AIP | Journal of Mathematical Physics

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Citation: Journal of Mathematical Physics **21**, 2114 (1980); doi: 10.1063/1.524719
View online: http://dx.doi.org/10.1063/1.524719
View Table of Contents: http://scitation.aip.org/content/aip/journal/jmp/21/8?ver=pdfcov
Published by the AIP Publishing

Nth-order multifrequency coherence functions: A functional path integral approach. II [a)]

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(Received 17 December 1979; accepted for publication 29 February 1980)

Nth-order multifrequency coherence functions arising in beam propagation through focusing media with random-axis misalignments and focusing media with additive statistical fluctuations are computed. The analysis is carried out by means of a simple formula which yields exact algorithmic solutions to a class of canonical path integrals.

1. INTRODUCTION

In a previous paper,[1] referred to in the sequel as Paper I, a functional (or path) integral applicable to a broad class of randomly perturbed media was constructed for the n^{th}-order multifrequency coherence function, a quantity intimately linked to n^{th}-order pulse statistics. This path integral was subsequently carried out explicitly in the case of a nondispersive, deterministically homogeneous medium characterized by a simplified (quadratic) Kolmogorov spectrum. It is our purpose in this paper to lift the restriction of a background flat medium. Specifically, we shall compute n^{th}-order multifrequency coherence functions arising in beam propagation through focusing media with random-axis misalignments, and focusing media with additive statistical fluctuations. We shall carry out this task by means of a simple basic formula which allows exact algorithmic evaluations to a class of canonical path integrals.

Our work in Paper I was based on the stochastic Cauchy problem

$$\frac{i}{k}\frac{\partial}{\partial z}\psi(\underline{x},z,w;\alpha) = H_{\text{op}}\left(\underline{x}, -\frac{i}{k}\nabla_{\underline{x}}, z,w;\alpha\right)\psi(\underline{x},z,w;\alpha), \tag{1.1a}$$

$\underline{x}\in R^2, \quad z>0,$

$$H_{\text{op}}\left(\underline{x}, -\frac{i}{k}\nabla_{\underline{x}},z,w;\alpha\right) = -\frac{1}{2k^2}\nabla^2_{\underline{x}} + V(\underline{x},z,w;\alpha), \tag{1.1b}$$

$$\psi(\underline{x},0,w;\alpha) = \psi_0(\underline{x},w). \tag{1.1c}$$

The "Hamiltonian" H_{op} is a self-adjoint stochastic operator depending on a parameter $\alpha\in A$, (A,F,P) being an underlying probability measure space. In addition, w in (1.1) denotes the radian frequency, $k\equiv k(w)$ the wave number, $\psi(\underline{x},z,w;\alpha)$ the complex random wavefunction, and $V(\underline{x},z,w;\alpha)$, the "potential" field which is assumed to be a real random function. The initial condition $\psi_0(\underline{x},w)$ incorporates all the information concerning the temporal frequency spectrum and the spatial distribution of the source at the initial plane $z=0$.

In the course of this work we shall deal explicitly with the following two distinct categories of the potential field $V(\underline{x},z,w;\alpha)$ entering into (1.1b):

$$\text{(i)}\; V(\underline{x},z,w;\alpha) = \tfrac{1}{2}g^2[x - a\underline{H}(z;\alpha)]^2, \tag{1.2a}$$

$$\text{(ii)}\; V(\underline{x},z,w;\alpha) = \tfrac{1}{2}g^2 x^2 - \tfrac{1}{2}\epsilon_1(\underline{x},z;\alpha), \tag{1.2b}$$

where $x=|\underline{x}|$, a is a constant, and g is a spatial frequency (units: radians/meter). The first category corresponds to a parabolically focusing medium whose equilibrium axis is perturbed via the zero-mean, range-dependent, vector-valued, random function $\underline{H}(z;\alpha)$; on the other hand, the second category represents a medium whose parabolically graded deterministic profile is additively perturbed by the zero-mean random function $\epsilon_1(\underline{x},z;\alpha)$. The absence of the angular frequency w in the right-hand sides of (1.2) signifies that the media are assumed to be nondispersive.[2]

Besides their generic significance in quantum mechanics,[3] Schrödinger-like equations of the form (1.1) and (1.2) play a significant role in plane and beam electromagnetic and acoustic wave propagation. They are usually derived from a scalar Helmholtz equation within the framework of the parabolic (or small-angle) approximation. In this context, the complex stochastic parabolic equation (1.1) with potentials (i) and (ii) provides a good description of the forward propagation of low-order modes in a fiber lightguide having a randomly perturbed parabolically graded refractive index. It can also give some insight into the problem of forward propagation of low-order acoustic modes near an idealized, randomly perturbed, underwater, sound channel axis.

The problem under consideration in this paper, that is the study of n^{th}-order pulse statistics associated with (1.1) and (1.2), can be made more specific as follows: Let $G(\underline{x},\underline{x}',z,w;\alpha)$ denote the fundamental solution (referred to alternatively as the propagator) of the stochastic complex parabolic equation (1.1). It follows, then, from the discussion in Paper I that the examination of pulse propagation in a random medium requires knowledge of the n^{th}-order coherence functions $E\{G^{(n)}(\underline{X},\underline{X}',z,\underline{w};\alpha\}$–at different frequencies and different transverse (with respect to z) coordinates. Here, the operator $E\{\cdot\}$ signifies ensemble averaging, the index n is assumed to be an even integer, $\underline{X}=(\underline{x}_1,\underline{x}_2,\cdots,\underline{x}_n)$

[a)] Research supported in part by the Office of Naval Research under contract No. N00014-76-C-0056.

$\in R^{2n}$, $\underline{X}' = (\underline{x}'_1, \underline{x}'_2, \cdots, \underline{x}'_n) \in R^{2n}$, and $\underline{w} = (w_1, w_2, \cdots, w_n) \in R^{n}$; finally, the nth-order propagators $G^{(n)}$ are defined in terms of G as follows:

$$G^{(n)}(\underline{X}, \underline{X}', z, \underline{w}; \alpha) = \prod_{p=1}^{n/2} G^*(\underline{x}_{2p}, \underline{x}'_{2p}, z, w_{2p}; \alpha) \times G(\underline{x}_{2p-1}, \underline{x}'_{2p-1}, z, w_{2p-1}; \alpha). \tag{1.3}$$

The fundamental solution $G(\underline{x}, \underline{x}', z, w; \alpha)$ to (1.1) can be expressed as a continuous functional path integral. This can be used, in turn, as a basis for constructing a path integral representation for the nth-order quantity $G^{(n)}(\underline{X}, \underline{X}', z, w; \alpha)$; specifically, $G^{(n)}(\underline{X}, \underline{X}', z, \underline{w}; \alpha)$

$$= \int d[\underline{X}(\zeta)] \exp\left\{\frac{i}{2} \sum_{p=1}^{n} \xi_p k_p \times \int_0^z d\zeta \, \{\dot{\underline{x}}_p^2(\zeta) - 2V[\underline{x}_p(\zeta), \zeta, w_p; \alpha]\}\right\}, \tag{1.4}$$

where $d[\underline{X}(\zeta)] = d[\underline{x}_1(\zeta)] d[\underline{x}_2(\zeta)] \cdots d[\underline{x}_n(\zeta)]$ is the usual Feymann path differential measure; $\xi_p = 1, p$ odd, $\xi_p = 1, p$ even; $k_p = k(w_p)$; and the integration is over all paths $\underline{x}_p(\zeta)$ $p = 1, 2, \ldots, n$, subject to the boundary conditions $\underline{x}_p(0) = \underline{x}'_p, x_p \; \underline{x}_p(z) = \underline{x}_p$.

The ensemble-averaged version of the path integral (1.4) for potentials (i) and (ii) [cf. Eq. (1.2)] will be evaluated in Secs. 3 and 4, respectively. These computations will be made on the basis of a simple formula which will be derived in the next section.

2. DERIVATION OF A BASIC FORMULA

A. Focusing Medium with Random-axis Misalignments: First-order Moment

Consider the stochastic parabolic equation (1.1) with a slightly modified potential of type (i) [cf. Eq. (1.2a)], viz.,

$$\frac{i}{k}\frac{\partial}{\partial z}\psi(x, z, w; \alpha) = -\frac{1}{2k^2}\nabla^2_{\underline{x}}\, \psi(\underline{x}\, z, w; \alpha) + \tfrac{1}{2} g^2 \underline{x}^2 \psi(\underline{x}, z, w; \alpha) - g^2 \bar{a}\underline{x}\cdot\underline{H}(z;\alpha)\psi(\underline{x}, z, w; \alpha) + \tfrac{1}{2}\lambda g^2 \bar{a}^2 H^2(z;\alpha) \times \psi(\underline{x}, z, w; \alpha). \tag{2.1}$$

The extra multiplicative factor λ appearing in (2.1) should be set equal to unity in order to maintain a correspondence between (2.1) and (1.1), with potential (i) given in (1.2a). The importance of this factor will be made clear in the sequel. It should also be noted that the constant factor "a" incorporated in (1.2a) has been changed to "$\bar{a}$" in (2.1), again for reasons which will be explained later on.

We set as our immediate goal the derivation of an equation for the first moment of the wavefunction $\psi(\underline{x}, z, w; \alpha)$. In order to carry out this task, we ensemble-average both sides of (2.1) over the realizations $\alpha \in A$:

$$\frac{i}{k}\frac{\partial}{\partial z}E\{\psi(\underline{x}, z, w; \alpha)\} = \frac{1}{2k^2}\nabla^2_{\underline{x}}\, E\{\psi(\underline{x}, z, w; \alpha)\} + \tfrac{1}{2}g^2 x^2 E\{\psi(\underline{x}, z, w; \alpha)\} - g^2\bar{a}x\cdot E\{H(z;\alpha)\,\psi(\underline{x}, z, w; \alpha)\} + \tfrac{1}{2}\lambda g^2 \bar{a}^2 E\{H^2(z;\alpha)\psi(\underline{x}, z, w; \alpha)\}. \tag{2.2}$$

To proceed further, we shall need expressions for the last two terms on the right-hand side of (2.2), viz.,

$$\underline{x}\cdot E\{\underline{H}(z;\alpha)\psi(\underline{x}, z, w; \alpha)\} = \sum_{j=1}^{2} x_j \, E\{H_j(z;\alpha)\psi(\underline{x}, z, w; \alpha)\}, \tag{2.3a}$$

$$E\{H^2(z;\alpha)\psi(\underline{x}, z, w; \alpha)\} = \sum_{j=1}^{2} E\{H_j^2(z;\alpha)\psi(\underline{x}, z, w; \alpha)\}, \tag{2.3b}$$

in terms of the first- and higher-order moments of $\psi(\underline{x}, z, w; \alpha)$. This "closure" problem will be examined next for a special class of random functions $\underline{H}(z;\alpha)$.

Let $\underline{H}(z;\alpha)$ be a zero-mean, wide-sense stationary, Gaussian random process with autocovariance tensor

$$E\{H_j(z;\alpha)H_k(z';\alpha)\} = \Gamma_{jk}(z - z'), \; j, k = 1, 2. \tag{2.4}$$

It follows, then, from the Donsker–Furutsu–Novikov functional formalism[4-6] that

$$E\{H_j(z;\alpha)\psi(\underline{x}, z, w; \alpha)\} = \sum_{k=1}^{2}\int_0^z dz' \, \Gamma_{jk}(z - z') E\{\delta\psi(\underline{x}, z, w; \alpha)/\delta H_k(z';\alpha)\}, \tag{2.5a}$$

$$E\{H_j^2(z;\alpha)\psi(\underline{x}, z, w; \alpha)\} = \Gamma_{jj}(0)\, E\{\psi(\underline{x}, z, w; \alpha)\} + \sum_{k=1}^{2}\int_0^z dz' \, \Gamma_{jk}(z - z')\, E\{H_j(z;\alpha) \times [\delta\psi(\underline{x}, z, w; \alpha)/\delta H_k(z';\alpha)]\}, \tag{2.5b}$$

where $\delta(\cdot)/\delta(\cdot)$ denotes functional differentiation.

Unless further restrictions are imposed on the process $\underline{H}(z;\alpha)$, it turns out that the computation of the functional derivatives together with the performance of the ensemble averaging entering into (2.5) lead to an infinite hierarchy,[7] which, in turn, exhibits the impossibility of "closing" Eq. (2.2) for $E\{\psi(\underline{x}, z, w; \alpha)\}$. Closure may be effected by truncating this infinite hierarchy. Such a truncation leads to well-known statistical approximations (e.g., the first-order smoothing and the direct-interaction approximation).

In the following discussion we shall eliminate altogether the aforementioned closure difficulties by restricting the process $\underline{H}(z;\alpha)$ to be both isotropic and δ-correlated, viz.,

$$\Gamma_{jk}(z - z') = (\sigma^2/2)\delta_{jk}\delta(z - z'),$$

where σ is a constant. With this assumption, (2.5a) and (2.5b) simplify to[8,9]

$$E\{H_j(z;\alpha)\psi(\underline{x}, z, w; \alpha)\} = (\sigma^2/2) E\{\delta\psi(\underline{x}, z, w; \alpha)/\delta H_j(z;\alpha)\}, \tag{2.6a}$$

$$E\{H_j^2(z;\alpha)\psi(\underline{x}, z, w; \alpha)\} = \Gamma_{jj}(0)\, E\{\psi(\underline{x}, z, w; \alpha) + (\sigma^2/2)\, E\{H_j(z;\alpha)\,[\delta\psi(\underline{x}, z, w; \alpha)/\delta H_j(z;\alpha)]\}. \tag{2.6b}$$

The functional derivative $\delta\psi/\delta H_j$ required in (2.6) can be found from the original stochastic complex parabolic equation (2.1). Omitting intermediate steps, we present here the final result:

$$\frac{\delta\psi(\underline{x},z,w;\alpha)}{\delta H_j(z;\alpha)} = ikg^2\bar{a}x_j\psi(\underline{x},z,w;\alpha) - ik\lambda g^2\bar{a}^2H_j(z;\alpha)\psi(\underline{x},z,w;\alpha). \quad (2.7)$$

Equation (2.7) together with (2.6) result in closed form solutions for $E\{H_j\psi\}$ and $E\{H_j^2\psi\}$ in terms of $E\{\psi\}$. When these two expressions are used then in conjunction with (2.3) and (2.2), the desired equation for the first statistical moment of ψ is obtained; specifically,

$$\frac{i}{k}\frac{\partial}{\partial z}E\{\psi(\underline{x},z,w;\alpha)\} = -\frac{1}{2k^2}\nabla^2_{\underline{x}}E\{\psi(\underline{x},z,w;\alpha)\} + V_e(\underline{x})E\{\psi(\underline{x},z,w;\alpha)\},\ z>0, \quad (2.8a)$$

$$E\{\psi(\underline{x},0,w;\alpha)\} = \psi_0(\underline{x},w), \quad (2.8b)$$

with the effective potential $V_e(\underline{x})$ given as

$$V_e(\underline{x}) = B(\lambda;k;\bar{a}) + \tfrac{1}{2}\Omega^2(\lambda;k;\bar{a})x^2, \quad (2.8c)$$

where

$$B(\lambda;k;\bar{a}) = \tfrac{1}{4}\lambda g^2\bar{a}^2\sigma^2[1+ik\lambda g^2\bar{a}^2(\sigma^2/2)]^{-1}\sum_{j=1}^{2}\Gamma_{jj}(0), \quad (2.8d)$$

$$\Omega^2(\lambda;k;\bar{a}) = g^2\{1 - ikg^2\bar{a}^2\sigma^2[1+k\lambda g^2(\sigma^2/4)] \times [1+ik\lambda g^2\bar{a}^2(\sigma^2/2)]^{-2}\}. \quad (2.8e)$$

B. The Propagator of Eq. (2.8)

Let $G(\underline{x},\underline{x}',z,w;\alpha)$ denote the propagator of the stochastic parabolic equation (2.1). It is defined by means of the relationship

$$\psi(\underline{x},z,w;\alpha) = \int_{R^2}d\underline{x}'\,G(\underline{x},\underline{x}',z,w;\alpha)\,\psi_0(\underline{x}',w). \quad (2.9)$$

Averaging both sides of the last equation gives rise to the expression

$$E\{\psi(\underline{x},z,w;\alpha)\} = \int_{R^2}d\underline{x}'\,E\{G(\underline{x},\underline{x}',w;\alpha)\}\psi_0(\underline{x}',w). \quad (2.10)$$

The quantity $E\{G(\underline{x},\underline{x}',w;\alpha)\}$ in (2.10) is clearly the propagator of (2.8). It can be written as a continuous path integral, viz.,

$$E\{G(\underline{x},\underline{x}',w;\alpha)\} = \exp[-ikB(\lambda;k;\bar{a})z] \times\int d[\underline{x}(\zeta)]\exp\{ik\int_0^z d\zeta\,[\tfrac{1}{2}\dot{\underline{x}}^2(\zeta) - \tfrac{1}{2}\Omega^2(\lambda;k;\bar{a})\underline{x}^2(\zeta)]\}, \quad (2.11)$$

which can be carried out explicitly,[10] resulting in the expression

$$E\{G(\underline{x},\underline{x}',w;\alpha)\} = \exp[-ikB(\lambda;k;\bar{a})z] \times(k\Omega/2\pi i\sin\Omega z)\exp\{(ik\Omega/2\sin\Omega z) \times[(x^2+x'^2)\cos\Omega z - 2\underline{x}\cdot\underline{x}']\}. \quad (2.12)$$

C. The Propagator of Eq. (2.1)

The fundamental solution (or propagator) $G(\underline{x},\underline{x}',z,w;\alpha)$ of (2.1) [cf. also the definition in (2.9)] can be expressed as a continuous path integral; specifically,

$$G(\underline{x},\underline{x}',z,w;\alpha) = \exp\left\{-\frac{i}{2}kg^2\bar{a}^2\int_0^z d\zeta\,H^2(\zeta;\alpha)\right\} \times\int d[\underline{x}(\zeta)]\exp\left\{+\frac{i}{2}k\int_0^z d\zeta\left[\dot{\underline{x}}^2(\zeta) - g^2\underline{x}^2(\zeta) + 2g^2\bar{a}\underline{H}(\zeta;\alpha)\cdot\underline{x}(\zeta)\right]\right\}. \quad (2.13)$$

This path integral can be performed without difficulty[11]:

$$G(\underline{x},\underline{x}',z,w;\alpha) = (kg/2\pi i\sin gz) \times\exp\{(ikg/2\sin gz)[(x^2+x'^2)\cos gz - 2\underline{x}\cdot\underline{x}'] + (ik\bar{a}g^2/\sin gz)\int_0^z d\zeta\,[\underline{x}'\sin g(z-\zeta) + \underline{x}\sin g\zeta]\cdot\underline{H}(\zeta;\alpha) - (ik\bar{a}^2g^3/\sin gz)\int_0^z d\zeta\int_0^\zeta d\zeta' \times\sin g(z-\zeta)\sin g\zeta'\,\underline{H}(\zeta;\alpha)\cdot\underline{H}(\zeta';\alpha) - (ikg^2\bar{a}^2/2)\int_0^z d\zeta\,H^2(\zeta;\alpha)\}. \quad (2.14)$$

D. Basic Formula

Clearly, the quantity $E\{G(\underline{x},\underline{x}',z,w;\alpha)\}$ obtained by formally averaging both sides of (2.14) must be the same with the result derived in Sec. 2B [cf. Eq. (2.12)]. This observation leads to the following relationship:

$$E\left\{\exp\left\{(ik\bar{a}g^2/\sin gz)\int_0^z d\zeta\,[\underline{x}'\sin g(z-\zeta) + \underline{x}\sin g\zeta]\cdot\underline{H}(\zeta;\alpha) - (ik\bar{a}^2g^3/\sin gz)\int_0^z d\zeta \times\int_0^\zeta d\zeta'\sin g(z-\zeta)\sin g\zeta'\,\underline{H}(\zeta;\alpha)\cdot\underline{H}(\zeta';\alpha) - (ik\lambda g^2\bar{a}^2/2)\int_0^z d\zeta\,H^2(\zeta;\alpha)\right\}\right\}$$
$$= (\Omega/2\pi i\sin\Omega z)(g/2\pi i\sin gz)^{-1}\exp(-ikBz) \times\exp\{(kg/2i\sin gz)[(x^2+x'^2)\cos gz - 2\underline{x}\cdot\underline{x}']\} \times\exp\{(ik\Omega/2\sin\Omega z)[(x^2+x'^2)\cos\Omega z - 2\underline{x}\cdot\underline{x}']\}. \quad (2.15)$$

It should be noted in connection with formula (2.15) that the various constants, as well as $\underline{x}$ and $\underline{x}'$, need not be those associated with the original set in (2.14). Performing the averaging of (2.14) requires finding the $\underline{H}(z;\alpha)$ that makes the argument of the exponential an extremum. This $\underline{H}(z;\alpha)$, however, does not depend on the specific values of quantities such as $\underline{x}$ and $\underline{x}'$ which are defined at the end points only. Thus, $\underline{x}$ and $\underline{x}'$ may be any functions of the endpoints (within reason), and (2.15) will still hold.

The above important observation will be used in the following two sections in order to compute a series of nth-order multifrequency moments.

3. FOCUSING MEDIUM WITH RANDOM-AXIS MISALIGNMENTS: Nth-ORDER MULTIFREQUENCY MOMENTS

Substituting the potential field given in (1.2a) into (1.4) we obtain

$$G^{(n)}(\underline{X},\underline{X}',z,\underline{w};\alpha) = \int d[\underline{X}(\zeta)] \exp\Big\{\frac{i}{2}\sum_{p=1}^{n}\xi_p k_p \int_0^z d\zeta\, [\dot{\underline{x}}_p^2(\zeta) - g^2 x_p^2(\zeta) + 2g^2 a \underline{x}_p(\zeta)\cdot\underline{H}(\zeta;\alpha) - \tfrac{1}{2}g^2a^2H^2(\zeta;\alpha)]\Big\}. \quad (3.1)$$

This path integral can be carried out explicitly [cf. definition (1.3); also, (2.13) and (2.14) with $\bar{a}\to a$, $\lambda\to 1$]:

$$G^{(n)}(\underline{X},\underline{X}',z,\underline{w};\alpha) = \Big(\prod_{i=1}^{n}\xi_i k_i\Big)(g/2\pi i \sin gz)^n \exp\Big\{(ig/2\sin gz) \times \sum_{p=1}^{n}\xi_p k_p\,[(x_p^2 + x_p'^2)\cos gz - 2\underline{x}_p\cdot\underline{x}_p'] + \Big\{(iag^2/\sin gz)\int_0^z d\zeta\Big[\Big(\sum_{p=1}^{n}\xi_p k_p \underline{x}_p'\Big)\sin g(z-\zeta) + \Big(\sum_{p=1}^{n}\xi_p k_p \underline{x}_p\Big)\sin g\zeta\Big]\cdot\underline{H}(\zeta;\alpha) - (ia^3g^3/\sin gz)\Big(\sum_{p=1}^{n}\xi_p k_p\Big)\int_0^z d\zeta\int_0^{\zeta} d\zeta' \times \sin g(z-\zeta)\sin g\zeta'\,\underline{H}(\zeta;\alpha)\cdot\underline{H}(\zeta';\alpha) - (ig^2a^2/2)\Big(\sum_{p=1}^{n}\xi_p k_p\Big)\int_0^z d\zeta\, H^2(\zeta;\alpha)\Big\}\Big\}. \quad (3.2)$$

In order to evaluate the desired nth-order multifrequency coherence quantity $E\{G^{(n)}(\underline{X},\underline{X}',z,\underline{w};\alpha)\}$ from (3.2), we must first compute the following statistical average:

$$E\Big\{\exp\Big\{(iag^2/\sin gz)\int_0^z d\zeta\Big[\Big(\sum_{p=1}^{n}\xi_p k_p \underline{x}_p'\Big)\sin g(z-\zeta) + \Big(\sum_{p=1}^{n}\xi_p k_p \underline{x}_p\Big)\sin g\zeta\Big]\cdot\underline{H}(\zeta;\alpha) - (iag^2\sin gz)\Big(\sum_{p=1}^{n}\xi_p k_p\Big)\int_0^z d\zeta\int_0^{\zeta} d\zeta' \times \sin g(z-\zeta)\sin g\zeta'\,\underline{H}(\zeta;\alpha)\cdot\underline{H}(\zeta';\alpha) - (ig^2a^2/2)\Big(\sum_{p=1}^{n}\xi_p k_p\Big)\int_0^z d\zeta\, H^2(\zeta;\alpha)\Big\}\Big\}. \quad (3.3)$$

This expression, however, can be brought into a one-to-one correspondence with the left-hand side of (2.15) provided that the following changes are made in the latter:

$$\lambda\to 1, \quad (3.4)$$

$$\underline{x}\to \sum_{p=1}^{n}\xi_p k_p \underline{x}_p, \quad (3.5a)$$

$$\underline{x}'\to \sum_{p=1}^{n}\xi_p k_p \underline{x}_p', \quad (3.5b)$$

$$\bar{a}\to a\Big(\sum_{p=1}^{n}\xi_p k_p\Big), \quad (3.6a)$$

$$k\to \Big(\sum_{p=1}^{n}\xi_p k_p\Big)^{-1}. \quad (3.6b)$$

It follows, therefore, that the statistical average (3.3) is equal to the right-hand side of the basic formula (2.15) if $\underline{x}$ and $\underline{x}'$ are those given in (3.5), and the quantities $kB(\lambda;k;\bar{a}) \equiv B'(\lambda;k;\bar{a})$ and $\Omega(\lambda;k;\bar{a})$ [cf. Eq. (2.8)] are evaluated at the arguments λ, $\bar{a}$, and k given in (3.4) and (3.6).

We present next the final form of the main result of this section:

$$E\{G^{(n)}(\underline{X},\underline{X}',z,\underline{w};\alpha)\} = \Big(\prod_{p=1}^{n}\xi_p k_p\Big)(g/2\pi i \sin gz)^{n-1}(\Omega/2\pi i\sin\Omega z) \times \exp(-iB'z)\exp\Big\{(i\Omega/2\sin\Omega z)\Big(\sum_{p=1}^{n}\xi_p k_p\Big)^{-1} \times \Big\{\Big[\Big(\sum_{p=1}^{n}\xi_p k_p \underline{x}_p\Big)^2\Big] + \Big(\sum_{p=1}^{n}\xi_p k_p \underline{x}_p'\Big)^2\Big\}\cos\Omega z - 2\Big(\sum_{p=1}^{n}\xi_p k_p \underline{x}_p\Big)\cdot\Big(\sum_{q=1}^{n}\xi_q k_q \underline{x}_q'\Big)\Big\}\Big\} \times \exp(ig/2\sin gz)\Big(2\sum_{p=1}^{n}\xi_p k_p\Big)^{-1} \times \sum_{p=1}^{n}\sum_{q=1}^{n}\xi_p\xi_q k_p k_q\,[(\underline{x}_p-\underline{x}_q)^2 + (\underline{x}_p'-\underline{x}_q')^2] \times \cos gz - 2(\underline{x}_p-\underline{x}_q)\cdot(\underline{x}_p'-\underline{x}_q')\}. \quad (3.7)$$

4. FOCUSING MEDIUM WITH ADDITIVE STATISTICAL FLUCTUATIONS: *N*th-ORDER MULTIFREQUENCY MOMENTS

Under the influence of the potential field (1.2b), the average of (1.4) over the realizations $\alpha\in A$ yields the expression

$$E\{G^{(n)}(\underline{X},\underline{X}',z,w;\alpha)\} = \int d[\underline{X}(\zeta)]\exp\{\frac{i}{2}\sum_{p=1}^{n}\xi_p k_p \times \int_0^z d\zeta\,[\dot{\underline{x}}_p^2(\zeta) - g^2\underline{x}_p^2(\zeta)]\} \times E\{\exp\{\frac{i}{2}\sum_{p=1}^{n}\xi_p k_p\int_0^z d\zeta\,\epsilon_1[\underline{x}_p(\zeta),\zeta;\alpha]\}\}. \quad (4.1)$$

To proceed further, we need to specify the structure of $\epsilon_1[\underline{x}_p(\zeta),\zeta;\alpha]$. If the latter is assumed to be a Gaussian random process, the statistical averaging appearing in (4.1) can be carried out explicitly, with the result

$$I_1 \equiv E\{\exp\{\frac{i}{2}\sum_{p=1}^{n}\xi_p k_p\int_0^z d\zeta\,\epsilon_1[\underline{x}_p(\zeta),\zeta;\alpha)]\}\} = \exp\{-\frac{1}{8}\sum_{p=1}^{n}\sum_{q=1}^{n}\xi_p\xi_q k_p k_q \times \int_0^z d\zeta\int_0^{\zeta} d\zeta'\,\gamma[\underline{x}_p(\zeta),\underline{x}_q(\zeta'),\zeta,\zeta']\}, \quad (4.2)$$

where γ is the correlation function of the random process ϵ_1, viz.,

$$\gamma[\underline{x}_p(\zeta),\underline{x}_q(\zeta'),\zeta,\zeta'] = E\{\epsilon_1[\underline{x}_p(\zeta),\zeta;\alpha]\epsilon_1[\underline{x}_q(\zeta'),\zeta';\alpha]\}. \quad (4.3)$$

We resort, next, to the usual Markovian approximation (cf., also, Paper I, Sec. 3B), i.e., we assume that the process ϵ_1 is δ-correlated along the longitudinal direction of propagation. We have, then, in the place of (4.3)

$$\gamma[\underline{x}_p(\zeta),\underline{x}_p(\zeta'),\zeta,\zeta'] = A[\underline{x}_p(\zeta),\underline{x}_q(\zeta')]\delta(\zeta-\zeta'). \quad (4.4)$$

The corresponding expression for I_1 is given as follows:

$$I_1 = \exp\left\{ -\frac{1}{8} \sum_{p=1}^{n} \sum_{q=1}^{n} \xi_p \xi_q k_p k_q \times \int_0^z d\zeta\, A\, [\underline{x}_p(\zeta), \underline{x}_q(\zeta)] \right\}. \tag{4.5}$$

In many physical problems, the transverse correlation $A[\underline{x}_p(\zeta), \underline{x}_q(\zeta)]$ is homogeneous, isotropic, and of a power-law type (cf. Paper I and references therein), viz.,

$$A[\underline{x}_p(\zeta), \underline{x}_q(\zeta)] = A(0)\left\{ 1 - \frac{1}{2}\left[\frac{1}{L_0}\left|\underline{x}_p(\zeta) - \underline{x}_q(\zeta)\right|\right]^{\beta}\right\}, \tag{4.6}$$

where L_0 is a characteristic length, and the parameter β is usually within the range $1 < \beta < 4$.

Even under the restrictive assumptions made so far about the statistical characteristics of the random process ϵ_1, it is impossible to evaluate $E\{G^{(n)}(\underline{X}, \underline{X}', z, \underline{w}; \alpha)\}$ exactly unless the parameter β introduced in (4.6) is equal to 2. For values of β different from 2, the most comprehensive contribution to the evaluation of n th-order multifrequency coherence functions can be found in the recent work by Dashen,[12] whereby $E\{G^{(n)}(\underline{X}, \underline{X}', z, \underline{w}; \alpha)\}$, with n even, is asymptotically expressible in terms of two-frequency mutual coherence functions $E\{G^{(2)}(\underline{X}, \underline{X}', z, \underline{w}; \alpha)\}$. It should be pointed out, however, that, in contradistinction to single-frequency mutual coherence functions, the exact integration of the two-frequency quantities $E\{G^{(2)}(\underline{X}, \underline{X}', z, \underline{w}; \alpha)\}$ still constitutes an open problem. The only recourse presently is to rely on approximate techniques. An excellent contribution along this direction was made recently by Furutsu[13] who examined second-order pulse statistics for an initially pulsed planar source distribution propagating in a channel devoid of a deterministic background profile.

In the following we shall restrict the discussion to the case $\beta = 2$ (simplified or quadratic Kolmogorov spectrum.)[14] Under this assumption, (4.1) assumes the form

$$\begin{aligned} E\{G^{(n)}(\underline{X}, \underline{X}', z, \underline{w}; \alpha)\} &= \exp\Big\{ -\frac{1}{8} A(0) \Big(\sum_{p=1}^{n} \sum_{q=1}^{n} \xi_p \xi_q k_p k_q\Big) z\Big\} \\ &\times \int d[\underline{X}(\zeta)] \exp\left\{\frac{i}{2} \sum_{p=1}^{n} \xi_p k_p \right. \\ &\times \int_0^z d\zeta \left[\dot{\underline{x}}_p^2(\zeta) - g^2 \underline{x}_p^2(\zeta) \right. \\ &\left.\left. - \frac{i}{4} D \sum_{q=1}^{n} \xi_q k_q [\underline{x}_p(\zeta) - \underline{x}_q(\zeta)]^2 \right]\right\}, \end{aligned} \tag{4.7}$$

where $D = A(0)/2L_0^2$. With $g = 0$, this is precisely the path integral evaluated in Paper I. It is possible to modify the technique developed in that paper so that it can accommodate the presence of a focusing background profile ($g \neq 0$). Instead of following such a procedure in this paper, however, we shall recast (4.7) in a form which, when used in conjunction with the basic formula (2.15), will yield a solution for $E\{G^{(2)}(\underline{X}, \underline{X}', z, \underline{w}; \alpha)\}$ in a straightforward manner.

We begin by expanding the quadratic form entering into (4.7) and recombining terms:

$$\begin{aligned} &-\frac{i}{2} \sum_{p=1}^{n} \xi_p k_p \left\{ \frac{i}{4} D \sum_{q=1}^{n} \xi_q k_q [\underline{x}_p(\zeta) - \underline{x}_q(\zeta)]^2 \right\} \\ &= -\frac{i}{2} \sum_{p=1}^{n} \xi_p k_p \left\{ \frac{i}{2} D \Big(\sum_{q=1}^{n} \xi_q k_q\Big) \underline{x}_p^2(\zeta) \right. \\ &\left. - \frac{i}{2} D \Big(\sum_{r=1}^{n} \xi_r k_r\Big)^{-1} \left[\sum_{q=1}^{n} \xi_q k_q \underline{x}_q(\zeta)\right]^2 \right\}. \end{aligned} \tag{4.8}$$

With this change, (4.7) becomes

$$\begin{aligned} E\{G^{(n)}(\underline{X}, \underline{X}', z, \underline{w}; \alpha)\} &= \exp\Big\{ -\frac{1}{8} A(0) \Big(\sum_{p=1}^{n} \sum_{q=1}^{n} \xi_p \xi_q k_p k_q\Big) z\Big\} \\ &\times \int d[\underline{X}(\zeta)] \exp\left\{\frac{i}{2} \sum_{p=1}^{n} \xi_p k_p \right. \\ &\times \int_0^z d\zeta \left[\dot{\underline{x}}_p^2(\zeta) - \bar{g}^2 \underline{x}_p^2(\zeta) \right. \\ &\left.\left. + \frac{i}{2} D \Big(\sum_{r=1}^{n} \xi_r k_r\Big)^{-1} \left[\sum_{q=1}^{n} \xi_q k_q \underline{x}_q(\zeta)\right]^2 \right]\right\}, \end{aligned} \tag{4.9}$$

where the complex-valued spatial frequency $\bar{g}$ is defined as follows:

$$\bar{g} = \left[g^2 + i(D/2)\Big(\sum_{q=1}^{n} \xi_q k_q\Big)\right]^{1/2}. \tag{4.10}$$

We introduce next a fictitious zero-mean, Gaussian, vector-valued, real process $\underline{F}(z; \alpha)$ with autocovariance tensor $E\{F_i(z; \alpha) F_j(z'; \alpha)\} = (\frac{1}{2})\delta_{ij}\delta(z - z')$, $i, j = 1, 2$. It is well-known in this case that

$$\begin{aligned} &\int d[\underline{F}(\zeta; \alpha)] \exp\left\{ i \int_0^z d\zeta\, \underline{y}(\zeta) \cdot \underline{F}(\zeta; \alpha) - \tfrac{1}{2} \int_0^z d\zeta\, \underline{F}^2(\zeta; \alpha) \right\} \\ &= \exp\left\{ -\tfrac{1}{2} \int_0^z d\zeta\, \underline{y}^2(\zeta) \right\}. \end{aligned} \tag{4.11}$$

With the specific choice

$$\underline{y}(\zeta) = (D/2)^{1/2} \sum_{p=1}^{n} \xi_p k_p \underline{x}_p(\zeta), \tag{4.12}$$

it follows from (4.9) and (4.11) that

$$\begin{aligned} E\{G^{(n)}(\underline{X}, \underline{X}', z, \underline{w}; \alpha)\} &= \exp\left\{ -\tfrac{1}{8} A(0) \left(\sum_{p=1}^{n} \sum_{q=1}^{n} \xi_p \xi_q k_p k_q\right) z\right\} \\ &\times \int d[\underline{X}(\zeta)] \int d[\underline{F}(\zeta; \alpha)]\, P[\underline{F}(\zeta; \alpha)] \\ &\times \exp\left\{ \frac{i}{2} \sum_{p=1}^{n} \xi_p k_p \int_0^z d\zeta\, [\dot{\underline{x}}_p^2(\zeta) \right. \\ &\left. - \bar{g}^2 \underline{x}_p^2(\zeta) + 2(D/2)^{1/2} \underline{x}_p(\zeta) \cdot \underline{F}(\zeta; \alpha)] \right\}, \end{aligned} \tag{4.13}$$

where $P[\underline{F}(z; \alpha)]$ is the probability distribution functional of the process $\underline{F}(z; \alpha)$.

The path integral with respect to $\underline{X}(\zeta)$ in (4.13) is isomorphic to n uncoupled quantum mechanical harmonic oscillators; it can, therefore, be performed easily. The final result is

$$\begin{aligned} &E\left\{G^{(n)}(\underline{X}, \underline{X}', z, \underline{w}; \alpha)\right. \\ &= \exp\left\{ -\tfrac{1}{8} A(0) \left(\sum_{p=1}^{n} \sum_{q=1}^{n} \xi_p \xi_q k_p k_q\right) z\right\} \end{aligned}$$

$$\times\left(\prod_{p=1}^{n}\xi_p k_p\right)(\bar g/2\pi i\ \sin\bar g z)^n \exp\{(i\bar g/2\sin\bar g z)$$
$$\times\sum_{q=1}^{n}\xi_q k_q[(\underline{x}_q^2+\underline{x}_q'^2)\cos\bar g z-2\underline{x}_q\cdot\underline{x}_q']\}I_2, \qquad (4.14)$$

where

$$I_2=E\left\{\exp\left\{[i(D/2)^{1/2}/\sin\bar g z\int_0^z d\zeta\left[\left(\sum_{p=1}^{n}\xi_p k_p\underline{x}_p'\right)\right.\right.\right.$$
$$\times\sin\bar g(z-\zeta)+\left(\sum_{p=1}^{n}\xi_p k_p\underline{x}_p\right)\sin\bar g\zeta\Big]$$
$$\cdot\underline{F}(\zeta;\alpha)-(iD/2\bar g\sin\bar g z)\left(\sum_{p=1}^{n}\xi_p k_p\right)$$
$$\times\int_0^z d\zeta\int_0^\zeta d\zeta'\sin\bar g(z-\zeta)\sin\bar g\zeta'\underline{F}(\zeta;\alpha)\cdot\underline{F}(\zeta;\alpha)\Big\}\Big\}. \qquad (4.15)$$

We are now in a position to use the basic formula derived in Sec. 2D. We note that the expression for I_2 given (4.15) can be brought into a direct correspondence with the left-hand side of the basic formula (2.15) by means of the following changes:

$$\lambda\to 0, \qquad (4.16a)$$
$$\sigma\to 1, \qquad (4.16b)$$
$$g\to\bar g, \qquad (4.16c)$$
$$\underline{x}\to\sum_{p=1}^{n}\xi_p k_p\underline{x}_p, \qquad (4.16d)$$
$$\underline{x}'\to\sum_{p=1}^{n}\xi_p k_p\underline{x}_p', \qquad (4.16e)$$
$$\bar a\to(D/2)^{1/2}\bar g^{-2}\sum_{p=1}^{n}\xi_p k_p, \qquad (4.16f)$$
$$k\to\left(\sum_{p=1}^{n}\xi_p k_p\right)^{-1}. \qquad (4.16g)$$

As a consequence, the statistical average (4.15) is equal to the right-hand side of the basic formula (2.15). In the latter, g, $\underline{x}$ and $\underline{x}'$ are those given in (4.16c)–(4.16e); B must be set equal to zero by virtue of (4.16a); finally, Ω must be evaluated at the values of λ, σ, g, $\bar a$, and k specified above. (When this is done, one has the simple relationship $\Omega\to g$).

We present next the solution of the problem under consideration in this section:

$$E\{G^{(n)}(\underline{X},X',z,\underline{w};\alpha)\}$$
$$=\exp\left\{-\tfrac{1}{8}A(0)\left(\sum_{p=1}^{n}\sum_{q=1}^{n}\xi_p\xi_q k_p k_q\right)z\right\}$$
$$\times\left(\prod_{p=1}^{n}\xi_p k_p\right)(2\pi i)^{-n}(\bar g/\sin\bar g z)^{n-1}(g/\sin gz)$$
$$\times\exp\left\{(i\bar g/2\sin\bar g z)\left(2\sum_{r=1}^{n}\xi_r k_r\right)^{-1}\sum_{p=1}^{n}\sum_{q=1}^{n}\xi_p\xi_q k_p k_q\right.$$
$$\times\{[(\underline{x}_p-\underline{x}_q)^2+(\underline{x}_p'-\underline{x}_q')^2]\cos\bar g z-2(\underline{x}_p-\underline{x}_q)$$
$$\cdot(\underline{x}_p'-\underline{x}_q')\}\Big\}\times\exp\left\{(ig/2\sin gz)\left(\sum_{r=1}^{n}\xi_r k_r\right)^{-1}\right.$$
$$\times\left\{\left[\left(\sum_{p=1}^{n}\xi_p k_p\underline{x}_p\right)^2+\left(\sum_{p=1}^{n}\xi_p k_p\underline{x}_p'\right)^2\right]\cos gz\right.$$
$$-2\left(\sum_{p=1}^{n}\xi_p k_p\underline{x}_p\right)\cdot\left(\sum_{q=1}^{n}\xi_q k_q\underline{x}_q'\right)\Big\}\Big\}. \qquad (4.17)$$

In the absence of a focusing background channel, i.e., $g=0$, (4.17) coincides with the main result in Paper I [cf. Eq. (6.36)].

5. CONCLUDING REMARKS

Our main contribution in this paper is the computation of a set of nth-order multifrequency statistical moments $E\{G^{(n)}(\underline{X},\underline{X}',z,\underline{w};\alpha)\}$ by means of a simple formula [cf. Eq. (2.15)] which provides straightforward solutions to a class of canonical path integrals. Alternative methods for obtaining these coherence functions, such as direct evaluation of the corresponding path integrals (cf. Paper I), or integration of the local transport equations satisfied by $E\{G^{(n)}(\underline{X},\underline{X}',z,\underline{w};\alpha)\}$, are more difficult.

Once the quantities $E\{G^{(n)}(\underline{X},\underline{X}',z,\underline{w};\alpha)\}$ are known, physically measurable pulse statistics are contained in the nth-order moments given in Eq. (2.11) of Paper I, viz.,

$$E\left\{\prod_{p=1}^{n}u_r(\underline{x}_p,z,t_p;\alpha)\right\}$$
$$=\frac{1}{(2\pi)^n}\int_{R^n}d\underline{w}\int_{R^{2n}}d\underline{X}'\,E\{G^{(n)}(\underline{X},\underline{X}',z,\underline{w};\alpha)\}F_r^{(n)}(\underline{w})$$
$$\times\psi_0^{(n)}(\underline{X}',\underline{w})\exp\left\{\sum_{p=1}^{n}(-i)\xi_p[w_p t_p-k(w_p)z]\right\}. \qquad (5.1)$$

Here, $u_r(\underline{x},z,t;\alpha)$ is a real field of radiation (acoustic or electromagnetic) whose time Fourier transform $U_r(\underline{x},z,w;\alpha)$ is linked to the wavefunction $\psi(\underline{x},z,w;\alpha)$ [cf. Eq. (1.1)] by the relation $U_r(\underline{x},z,w;\alpha)=\psi(\underline{x},z,w;\alpha)\exp[ik(w)z]$; furthermore, $F_r^{(n)}(\underline{w})=F_r^*(w_2)F_r(w_1)\cdots F_r^*(w_n)F_r(w_{n-1})$, where $F_r(w)$ is the temporal spectrum—usually a bandpass function of w—characterizing the receiver at range z, and $\psi_0^{(n)}(\underline{X}',\underline{w})=\psi_0^*(\underline{x}_2,w_2)\psi_0(\underline{x}_1,w_1)\cdots\psi_0^*(\underline{x}_n,w_n)\psi_0(\underline{x}_{n-1},w_{n-1})$, where $\psi_0(\underline{x},w)$ is the initial distribution associated with the stochastic parabolic equation (1.1).

It was demonstrated in Paper I that under very restrictive assumptions ($g=0$, a broadband receiver, i.e., $F_r^{(2)}(w)\simeq 1$, and an impulsive planar source intensity), (5.1), with $n=2$ and $\underline{x}_2=\underline{x}_1$, can be evaluated exactly. If these conditions are relaxed, however, the nth-order moments shown in (5.1) can be computed only asymptotically (e.g., by the method of steepest descent), or numerically.

[1]C.M. Rose and I.M. Besieris, J. Math. Phys. 20, 1530 (1979).
[2]This statement is incorrect unless the wave number $k(w)$ is a linear function of the radian frequency w.
[3]I.M. Besieris, W.B. Stasiak, and F.D. Tappert, J. Math. Phys. 19, 359 (1978).
[4]M.D. Donsker, "On Function Space Integrals," in *Analysis in Function Space*, edited by W.T. Martin and I. Segal (M.I.T., Cambridge, Massachusetts, 1964).
[5]K. Furutsu, J. Res. Natl. Bur. Stand. D 67, 303 (1963).
[6]E.A. Novikov, Zh. Eksp. Teor. Fiz. 47, 191 (1964) [Sov. Phys. JETP 20, 1290 (1965)].
[7]V.I. Klyatskin and V.I. Tatarskii, Izv. Vyssh. Uchebn. Zaved. Radiofiz. 14, 1400 (1971).
[8]For practical considerations, it is only meaningful to specify a sufficiently wideband process $\underline{H}(z;\alpha)$ with finite power $\Sigma_{j=1}^{n}\Gamma_{jj}(0)$. This point is discussed further by Mazychuk (cf. Ref. 9).

[9]A.V. Mazychuk, Izv. Vyssh. Uchebn. Zaved. Radiofiz. **21**, 217 (1978).
[10]R.P. Feynman and A.R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw–Hill, New York, 1065). p. 63.
[11]Cf. Ref. 10, p. 64.
[12]R. Dashen, J. Math, Phys. **20**, 894 (1979).
[13]K. Furutsu, J. Math. Phys. **20**, 617 (1979).
[14]It should be noted that this model must be used with care since it behaves in a nonphysical manner for large spatial separations.