THE REFLECTION METHOD IN THE BENDING OF BEAMS AND PLATES

by

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a	Arbitrary Length
q	Intensity of a continuously distributed load
r, θ	Polar coordinates
t	Thickness of a plate
w	Displacement component in the z direction
$\bar{w}_i$	Deflection induced at the i^{th} point on the boundary by the applied loads
x, y, z	Rectangular coordinates
D	Flexural rigidity of a plate
$\bar{D}$	Retracted distance
E	Modulus of elasticity
I	Rectangular moment of inertia
M	Bending moment per unit length
M_r, M_θ, $M_{r\theta}$	Radial, transverse, and twisting moments when using polar coordinates
M_N, M_{Nt}	Bending and twisting moments per unit length of a section of a plate perpendicular to the n direction
$\bar{M}_N$	Normal moment induced at the i^{th} point on the boundary by the applied loads
N	Number of points on the boundary where the boundary conditions are satisfied
P	Single load
Q	Concentrated equivalent force
Q_r, Q_θ	Radial and transverse shearing forces

Q_n	Shearing force parallel to z axis per unit length of a section of a plate perpendicular to n direction
S_i	Normal slope induced at the i^{th} point on the boundary by all equivalent loads
$\bar{S}_i$	Normal slope induced at the i^{th} point by the applied loads
$\bar{V}_N$	Kirchhoff shear induced at the i^{th} point by the applied loads
α	Angle measured counterclockwise from the inward normal with respect to the boundary
μ	Poisson's ratio
$(\ )'$	Differentiation with respect to x

I. INTRODUCTION

The problem of determining the deflection and stress in a plate under transverse loading can be approached by first considering the plate to be a portion of an infinite plate, ignoring the prescribed boundary conditions. The boundary of the desired plate is then prescribed in the infinite plate with reference to any arbitrary origin. Once the boundary of the plate has been defined, the desired loadings can be placed on the infinite plate at their proper places within this boundary. In any given problem, two conditions are specified at every point on the boundary of the plate in question. These may involve the deflection, the normal slope, the bending moment, and the Kirchhoff shear. If the distributions of transverse displacement, bending and twisting moments, and shears due to the applied loads are known for an infinite plate, then those quantities usually involved in the boundary conditions can be calculated at every point on the prescribed boundary. In order that some point on the boundary satisfy the conditions specified in the given problem, a concentrated force and a concentrated moment will be applied at some arbitrary point in the infinite plate and with magnitudes such that the point on the boundary is forced to conform to the initial boundary conditions. Then, for every point on the boundary which is forced to conform to the specified boundary conditions, a pair of the above loads, called equivalent loads, must be applied at some point in the infinite plate. Thus, an infinite number of pairs of equivalent loads must be applied in order to achieve an exact solution. However, an approximate solution can be obtained if only a finite number of points are used.

A solution for a concentrated force and concentrated moment on an infinite plate can be found in [1][1]. From these solutions, it is easily shown that the slope due to a concentrated moment and the moment due to a concentrated force become infinite at the pcint of application. Therefore, the equivalent loads must be applied at some finite distance from the boundary. This distance is called the retracted distance. Normally, the equivalent loads would be applied at points on a curve which is called the retracted boundary.

Figure 1 shows the desired boundary of a finite plate in an infinite plate. N points on the boundary are to be forced to conform to the specified boundary conditions. Therefore, N pairs of equivalent loads are to be applied at N points on the retracted boundary. The magnitudes of the equivalent loads are determined by specifying that the combined effects of all of the equivalent loads and applied

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[1][ ] refers to references on page

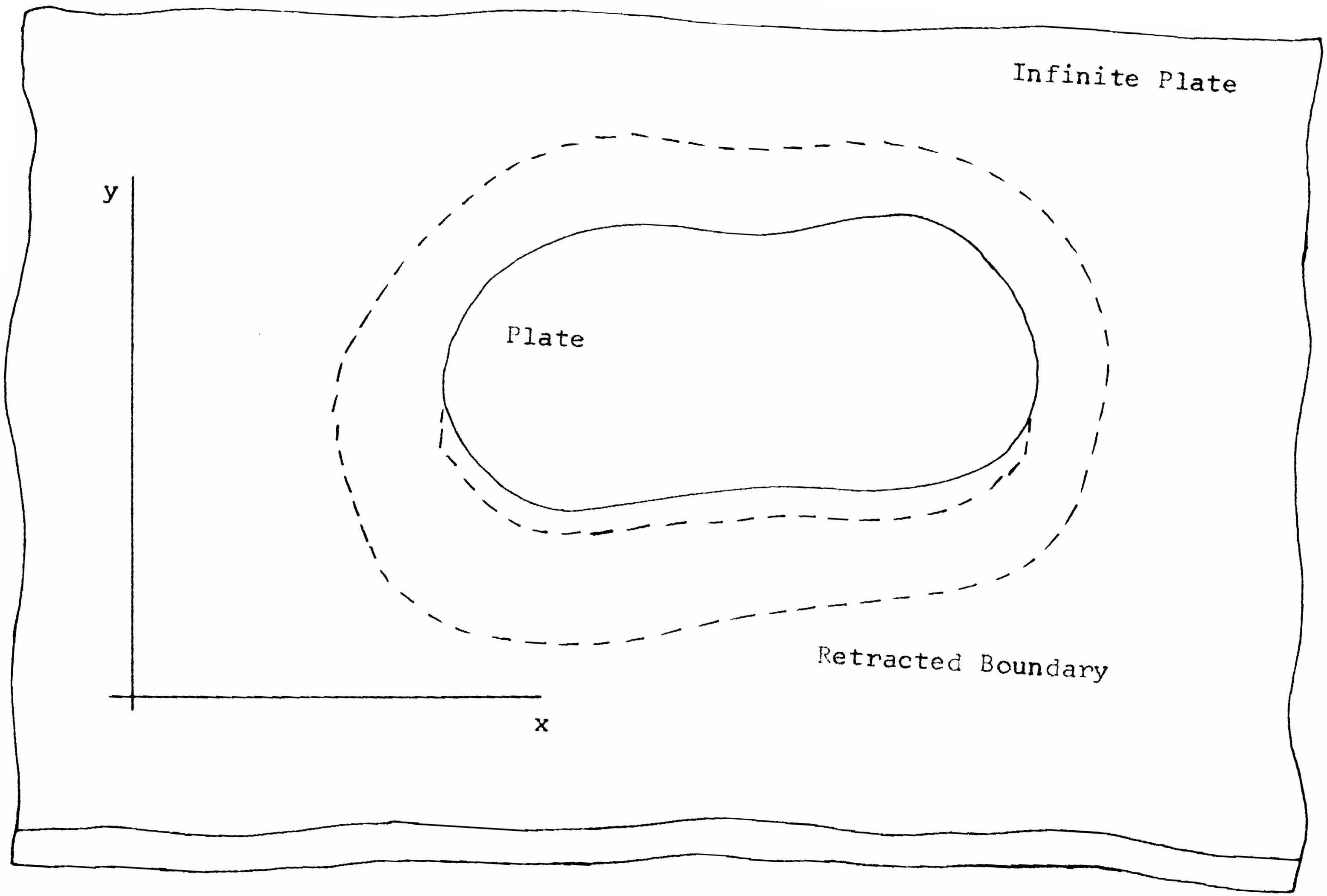


Figure 1. Finite Plate as Part of an Infinite Plate

loads be such that the boundary conditions are satisfied at each point. Since there are two conditions for each point, this leads to a system of 2N simultaneous equations for the 2N equivalent loads. If N is chosen sufficiently large, then the values of the transverse deflection, normal slope, bending moment and Kirchhoff shear existing on that portion of the boundary between the N points will deviate only slightly from the required boundary conditions and the solution will approximate on exact solution. The procedure will first be illustrated for beams and then extended for plates.

II. REVIEW OF THE LITERATURE

Massonet [2] developed an approach of this type for the solution of three dimensional elasticity problems. In this formulation, the elastic body is embedded in an infinite elastic space and the stresses are specified over the entire boundary.

A discussion of the reflection method for the case of plane elasticity problems is presented in a paper by Dr. R. Chicurel and Prof. E. W. Suppiger [3]. In this paper, the authors also indicate how this method may be applied to the problem of transverse loading of finite plates.

Avzas [4], using an iterative procedure, has applied the plane elasticity solution outlined in [3] to a number of problems with known solutions.

Recently, a method for satisfying the boundary conditions for only a finite number of points on the boundary using the governing differential equation has been employed by Conway [5]. A comparison of the results obtained by Conway to the results obtained using the reflection method will be discussed in Chapter IV, paragraph d, for the problem of a uniformly loaded, simply supported triangular plate.

III. APPLICATION OF THE REFLECTION METHOD TO BEAMS

1. Assumptions and Basic Equations

Consider a beam of infinite length for which the following assumptions are made.

a. The beam is perfectly straight.

b. The material from which the beam is fabricated is perfectly elastic and obeys Hooke's law under the applied loads.

c. Plane sections initially perpendicular to the axis of the beam remain plane in a bent beam.

d. Only deformations due to bending are considered.

e. Small deflections occur in the region of interest.

Next, consider a section of the infinite beam loaded as shown in Figure 2 by one concentrated force P.

For $x > 0$, the shear is equal to P/2. Thus,

$$EIy''' = \frac{P}{2} .$$

Also,

$$EIy'' = \frac{Px}{2} + C_1 ,$$

$$EIy' = \frac{Px^2}{4} + C_1x + C_2 ,$$

and

$$EIy = \frac{Px^3}{12} + \frac{C_1x^2}{2} + C_2x + C_3 .$$

For $x < 0$, the shear is equal to -P/2. Then

$$EIy''' = \frac{-P}{2} ,$$

$$EIy'' = \frac{-Px}{2} + C_4 \quad ,$$

$$EIy' = \frac{-Px^2}{4} + C_4 x + C_5 \quad ,$$

and

$$EIy = \frac{-Px^3}{12} + \frac{C_4 x^2}{2} + C_5 x + C_6 \quad .$$

Due to symmetry, the slope is zero at the origin. Therefore,

$$C_2 = C_5 = 0 \quad .$$

Also at the origin, the deflection and moment must be continuous. Thus,

$$C_1 = C_4 \qquad \text{and} \qquad C_3 = C_6 \quad .$$

The expressions for the deflection of an infinite beam loaded with a concentrated moment can be obtained by superposing the results obtained for two concentrated loads at a vanishingly small separation. From Figure 3, the deflection at any point $x > 0$ is given by the expression

$$EIy = \frac{Px^3}{12} + \frac{C_1 x^2}{2} + C_3 - [\frac{P}{12}(x-e)^3 + \frac{C_1}{2}(x-e)^2 + C_3] \quad ,$$

or

$$EIy = \frac{P}{12}[x^3 - (x-e)^3] + \frac{C_1}{2}[x^2 - (x-e)^2] \quad .$$

Now let e approach zero and Pe approach M, the value of the concentrated moment. Thus we get

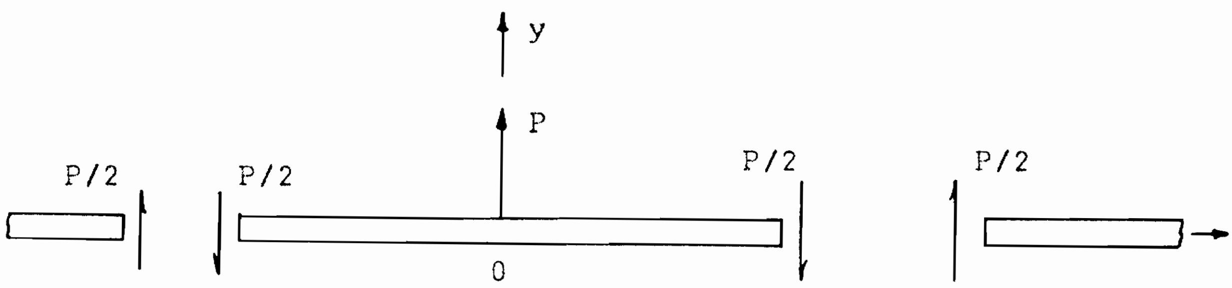


Figure 2. Shear in an Infinite Beam. (Bending moments not shown.)

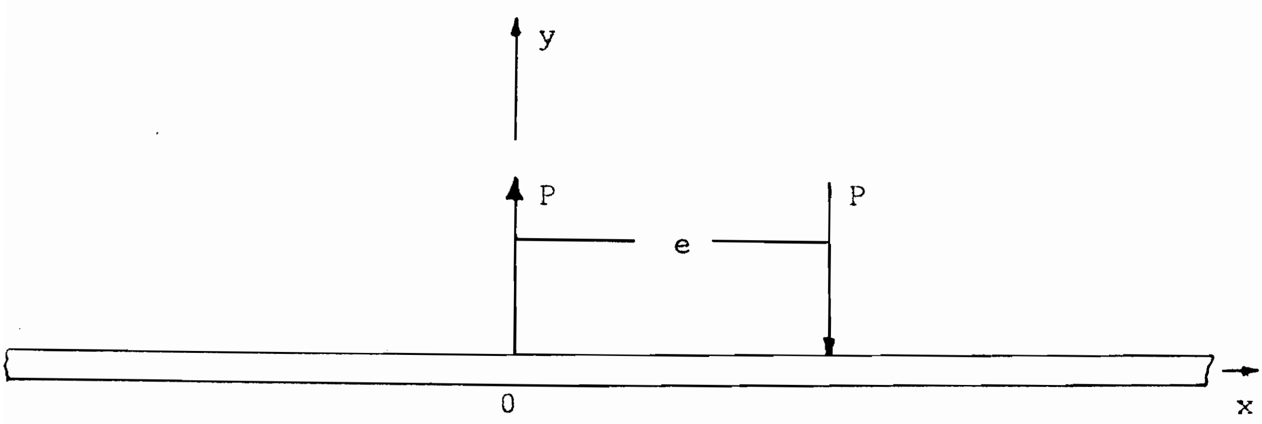


Figure 3. Two forces on an Infinite Beam.

$$EIy = \lim_{e \to 0} \left[\frac{Pe}{12} \frac{(x^3 - (x-e)^3)}{e}\right] + \lim_{e \to 0} \left[\frac{C_1}{2} (x^2 - (x-e)^2)\right]$$

$$= \frac{Mx^2}{4} \quad .$$

For $x < 0$, $EIy = \frac{-Mx^2}{4}$.

The expressions for the slope, moment and shear can easily be obtained from the above by differentiation.

2. Simultaneous Solution for Equivalent Loads

Let us consider an as application of the reflection method the problem of determining the expression for the deflection of a simply supported beam loaded with a concentrated force at the center of the beam. Let the length of the beam be 2L. Then, two pairs of equivalent loads must be applied somewhere outside the interval 2L such that the boundary conditions are satisfied at each end of the beam. The retracted distance may be infinitesimal in this case since all beam functions are finite at the point of application. The locations of the equivalent loads are shown in Figure 4. The boundary conditions are

$$y_A = y_B = 0 \quad ,$$

and

$$y''_A = y''_B = 0 \quad .$$

From symmetry,

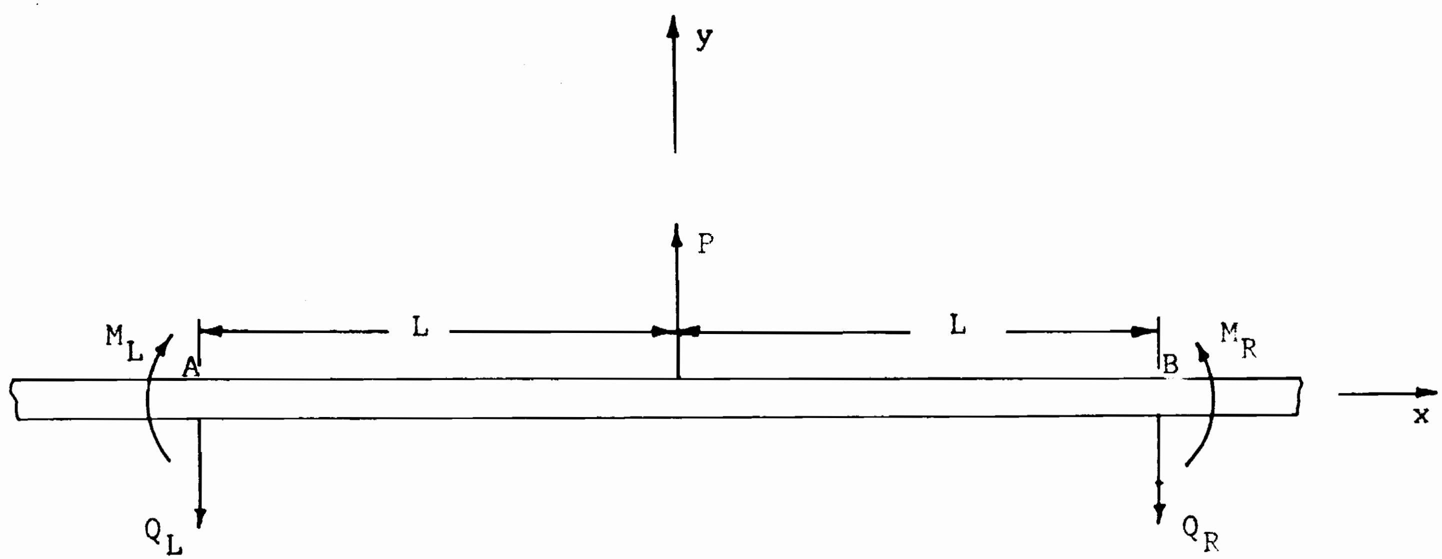


Figure 4. Equivalent loads on the infinite beam.

$$Q = Q_L = Q_R \quad ,$$

and ------------- (1)

$$M = M_L = M_R \quad .$$

The deflections at A due to the various loads are listed below.

$$P \quad : \quad EIy_A = \frac{-P(-L)^3}{12} + \frac{C_4(-L)^2}{2} + C_6 \quad .$$

$$Q_L \quad : \quad EIy_A = -C_3 \quad .$$

$$Q_R \quad : \quad EIy_A = -[\frac{-Q_R(2L)^3}{12} + \frac{C_4}{2}(-2L)^2 + C_6] \quad .$$

$$M_L \quad : \quad EIy_A = 0 \quad .$$

$$M_R \quad : \quad EIy_A = -[\frac{-M_R}{4}(-2L)^2] \quad .$$

Since $y_A = 0$, we obtain, using equation (1), the following equation for Q and M.

$$\frac{PL^3}{12} - \frac{3}{2}C_4L^2 - 8\frac{QL^3}{12} - C_6 + ML^2 = 0 \quad . \qquad \text{--------} \quad (2)$$

The moments at A due to the various loads are now listed.

$$P \quad : \quad EIy''_A = \frac{-P}{2}(-L) + C_4 \quad .$$

$$Q_L \quad : \quad EIy''_A = -C_1 \quad .$$

$$Q_R \quad : \quad EIy''_A = -[\frac{-Q_R(-2L)}{2} + C_4] \quad .$$

$$M_L \quad : \quad EIy''_A = \frac{M}{2}L$$

$$M_R \quad : \quad EIy''_A = -[\frac{-M_R}{2}] \quad .$$

From the condition that the moment at A is zero and the use of equation (1), we obtain

$$\frac{PL}{2} - Q - C_4 + M = 0 \quad . \qquad (3)$$

Q and M are obtained by solving equations (2) and (3) simultaneously.

$$Q = \frac{5}{4}P + \frac{3}{2}\frac{C_4}{L} + \frac{3C_6}{L^3} \quad , \qquad (4)$$

and

$$M = \frac{3}{4}PL + \frac{3C_6}{L^2} + \frac{5}{2}C_4 \quad . \qquad (5)$$

In order to obtain an expression for the deflection, consider the effect of all the loads on a point taken an x distance from the left end of the beam shown in Figure 4. Then for $0 < x < L$,

$$EIy = \frac{-P(L-x)^3}{12} + \frac{C_4}{2}(x-L)^2 + C_6 - [\frac{Qx^3}{12} + \frac{C_1x^2}{2} + C_3]$$

$$-[\frac{-Q(x-2L)^3}{12} + \frac{C_4(x-2L)^2}{2} + C_6] + \frac{Mx^2}{4} - [\frac{-M}{4}(x-2L)] \quad ,$$

or

$$EIy = \frac{Px}{12}(3L^2 - x^2) \quad ,$$

which agrees with the well known result.

Notice that the constants C_1, C_3, C_4, and C_6 are arbitrary and do not appear in the solution. Therefore, they could have been assumed zero and the solution greatly simplified.

3. Iterative Procedure

Instead of using simultaneous equations to solve for the equivalent loads in the simple beam problem, an iterative procedure can be employed instead. First, the deflection and moment at points A and B in Figure 4 due to the applied load P will be calculated.

$$\text{Deflection at } A = \frac{PL^3}{12EI} \;,$$

$$\text{Moment at } A = \frac{PL}{2} \;,$$

where

$$C_1 = C_3 = C_4 = C_6 = 0 . \qquad (6)$$

Next, a pair of balancing loads Q_1 and M_1 will be applied at point B to force the boundary conditions at A to be satisfied. (See Figure 5.) Notice that, due to the assumption of equation (6), the balancing loads cannot be applied at the point where you wish to balance the effect of the applied load. Thus, since the deflection and moment at A must be zero, we obtain the following two equations.

$$\frac{PL^3}{12} - [-Q_1 \frac{(-2L)^3}{12}] - [\frac{-M_1(-2L)^2}{4}] = 0 \;, \qquad (7)$$

and

$$\frac{PL}{2} - [-Q_1 \frac{(-2L)}{2}] - [\frac{-M_1}{2}] = 0 \;. \qquad (8)$$

By a simultaneous solution of equations (7) and (8) we obtain

$$Q_1 = \frac{11}{16}P \quad , \qquad \text{-----------------------} \quad (9)$$

and

$$M_1 = \frac{3}{8}PL \quad . \qquad \text{-----------------------} \quad (10)$$

However, at point B,

$$\text{Deflection at B} = \frac{PL^3}{12EI} \quad ,$$

and

$$\text{moment at B} = \frac{PL}{2} + \frac{M_1}{2} \quad .$$

Applying the balancing loads as shown in Figure 6 and using the condition that the deflection and moment at B must be zero, we obtain the following equations.

$$\frac{PL^3}{12} - \frac{8Q_2L^3}{12} + \frac{M_2(2L)^2}{4} = 0 \quad , \qquad \text{----------------} \quad (11)$$

and

$$\frac{PL}{2} + \frac{M_1}{2} + \frac{M_2}{2} - \frac{Q_2}{2}(2L) = 0 \quad . \qquad \text{----------------} \quad (12)$$

Solving equations (11) and (12) simultaneously, we obtain

$$Q_2 = \frac{31}{32}P \quad , \qquad \text{-----------------------} \quad (13)$$

and

$$M_2 = \frac{9}{16}PL \quad . \qquad \text{-----------------------} \quad (14)$$

Repeating the procedure two more times for each end

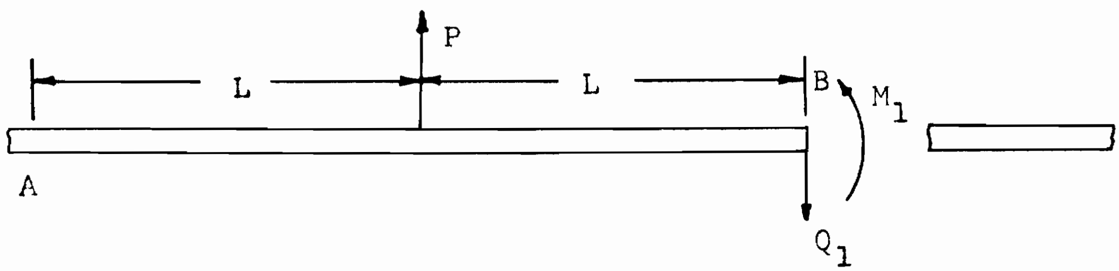


Figure 5. First Balancing Loads on the Infinite Beam.

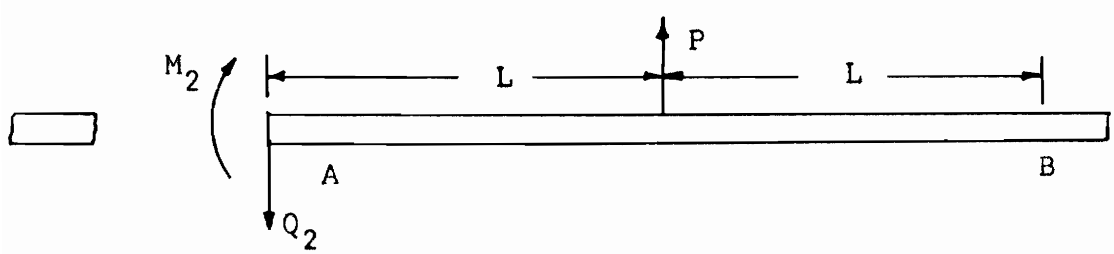


Figure 6. Second Balancing Loads on the Infinite Beam.

results in the following expressions for the balancing loads at points A and B.

$$Q_A = \frac{31}{32}P + \frac{27}{128}F + \frac{27}{16(32)}P \quad ,$$

$$= 1.23P \quad . \qquad \text{-------------------------} \qquad (15)$$

$$M_A = \frac{9}{16}PL + \frac{27}{192}PL + \frac{27}{24(32)}PL \quad ,$$

$$= .73PL \quad . \qquad \text{-------------------------} \qquad (16)$$

$$Q_B = 1.21P \quad . \qquad \text{-------------------------} \qquad (17)$$

$$M_B = .726PL \quad . \qquad \text{-------------------------} \qquad (18)$$

From equations (4) and (5) and using equation (6), we have

$$Q = 1.25P \quad , \qquad \text{-------------------------} \qquad (19)$$

and

$$M = .75PL \quad . \qquad \text{-------------------------} \qquad (20)$$

Comparing the values of Q given by equations (15) and (17) to the exact value given by equation (19), we see that the iterative procedure used here quickly converges on the exact value. A similar observation can be made for the moment by comparing equations (16) and (18) to equation (20).

IV. APPLICATION OF THE REFLECTION METHOD TO PLATES

1. Assumptions and Basic Equations

Consider a plate of infinite length and width for which we make the following assumptions.

a. The deflection of the middle plane is small compared to the thickness of the plate.

b. The normals of the middle plane before bending are deformed into the normals of the middle plane after bending.

c. The stress σ_z, where the z axis is normal to the middle plane, is small compared with the other stress components.

d. The middle plane remains unstrained after bending.

e. The material from which the plate is fashioned is perfectly elastic and obeys Hooke's law under the applied loads.

From equation 206 of [1], the deflection of an infinite plate due to a single concentrated force P is

$$w = \frac{P}{8\pi D} r^2 \log \frac{r}{a} \quad , \qquad (21)$$

where r is the radial distance from the force and a is an arbitrary length.

Also from [1], the deflection of an infinite plate due to a concentrated moment M is

$$w = \frac{M}{4\pi D} r \log \frac{r}{a} \cos \theta \quad , \qquad (22)$$

where r and θ are polar coordinates centered at the point of application of the concentrated moment which is oriented with respect to the coordinate axes as shown in Figure 7.

Finally, from equation 60 of [1], the deflection of a uniformly loaded infinite plate is

$$w = \frac{qr^4}{64D} + \frac{C_1 r^2}{4} + C_2 \log \frac{r}{a} + C_3 \quad , \qquad (23)$$

where r is the radial distance from some arbitrary point in the plate and q is the intensity of the load. C_1, C_2 and C_3 are arbitrary constants and can be taken as zero.

Thus equation (23) becomes

$$w = \frac{qr^4}{64D} \quad . \qquad (24)$$

In order to calculate the effect of an applied load on any point along some arbitrary boundary in the infinite plate, consider the point to be located at a distance r from an origin taken at the load and the normal to the boundary inclined at an angle α to r as shown in Figure 8.

The normal moment is given by the expression

$$M_N = M_r \cos^2\alpha + M_\theta \sin^2\alpha - 2M_{r\theta} \sin\alpha\cos\alpha \quad . \qquad (25)$$

Also,

$$M_{Nt} = \frac{1}{2}(M_r - M_\theta) \sin 2\alpha + M_{r\theta} \cos 2\alpha \quad . \qquad (26)$$

The slope normal to the boundary shown in Figure 8 and expressed in polar coordinates is

$$\frac{\partial w}{\partial n} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial n} + \frac{\partial w}{\partial \theta}\frac{\partial \theta}{\partial n} \quad .$$

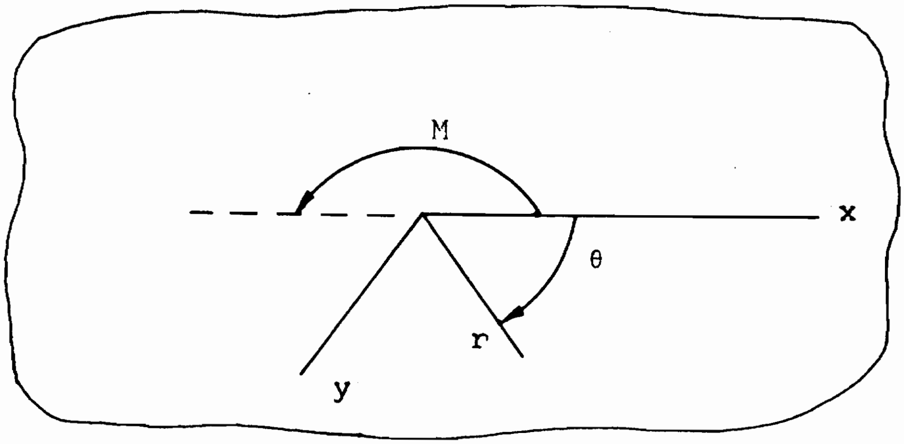


Figure 7. Concentrated Moment on an Infinite Plate.

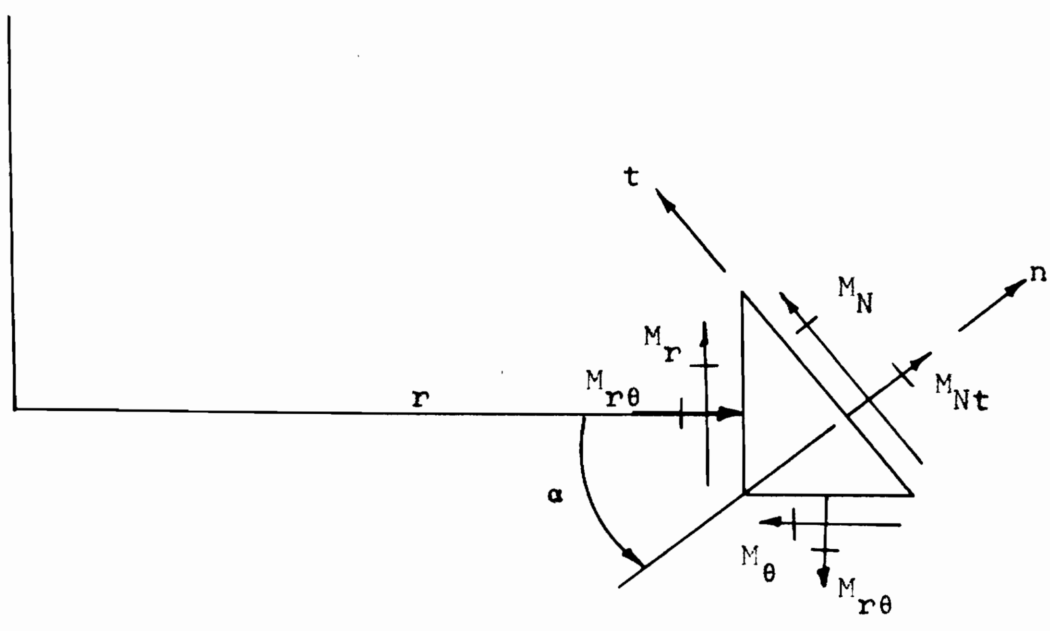


Figure 8. Element Cut from an Infinite Plate.

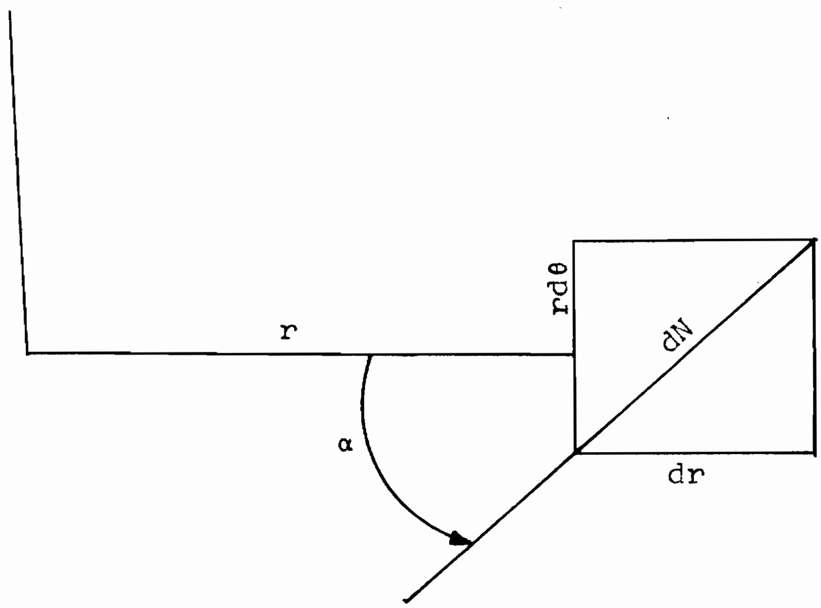


Figure 9. Differential Relationships on the Boundary.

From Figure 9,

$$\frac{dr}{dn} = \cos\alpha \qquad \text{and} \qquad \frac{d\theta}{dN} = \frac{1}{r}\sin\alpha \ .$$

Therefore, the equation for the normal slope becomes

$$\frac{\partial w}{\partial n} = \frac{\partial w}{\partial r}\cos\alpha + \frac{\partial w}{\partial \theta}\frac{\sin\alpha}{r} \ . \qquad \text{------------------} \quad (27)$$

From [1], the Kirchhoff shear at the point on the boundary shown in Figure 8 is

$$V_N = Q_N - \frac{\partial M_{Nt}}{\partial t} \ .$$

But

$$Q_N = Q_r \cos\alpha + Q_\theta \sin\alpha \ .$$

Therefore,

$$V_N = Q_r \cos\alpha + Q_r \sin\alpha + \frac{\partial M_{Nt}}{\partial r}\sin\alpha - \frac{\partial M_{Nt}}{\partial \theta}\frac{\cos\alpha}{r} \ .$$

-------- (28)

Also from [1], the following relations, given in polar coordinates, can be obtained.

$$M_r = -D\left[\frac{\partial^2 w}{\partial r^2} + \mu\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\right] \ .$$

$$M_\theta = -D\left[\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \mu\frac{\partial^2 w}{\partial r^2}\right] \ .$$

----- (29)

$$M_{r\theta} = (1-\mu)\, D\left(\frac{1}{r}\frac{\partial^2 w}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial w}{\partial \theta}\right) \ .$$

$$Q_r = -D\frac{\partial}{\partial r}(\nabla^2 w) \ .$$

$$Q_\theta = -D\frac{1}{r}\frac{\partial}{\partial \theta}(\nabla^2 w) \ .$$

Now, let us consider the effect on an infinite plate of each of the previously discussed applied loads.

i) Concentrated Load

The deflection due to a concentrated load is given by equation (21).

$$w = \frac{P}{8\pi D} r^2 \log \frac{r}{a} \quad .$$

Therefore,

$$\frac{\partial w}{\partial r} = \frac{P}{8\pi D} (2r \log \frac{r}{a} + r) \quad ,$$

$$\frac{\partial^2 w}{\partial r^2} = \frac{P}{8\pi D} (2\text{Log} \frac{r}{a} + 3) \quad ,$$

and

$$\frac{\partial^3 w}{\partial r^3} = \frac{P}{4\pi D r} \quad .$$

Substituting these expressions into equations (29), we obtain

$$M_r = \frac{-P}{8\pi} [2\log \frac{r}{a}(1+\mu) + 3 + \mu] \quad ,$$

$$M_\theta = \frac{-P}{8\pi} [2\log \frac{r}{a}(1+\mu) + 1 + 3\mu] \quad ,$$

$$M_{r\theta} = 0 \quad ,$$

$$Q_r = \frac{-P}{2\pi r} \quad ,$$

and

$$Q_\theta = 0 \quad .$$

Substitution of these values into equations (25), (26),

(27) and (28) results in the following expressions for the slope, bending moments, and Kirchhoff shear in an infinite plate due to a concentrated force P centered at the origin of our coordinate system.

$$\frac{\partial w}{\partial N} = \frac{P}{8\pi D}\left[2r \log \frac{r}{a} + r\right] \cos\alpha \quad . \qquad (30)$$

$$M_N = \frac{-P}{8\pi}\left[2(1+\mu)\log \frac{r}{a} + (3+\mu)\cos^2\alpha + (1+3\mu)\sin^2\alpha\right] \quad . \qquad (31)$$

$$M_{Nt} = \frac{-P}{8\pi}(1+\mu)\sin^2\alpha \quad . \qquad (32)$$

$$V_N = \frac{-P}{2\pi r}\cos\alpha \quad . \qquad (33)$$

ii) Concentrated Moment

The deflection due to a concentrated moment is given by equation (22)

$$w = \frac{M}{4\pi D} r \log \frac{r}{a} \cos\theta \quad .$$

Therefore,

$$\frac{\partial w}{\partial r} = \frac{M}{4\pi D}\left[\log \frac{r}{a} + 1\right]\cos\theta \quad , \qquad \frac{\partial^2 w}{\partial r^2} = \frac{M \cos\theta}{4\pi D r} \quad ,$$

$$\frac{\partial^3 w}{\partial r^3} = \frac{-M \cos\theta}{4\pi D r^2} \quad , \qquad \frac{\partial w}{\partial \theta} = \frac{-Mr}{4\pi D}\log \frac{r}{a}\sin\theta \quad ,$$

$$\frac{\partial^2 w}{\partial \theta^2} = \frac{-Mr}{4\pi D}\log \frac{r}{a}\cos\theta \quad , \qquad \frac{\partial^3 w}{\partial \theta^3} = \frac{Mr}{4\pi D}\log \frac{r}{a}\sin\theta \quad ,$$

$$\frac{\partial^2 w}{\partial r \partial \theta} = \frac{-M}{4\pi D}\left[\log \frac{r}{a} + 1\right]\sin\theta \quad ,$$

$$\frac{\partial^3 w}{\partial r \partial \theta^2} = \frac{-M}{4\pi D} [\log \frac{r}{a} + 1] \cos\theta \quad ,$$

$$\frac{\partial^3 w}{\partial r^2 \partial \theta} = \frac{-M \sin\theta}{4\pi D r} \quad .$$

The above expressions are then substituted into equations (29) and the results tabulated below.

$$M_r = \frac{-M(1+\mu) \cos\theta}{4\pi r} \quad .$$

$$M_\theta = \frac{-M(1+\mu) \cos\theta}{4\pi r} \quad .$$

$$M_{r\theta} = \frac{-M(1-\mu) \cos\theta}{4\pi r} \quad .$$

$$Q_r = \frac{M \cos\theta}{2\pi r^2} \quad \text{and} \quad Q_\theta = \frac{M \sin\theta}{2\pi r^2}$$

These values are then substituted into equations (25), (26), (27) and (28) and the following expressions are obtained for the slope, bending moments, and Kirchhoff shear in an infinite plate due to a concentrated moment M which is oriented with respect to the coordinate axes as shown in Figure 7.

$$\frac{\partial w}{\partial n} = \frac{M}{4\pi D} [\log \frac{r}{a} \cos(\theta+\alpha) + \cos\theta\cos\alpha] \quad . \text{ -----} \quad (34)$$

$$M_N = \frac{-M}{4\pi r} [(1+\mu) \cos\theta - (1-\mu) \sin\theta\sin2\alpha] \quad . \text{ --} \quad (35)$$

$$M_{Nt} = \frac{-M}{4\pi r} [(1-\mu) \sin\theta\cos2\alpha] \quad . \text{ --------------} \quad (36)$$

$$V_N = \frac{M}{4\pi r^2} [2 + (1-\mu) \cos2\alpha] \cos(\theta-\alpha) \quad . \text{ -----} \quad (37)$$

iii) Uniform Load

Proceeding as before, we have from equation (24),

$$w = \frac{qr^4}{64D} \ .$$

Also,

$$\frac{\partial w}{\partial r} = \frac{qr^3}{16D} \ , \qquad \frac{\partial^2 w}{\partial r^2} = \frac{3qr^2}{16D} \ ,$$

and

$$\frac{\partial^3 w}{\partial r^3} = \frac{3qr}{8D} \ .$$

Substituting these values into equations (29), we obtain

$$M_r = \frac{-qr^2(3+\mu)}{16} \ ,$$

$$M_\theta = \frac{-qr^2(1+3\mu)}{16} \ ,$$

$$Q_r = \frac{-qr}{2} \ ,$$

and

$$Q_\theta = M_{r\theta} = 0 \ .$$

Once again, substituting these values into equations (25), (26), (27) and (28), we obtain the following expressions for a uniform load.

$$\frac{\partial w}{\partial n} = \frac{qr^3}{16D} \cos\alpha \ . \qquad \text{-----------------------} \quad (38)$$

$$M_N = \frac{-qr^2}{16} [(3+\mu) \cos^2\alpha + (1+3\mu) \sin^2\alpha] \ . \quad \text{---} \quad (39)$$

$$M_{Nt} = \frac{-qr^2}{16} (1-\mu) \sin 2\alpha \quad . \text{ ------------------ } (40)$$

$$V_N = \frac{-qr}{8} [4\cos\alpha + (1-\mu) \sin\alpha \sin 2\alpha] \quad . \text{ ------- } (41)$$

The equations derived in the preceding paragraphs are sufficient to solve many different plate problems where the boundary conditions specify either fixed, free or simply supported edges. Some examples follow.

2. Simultaneous Solution for Equivalent Loads

a. Circular Plate with Concentrated Load and Simply Supported

As a first example, we seek a solution for the deflection of a circular plate of radius R, simply supported along the boundary and loaded with a concentrated force P at the center of the plate. As before, the number of points on the boundary where the boundary conditions are to be satisfied is N. Choose for the retracted boundary a circle of radius $R + \overline{D}$, centered at the same point in the infinite plate as the circular plate. Let the points of application of the equivalent loads be determined by the intersections of the retracted boundary and the extensions of the radial lines drawn through the N points along the boundary. Choose the following values for the first example.

$R = 100$ inches	$\overline{D} = 50$ inches
$N = 8$	$P = 10$ pounds

$E = 30 \times 10^6$ psi $\qquad a = 500$ inches

$\mu = 1/4$

Let the eight points be equally spaced around the boundary as shown in Figure 10. Also, let the points of application of the equivalent loads be denoted by N'. The center of the plate is placed at the point (500,500) with respect to the arbitrary axes shown. The boundary conditions of the problem specify that at each of the eight points on the boundary, the deflection and the moment normal to the boundary must be zero. The deflection and normal moment due to the applied load can be calculated at each of the N points by using equations (21) and (31), where r = 100 in. and $\alpha = 0$ for all points. Next, a pair of equivalent loads is applied at each of the N' points on the retracted boundary.

From Figure 11, the effect on the deflection and normal moment at point i due to the unknown equivalent loads Q_j and M_j applied at point j can be calculated using equations (21) and (31) and equations (22) and (35) as follows.

The deflection at point i due to the equivalent loads Q_j and M_j applied at point j is

$$w_{ji} = [\frac{r_{ji}^2}{8\pi D} \log \frac{r_{ji}}{a}]Q_j + [\frac{r_{ji}}{4\pi D} \log (\frac{r_{ji}}{a}) \cos\theta_{ji}]M_j \quad .$$

Then, the deflection at i due to all equivalent loads is found by summation.

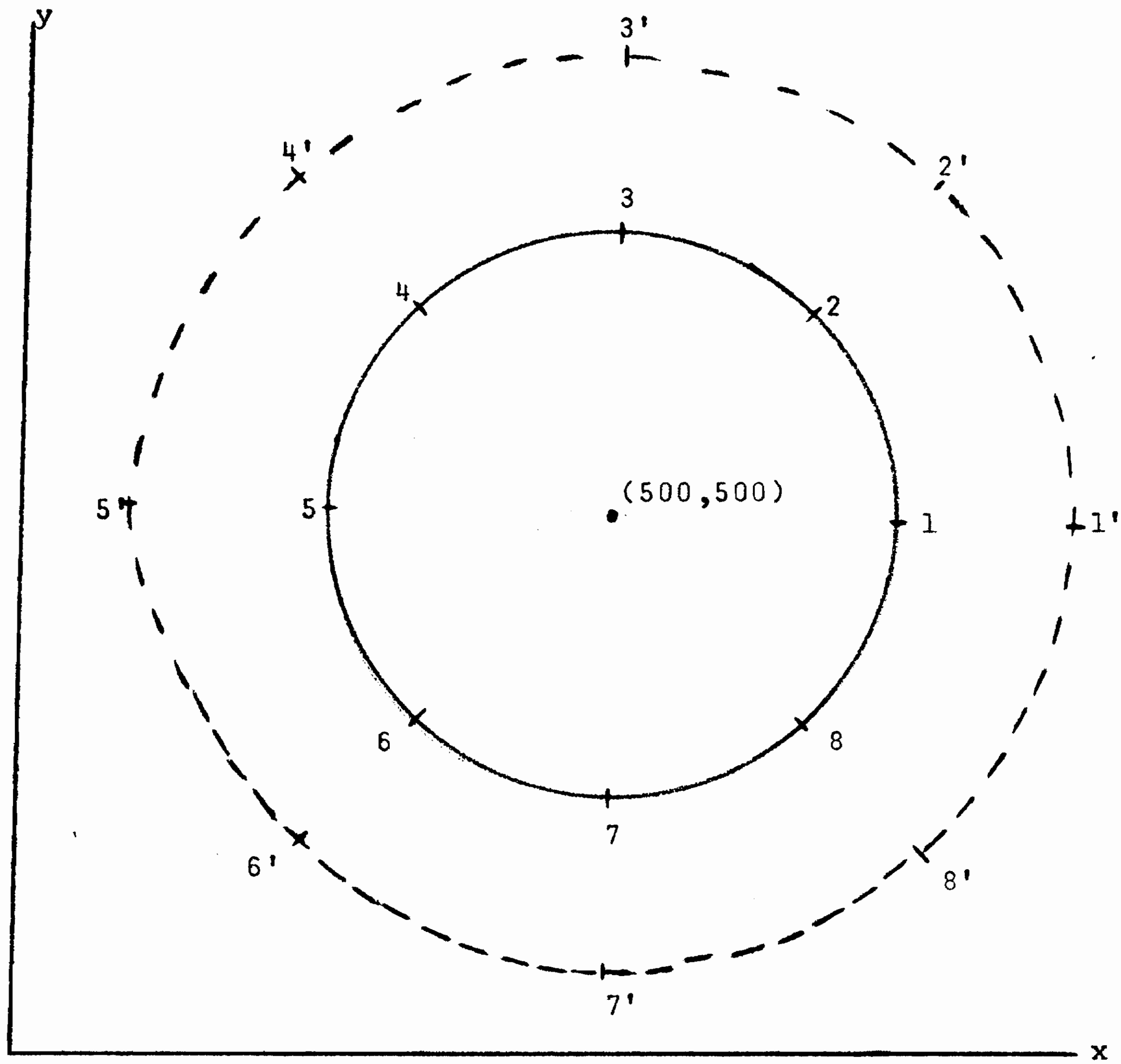


Figure 10. Boundary and Retracted Boundary for a Circular Plate.

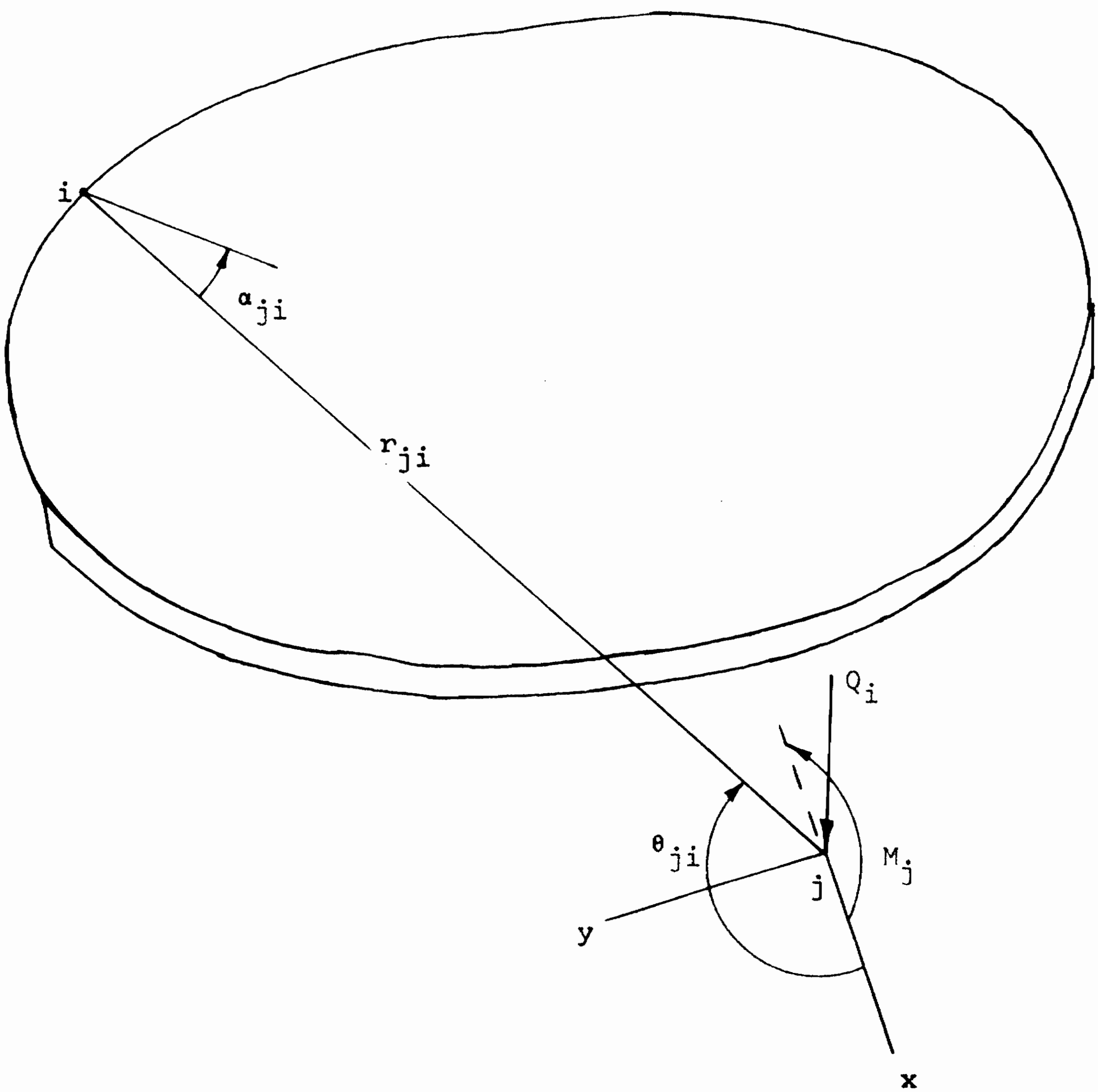


Figure 11. Orientation with Respect to the i^{th} Point on the Boundary of the Equivalent Loads Applied at the j^{th} Point on the Retracted Boundary.

$$w_i = \sum_{j=1}^{8} w_{ji} \quad . \qquad (42)$$

Also, the normal moment at point i due to the equivalent loads at point j is

$$M_{ji} = -[2(1+\mu) \log \frac{r_{ji}}{a} + (3+\mu) \cos^2\alpha_{ji}]\frac{Q_j}{8\pi}$$

$$-[(1+\mu) \cos\theta_{ji} - (1-\mu) \sin\theta_{ji}\sin 2\alpha_{ji}]\frac{M_j}{4\pi r_{ji}} \quad ,$$

and the moment due to all equivalent loads is found by summation.

$$M_j = \sum_{j=1}^{8} M_{ji} \quad . \qquad (43)$$

At each of the eight points on the boundary, let the deflection and moment due to the applied load be represented by $\overline{w}_i$ and $\overline{M}_i$ respectively. Then, since the resultant deflection and moment must be zero at each of the eight points on the boundary, we obtain a set of 16 simultaneous equations of the form

$$w_i + \overline{w}_i = 0 \quad , \qquad (44)$$

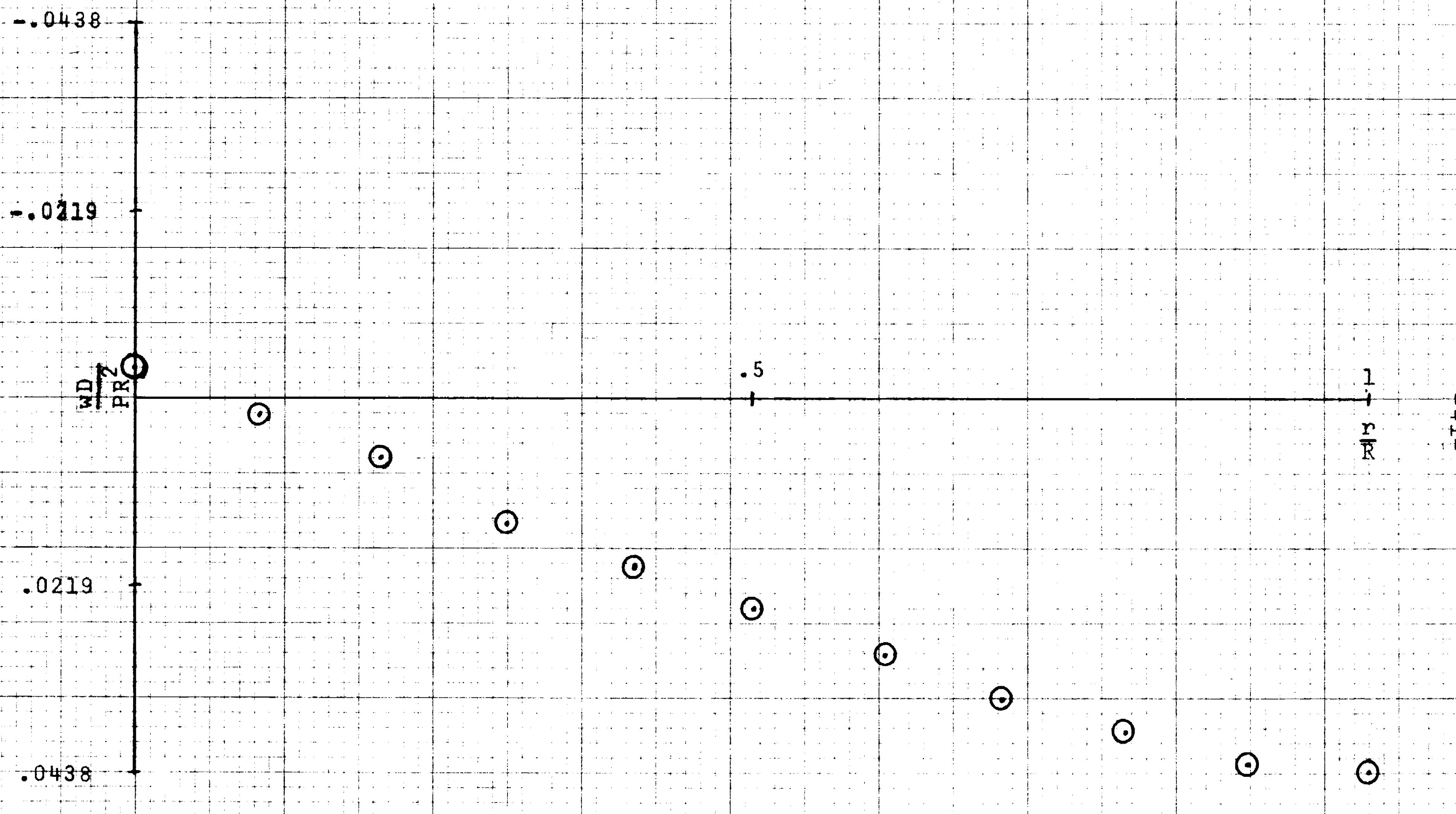
and

$$M_i + \overline{M}_i = 0 \quad , \qquad (45)$$

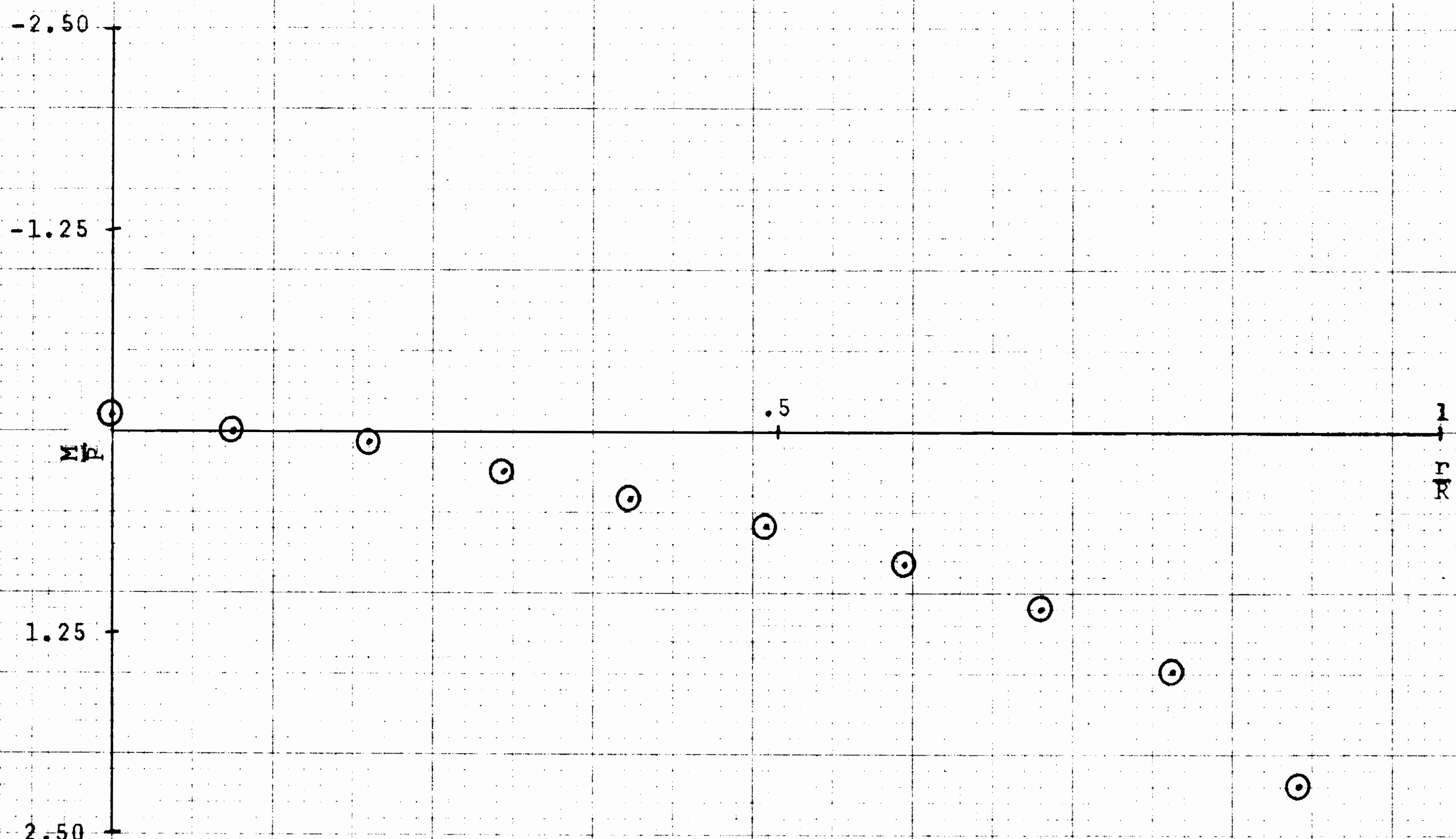
where

$i = 1, 2, \ldots, 8.$

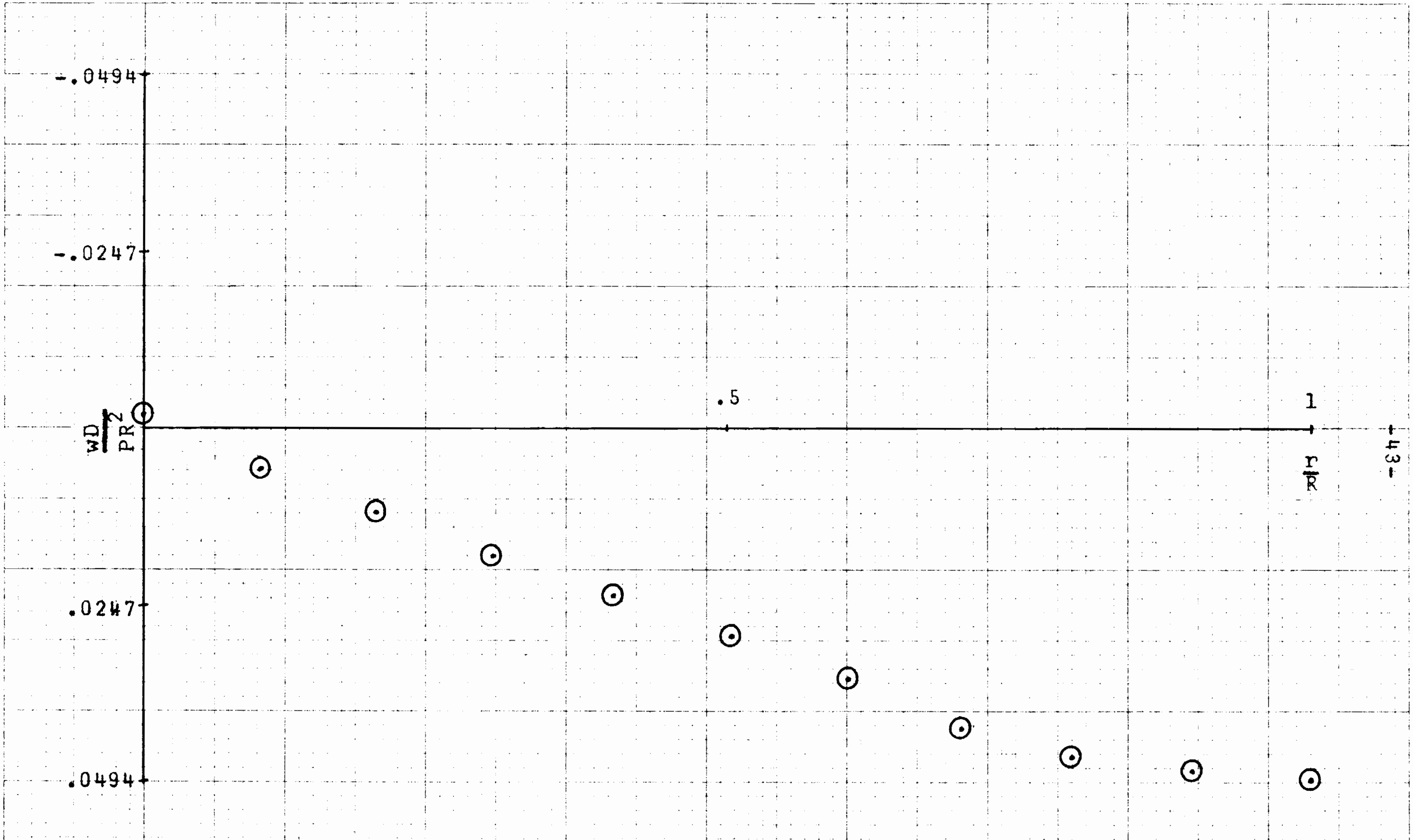
The values of the unknown Q_j's and M_j's were obtained by solving the above equations using an IBM 1620 computer



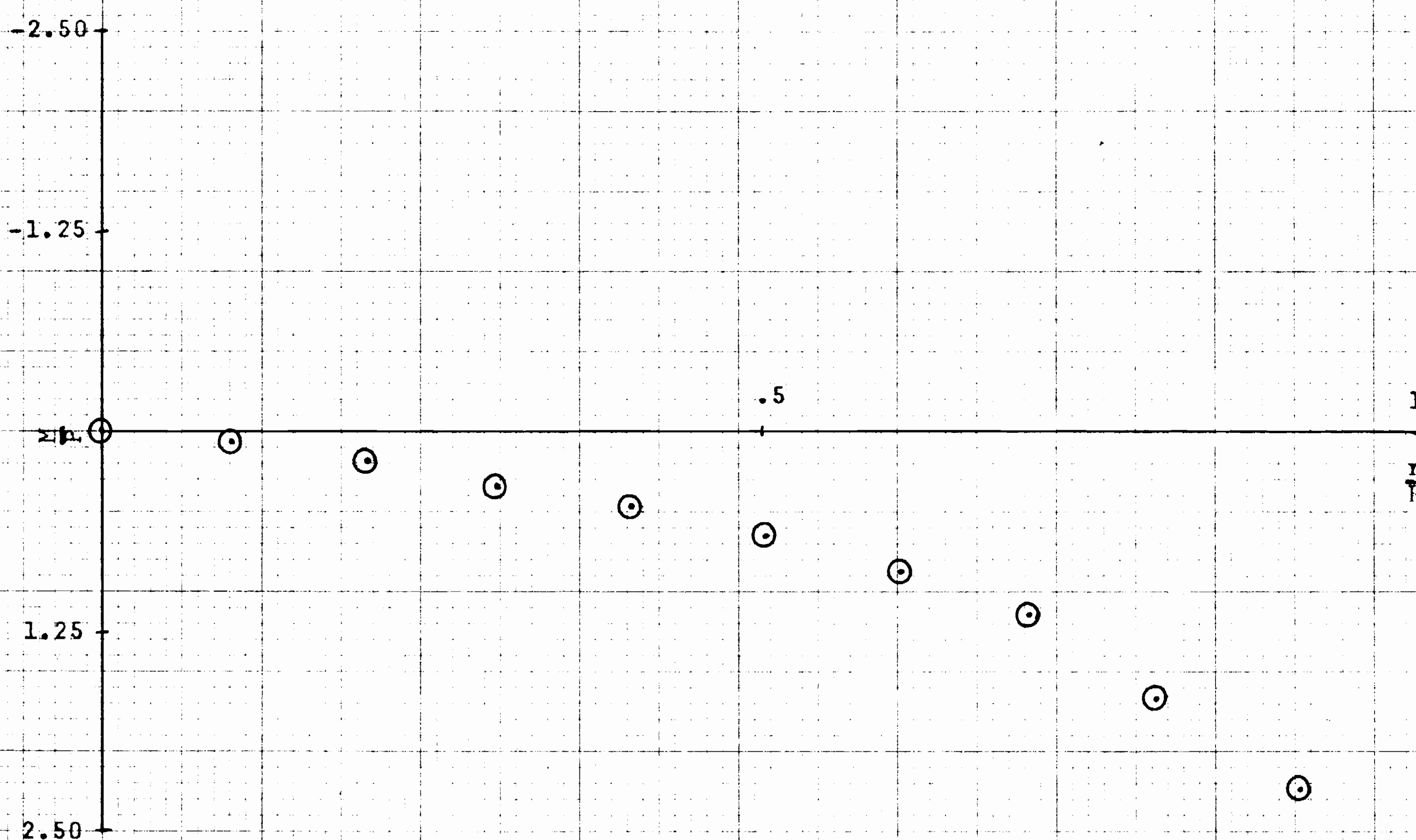
Graph 1. Deflection along the Radius of a Simply Supported Circular Plate with N = 8.



Graph 2. Moment along the Radius of a Simply Supported Circular Plate with N = 8.



Graph 3. Deflection along the Radius of a Simply Supported Circular Plate with N = 16.



Graph 4. Moment along the Radius of a Simply Supported Circular Plate with N = 16.

and plots of the deflection and moment along a radial line were obtained. In this case, the radial line extends to a point on the boundary midway between two balance points. These plots are shown in Graphs 1 and 2.

As a second example, consider the same plate with N increased to 16. The length a is changed to 25, simply to exhibit the arbitrariness of its value. A solution was obtained as before and plots for the deflection and moment along the same radial line are shown in Graphs 3 and 4.

From [1], the deflection at the center is .0155 inches. Thus, for N = 8, a 15.5% error exists in the maximum deflection. For N = 16, the error is 4.5%.

For the moments, the value from [1] is .0690P at a distance of 50 inches from the center. For the first example, the error is 13% and for the second example, the error is 4.33%.

The effect on the solution of different values for a and different amounts of retraction will be discussed later.

b. Circular Plate with a Concentrated Load and Clamped Edge

Now, consider the circular plate used in paragraph a with N = 8 but clamped along the boundary instead of simply supported. Then, in addition to zero deflection at each of the eight points on the boundary, the normal slope must also be zero.

The normal slope at each point on the boundary due to the applied load can be calculated using equation (34), and its value at the i^{th} point on the boundary denoted by $\overline{S}_i$.

From Figure 11, the normal slope at point i due to the equivalent loads at point j is

$$S_{ji} = [2r_{ji} \log \frac{r_{ji}}{a} + r_{ji}]\frac{Q_j}{8\pi D} \cos\alpha_{ji}$$

$$+ [\log \frac{r_{ji}}{a} \cos(\theta_{ji}+\alpha_{ji}) + \cos\theta_{ji}\cos\alpha_{ji}]\frac{M_j}{4\pi D} \quad .$$

As before, the resultant effect due to all equivalent loads is found by summation.

$$S_i = \sum_{j=1}^{8} S_{ji} \quad .$$

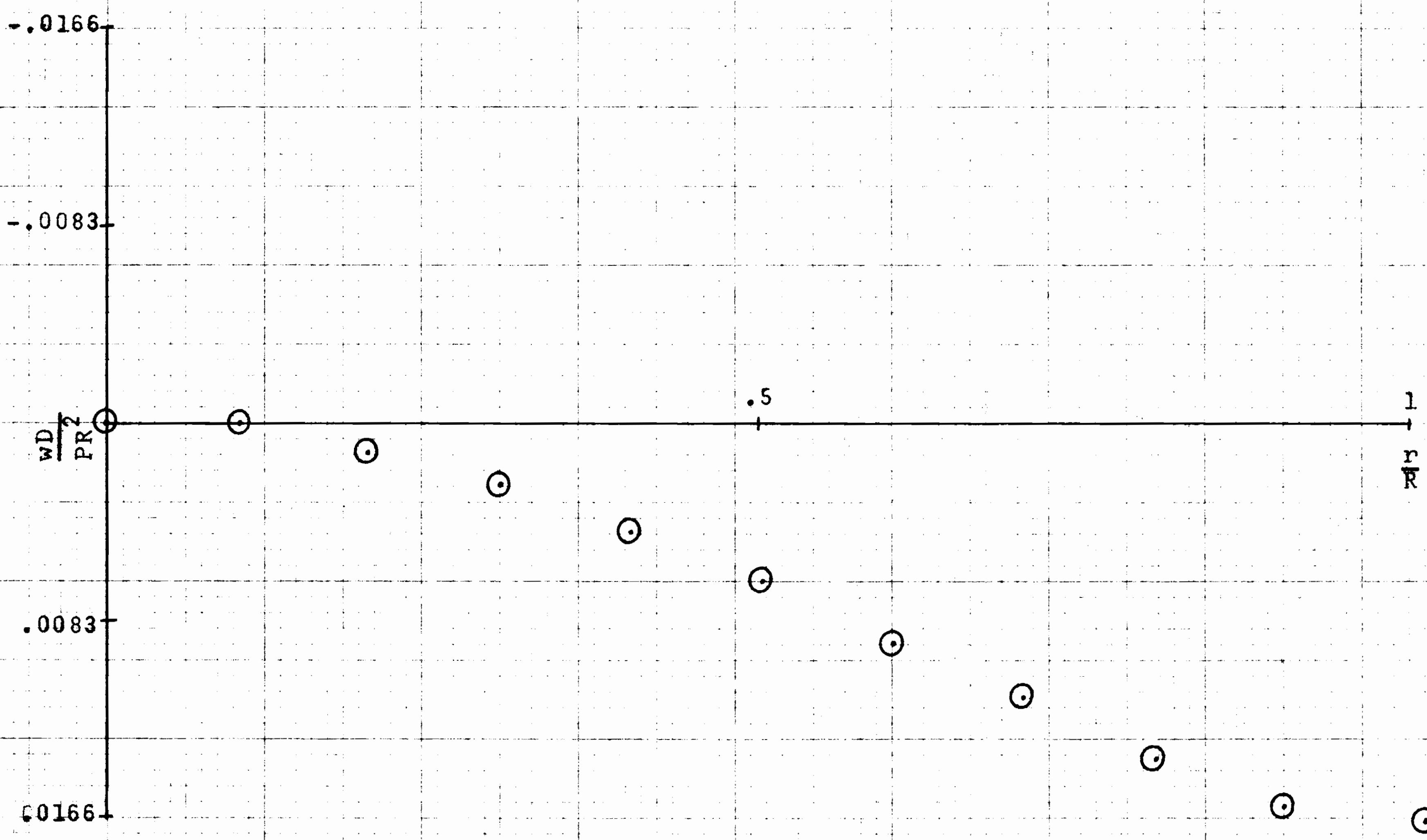
Thus, from the condition that the normal slope be zero at each of the N points on the boundary, we obtain the following equations.

$$\overline{S}_i + S_i = 0 \quad , \text{ where } i = 1, \ldots, 8.$$

The equations for the deflection are the same as those of equation (44) in paragraph a.

$$w_i + \overline{w}_i = 0 \quad , \text{ where } i = 1, \ldots, 8.$$

The above set of 16 simultaneous equations were solved for Q_i and M_i as before and a plot of the deflection along a radial line extending to a point on the boundary where the boundary conditions are satisfied, was obtained. See Graph 5.



Graph 5. Deflection along the Radius of a Clamped Circular Plate with N = 8.

From [1], the deflection at the center is .00597 inches. Therefore, a 16.7% error exists in the maximum deflection with N = 8.

c. Rectangular Plate with a Concentrated Load and Simply Supported

Next, consider a rectangular plate, simply supported, and located with respect to an arbitrary set of axes by the coordinates shown in Figure 12. A concentrated load of 10 lbs. is placed in the center of the plate. The same material properties and other constants used in paragraph a are assumed for this problem with N = 16. Four points, evenly spaced, are placed on each side.

The resulting 32 simultaneous equations were solved for the equivalent loads and plots of the deflection and moment along the centerline and the deflection along one edge were obtained. These are presented in Graphs 6, 7, and 8.

The maximum deflection was 1% larger than the value obtained from [1] for this problem.

d. Triangular Plate with a Uniform Load and Simply Supported Edges

In this example, we consider a simply supported triangular plate, located with respect to arbitrary axes in the infinite plate as shown in Figure 13 and loaded with a

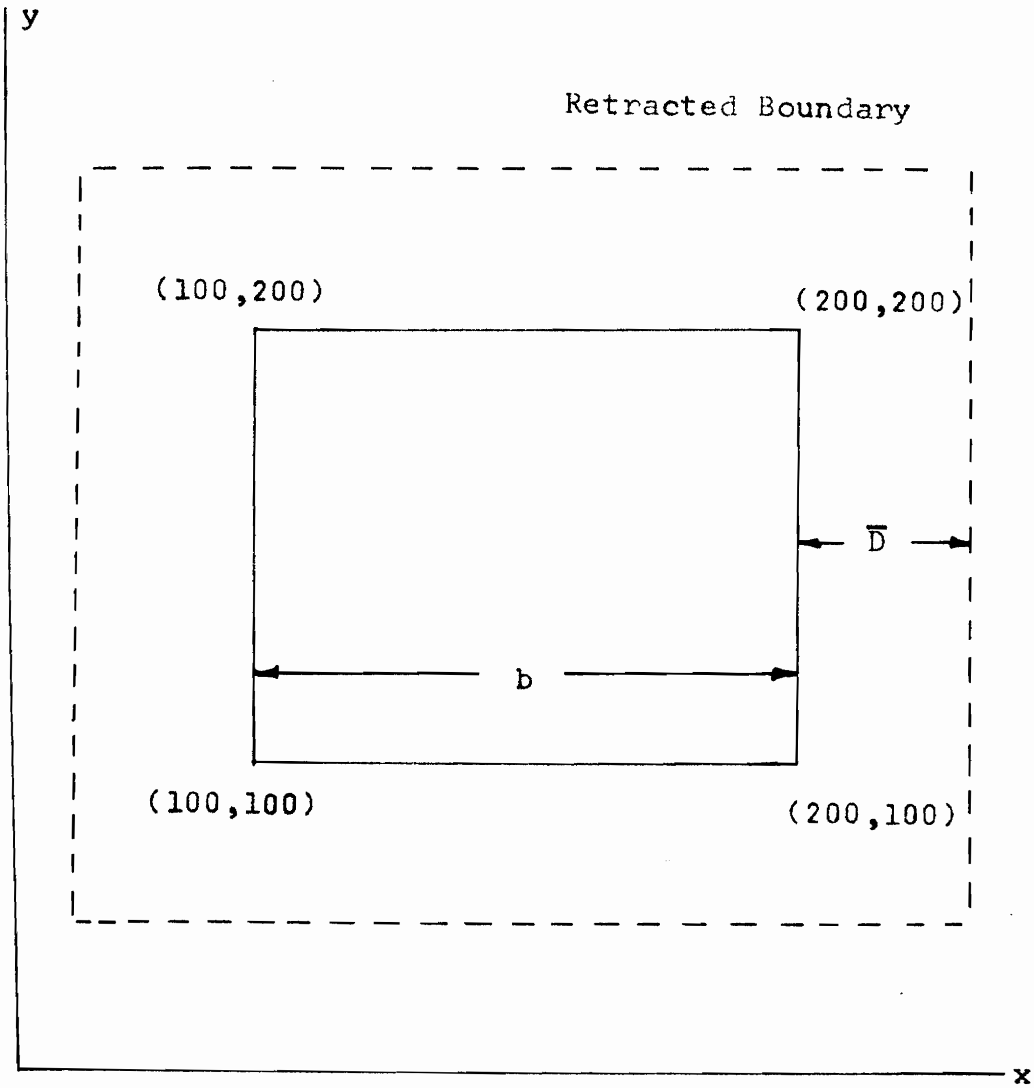
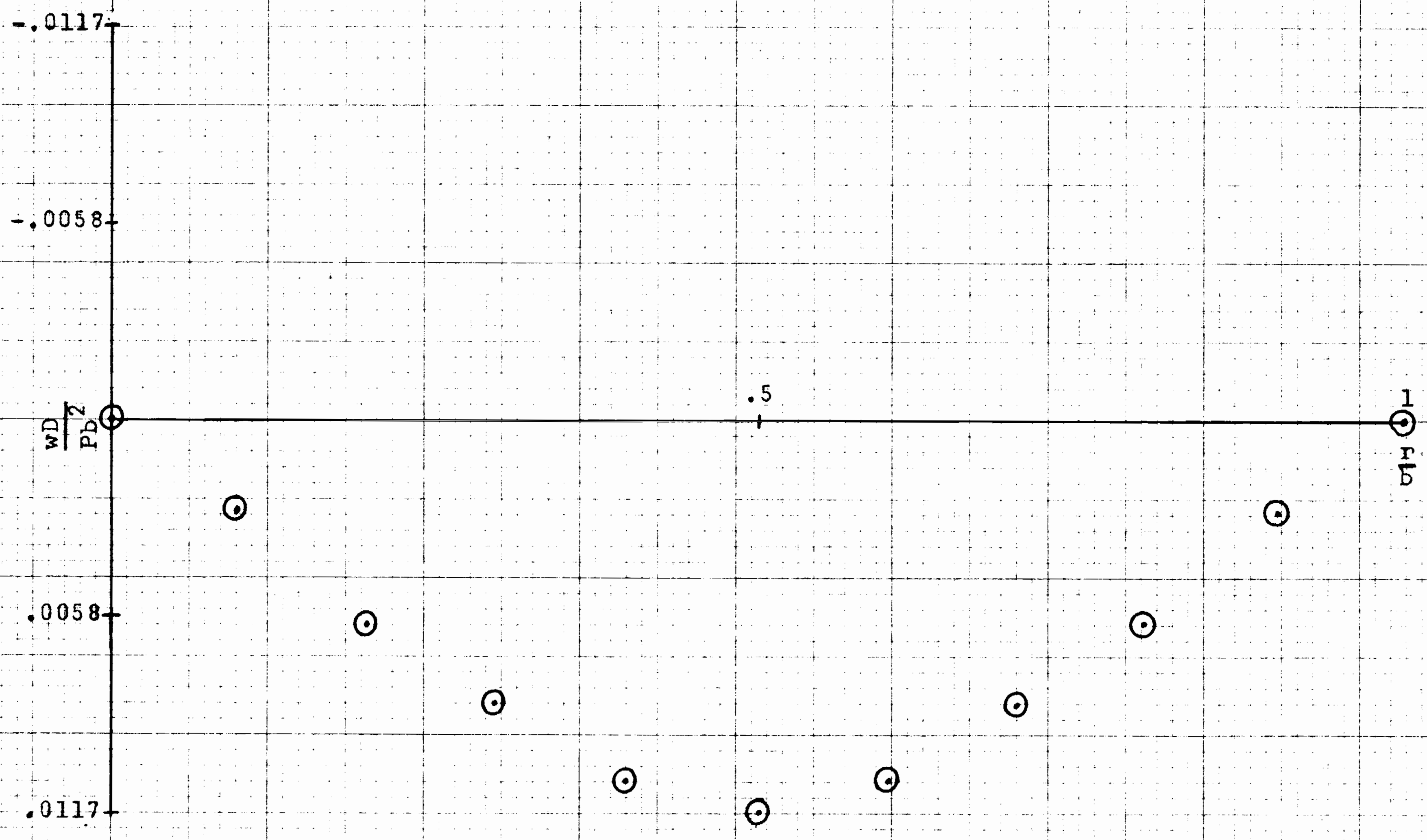
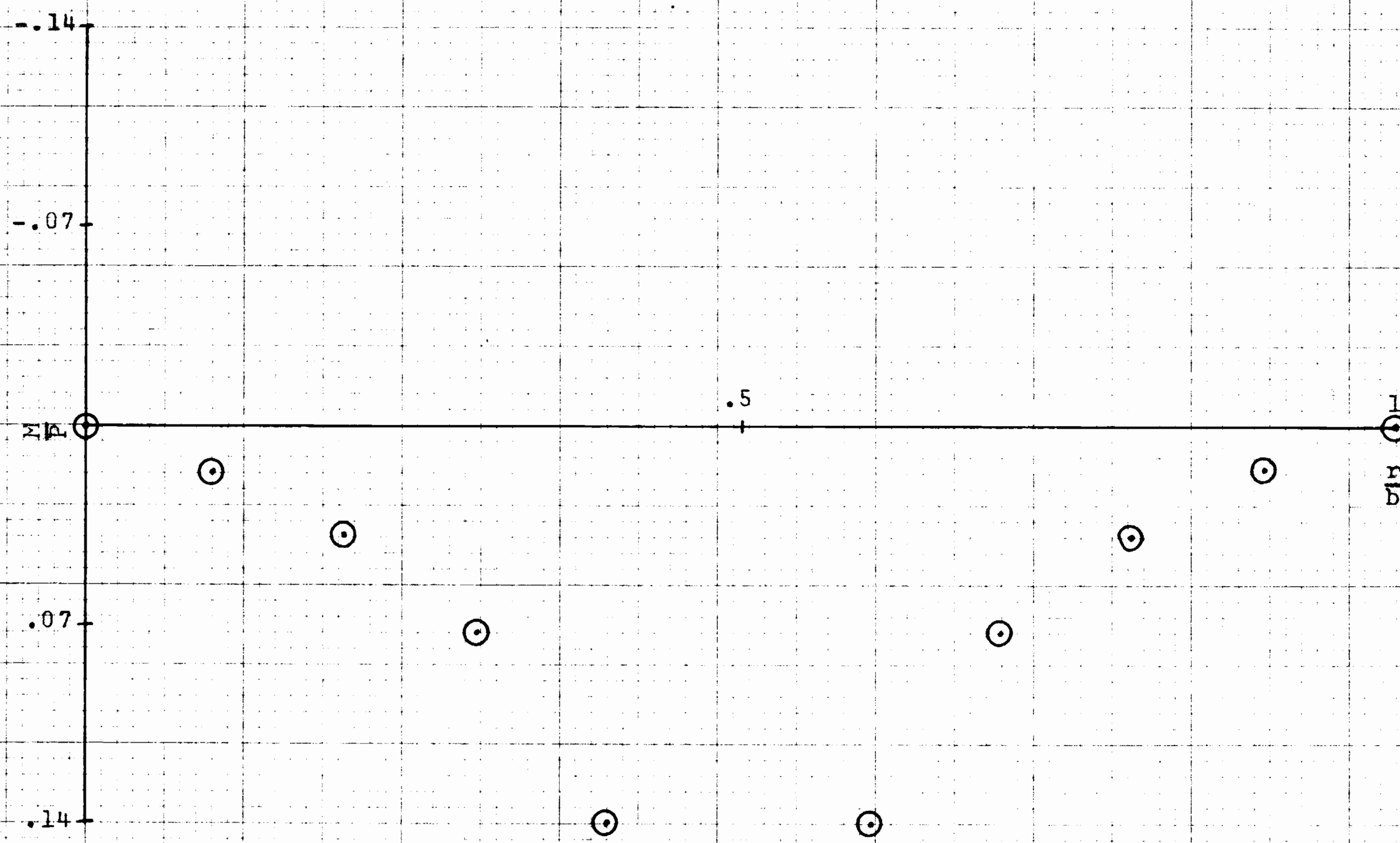


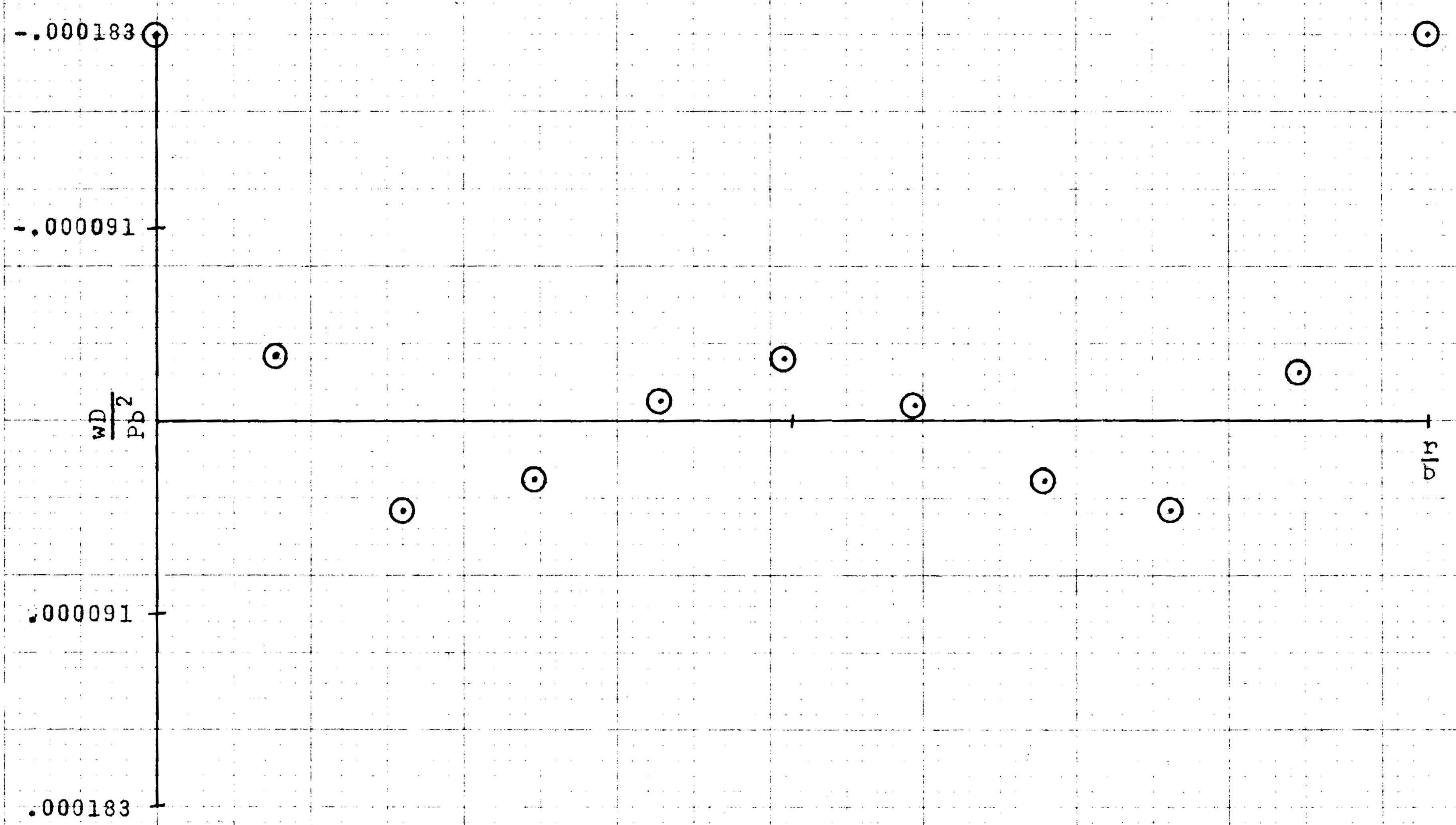
Figure 12. Rectangular Plate as Part of an Infinite Plate.



Graph 6. Deflection along the Centerline of a Simply Supported Rectangular Plate with N = 16.



Graph 7. Moment along the Centerline of a Simply Supported Rectangular Plate with N = 16.



Graph 8. Deflection along the Edge of a Simply Supported Rectangular Plate with N = 16.

uniform load of .01 pounds per square inch. The following values were assumed for the material properties and other constants.

$\mu = 1/4$, $t = 1/2$ in. , $E = 30 \times 10^6$ psi ,

$a = 1$ in. , $\bar{D} = 50$ in. , $N = 14$.

The 14 points were distributed as follows. Five points equally spaced between the corners of the sides of equal length and four points equally spaced along the third side. A plot of the deflection from the point (100,200) to the point (200,200) was obtained. From Graph 9, the maximum deflection occurs at the point (200,200).

In order to correct this anomalous result an additional point was placed at (200,200) by the following method. The triangle was truncated as shown in Figure 14 and one point placed midway between the points A and B. Also, one extra point was added along the side from point (100,100) to point (100,300). New plots using the two additional points were obtained for the deflection and moment along the center-line. These are shown in Graphs 10 and 11.

The maximum deflection and moment obtained by the reflection method as compared to those obtained by Conway [5] for this problem are shown in Table 1.

e. Cantilevered Semicircular Plate

Consider a semicircular plate located as shown in Figure 15. Let the boundary conditions be prescribed as

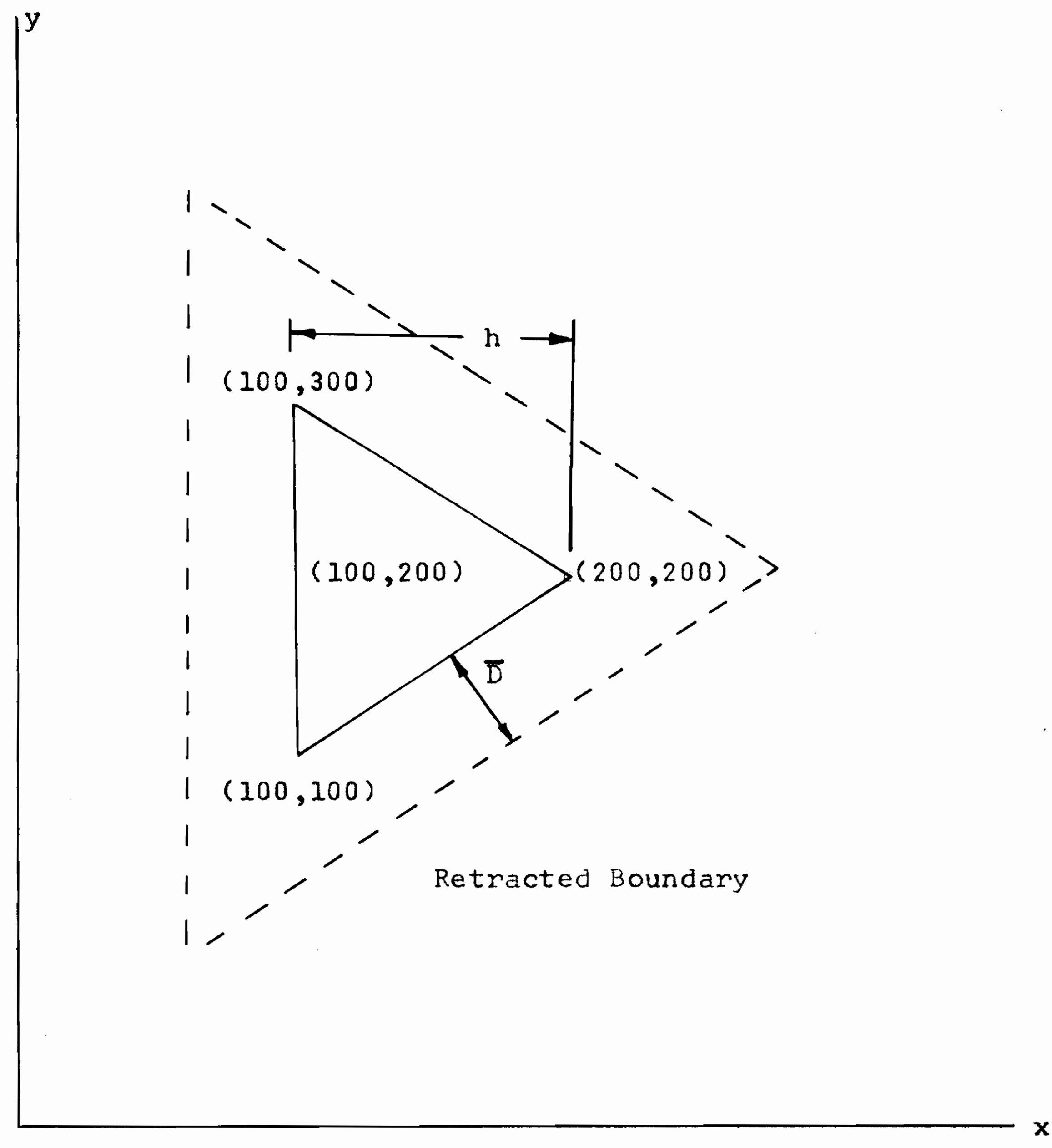


Figure 13. Triangular Plate as Part of an Infinite Plate.

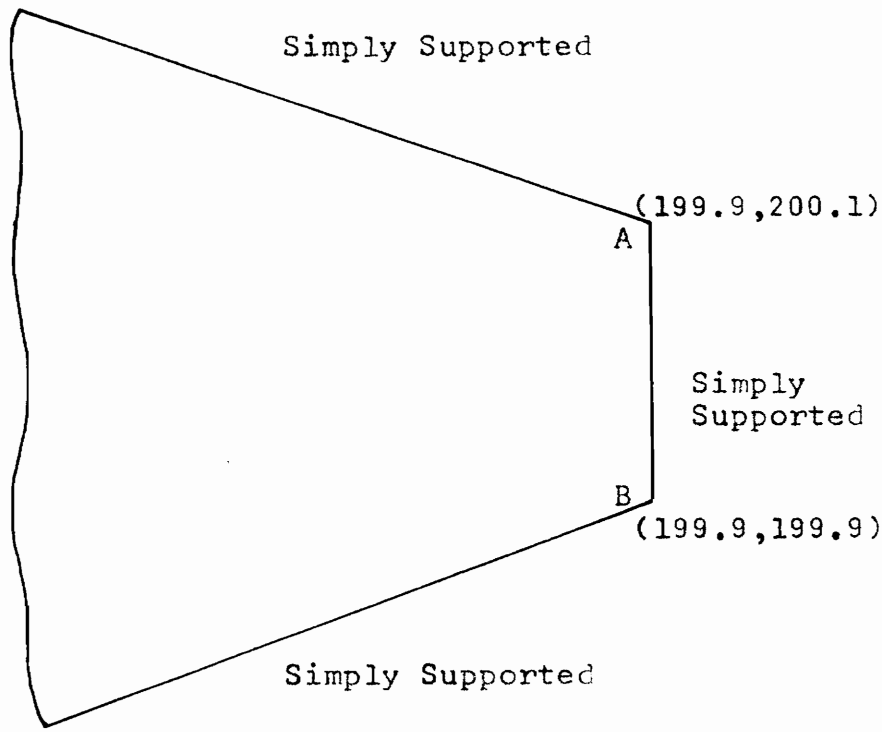
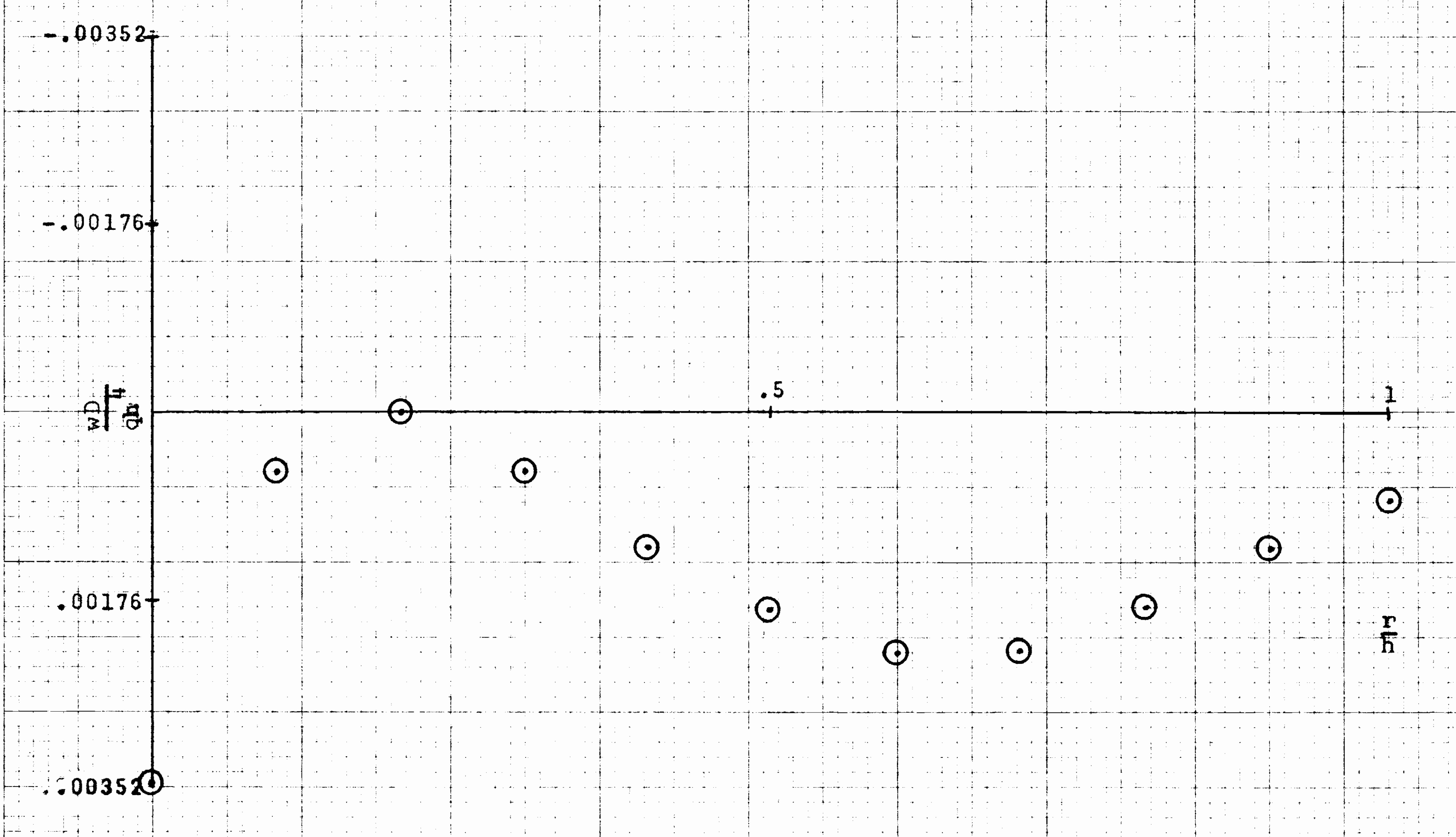
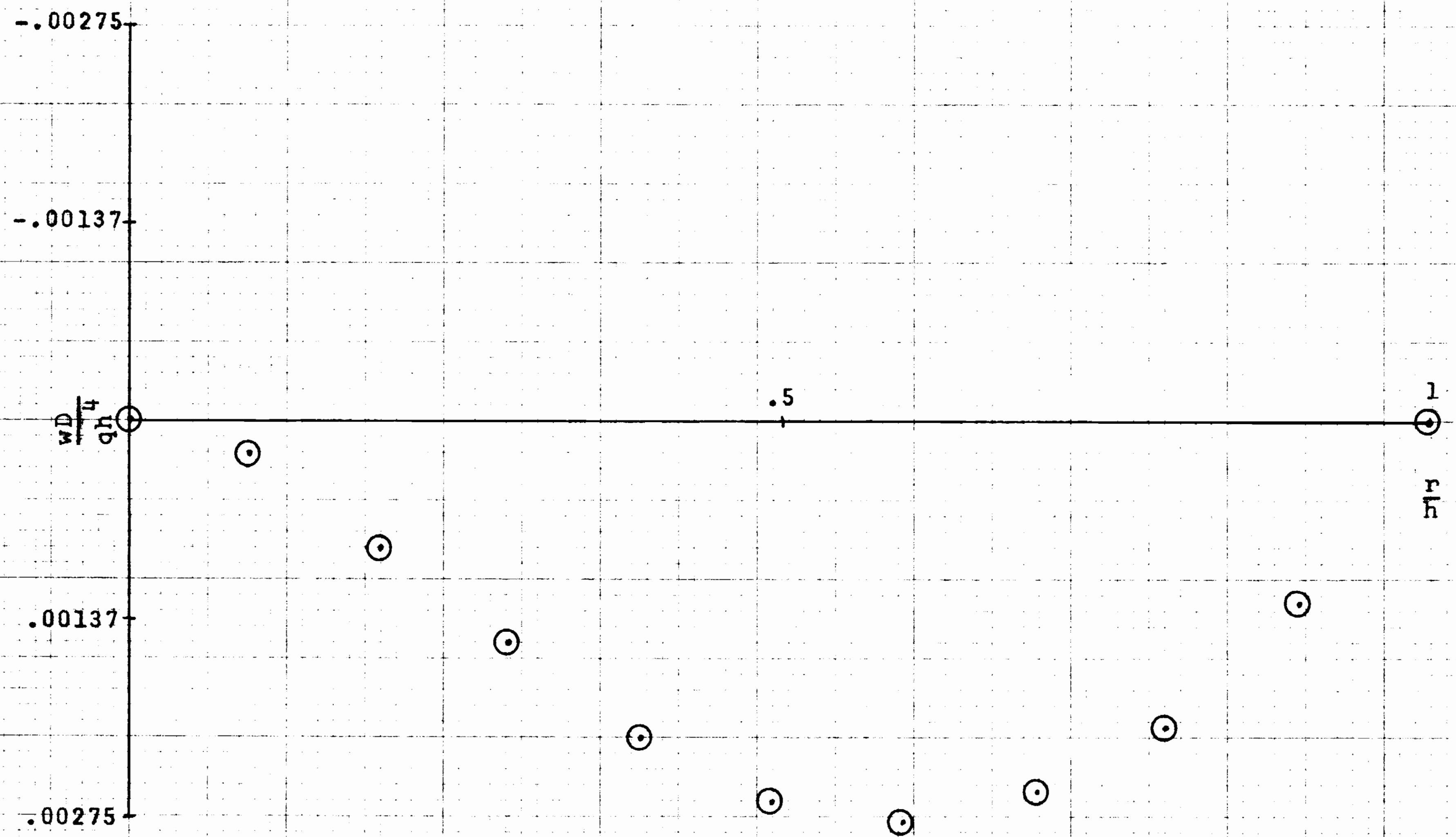


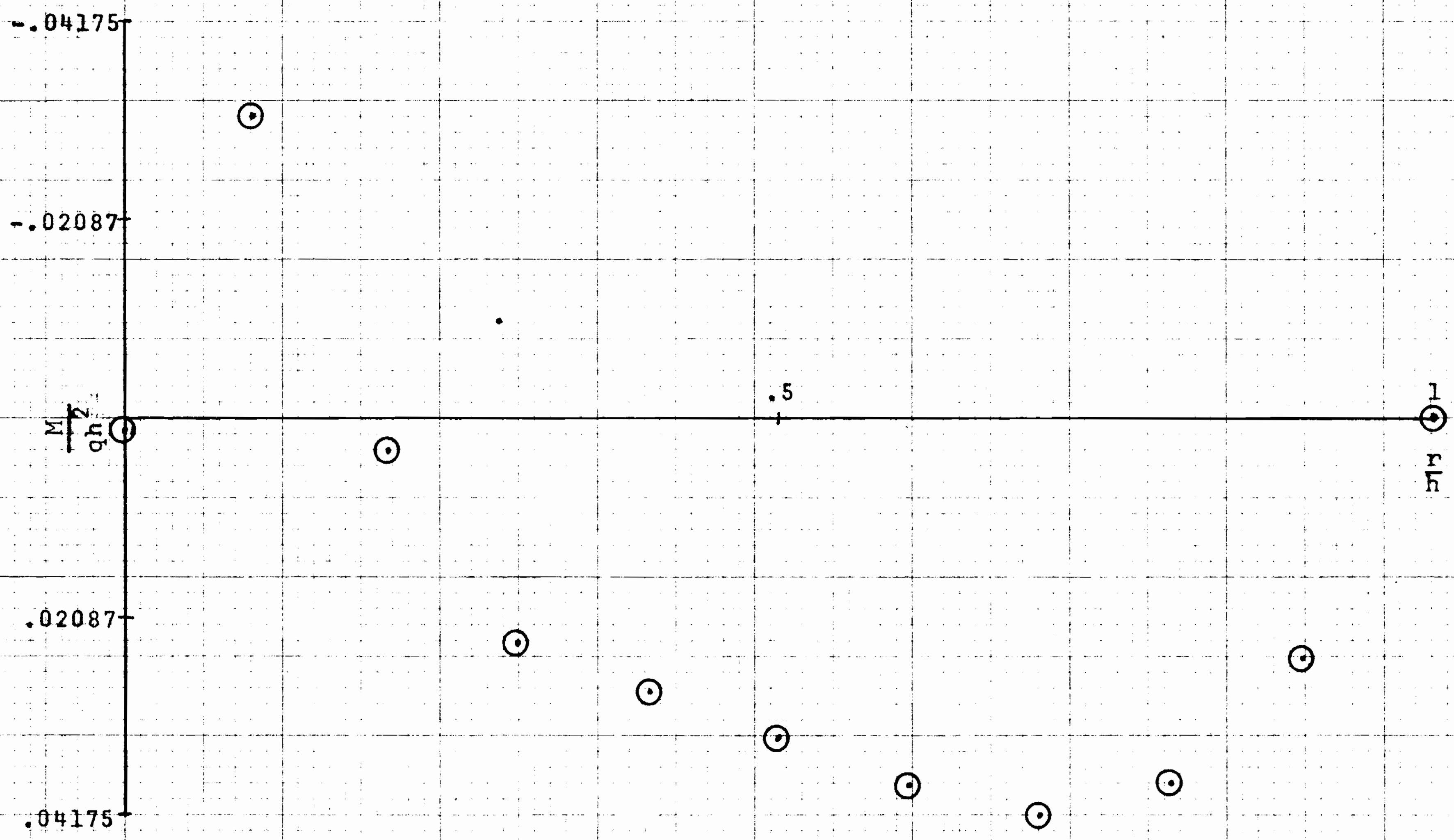
Figure 14. Location of Corner Balancing Points on a Triangular Plate.



Graph 9. Centerline Deflection for a Simply Supported Triangular Plate with N = 14.



Graph 10. Centerline Deflection for a Simply Supported Triangular Plate with N = 16.



Graph 11. Centerline Moment for a Simply Supported Triangular Plate with N = 16.

TABLE 1

Comparison of Results Obtained by the Point Matching and Reflection Methods

	$\frac{w_{Max} D}{qh^4}$	Max Defl. Radius*
Point Matching	.002628	.40 h
Reflection	.002743	.40 h

	$\frac{M_{Max}}{qh^2}$	Max Mom Radius*
Point Matching	.0439	.30 h
Reflection	.0418	.30 h

*Radius equals distance from point (100,200) along centerline.

follows: fixed along the line AB and free along the semicircle ABC. Choose the same values for E, μ, t, a and $\overline{D}$ as in the previous example. Let N = 16. Now, load the infinite plate with a uniform load of .01 lb/in^2. Place the 16 points as follows: four points evenly spaced along the fixed boundary and twelve points evenly spaced between A and C along the semicircular boundary. Number the points in the manner indicated by Figure 15. Then, the boundary conditions prescribe that for points 1 through 12, the normal moment and Kirchhoff shear are zero, and for points 12 through 16, the deflection and normal slope are zero.

Now, center the uniform load at the point (100,200). The bending moment $(\overline{M}_N)_i$ and Kirchhoff shear $(\overline{V}_N)_i$ at points 1 through 12 due to the uniform load can be calculated using equations (39) and (41).

The deflections $\overline{w}_i$ and slopes $\overline{s}_i$ at points 12 through 16 are calculated using equations (24) and (38).

From Figure 11, the normal moment at point i due to the equivalent loads at point j is

$$(M_N)_{ji} = -[2(1+\mu) \log \frac{r_{ji}}{a} + (3+\mu) \cos^2\alpha_{ji}$$

$$+ (1+3\mu) \sin^2\alpha_{ji}]\frac{Q_j}{8\pi} - [(1+\mu) \cos\theta_{ji}$$

$$- (1-\mu) \sin\theta_{ji}\sin2\alpha_{ji}]\frac{M_j}{4\pi r_{ji}} .$$

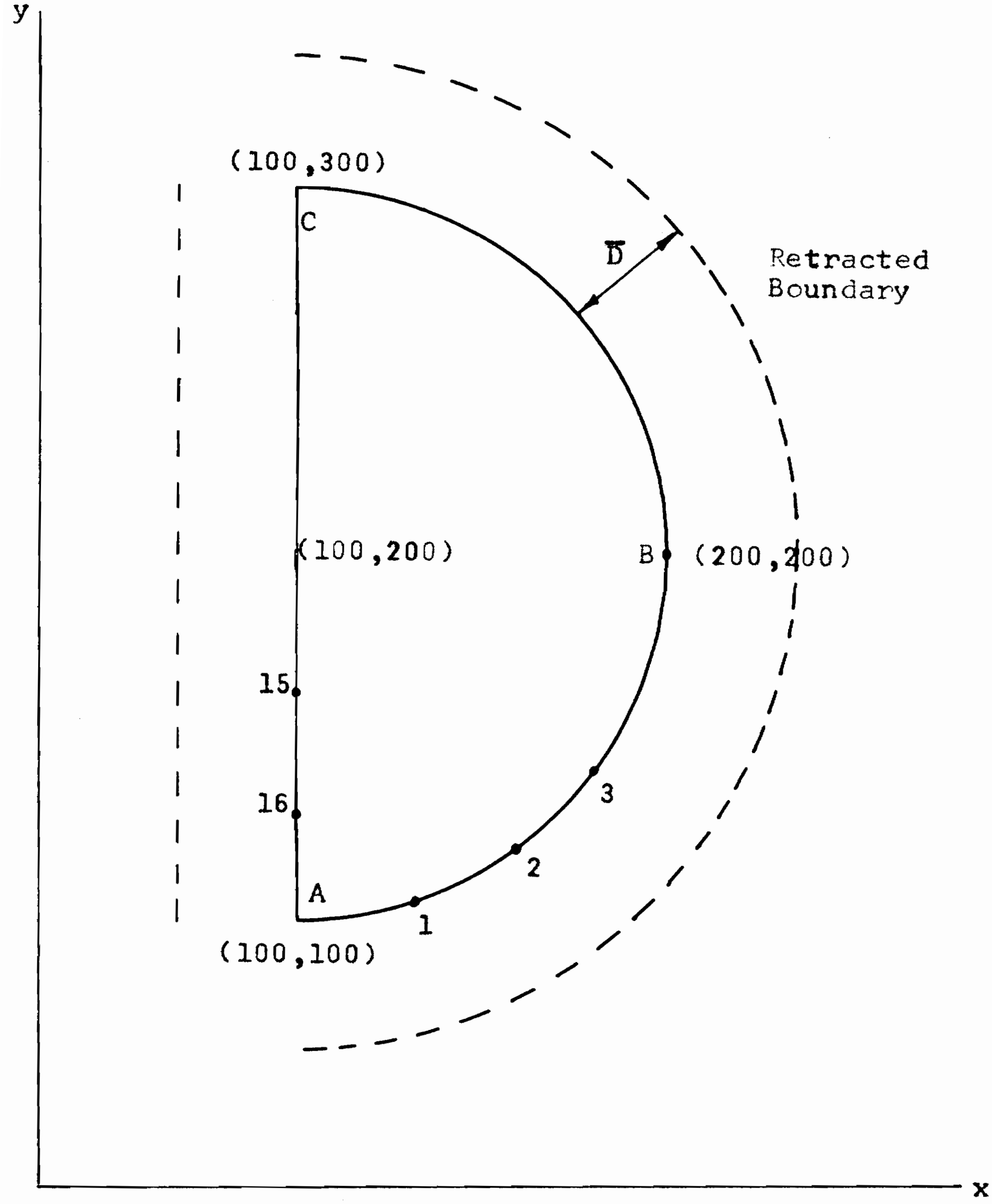


Figure 15. Semicircular Plate as Part of an Infinite Plate

Thus,

$$(M_N)_i = \sum_{j=1}^{16} (M_N)_{ji} \quad .$$

Also from Figure 11, the Kirchhoff shear at point i due to the equivalent load at point j is

$$(V_N)_{ji} = \frac{-Q_j \cos\alpha_{ji}}{2\pi r_{ji}} + [(2+(1-\mu)\cos^2\alpha_{ji})\cos(\theta_{ji}-\alpha_{ji})]\frac{M_j}{4\pi r_{ji}^2}$$

and

$$(V_N)_i = \sum_{j=1}^{16} (V_N)_{ji} \quad .$$

From the boundary conditions of the problem, the following simultaneous equations arise.

$$(V_N)_i + (\overline{V}_N)_i = 0 \quad ,$$

and

$$(M_N)_i + (\overline{M}_N)_i = 0 \quad \text{where } i = 1, \ldots, 12.$$

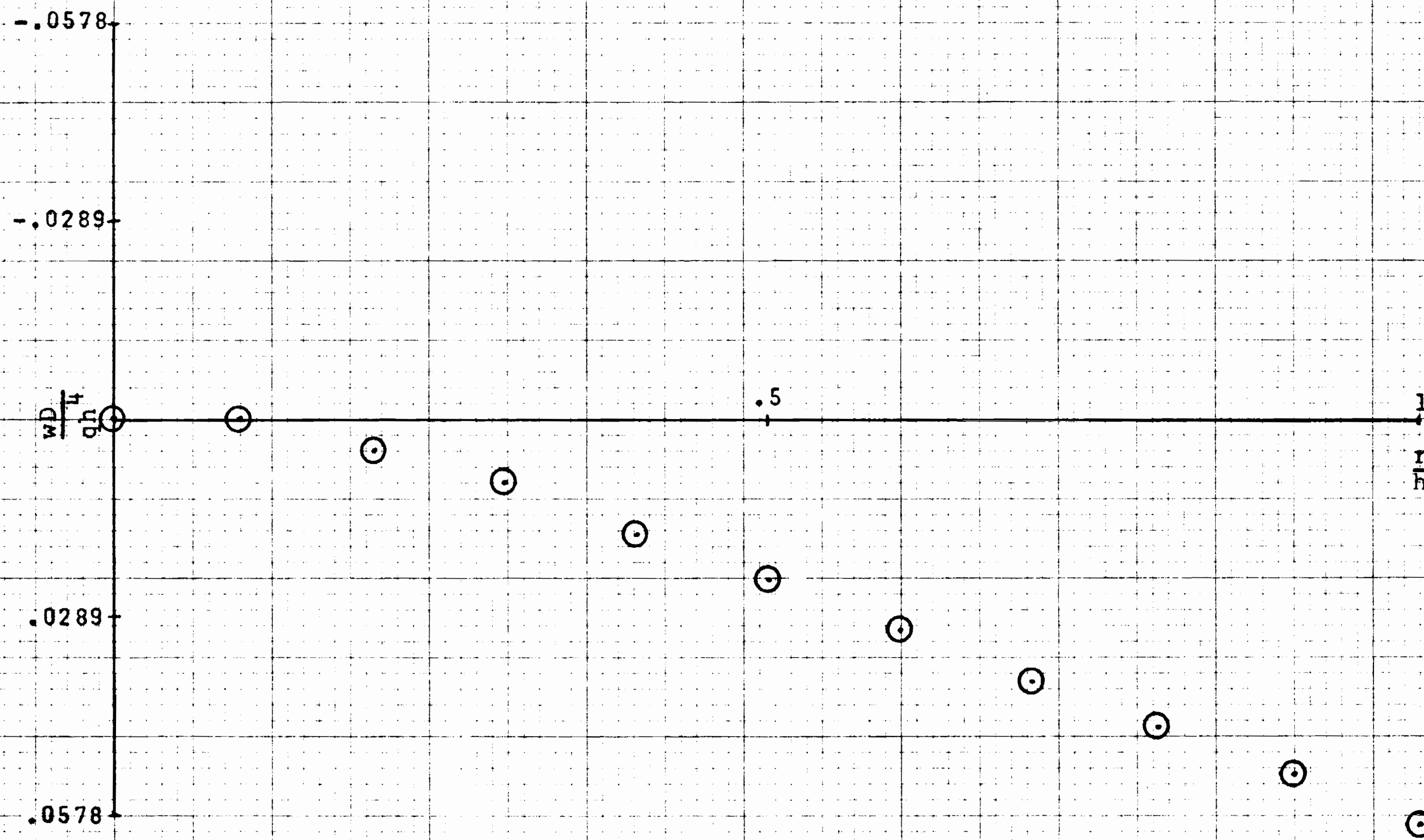
$$w_i + \overline{w}_i = 0 \quad ,$$

and

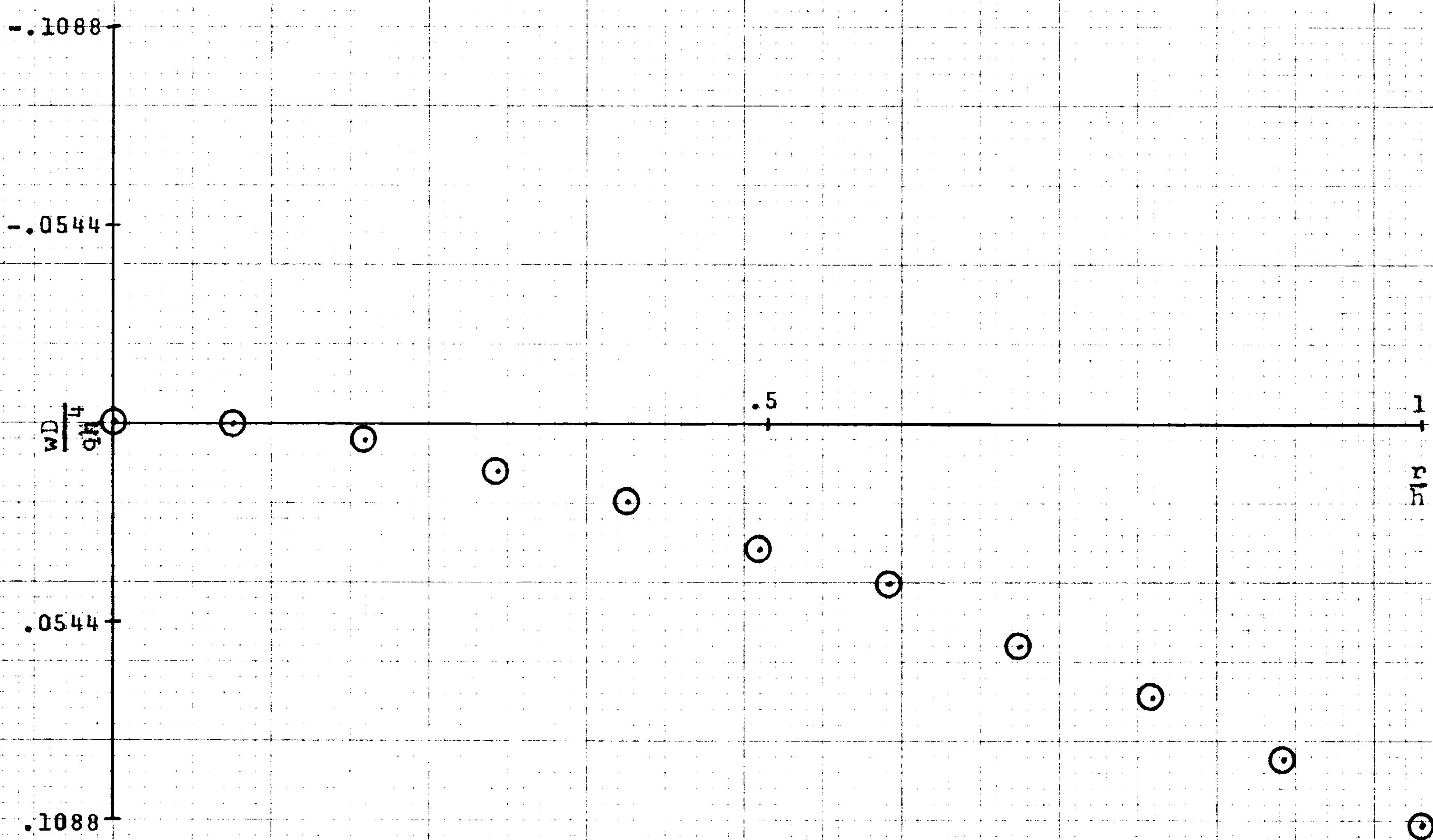
$$S_i + \overline{S}_i = 0 \quad \text{where } i = 13, \ldots, 16.$$

The result of the solution of the above equations is shown by plotting the deflection from point (100,200) to point (200,200). See Graph 12.

The above problem was reworked using a concentrated force of magnitude 10 lbs. applied at point (200,200) instead of the uniform load. Once again, the deflection curve is



Graph 12. Centerline Deflection for a Uniformly Loaded Cantilevered Semicircular Plate with N = 16.



Graph 13. Centerline Deflection for a Cantilevered Semicircular Plate with N = 16 and Loaded with a Concentrated Force.

plotted between the same points and presented as Graph 13.

f. Cantilevered Triangular Plate

As a final example, consider the triangular plate shown in Figure 13, loaded at the point (200,200) with a concentrated force of magnitude 10 lbs. and subject to the following boundary conditions. The portion of the boundary from the point (100,100) to the point (100,300) is fixed and the remainder of the boundary is free. All constants have the same value as in paragraph d except N. The equations for M_i and Q_i are obtained in a manner similar to that used in paragraph e.

Beginning with N = 10, we increase N until no appreciable change occurs in the maximum deflection. The N points are located as follows: four points equally spaced along the fixed edge and the rest divided equally between the free edges. Table 2 compares the maximum deflection for N = 10, 12, 14 and 16. Since little change occurs in the deflection for N = 14 and N = 16, values of N greater than 16 were not used.

The plot of the centerline deflection with N = 16 (Graph 14) is followed by plots of the centerline moment and of the deflections and slopes along the fixed end and moments and shears along one free edge.

Using a strength of materials approach, the maximum deflection was estimated to be .08 inches, which is over 50% less than the value predicted by the Reflection method.

3. Plates with Holes

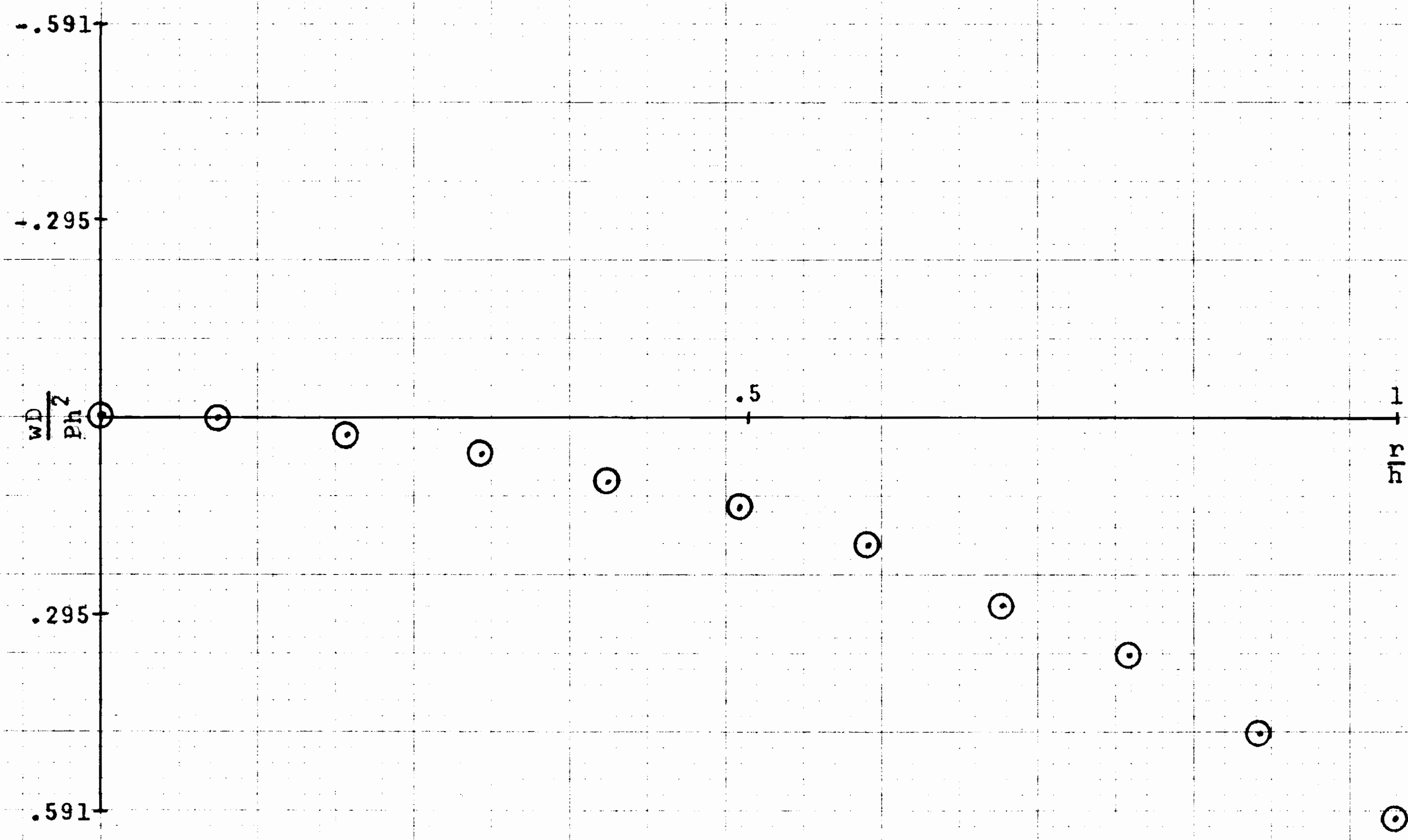
The Reflection method, as outlined in the previous paragraphs, can be used, with only slight modification, to solve problems of plates with holes. For such a plate, the retracted boundary would no longer be a continuous curve, but would be broken into a finite number of curves, depending on the number of holes.

Consider the plate shown in Figure 16. Let N_1 be the number of points on the outer boundary and N_2 the number of points on the inner boundary where the boundary conditions are to be satisfied. Then the retracted boundary can be broken into two segments (see Figure 16) where N_1 equivalent loads are to be applied along the outer retracted boundary and N_2 equivalent loads along the inner retracted boundary. The effect on the deflection, slope, moment and Kirchhoff shear at points along both boundaries of the plate due to any applied load can be calculated using the previous formulas where the angle α_{ij} is defined as the angle between the radius r_{ij} and the inward normal with respect to the plate, as shown in Figure 17. The deflection, slope, moment and Kirchhoff shear induced at points on both boundaries by equivalent loads applied along the outer retracted boundary can also be calculated using the previous formulas where

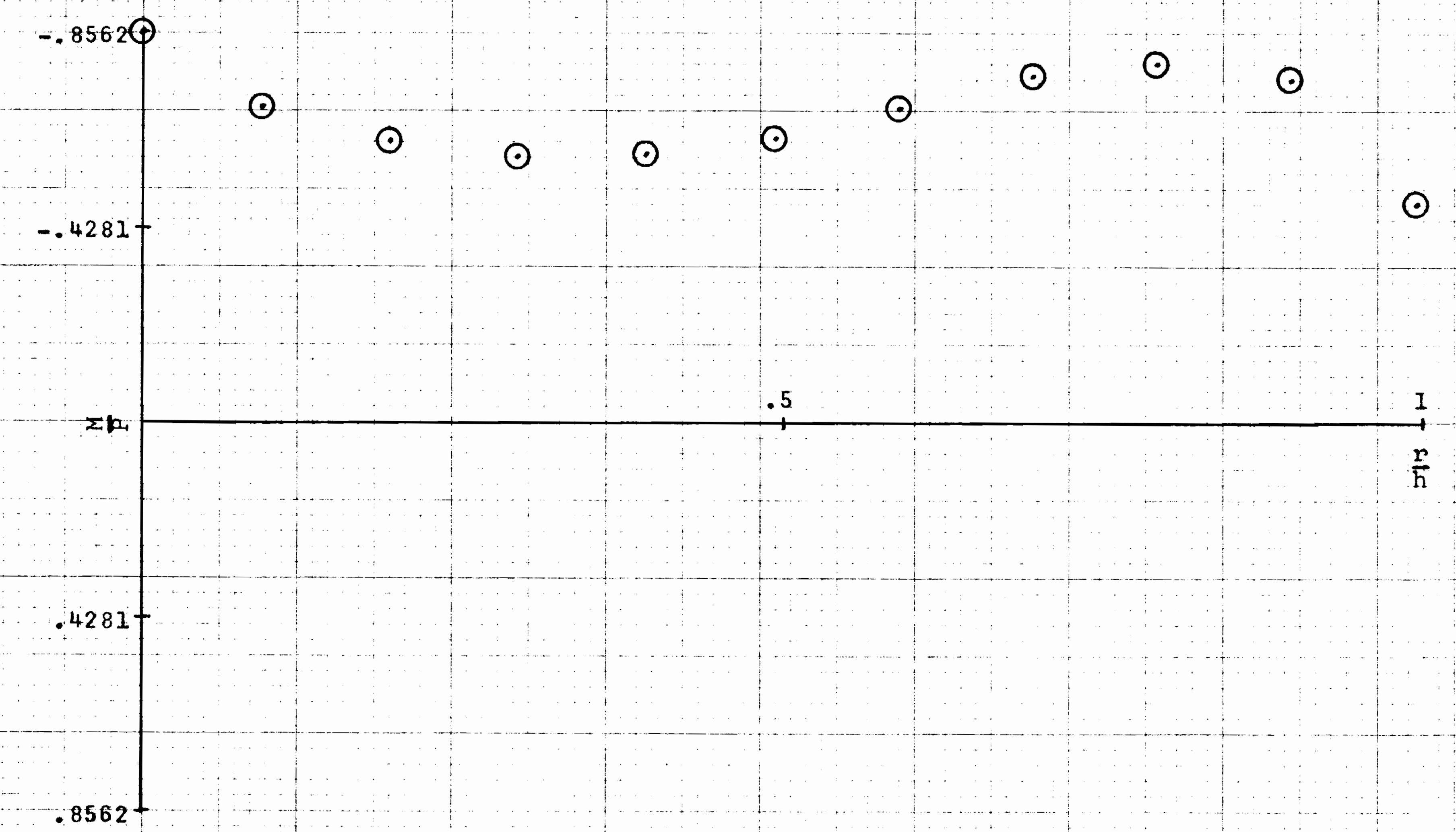
TABLE 2

Maximum Deflection of a Cantilevered Triangular Plate for Different Values of N

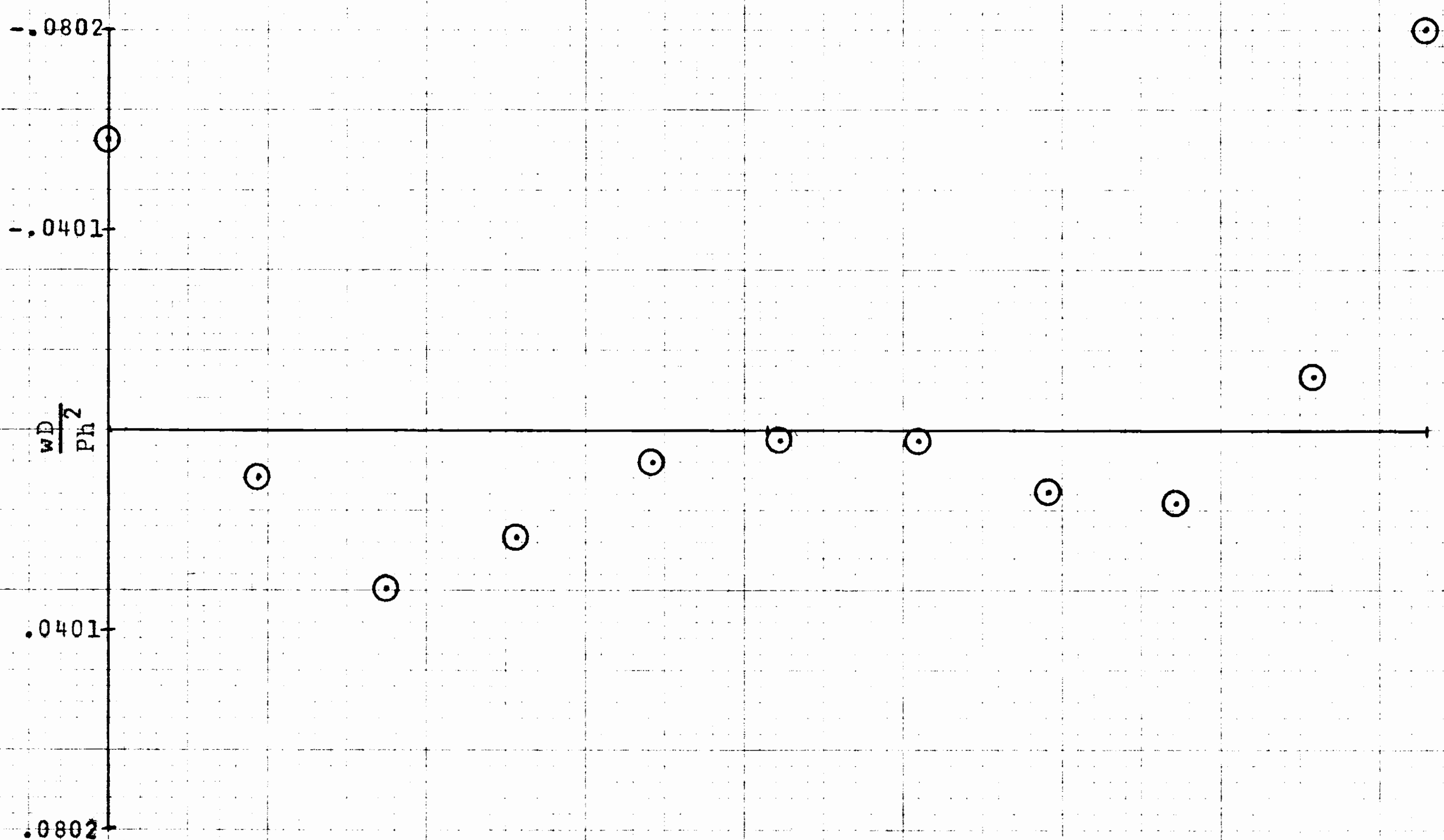
N	Maximum Deflection
10	.1007
12	.1645
14	.1778
16	.1773



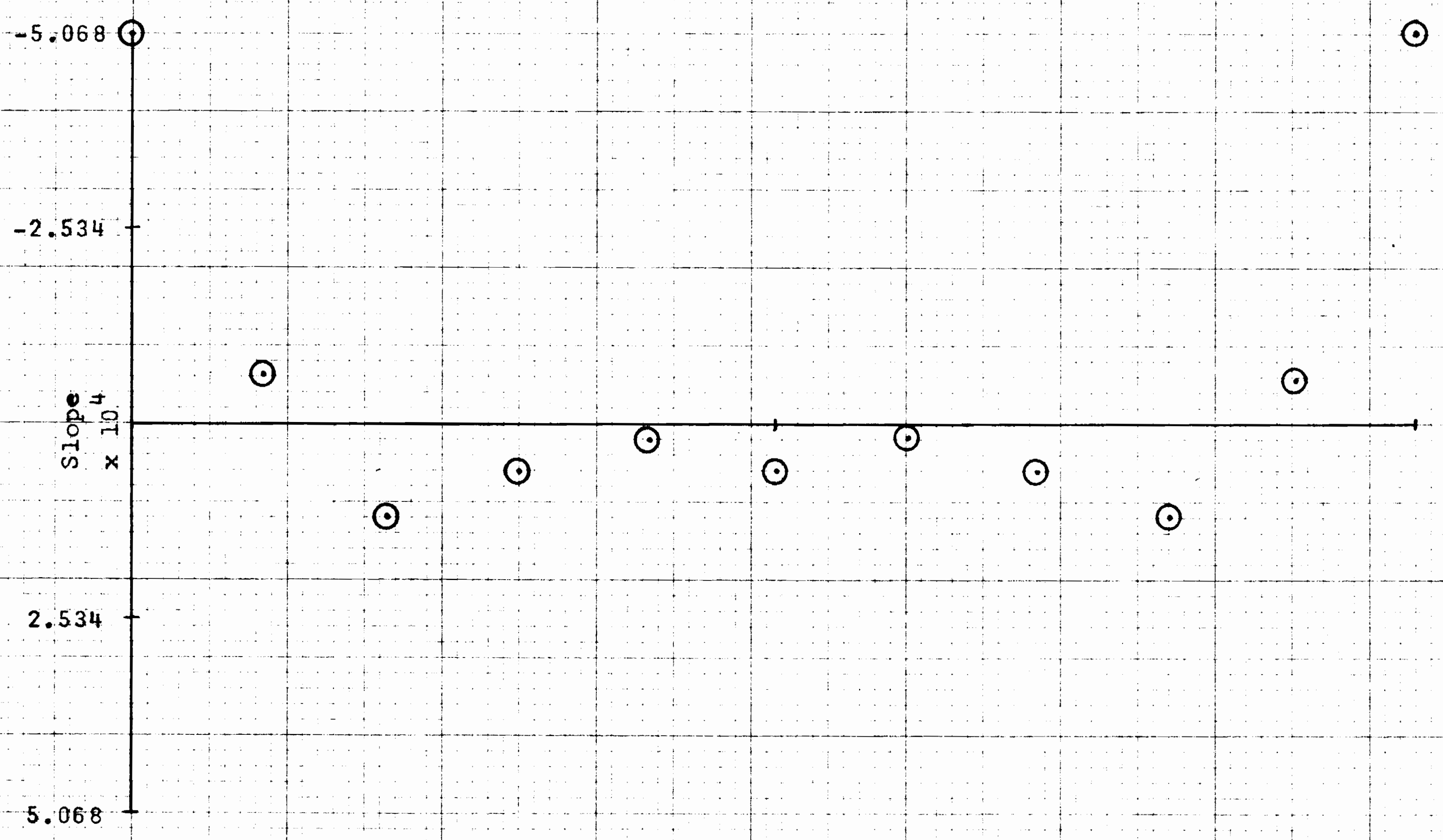
Graph 14. Centerline Deflection for a Cantilevered Triangular Plate with N = 16.



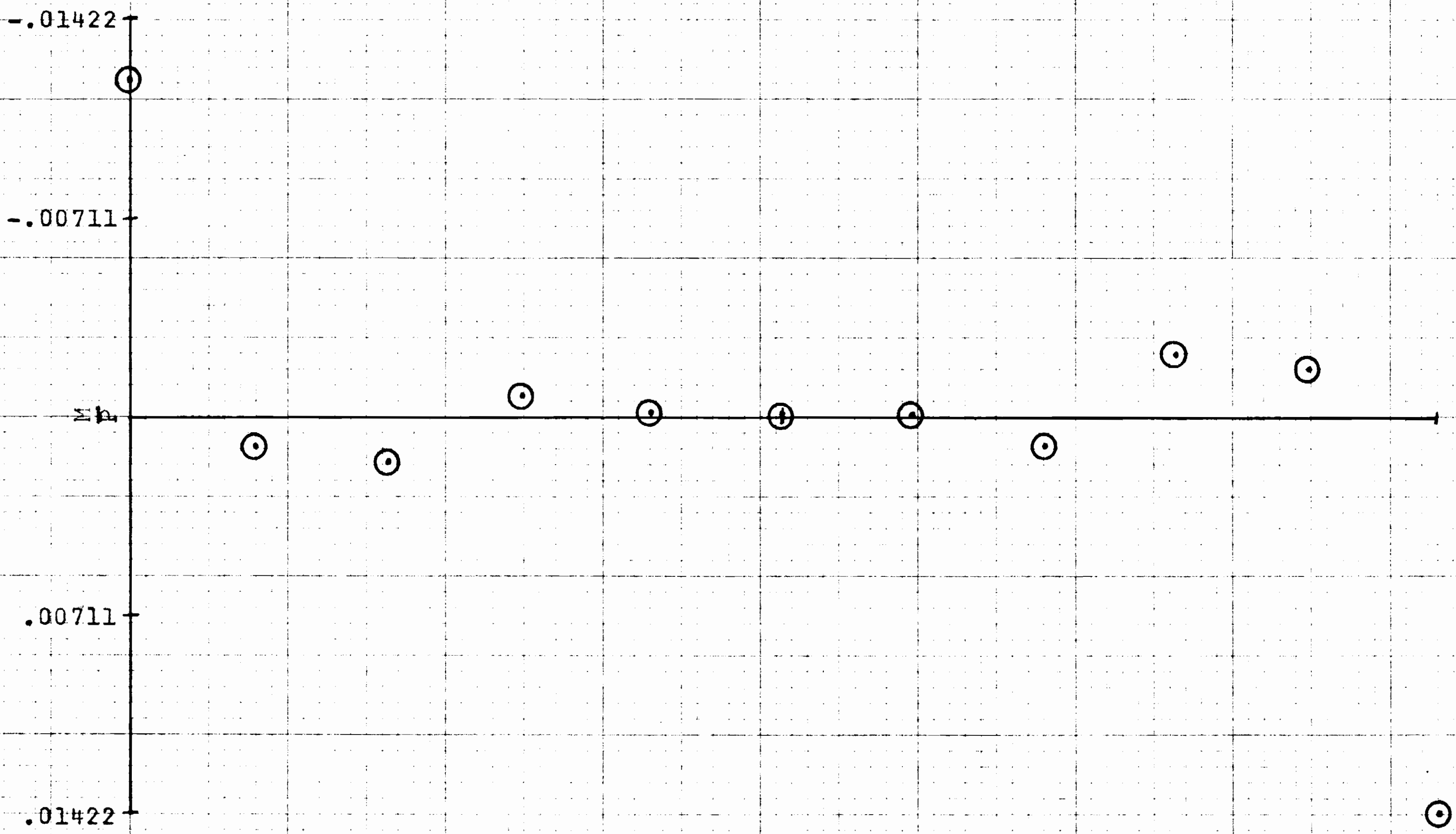
Graph 15. Centerline Moment for a Cantilevered Triangular Plate with N = 16.



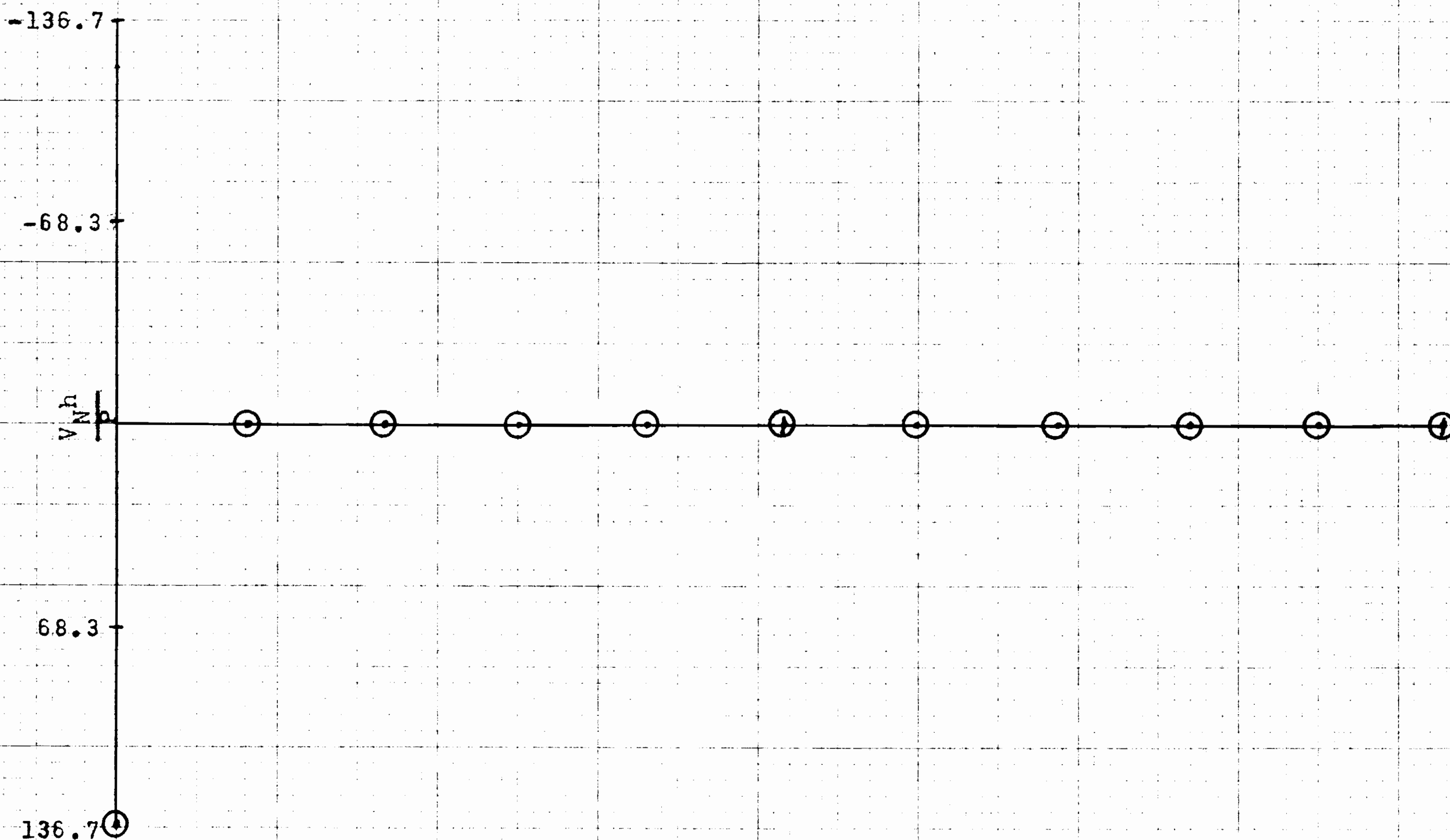
Graph 16. Deflection along the Fixed Edge for the Cantilevered Triangular Plate.



Graph 17. Normal Slope along the Fixed Edge for the Cantilevered Triangular Plate.



Graph 18. Normal Moment along the Free Edge for the Cantilevered Triangular Plate.



Graph 19. Kirchhoff shear along the Free Edge for the Cantilevered Triangular Plate.

α_{ij} is defined as shown in Figure 18. For equivalent loads applied along the inner retracted boundary, the angle α_{ij} is defined in Figure 19.

With the above definitions, and proceeding as in the previous paragraphs, we could obtain a set of $2(N_1 + N_2)$ simultaneous equations for the unknown equivalent loads.

For more than one hole, the same procedure of placing the retracted boundaries inside the holes and for defining the angles α_{ij} would be followed.

4. Effect of Retraction

The effect on the solution of different amounts of retraction was investigated for the circular plate problem discussed in paragraph a with N = 8. Five different values of the retracted distance $\overline{D}$ were used. The maximum deflection and the deflection and moment on the boundary at a point midway between two balance points were obtained using each value of $\overline{D}$. The results are presented in Table 3.

The only appreciable difference in results occurs when $\overline{D}$ = 5 inches. In this case, an examination of the values for the deflection and moment at the boundary indicates that the equivalent loads have been placed too close to the boundary for only eight balance points. For example, the moment on the boundary for $\overline{D}$ = 5 is 40 times as large as the moment for $\overline{D}$ = 500.

The maximum value of $\overline{D}$ appears to be limited only by

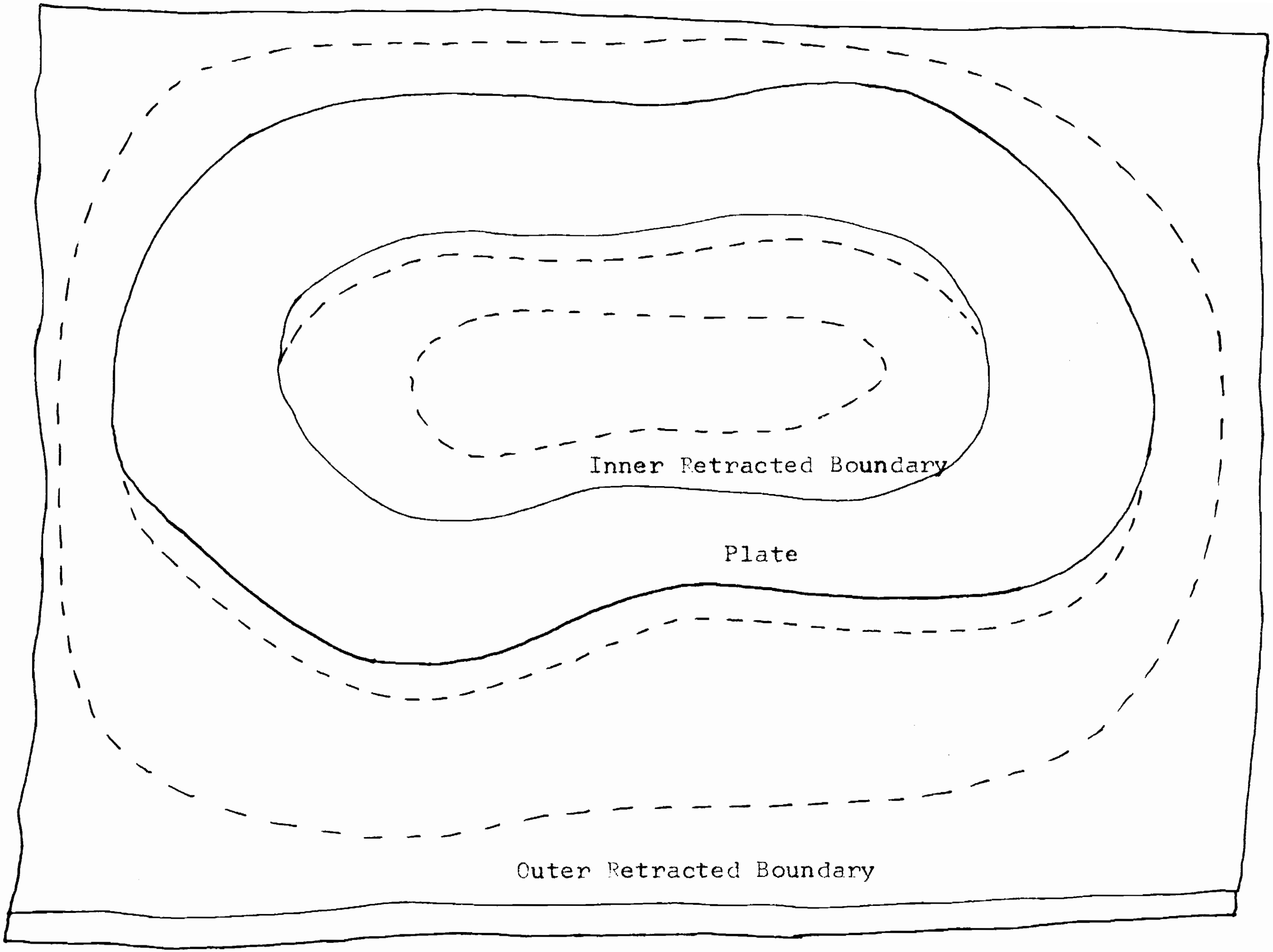


Figure 16. Plate with a Hole as Part of an Infinite Plate.

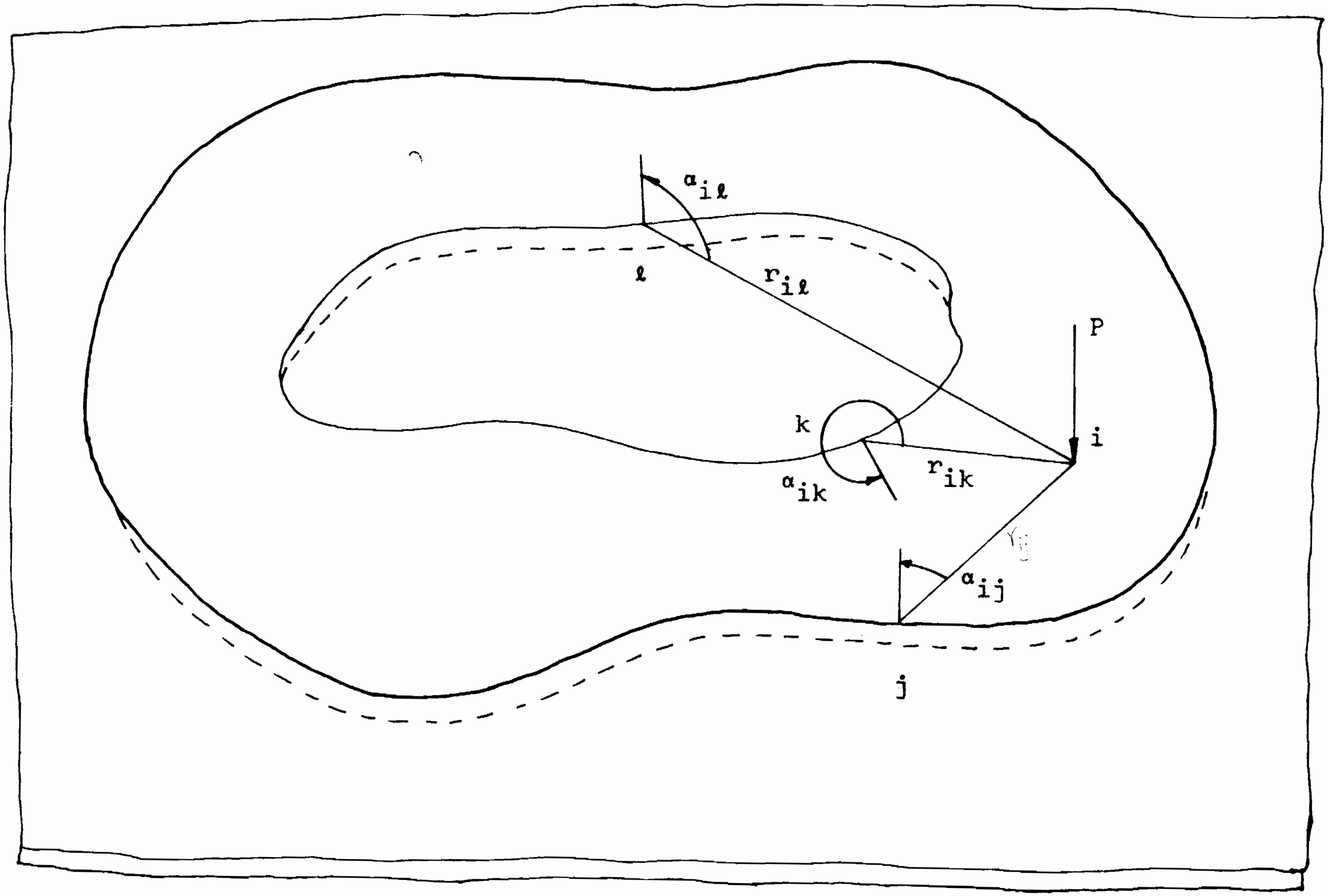


Figure 17. Concentrated Force Acting on a Plate with a Hole Which is Part of an Infinite Plate.

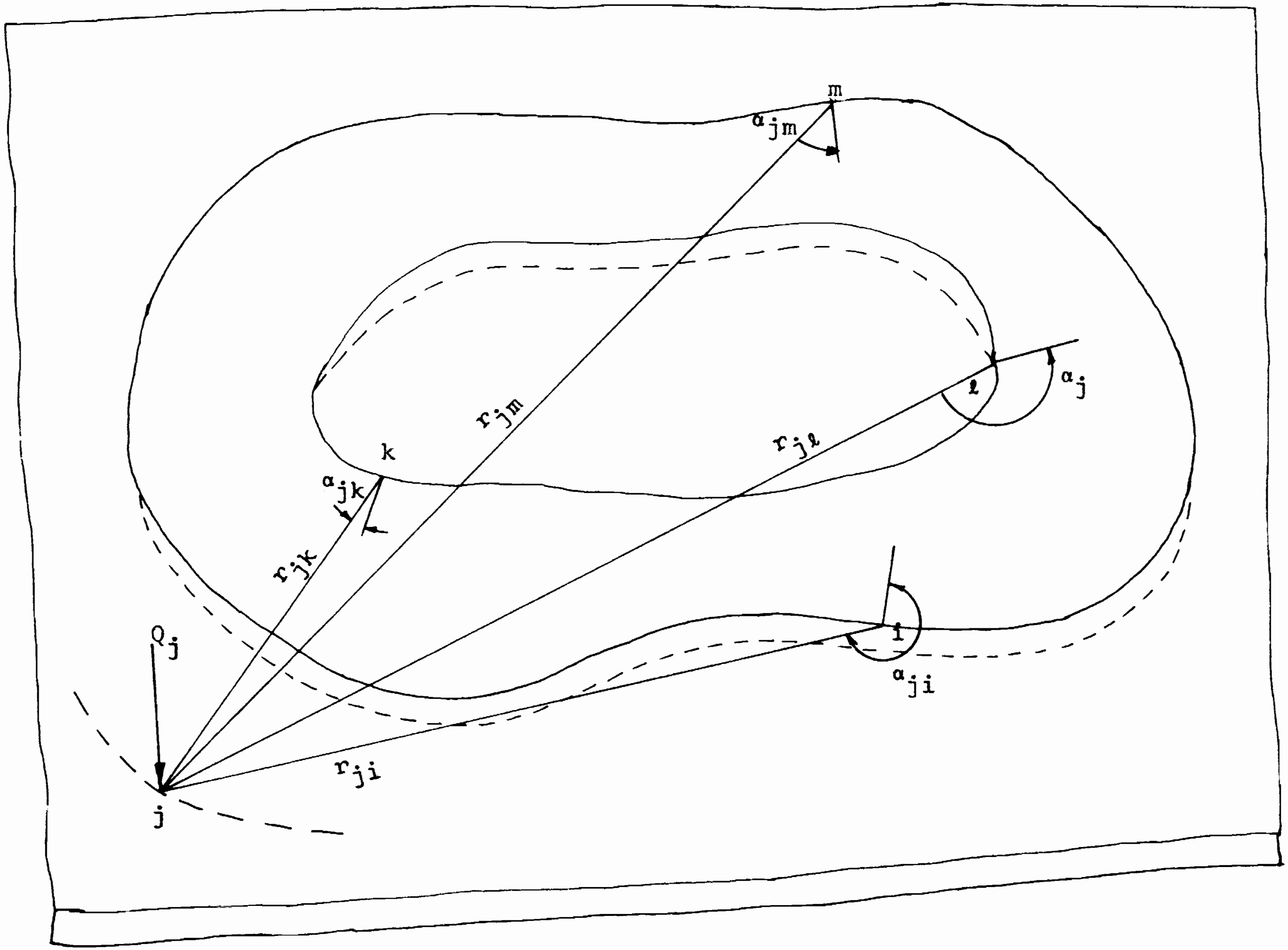


Figure 18. Equivalent Loads Applied on the Outer Retracted Boundary.

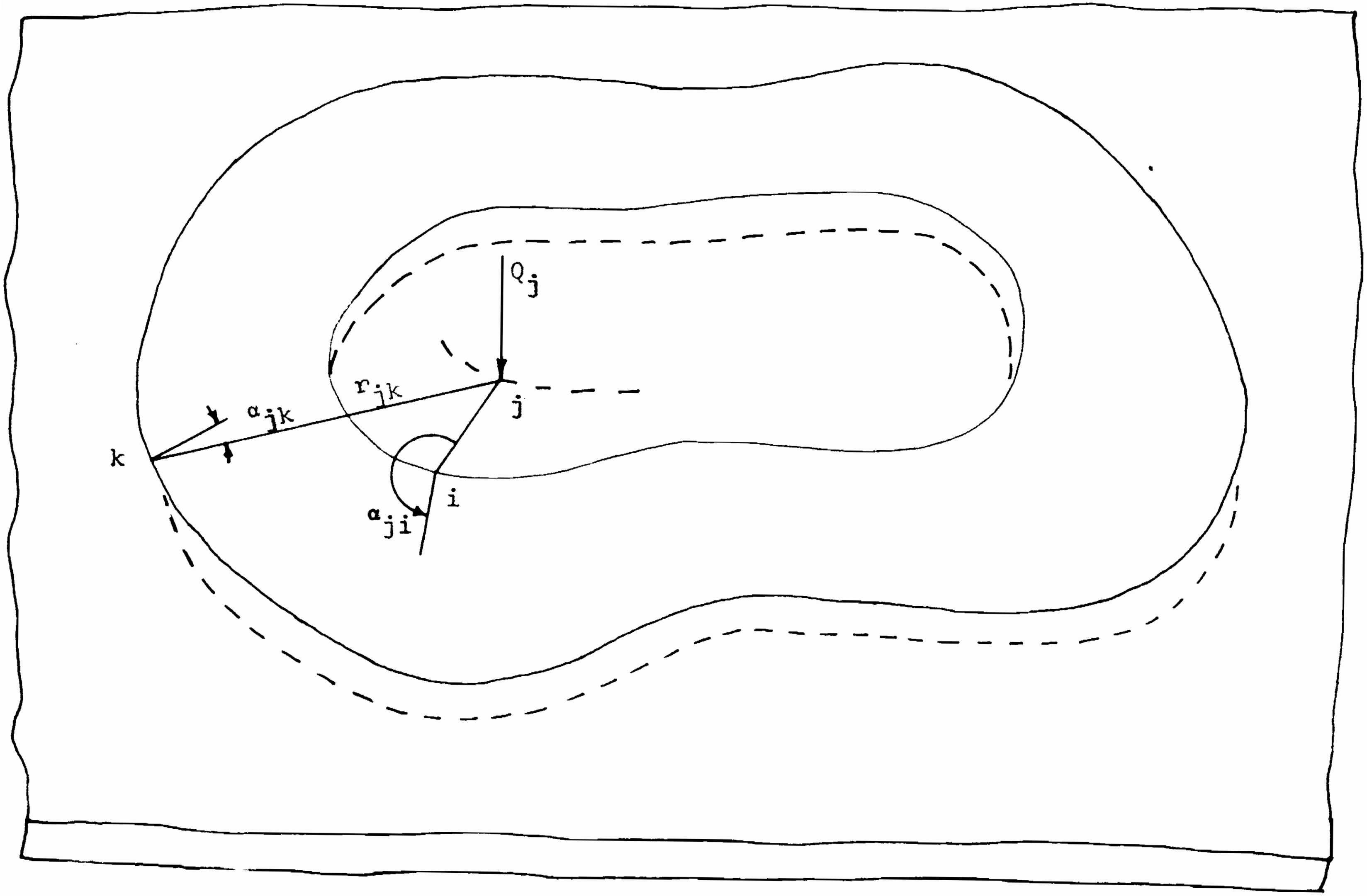


Figure 19. Equivalent Loads Applied on the Inner Retracted Boundary.

the number of significant figures retained in the numerical computations.

TABLE 3

Effect of Retraction on the Maximum Deflection and on the Deflection and Moment at the Boundary for a Circular Plate with N = 8

$\overline{D}$	w max	w boundary	M boundary
5	.03431	-.00132	3.149
25	.01347	-.00115	+.0977
200	.01311	-.00127	-.114
500	.01324	-.00132	-.0787
5000	.01325	-.00132	-.0787
inches	inches	inches	in.-lb.

V. DISCUSSION

The Reflection method, as outlined in Chapter IV, presents a new approach to the problem of obtaining numerical solutions for plates subjected to transverse loadings. The method of solution can be applied to plates with any arbitrary shape and applied with equal facility to either simple or complex shapes. The boundary conditions can be specified as desired along any portion of the boundary.

The accuracy of the solution depends on how well the boundary conditions are satisfied, which, in turn, depends on the magnitude of N. Since the coefficient matrix of the resulting 2N simultaneous equations is very ill-conditioned, large values of N should be avoided unless absolutely necessary. Where necessary, a large number of significant figures must be carried in the solution in order to insure accurate results. For the examples in Chapter IV, 14 significant figures were carried in the solution of the simultaneous equations.

In an attempt to avoid solving a large number of simultaneous equations, an iterative procedure similar to the one employed for beams was developed. Unfortunately, convergence of the iterative procedure could not be obtained despite several variations in approach. These procedures were checked and found to be satisfactory and rapidly convergent for better conditioned matrices than those which arose from the example problems.

In most problems where the plate can be represented as a simply connected region in the infinite plate, a value for N in the range 16 to 20 would be sufficient to achieve a close approximation of the required boundary conditions along the entire boundary. In general, the greatest difficulty in satisfying the boundary conditions usually occurs at sharp corners, similar to the problem encountered in the triangular plate problem discussed in paragraph d of Chapter IV. By dulling the corner as explained in the paragraph, a minimum increase in N alleviates the problem.

Sometimes, as in the rectangular plate problem of paragraph c, a good approximation of the boundary conditions can be achieved without special attention being paid to the corners. Thus, the determining factor in selecting a value for N is that minimum number which produces acceptable conditions along the boundary of the plate. Plots of measures of deviation from the boundary conditions can easily be obtained and any necessary increase in the value of N, and the most effective location for the additional points, can be determined from those plots.

The value of the constant a was found to be completely arbitrary since the use of different values had no effect upon the solution.

The magnitude of the retracted distance $\overline{D}$ exerts a varying influence on the solution as demonstrated in Section 4 of Chapter IV. If $\overline{D}$ is taken too small, then the influence of the equivalent loads on the portion of the boundary between the balance points can be significant and a large value for N would be necessary to insure close satisfaction of the boundary conditions. This undesired influence decreases rapidly as $\overline{D}$ increases. Thus, from Table 3, while an increase in $\overline{D}$ from 5 to 25 inches had a very noticeable effect on the solution, an increase from 25 to 5000 inches produced almost no variation in results. Therefore, while $\overline{D}$ is arbitrary, care should be taken not to place the equivalent loads too close to the boundary for the particular value chosen for N or too far from the boundary so as to greatly increase the number of significant figures necessary for a solution.

In paragraph d of Chapter IV, the solution by the reflection method was compared to a solution of the same problem by Conway [5]. The point matching method, as used by Conway, requires functions which satisfy the governing fourth order differential equation for thin plates and also satisfied exactly the boundary conditions along one edge. This method appears to be limited as to the type of loading, shape of the plate and to problems with constant boundary conditions along at least one edge.

One of the most powerful methods of obtaining numerical solutions is the method of finite differences [6]. In this method, the domain of the plate is divided into a net with n inner points and n finite difference equations are obtained which must be solved simultaneously. Usually, the number of finite difference equations would be approximately the same as the number of equations obtained by the Reflection method. Also, finite difference equations usually produce well conditioned coefficient matrices which are readily solvable. However, considerable difficulty is introduced into the finite difference method by irregular boundaries, which is not present in the Reflection method. Moreover, a single computer program, such as presented in the appendix, which is capable of developing and solving the equations for a large variety of problems, is not feasible for the finite difference method.

In the problems worked in Chapter IV, the advantages of symmetry were not used, although they would have greatly reduced the number of equations in each case. However, if symmetry is used, a different computer program would be required for each problem, but the number of equations is greatly reduced.

Based on the results of this thesis, a single computer program, similar to the one in the appendix, could be written for a large storage capacity digital computer which would be capable of solving almost any plate problem, limited only by the assumptions in Section 1 of Chapter IV.

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VIII. VITA

The author was born in Walterboro, South Carolina, on August 21, 1937. He graduated from Hempstead High School, Hempstead, New York, on June 24, 1955. In September 1955, he entered Virginia Polytechnic Institute and received his B.S. degree in Engineering Mechanics in June 1959. Since September 1959, the author has been a one-half time instructor in Engineering Mechanics at Virginia Polytechnic Institute. He received his M.S. degree in Engineering Mechanics in June 1963 and has since been pursuing courses leading to the degree of Doctor of Philosophy in Engineering Mechanics.

APPENDIX

Computer Program and Discussion

The following program, written in Fortran II for the IBM 1620 digital computer, can be used to solve many different plate problems of the type illustrated in Part 2 of Chapter IV.

The program is divided into three parts or passes. Pass 1, as presented in this appendix, sets up the simultaneous equations and punches them out on cards. Pass 2 reads in the simultaneous equations and solves them for the equivalent loads [8]. Pass 3 reads in the equivalent loads and can calculate the deflection, slope, moment or Kirchhoff shear at any point [8].

For any particular problem, the following data must be supplied in Pass 1.

N - number of balance points

NC - number of corners on the boundary

NBC - number of different boundary conditions

AL - arbitrary length a

D - retracted distance

XC,XY - coordinates of arbitrary center of a unifrom load

M - number of applied loads

Q - intensity of uniform load

E - modulus of elasticity

ANU - Poission's ratio

H - plate thickness

PL(I) - magnitudes of applied forces

CM(I) - magnitudes of applied moments

XL(I),YL(I) - coordinates of applied loads

LBC - last point on the boundary where a certain boundary condition applies

KBC - kind of boundary condition for each region of the boundary described by LBC(I)

ND - number of balance points between each corner

XR(I),YR(I) - coordinates of the corners

```
C     PASS I A
      DIMENSION X(17),Y(17),PL(5 ),CM(5 ),XL(5 ),YL(5 ),A(32,33),B(32),
     1WI(16),BI(16),PT(16),TM(16)    ,LBC(8) , KBC(8)
      COMMONX,Y,AL,BL,D,BETI,BETJ,ALP,THE,FR,N,R,ANU,Q,PI,A,B,IS,XC,YC,
     1PL,CM,XL,YL,M,LBC,KBC,NBC,            MNP,MN,WI,BI,PT,TM
      PI = 3.1415927
      CALL ASSX
      CALL SIML
      PUNCH 1, (I,B(I),I = 1,MN)
    1 FORMAT(I3,E14.8)
      STOP
      END
      SUBROUTINE ASSX
      DIMENSION X(17),Y(17),PL(5 ),CM(5 ),XL(5 ),YL(5 ),A(32,33),B(32),
     1WI(16),BI(16),PT(16),TM(16)    ,LBC(8) , KBC(8)
      COMMONX,Y,AL,BL,D,BETI,BETJ,ALP,THE,FR,N,R,ANU,Q,PI,A,B,IS,XC,YC,
     1PL,CM,XL,YL,M,LBC,KBC,NBC,            MNP,MN,WI,BI,PT,TM
      READ 48,N,NC,NBC,AL,D,XC,YC
   48 FORMAT(3I4,4E10.4)
      READ 49, M,Q,E,ANU,H
   49 FORMAT(I4,4E10.4)
      READ 4,(PL(I),CM(I),XL(I),YL(I),I = 1,M)
    4 FORMAT(4E10.4)
      READ 5,(LBC(I),KBC(I),I = 1,NBC)
      NR= NC+1
      READ 5 (ND(I),I=1,NC)
    5 FORMAT(17I4)
      READ 6,(XR(I),YR(I),I=1,NR)
     6 FORMAT (2E10.4)
      FR = E*H**3/(12.0*(1.0-ANU**2))
      JPREV = 0
      DO 20 I = LA,LB
      BETA = ANG(Y2(I),Y1(I),X2(I),X1(I),0)
      DIST = SQRTF((X2(I)-X1(I))**2+(Y2(I)-Y1(I))**2)
      ND1 = ND(I)+1
      DIV = ND(I)
      DIV = DIST/DIV
      DO 21 J = 1,ND1
      K = J+JPREV
      V = J-1
      FRAC = V*DIV
      X(K) = X1(I)+FRAC*COSF(BETA)
   21 Y(K) = Y1(I)+FRAC*SINF(BETA)
   20 JPREV = K-1
      RETURN
      END
      FUNCTION ANG(T2,T1,B2,B1,NORM)
      PI = 3.1415927
      IF(NORM)1,2,1
    1 A = B2-B1
      B = T1-T2
      GO TO 3
    2 A = T2-T1
      B = B2-B1
    3 IF(A)10,14,4
    4 IF(B)5,6,7
   10 IF(B)5,8,9
   14 IF(B)13,13,11
```

```
 13 ANG = PI
    RETURN
 11 ANG = 0.0
    RETURN
  5 ANG = PI  + ATANF(A/B)
    RETURN
  6 ANG = PI/2.0
    RETURN
  7 ANG = ATANF(A/B)
    RETURN
  8 ANG = 3.0*PI/2.0
    RETURN
  9 ANG = 2.0*PI+ATANF(A/B)
    RETURN
    END
    SUBROUTINE SIML
    DIMENSION X(17),Y(17),PL(5 ),CM(5 ),XL(5 ),YL(5 ),A(32,33),B(32),
   1WI(16),BI(16),PT(16),TM(16)    ,LBC(8) , KBC(8)
    COMMONX,Y,AL,BL,D,BETI,BETJ,ALP,THE,FR,N,R,ANU,Q,PI,A,B,IS,XC,YC,
   1PL,CM,XL,YL,M,LBC,KBC,NBC,           MNP,MN,WI,BI,PT,TM
    MN = 2*N
    MNP = MN+1
    DO 100 I = 1,MN
    DO 100 J = 1,MNP
100 A(I,J) = 0.0
    CALL ARRAY
    IF(SENSE SWITCH 2)6,1
  6 IF(SENSE SWITCH 3)4,2
  2 READ 3,((K,L,A(I,J),J = 1,MN),I = 1,MN)
  3 FORMAT(2I3,E14.8)
    GO TO 1
  4 READ 5,((A(I,J),J = 1,MN),I = 1,MN)
  5 FORMAT(5E14.8)
  1 CALL TRI(1,MN-1,MN-1,MNP)
    CALL SOLB
    RETURN
    END
    SUBROUTINE ARRAY
    DIMENSION X(17),Y(17),PL(5 ),CM(5 ),XL(5 ),YL(5 ),A(32,33),B(32),
   1WI(16),BI(16),PT(16),TM(16)    ,LBC(8) , KBC(8)
    COMMONX,Y,AL,BL,D,BETI,BETJ,ALP,THE,FR,N,R,ANU,Q,PI,A,B,IS,XC,YC,
   1PL,CM,XL,YL,M,LBC,KBC,NBC,           MNP,MN,WI,BI,PT,TM
    DO 1 I = 1,N
    IN = I+N
    BETI = ANG(Y(I),Y(I+1),X(I),X(I+1),1)
    CALL CORD(XPI,YPI,I,PXI,PYI,BETI)
    LM = NBC+1
    DO 3 L = 1,NBC
  3 LM = LM-1/(1+I/LBC(L))
    IS = KBC(LM)
    IF(SENSE SWITCH 2)8,9
  9 DO 2 J = 1,N
    JN = J+N
    BETJ = ANG(Y(J+1),Y(J),X(J+1),X(J),1)
    CALL CORD(XPJ,YPJ,J,PXJ,PYJ,BETJ)
    CALL RAD(PXJ,PYJ,XPI,YPI)
  2 CALL COEF(1.,1.,R,ALP,THE,IS,A(I,J),A(I,JN),A(IN,J),A(IN,JN))
```

```
 8 BETJ = 0.0
   CALL LOADI(XPI.YPI.I.IS)
   A(I.MNP) = -WI(I)
 1 A(IN.MNP) = -BI(I)
   IF(SENSE SWITCH 1)11,7
11 IF(SENSE SWITCH 2)10,6
 6 IF(SENSE SWITCH 3)12,13
12 PUNCH 14.((A(I,J).J = 1.MN).I = 1.MN)
14 FORMAT(5E14.8)
   GO TO 10
13 PUNCH 4.((I.J.A(I.J),J = 1.MN).I = 1.MN)
 4 FORMAT(2I3.E14.8)
10 PUNCH 5.(I.A(I.MNP).I = 1.MN)
 5 FORMAT(I3.E14.8)
 7 RETURN
   END
   SUBROUTINE CORD(XP.YP.I.PX.PY.BET)
   DIMENSION X(17),Y(17),PL(5 ),CM(5 ),XL(5 ),YL(5 ).A(32,33),B(32),
  1WI(16).BI(16).PT(16),TM(16)     ,LBC(8) , KBC(8)
   COMMONX.Y.AL.BL,D,BETI,BETJ.ALP.THE,FR,N,R,ANU,Q,PI,A,B,IS,XC,YC,
  1PL.CM.XL,YL.M,LBC,KBC.NBC.            MNP,MN,WI.BI,PT,TM
   KI = I+1
   XP = (X(I)+X(KI))/2.0
   YP = (Y(I)+Y(KI))/2.0
   PX = XP-D*COSF(BET)
   PY = YP-D*SINF(BET)
   RETURN
   END
   SUBROUTINE RAD(PX,PY,XP,YP)
   DIMENSION X(17),Y(17),PL(5 ),CM(5 ).XL(5 ).YL(5 ).A(32,33),B(32),
  1WI(16).BI(16).PT(16).TM(16)     .LBC(8) , KBC(8)
   COMMONX.Y.AL.BL.D.BETI,BETJ.ALP.THE.FR.N.R,ANU,Q,PI.A.B,IS.XC,YC,
  1PL.CM.XL.YL.M,LBC.KBC.NBC.            MNP.MN.WI.BI,PT.TM
   R = SQRTF((PX-XP)**2+(PY-YP)**2)
   RLM = ANG(YP.PY.XP.PX.0)
   ALP = BETI-RLM
   THE = RLM-BETJ
   RETURN
   END
   SUBROUTINE COEF(P.BM.S.ALT.PHE.IS.F1.F2,F3.F4)
   DIMENSION X(17),Y(17),PL(5 ).CM(5 ),XL(5 ),YL(5 ),A(32,33),B(32),
  1WI(16),BI(16).PT(16),TM(16)     .LBC(8) , KBC(8)
   COMMONX,Y.AL,BL,D,BETI,BETJ.ALP,THE.FR.N.R,ANU,Q.PI,A,B,IS,XC,YC,
  1PL,CM.XL,YL.M.LBC,KBC,NBC,            MNP,MN,WI.BI,PT.TM
   GO TO(1,1.2),IS
 1 F1 = DEFP(P.S.AL.FR)
   F2 = DEFM(BM,S.AL.PHE.FR)
   GO TO(3.4).IS
 3 F3 = SLOP(P.S.AL.ALT.FR)
   F4 = SLOM(BM.S.AL.ALT.PHE.FR)
   GO TO 5
 2 F1 = SHEP(P.S.ALT)
   F2 = SHEM(BM,S.ALT.PHE.ANU)
 4 F3 = BENP(P.S.AL,ALT.ANU)
   F4 = BENM(BM,S.PHE,ALT.ANU)
 5 RETURN
   END
   CALL LOADI(XS.YS.J.IS)
```

```
      CALL LOADB(XS,YS,J,IS,PT,TM)
    3 CONTINUE
      CALL PLOT(WI,RA,1,11,4)
      PUNCH 158
      PUNCH 160, BETA
      CALL PLOT(BI,RA,1,11,4)
      PUNCH 159
      PUNCH 160, BETA
  160 FORMAT(11HORIENTATIONF9.4,7HRADIANS/////)
   20 CONTINUE
      RETURN
      END
```

ABSTRACT

The problem of determining the deflection and stress in a plate under transverse loading can be approached by first considering the plate to be a portion of an infinite plate, ignoring the prescribed boundary conditions. The known loads are then applied to the infinite plate and their effects are calculated at those points which correspond to the boundary of the original plate. A system of suitably chosen loads and moments is then applied on the infinite plate at points beyond the boundary of the original plate such that the prescribed boundary conditions are satisfied.

For an exact solution, the number of external loads and moments would have to be infinite. However, in order to deal with the problem numerically, only a finite number of each are considered. Thus, solutions are obtained by satisfying the boundary conditions at only a finite number of points. The method is illustrated for beams and then extended to plates.

Several problems with known solutions are solved and the results compared with the exact values. Also, plots of the deflection and moment along the centerline of a cantilevered triangular plate are presented.

Discussions of the problem of plates with holes and the effect on the solution of various placements of the balancing loads are also presented.

An IBM 1620 digital computer is used to facilitate calculations.