A SYSTEMS APPROACH TO REPLACEMENT

by

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CHAPTER I

INTRODUCTION

General

The subject of this research is replacement economy. Its objective is to develop a systems approach to replacement problems. Replacement has been defined by many authors (15, 23, 24) and has been given as many definitions. In general, and in the most liberal interpretation it means that a system,[1] be it an entire process, a machine or a component, has been displaced from service. There is no implication of functional demise, nor of the dissolution of ownership, but merely the fact that the system has been displaced from its past position of rendering service for economic or utility reasons. Utility reasons will not be treated in this investigation.

It should then be surmised that replacement economy, as approached by this investigator, is concerned with that body of knowledge which relates to factors involved in the displacement of systems from service for economic reasons.

Terborgh states the following:

> It follows that replacement policy is much broader than acquisition policy. Its task is not simply the procurement of new facilities which can economically take

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[1]System will be used with reference to any machine, component, etc.

> over the functions of existing equipment; it is the assignment of the existing equipment itself to secure the highest service at the lowest cost. Replacement policy should insure that all facilities in service are able to defend their functions against economical displacement by any challenger, whether inside or outside the same ownership.[2]

Major Problem Areas

Two major problem areas in replacement economy are determining the economic life of an asset and identifying and selecting the equipment with which replacement is to be made.

Economic life. Many problems are encountered in determining the economic life of an asset. Most of these are concerned with the inability of man to predict the future. Accurate forecasting is essential in the determination of economic lives of equipment. When forecasting is poor, when there is no forecasting, or when predicted trends fail to follow forecasts, difficulties sometime occur. These difficulties normally have the effect of shortening the economic life of an asset. The result of this change in economic life is that an additional cost has occurred which was not originally recognized. This problem has not been properly recognized by most. Some writers say that any past cost is a sunk cost and as such is not relevant. This is so to a point, but only so many sunk costs can be absorbed before operations will have to be discontinued for lack of capital. This problem with sunk costs is caused by not properly forecasting such things as operating

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[2]George Terborgh, Dynamic Equipment Policy, McGraw-Hill, New York, 1949, p. 25.

inferiority of equipment and technological improvement in the area. Failure to forecast these properly, or in most cases failure to account for them at all, can leave a decision maker with the problem of a continuing string of sunk costs. This research will be based upon the assumption that data input to models is obtained by economically sound forecasting techniques. It is realized that elements which effect the overall outcome of an investment are numerous. Even the well educated and skilled economists which staff the offices of leading corporations and the agencies of government are often deceived by clouds of uncertainty on the prediction horizon.

Equipment selection. In the area of equipment selection, a major problem is that of identifying viable alternatives for consideration. This problem is due to the tremendous volume of machinery production in this country and abroad. A second problem is the determination of the point in time at which to begin a replacement study. After the identification of alternatives for inclusion in a decision process, one is then faced with the difficulty of reducing the various attributes of the alternatives to a single common denominator for analysis. Many multi-dimensional ranking schemes have been developed for this purpose. Most ranking procedures are very subjective. They have not received popular acceptance in the field. This investigator does not advocate the use of any single ranking scheme. An assumption is made that a choice can be made to determine the best challenger for the function of any defender.

This discussion of problems should indicate to the reader that the proper application of replacement theory is no easy task. Many problems exist which will provide challenges to researchers into the distant future.

Types of Replacement Situations

Replacement problems can normally be classified into the following categories:

1. Replacement of items which fail suddenly.
2. Like-for-like replacement.
3. Military aircraft type replacement.
4. Replacement of items which deteriorate and become obsolete with time.

These problem areas will be further discussed in order to clarify their meaning.

Items which fail suddenly. This type of replacement is of greatest concern in the area of electronic components. Its problems have lead to the development of an area of study, reliability engineering. This area is concerned with the development of policies and procedures which in general have as an objective, the minimization of two types of costs. These costs are: (1) equipment and installation costs and (2) cost of loss of service. The general procedure is to develop a trade-off between these costs to determine an optimum policy. When cost of lost service is prohibitive, considerations are made for equipment redundancy. In cases where weight of equipment is important, the trade-offs are often between weight and system reliability with costs being a tertiary concern. An example of an item which fails suddenly is a light bulb.

Like-for-like replacement. This category of replacement is one in which equipment is replaced by identical equipment. It is practically nonexistent. Its main use is a philosophical one; that of an assumed approach in illustrative problem situations. This is so if we make the strictest semantic interpretation of the title. However, if we make a more liberal interpretation of the meaning and assume that like in function is sufficient to satisfy the definition, we find this type of replacement common in industries which employ machine tools in their operations. It is evident that in any area where the machines are fairly standard and where they have been "state-of-the-art" for years that the assumption of like-for-like replacement will not be in error. This category of replacement is a subset of replacement of items which deteriorate and become obsolete with time. It is given separate ranking because of its common acceptance by practitioners.

Military aircraft type replacement. This replacement area is one in which the decision maker is faced with problems of analysis related to factors other than the system which is in service. Here the concern is with instant obsolescence. The item may display characteristics of other replacement categories and have their problems but when a superior unit is developed it is obsolete for first grade service. It may be functionally downgraded a number of times but its value for main line service has been lost. Military aircraft normally experience this type of replacement. Once the unit has become obsolete in the area, i. e., the enemy has developed a superior model, it has lost its value for first line service. It is no longer the deterrent that it was earlier. In the business world, electronic computers would be a close analogy to military aircraft in national defense.

Replacement of items which deteriorate and become obsolete with time. This is the most common category of replacement and the one which has the greatest effect on the economic well-being of most industries and most countries. This category of replacement is concerned with items which have design weaknesses. Machines wear and with this wear sometimes comes poor quality production. When quality drops, inspection costs increase. When the poor quality production is attributed to a machine, reasons are sought for the change in quality. Discovery of an assignable cause for poor quality would probably lead to decreasing machine speed or altering some variable which might enable the machine to hold process tolerances. This change normally leads to increased unit cost. The continuation of such changes, together with decreasing power efficiency, increasing maintenance cost due to aging, and profit loss from inferior products indicates that the machine is deteriorating.

The point in time at which the machine becomes obsolete is related to deterioration of the defender and the technological advances made in challengers. This area of replacement is one in which the state-of-the-art is dynamic.

Approach of Investigation

As is evident, the area of replacement economy is very broad and encompasses consideration of many factors. To narrow the topic of this thesis somewhat, it will be concerned specifically with the areas covered by categories two and four discussed earlier. These categories, (1) like-for-like replacement and (2) replacement of items which deteriorate and become obsolete with time, constitute a major portion of the replacement situations in industry. It is believed that a more realistic approach to these problems would constitute a major contribution to the replacement literature.

The following is a chapter by chapter outline of the development of a systems approach to replacement:

Chapter II. A review of the replacement literature including discrete and continuous models, the MAPI approach, and existing mathematical programming approaches.

Chapter III. A selected review of capital budgeting literature and mathematical programming techniques as applied to capital budgeting.

Chapter IV. The systems approach to the replacement problem. Formulation of process relationships and constraints for adaptation into the mathematical programming models.

Chapter V. Formulation of mathematical programming models of replacement situations which include process relationships.

Chapter VI. Dimensionality and model analysis.

Chapter VII. Further analysis, model reduction and formulation of example problem.

Chapter VIII. Recommendations and conclusions.

CHAPTER II

SURVEY OF THE REPLACEMENT LITERATURE

General

The basic work of replacement theory has been incorporated in engineering economy texts. Most texts deal with the one machine or one system case. Some of these texts, within subject matter other than replacement treat such problems as budgeting under limitations of funds and the choice between alternatives, some of which may be mutually exclusive. Much other information is to be found in the literature. Basic texts together with material from other literature will be covered in this survey.

Reasons for replacement. Thuesen and Fabrycky (25, p. 261) point out the two basic reasons for replacement or for the consideration of replacing an asset. These are physical impairment and obsolescence. Physical impairment is described by Masse (20, pp. 44-51) as being a change in the "function" of an asset due to a design weakness. Obsolescence, of course, has nothing to do with the asset itself but rather the environment external to the asset--the world of the assets which are in functional competition for the "job" of the asset presently being used.

Depreciation and sunk cost. Most basic texts in the area of engineering economy give adequate coverage to the subject of depreciation and its proper relation to replacement theory. Sunk costs are discussed by the authors and methods for their treatment are discussed. However, few authors point out the relevance of a series of sunk costs or possible reasons for the occurrence of these costs. Most merely indicate that they occur due to mistakes made in the past. Undoubtedly these costs occur because of three factors. These factors are: (1) failure to take into account an obsolescence gradient cost, (2) use of overly optimistic machine operating costs, and (3) poor forecasting in general. It is believed that an improvement can be made on each of these factors by stressing their importance to young engineers who are about to enter the profession. The use of data in analysis which takes into account an obsolescence cost could possibly reduce the number of mistakes. The second and third points are, of course, both related to forecasting and could be improved upon by use of good forecasting techniques.

Inflationary effects. Ghare and Torgersen (14) point out the effects of inflation on engineering economy decisions. This is indeed a relevant factor which has been discounted or omitted by some authors. The factor is assumed by most to be self-cancelling in a comparison of alternatives due to the factor having the same effect on each alternative. However, where the distributions of cash flows through time are significantly different, inflation can have influencing terminal effects. It is believed that this factor can be properly incorporated in a forecasting model to give better data for analysis.

Survey approach. This literature survey will be confined to deterministic models since these are the most relevant to the material to be encompassed within this investigation. The investigator seeks partial justification in this treatment from Tryvge Haavelmo who said, "We are . . . far from having exhausted the amount of clarification and insight that can be gained from the study of exact models. We shall find more than enough to do even in a hypothetical world of non-stochastic models."[3] Even though this statement was written some ten years ago, this writer believes that it is still valid in the area of replacement economy. Hopefully, this belief will be supported by material introduced here and in subsequent chapters.

Text Book Approach

In the basic engineering economy texts, the material on replacement is limited. In most cases, an adequate discussion is given of the problems which are encountered in the application of economic principles. Then students are presented with special problems which illustrate the desired points. Most problems reduce to a simple annual cost or present worth comparison.

Terborgh's MAPI Methods

These methods were developed over a twenty-year period and published in four books; Dynamic Equipment Policy (1949), MAPI

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[3] Tryvge Haavelmo, A Study in the Theory of Investment, University of Chicago Press, Chicago, 1960, p. 17-21.

Replacement Manual (1950), Business Investment Policy (1958), and Business Investment Management (1967). The first two books deal with the equipment replacement problem. The latter extend the principles set forth in the first books to other types of investments.

MAPI basics. The basic upon which the MAPI system is built is a rate of return analysis computed from the equivalence of annual costs. It is, however, more complete than this statement would indicate. In the basic form, it is a "rate-of-return for holding one more year" type of analysis. In this analysis, opportunity costs are taken into account and costs of not waiting are capitalized and added to other costs of the challenger.

MAPI variations. The various MAPI models are developed to take into account such things as income tax, different forms of depreciation, various forms of income series, and various rates of annual accumulation of inferiority. Analysis can also be made for situations in which the defender will be retained for more than one year. Although Terborgh limited, by necessity, the number of variants of the factors described above, modifications can be made to the procedures to take into account alternatives which he omitted in the categories. For example; the method could be modified for depreciation methods not covered by Terborgh.

Terborgh developed the MAPI system as a workable one which could be applied by practitioners in industry and, hence, went into great detail in so doing. His system is built upon a collection of charts which simplify the calculation of information needed. It also is formulated around his MAPI worksheets for total coverage of the analysis. He expanded the coverage of the basic system in each work that he published but has not accounted for some important factors in the replacement area as this investigator would prefer to define it. Two factors unaccounted for are comparisons of new pieces of equipment and comparisons of expansionary investments. Expansionary investment is investment in new assets or systems to expand existing capacity.

Morris' Models

Several models for replacement situations are developed by Morris (21, pp. 194-221). These models include considerations for one or more of the factors planning horizon, technology, cost patterns and zero or non-zero interest rate, together with assumptions with regard to the presence or absence of asset salvage value.

Determination of economic life. Morris begins with a simple problem of determining the economic service life of an asset. The total cost for n years service from an asset is given by

$$T\ C\ (n) = I + \sum_{j=1}^{n} C_j$$

where I is the initial investment required and C_j is a monotonically increasing series of maintenance and operating costs. Dividing the above through by n, the number of years, yields

$$AC(n) = \frac{I}{n} + \frac{1}{n} \sum_{j=1}^{n} C_j.$$

If N is the value of n which minimizes the above expression, then it holds that

$$AC(N+1) - AC(N) \geq 0$$

and

$$AC(N-1) - AC(N) \geq 0.$$

Substituting the expression for AC(n) into the first expression and simplifying gives the following:

$$AC(N) \leq C_{N+1}.$$

Making the same evaluation with the second expression gives

$$AC(N) \geq C_N.$$

The combination of the above two conditions results in

$$C_N \leq AC(N) \leq C_{N+1}.$$

This gives a condition for replacement which can be stated as a marginal evaluation principle. As long as the marginal cost is lower than the average period cost to date, do not replace. When the marginal cost for the next period will exceed the average period cost to date, replace.

Morris developed a model for an indefinite sequence of identical machines, interest and salvage value. This model will not be covered since it is a case which has little application in practice where people are concerned with a definite planning horizon.

<u>Improved candidate for replacement.</u> Morris' model of an improved candidate for replacement, salvage value, interest and a finite planning horizon is the one deemed most relevant in actual application by this investigator.

Morris defined terms used in the model development as follows:

C_{0j} = the operating and maintenance cost for the present machine during the jth additional year of use (monotonically increasing).

S_{0j} = the salvage value of the present machine at the end of the jth additional year of use.

I_1 = the investment required to obtain the newer machine.

C_{1j} = the operating and maintenance cost for the newer machine during the jth year of use (monotonically increasing).

S_{1j} = the salvage of the newer machine at the end of the jth year of use.

The problem is to determine the year n at which to replace the present machine in order to minimize the present worth of the cost of the service over the planning horizon. The total cost in terms of present worth is given by

$$TC\ (n) = I_0 + \sum_{j=1}^{n} \frac{C_{0j}}{(1+i)^j} - \frac{S_0}{(1+i)^n} + \frac{I_1}{(1+i)^n} + \sum_{j=1}^{T-n} \frac{C_{1j}}{(1+i)^{n+j}} - \frac{S_{1T-n}}{(1+i)^T}$$

Designating N as the optimal value of n, the following conditions must hold.

$$TC\ (N+1) - TC\ (N) \geq 0$$

$$TC\ (N-1) - TC\ (N) \geq 0.$$

Following the procedure of substitution of the expression for TC (n) into these expressions developed above the conditions may be expressed as

$$C_{0N+1}+(1+i)\,S_{0N}-S_{0N+1}\geq I_1 i-\frac{S_{1T-N}-S_{1T-N-1}}{(1+i)^{T-N-1}}+\sum_{j=1}^{T-N}\frac{C_{1j}}{(1+i)^{j-1}}-\sum_{j=1}^{T-N-1}\frac{C_{1j}}{(1+i)^{j}}$$

and

$$C_{0N}+(1+i)\,S_{0N-1}-S_{0N}\leq I_1 i-\frac{S_{1T-N+1}-S_{1T-N}}{(1+i)^{T-N}}+\sum_{j=1}^{T-N}\frac{C_{1j}}{(1+i)^{j}}-\sum_{j=1}^{T-N+1}\frac{C_{1j}}{(1+i)^{j+1}}.$$

Morris states the principle of replacement here to be,

> "As long as the cost of one additional year of use for the present machine is less than the savings resulting from postponing the purchase of the new machine one year, do not replace; when the cost of extending the use of the present machine for an additional year exceeds the savings resulting from postponing the purchase of the new machine, then the new machine should be purchased."[4]

Obsolescence model. Morris also developed a model for replacement in which obsolescence was taken into account. He assumed that operating and maintenance costs increased linearly with age. He also assumed that the first year operating cost decreased linearly with calendar time.

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[4]W. T. Morris, Analysis of Management Decisions, R. D. Irwin, Homewood, Illinois, 1964, p. 204.

A machine, purchased new at the end of the Kth year of a period under consideration, would have an operating and maintenance cost in year j from the time of purchase which is given by

$$C_j = c + (j-1)\,a - (K-1)\,b, \quad \text{for} \quad K \leq j$$

where c is the operating and maintenance cost of the machines in use at the beginning of the period, j is the period after purchase for which the cost determination is to be made, K is the period at the end of which the new machine was adopted. The yearly increase in operating and maintenance costs is a, and b is the yearly change in operating advantage.

The total cost for replacement of a machine at the end of each n years is given by

$$TC\ (n) = I + \sum_{j=1}^{n} \frac{c+(j-1)a}{(1+i)^j} + \frac{I}{(1+i)^n} + \sum_{j=1}^{n} \frac{c+(j-1)a - nb}{(1+i)^{n+j}}$$

$$+ \frac{I}{(1+i)^{2n}} + \sum_{j=1}^{n} \frac{c+(j-1)a - 2nb}{(1+i)^{2n+j}} + \frac{I}{(1+i)^{3n}} + \ldots$$

which can be reduced to

$$TC\ (n) = \left[I + \sum_{j=1}^{n} \frac{c+(j-1)a}{(1+i)^j} \right] \frac{(1+i)^n}{(1+i)^n - 1} - \sum_{k=1}^{\infty} \sum_{j=1}^{n} \frac{knb}{(1+i)^{kn+j}} .$$

Converting the above to an annual cost yields

$$AC\ (n) = \left[I + \sum_{j=1}^{n} \frac{c+(j-1)a}{(1-i)^j} \right] \frac{i\ (1+i)^n}{(1+i)^n - 1} - \left[\sum_{k=1}^{\infty} \sum_{j=1}^{n} \frac{knb}{(1+i)^{kn+j}} \right] i .$$

Morris next solved a substitute problem by reasoning that the total cost of operating a machine is the sum of the operating and maintenance costs and the opportunity costs foregone by not replacing it with the best alternative available. In this situation the total cost in any year from the beginning of a machine's operation is given by

$$C = c + (j-1)(a+b).$$

Here j is the year for which the total operating cost is to be determined and other terms are as previously defined. By setting up the annual costs in this form, it can be shown (21, pp. 524-525) that the difference in costs between the two expressions is a constant which is equal to the capitalized value of the annual operating advantage. Since the expressions differ only by a constant, then their optimal values will be the same.

The conditions for an optimum solution to this problem are as follows:

$$AC\,(N) \leq c + N\,(a+b)$$

$$AC\,(N-1) \geq c + (N-1)(a+b).$$

It is believed that Morris covers points which are indeed relevant, namely, those of technological advancement and increasing costs of operation and maintenance. He also develops various models which incorporate presence or absence of interest and salvage value. His work provides a basis from which many models of replacement situations could be constructed.

Masse's Models

Masse (20, pp. 42-83) extended models similar to those of Churchman, Ackoff, and Arnoff (6) to include continuous interest and components which had been used as constants as a function of time. A similar treatment of the continuous interest approach was also done by Reisman and Buffa (22). The work of Masse is, however, of greater interest.

Elementary model. Masse's first model is as shown below:

$$B(T)=\int_0^T Q(t)e^{-I(t)}dt+S(T)e^{-I(T)} - C.$$

B(T) is the total discounted profit realized from an investment, T is the replacement time, Q(t)dt is gross income earned from the project in the time interval (t, t+dt), C is the initial investment, S(T) the salvage value of the project at time T, and i(t) the continuous interest rate which is assumed to vary with time t. Masse assumes that the gross income is the excess of sales over variable costs and does not include interest or depreciation expense. He also assumes that

$$\int_0^t i(t)dt=I(t).$$

The optimum interval of replacement, T, is obtained by differentiating the expression for B(T) with respect to T. This is given by

$$0=\frac{dB}{dT}=\Big[Q(T) - i(T)\ S(T)+S'(T)\Big]\ e^{-I(T)}.$$

Masse's interpretation of the preceding condition was as follows:

> Condition (II. 2) means that at the end of the optimum length of service T, the interest on the retirement value equals the potential profit on continued operation minus the loss in retirement value, all these magnitudes expressed per unit of time.[5]

<u>Partial Replacement.</u> Masse also developed a model for partial replacement. In this model he assumed that there were two decision variables under the control of management. These were the time of partial replacement θ and the time of retirement T'. His model was as shown below:

$$B' = \int_0^{T'} Q(t)e^{-I(t)}dt - ce^{-I(\theta)} + \int_\theta^{T'} q(\theta, t)e^{-I(t)}dt.$$

Taking the partial derivatives of the above expression with respect to θ and T' and setting them equal to zero gives the following:

$$0 = \frac{\partial B'}{\partial \theta} = ci(\theta)e^{-I(\theta)} - q(\theta, \theta)e^{-I(\theta)} + \int_\theta^{T'} \frac{\partial q}{\partial \theta}(\theta, t)e^{-I(t)}dt,$$

and

$$0 = \frac{\partial B'}{\partial T'} = Q(T')e^{-I(T')} + q(\theta, T')e^{-I(T')}.$$

The first expression gives the optimum partial replacement interval for a given T'. This can be substituted into the second expression in order to find T'. Then, θ can be evaluated from the first equation.

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[5] Pierre Masse', <u>Optimal Investment Decisions</u>, Prentice-Hall, Englewood Cliffs, New Jersey, 1962, p. 55.

Masse developed other models which were of less significance and also used a continuous approach to develop the conditions for the adverse minimum of Terborgh's MAPI models. His work is refreshing and is of considerable value to the body of replacement.

Dynamic Programming Formulations

Bellman and Dreyfus (2, 3, 9) developed the dynamic programming model for the replacement problem. They assumed certain patterns for costs associated with the maintenance operation and replacement of equipment. The patterns and cost considered were as shown in Figures 1, 2, and 3.

This approach was developed for a one machine or one system consideration. It was assumed that decisions would be made only at discrete intervals of time. The decisions would be either to keep the present machine or to dispose of it and replace it with the challenger. These alternatives were designated by K for "keep" and P for "purchase". A function f(t) was introduced to represent the return over the period that the service was required assuming that the machine now in service was of age t. The value given by f(t) was also assumed to be the best value obtainable, i.e., that given by an optimal policy. A discounting factor was introduced in order to take the time value of money into account and also to keep the return finite.

Functional equations were developed. These were:

$$f(t) = \max \begin{Bmatrix} P: & r(0) - U(0) - C(t) + af(1) \\ K: & r(t) - U(t) + af(t+1) \end{Bmatrix}.$$

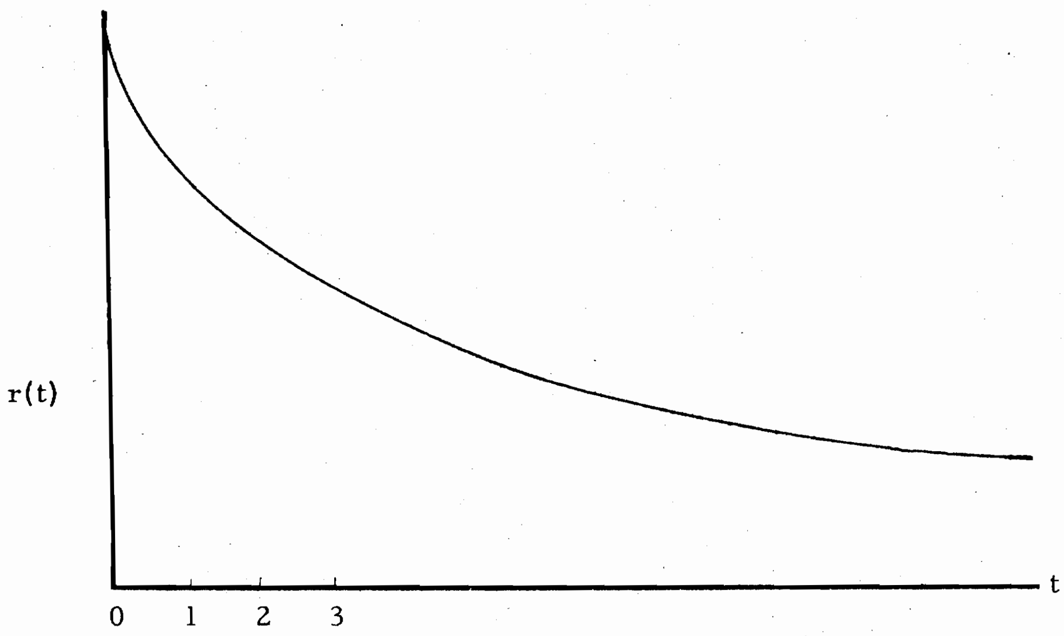

FIGURE 1

ANNUAL RETURN FROM MACHINE VERSUS AGE OF MACHINE

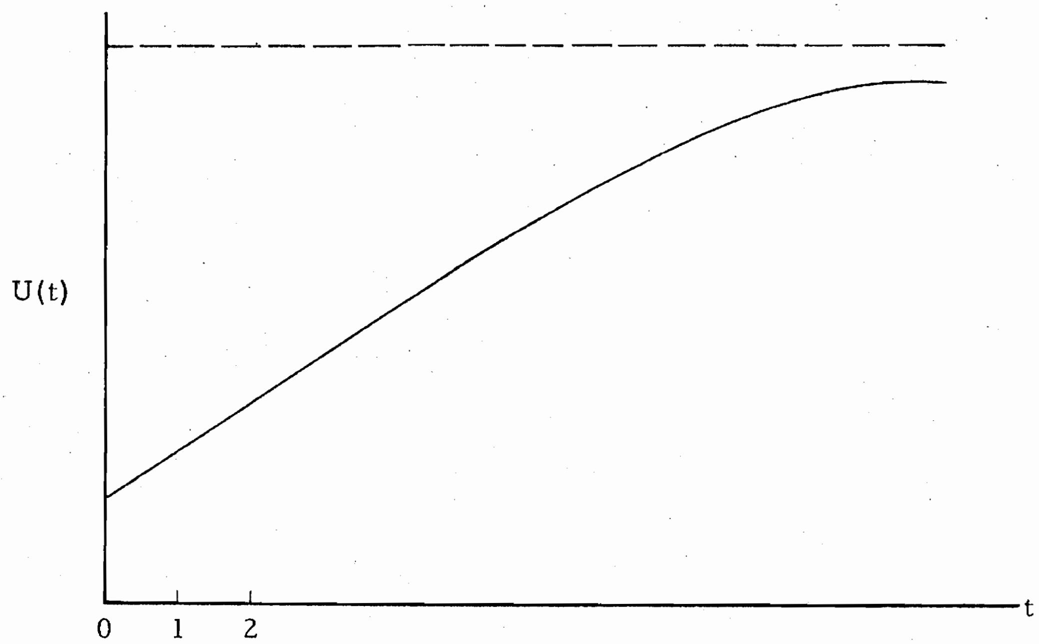

FIGURE 2

ANNUAL OPERATING EXPENSE VERSUS AGE OF MACHINE

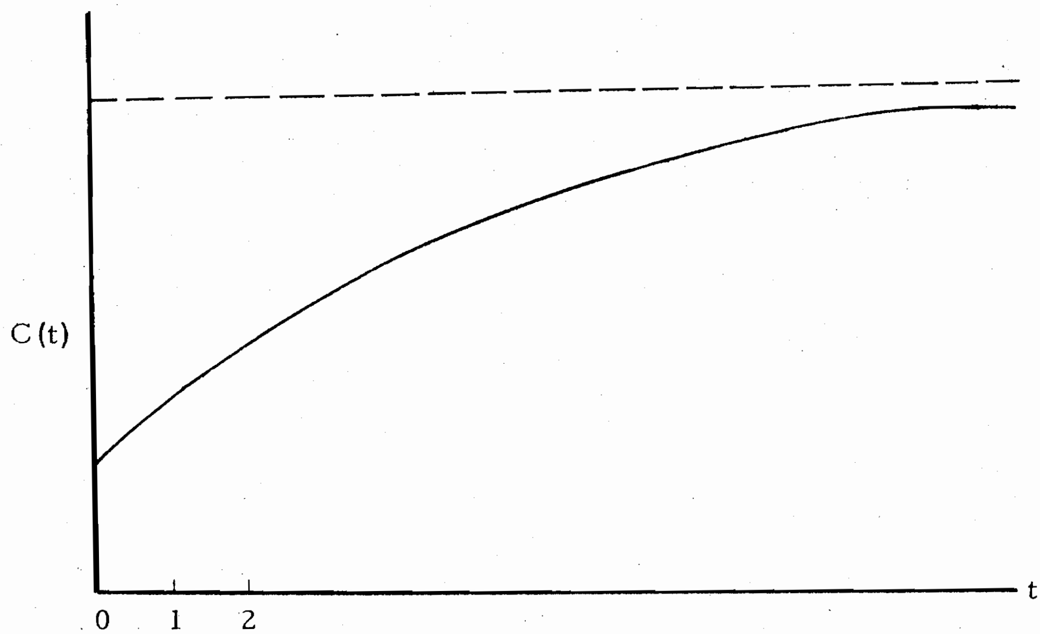

FIGURE 3

REPLACEMENT COST VERSUS AGE OF MACHINE

The preceding equations describe an infinite process. It is evident that an optimal policy will be to keep a machine or system until year "X" and then replace it. If year "X" is past the end of the planning horizon, the machine will not be replaced.

To simplify the model n(t) was designated as the difference between the yearly return, r(t), and the yearly upkeep, u(t).

To solve this problem analytically, a system of equations was developed.

$$f(0) = n(0) + af(1)$$

$$f(1) = n(1) + af(2)$$

$$\vdots$$

$$f(T-1) = n(T-1) + af(T)$$

$$f(T) = -C(T) + n(0) + af(1)$$

where T is the year of replacement and C(T) is the cost of replacing the machine or system at that time. The above system of equations can be solved for f(1) to yield

$$f(1) = \frac{[n(1) + an(2) + a^2 n(3) + \ldots + a^{T-2} n(T-1) + n(0)a^{T-1}] - a^{T-1} C(T)}{1 - a^T}.$$

A value of T is then determined which will maximize f(1). This value will maximize f(0) also.

It should be noted that if the machine in service is older than T when the study is begun, it is not clear what to do.

Bellman and Dreyfus make the dynamic programming formulation as follows:

Let us introduce the functions

1. $fN(t)$ = the value at year N of the overall return from a machine which is t years old, where an optimal replacement policy is employed for the remainder of the process.

The future is discounted as before. However, we also assume that the process lasts N_0 stages, and then stops. Hence $f_{N_0+1}(t)=0$.

Since the overall return associated with purchase at year N is

2. $f_N^{(P)}(t) = r_N(0) - U_N(0) - C_N(t) + af_{N+1}(1)$,

and the return from a decision to keep is

3. $f_N^{(K)}(t) = r_N(t) - U_N(t) + af_{N+1}(t+1)$,

we obtain the equation

4. $f_N(t) = \max \left\{ f_N^{(P)}(t),\ f_N^{(K)}(t) \right\}$,

or

5. $$f_N(t) = \max \begin{Bmatrix} P: & r_N(0) - U_N(0) - C_N(t) + af_{N+1}(1) \\ K: & r_N(t) - U_N(t) + af_{N+1}(t+1) \end{Bmatrix}.$$

The function $f_N(t)$ is taken to be zero for $N \geq N_0+1$.

Letting N assume the value N_0 in equation (5), we obtain an expression for $f_{N_0}(t)$ in terms of known functions. Hence, we can solve for $f_{N_0}(t)$, for all admissible t.

. . . Having constructed the function $f_{N_0}(t)$, we can use equation (5) to determine the function $f_{N_0-1}(t)$. Continuing this sequence, we obtain $f_1(t)$, the optimal return for a process starting in year 1. Recording the policy used in the maximization of equation (5), we have the replacement policy which yields the optimal return.[6]

[6] Richard E. Bellman and Stewart E. Dreyfus, Applied Dynamic Programming, Princeton University Press, Princeton, New Jersey, 1962, pp. 118-119.

Bellman and Dreyfus point out that the technique used is sufficiently flexible to apply to a variety of problems in the replacement area.

Hanssmann's Models

Hanssmann (16, pp. 98-106) did some of the work in replacement economy most relevant to the approach of this paper. His models were very basic but, indeed, of value for further development.

One machine model. Hanssmann developed a one machine replacement model for a finite planning horizon. He chose to make the life of the machine coincident with that horizon. His assumption was that there would be no more than one replacement during the planning period. He assumed only one "challenger" for consideration. His problem was then to find the optimum time at which to replace in order to minimize the sum of the non-discounted cost for maintaining the required production capability over the time horizon. His model was as shown below.

$$C(r) = \sum_{t=1}^{r} \left[d(t) + e(t) \right] + (T-r) \left[\frac{A-S+E}{L} \right], \quad r = 0, \ldots, T$$

where r is the year of replacement, T is the planning horizon, d(t) is the depreciation in actual value during year t, e(t) is the operating costs in year t, A is the first cost, S is the salvage value, E is the total current expense over the project life and L is the economic life of the challenger.

Noticeable in this model is the absence of consideration of the time value of money and the presence of a terminal value assumption. This model is identical in function to Morris' improved candidate for replacement with a finite planning horizon, if Morris had neglected the time value of money.

Hanssmann next reasons as follows:

> A problem that remains outside these replacement models is that of flexibility. For example, minimization of costs during the horizon may require the use of a machine with relatively long economic life. Therefore, this solution may prevent us from making the best use of future technological alternatives that may become available after the horizon. In our present treatment flexibility should be considered an intangible that must be weighed against the economic evaluations obtained from the replacement model.[7]

Model for several machines. Hanssmann next looked at the replacement problem for several independent machines and imposed a first year budget, B. Using the model above, he calculated the optimal replacement time for each machine.

Problems occur only if more machines are to be replaced in the budget year than available capital permits. His assumption was that if a machine cannot be replaced in the first budget year then it can be replaced in the second. The cost of not replacing a machine immediately is the difference between the costs of the earlier model evaluated at time one and at time zero. This can be expressed as

$$E_j = C_j(1) - C_j(0)$$

---

[7] Fred Hanssmann, Operations Research Techniques for Capital Investment, John Wiley and Sons, Inc., New York, 1968, p. 100.

where E_j is the cost of postponing the replacement of machine j for the first period. $C_j(r)$ is the total cost for the planning horizon if machine j is replaced at time r. The quantity E_j can also be considered as the gain from immediate replacement of the asset j. This replacement situation can then be formulated as follows:

maximize

$$E = \sum_{j=1}^{m} E_j X_j,$$

subject to

$$\sum_{j=1}^{m} I_j X_j \leq B,$$

$$X_j = 0,\ 1$$

where I_j represents the investment required for machine j and X_j is a zero-one variable for project selection.

Next, Hanssmann modified his model to include a yearly budget, B_t, over a planning horizon, T. He again limited each machine to one challenger and to one replacement during the horizon. The decision variable X_{it} now represented whether or not machine i is replaced in year t. The problem then became

minimize

$$C = \sum_{i=1}^{n} \sum_{t=1}^{T} C_{it} X_{it},$$

subject to

$$\sum_{i=1}^{n} I_i X_{it} \leq B_t, \qquad t = 1, \ldots, T,$$

and

$$\sum_{t=1}^{T} X_{it} = 1, \qquad (i = 1, \ldots, n).$$

Expansionary Investment. Hanssmann also treated the case of replacement which he calls "expansionary investment." This is the case where an alternate method of performing a task is included in a formulation such as that shown above. The reason for not including it under replacement is that the alternative for consideration may require additional capital and may not be like in function to the old asset. It also may increase the capacity of the process, hence, this accounts for the separate treatment.

Here, Hanssmann takes the case of a one product firm and assumes the same finite planning horizon. He designates r_j as the forecasted requirement for the jth year of the planning horizon. A simple process model used is as shown in Figure 4.

With reference to Figure 4 as a network flow, it can be seen that the following must hold:

$$U_j + V_j = r_j$$

$$U_j^1 + U_j^2 = U_j$$

Facilities will be excluded by not allocating units of production to them. Each machine center must have adequate capacity to service the allocated work. Hence, the following must hold.

$$C_1 X_j^1 \geq U_j$$

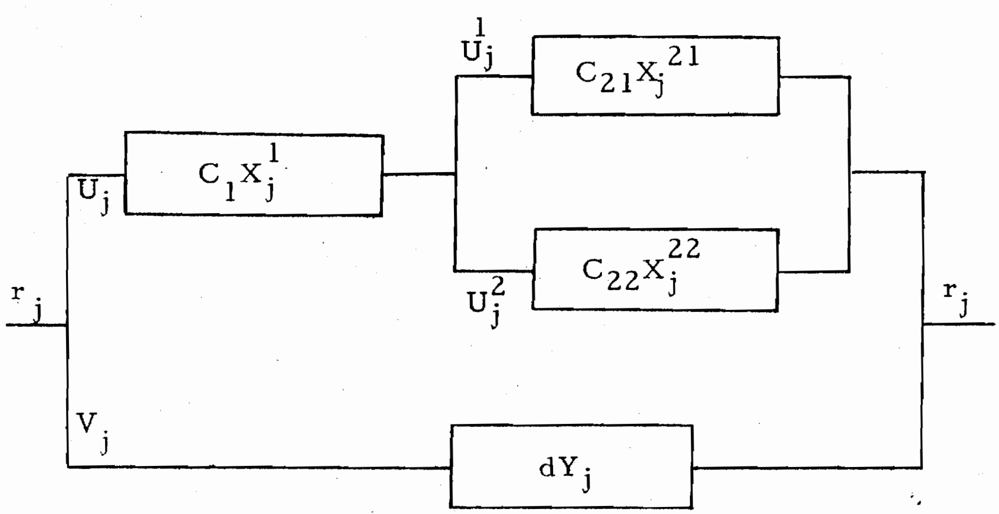

FIGURE 4

MODEL OF PRODUCTION PROCESS USED BY HANSSMANN

Note: The coefficients of X and Y in the figure represent machine capacities.

$$C_{21}X_j^{21} \geq U_j^1$$

$$C_{22}X_j^{22} > U_j^2 \qquad (j=1, \ldots, T)$$

$$dY_j \geq V_j$$

Here, the number of machines X_j and Y_j are decision variables.

At this point no objective function has been specified. Hanssmann states the following:

> The proper measure of input is the incremental cost of owning and operating all machines during the horizon. Consider the cost contribution in year j of one of the machine inventories, say Y_j.
>
> . . . we know that the cost contributions of old machines (already installed at the beginning of the horizon) and new machines (installed at some time during the horizon) must be treated differently. We therefore decompose Y_j into Y_{j0} of old machines held over from the beginning of the horizon and the number Y_{jn} installed later on and kept during period j. Thus we have
>
> $$Y_j = Y_{j0} + Y_{jn} \qquad (j=1, \ldots, T).$$
>
> Clearly, the old machines can only decrease in time through retirement.
>
> $$Y_{j0} \leq Y_{j-1,0} \qquad (j=1, \ldots, T).$$
>
> Let b_{j0} be the cost contribution of an old machine in year j, including decline of salvage value and current maintenance expense. These costs are assumed independent of the actual utilization of the facility for production. Let b_n be the annual cost contribution of a new machine. . . . Finally, let B be the operating cost per unit of product actually manufactured on a facility of type Y. Then it is clear that the total cost contribution of machine center Y in period j can be written as
>
> $$C_Y = b_{j0}Y_{j0} + b_N Y_N + BV_j.$$ [8]

---

[8] Ibid., pp. 103-104.

Hanssmann next extends this concept over all machine centers and the resulting objective function becomes

$$C = \sum_{j=1}^{T} (a_{j0}^{1} X_{j0}^{1} + a_{j0}^{21} X_{j0}^{21} + a_{j0}^{22} X_{j0}^{22} + b_{j0} Y_{j0}$$

$$+ a_{n}^{1} X_{jn}^{1} + a_{n}^{21} X_{jn}^{21} + a_{n}^{22} X_{jn}^{22} + b_{n} Y_{jn}$$

$$+ A_{1} U_{j} + A_{21} U_{j}^{1} + A_{22} U_{j}^{2} + BV_{j}),$$

which is to be minimized subject to the previously developed constraints in addition to

$X, Y, U, V \geq 0$ (X, Y integers).

Hence, it can be seen that this work by Hanssmann is the most complete of the deterministic models of replacement theory. It does not, however, encompass many important principles of engineering economy, however, and leaves much room for further or alternative developments.

Discussion of Models

It is evident from this survey of the literature that much work has been done in the replacement area and that there is much which remains to be done. Terborgh's MAPI method is one which is applied by some practitioners. It takes into account the possibilities of improvements in technology and the expected increases in operating costs. The models of Masse, Morris, and Bellman and Dreyfus are structured so that these factors can be considered. Interest rates are provided for in all of the models as in the presence or absence of a salvage value. None of the models, with the possible exception of Hanssmann's, provide any relationship between the single machine or system in consideration and other projects being considered within the same firm. The models do not account for the fact that there is some change in the productivity of most machines with aging. In general, they overlook some very important considerations in decision making. These considerations will be discussed in detail later.

Replacement Versus Capital Budgeting

The problems confronted by those concerned with capital budgeting are much the same as those encountered by individuals in the application of replacement economics. Capital budgeting is normally a problem with fewer constraints at a much higher level of activity. Capital budgeting normally concerns itself with new projects and might be considered a macro-budgeting. Replacement is an activity which is carried on, in general, at a lower level than capital budgeting. It is concerned with taking funds allocated from the capital budgeting process and allocating them as scarce resources in the production process, or contrawise, justifying the allocation of scarce resources in the processes of manufacture to those responsible for capital budgeting. Hence, replacement economics might be called an application of micro-budgeting.

It is believed that a review of literature of capital budgeting might be highly relevant since much more time has been devoted to that problem than to that of replacement economy.

CHAPTER III

CAPITAL BUDGETING AND SELECTED CAPITAL BUDGETING MODELS

General

Lorie and Savage (19) defined three tasks which managers encounter in achieving good financial management. The first task was preparing and reviewing capital budgets, delegating authority and assigning responsibility for money expended, and reviewing investments made. The second task was forecasting future cash flows with accuracy. Rationing of capital was listed as the third task. This latter task was broken down into the problems listed below:

> 1. Given a firms cost of capital and a management policy of using this cost to identify acceptable investment proposals, which group of "independent" investment proposals should the firm accept? . . .
>
> 2. Given a fixed sum of money to be used for capital investment, what group of investment proposals should be undertaken? . . .
>
> 3. How should a firm select the best among mutually exclusive alternatives?[9]

The Lorie and Savage Problem

The primary item of interest in the work of Lorie and Savage (19) is their method developed for the ranking of investment proposals, given

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[9]James H. Lorie and Leonard J. Savage, "Three Problems in Capital Rationing," Journal of Business, Vol. 28, October 1955, pp. 229-239.

the firm's interest rate. This procedure was one of trial and error. It gives a feasible but not necessarily optimal solution.

Problem approach. The problem approach is simple. Given the net present value (hereafter NPV) of project j as Y_j, the NPV of the outlays for project j in period i as C_{ij}, the NPV of the capital budgeted in period i as P_i, and the available capital as C_i, the problem becomes one of maximizing the NPV of the investments subject to the period by period budgets. First a one year planning horizon will be examined. The procedure is to rank the projects in decreasing order of Y_j/C_{ij}. Intuitively,

$$\sum_{j=1}^{K} C_{1j} \leq C_1,$$

where C_1 is the budget for year 1 and K is the last project chosen before the capital budget would be exceeded. At this point the quantity

$$Y_j - A_1 C_{1j}$$

is positive or zero for all projects chosen where

$$A_1 = \frac{Y_K}{C_{1K}} .$$

This multiplier, A_1, can be determined by trial and error to obtain a workable solution. Lorie and savage believed it related to a Lagrange multiplier but gave no proof. A simple example will illustrate the application of this method.

Project K	Y_j	C_{1j}	Y_j/C_{1j}
1	25	5	5
2	10	2.5	4
3	15	5	3
4	5	2.5	2

If the capital budget C_1 is assumed to be 9, K must then equal 2. Then $A_1 = 4$ and upon calculating $Y_j - A_1C_{1j}$ for each project the conditions as described previously are met. If the condition A_1 had not been known, it could have been determined by trial and error as follows:

Assume $A_1 = 5$ and calculate $Y_j - A_1C_{1j}$ for each project.

Project	$Y_j - A_1C_{1j}$
1	0
2	- 2.5
3	-10.0
4	- 7.5

This would indicate that project 1 should be chosen. Upon evaluation, it is found that the capital budget is still not expended. Hence, the value chosen for A_1 was too large.

Choose $A_1 = 4$ and evaluate the projects again in the same manner. This will give the solution obtained earlier. It should be noted, however, that the solution is not optimal since if projects 1 and 3 were chosen the budget would be exhausted and the NPV of the investments chosen would be maximized. It is interesting to note that, as the value of A_1 is decreased toward zero, then at the value $A_1 = 2$ projects 2 and

3 both have a $Y_j - A_1C_{1j}$ value of 5. This could be an indication of the fact that project 3 could be substituted for project 2.

Lorie and Savage used an example which illustrated the application of the method in a two period situation. It was as shown below:

Investment	Expense in Period 1 (C_{1j})	Expense in Period 2 (C_{2j})	NPV of Project
1	12	3	14
2	54	7	17
3	6	6	17
4	6	2	15
5	30	35	40
6	6	6	12
7	48	4	14
8	36	3	10
9	18	3	12

To apply the method, values must be assigned to A_1 and A_2, the multipliers. Note that no ranking is required to begin as was previously assumed. Assign various values to the multipliers and evaluate the following expression for each alternative.

$$Y_j - A_1C_{1j} - A_2C_{2j}$$

These values for some combinations are as shown in the following.

Value of $Y_j - A_1C_{1j} - A_2C_{2j}$ for given A_1, A_2

Project	$A_1=3,\ A_2=1$	$A_1=1,\ A_2=1$	$A_1=.5,\ A_2=1$
1	- 25	- 1	5*
2	-152	-44	-17
3	- 7	5*	8*
4	- 5	7*	10*
5	- 85	-25	-10
6	- 12	0*	3*
7	-138	-38	-14
8	-101	-29	-11
9	- 45	- 9	0*

* indicates which projects should be chosen under the given conditions.

By studying the table, it can be determined that, as the value of A_1 was decreased, more projects became acceptable by the desirability criterion Y_j/C_{ij}. Budget requirements for the projects chosen are zero for the first case, $P_1=18$ and $P_2=14$ for the second case, and $P_1=48$ and $P_2=20$ for the third case. In the third case the budget for the second period is expended and projects 1, 3, 4, 6, and 9 chosen for investment. The maximum present value was \$70. Weingartner (27) studied this approach to the problem and provided an explanation of the ranking or selection procedure in terms of the "shadow prices" of the dual problem.

Weingartner's formulations of the Lorie and Savage problem. Weingartner (27, pp. 16-19) first formulated the Lorie and Savage problem as a simple linear programming problem. This formulation was as follows:

maximize

$$\sum_{j=1}^{n} b_j X_j,$$

subject to

$$\sum_{j=1}^{n} C_{tj} X_j \leq C_t,$$

$$0 \leq X_j \leq 1,$$

where b_j is the present value of an individual project, C_{tj} is the amount of capital required by project X_j in period t, C_t is the budget constraint for period t, and X_j is the decision variable which represents what portion of a project will be chosen. Weingartner acknowledged that this formulation would provide a result which involved fractional projects in most cases. He formulated a theorem for the maximum number of fractional projects which would be involved.

His second approach was to formulate the problem as one in integer programming. This problem formulation is the same as that given above with the addition of the requirement that X_j be an integer. Weingartner's work is very rigorous and his study of the problems thorough. However, he spends very little time on formulations and devotes most of his work to the interpretation of the problems in terms of primal and dual variables and shadow prices of the primal and dual problems. His work is an approach to the capital budgeting problem from a theoretical mathematical programming approach.

Unger's Models for Capital Budgeting

Unger (26) formulated several interesting models for capital budgeting situations. One of his problem situations was as follows: A firm is faced with n independent investment projects which can be accepted or rejected. It has a limited budget for operating and uses no outside financing. Any returns from investments adopted can be used in funding new investments. Funds which are unused in a period can be carried over to the following period. The objective of the firm is to maximize the present value of the dividends paid to stockholders.

Unger defined the terms below:

A_{tj} = the net cash flow from the jth project in the tth time period.

E_t = the net cash flow in the tth period from projects adopted prior to the 1st period.

Y_t = the dividends paid at time t.

P_t = the interest rate at which dividends paid at time t are discounted.

S_t = the amount of cash carried over from the tth to the (t+1)st period.

A planning horizon of T years is assumed. The problem statement becomes

maximize

$$\sum_{t=1}^{T} P_t Y_t$$

subject to

$$Y_t - S_{t-1} + S_t - \sum_{j=1}^{n} A_{tj}X_j \leq E_t, \quad t=1, \ldots, T,$$

$$X_j = \begin{cases} 0 \\ 1 \end{cases}, \quad j=1, \ldots, n, \; S_0=0,$$

$$Y_t, \; S_t \geq 0, \quad t=1, \ldots, T.$$

The variable X_j is zero-one depending upon acceptance or rejectance of the project. This problem is a mixed zero-one integer programming problem.

Unger assumes that the interest rate, P_t, is positive in every period. He then reasons that the constraints can be treated as equality constraints because any excess money in a period would be paid out as a dividend.

Next, he assumes no funds can be carried forward from one period to the next and develops a second formulation as given below.

The first constraint type becomes

$$Y_t - \sum_{j=1}^{n} A_{tj}X_j = E_t$$

which can be written as

$$Y_t = E_t + \sum_{j=1}^{n} A_{tj}X_j.$$

Substituting the above quantity in the previously formulated objective function and restructuring the problem yields:

maximize

$$\sum_{t=1}^{T} P_t \left(E_t + \sum_{j=1}^{n} A_{tj}X_j\right)$$

subject to

$$E_t + \sum_{j=1}^{n} A_{tj}X_j \geq 0, \quad t=1, \ldots, T,$$

$$X_j = \begin{cases} 0 \\ 1, \end{cases} \quad j=1, \ldots, n.$$

The problem has now become a pure zero-one integer programming problem.

Unger made one additional problem formulation. The assumptions were the same as the first problem with the exception of the planning horizon which was changed to infinity. He then used specific assumptions to derive terminal effects at the end of a definite time horizon. The problem was then reduced to the first problem formulation with minor modifications.

Bernhard's General Model for Capital Budgeting

Bernhard (4) developed a general model for capital budgeting. His interpretation of the economic implications of the optimal solution to the model in terms of the Kuhn-Tucker conditions was the best found in the literature.

The definition of terms used by Bernhard was as follows:

A_{tj} = cash flow from one unit of project j at time t. (may be either + or -)

A_j = present worth at time T of all returns to project j after time T.

M_t' = funds available from outside projects at time t.

M' = present worth at time T of cash flows from all outside projects.

l_t = $1+r_{lt}$, where r_{lt} is the lending interest rate in period $t+1$.

$b_t = \begin{cases} 1+r_{bt}, \text{ where } r_{bt} \text{ is the rate of interest on borrowed funds in period } t+1. \\ l_t \text{ for } t=1, \ldots, T-1, T, T+1, \ldots . \end{cases}$

B_t = maximum value of W_t.

d = total supply of scarce resource.

d_j = consumption of scarce resource by unit of product j.

X_j = units of project j to be undertaken.

W_t = dividends paid at time t.

w_t = cash to be borrowed from time t to time $t+1$.

V_t = cash to be loaned out in period $t+1$.

G = the terminal wealth of the firm at time T after payment of dividends for period T.

Bernhard's objective function then, in general terms, is to maximize the wealth of the stockholders or owners at some point in time. In general terms, this could be expressed as

maximize $f(W_1, W_2 \ldots, W_T, G)$,

(where the actual function can be determined to suit the specific case) subject to many variations in constraints, some of which are included below.

The firm must maintain a cash balance during period $t+1$ of at least $(C_t+c_tw_t)$, where C_t and c_t are constants with $C_t \geq 0$ and $0 \leq c_t \leq 1$. This amount carried is a linear function of the outstanding debt and earns interest at the lending rate. The general form of this liquidity constraint is

$$- \sum_{j=1}^{n} A_{tj}X_j - l_{t-1}\left[V_{t-1}+c_{t-1}\, w_{t-1}+C_{t-1}\right]$$

$$+ \left[V_t+c_t\, w_t+C_t\right] \quad b_{t-1}\, w_{t-1}$$

$$-W_t+w_t \le M_t' , \qquad \text{for } t=1,\ 2,\ \ldots,\ T.$$

Bernhard explained this constraint as follows:

> Taking the terms in order, this says that at time t, the net cash outflow to projects, minus the cash inflow from time t-1 loans, plus the cash outflow for time t loans, plus the cash outflow for repayment of time t-1 borrowing, minus the cash inflow from time t borrowing, plus the cash outflow for time t dividends, must be less than or equal to the cash available from outside sources at time t.[10]

The next constraint is for terminal wealth.

$$G=M' + \sum_{j=1}^{n} A_jX_j+V_T+ c_T\, w_T+C_T - w_T$$

This constrains the terminal wealth of the firm at time T to be equal to the present worth at time T of cash flows from all outside projects plus the present worth of the cash flows after time T of projects chosen from the current analysis, plus the sum of all money loaned at time T minus all money owed at time T.

A scarce resource constraint is formulated as

$$\sum_{j=1}^{n} d_jX_j \le d.$$

This constraint requires that the amount of material d used be less than

[10] Richard H. Bernhard, "Mathematical Programming Models for Capital Budgeting--A Survey, Generalization, and Critique," Tech. Rept. #57, Dept. of O. R. College of Engineering, Cornell University, Ithica, New York, November 1968, pp. 5-6.

or equal to the supply available. Constraints of this type are used to ascertain that no projects will be chosen which are infeasible from a standpoint other than capital budgeting.

Bernhard formulated other constraints for limiting the borrowing in any period, for prohibiting multiple projects and for group payback restrictions. His non-negativity constraint required the variables X_j, W_t, w_t, and V_t to be greater than or equal to zero. Bernhard formulated one constraint on terminal wealth which was not necessarily linear. He also pointed out that the objective function may not be linear. Bernhard's work stressed the simplification of his general model to take on the form of the models developed by others and the fact that his model incorporated many features that were omitted by people such as Weingartner or Baumol and Quandt.

Kendrick's Intertemporal Planning Models

Kendrick (17) formulated economic planning models for the development of markets and industries. The result of the application of these models is used in capital budgeting but at a much larger scale, namely, that of a national economy or a segment of the economy.

He assumed that the industry of concern produced R=1, . . . , r intermediate goods and R=r+1, . . . , K final goods in e=1, 2, . . . , E productive units at each plant. He used the symbol a_i^{ke} to denote the productive capacity of unit e at plant i to produce a unit of product k. If a product does not require processing through a specific unit, then $a_i^{ke}=0$ for that unit. At any stage of a manufacturing process, the total product produced must be equal to that product passed to the next stage of the process, plus that product sent out of the plant as intermediate goods, plus that part of the product sent out of the plant as final goods for consumption.

Productive capacity of a unit may be recognized as the factor which limits production inside a given plant. This problem can be altered by shifting goods between plants. Since there are many products and plants to be considered, Kendrick defined additional terms as follows:

S_j^k = the demand for product k in market j.

g_i^e = the capacity of productive unit e in plant i.

C_{ij}^k = the cost of transporting a unit of product k from location i to location j.

W_{ij}^k = the units of product k transferred from location i to location j.

He then formulated the model as

minimize

$$\sum_i \sum_j \sum_{k=r+1}^{K} C_{ij}^k W_{ij}^k$$

subject to capacity utilization constraints,

$$\sum_j \sum_{k=r+1}^{K} a_i^{ke} W_{ij}^k \leq g_i^e, \quad \text{all } i, e,$$

market requirement constraints,

$$\sum_i W_{ij}^k \geq S_j^k, \quad \text{all } j, k=r+1, \ldots, K,$$

and non-negativity restrictions,

$W_{ij}^{k} \geq 0$, all i, j, k=r+1, . . ., K.

This problem is one in ordinary linear programming. Its objective is to minimize the transportation cost of meeting a market requirement in an industry. It could also be modified to take into account the costs of transporting intermediate products in the industry to eliminate production bottlenecks and increase the overall output and efficiency. Kendrick defined a new variable u_{il}^{k} which represented the units of intermediate product k shipped from plant i to plant l. He then added the terms

$$\sum_{i} \sum_{l \neq i} \sum_{k=1}^{n} h_{il}^{k} u_{il}^{k}$$

where h_{il}^{k} is the cost of moving a unit of product k from location i to location l.

The effect of this new freedom on the capacity constraints may be accounted for by adding the following term:

$$\sum_{l \neq i} \sum_{k=1}^{r} a_{i}^{kl} u_{il}^{k} - \sum_{l \neq i} \sum_{k=1}^{n} a_{l}^{kl} u_{il}^{k}, \qquad \text{all i, e.}$$

The first term has the effect of using capacity since it is units taken from this process and sent elsewhere. The second term eases the capacity since it represents units brought in from other facilities.

Kendrick also treats the effects of imports and exports on the overall model. Kendrick's models were used in studying the effects of various locations considered for the location of a new automobile plant in Brazil. It was also used in the determination of the location at which plant capacity should be constructed. In this light it can be considered as a tool for capital budgeting.

CHAPTER IV

THE SYSTEMS APPROACH TO THE REPLACEMENT PROBLEM: FORMULATION OF GENERAL CONSTRAINTS

General

As shown in Chapter II, most of the models developed for replacement problems consider a one machine case. These models all make the assumption that the machine will perform satisfactorily or produce the needed output over its life. In reality, the output of the machine will probably decrease with time or at best its quality will decrease. Hence, over the life of a machine there is some difference in output, either quality wise or quantity wise. Decision makers realize this difference and take into account such things as over capacity in early years in order to meet the capacity requirements toward the end of the machine's useful life. These decisions are predicated upon certain assumptions and when the decisions are made by more than one decision maker the assumptions may not be valid. One decision maker's assumptions may conflict with those of another. Except in rare circumstances, there will be more than one machine to be considered. If a manufacturing facility has only one machine, the approach generally taken may be proper, but given the normal situation, the solution obtained by application of the basic models will be a feasible one only.

To take into account the important considerations one must realize that a machine is a subsystem, a part of an overall manufacturing system. Optimization of results at the machine level in a system will not necessarily result in the optimization of results at the system level. To obtain the best results the consideration for the optimal allocation of capital should be made at the highest level practical for a given situation at hand.

The approach taken by Hanssmann (16) is one which takes into account the machine as the part of a system. His treatment of the replacement problem was done superficially. His idea, however, was a good one which was worthy of much further study. The investigator chose to pursue that idea.

The Production System as a Network

Any production process can be represented by a network. The complexity of these networks will vary with the complexity of the product produced. Any production facility can also be represented by one or more networks. In some multi-product facilities, there will be only one network due to the fact that one machine is used to produce a component of each product and, therefore, ties the otherwise unrelated networks together. This work will treat the elements of a network approach to the replacement problem by examining simple models of production processes. It is realized that as decisions are made and replacements occur the network may become more complex.

It will also be acknowledged that processes already in use may be much more complex than these simple cases but can be synthesized from these cases.

Figure 5 represents a network model of an m stage production process with K_i machines per stage.

The arcs of the network represent machines which have the capacity required for this section of the process. Each also has a cost associated with its purchase and operation. The dotted arcs in Figure 6 represent alternatives to be considered in the replacement or addition to the network.

Looking at a given stage of the process, we find the situation as shown in Figure 7.

The solid lines represent equipment in use, and the dotted lines represent the alternatives for replacement. The alternatives for replacement might all be the same (replications of the same machine) if it were found to dominate all alternatives considered for this stage of the process. If, however, alternatives for possible selection in this stage did not present one superior candidate, the number of possibilities could be as many as kn where k equals the number of machines in the stage and n is the number of alternatives which are alternatively dominant as the various characteristics of the machines are examined.

Examining the process as a whole, if j units are to be completed each period, then j units on the average must pass through each stage

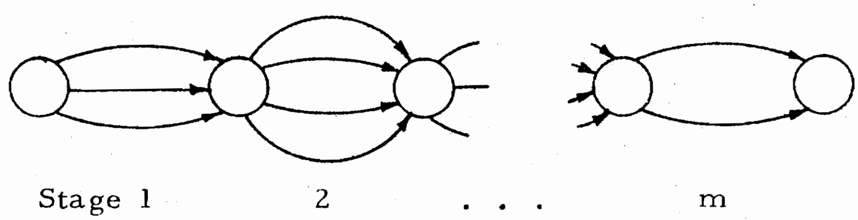


FIGURE 5

MODEL OF AN "M" STAGE PRODUCTION PROCESS

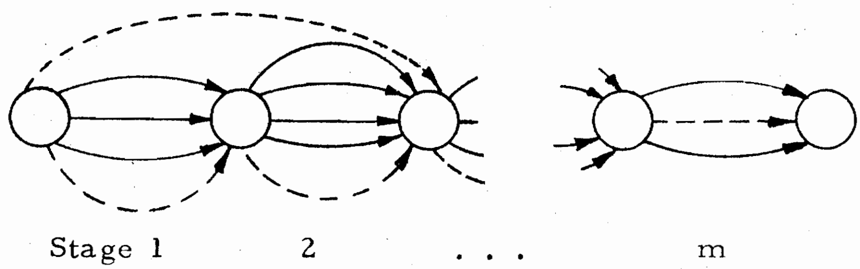


FIGURE 6

MODEL OF AN "M" STAGE PRODUCTION PROCESS
WITH ALTERNATIVES SUPERIMPOSED

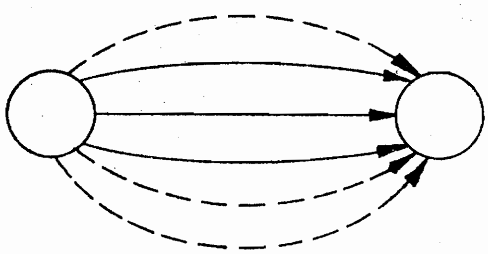


FIGURE 7

DETAIL OF A SINGLE STAGE OF A PROCESS WITH ALTERNATIVES SUPERIMPOSED of the production process each period. Therefore, the minimum network cut must be equal to j.

It is possible to pose alternatives for selection which would perform the operations of more than one stage in the network. This is the situation as shown in Figure 8 by the dotted line. The net effect of such an alternative is an increase in the capacity of the intervening stages.

Another possibility for consideration is the selection of a piece of equipment which would decrease the required capacity of a following or preceding stage of manufacture. This might be as shown in Figure 9.

Here the selection of a in stage one has a secondary effect on stage two which can be represented by a'. These are, therefore, dependent alternatives and a' cannot be attained individually. The same situation might also occur in reverse where the selection of a might have a relationship which preceded a in the process. An example of this might be that, due to the capacity of a, some work could be left undone at a preceding operation and passed onto a and, thereby, reduce a bottleneck.

Other considerations might be included such as a policy constraint. For example, no machine will be replaced before it has been in service more than x years. In the long run, this policy has the effect of decreasing the size of the problem, but it does not aid optimality of the resulting solution.

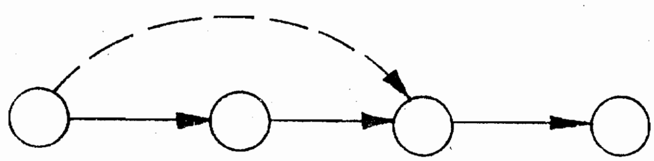


FIGURE 8

PROCESS OF SEVERAL STAGES WITH AN ALTERNATIVE WHICH WOULD REPLACE TWO STAGES

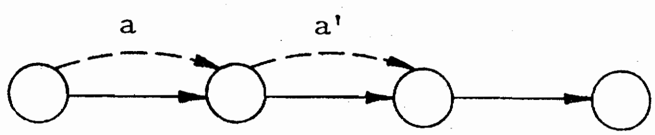


FIGURE 9

PROCESS OF SEVERAL STAGES WITH ALTERNATIVE WHICH HAS INSEPARABLE STAGE EFFECTS

Network relationships. Some of the possibilities for process relationships are listed below:

1. Either alternative a or alternative b must be chosen but not both.

2. Either alternatives a and b can be chosen together, or they can both be rejected, or alternative a can be chosen by itself. Alternative b cannot be chosen by itself.

3. Both projects can be chosen or both rejected.

4. If project X_i is not chosen then projects X_j, X_h, and X_l must be chosen.

5. Once a project has left the process for a period, it cannot be reconsidered as an alternative.

6. Project a requires funding in advance.

7. Alternative a will increase or decrease the capacity of stage X, Y, and Z.

8. Alternative a will lower the operating costs on machines j, k, and l by X% or conversely raise the operating costs.

9. If alternative a is not chosen, then a new alternative is presented.

10. The distribution of funds required in the planning horizon is . . . or the choice of certain alternatives would exceed the budget for period j.

Formulation of System Relationships as Constraints

Constraints will be formulated here for certain elementary cases. In special problems or applications the formulation of these simple constraints will become much more complex.

Mutually exclusive project constraint. Either project X_i or project X_j must be chosen but only one of the two. (Until qualified further all variables will be zero-one.)

$X_i + X_j = 1$

Project contingency constraint--type 1. Either project X_i or projects X_i and X_j may be chosen but not X_j exclusively.

$X_i - X_j \geq 0$

Project contingency constraint--type 2. This constraint type requires that both projects are chosen or both are rejected.

$X_i - X_j = 0$

Here, both X_i and X_j must equal zero or both equal one for the constraint to hold.

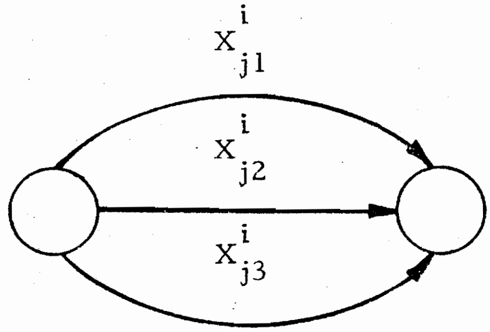
Project contingency constraint--type 3. If project X_i is not chosen, then X_j, X_k, and X_l must be chosen. This type of constraint requires the combination of a series of other constraint types. This can be accomplished as follows:

$X_i + X_j = 1$

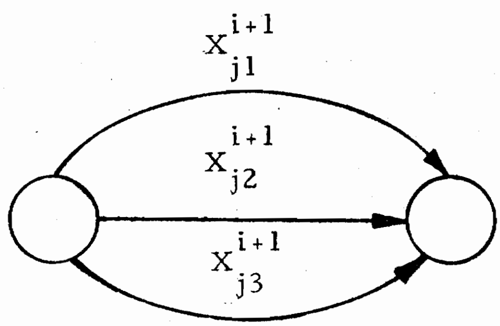
$X_j - X_k = 0$

$X_k - X_l = 0$

Project re-entry constraint. Once a project has entered the system, it must continue in the process in period i to be a candidate for continuing in period i+1. Refer to Figure 10 for the representation of the jth stage of a manufacturing process in two subsequent periods,



a.



b.

FIGURE 10

REPRESENTATION OF ALTERNATIVES IN SUBSEQUENT PERIODS i and i+1. X_{jk}^{i} represents machine number k in the jth stage of the process for the ith period of consideration. For X_{jl}^{i+1} to be considered, it must have been included in the previous period, otherwise, the assumption would be the dissolution of ownership. This type constraint is made by tying other constraints together with contingency constraint--type 1.

<u>Constraints for project with lead time</u>. Realistically, most projects in industry require lead time. This time is required as either order time or construction time. In the case of order lead time, capital may or may not be required until the delivery date. In the case of construction time, capital will most certainly be required in advance. Development of a constraint for a project of this type requires only that the capital requirement be accounted for in the proper periods in advance of any production benefits gained from the process, machine, or system.

<u>Constraints for capacity effects</u>. Here each stage has to produce some given number of units. The constraints for each stage would be of the type

$$\sum_{j} c_j X_{ij} \geq C_i, \qquad \text{for all } i,$$

where X_{ij} is an alternative for stage i, c_j is the capacity added by the alternative, and C_i is the required stage capacity. An alternative which would increase the capacity of stages 1, 2, and 3 would be treated as follows in the constraint where X_k is the alternative:

$$\sum_{j} c_j X_{ij} + d_i X_k \geq C_i, \qquad \text{for } i=1, 2, 3,$$

where c_j, X_{ij}, X_k, and C_i are as defined earlier and d_i is the effect of alternative X_k on the ith stage capacity for i=1, 2, 3. This "stage effect constraint" could be combined with any of the project type constraints to give additional relationships.

Constraint for cost effects. A constraint of this type is very similar to that for capacity effects. The difference being that the effect will be in a budget constraint rather than in a capacity constraint. The situation where such a constraint would be applicable would be one in which the introduction of a machine into a process would have some "edge effects" which might permeate other adjacent machines, i, e., increase or decrease their capacities or work loads or alter them in some way.

Functional degradation type constraint. For a machine to be functionally degraded, it must be removed from service somewhere in the process and placed in a lower grade service elsewhere in the process. If the use of X_k in the downgraded service is represented by $X_{k'}$, then

$$X_k + X_{k'} = 1.$$

It is seen that functional degradation of a machine is no more than treating it as two mutually exclusive alternatives in the same process.

Capital constraints. The general heading, capital constraints, is so designated in order to take care of several situations. In some cases, it will be desirable to shift budgeted funds from one period to the next. In other cases, it might be desirable to allow the budget to be exceeded if borrowing funds could be profitable.

The most elementary formulation of a budget constraint would be as follows:

$$\sum_{i=1}^{m} d_{ij}X_i \leq B_j, \qquad \text{for all } j,$$

where m is the number of alternatives available, d_{ij} is the cost of project X_i in period j, and B_j is the budget in period j.

To be more realistic the constraint might be formulated in the manner shown below.

$$\sum_{i=1}^{m} d_{i1}X_i - S_1 = B_1, \qquad j=1,$$

$$\sum_{i=1}^{m} d_{ij}X_i + S_{i-1} - S_i = B_j, \qquad \text{for } j=2, \ldots, n-1,$$

$$\sum_{i=1}^{m} d_{in}X_i + S_{n-1} = B_j, \qquad \text{for } j=n,$$

or

$$\sum_{i=1}^{m} d_{in}X_i + S_{n-1} - S_n = B_n, \qquad \text{for } j=n.$$

A slack variable S_i is added in each period to carry funds forward to the following period. In the first period no funds are brought forward and in the last funds may or may not be carried forward. It would be logical to allow this "carry-forward" in the last period, and to place the variable S_n, which represents funds carried forward in the last period, in the objective function with the proper sign. If an equality were forced, i. e., no "carry-forward" in the last period was allowed, the resulting solution might not be as good as could be obtained otherwise.

Constraints of the type used above lack some realism. Funds have some time value. There is also the possibility of borrowing from future budgets. Figure 11 gives an idea of the fund's balance for a given period. The first formulation above allowed for only (1) and (6) as shown in the figure. The second formulation allowed for (1), (3), (4), and (6). Before incorporating (2) and (5) into a constraint formulation, it is realized that the time value of money must be accounted for.

With reference to Figure 11, it can be seen that the funds available in any period are equal to the funds budgeted, plus the funds borrowed, plus the funds brought forward from previous periods, less the funds carried forward to the next period, less the funds borrowed from this period. Letting these be represented by Y_1 through Y_5, respectively, and Y_6 represent the funds used in a period the following must hold:

$$Y_{1j}+Y_{2j}+Y_{3j} - Y_{4j} - Y_{5j} - Y_{6j}=0, \qquad \text{for all } j,$$

where j is a budget period.

This could be handled easier if the time value of money was not of concern. In the formulation of the budget constraints, Y_{6j} would be partitioned out into the requirements for each project thereby adding

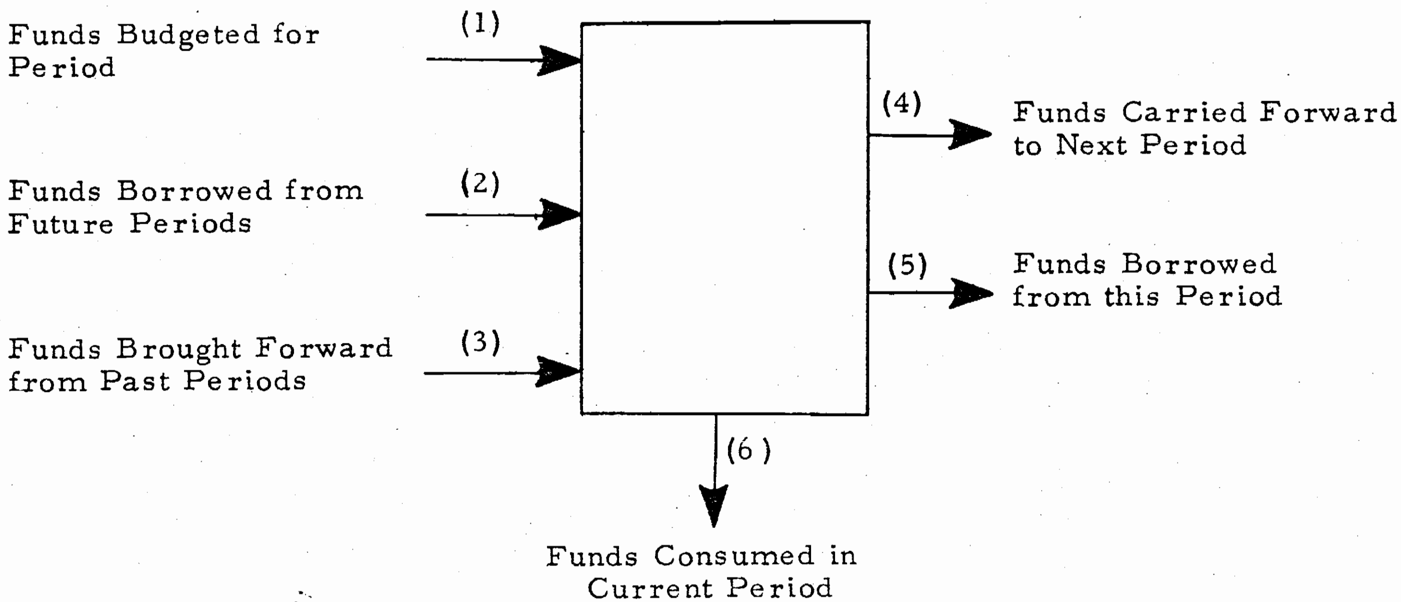


FIGURE 11

FUNDS BALANCE FOR A PERIOD

more variables to the constraint. These constraints would then be tied together by additional constraints which would account for the interest rates related to carrying money forward or backward in time. For example, two constraints might be

$$Y_{1j}+Y_{2j}+Y_{3j} - Y_{4j} - Y_{5j} - Y_{6j}=0$$

and

$$Y_{1,\ j+1}+Y_{2,\ j+1}+Y_{3,\ j+1} - Y_{4,\ j+1} - Y_{5,\ j+1} - Y_{6,\ j+1}=0.$$

These would be tied together by two additonal constraints which would be written as below:

$$(1+i_f)\ Y_{4j}=Y_{3,\ j+1}.$$

This constraint requires that funds not used in period j grow at an interest rate i_f in passing to another period.

$$Y_{2j}=\left(\frac{1}{1+i_b}\right)\ Y_{5,\ j+1}.$$

This constraint requires that no more can be borrowed in period j than can be paid back in period j+1, or that the limit on borrowing in period j is equal to the present worth of what can be paid back in period j+1 discounted at the borrowing rate i_b.

Conclusions from Constraint Formulation

Using the constraints that have been formulated in this chapter, and variations thereof, it is believed that practically any production system can be modeled for a replacement analysis using a mathematical programming approach. In the following chapter example situations will be modeled.

CHAPTER V

FORMULATION OF REPLACEMENT MODELS

General

In the previous chapter, various constraints were formulated which could be used, with modifications, in describing process relationships and requirements. The discussion of objective functions was omitted from previous material. Before formulating models from problem situations in this chapter, the subject of objective functions must be discussed.

Objective Functions for Replacement Models

The normal objective function, either one for maximization or minimization, was used for the replacement model. If the level of modeling for the process was at the same level as capital budgeting for the firm, then these two might possibly be merged as one and the model be constructed of the entire firm. Such a model would include replacement and expansionary investment. The objective function could then be formulated as one of profit maximization.

In other situations, the person responsible for replacement decisions may not have visibility over the entire firm. The concern of the responsible individual then becomes the optimal allocation of his budgeted capital to accomplish assigned objectives. In this situation a cost minimization model would be most appropriate. It is believed that this second case would be the one most frequently encountered.

Several possibilities exist for formulation of the objective function. The approach chosen here was for a finite planning horizon which was used in all model formulations. The treatment was to minimize the present worths of the costs of the alternatives chosen for accomplishment of the assigned objectives. Since income tax is a relevant consideration in every case, it was included. (For treatment of tax and determination of costs used, see the Appendix.)

Hence, the objective functions for the cost minimization models were of the form

$$\sum_{i=1}^{n} C_i X_i$$

where C_i is the after tax present worth cost of the alternative X_i, and n, the number of alternatives. The objective was to minimize this function. For a profit maximization type model, the objective function would be of the same form. The coefficients C_i would represent profits in this case.

Any deviations from these forms will be explained when they are used in this investigation. Other relevant explanations will be given for objective functions when the occasion demands.

Formulation of a Model for a Two Stage Process

The process as shown in Figure 12 was modeled to illustrate a simple application of the constraints developed in Chapter IV. The

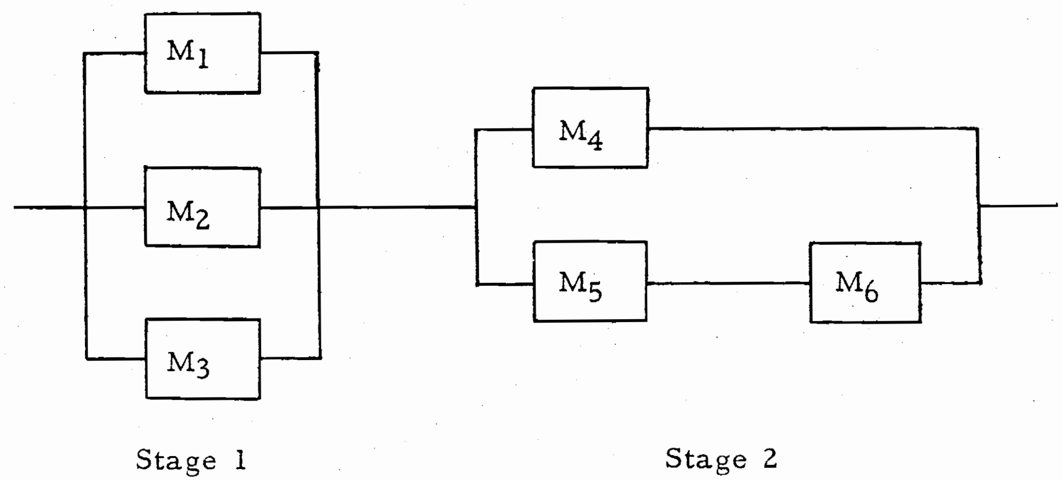


FIGURE 12

A TWO STAGE PRODUCTION PROCESS

following problem statement applies to Figure 12.

In the first stage machines 1, 2, and 3 perform identical jobs. The second stage consists of three machines. Two of these machines, 5 and 6, are required to perform the task which is performed by machine 4. They operate in parallel with machine 4.

Management wishes to make an analysis in order to choose alternatives for replacement with the objective of minimizing the cost of meeting a forecasted output over the four period planning horizon.

The following data is available from forecasts and current market information:

1. The expected operating costs of each machine in the process and of those being considered as alternatives for the process.

2. Machine capacities.

3. The present market value and forecasted year by year market values and salvage values for machine being considered.

4. The forecasted demand for output of the system over the planning horizon (period by period).

5. The budget for operating costs and replacement costs over the planning horizon (period by period).

The following assumptions will be made:

1. Machines 1 and 2 will not be considered for replacement.

2. Machines 5 and 6 will be considered for replacement by a new machine, but machine 4 will not be considered for replacement.

3. Machine capacities will be considered constant for all machines over the planning horizon.

4. There will be no provision for "carry-forward" or "carry-backward" of funds.

5. Only the superior challenger will be considered in each stage.

The problem can now be reduced to that shown by the disjoint network in Figure 13. Machines 5 and 6 are combined into a single machine M_5^*. M_3' and $M_5'^*$ represent the best alternatives which are to be considered for the process.

For the problem formulation the process output was reduced at each stage by the capacity of each machine which is not considered for replacement. The budget was also reduced by the amounts required in each period for operating any machine whose acceptance was certain. This reduction is made because that is the problem which we are to solve. Hence, the problem was reduced to the situation shown in Figure 13.

<u>Definition of terms to be used in the formulation.</u>

X_{ijkl} = alternative k of the ith stage during the jth period which was first introduced in period l.

C_{ikl} = the present worth cost associated with the adoption of alternative k of stage i in period l.

B_j = the amount of budgeted funds available in period j.

b_{ijkl} = funds required for X_{ijkl}.

K_{ij} = capacity required from stage i in period j.

$\overline{k}_{ijkl}$ = capacity of X_{ijkl}.

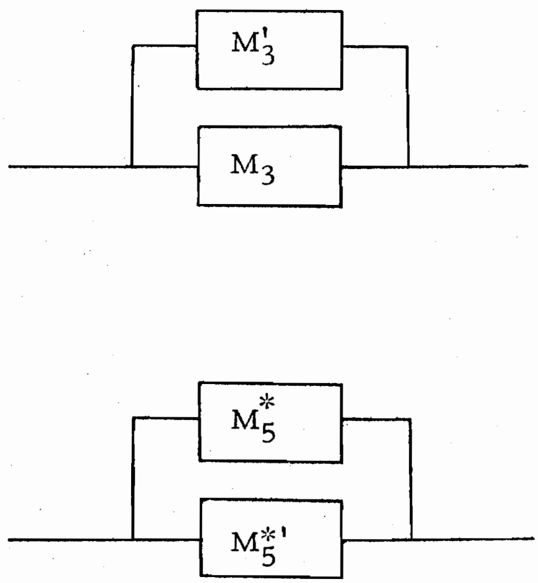


FIGURE 13

REDUCED TWO STAGE PROCESS

Formulation of the objective function. The alternative presently in service in stage 1, M_3, is represented as

$$X_{1,1,1,1}.$$

M_5^*, which is in stage two, was represented as

$$X_{2,1,1,1}$$

in the first period. Referring to M_3, if it is continued in service throughout the planning horizon, the variables in subsequent periods will be

$$X_{1,2,1,1},\ X_{1,3,1,1},\ \text{and}\ X_{1,4,1,1}.$$

If the choice is made to replace the machine (defender) in the first period, the variables involved will be

$$X_{1,1,2,1},\ X_{1,2,2,1},\ X_{1,3,2,1},\ X_{1,4,2,1}.$$

If the choice is made to replace the defender in the nth period, the variables involved will be

$$X_{1,n,2,1}\ \ldots,\ X_{1,n,2,4+1-n}.$$

These variables are shown in Table I.

An examination of Table I indicates that the appearance of certain variables in the solution show which alternative had been chosen. These variables would be

$$X_{1,4,1,1},\ X_{1,4,2,1},\ X_{1,4,2,2},\ X_{1,4,2,3},\ \text{and}\ X_{1,4,2,4}.$$

$X_{1,4,1,1}$ would indicate that the defender had not been replaced during the planning horizon. $X_{1,4,2,1}$ would indicate that the defender had

TABLE I

VARIABLES IN SOLUTION FOR ALTERNATIVE CASES

Not Replaced	Replaced At 1st	Replaced At 2nd	Replaced At 3rd	Replaced At 4th
$X_{1,1,1,1}$	$X_{1,1,2,1}$			
$X_{1,2,1,1}$	$X_{1,2,2,1}$	$X_{1,2,2,2}$		
$X_{1,3,1,1}$	$X_{1,3,2,1}$	$X_{1,3,2,2}$	$X_{1,3,2,3}$	
$X_{1,4,1,1}$	$X_{1,4,2,1}$	$X_{1,4,2,2}$	$X_{1,4,2,3}$	$X_{1,4,2,4}$

been replaced in the first period. $X_{1,4,2,n}$ would indicate that the defender had been replaced in the nth period. Hence, that part of the objective function related to the first stage would be

$$C_{1,1,1}X_{1,4,1,1}+C_{1,2,1}X_{1,4,2,1}+C_{1,2,2}X_{1,4,2,2}$$
$$+C_{1,2,3}X_{1,4,2,3}+C_{1,2,4}X_{1,4,2,4}$$

where the $C_{i,k,1}$ would be determined by the method shown in the Appendix for an alternative cost.

<u>Budget constraint (B).</u> The total costs in each period must be equal to or less than the budget for the period.

$$b_{1,1,1,1}X_{1,1,1,1}+b_{1,1,2,2}X_{1,1,2,1}+b_{2,1,1,1}X_{2,1,1,1}$$
$$+b_{2,1,2,1}X_{2,1,2,1}\leq B_1,$$

or

$$\sum_i \sum_k b_{i,j,k,1}X_{i,1,k,1}\leq B_1.$$

A general constraint was formulated as

$$\sum_{i=1}^{M} \sum_{k=1}^{2} \sum_{1=1}^{j} b_{i,j,k,1}X_{i,j,k,1}\leq B_j, \text{ for } j=1, \ldots, N.$$

<u>Capacity constraints (K).</u> The machines in service at any future date in any stage of the process must provide the volume required from that stage of the process.

<u>For stage 1.</u>

$$\bar{k}_{1,1,1,1}X_{1,1,1,1}+\bar{k}_{1,1,2,1}X_{1,1,2,1}\geq K_{1,1}$$

$$\bar{k}_{1,2,1,1}X_{1,2,1,1}+\bar{k}_{1,2,2,1}X_{1,2,2,1}+\bar{k}_{1,2,2,2}X_{1,2,2,2}$$
$$\geq K_{1,2}$$

$$\bar{k}_{1,3,1,1}X_{1,3,1,1}+\bar{k}_{1,3,2,1}X_{1,3,2,1}+\bar{k}_{1,3,2,2}X_{1,3,2,2}$$
$$+\bar{k}_{1,3,2,3}X_{1,3,2,3}\geq K_{1,3}$$

$$\bar{k}_{1,4,1,1}X_{1,4,1,1}+\bar{k}_{1,4,2,1}X_{1,4,2,1}+\bar{k}_{1,4,2,2}X_{1,4,2,2}$$
$$+\bar{k}_{1,4,2,3}X_{1,4,2,3}+\bar{k}_{1,4,2,4}X_{1,4,2,4}\geq K_{1,4}$$

The constraints can be formulated in a similar manner for stage 2, but in general the constraints are given by

$$\sum_{l=1}^{j}\sum_{k=1}^{2}\bar{k}_{ijkl}X_{ijkl}\geq K_{ij},\quad \text{for } i=1,\ldots, M \text{ and } j=1,\ldots, N.$$

Process relationship constraints. These constraints are included to ascertain that feasible solutions are obtained. Once a machine has been replaced during the planning horizon, it cannot be replaced again. Until the time of replacement, it faces new and different machines as challengers in each period.

To formulate constraints of this type, it was useful to array the alternatives. This was done in Table II. Observing this table it was noted that n+1 possibilities exist at period n. With the additional requirement that only one replacement be allowed during the planning horizon, the number of alternatives was reduced to a total of n+1. These n+1 alternatives can be constrained to allow the selection of one alternative from the group of five.

The period alternatives were formulated from observation of Table II. In period one there were only two possibilities from which to choose. Since variables representing the alternatives are zero-one,

TABLE II

VARIABLES WHICH APPEAR IN THE PROCESS RELATIONSHIP CONSTRAINTS

Period					
1	$X_{1,1,1,1}$	$X_{1,1,2,1}$	$X_{1,1,1,1}$	$X_{1,1,1,1}$	$X_{1,1,1,1}$
2	$X_{1,2,1,1}$	$X_{1,2,2,1}$	$X_{1,2,2,2}$	$X_{1,2,1,1}$	$X_{1,2,1,1}$
3	$X_{1,3,1,1}$	$X_{1,3,2,1}$	$X_{1,3,2,2}$	$X_{1,3,2,3}$	$X_{1,3,1,1}$
4	$X_{1,4,1,1}$	$X_{1,4,2,1}$	$X_{1,4,2,2}$	$X_{1,4,2,3}$	$X_{1,4,2,4}$

it follows that

$$X_{1,1,1,1} + X_{1,1,2,1} = 1.$$

In the second period there are three possibilities,

$$X_{1,2,1,1} + X_{1,2,2,1} + X_{1,2,2,2} = 1.$$

Similarly, in the third and fourth periods there are four and five alternatives, respectively,

$$X_{1,3,1,1} + X_{1,3,2,1} + X_{1,3,2,2} + X_{1,3,2,3} = 1,$$

and

$$X_{1,4,1,1} + X_{1,4,2,1} + X_{1,4,2,2} + X_{1,4,2,3} + X_{1,4,2,4} = 1.$$

Once an alternative has been chosen, it must continue in service throughout the planning horizon. Constraints for this were formulated in the manner shown below; if machine M'_3 is chosen in the first period, it must continue throughout the four year planning horizon or

$$X_{1,1,2,1} - X_{1,2,2,1} = 0,$$

$$X_{1,2,2,1} - X_{1,3,2,1} = 0,$$

and

$$X_{1,3,2,1} - X_{1,4,2,1} = 0.$$

If machine M_3^* is replaced by the best challenger in the second period, these conditions must hold

$$X_{1,2,2,2} - X_{1,3,2,2} = 0,$$

and

$$X_{1,3,2,2} - X_{1,4,2,2} = 0.$$

For the third period it follows:

$$X_{1,3,2,3} - X_{1,4,2,3} = 0.$$

No constraint is required for the fourth period. This was done for each series of alternatives in the process and will satisfy the process relationship requirement.

Mathematical Model of the Replacement Situation

The model can be expressed as shown below.

minimize $Z =$

$$C_{1,1,1}X_{1,4,1,1} + C_{1,2,1}X_{1,4,2,1} + C_{1,2,2}X_{1,4,2,2}$$
$$+ C_{1,2,3}X_{1,4,2,3} + C_{1,2,4}X_{1,4,2,4} + C_{2,1,1}X_{2,4,1,1}$$
$$+ C_{2,2,1}X_{2,4,2,1} + C_{2,2,2}X_{2,4,2,2} + C_{2,2,3}X_{2,4,2,3}$$
$$+ C_{2,2,4}X_{2,4,2,4}$$

subject to the budget constraints,

$$\sum_{i=1}^{2} \sum_{k=1}^{2} \sum_{l=1}^{j} b_{ijkl}X_{ijkl} \leq B_j, \text{ for all } j,$$

the capacity constraints,

$$\sum_{k=1}^{2} \sum_{l=1}^{4} \bar{K}_{ijkl}X_{ijkl} \geq K_{ij}, \quad \text{for all } i,\ j,$$

the process relationship constraints,

$$X_{1,1,1,1} + X_{1,1,2,1} = 1,$$

$$X_{1,1,2,1} + X_{1,2,2,1} + X_{1,2,2,2} = 1,$$

$$X_{1,3,1,1} + X_{1,3,2,1} + X_{1,3,2,2} + X_{1,3,2,3} = 1,$$

$$X_{1,4,1,1} + X_{1,4,2,1} + X_{1,4,2,2} + X_{1,4,2,3} + X_{1,4,2,4} = 1,$$

$X_{1,1,2,1} - X_{1,2,2,1} = 0,$

$X_{1,2,2,1} - X_{1,3,2,1} = 0,$

$X_{1,3,2,1} - X_{1,4,2,1} = 0,$

$X_{1,2,2,2} - X_{1,3,2,2} = 0,$

$X_{1,3,2,2} - X_{1,4,2,2} = 0,$

$X_{1,3,2,3} - X_{1,4,2,3} = 0,$

$X_{2,1,1,1} + X_{2,1,2,1} = 1,$

$X_{2,1,2,1} + X_{2,2,2,1} + X_{2,2,2,2} = 1,$

$X_{2,3,1,1} + X_{2,3,2,1} + X_{2,3,2,2} + X_{2,3,2,3} = 1,$

$X_{2,4,1,1} + X_{2,4,2,1} + X_{2,4,2,2} + X_{2,4,2,3} + X_{2,4,2,4} = 1,$

$X_{2,1,2,1} - X_{2,2,2,1} = 0,$

$X_{2,2,2,1} - X_{2,3,2,1} = 0,$

$X_{2,3,2,1} - X_{2,4,2,1} = 0,$

$X_{2,2,2,2} - X_{2,3,2,2} = 0,$

$X_{2,3,2,2} - X_{2,4,2,2} = 0,$

$X_{2,3,2,3} - X_{2,4,2,3} = 0,$

and finally the zero-one restriction,

$X_{ijkl} = \begin{cases} 0 \\ 1 \end{cases},$ for all i, j, k and l.

Formulation of Model for a Three Stage Process

The problem, as shown by Figure 14, is to minimize the cost of providing the service in accordance with the following problem statement:

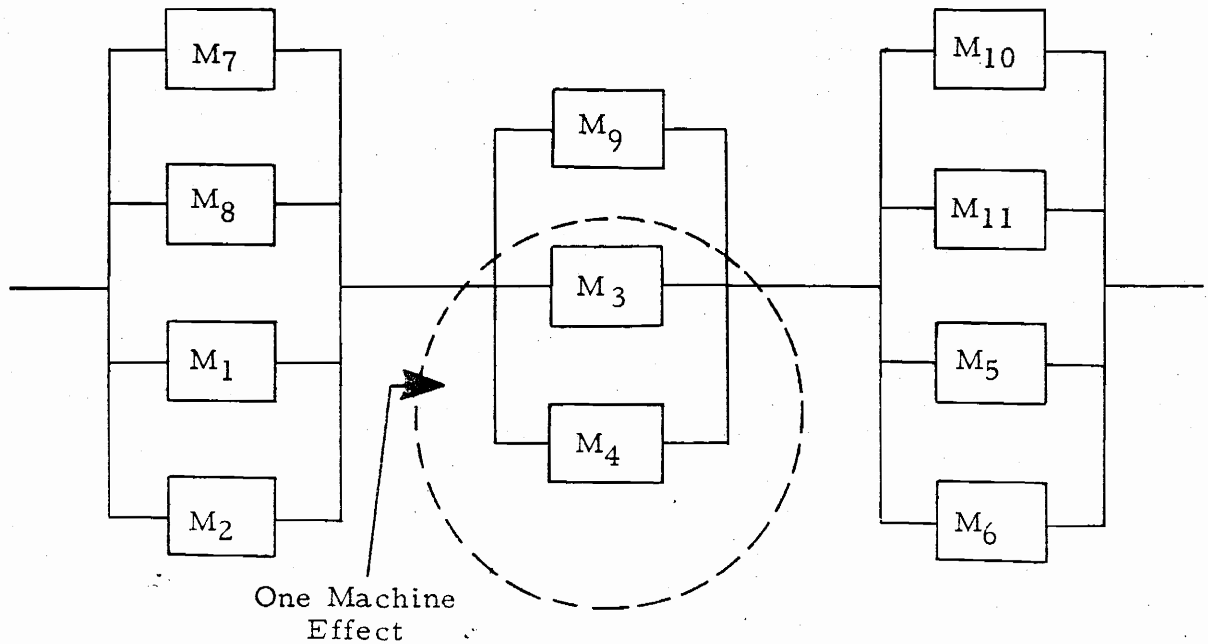


FIGURE 14

A THREE STAGE PRODUCTION PROCESS

Problem statement. All machines will be considered for replacement in the present period only. M_1 and M_2 have challengers for replacement, M_7 and M_8, which are repetitions of the same machine. If M_9 is chosen, it will replace both M_3 and M_4. M_{10} will replace M_5 only, and M_{11} will replace M_6 only. If M_{10} is chosen, it will have a secondary effect on the first stage of the process of increasing its capacity by 50 units per year. Although the decision concerns what to do in allocating capital to the process in a single period, it is desired that the decisions provide a process that will meet forecasted demands for a three year planning horizon.

Definitions of terms to be used in the formulation.

X_{ijk} = machine k of the ith stage during the jth period.

C_{ijk} = the present worth cost associated with the acceptance of alternative X_{ijk}.

B_j = the budgeted funds available in period j.

b_{ijk} = funds required for X_{ijk}.

K_{ij} = capacity required from stage i in period j.

$\overline{k}_{ijk}$ = capacity of X_{ijk}.

Model formulation. The problem then becomes the following:

minimize

$$\sum_{i=1}^{3} \sum_{k} C_{i1k} X_{i1k},$$

subject to the following constraints:

capacity constraints,

$$\sum_k \bar{k}_{ijk} X_{ijk} \geq K_{ij}, \qquad \text{for } i=1,\ 2, \text{ and all } j,$$

$$\sum_k \bar{k}_{3jk} X_{3jk} + 50X_{3,1,11} \geq K_{3j}, \qquad \text{for } i=3, \text{ and all } j;$$

budget constraints,

$$\sum_i \sum_k b_{ijk} X_{ijk} \leq B_j, \qquad \text{for all } j;$$

process relationship constraints,

$$X_{1,1,1} + X_{1,1,7} = 1$$

$$X_{1,1,7} - X_{1,2,7} = 0$$

$$X_{1,2,7} - X_{1,3,7} = 0$$

$$X_{1,1,2} + X_{1,1,8} = 1$$

$$X_{1,1,8} - X_{1,2,8} = 0$$

$$X_{1,2,8} - X_{1,3,8} = 0$$

$$X_{2,1,3} + X_{2,1,9} = 1$$

$$X_{2,1,9} - X_{2,2,9} = 0$$

$$X_{2,2,9} - X_{2,3,9} = 0$$

$$X_{3,1,5} + X_{3,1,10} = 1$$

$$X_{3,1,10} - X_{3,2,10} = 0$$

$$X_{3,2,10} - X_{3,3,10} = 0$$

$$X_{3,1,6} + X_{3,1,11} = 1$$

$$X_{3,1,11} - X_{3,2,11} = 0$$

$$X_{3,2,11} - X_{3,3,11} = 0.$$

Formulation of a General Systems Replacement Model for a Finite Planning Horizon of N Periods

This model was generalized from the process relationships and process requirements covered in the two previous formulations. It was assumed that one alternative was available in each period for each machine in each stage of the process. Only one replacement was allowed during the planning horizon.

The terms used in the formulation were the same as those used in the Model I formulation with the following added:

X_{ijklm} = represents the kth alternative for machine m in stage i during period j which was originally adopted at period l.

C_{iklm} = the cost of adopting X_{iNklm}.

Y_{rj} = fund flow of type r into period j.

$\overline{k}_{ijklm}$ = the capacity contributed to stage i of the process in period j by alternative k for machine m which was adopted in period l.

The objective function of the model was constructed in the same manner as the Model I objective function. In the Nth period, since only one replacement was allowed during the planning horizon, the variables present will indicate which alternatives were chosen for the process. Since there were only two alternatives for each process subsystem or machine, it was reasoned that one of these alternatives will be included in the solution. $X_{iN,1,1,1}$ represents the original machine used for machine 1 in stage i at the outset. $X_{iN,2,1,2}$ represents the choice of alternative 2 for machine 2 in the ith stage of the process which was originally chosen in period 1. Hence, the objective function was developed as follows:

$$\sum_i \sum_j \sum_k \sum_l \sum_m X_{ijklm}$$

would represent all possible X terms used in the problem formulation. Since the examination was to be made of the terms in solution in the last period only, the formulation can be reduced to

$$\sum_i \sum_k \sum_l \sum_m X_{iNklm}.$$

There were only two alternatives for each machine in a stage. The above was separated into

$$\sum_i \sum_l \sum_m X_{iN,2,lm} + \sum_i \sum_l \sum_m X_{iN,1,lm},$$

but the first alternative was introduced in period 1 in every case, so the above was rewritten as:

$$\sum_i \sum_l \sum_m X_{iN,2,lm} + \sum_i \sum_m X_{iN,1,lm}.$$

The above represents the total alternatives available during the planning horizon expressed in terms of variables appearing in the solution of the problem at the Nth period. To convert it to an objective function, constants representing the cost of each alternative must be added. This was expressed as

$$\sum_i \sum_l \sum_m C_{iklm} X_{iN,2,lm} + \sum_i \sum_m C_{iklm} X_{iN,1,lm}.$$

The above is the objective function for the general model.

The budget constraints were formulated for the general case with the possibility of funds being carried forward or backward through time. The fund balance in any period was equal to the budgeted funds for the period, plus funds brought forward from previous periods, plus funds borrowed from future periods, less operating expenses and capital disbursements for the present period, less funds carried forward to the next period, less funds carried back to the previous period. (It was assumed that funds are carried forward at an interest rate, i_f, and backward at an interest rate, i_b.) The amount of each of these six possibilities was represented by Y_1, Y_2, Y_3, Y_4, Y_5, and Y_6, respectively. The funds balance for any period was then given by

$$Y_{1j} + Y_{2j} + Y_{3j} - Y_{4j} - Y_{5j} - Y_{6j} = 0.$$

It was clear that

$$Y_{1j} = B_j.$$

It was reasoned that the money consumed by the projects adopted was

$$\sum_i \sum_k \sum_l \sum_m b_{ijklm} X_{ijklm} = Y_{4j}, \qquad \text{for } j=1, \ldots, N.$$

In terms of present value in period j, the Y_j were given by

$$Y_{ij} = B_j, \qquad \text{for } j=1, \ldots, N,$$

$$Y_{2j} = Y_{5,j-1}(1+i_f), \qquad \text{for } j=2, \ldots, N,$$

$$Y_{3j} = Y_{6,j+1}\left(\frac{1}{1+i_b}\right), \qquad \text{for } j=1, \ldots, N-1,$$

$$Y_{4j} = Y_{4j}$$

$Y_{5j} = Y_{5j}$

$Y_{6j} = Y_{6j}$

In general, then the constraints for budgets were of the following form:

$$Y_{1j} + Y_{2j} + Y_{3j} - Y_{4j} - Y_{5j} - Y_{6j} = 0, \quad \text{for } j = 1, \ldots, N,$$

with the additional requirements that

$$Y_{2j} = Y_{5,\, j-1}(1 + i_f), \quad \text{for } j = 2, \ldots, N,$$

and

$$Y_{3j} = Y_{6,\, j+1}\left(\frac{1}{1 + i_b}\right), \quad \text{for } j = 1, \ldots, N-1,$$

where the Y_{rj} were as defined previously.

The capacity constraints were formulated by reasoning that the capacity of each stage must be equal to or greater than the required stage capacity for that stage and period. This was written as

$$\sum_{j} \sum_{k} \sum_{l} \sum_{m} \bar{k}_{ijklm} X_{ijklm} \geq K_{ij}, \quad \text{for all } i,\ j.$$

Process relationships constraints are unique. They cannot be generalized. The relationship for a single replacement during a finite planning horizon is formulated here. It is not intended to infer that this represents all possible process relationships.

For a given machine (M) in stage a of the process it is necessary that the following be true in the first period:

$$X_{a,1,1,1,M} + X_{a,1,2,1,M} = 1.$$

If $X_{a,1,2,1,M}$ is chosen, it must be kept in service throughout the planning horizon. Hence,

$$X_{a,1,2,1,M} - X_{a,2,2,1,M} = 0$$

$$X_{a,2,2,1,M} - X_{a,3,2,1,M} = 0$$

$$\vdots \qquad\qquad \vdots$$

$$X_{a,N-1,2,1,M} - X_{a,N,2,1,M} = 0.$$

In period two it must hold that

$$X_{a,2,1,1,M} + X_{a,2,2,1,M} + X_{a,2,2,2,M} = 0.$$

In other words, the only possibilities are that the original machine is still in service or that the new machine was put in service in period 1 or that the new machine was put in service in period 2. Upon study, it can be seen that this is compatible with the alternatives presented. If in period 1 it was decided to replace the old machine, then $X_{a,2,2,1,M}$ will be equal to one. If not, then in period two the choice between $X_{a,2,1,1,M}$ and $X_{a,2,2,2,M}$ must be made. This can be generalized to N periods to provide the following constraints:

$$X_{a,j,1,1,M} + \sum_{1=1}^{j} X_{a,j,2,1,M} = 1, \qquad \text{for } j=1, \ldots, N, \quad i=1, \ldots, M, \quad \text{and } m=1, \ldots, K_i,$$

where k_i is the number of machines in a stage

$$(X_{a,j,2,1,m} - X_{a,j\ \ 1,2,1,M}) = 0, \qquad \text{for } 1=1, \ldots, N-1, \quad i=1, \ldots, M, \quad j=1, \ldots, N-1, \quad \text{and } m=1, \ldots, K_i.$$

This problem can be summarized as follows:

minimize

$$\sum_{i}\sum_{l}\sum_{m} C_{iklm}X_{i,N,2,1,M}+\sum_{i}\sum_{M} C_{iklm}X_{i,N,1,1,M},$$

subject to

$$Y_{1j}+Y_{2j}+Y_{3j}-Y_{4j}-Y_{5j}-Y_{6j}=0, \qquad \text{for } j=1, \ldots, N,$$

$$Y_{2j}=Y_{5,j-1}(1+i_f), \qquad \text{for } j=2, \ldots, N,$$

$$Y_{3j}=Y_{6,j+1}\left(\frac{1}{1+i_b}\right), \qquad \text{for } j=1, \ldots, N-1,$$

$$\sum_{j}\sum_{k}\sum_{l}\sum_{m} \bar{K}_{ijklm}X_{ijklm}\geq K_{ij}, \qquad \text{for all } i,\ j,$$

$$X_{i,j,1,1,m}+\sum_{l=1}^{j} X_{i,j,2,1,m}=1, \qquad \text{for } j=1, \ldots, N,\ i=1, \ldots, M, \text{ and } m=1, \ldots, K_i,$$

and

$$X_{i,j,2,1,m}-X_{i,j\ \ 1,2,1,m}=0, \qquad \text{for } l=1, \ldots, N-1,\ j=1, \ldots, N-1,^{11}\ i=1, \ldots, M, \text{ and } m=1, \ldots, K_i,$$

$$X_{i,j,k,l,m}=\begin{cases}0\\1\end{cases}, \qquad \text{for all } i,\ j,\ k,\ l,\ m,$$

---

[11] Possibly this should be written as "for ((l=1, . . . , N-1), j=1, . . . , N-1)," since this is the idea to be conveyed.

$Y_{rj} \geq 0$, for all r, j, r = 1, . . . , 6,

and j = 1, . . . , N.

This problem formulation can be reduced into any of the simpler cases which have been formulated previously. It can also be reduced to give results similar to many given by the methods discussed in Chapters II and III. It is believed that this model for process replacement will give results superior to those given by most methods discussed previously. It is realized that this problem formulation is hardly applicable in most complex processes over long planning horizons since it will become unwieldly rapidly. However, for a small problem and a short planning horizon very good results could be expected. (This would depend, of course, upon the accuracy of forecasting techniques used to predict the future, or more specifically, upon how well the future developments followed the predicted pattern.)

Further analysis of the problem will be conducted in subsequent chapters.

CHAPTER VI

MODEL ANALYSIS AND DISCUSSION

Introduction

In the previous chapter a general model was formulated for a systems replacement situation over a finite planning horizon. In this chapter that model will be examined to determine its characteristics which were not previously discussed. Limiting cases will be examined in addition to the general model. Solution procedures will be discussed for the general model and for a special case.

Dimensionality

The size of the general replacement model should be determined in order to have some appreciation for the model. The number of zero-one variables in the problem is equal to the number of possible X_{ijklm}'s in the formulation. This can be determined as follows:
The total possibilities are given by

$$\sum_{i=1}^{M} \sum_{j=1}^{n} \sum_{k=1}^{2} \sum_{l=1}^{n} \sum_{m=1}^{K_i} X_{ijklm}.$$

This can be rewritten as

$$\sum_{i=1}^{M} \sum_{j=1}^{n} \sum_{l=1}^{n} \sum_{m=1}^{K_i} X_{ij1lm} + \sum_{i=1}^{M} \sum_{j=1}^{n} \sum_{l=1}^{n} \sum_{m=1}^{K_i} X_{ij2lm}.$$

Now alternative one was introduced in period 1 so

$$\sum_{i=1}^{M} \sum_{j=1}^{n} \sum_{m=1}^{K_i} X_{ijk1m} + \sum_{i=1}^{M} \sum_{j=1}^{n} \sum_{l=1}^{n} \sum_{m=1}^{K_i} X_{ij21m}.$$

Hence, the maximum of zero-one variables can be seen to be

$$m \, . \, n \, . \, K_{i\ max} + m \, . \, n^2 \, . \, K_{i\ max}$$

$$\text{or} \quad m \, n \, K_{i\ max} \, (1+n).$$

where $K_{i\ max}$ is the maximum number of machines in any stage.

The maximum number of continuous variables, those variables used to carry funds backward and forward through time, in the problem formulation is given by (6 . n) - 4 where n is the planning horizon of the project and 6 is the number of continuous variables associated with each year in the project life except the first and last. The four is deducted because in year one and year n two variables are not defined. Borrowing and carrying forward are not permitted in the last year. Carrying back and borrowing from earlier periods are not permitted in period one. These years have two less variables in each case. The total number of variables is equal to

$$(1+n) \, n \, m \, (K_{i\ max}) + 6n-4.$$

The maximum number of constraints associated with the problem can be determined as follows:

Budget constraints	n
Constraints for forward equivalence	n-1
Constraints for backward equivalence	n-1

Capacity constraints	Mn
Process relationship constraints	$Mn\,K_{i\ max}$ +
	$Mn\,\frac{(n-1)}{2}\,K_{i\ max}$
Total	$3n-2+Mn(1+K_{i\ max}+\frac{n-1}{2}\,K_{i\ max})$

To determine the number of constraints of the form $(X_{ijklm} - X_{ijklm}=0)$ for values of i, j, k, l and m the following procedure will be used. The relationship ((l=1, . . . , n-1), j=1, . . . , n-1) is of the form of a nested loop. It is desired that the total number of terms included in the nesting be found. The reasoning is when

l=1,

j=1, . . . , n-1

l=2,

j=2, . . . , n-1

.
.
.

l=n-1,

j=n-1

The number of terms in the 1st series is n-1.

The number of terms in the 2nd series is n-2.

. .
. .
. .

The number of terms in the n-1st series is 1.

Hence, the total number of constraints involved due to the indices is

$\frac{n(n-1)}{2}$, which is given by the sum of an arithmetic series. This can be multiplied by $M \cdot K_{i\ max}$ to give the total possible constraints of this type.

The total number of constraints is then given by the sum of the previously listed possibilities. This is

$$3n-2+Mn(1+K_{i\ max}+\frac{(n-1)}{2}K_{i\ max}).$$

In the case where a process has two stages, there is a four year planning horizon, and one machine per stage with no provision for funds being carried forward and backward, the model would be expected to be of the size 40 variables and 32 constraints. This can be verified by examining the first model formulated in Chapter V.

<u>Variables Versus Constraints</u>

It would be desirable to show that

$$3n-2+Mn(1+K_{i\ max}+\frac{(n-1)}{2}K_{i\ max}<(n-1)\ nM\ K_{i\ max}+6n-4$$

in order to know that the number of variables will always exceed the number of constraints. The expression will be simplified as follows:

The expression is:

$$Mn\ K_{i\ max}+Mn^2\ K_{i\ max}+6n-4>3n-2\ Mn+Mn\ K_{i\ max}+Mn\frac{(n-1)}{2}K_{i\ max}$$

which can be rewritten as,

$$Mn^2\ K_{i\ max}+6M-4>3n-2+Mn+Mn\ \frac{(n-1)}{2}K_{i\ max}.$$

Multiply both sides through by 2,

$$2Mn^2\ K_{i\ max}+12n-8>6n-4+2Mn+Mn(n-1)\ K_{i\ max}.$$

Simplifying the above yields,

$$Mn(n-1)\ K_{i\ max} + 2Mn\ K_{i\ max} + 12n - 8 > 6n - 4 + 2Mn.$$

At the lower limit, the values of n, M, and $K_{i\ max}$ would each equal one. It is evident that at this lower limit the condition holds. For any values of n, M, and $K_{i\ max}$ above the lower limits, a sufficient condition for the inequality to hold is that the partial derivatives with respect to n, M, and $K_{i\ max}$ of the left hand side be greater than the same partial derivatives of the right hand side. Since this condition is met, it can be said that the number of variables will always be greater than the number of constraints.

Possibility of Solution

The intent of the investigator at the outset was to formulate a general systems model for replacement which would take into account process relationships, equipment output requirements changing equipment availability and budgeted funds. Methods for solution of this problem, once formulated, would be available in the literature. There are many solution procedures available for small problems. However, this general replacement problem was much too large to be solved by hand with one of these procedures. One efficient algorithm for solution claimed that a fifteen variable integer problem could be solved in four hours by hand. The general replacement model for a six period planning horizon and a five stage process with three machines per stage would have 662 variables and 361 constraints. Six hundred and thirty of these variables would be zero-one while the remaining thirty-two would be continuous.

Hence, solution procedures that exist were not feasible for working a problem of this size. It was concluded that even though this model would give good results is applied in conjunction with accurate data, this was infeasible for other than a small problem.

CHAPTER VII

MODEL REDUCTION

General

In the previous chapter reference was made to a problem with a six period planning horizon and a five stage process with three machines per stage. Each machine was to be considered for replacement in each period of the planning horizon by constantly improving challengers. A problem of this size is one that would cover many situations encountered in replacement applications. When formulated into the general model, however, it involves the solution of a mixed zero-one programming problem with 662 variables and 361 constraints. No literature reference could be found related to mixed zero-one problems of that size.

Imposed Limitations

Consultation (13) and further surveys of the literature on solution algorithms (7, 10, 12) substantiated the belief that the general model for a problem of that size would be unsolvable with currently available algorithms.

Fleischmann (11) experimented with Balas' additive algorithm and found that problems of the size 80 rows by 150 columns of the zero-one type could be solved by a computer in a feasible amount of time. Ghare (13) indicated the feasibility of larger zero-one problems. Lemke and Speilberg (18) discussed the solution of problems of the order 30 constraints and 90 variables in their article related to zero-one and mixed programming.

The investigator concluded that zero-one problems of the size 90 to 100 constraints and 150 to 200 variables are solvable with algorithms currently available in the literature. Mixed zero-one problems can be solved if the number of constraints does not exceed 20 to 30.

This would indicate that solution of the general model for a practical problem with algorithms currently developed is impractical.

Consideration of Linear Programming

The use of linear programming to solve such a problem cannot be deemed altogether infeasible. Weingartner (27) pointed out by way of a proof that the maximum number of fractional projects in his linear programming solution to the integer formulation of a capital budgeting problem was equal to the number of budget constraints. Cabot and Hurter (5) formulate a zero-one program as a linear programming and point out that the linear programming solution to the special problem yields a zero-one solution to the original problem.

Although at first glance linear programming seems infeasible, an approximate solution to the problem might be equally desirable to the exact solution and indeed would be much superior to no solution at all. The linear programming formulation of the general replacement model can be solved with currently available algorithms.

Reduction Possibilities

Since it seemed impossible to solve the problem exactly as formulated, it now became desirable to consider trade-offs which could be made. Possibly some reality included in the problem could be traded-off to obtain a solvable problem.

Referring to Chapter VI, the number of constraints was given as shown below:

Budget constraints	n
Equivalence constraints	
forward	n-1
backward	n-1
Capacity constraints	Mn
Process relationship constraints	$Mn\ K_{i\ max}\ (1+\frac{n-1}{2})$.

The number of variables is given by $Mn\ K_{i\ max}\ (1+n)+6n-4$.

Budget constraint reduction. It seemed infeasible to consider deleting budget constraints since budgets are a primary factor of concern in the modeling procedure.

Equivalence constraint deletion. The equivalence constraints could be deleted without loss of model fidelity. However, one of the benefits to be obtained from this modeling was to be a schedule of funds desired for future projects for budgeting purposes. In reality, funds are normally budgeted year by year with no provisions for carrying funds forward or backward. This would not appreciably reduce the problem size since the reduction would involve only 2n-2 constraints. It would, however, convert the problem to a pure zero-one programming problem. For the reference problem the size would be reduced to 662 variables and 351 constraints. This is roughly a three per cent reduction in the number of constraints.

Capacity constraint deletion. Capacity constraints could be deleted by assuming some constant capacity for each machine over the planning horizon. This results in a reduction of "Mn" constraints. For the example problem, this represents a reduction to a problem of 662 variables and 331 constraints. This was roughly a nine per cent reduction in the original number of constraints.

Process relationship constraint deletion. Dropping the process relationships constraints was infeasible because of the way the problem had been formulated. The objective function depended upon the presence of certain variables in the final solution. If these variables were not forced to appear in the final solution, a meaningless solution would occur at a zero total cost.

Assumption of a single interest rate. Further study revealed that if a single interest rate could be assumed, a large scale reduction in problem size would occur. The equivalence relationships required the use of two interest rates. One of these was related to carrying funds forward and the other to bringing funds backward through time. This concept was related to investing or borrowing of funds. Invested funds normally return a higher rate of interest than is charged on borrowed funds.

A conservative assumption was made at this point. If the rate-of-return on invested funds was assumed equal to the cost of borrowed money, an equivalence relationship could be used for moving funds forward and backward through time. With this change, the problem was formulated as a single period (n=1) problem with all dollar values in terms of present dollars. The effect of this change on problem size was significant. The number of constraints and the number of variables of the reduced problem are as shown below:

<u>Constraints</u>			
Budget constraints	n	1	(1)
Equivalence constraints			
forward	$n-1$	0	(0)
backward	$n-1$	0	(0)
Capacity constraints	Mn	5	(30)
Process relationship constraints	$Mn\,K_{i\,max}\left(\frac{n-1}{2}\right)$	15	(15)
		21	(46)
<u>Variables</u>			
$Mn\,K_{i\,max}\,(1+n^2)$		30	(105)

The numbers in parentheses represent the value for the number of constraints or variables obtained from the actual problem. The difference in values will be explained in subsequent paragraphs. The example problem was reduced from a mixed zero-one problem with 662 variables and 361 constraints to a zero-one problem with 105 variables and 46 constraints. This problem is of a size amenable to solution by algorithms currently available in the literature.

To understand the apparent difference in what is obtained by substituting n=1 into the expression for the number of constraints or the number of variables and the value used in the reduced problem analysis requires some study.

Only one budget constraint is needed because all required funds will be represented by their present values. Equivalence constraints are dropped from the problem since it is to be formulated as a single period problem. Capacity constraints do not benefit from the reduction necessarily. The reason for this is that the equivalence reduction which occurs in the monetary constraints does not apply in capacity constraints. This could be reduced only if capacity were assumed constant for each machine in each period. This should indicate to the observer that the term "n" must have two different contexts in the formulas; one as the number of periods in the capital horizon and the second as the number of periods in the formulation or planning horizon. These will be designated as n_c and N, respectively. These can be identical or the capital horizon can be reduced to a single period problem as in this analysis. In the expression for the number of variables the first "n" represents the capital horizon, n_c, and the second "n", N, the problem horizon.

When the capital horizon is one, a zero-one variable can be used to represent an alternative in the formulation. In the old formulation a zero-one variable represented a machine in a period.

General expressions for the number of variables and constraints in the new problem were as follows:

$C = 1 + MN + M\ K_{i\ max}$,

$V = M\ K_{i\ max}\ (N+1)$,

where C represents the maximum number of constraints, V represents the maximum number of variables, N represents the planning horizon, M represents the number of stages in the process being modeled, and $K_{i\ max}$ represents the maximum number of machines per stage.

Observation should indicate that if the stages had varying numbers of machines or if some machines were not considered for replacement in every period, the problem would become somewhat smaller.

Dimensions of Solvable Problems

Assuming that problems of the size 80 to 100 constraints and 150-200 variables are solvable this could allow for many possible variations in problem size. Table III contains many combinations of M, N and $K_{i\ max}$ and indicates the corresponding maximum problem size in terms of the number of constraints, C, and the number of variables, V.

Formulation of Example Problem

An example of the zero-one formulation has been constructed for the following problem statement.

Problem statement. A company wishes to determine an optimum equipment replacement policy over a three year planning horizon. Management realizes that this policy is optimum only with respect to

TABLE III

DIMENSIONS OF MAXIMUM PROBLEM SIZE FOR VARIOUS VALUES OF N, M AND K_{max}

N	$K_{max} = 3$						$K_{max} = 4$					
	M	C	V	M	C	V	M	C	V	M	C	V
	2	13	24	7	43	84	2	15	32	7	50	112
	3	19	36	8	49	96	3	22	48	8	57	128
3	4	25	48	9	55	108	4	29	64	9	64	144
	5	31	60	10	61	120	5	36	80	10	71	160
	6	37	72	11	67	132	6	43	96	11	78	176
	2	15	30	7	50	105	2	17	40	7	57	140
	3	22	45	8	57	120	3	25	60	8	65	160
4	4	29	60	9	64	135	4	33	80	9	73	180
	5	36	75	10	71	150	5	41	100	10	81	200
	6	43	90	11	78	165	6	49	120	11	89	220
	2	17	36	7	57	126	2	19	48	7	64	168
	3	25	54	8	65	144	3	28	72	8	73	192
5	4	33	72	9	73	162	4	37	96	9	82	216
	5	41	90	10	81	190	5	46	120	10	91	240
	6	49	108	11	89	208	6	55	144	11	100	264
	2	19	42	7	64	147	2	22	56	7	72	196
	3	28	63	8	73	168	3	32	84	8	82	224
6	4	37	84	9	82	189	4	42	112	9	92	252
	5	46	105	10	91	210	5	52	140	10	102	280
	6	55	126	11	100	231	6	62	168	11	112	308

the data input and further realizes that the extent to which it is accurate will depend upon how well outcomes follow their forecasted behavior.

The company has a production process which is composed of three stages with three machines in parallel in each stage. Each machine will be considered for replacement in each period of the planning horizon until it is replaced. Then, no further consideration for replacement will be made during the planning horizon.

It is assumed that one superior challenger has been found for each defender. When, and if, the defender is replaced, it will be with the challenger. This challenger is "state-of-the-art" and is expected to remain so throughout the horizon.

Salvage values are assumed to be represented accurately by book values. The companies effective tax rate is assumed to be fifty per cent. For purposes of this illustration the dollar values given in the tables are assumed to be present worths. The capital and operating budget is assumed to be twelve thousand dollars for the period and the required output is 30, 34 and 38 units in the respective periods. Other data related to the problem is given in Tables IV through IX.

<u>Representation of Alternatives.</u> In this formulation X_{ijk} will represent selection or rejection of alternative k for the jth machine in the ith stage of the process.

TABLE IV

FORECASTED SALVAGE VALUES FOR EXISTING MACHINES

	Stage #								
	1			2			3		
	Machine #			Machine #			Machine #		
Time	1	2	3	1	2	3	1	2	3
0	400	500	600	800	800	700	700	500	400
1	300	300	450	600	700	600	600	400	300
2	200	200	300	400	600	500	500	300	200
3	100	100	150	200	500	400	400	200	100

TABLE V

FORECASTED SALVAGE VALUES FOR CHALLENGERS

	Stage # 1			Stage # 2			Stage # 3		
	Machine #			Machine #			Machine #		
Age	1	2	3	1	2	3	1	2	3
0	1200	1200	1200	1400	1400	1400	1600	1600	1600
1	1000	1000	1000	1200	1200	1200	1200	1200	1200
2	800	800	800	1000	1000	1000	1000	1000	1000
3	600	600	600	800	800	800	800	800	800

TABLE VI

FORECASTED CAPACITY FOR EXISTING MACHINES

	Stage # 1			Stage # 2			Stage # 3		
	Machine #			Machine #			Machine #		
Year	1	2	3	1	2	3	1	2	3
1	10	10	10	5	10	15	12	8	10
2	10	10	9	4	9	12	12	8	10
3	9	10	9	3	9	12	11	8	10

TABLE VII

FORECASTED CAPACITY FOR CHALLENGERS

	Stage #								
	1			2			3		
Year	Machine #			Machine #			Machine #		
of Use	1	2	3	1	2	3	1	2	3
1	15	15	15	15	20	20	14	10	15
2	15	15	15	15	20	20	14	10	15
3	15	15	15	15	20	20	14	10	15

TABLE VIII

FORECASTED OPERATING COSTS FOR EXISTING MACHINES

	Stage #								
	1			2			3		
	Machine #			Machine #			Machine #		
Year	1	2	3	1	2	3	1	2	3
1	190	210	200	100	110	100	150	180	150
2	210	210	200	120	120	110	160	180	180
3	230	250	230	140	120	120	160	180	200

TABLE IX

FORECASTED OPERATING COSTS FOR CHALLENGERS

	Stage # 1			Stage # 2			Stage # 3		
Year of Use	Machine # 1	2	3	Machine # 1	2	3	Machine # 1	2	3
1	100	100	110	80	80	60	120	120	130
2	100	100	120	80	80	60	120	140	130
3	110	120	120	80	90	60	120	140	140

Each machine in the process will present four alternatives. These will be (1) to replace immediately, (2) to replace at the end of the first year, (3) to replace at the end of the second year, and (4) not to replace.

Model Formulation

As described previously, this model was formulated by alternatives. Each zero-one variable represents an alternative for selection or rejection. Terms used in the formulation are as follows:

X_{ijk} = Alternative k for the jth machine in the ith stage of the process.

C_{ijk} = Present worth cost associated with X_{ijk}.

$\overline{k}_{ijl}$ = Capacity of machine j of the ith stage during period l.

K_l = Required stage capacity during period l.

D_{ijk} = Cash flow consumption of alternative X_{ijk}.

The objective function is then to minimize

$$\sum_{i=1}^{M} \sum_{j=1}^{K_i} \sum_{k=1}^{N+1} C_{ijk} X_{ijk},$$

where K_i is the number of machines in stage i, M is the number of stages and N is the planning horizon of the problem. Other terms are as defined in the previous paragraph.

The budget constraint is as follows:

$$\sum_{i=1}^{M} \sum_{j=1}^{K_i} \sum_{k=1}^{N+1} D_{ijk} X_{ijk}$$

where all terms are as previously defined. This is very similar to the objective function. D_{ijk}, however, represents cash flow rather than cost.

Capacity constraints are formulated in the following manner:

$$\sum_{j=1}^{k_i} \left(\sum_{k=1}^{l} \bar{k}_{ijl} X_{ijk} + \sum_{k=l+1}^{N+1} \bar{k}_{ijl} X_{ijk} \right) \geq k_l, \quad \text{for } i=1, \ldots, M \text{ and } l=1, \ldots, N.$$

Process relationship constraints require that only one alternative can be chosen from the group of N+1 available for each machine. This is accomplished as follows:

$$\sum_{k=1}^{N+1} X_{ijk} = 1, \quad \text{for } i=1, \ldots, M \text{ and } j=1, \ldots, K_i.$$

In addition, it is required that all variables be zero-one.

$$X_{ijk} = \begin{cases} 0 \\ 1 \end{cases}, \quad \text{for all } i, j, k.$$

This model was used to formulate the example which follows this section.

Problem Formulation

The problem formulation is as follows:

minimize the objective function

$$\begin{aligned} Z = {} & 455\,X_{111} + 445\,X_{112} + 450\,X_{113} + 465\,X_{114} \\ & + 460\,X_{121} + 505\,X_{122} + 510\,X_{123} + 535\,X_{124} \\ & + 475\,X_{131} + 490\,X_{132} + 505\,X_{133} + 540\,X_{134} \\ & + 420\,X_{211} + 430\,X_{212} + 450\,X_{213} + 480\,X_{214} \end{aligned}$$

$$+425\ X_{221}+335\ X_{222}+355\ X_{223}+325\ X_{224}$$
$$+390\ X_{231}+360\ X_{232}+335\ X_{233}+315\ X_{234}$$
$$+580\ X_{311}+545\ X_{312}+265\ X_{313}+335\ X_{314}$$
$$+600\ X_{321}+570\ X_{322}+540\ X_{323}+420\ X_{324}$$
$$+600\ X_{331}+555\ X_{332}+530\ X_{333}+415\ X_{334},$$

subject to several different constraints. The first is the cash flow constraint which is

$$1510\ X_{111}+1690\ X_{112}+1900\ X_{113}+1030\ X_{114}$$
$$+1520\ X_{121}+1810\ X_{122}+2020\ X_{123}+1170\ X_{124}$$
$$+1550\ X_{131}+1780\ X_{132}+2010\ X_{133}+1230\ X_{134}$$
$$+1640\ X_{211}+1860\ X_{212}+2100\ X_{213}+1160\ X_{214}$$
$$+1650\ X_{221}+1770\ X_{222}+1910\ X_{223}+1150\ X_{224}$$
$$+1580\ X_{231}+1720\ X_{232}+1870\ X_{233}+1030\ X_{234}$$
$$+1960\ X_{311}+2090\ X_{312}+2230\ X_{313}+1170\ X_{314}$$
$$+2000\ X_{321}+2140\ X_{322}+2280\ X_{323}+1040\ X_{324}$$
$$+2000\ X_{331}+2110\ X_{332}+2260\ X_{333}+930\ X_{334}$$
$$\leq 17,400.$$

Secondly, for each stage and period the required capacity must be available. Constraints by stage and period are as follows:

stage one, period one,

$$15\ X_{111}+10\ (X_{112}+X_{113}+X_{114})$$

$$+\ 15\ X_{121}+10\ (X_{122}+X_{123}+X_{124})$$

$$+15\ X_{131}+10\ (X_{132}+X_{133}+X_{134})$$

$$\geq 30,$$

stage one, period two,

$$15\ (X_{111}+X_{112})+10\ (X_{113}+X_{114})$$

$$+\ 15\ (X_{121}+X_{122})+10\ (X_{123}+X_{124})$$

$$+\ 15\ (X_{131}+X_{132})+10\ (X_{133}+X_{134})$$

$$\geq 34,$$

stage one, period three,

$$15\ (X_{111}+X_{112}+X_{113})+10\ X_{114}$$

$$+15\ (X_{121}+X_{122}+X_{123})+10\ X_{124}$$

$$+15\ (X_{131}+X_{132}+X_{133})+10\ X_{134}$$

$$\geq 38,$$

stage two, period one,

$$15\ X_{211}+5\ (X_{212}+X_{213}+X_{214})$$

$$+\ 20\ X_{221}+10\ (X_{222}+X_{223}+X_{224})$$

$$+\ 20\ X_{231}+15\ (X_{232}+X_{233}+X_{234})$$

$$\geq 30,$$

stage two, period two,

$$15\ (X_{211}+X_{222})+10\ (X_{223}+X_{224})$$

$+ 20 (X_{221}+X_{222})+10 (X_{223}+X_{224})$

$+ 20 (X_{231}+X_{232})+15 (X_{233}+X_{234})$

$\geq 34,$

stage two, period three,

$15 (X_{211}+X_{212}+X_{213})+5 X_{214}$

$+ 20 (X_{221}+X_{222}+X_{223})+10 X_{224}$

$+20 (X_{231}+X_{232}+X_{233})+15 X_{234}$

$\geq 38,$

stage three, period one,

$14 X_{311}+12 (X_{312}+X_{313}+X_{314})$

$+ 10 X_{321}+8 (X_{322}+X_{323}+X_{324})$

$+ 15 X_{331}+10 (X_{332}+X_{333}+X_{334})$

$\geq 30,$

stage three, period two,

$14 (X_{311}+X_{312})+12 (X_{313}+X_{314})$

$+10 (X_{321}+X_{322})+8 (X_{323}+X_{324})$

$+15 (X_{331}+X_{332})+10 (X_{333}+X_{334})$

$\geq 34,$

and finally stage three, period three,

$14 (X_{311}+X_{312}+X_{313})+12 X_{314}$

$+10 (X_{321}+X_{322}+X_{323})+8 X_{324}$

$$+15\,(X_{331}+X_{332}+X_{333})+10\,X_{334}$$

$$\geq 38.$$

Process relationship constraints require that only one alternative be selected for each machine. By individual machines and stage, these are as follows:

$$X_{111}+X_{112}+X_{113}+X_{114}=1,$$

$$X_{121}+X_{122}+X_{123}+X_{124}=1,$$

$$X_{131}+X_{132}+X_{133}+X_{134}=1,$$

$$X_{211}+X_{212}+X_{213}+X_{214}=1,$$

$$X_{221}+X_{222}+X_{223}+X_{224}=1,$$

$$X_{231}+X_{232}+X_{233}+X_{234}=1,$$

$$X_{311}+X_{312}+X_{313}+X_{314}=1,$$

$$X_{321}+X_{322}+X_{323}+X_{324}=1,$$

$$X_{331}+X_{332}+X_{333}+X_{334}=1.$$

In addition, it is required that each of the variables be zero-one.

$$X_{ijk}=\begin{cases}0\\1\end{cases}\quad \text{for all i, j, k.}$$

Cost Coefficients of Objective Function

The cost coefficients of the objective function were obtained in the manner shown in the Appendix. In this case, all dollar values given in Tables IV, V, VIII, and IX were assumed to be present values in order to expedite preparation.

Capacity Constraints

Capacity constraints were formulated by alternatives. Each constraint is related to a single period and stage of the process. An alternative consists of either one or two machines. Variable X_{222} represents the second alternative for machine two in the second stage of the process. This alternative was that of replacing the old machine at the end of the first period. Hence, when a first period constraint is formulated for the second stage of the process the coefficient of X_{222} must be equal to the production capacity of the old machine. In the second or third periods, the coefficient of X_{222} would be equal to the production capacity of the new machine. Similarly, the process is followed for all stages and periods.

To formulate the capacity constraint the possibilities for production capacity in each stage for each period are arrayed. There are three machines per stage and four alternatives per machine, hence, twelve variables per constraint. The proper capacity coefficient is assigned to the alternatives and periods. This is illustrated below for the first stage and first period:

X_{111}	X_{112}	X_{113}	X_{114}
X_{121}	X_{122}	X_{123}	X_{124}
X_{131}	X_{132}	X_{133}	X_{134}

In the first period X_{111}, X_{121} and X_{131} would produce 15 units. In the first period X_{112}, X_{113}, X_{114}, X_{122}, X_{123}, X_{124}, X_{132}, X_{133}

and X_{134} produce 10 units each. The total capacity of the stage in this period must exceed 30 units.

Process Relationship Constraints

These constraints require that only one alternative be selected for each machine during the planning horizon. An alternative consists of a specific defender-challenger time relationship during the planning horizon. For example; replace immediately, at time one, . . . do not replace, are examples of alternatives.

Budget Constraint

This constraint is more difficult to understand. The total resources available during the period are equal to the present machines plus the capital and operating budget. This may be summed up by adding the total value of present machines to the capital and operating budget. Each alternative for implementation will consume a certain amount of cash flow. The coefficient of X_{ijk} in the budget constraint is equal to the cash flow consumed by the alternative, plus the residual remaining at the end of the planning horizon in the form of a salvage value.

The objective function and restrictions together with the zero-one requirement for the variables constitute the problem formulation.

CHAPTER VIII

RECOMMENDATIONS AND CONCLUSIONS

General

At the outset of this writing an idea was proposed to develop a systems model for replacement.

Chapter I introduced the replacement concept along with ideas which have been set forth by many knowledgeable writers in the area. It discussed the major areas of replacement theory and pointed out the general topics to be covered in this investigation.

Chapter II introduced the reader to the replacement literature. It contained a survey of many models for replacement situations. It also introduced the idea of a systems approach to the replacement problem.

Chapter III illustrated some models of capital budgeting and their treatment as mathematical programming problems.

Chapter IV developed general process and problem relationships and formulated these relationships as constraints for mathematical programming problems.

Chapter V was devoted to the development of two special case problems and a general model for systems replacement.

Chapter VI was an analysis of the general model formulated in Chapter V.

Chapter VII presented a slightly modified general systems replacement model. It included a formulation of the model of a production process as a zero-one programming problem. Also included was an example problem with an analysis of problem size.

It is believed by the investigator that the objectives of this research, as originally outlined, have been accomplished.

Recommendations

It is recommended that further study of this problem area be directed as follows:

1. The development of a sufficiently large horizon model for optimality under the conditions of an infinite planning horizon. Possibly conditions can be derived for an adequate planning horizon which would give optimal conditions for the infinite horizon model.
2. The development of additonal models or constraints which would be applicable in this replacement area.
3. Formulation of a profit maximization model for a situation similar to that of the models in Chapter V.
4. Formulation of a model for alternatives rather than a model for machines such as those covered in this writing.

5. Study of a method for including secondary process effects in the formulation by alternatives.

Conclusions

It is believed by the investigator that the idea of the systems approach to the replacement problem is sound. It is also believed that this is a correct approach to the problem. It has been stated conclusively that this model can be reduced to show that most of the single machine replacement models are special cases of the general model presented in Chapter V.

In most machine replacement situations the decision maker is confronted with a variety of opportunities for investment of his capital. It is very difficult, if not impossible, for a decision maker to consider all possible process relationships in allocating capital. Hence, any procedure for replacement used should take into account machine and process relationships, forecasted demand for future output, machine capacity, salvage and replacement costs. The procedural model developed in Chapter V takes these points into account. The fact that the problem size expands rapidly is only incidental. With the advent of better procedures for solution of large scale programs and with the development of larger computers the economic feasibility for solutions of this model may soon be achieved. Until that time, at which sensitivity analysis can be applied to the general model, better results should be obtainable with the reduced model as shown in Chapter VII.

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APPENDIX

DETERMINING CONSTANTS FOR USE IN OBJECTIVE FUNCTION

APPENDIX

This appendix illustrates the approach used to determine constants used as the coefficients of the variables appearing in the objective function. The approach was to determine the cost of the alternative represented by the variable.

For each alternative a determination of the after tax cash flows of costs associated with the implementation of the alternative was made. Next, the present worth discounted at the firm's minimum attractive rate of return was determined. This present worth was used as the coefficient of the variable representing the alternative.

Table X gives data which is analyzed for an example. The cash flow analysis is given in Table XI.

Discounting of the after tax cash flow given in Table XI at an interest rate of 10 per cent was accomplished in the following manner:

$$\begin{aligned} DCF = & -1000 - 400 \left(\begin{matrix} PF\text{-}10\%\text{-}1 \\ .90909 \end{matrix} \right) \\ & -10,900 \left(\begin{matrix} PF\text{-}10\%\text{-}2 \\ .82645 \end{matrix} \right) \\ & + 300 \left(\begin{matrix} PA\text{-}10\%\text{-}3 \\ 2.4869 \end{matrix} \right) \left(\begin{matrix} PF\text{-}10\%\text{-}2 \\ .82645 \end{matrix} \right) \\ & + 7000 \left(\begin{matrix} PF\text{-}10\%\text{-}5 \\ .62092 \end{matrix} \right) \end{aligned}$$

$$DCF = -5408.89$$

where DCF was the discounted cash flow for the analysis.

TABLE X

DATA FOR DETERMINATION OF COST COEFFICIENT FOR VARIABLE REPRESENTING ALTERNATIVE

	Current Machine	Challenger in Second Year
Realizable Value at Time 2	1,000	10,000
Book Value at Time 2	2,000	10,000
Method of Depreciation	straight line	straight line
Annual Operating Cost	1,000	400
Yearly Depreciation Cost	200	1,000
Planning Horizon	5 years	
Estimated Salvage at Time 5		7,000
Assumed Tax Rate	50%	50%
Life of Machine	5	10

TABLE XI

CASH FLOW ANALYSIS FOR DETERMINATION OF COST COEFFICIENT FOR VARIABLE IN THE OBJECTIVE FUNCTION

Time	Cash Flow	Deprecia- tion	Operating Expense	Taxable Income	Income Tax	Cash Flow After Tax
0	- 1,000	-	-	-	-	- 1,000
1	-	- 200	-1,000	-1,200	+600	- 400
2	-	- 200	-1,000	-1,200	+600	- 400
2	-	-1,000*	-	-1,000	+500	- 500
2	-10,000	-	-	-	-	-10,000
3	-	-1,000	- 400	-1,400	+700	+ 300
4	-	-1,000	- 400	-1,400	+700	+ 300
5	-	-1,000	- 400	-1,400	+700	+ 300
5	+ 7,000	-	-	-	-	+ 7,000

* Loss on disposal of asset.

The value obtained would represent the present worth of the after tax cost of the alternative replace machine "X" at the end of year two with the challenger.

Similarly, other alternatives could be priced to make up the cost coefficient of variables appearing in the objective function.

VITA

Thomas Gerald Ray, son of the late Mr. and Mrs. William L. Ray, was born on June 11, 1940, in Bennington, Oklahoma. He graduated from Friendswood High School, Friendswood, Texas, in May 1958.

He received the Bachelor of Science degree in Industrial Engineering from Mississippi State University in June 1966, after completing undergraduate study which was interrupted by a three year tour of duty in the United States Army.

In May 1966, he married Andre Carolyn Whitaker. During the summer of 1966, he was employed by Owens-Corning Fiberglas. In September 1966, he returned to Mississippi State University and completed the requirements there for the degree of Master of Science in Engineering in June 1968. During that time he served one year as a graduate research assistant and one academic year as an instructor. During the summer of 1968, he was employed by the National Aeronautics and Space Administration and in September 1968, began studies for the Ph. D. degree at Virginia Polytechnic Institute. During his stay at Virginia Polytechnic Institute he served the Department of Industrial Engineering and Operations Research as a graduate teaching assistant. He was absent the summer of 1969 at which time he was again employed by the National Aeronautics and Space Administration.

In September 1970, he began a job as Assistant Professor in the Mechanical, Aerospace and Industrial Engineering Department of Louisiana State University at Baton Rouge, Louisiana.

He completed the requirements for a Doctor of Philosophy Degree in Industrial Engineering and Operations Research in August 1971.

He is a member of Phi Eta Sigma, Tau Beta Pi, Phi Kappa Phi and the American Society for Engineering Education.

He is the father of two wonderful children, Tommy, age 3, and Cathy, age 2.

ABSTRACT

A SYSTEMS APPROACH TO REPLACEMENT

by

Thomas Gerald Ray

The object of this study is a new approach to the problem of replacement of assets which deteriorate and become obsolete with time. The investigator views replacement theory as a subset of capital budgeting. Capital budgeting has received much more attention, has benefited from many advances in mathematical programming techniques, and in general has been advanced to a much more sophisticated state of the art than has replacement theory.

The approach taken in this work is to point out this divergence in advances in these areas by surveying the literature in each. Next a new approach to the replacement problem is presented. This approach is new in that it attacks the replacement problem as a system of interacting components rather than take the normal replacement approach to a single machine. The production process is modeled as a network in which each machine is represented by an arc. A single machine or two or more machines in parallel compose a stage of the process. Several stages are combined to complete the network.

The general model is formulated as a mixed zero-one programming problem for a finite planning horizon. This model can be further modified by adding specialized constraints to make it fit more

specific cases. The general model has provisions for equivalence relationships to carry funds forward or backward through time. It also takes into account such items as process requirements and machine capacities.

Difficulties are encountered in that a normal problem is too large to solve. Further study reveals that by making a conservative assumption of using a single interest rate the problem can be reduced to a much smaller zero-one programming problem. This formulation for a reasonably sized production process is small enough to be solved by zero-one algorithms available in the literature.

An example problem is formulated for illustrative purposes.