities increase the use of I_1 which uses the shorter time interval d_1. This will lead to faster detection of the shift. The new time to signal is

$$ATS(\lambda_1{=}4.5) = \frac{0.2(0.8093285) + 4.4487(0.1735780)}{0.01707324(1-0.01070324)} = 55.60$$
hours.

Note that the $ATS(\lambda_1 = 4.5)$ for the fixed interval chart = $\frac{2.0}{0.01070324}$ = 117.00 hours.

This means that it will require an average of approximately 56 hours for the VSI c-chart to signal a 50% shift from 3.0 to 4.5 and an average of approximately 117 hours for the FSI c-chart to signal a 50% shift from 3.0 to 4.5.

Additional examples of comparing VSI c-charts to FSI c-charts are contained in tables (2.4) - (2.6) located at the end of the chapter. All comparisons are based on both the VSI c-chart and the FSI c-chart having the same in control ATS.

2.4 Adjusted time to signal

Thus far, all computations done for $\lambda \neq \lambda_0$ are based on the assumption that the shift in count happens at time zero. In actual practice though, the shift may occur at a random time in the future. If a large shift occurs early in a VSI long interval, it may not be detected for a much longer time than if an FSI had been used. Reynolds et al(1987) developed a model that allows the shift to occur randomly. The adjusted ATS was formulated by using

T^* = adjusted time to signal

= time from shift to signal

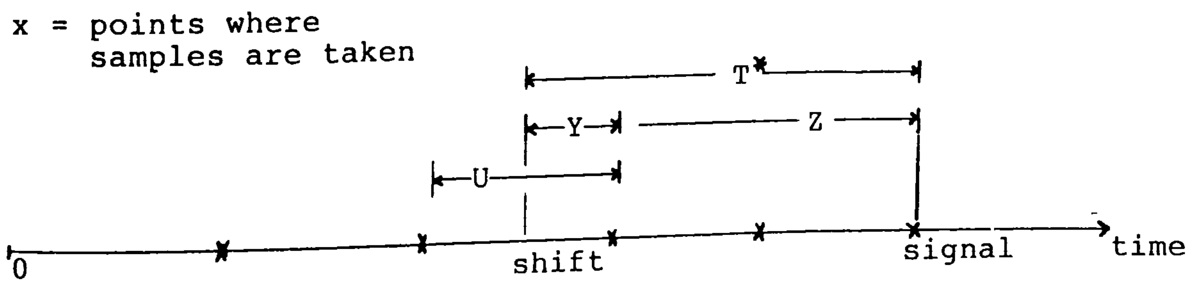
U = length of interval in which shift occurs

Y = time from process shift until next sample

Z = time from the next sample after the process shift until a signal

N = number of samples after the shift until a signal

This can be graphically represented as:



The new adjusted ATS can be written as

$$E(T^*) = E(Y) + E(Z).$$

Since Z has the same distribution as $\sum_{i=1}^{N-1} R_i$,

$$E(T^*) = E(Y) + E(N\text{-}1)E(R_i).$$

Using the results in Reynolds et al(1987) the adjusted ATS is

$$E(T^*) = \frac{d_1^2 p_1 + d_2^2 p_2}{2(d_1 p_1 + d_2 p_2)} + \frac{d_1 p_1^* + d_2 p_2^*}{q^*},$$

where p_1 and p_2 are the probabilities of d_1 and d_2 when $\mu=\mu_0$ and p_1^*, p_2^*, and q^* are the probabilities of d_1, d_2, and a signal when $\lambda = \lambda_1$. Tables (2.4) - (2.6) give the adjusted ATS for examples given in Section 2.2.

Table 1. ATS values of ASI and VSI charts with $\lambda_0 = 3.0$, $d = 2.0$, $I_1 = [3,9]$, $I_2 = [0,2]$, $d_1 = 0.1$, and $d_2 = 4.45$.

Shift	$P(I_1)$	$P(I_2)$	P(signal)	FSI ATS	VSI ATS	Adj.FSI ATS	Adj.VSI ATS
0%=3.0	0.5757	0.4231	0.001103	1813.7	1813.7	1813.7	1813.7
25%=3.75	0.7176	0.2771	0.005308	376.8	260.6	377.8	261.4
50%=4.5	0.8093	0.1735	0.017093	117.0	55.6	118.0	56.7
100%=6.0	0.8541	0.0619	0.083924	23.8	5.8	24.8	7.4
150%=7.5	0.7562	0.0203	0.223593	8.9	1.4	9.9	3.2
200%=9.0	0.5816	0.0062	0.412591	4.8	0.6	5.8	2.4

Table 2. ATS values of FSI and VSI charts with $\lambda_0 = 1.0$, $d = 1.0$, $I_1 = [2,5]$, $I_2 = [0,1]$, $d_1 = 0.1$, and $d_2 = 1.32$.

Shift	$P(I_1)$	$P(I_2)$	P(signal)	FSI ATS	VSI ATS	Adj.FSI ATS	Adj.VSI ATS
0%=1.0	0.2636	0.7357	0.000594	1682.6	1682.6	1682.6	1682.6
25%=1.25	0.3535	0.6446	0.001838	542.0	483.9	544.5	483.6
50%=1.5	0.4377	0.5578	0.004456	224.4	176.2	224.9	176.0
100%=2.0	0.5447	0.4060	0.016563	60.4	36.5	60.9	36.5
150%=2.5	0.6706	0.2872	0.042022	23.8	11.1	24.3	11.3
200%=2.0	0.7169	0.1991	0.083918	11.9	6.6	12.4	4.6

Table 3. ATS values of FSI and VSI charts with $\lambda_0 = 5.0$, $d = 3.0$, $I_1 = [5,13]$, $I_2 = [0,4]$, $d_1 = 0.1$, and $d_2 = 6.43$.

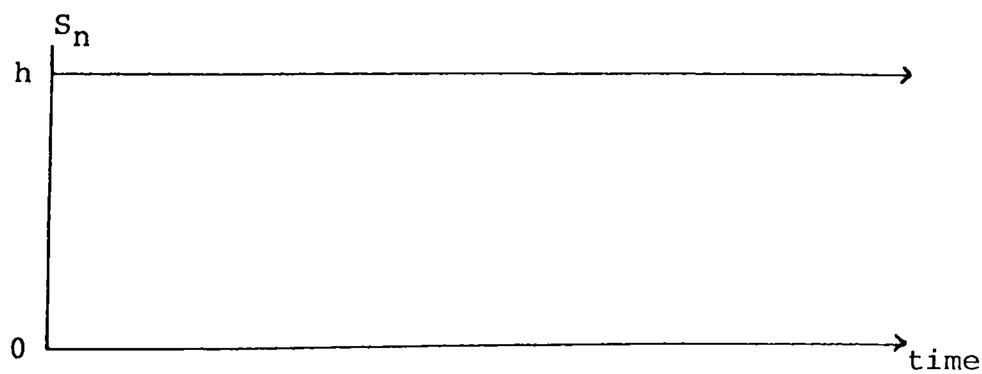
Shift	$P(I_1)$	$P(I_2)$	P(signal)	FSI ATS	VSI ATS	Adj.FSI ATS	Adj.VSI ATS
0%=5.0	0.5588	0.4404	0.0006989	4292.9	4292.9	4292.9	4292.9
25%=6.25	0.7419	0.2529	0.0051322	584.5	361.9	586.0	363.1
50%=7.5	0.8464	0.1321	0.0215654	139.1	52.2	140.6	54.2
100%=10.0	0.8352	0.0292	0.1355361	22.1	3.7	23.6	6.3
150%=12.5	0.6225	0.0053	0.3721649	8.1	0.9	9.6	3.6
200%=15.0	0.3624	0.0008	0.6367822	4.7	0.5	6.2	3.2

Chapter III

Variable Sampling Interval Cusum Chart

3.1 Introduction to the Poisson cusum chart

The cumulative sum (cusum)chart detailed by Lucas(1985) detects a process shift by plotting the cumulative sum of the statistic $x_i - k$. The term x_i is the count in sample i and k is a function of the target number of defects per sample (λ_0) and the shift in defects to be detected (λ_1). The fixed sampling interval (FSI) upper one-sided cusum chart looks as follows:



where h is the upper control limit indicating a signal.

A cusum chart works by plotting the control statistic $S_n = \sum_{i=1}^{n}(x_i - k)$. For example, if the count for $x_1 = j$ and the count for $x_2=m$, then $S_1=j-k$ and $S_2=(j-k)+(m-k)=j+m-2k$ are the points plotted on the chart. Whenever S_n crosses the control limit h the chart signals, and whenever S_n goes below zero the chart restarts at zero. By resetting at zero whenever S_n is plotted below zero, the cusum chart can be considered a sequence of probability ratio tests (SPRT's). The SPRT uses critical inequality $0 \leq \Sigma z_i \leq a$ where $z_i = \ln[f(x_i|\lambda_1)/f(x_i|\lambda_0)]$. When the Poisson distribution is used, the SPRT reduces to

$$0 \leq \sum_{i=1}^{n}[x_i - \frac{\lambda_1 - \lambda_0}{\ln\frac{\lambda_1}{\lambda_0}}] \leq \frac{a}{\ln\frac{\lambda_1}{\lambda_0}},$$

where

$$k = \frac{\lambda_1 - \lambda_0}{\ln\frac{\lambda_1}{l_0}},$$

$$\sum_{i=1}^{n}[x_i - \frac{\lambda_1 - \lambda_0}{\ln\frac{\lambda_1}{\lambda_0}}] = \sum_{i=1}^{n}[x_i - k] = S_n \quad \text{(the statistic to be charted)},$$

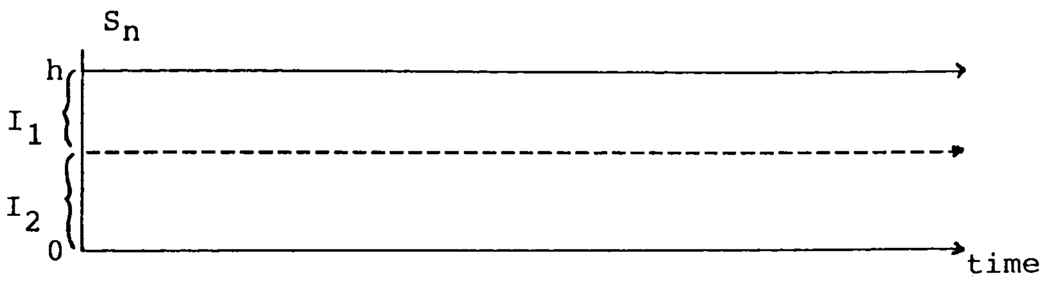
and

$$\frac{a}{\ln\frac{\lambda_1}{\lambda_0}} = h \quad \text{(the upper control limit).}$$

Note that the variable "a" can be changed to produce an h that will give the desired signal probabilities. Details on the derivation of the discrete cusum chart can be found in Lucas(1985).

In an FSI upper one-sided cusum chart, one sampling interval is used so a sample is taken every d time units. In the variable sampling interval (VSI) upper one-sided cusum chart, $I_1, I_2, \ldots I_n$ sampling intervals are used.

Which sampling interval is used is dependent on where S_n is plotted on the cusum chart. An example of a VSI cusum chart with two sampling intervals follows.



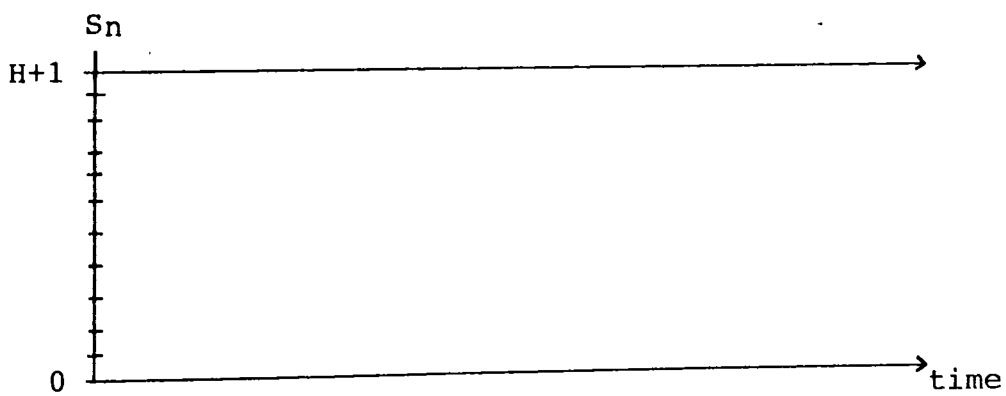
3.2 Time to Signal

As in Shewhart charts, cusum charts are evaluated by how quickly they detect a shift in the variable being monitored. Since cusum charts are designed using a specific shift to be detected, it will be most efficient at detecting shifts near the design shift. However, no matter what the design shift, a cusum chart is evaluated by its effectiveness over a wide range of shifts.

How long a VSI control chart takes to signal is based on N, the number of samples to signal, and the time interval used at each sample. Which time interval, d_i, is used depends on which interval I_i the control statistic S_n is plotted. The time to signal is then the sum over all i of the number of samples falling in interval I_i multiplied by its corresponding sampling interval d_i.

The object is to determine how many times each interval I_i is used before a signal. Since N is the total number of samples to signal, it must be known how N is distributed among the $I_1, I_2, \ldots I_n$ intervals. Brook and Evans(1972) first used a Markov chain approach to solve the problem of determining the average number of samples to signal (ANSS) for a fixed sampling interval (FSI) cusum chart. This approach will yield an exact ANSS for discrete distributions such as the Poisson and a good approximation of the ANSS for continuous distributions. Amin et al(1987,1988) extended this approach to include the VSI cusum chart.

In order to use a Markov chain approach, the area $(-\infty,+\infty)$ must be partitioned into discrete states. Since an upper one-sided cusum chart cannot be negative, let the interval $(-\infty,0]$ be the 0 state of the chain. Also, since $[h,+\infty)$ is the signal region of the cusum chart let it be the absorbing state A. It is now left to partition the remaining interval (0,h). How this is done depends on the variable k. Assuming k is a rational number, it can be written as the ratio r_1/r_2 where r_1 and r_2 are integers. The area (0,h) can then be divided into $(r_2*h)-1$ states. Each state corresponds to an interval of length $1/r_2$. Adding the $(-\infty,0]$ state and the $[h,+\infty)$ absorbing state, there are a total of $(r_2*h)+1$ states in the Markov chain. Let H be the notation for r_2*h so that the total number of states is H+1. An example of the partitioned cusum chart is:



where each state $C_w = w/r_2$ is a possible value of the statistic x_i-k.

The next step is to compute the transition matrix P. P will be an $(H+1)\times(H+1)$ matrix with elements p_{uv}. Each p_{uv} is computed as the probability of the process going from state u to v or the probability of the statistic x_i-k changing from u/r_2 to v/r_2. This is written as

$$p_{uv} = P(x_i-k=(v-u)/r_2 \mid \lambda)$$

where λ is the mean count per sample for which the probabilities are being computed. Recall that the 0 state of the Markov chain represents the partition $(-\infty,o]$. Therefore, all of the p_{u0} probabilities are actually

$$p_{u0} = P(x_i - k \leq (0-u)/r_2 \mid \lambda) \text{ for } 0 \leq u \leq H.$$

Also, recall that state A is the absorbing state of $[H,+\infty)$. Therefore, all of the p_{uH} probabilities are actually

$$p_{uH} = P(x_i - k \geq (H-1-u)/r_2 \mid \lambda) \text{ for } 0 \leq u \leq H.$$

Writing k as r_1/r_2 and simplifying, the p_{uv}'s become

$$p_{uv} = \begin{cases} P(x_i \leq \frac{v-u+r_1}{r_2} \mid \lambda) & \text{for } 0 \leq u \leq H,\ v = 0 \\ P(x_i = \frac{v-u+r_1}{r_2} \mid \lambda) & \text{for } 0 \leq u \leq H \text{ and } 0 \leq v \leq H-1 \\ P(x_i = \frac{v-u+r_1}{r_2} \mid \lambda) & \text{for } 0 \leq u \leq H,\ v = H. \end{cases} \qquad (3.1)$$

The transition matrix is written as

state	0	1	2	3	....	H
contol statistic value =	0	$\frac{1}{r_2}$	$\frac{2}{r_2}$	$\frac{3}{r_2}$	...	h

$$
\begin{array}{c}
0=0 \\ 1=\frac{1}{r_2} \\ 2=\frac{2}{r_2} \\ 3=\frac{3}{r_2} \\ \cdot \\ \cdot \\ \cdot \\ H=h
\end{array}
\begin{bmatrix}
p_{00} & p_{01} & p_{02} & p_{03} & \cdots & p_{0H} \\
p_{10} & p_{11} & p_{12} & p_{13} & \cdots & p_{1H} \\
p_{20} & p_{21} & p_{22} & p_{23} & \cdots & p_{2H} \\
p_{30} & p_{31} & p_{32} & p_{33} & \cdots & p_{3H} \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\
p_{H0} & p_{H1} & p_{H2} & p_{H3} & \cdots & p_{HH}
\end{bmatrix} .
$$

By computing the fundamental matrix related to P, the expected number of times the statistic will be in a given state can be found. The fundamental matrix is computed by first deleting the column and row corresponding to state A=H. The resulting HxH matrix is called Q. The matrix M = $(I_H - Q)^{-1}$, where I_H is an HxH identity matrix, is then computed. M is the fundamental matrix. If the process being monitored starts in a given state j, the ANSS is the sum of the elements in the j^{th} row of M. That is,

$$\text{ANSS}(j) = \sum_{v=0}^{H} m_{jv} .$$

All examples and charts contained in this paper assume that the starting state is 0. Therefore, ANSS(0) will be written

$$\text{ANSS}(0) = \text{ANSS} = \sum_{v=0}^{H} m_{0v}.$$

Once the ANSS has been found the average time to signal (ATS) can be computed. For an FSI cusum chart that takes a sample every d time units, the ATS = ANSS*d. For a VSI cusum chart the ATS is

$$\sum_{v=0}^{H-1} M_{0v} b_v,$$

where b_v is the time interval used when the Markov chain is in state v. The vector $\underline{b}$ is composed of the time intervals $d_1, d_2, \ldots, d_n$ where each d_i can be used for more than one state.

An example of computing the ATS for a VSI cusum chart follows:

In order to keep the matrix size down, the numbers chosen in this example will be kept small. Let the mean number of defects per sample be $\lambda_0=1.0$. Let the value of k=1/2 and the control limit be h=2. The statistic x_i-k is now $x_i-1/2$. We find the ATS for $\lambda=\lambda_0$, which is the in control ATS.

The first step is to generate the transition matrix. Using equation (3.1), the matrix P is

state		0	1	2	3	4
	=	0	1/2	1	3/2	2
0	0	$P(x_i \leq \frac{1}{2})$	$P(x_i=1)$	$P(x_i=\frac{3}{2})$	$P(x_i=2)$	$P(x_i \geq \frac{5}{2})$
1	1/2	$P(x_i \leq 0)$	$P(x_i=\frac{1}{2})$	$P(x_i=1)$	$P(x_i=\frac{3}{2})$	$P(x_i \geq 2)$
2	1	$P(x_i \leq -\frac{1}{2})$	$P(x_i=0)$	$P(x_i=\frac{1}{2})$	$P(x_i=1)$	$P(x_i \geq \frac{3}{2})$
3	3/2	$P(x_i \leq -1)$	$P(x_i=-\frac{1}{2})$	$P(x_i=0)$	$P(x_i=\frac{1}{2})$	$P(x_i \geq 1)$
4	2	$P(x_i \leq -\frac{3}{2})$	$P(x_i=-1)$	$P(x_i=-\frac{1}{2})$	$P(x_i=0)$	$P(x_i \geq \frac{1}{2})$

Note that since the Poisson distribution is discrete, only probabilities involving integers will have a value greater than zero. Using this fact, the transition matrix becomes

$$P = \begin{bmatrix} .3679 & .3679 & 0 & .1839 & .0803 \\ .3679 & 0 & .3679 & 0 & .2642 \\ 0 & .3679 & 0 & .3679 & .2642 \\ 0 & 0 & .3679 & 0 & .6321 \\ 0 & 0 & 0 & .3679 & .6321 \end{bmatrix}$$

The second step is to compute the fundamental matrix $M=(I_4-Q)^{-1}$ where I_4 and Q are as previously defined. The matrix M is computed to be

$$\begin{bmatrix} 2.179 & 1.024 & 0.606 & 0.624 \\ 0.950 & 1.632 & 0.769 & 0.458 \\ 0.404 & 0.694 & 1.483 & 0.620 \\ 0.149 & 0.255 & 0.546 & 1.228 \end{bmatrix} .$$

The in control ANSS for the example is 2.179+1.024 +0.606+0.624=4.433 samples. Note that this is a small number. However, the value of h can be changed to achieve a desired ANSS. The larger the value of h, the larger the ANSS. The ATS for an FSI cusum chart would be 4.433*d. The VSI ATS would be

$$[2.179 \;\; 1.024 \;\; 0.606 \;\; 0.624] \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} =$$

$$2.179b_1+1.024b_2+0.606b_3+0.624b_4$$

where b_i is the time delay used for state i.

3.3 Comparing cusum charts

Comparing the VSI cusum chart to the FSI cusum chart can be done by comparing their respective ATS's for a given shift. As in comparing the standard c-chart to the VSI c-chart, this is done by matching the in control FSI ATS and the VSI ATS and then comparing their respective ATS's for given shifts. The ANSS is the same for both cusum charts as long as the values for λ_0,k,and h are the same. Therefore, if the FSI cusum chart uses a time interval of d and the VSI cusum chart uses an interval vector $\underline{b}$ containing the $d_1, d_2, \ldots, d_n$ time intervals, the two charts will have the same ATS at $\lambda=\lambda_0$ if

$$d \sum_{j=0}^{H-1} m_{ij} = \sum_{j=0}^{H-1} m_{ij} b_j, \tag{3.2}$$

where i is the starting state and m_{ij} is an element of matrix M. Recall that examples and charts contained here use a starting state of 0. The values of $d_1, d_2, \ldots, d_n$ are to be decided next. Theoretical results in Reynolds(1988a,1988b) and numerical results in Amin(1987) showed that for the case of the normal mean, the optimal number of time intervals to use is two and that the upper interval d_1 should be as short as possible. Also, Amin et al(1987) showed that for the VSI $\bar{x}$-chart, the properties and results are optimal when d_1 and d_2 are used approximately the same number of times when $\mu=\mu_0$. This extends to the VSI cusum chart by dividing the in control ANSS into two sets, $(-\infty,g]$ and $(g,H-1]$, where the expected number of samples in $(-\infty,g]$ is approximately equal to the expected number of samples in $(g,H-1]$. This means that if a statistic S_n is plotted in state g or less, use time interval d_2, and if S_n is plotted in a state greater than g use time interval d_1. The vector $\underline{b}$ is now comprised of g+1 elements equal to d_2 and H-g elements that are equal to d_1. The VSI ATS can now be computed as

$$\text{VSI ATS} = d_2 \sum_{j=0}^{g} m_{ij} \quad + \quad d_1 \sum_{j=g+1}^{H-1} m_{ij}.$$

Using d_1=0.1*d as was done in the VSI c-chart, equation 3.2 can be used to solve for d_2. When states 0 to g use d_2 and states g+1 to H-1 use d_1, d_2 can be computed as

$$d_2 = \frac{d*\text{ANSS}(i) - d_1 \sum_{j=g+1}^{H-1} M_{ij}}{\sum_{j=0}^{g} M_{ij}} \quad ,$$

where i is the starting state. In this thesis, examples assume a starting state equal to 0, and therefore

$$d_2 = \frac{d*\text{ANSS}(0) \; - \; d_1 \sum_{j=g+1}^{H-1} M_{0j}}{\sum_{j=0}^{g} M_{0j}}.$$

When d_1 and d_2 have been chosen, the VSI ATS can be found for different shifts in λ. The following tables use d_1 and d_2 found in this manner. However, in practice d_1 and d_2 can be chosen as any values convenient to the process being monitored. Table (3.3) at the end of this chapter shows ATS comparisons between the FSI and VSI cusum charts.

3.4 Adjusted ATS

As in VSI Shewhart charts, the VSI cusum chart has a need for an adjusted ATS. An adjusted ATS is needed because of the possibility of a process shift occurring at the beginning of a long time interval and because S_n may not be zero when the shift occurs. Are VSI cusum charts as effective as FSI cusum charts when a process shift occurs at some random time in the future? This question is answered by comparing adjusted ATS's for both the FSI and VSI cusum charts.

In order to compute the adjusted ATS, a distribution must be found for the point in the sampling interval where the shift occurs. Also, because the cumsum chart may not be in its starting state when the shift occurs, a distribution for where the cusum statistic is at the time of the shift needs to be known. Assuming there have been no false alarms and that the process has been running for a significant amount of time, an approach using the steady state distribution can be used. Amin et al(1988) recommend this approach which was first investigated by Yashchin(1985). The term steady state ATS was suggested as an alternative to adjusted ATS by Amin it al(1988) because of the method's dependency on the steady state distribution of the Markov chain.

The idea is that once the Markov chain has attained its steady state distribution $\underline{\pi}=(\pi_0,\pi_1,\pi_3,\ldots\pi_H)$, the probability that the shift falls in a given sampling interval I_i depends on the frequency of use of the states within I_i and the length d_i of I_i. Using the reasoning that the probability of a shift occurring immediately after state i is proportional to the length of the sampling interval containing state i and frequency of use of the interval, a probability α_i can be defined as

$$\alpha_i = \frac{\pi_i b_i}{\sum_{j=0}^{H} \pi_i b_i} = \frac{\pi_i b_i}{\pi b},$$

where α_i is the probability that the Markov chain is in state i in the sample immediately before the shift occurred. It is also assumed that the position in the interval which the shift occcurs is uniformly distributed over the interval length. Thus the expected time from the shift to the next sample is $b_i/2$. Therefore, if the Markov chain is in state j immediately following the shift, the ATS when the Markov chain was in state i before the shift is

$$(b_i/2) + \sum_{j=0}^{H} p_{ij} E(T_j),$$

where $E(T_j)$ is the expected time to detection given j is the state of the Markov chain immediately following the shift and p_{ij} is as defined earlier. The steady state ATS is now found by using $\underline{\alpha}$ to compute a weighted average. The steady state ATS is

$$E(T^*) = \underline{\alpha}[\underline{b}/2 + QE(\underline{T})] = \underline{\alpha}[I/2 + QM]\underline{b} = \underline{\alpha}[M - I/2]\underline{b}$$

In summary, Amin et al(1988) described the steady state ATS as "the steady state ATS averages the ATS over the possible states of the Markov chain and the possible positions of the shift within an interval." Table (3.3) compares all four ATS possibilities, the FSI ATS, VSI ATS, steady state FSI ATS, and the steady state VSI ATS. Note that the VSI ATS and the adjusted VST ATS are similar.

For this reason, the guidlines in Chapter IV will be based on the VSI ATS.

Table 4. Examples of the four possible ATS's for the cusum chart that uses the Poisson distribution.

		g	ANSS	FSI ATS	Adj.FSI ATS	VSI ATS	Adj.VSI ATS
1.0	1.0	2	61.50	61.50	61.50	61.50	61.50
1.0	1.5	2	13.63	13.63	12.44	10.26	8.17
1.0	2.0	2	7.49	7.49	6.61	5.60	4.01
1.0	3.0	2	4.06	4.06	3.39	3.27	2.10
3.0	3.0	5	92.19	92.19	92.19	92.19	92.19
3.0	4.5	5	10.54	10.54	9.16	7.62	5.37
3.0	6.0	5	5.60	5.60	4.73	4.15	2.27
3.0	9.0	5	3.03	3.03	2.42	2.43	1.46

Chapter IV

Guidelines to Discrete Control Charts

4.1 Introduction to guidelines for the c-chart

Chapter II showed how a VSI c-chart signals faster than a FSI c-chart. Chapter III showed how the VSI cusum chart signals faster than the FSI cusum chart. This thesis used recently investigated properties of VSI $\bar{X}$-charts and the assumption of a Poisson underlying distribution to compute the VSI ATS for the c-chart and cusum chart. It is thought, however, that VSI control charts have been used in practice without the benefit of theoretical backing. That is, VSI control chart parameters have been guessed at in the past and are now capable of being statistically computed.

In this chapter, some guidelines for setting up the VSI c-chart and VSI cusum chart will be discussed and illustrated via graphs for the c-chart and tables for the cusum chart.

4.2 Guidelines for the VSI c-chart

The following graphs have been set-up so that several VSI c-charts may be designed. The way to use a graph to design a VSI c-chart is as follows:

1) Choose a target count λ_0 . Notation for λ_0 on the graphs is L0.
2) Choose an average sampling interval and compute the upper and lower sampling intervals as follows:
   upper = D1 * D
   lower = D2 * D
   where D is the average sampling interval and D1 and D2 are the values given beneath the graph corresponding to the appropriate L0.
3) Use the given h and G as the signal limit and upper bound of the lower sampling interval respectively.
4) The ATS for a given shift can be found by
   a) Compute L1/L0 where L1 is the shift to be detected and L0 is as defined previously.
   b) From the plot of L0, find the corresponding LN(ASTS)/D and compute the VSI ATS(L1) by

$$e^{\ln(ATS)}/D .$$

5) The in control ATS can be found by the same method using L0/L1 = 1.0.

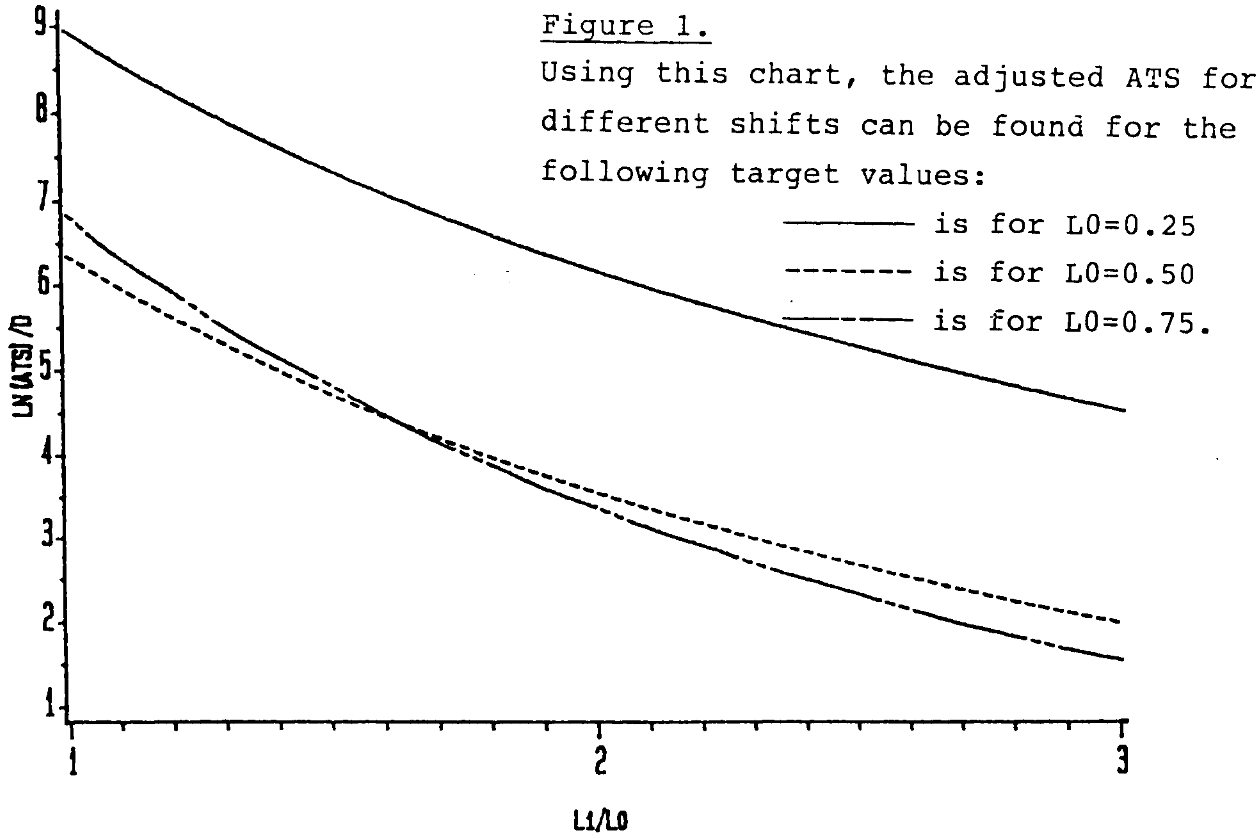


Figure 1.
Using this chart, the adjusted ATS for different shifts can be found for the following target values:

L0	0.25	0.50	0.75
D1	0.1/D	0.1/D	0.1/D
D2	1.3/D	1.6/D	2.0/D
G	0	0	0
H	4	4	5

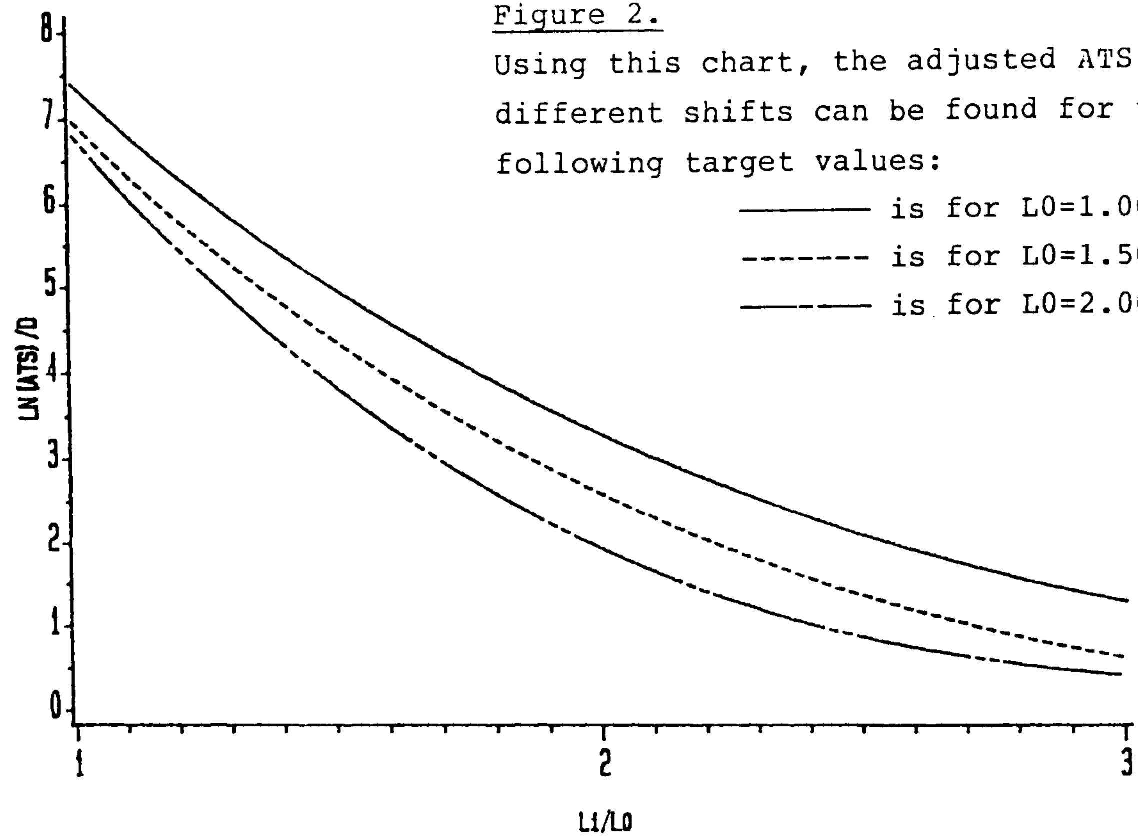


Figure 2.
Using this chart, the adjusted ATS for different shifts can be found for the following target values:
——— is for L0=1.00
------- is for L0=1.50
—-—-— is for L0=2.00.

LO	1.00	1.50	2.00
D1	0.1/D	0.1/D	0.1/D
D2	2.5/D	1.7/D	2.3/D
G	0	1	1
H	6	7	8

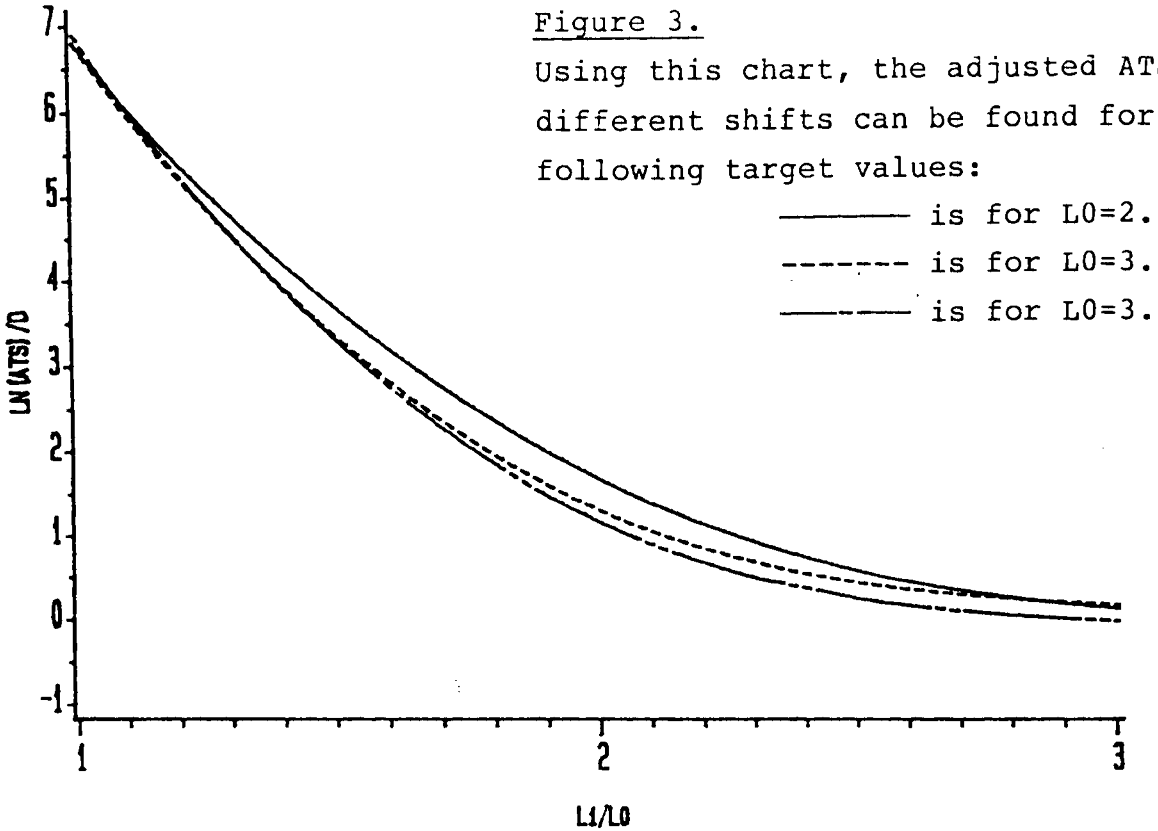


Figure 3.

Using this chart, the adjusted ATS for different shifts can be found for the following target values:

———— is for L0=2.50

-------- is for L0=3.00

—-—-— is for L0=3.50.

L0	2.50	3.00	3.50
D1	0.1/D	0.1/D	0.1/D
D2	1.8/D	2.2/D	1.8/D
G	2	2	3
H	9	10	11

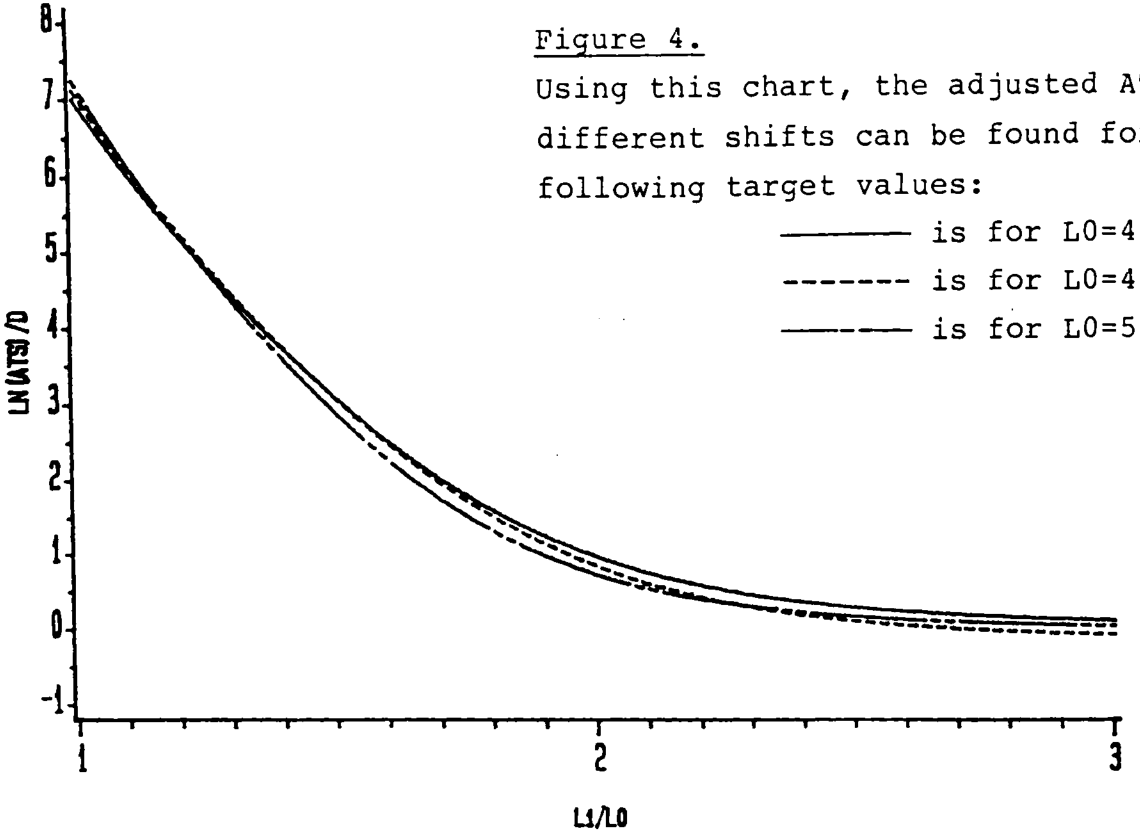


Figure 4.

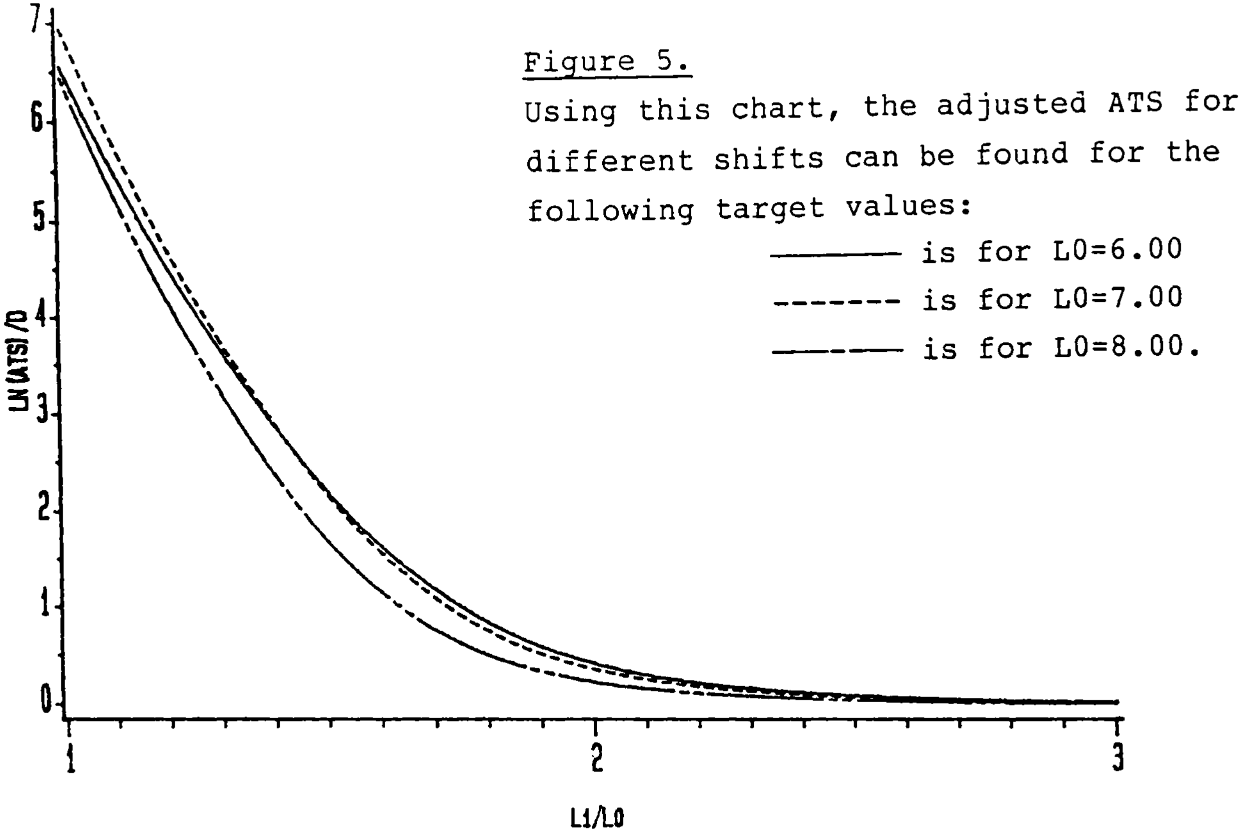
Using this chart, the adjusted ATS for different shifts can be found for the following target values:

——— is for L0=4.00

------- is for L0=4.50

—·—·— is for L0=5.00.

L0	4.00	4.50	5.00
D1	0.1/D	0.1/D	0.1/D
D2	2.2/D	1.8/D	2.1/D
G	3	4	4
H	12	13	14



L0	6.00	7.00	8.00
D1	0.1/D	0.1/D	0.1/D
D2	2.1/D	2.1/D	2.1/D
G	5	6	7
H	15	17	18

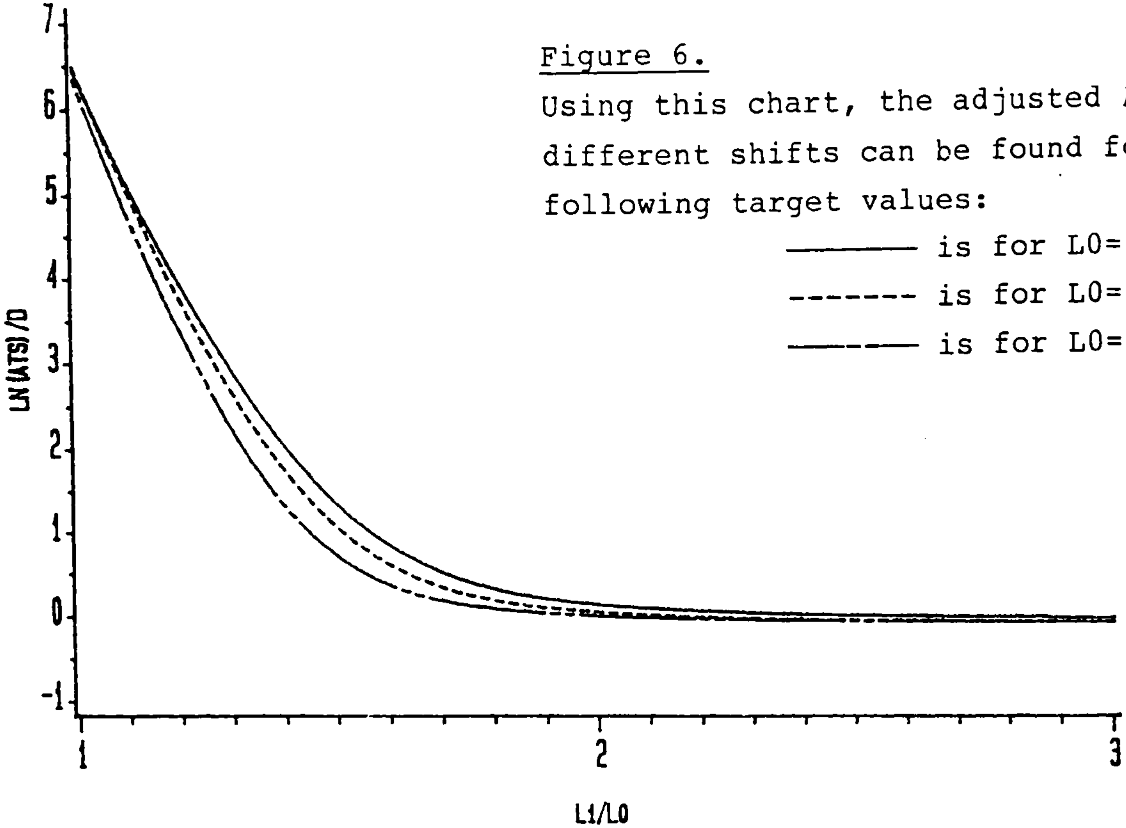


Figure 6.

Using this chart, the adjusted ATS for different shifts can be found for the following target values:

——— is for L0=10.00

------- is for L0=12.00

—— — — is for L0=15.00.

L0	10.00	12.00	15.00
D1	0.1/D	0.1/D	0.1/D
D2	2.1/D	2.0/D	2.0/D
G	9	11	14
H	21	24	28

4.3 Guidelines for the VSI cusum chart

The following table has been set-up so that the practictioner can design a VSI cusum chart. The procedure is as follows:

1) Choose an appropriate value of K (recall $K = (\lambda_1 - \lambda_0)/\ln(\lambda_1/\lambda_0)$) from the K's available in the tables.

2.) Compute L0/K and locate the corresponding column. L0/K is stated more generally in the tables as L/K.

3) Pick an H that results in an acceptable in-control ANSS. ANSS is computed by adding I1 + I2 for a corresponding L/K. That is I1 is the ANSS for the interval of the chart and I2 is the ANSS for the lower interval.

Note that K and h can be changed to attain an acceptable in control ANSS.

Once an h, K, and L0 are chosen, the VSI cusum chart can be designed using H as the upper control limit, K is as described in Chapter III, and G is the upper bound for the lower sampling interval. The ATS for the cusum chart is computed by

$$\text{VSI ATS} = I1 * d_1 + I2 * d_2 \tag{4.1}$$

where d_1 and d_2 are sampling intervals chosen by the designer. The ANSS for a shift L1 can be found by computing L1/K where K is the design parameter. Find the I1 and I2 that correspond to L1/D and the design h. I1+I2 is then the ANSS for a shift to L1.

TABLE 4 This table uses k = 0.25. For the given G, I1 is the ANSS for the upper interval and I2 is the ANSS for the lower interval. The numerical value of the upper bound for the lower interval I2 is N = G/4-0.25. Therefore, states 0-N are in the lower interval.

h		L/K 0.1	0.2	0.3	0.4	0.5	0.6	0.8	1.0	1.2	1.4	1.7	2.0	2.5	3.0	5.0
	G	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	I1	34.5	17.3	11.6	8.7	7.0	5.9	4.5	3.6	3.0	2.6	2.2	1.8	1.5	1.2	0.7
	I2	483.3	127.6	59.8	35.5	24.0	17.6	10.9	7.7	5.9	4.8	3.7	3.0	2.4	2.0	1.4
	G	1	1	1	1	1	1	1	2	2	2	3	3	3	3	3
2	I1	826.4	219.3	103.3	61.4	41.5	30.4	18.9	11.1	8.6	6.9	4.3	3.6	2.8	2.3	1.3
	I2	9993.2	1219.1	356.8	150.5	78.1	46.3	21.2	14.3	9.5	7.0	6.0	4.6	3.3	2.7	1.6
	G		1	1	1	1	1	2	3	4	4	4	4	4	4	4
3	I1	*	2828.6	819.9	346.7	182.2	110.2	45.4	23.2	12.7	9.9	7.3	5.8	4.2	3.3	1.7
	I2		15047.6	2573.1	724.7	272.1	124.4	45.6	25.3	18.4	12.5	8.3	6.2	4.4	3.5	2.1
	G				1	1	2	4	6	7	8	8	8	9	9	8
5	I1	*	*	*	9318.1	2556.1	794.4	141.7	52.7	29.2	18.7	13.0	9.9	6.5	5.0	2.8
	I2				18408.8	3396.1	991.5	190.5	67.1	34.2	22.8	13.9	10.0	7.5	5.7	3.0
	G					1	2	4	8	10	11	12	13	13	13	12
7	I1	*	*	*	*	32011.0	5581.6	471.3	104.9	48.7	30.3	18.7	13.0	9.1	7.0	3.9
	I2					41921.3	6633.5	487.0	118.2	51.2	31.0	19.6	14.9	10.2	7.7	3.9
	G							5	11	16	17	18	18	19	19	18
10	I1	*	*	*	*	*	*	1897.6	217.5	74.6	44.9	27.3	20.0	13.1	10.0	5.4
	I2							2046.8	220.7	83.1	46.3	28.2	20.0	14.2	10.8	5.4

TABLE 5 This table uses k = 0.5. For the given G, I1 is the ANSS for the upper interval and I2 is the ANSS for the lower interval. The numerical value of the upper bound for the lower interval I2 is N - G/2-0.5. Therefore, states 0-N are in the lower interval.

		L/K														
h		0.1	0.2	0.3	0.4	0.5	0.6	0.8	1.0	1.2	1.4	1.7	2.0	2.5	3.0	5.0
	G	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	I1	13.5	6.8	4.6	3.5	2.8	2.4	1.8	1.4	1.2	1.1	0.9	0.7	0.6	0.5	0.2
	I2	283.4	75.2	35.5	21.2	14.4	10.6	6.7	4.8	3.7	3.0	2.4	2.0	1.6	1.4	1.1
	G	1	1	1	1	1	1	1	1	1	1	3	3	1	1	1
2	I1	367.3	98.2	46.6	27.9	18.9	13.9	8.7	6.2	4.7	3.7	1.5	1.2	1.7	1.3	0.6
	I2	6534.0	789.9	229.7	96.4	49.8	29.5	13.5	7.7	5.2	3.8	4.1	3.2	1.7	1.4	1.1
	G		1	1	1	1	1	3	3	3	3	3	3	3	3	1
3	I1	*	1336.5	386.0	163.6	86.2	52.3	14.5	9.6	7.0	5.4	3.9	3.0	2.2	1.7	1.1
	I2		10257.8	1720.2	477.8	177.6	80.3	35.3	16.5	9.6	6.5	4.4	3.3	2.4	2.0	1.1
	G				1	1	1	3	5	7	7	7	7	7	7	7
5	I1	*	*	*	4436.3	1219.4	446.7	76.8	28.1	13.6	9.8	6.8	5.2	3.6	2.8	1.4
	I2				12216.5	2224.1	555.0	101.3	34.6	19.3	11.7	7.2	5.2	3.6	2.9	1.7
	G					1	1	3	7	9	11	11	13	13	11	11
7	I1	*	*	*	*	15250.8	3125.2	251.6	54.9	25.3	14.5	9.7	6.2	4.3	3.8	2.0
	I2					27402.3	3710.3	457.5	60.5	25.9	16.9	10.0	8.2	5.6	3.8	2.2
	G							3	6	8	9	9	9	10	9	9
10	I1	*	*	*	*	*	*	896.8	105.0	38.4	21.8	13.9	10.2	6.3	5.3	2.7
	I2							1187.5	119.4	41.8	24.6	14.3	10.2	7.6	5.3	2.9
	G							3	9	13	14	14	14	15	14	14
15	I1	*	*	*	*	*	*	8574.0	231.0	61.8	34.2	21.1	15.2	9.7	7.8	4.0
	I2							10359.3	255.1	67.9	37.2	21.5	15.2	11.0	9.8	4.2
	G								12	18	19	19	19	20	19	19
20	I1	*	*	*	*	*	*	*	405.9	86.2	46.8	28.2	20.2	13.0	10.3	5.2
	I2								441.7	93.5	49.7	28.6	20.2	14.3	10.3	5.4

TABLE 6 This table uses k = 1.0. For the given G, I1 is the ANSS for the upper interval and I2 is the ANSS for the lower interval. The numerical value of the upper bound for the lower interval I2 is N − G. Therefore, states 0-N are in the lower interval.

h		L/K 0.1	0.2	0.3	0.4	0.5	0.6	0.8	1.0	1.2	1.4	1.7	2.0	2.5	3.0	5.0
	G	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	I1	27.9	3.1	8.3	5.9	4.5	3.6	2.5	1.9	1.5	1.2	.9	.7	.5	.3	.1
	I2	5618.2	670.4	192.9	80.4	41.3	24.3	11.1	6.6	4.3	3.2	2.3	1.3	1.5	1.3	1.0
	G		1	1	1	1	1	1	1	1	1	1	1	1	1	1
3	I1	*	206.1	81.6	42.2	25.4	16.9	9.1	5.8	4.0	3.0	2.1	1.6	1.1	.8	.3
	I2		8967.3	1475.2	404.2	148.8	66.9	20.4	9.1	5.2	3.5	2.4	1.9	1.5	1.3	1.0
	G			1	1	1	1	1	2	2	2	2	2	2	2	1
5	I1	*	*	5303.4	1202.2	386.4	158.8	44.7	12.0	7.6	5.4	3.6	2.7	1.9	1.4	.7
	I2			91075.8	10330.4	1862.1	461.7	56.9	12.2	10.0	6.0	3.7	2.8	2.0	1.6	1.0
	G					1	1	1	2	3	3	3	4	3	3	3
7	I1	*	*	*	*	4865.0	1130.2	142.5	30.0	11.5	7.7	5.1	2.7	2.6	2.0	1.0
	I2					22900.5	3086.2	143.4	31.5	15.3	8.6	5.2	4.7	2.6	2.1	1.3
	G						1	2	3	4	5	5	5	5	5	4
10	I1	*	*	*	*	*	19528.0	393.5	56.0	20.2	10.2	6.5	4.7	3.3	2.5	1.5
	I2						52559.0	766.1	61.5	21.1	13.7	8.0	5.7	4.0	3.1	1.5
	G							2	5	7	7	7	7	7	7	7
15	I1	*	*	*	*	*	*	3774.0	110.0	29.6	17.6	10.8	7.7	5.3	4.0	2.0
	I2							6678.4	140.8	36.6	18.7	10.9	7.7	5.3	4.1	2.3
	G							2	6	9	10	10	10	10	10	9
20	I1	*	*	*	*	*	*	32611.4	210.0	44.2	22.6	13.6	9.7	6.6	5.0	2.7
	I2							56785.6	224.2	46.9	26.2	15.2	10.7	7.3	5.6	2.8

TABLE 7 This table uses k = 2.0. For the given G, I1 is the ANSS for the upper interval and I2 is the ANSS for the lower interval. The numerical value of the upper bound for the lower interval I2 is N − G. Therefore, states 0-N are in the lower interval.

		L/K														
h		0.1	0.2	0.3	0.4	0.5	0.6	0.8	1.0	1.2	1.4	1.7	2.0	2.5	3.0	5.0
	G	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	I1	19.1	9.0	5.7	4.0	3.0	2.4	1.6	1.1	.8	.6	.4	.3	.2	.1	0
	I2	17204.1	1195.9	258.7	89.5	40.0	21.2	8.4	4.5	2.9	2.2	1.6	1.3	1.2	1.1	1.0
	G		1	1	1	1	1	1	1	1	1	1	1	1	1	1
3	I1	*	116.5	48.4	25.4	15.2	10.0	5.2	3.2	2.2	1.6	1.0	.7	.4	.3	0
	I2		13545.8	1783.4	418.3	136.4	55.5	14.7	6.1	3.4	2.3	1.7	1.4	1.6	1.1	1.0
	G				1	1	1	1	2	2	2	2	2	1	1	1
5	I1	*	*	*	700.1	225.9	91.5	24.7	6.6	4.1	2.8	1.9	1.4	1.1	.7	.1
	I2				10403.2	1677.1	376.2	39.6	13.2	5.8	3.5	2.2	1.7	1.6	1.1	1.0
	G					1	1	1	2	3	3	3	3	3	2	1
7	I1	*	*	*	*	2848.2	648.2	77.7	16.1	6.1	4.1	2.6	1.9	1.3	1.1	.3
	I2					20688.9	2511.2	98.7	18.3	8.4	4.8	2.9	2.2	1.6	1.2	1.0
	G						1	1	3	4	5	5	5	5	4	1
10	I1	*	*	*	*	*	11097.1	355.5	29.6	10.5	5.3	3.3	2.4	1.7	1.5	.8
	I2						42423.4	369.0	34.1	11.3	7.3	4.3	3.2	2.3	1.5	1.0
	G							1	5	7	7	7	7	7	7	6
15	I1	*	*	*	*	*	*	3103.2	57.2	15.2	9.0	5.5	3.9	2.6	2.0	1.2
	I2							3202.0	75.3	19.0	9.8	5.8	4.2	2.9	2.3	1.2
	G							1	6	9	10	10	10	10	9	9
20	I1	*	*	*	*	*	*	26571.4	108.1	22.6	11.5	6.9	4.9	3.3	2.7	1.4
	I2							27128.0	118.2	24.2	13.6	7.9	5.9	3.9	2.8	1.6

TABLE 8 This table uses k = 3.0. For the given G, I1 is the ANSS for the upper interval and I2 is the ANSS for the lower interval. The numerical value of the upper bound for the lower interval I2 is N − G. Therefore, states 0-N are the lower interval.

h		L/K 0.1	0.2	0.3	0.4	0.5	0.6	0.8	1.0	1.2	1.4	1.7	2.0	2.5	3.0	5.0
	G	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	I1	15.7	7.5	4.7	3.3	2.4	1.9	1.2	.8	.6	.4	.3	.2	.1	0	0
	I2	62730.0	2470.6	399.7	114.8	45.2	21.8	7.6	3.8	2.4	1.8	1.4	1.2	1.1	1.0	1.0
	G		1	1	1	1	1	1	1	1	1	1	1	1	1	1
3	I1	*	82.8	35.6	19.1	11.5	7.6	3.9	2.3	1.5	1.0	.7	.4	.2	.1	0
	I2		24135.4	2486.3	501.1	146.4	54.7	12.8	5.0	2.8	1.9	1.4	1.2	1.1	1.0	1.0
	G				1	1	1	1	1	2	2	2	1	1	1	1
5	I1	*	*	*	490.5	163.8	66.8	17.7	7.3	2.8	1.9	1.2	1.1	.6	.3	0
	I2				11677.4	1704.6	362.7	33.8	7.4	4.3	2.6	1.7	1.2	1.1	1.0	1.0
	G					1	1	1	2	3	3	3	3	1	1	1
7	I1	*	*	*	*	2058.7	472.8	55.2	11.2	4.3	2.8	1.8	1.3	1.0	.7	.1
	I2					21423.1	2423.8	83.6	13.7	6.0	3.4	2.2	1.7	1.1	1.0	1.0
	G						1	1	3	4	5	5	5	4	3	1
10	I1	*	*	*	*	*	8129.7	236.9	20.5	7.2	3.6	2.3	1.6	1.3	1.1	.3
	I2						41153.5	311.3	24.6	8.0	5.1	3.1	2.3	1.5	1.1	1.0
	G							1	5	7	7	7	7	7	7	1
15	I1	*	*	*	*	*	*	2185.1	39.3	10.4	6.1	3.7	2.6	1.8	1.4	.8
	I2							2696.8	52.8	13.1	6.8	4.1	3.0	2.1	1.7	1.0
	G							1	6	9	10	10	9	9	9	8
20	I1	*	*	*	*	*	*	18663.8	73.7	15.3	7.8	4.6	3.6	2.4	1.8	1.1
	I2							22798.9	82.2	16.5	9.3	5.5	3.6	2.6	2.0	1.1

TABLE 9 **This table uses k = 5.0. For the given G, I1 is the ANSS for the upper interval I2 is the ANSS for the lower interval. The numerical value of the upper bound for the lower interval is N − G. Therefore, states 0-N are in the lower interval.**

h		L/K 0.1	0.2	0.3	0.4	0.5	0.6	0.8	1.0	1.2	1.4	1.7	2.0	2.5	3.0	5.0
	G		1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	I1	*	6.12	3.8	2.6	1.9	1.5	.9	.6	.4	.3	.1	.1	0	0	0
	I2		1195.8	1061.2	210.9	64.8	26.2	7.3	3.3	2.0	1.5	1.9	1.1	1.0	1.0	1.0
	G		1	1	1	1	1	1	1	1	1	1	1	1	1	1
	I2		56.2	24.9	13.6	8.3	5.5	2.7	1.5	.9	.6	.3	.2	.1	.0	.0
2	I1	*	95864.5	5657.6	821.6	193.0	61.5	11.6	4.0	2.2	1.5	1.2	1.1	1.0	1.0	1.0
	G				1	1	1	1	1	2	1	1	1	1	1	1
5	I1	*	*	*	289.7	105.5	44.7	11.8	4.7	1.8	1.5	.8	.5	.2	.1	0
	I2				15907.4	2068.1	383.7	29.3	5.7	3.1	1.6	1.2	1.1	1.0	1.0	1.0
	G					1	1	1	2	3	3	2	1	1	1	1
7	I1	*	*	*	*	1288.7	314.6	36.4	7.2	2.7	1.7	1.3	.9	.4	.2	.0
	I2					24713.1	2551.0	71.9	9.7	4.1	2.4	1.4	1.1	1.0	1.0	1.0
	G						1	1	3	4	4	4	4	1	1	1
10	I1	*	*	*	*	*	5447.1	155.0	13.0	4.5	2.7	1.7	1.2	.9	.5	.0
	I2						43628.6	266.6	16.7	5.2	2.8	1.8	1.4	1.0	1.0	1.0
	G							1	4	7	7	7	7	6	4	1
15	I1	*	*	*	*	*	*	1424.9	29.4	6.4	3.7	2.2	1.6	1.2	1.0	.1
	I2							2308.1	29.7	8.3	4.5	2.7	2.0	1.3	1.0	1.0
	G							1	6	9	9	9	9	9	8	1
20	I1	*	*	*	*	*	*	12245.1	45.9	9.4	5.2	3.1	2.2	1.5	1.2	1.5
	I2							19641.2	52.7	10.3	5.3	3.2	2.4	1.7	1.3	1.0

TABLE 10 This table uses k - 7.0. For the given G, I1 is the ANSS for the upper interval and I2 is the ANSS for the lower interval. The numerical value of the upper bound for the lower interval I2 is N - G. Therefore, states 0-N are in the lower interval.

h		L/K 0.1	0.2	0.3	0.4	0.5	0.6	0.8	1.0	1.2	1.4	1.7	2.0	2.5	3.0	5.0
	G		1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	I1	*	5.5	3.4	2.3	1.7	1.3	.8	.5	.3	.2	.1	0	0	0	0
	I2		60520.8	2947.0	403.2	96.7	32.9	7.4	3.0	1.8	1.3	1.1	1.0	1.0	1.0	1.0
	G			1	1	1	1	1	1	1	1	1	1	1	1	1
3	I1	*	*	20.3	11.2	6.9	4.5	2.7	1.2	.7	.4	.2	.1	0	0	0
	I2			14209.3	1450.1	271.4	73.9	11.4	3.6	1.9	1.4	1.1	1.0	1.0	1.0	1.0
	G				1	1	1	1	1	1	1	1	1	1	1	1
5	I1	*	*	*	200.2	77.6	34.1	9.1	3.5	1.8	1.0	.5	.3	.1	0	0
	I2				23617.4	2602.6	431.1	27.9	5.0	2.1	1.4	1.1	1.0	1.0	1.0	1.0
	G					1	1	1	2	2	2	1	1	1	1	1
7	I1	*	*	*	*	901.9	234.4	27.8	5.4	2.5	1.5	.9	.5	.2	0	0
	I2					29580.4	2797.0	67.7	8.0	2.7	18.6	1.1	1.0	1.0	1.0	1.0
	G						1	1	3	4	4	4	1	1	1	1
10	I1	*	*	*	*	*	3967.8	118.0	9.8	3.3	2.0	1.2	1.0	.4	.2	0
	I2						46755.7	250.4	13.2	4.0	2.4	1.5	1.0	1.0	1.0	1.0
	G							1	4	6	7	6	6	1	1	1
15	I1	*	*	*	*	*	*	1082.9	21.8	5.4	2.7	3.7	1.3	1.0	.6	0
	I2							2165.9	22.8	5.5	3.3	3.7	1.4	1.0	1.0	1.0
	G							1	6	9	9	9	9	8	1	1
20	I1	*	*	*	*	*	*	9306.4	33.7	6.8	3.8	2.2	1.6	1.2	1.0	.1
	I2							18437.1	39.7	7.6	4.0	2.5	1.8	1.3	1.0	1.0

TABLE 11 This table uses k = 10.0. For the given G, I1 is the ANSS for the upper interval and I2 is the ANSS for the lower interval. The numerical value of the upper bound for the lower interval I2 is N − G. Therefore, states 0-N are in the lower interval.

		L/K														
h		0.1	0.2	0.3	0.4	0.5	0.6	0.8	1.0	1.2	1.4	1.7	2.0	2.5	3.0	5.0
	G			1	1	1	1	1	1	1	1	1	1	1	1	1
2	I1	*	*	3.1	2.1	1.5	1.11	.63	.34	.2	.1	0.0	0	0	0	0
	I2			13955.31	1085.5	179.4	47.4	7.9	2.8	1.6	1.2	1.1	1.0	1.0	1.0	1.0
	G			1	1	1	1	1	1	1	1	1	1	1	1	1
3	I1	*	*	16.9	9.3	5.7	3.7	1.8	.9	.5	.3	.1	0	0	0	0
	I2			61023.9	3599.0	472.8	101.6	11.8	3.3	1.8	1.2	1.1	1.0	1.0	1.0	1.0
	G				1	1	1	1	1	1	1	1	1	1	1	1
5	I1	*	*	*	135.6	55.9	25.7	7.0	2.6	1.2	.7	.3	.1	0.0	0	0
	I2				47332.3	3915.7	540.0	27.5	4.3	1.8	1.3	1.1	1.1	1.0	1.0	1.0
	G					1	1	1	1	2	1	1	1	1	1	1
7	I1	*	*	*	*	2830.9	88.9	7.2	2.4	1.4	1.0	.5	.1	0	0	0
	I2					55437.9	243.2	10.4	3.1	1.8	1.0	1.0	1.0	1.0	1.0	1.0
	G							1	4	6	6	6	3	1	1	1
10	I1	*	*	*	*	*	*	815.2	15.9	3.9	2.2	1.3	1.0	.5	.2	0
	I2							2105.2	17.5	4.1	2.2	1.4	1.0	1.0	1.0	1.0
	G							1	6	9	9	9	8	1	1	1
15	I1	*	*	*	*	*	*	7081.6	24.5	4.5	2.7	1.6	1.2	.9	.5	0
	I2							18121.8	29.6	5.6	3.0	1.9	1.3	1.0	1.0	1.0
	G							1	6	9	9	9	9	8	1	1
20	I1	*	*	*	*	*	*	9306.4	33.7	6.8	3.8	2.2	1.6	1.2	1.0	.1
	I2							18437.1	39.7	7.6	4.0	2.5	1.8	1.3	1.0	1.0

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