ELECTROSTATIC OSCILLATIONS IN INHOMOGENEOUS PLASMAS

By

Leo Douglas Staton

Thesis submitted to the Graduate Faculty of the

Virginia Polytechnic Institute

in candidacy for the degree of

DOCTOR OF PHILOSOPHY

in

Physics

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It is done!

Clang of bell and roar of gun
Send the tidings up and down.
How the belfries rock and reel!
How the great guns, peal on peal,
Fling the joy from town to town!

\- John Greenleaf Whittier

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APPROVED:

Chairman, Dr. C.D. Williams

Dr. J.A. Jacobs

Dr. T.E. Gilmer, Jr.

Dr. R.L. Bowden, Jr.

Dr. S.T. Gormsen

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V. INTRODUCTION

5.1 Background

Interest in the theory of gaseous plasmas has increased sharply in recent years, primarily because of the impetus of practical problems in such fields as controlled thermonuclear fusion and advanced rocket propulsion systems, but also because of intrinsic interest in the plasma state and its applications in astrophysics and geophysics. In many of these applications the plasma may be considered to be sufficiently tenuous and/or sufficiently hot that collisions between individual particles may be neglected in analyzing its behavior. Thus the Vlasov-Maxwell equations, which describe the behavior of a collisionless gas of charged particles, have become the basis for a significant portion of the papers written in this field.

In order to cope with the extreme complexity of these equations it is necessary to make approximations which, it is hoped, will not sacrifice the central aim of the calculations. For problems with complicated geometries the general practice has been to calculate the first few velocity moments of the Vlasov equation and to postulate an equation of state to allow truncation of the set of moment equations at the third moment. The resulting hydrodynamic equations describe the plasma as a conducting fluid and do successfully provide pessimistic stability criteria for schemes for confining plasmas with magnetic fields (ref. 1) as well as predicting a wide class of low frequency hydromagnetic wave phenomena (ref. 2). The Vlasov equation however contains a great deal of physically interesting information which is not expressible in terms of a small number of moments, the most widely discussed example of which is the phenomenon of Landau damping (ref. 3). In an attempt to retain the possibility of describing this and other such phenomena requiring this wider information, many workers have followed the original lead of Landau in treating sufficiently simple geometries that the Vlasov-Maxwell equations may be treated in a more general way than that allowed in the hydrodynamic approximation.

Thus one finds that a very large number of papers have been written describing small amplitude electrostatic oscillations in homogeneous plasmas, which is the problem originally treated by Landau. Much analytical progress has been made in examining this problem, primarily because of its one-dimensional character, which in turn is due to the fact that such oscillations do not interact with an externally applied magnetic field if that field is directed along the gradient of the small oscillating electrostatic potential. It is true however that most laboratory and other applications of plasmas necessitate that the plasma be spatially inhomogeneous. In recent years then more attention has been turned to investigating inhomogeneous plasmas.

Some analytical progress has been made for three-dimensional inhomogeneities for the case where the wavelength of the small oscillations is small compared to the characteristic length of the inhomogeneity (ref. 4) and for the case of the cold plasma (ref. 5).

Rosenbluth, Krall, and Rostoker (ref. 6) and Krall and Rosenbluth (ref. 7) have treated oscillations both parallel and transverse to the direction of the inhomogeneity for frequencies small compared with the ion Larmor frequency.

For electrostatic oscillations, whose frequencies are of the order of the electron plasma frequency, directed along the plasma inhonogeneity, the situation is much more difficult. Analytical progress has been made in the limit of short wavelengths (compared with the plasma inhomogeneity) by Berk, Rosenbluth, and Sudan (ref. 8) and in the long wavelength limit by Jackson and Raether (ref. 9), but in all other cases resort has had to be made to numerical computations. Even so the range of problems which have been treated in this area is very limited. Integral equations for small electric field oscillations of this type have been developed by several workers. Pavkovitch (ref. 10) developed such an integral equation to compute numerically the response to an external radio frequency source of a plasma inhomogeneous to the extent that the static potential in the plasma consisted of a parabolic sheath adjoined to a homogeneous plasma. Leavens (ref. 11) also computed the response of such a parabolic sheath to an external source, and more recently (ref. 12) he has sought the complex normal mode oscillation frequencies (eigenfrequencies) of such a system by making use of the resonance curve resulting from driving the plasma over a range of externally imposed (real) frequencies. Harker (ref. 13) has treated the eigenfrequencies of a plasma with a parabolic static potential as has Watson (ref. 14). Both find that this special inhomogeneous potential will not allow Landau damping of the small potential oscillations superimposed upon it; i.e., the eigenfrequencies of the small oscillations are real. Detailed treatments for cases other than these parabolic cases are conspicuously lacking in the literature.

Very little attention has been paid in the literature to the question of whether the given inhomogeneous state, upon which small oscillations are to be allowed, is in fact a physically accessible state of the plasma. That is, if the plasma is initially prepared in some state other than the given inhomogeneous state, will the Vlasov-Maxwell equations allow it to pass to the latter state? In general, of course, this question must remain unanswered, since analytic, nonlinearized solutions of the Vlasov-Maxwell equations are unavailable. Armstrong (ref. 15), however, by numerically following the nonlinearized time development of a single spatial Fourier component of a small electric field perturbation in an initially homogeneous plasma, has found as a final result what appears to be an inhomogeneous stationary state.

5.2 Present Problem

In Chapter VI the Vlasov-Maxwell equations are introduced and reduced to the one-dimensional form appropriate for treating small electrostatic plasma oscillations in the same direction as the basic plasma inhomogeneity. A special solution, the rectangular energy distribution function, of the equations describing the basic inhomogeneous plasma is singled out as having unique properties as regards its accessibility from a more general state. A theorem is developed which shows that the stationary state corresponding to this distribution function corresponds to a minimum of the total system energy and that therefore such a distribution function is generally inaccessible from other states. Next an integral equation similar to those in the literature is developed in order to describe small electric field oscillations in such inhomogeneous plasmas. This equation is then applied to the special case of the rectangular energy distribution function and it is shown that for this special case the integral equation degenerates to a second-order differential equation having real eigenfrequencies.

The main effort of the present work is described in Chapters IX and X. A method is developed for calculating numerically the complex eigenfrequencies of oscillations of arbitrary wavelength in inhomogeneous one-dimensional plasmas. The method, based upon the development of the spatial dependence of the solutions in a Fourier series and the velocity dependence in a series of Hermite polynomials, contains the eigenfrequency calculations of Grant (ref. 16) as a special case. A small collision term is added to the equations in order to facilitate the calculation of the Landau (collisionless) damping. The result is a calculation for the first time of the effects of plasma inhomogeneities on the eigenfrequencies of plasma oscillations having fundamental wavelengths of the same order as the characteristic dimensions of the inhomogeneity. This is an important new contribution inasmuch as no analytical or numerical predictions in this important wavelength range have been given heretofore.

Another phase of the present work using the above method is the calculation for the first time of the effects of plasma inhomogeneity on the growth rates of the two-stream instability in hot plasmas. Also, the effect of adding a small collision term upon the growth rates of the two-stream instability in a homogeneous plasma is computed and helps to clarify some apparent contradictions on this point which have appeared in the literature.

VI. FUNDAMENTAL EQUATIONS

A central goal of the kinetic theory of plasmas has been to justify the existence of an equation of the form

$$\frac{\partial f_i}{\partial t} + \vec{v} \cdot \frac{\partial f_i}{\partial \vec{x}} + \frac{\vec{F}_i}{m_i} \cdot \frac{\partial f_i}{\partial \vec{v}} = \left(\frac{\partial f_i}{\partial t}\right)_c \qquad (6\text{-}1)$$

where $f_i = f_i(\vec{x},\vec{v},t)$ is the distribution function defining the number of particles of species i in the volume element $d\vec{x}\, d\vec{v}$ in phase space at time t, $\vec{F}_i = \vec{F}_i(\vec{x},t)$ is the force on a given particle of type i due to externally applied or other macroscopic fields, and $\left(\frac{\partial f}{\partial t}\right)_c$ is the rate of change of the distribution function due to collisions between particles. It has been shown by Rostoker and Rosenbluth (ref. 17) and in a much more elegant and convincing way by Balescu (ref. 18) that, for a plasma for which the quantity $\left(\frac{4\pi e^2}{kT}\right)^{3/2} n^{1/2} = \frac{1}{n\lambda_D^3} \equiv g$ (where n is the number density of particles, e is the charge on a particle, and kT is the absolute temperature in energy units) is small compared to unity, there is indeed an equation of the form (6-1) and that the right side is of order g compared with the other terms. The parameter λ_D is the Debye length, a measure of the range of two particle correlations in the plasma. For many cases of practical interest (e.g., the plasma in a controlled thermonuclear fusion device) g is very small indeed so that the right hand side of equation (6-1) may be set to zero. If the force $\vec{F}_i$ is the electromagnetic force then one must introduce Maxwell's equations in addition to (6-1) so that the equations describing the collisionless plasma are

$$\frac{\partial f_i}{\partial t} + \vec{v} \cdot \frac{\partial f_i}{\partial \vec{x}} + \frac{q_i}{m_i}\left(\vec{E} + \frac{1}{c}\vec{v} \times \vec{B}\right) \cdot \frac{\partial f_i}{\partial \vec{v}} = 0 \tag{6-2a}$$

$$\nabla \times \vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t} \tag{6-2b}$$

$$\nabla \cdot \vec{E} = 4\pi\rho \tag{6-2c}$$

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \tag{6-2d}$$

$$\nabla \cdot \vec{B} = 0 \tag{6-2e}$$

Equation (6-2a) is the Vlasov equation (or collisionless Boltzmann equation) for the distribution function $f_i(\vec{x},\vec{v},t)$ of species i. $\vec{E}$ and $\vec{B}$ are the macroscopic electric and magnetic fields existing in the plasma, and q_i and m_i are respectively the charge and mass of a particle of species i. Equations (6-2b)-(6-2e) are the Maxwell equations for the electromagnetic fields in the plasma. The current and charge densities $\vec{J}(\vec{x},t)$ and $\rho(\vec{x},t)$ are related to the distribution functions of the various species making up the plasma through the equations

$$\vec{J}(\vec{x},t) = \sum_{j=1}^{N} q_j \iiint \vec{v}\, f_j(\vec{x},\vec{v},t)\, d\vec{v} \qquad (6\text{-}3a)$$

$$\rho(\vec{x},t) = \sum_{j=1}^{N} q_j \iiint f_j(\vec{x},\vec{v},t)\, d\vec{v} \qquad (6\text{-}3b)$$

where N is the number of different species present.

Equations (6-2) and (6-3) then, along with appropriate initial and boundary conditions, form a closed set of equations for the f_i. Knowledge of the f_i, of course, allows computation of macroscopic quantities of interest since these quantities are related to the moments of the f_i. Unfortunately, these equations are completely intractable as they stand, and drastic simplifications have had to be made by every worker in the field in order to make even slight progress. In this work the simplifications are to be directed toward the study of small amplitude electrostatic oscillations in which the charge density gradients exist in the same direction as and are superimposed upon a static charge density distribution.

To this end equations (6-2) and (6-3) are for the present simplified in the following way. First, it is assumed that the plasma has a single species of particles, consisting of electrons immersed in a uniform positively charged background of constant charge density $n_o e$, where e is the magnitude of the electronic charge. Second, the distribution function of the electrons, $f(\vec{x},\vec{v},t)$, is taken as depending on position in the x direction only and on the x component of velocity only. Last, any magnetic fields due to currents in the plasma are ignored and any external magnetic field present must be homogeneous and directed along the density gradient. The pertinent equations now become

$$\frac{\partial f}{\partial t} + v\,\frac{\partial f}{\partial x} - \frac{e}{m}\,E(x,t)\,\frac{\partial f}{\partial v} = 0 \tag{6-4a}$$

$$4\pi J + \frac{\partial E}{\partial t} = 0 \tag{6-4b}$$

$$\frac{\partial E}{\partial x} = 4\pi\rho \tag{6-4c}$$

$$\rho(x,t) = e\left[n_o - \int_{-\infty}^{\infty} f(x,v,t)\,dv\right] \tag{6-4d}$$

$$J(x,t) = -\,e\int_{-\infty}^{\infty} vf(x,v,t)\,dv \tag{6-4e}$$

where e and m are the respective magnitudes of the electronic charge and mass. Equations (6-4) form the basis for the entire remainder of this work with the exception that at times n_o will be allowed to be spatially dependent.

The next step is to examine the time independent solutions of equations (6-4), since in any linearized analysis it is these solutions which play the central role. Thus one is led to consider the equations

$$v \frac{\partial f}{\partial x} - \frac{e}{m} E \frac{\partial f}{\partial v} = 0 \tag{6-5a}$$

$$\frac{dE}{dx} = - \frac{d^2\varphi}{dx^2} = 4\pi e \left[n_o - \int_{-\infty}^{\infty} f(x,v)\, dv \right] \tag{6-5b}$$

where $\varphi(x)$ is the electric potential.

If one sets $U = \frac{1}{2} mv^2 - e\varphi(x)$, then it can be readily verified that the most general solution of (6-5a) is any function $F(U)$. The electric potential corresponding to such an $F(U)$ is then described by

$$\frac{d^2\varphi}{dx^2} = - 4\pi e \left[n_o - \int_0^{\infty} \frac{F(U)}{\sqrt{2m(U + e\varphi(x))}}\, dU \right] \tag{6-6}$$

It is easily seen then that there is an extremely broad range of solutions of equations (6-5). Bernstein, Greene, and Kruskal have written a now classic paper (ref. 19) describing these one-dimensional stationary solutions and have shown that under very broad conditions indeed it is possible to find $F(U)$ corresponding to a given $\varphi(x)$ and conversely. Further, the transformation $x \rightarrow x - Vt$ will, for any such $\varphi(x)$, produce a travelling wave whose wavelength and wave velocity (V) are unrelated.

Very little work has been done toward examining the stability of these so-called B-G-K modes or toward determining whether such modes are accessible states in the time evolution of a plasma initially prepared in some other state. Armstrong and Montgomery (ref. 15) have reported on the integration of the equivalent of equations

(6-4a) and (6-4c) above for an initially homogeneous plasma which was subject to the two-stream instability and found that after a long time the plasma tended to one of the stationary B-G-K modes. If this result is reliable it indicates that at least one of these modes is accessible in the sense described above. It will be seen in the following chapter however that there exists a simply constructed class of stationary states which is not accessible.

VII. THE RECTANGULAR ENERGY DISTRIBUTION FUNCTION

One of the simplest of the possible stationary distribution functions satisfying equation (6-5a) is the following.

$$F(U) = \begin{cases} F_o, \text{ a constant, for } & 0 \le U \le U_o \\ 0 \quad \text{for} \quad U > U_o \end{cases} \tag{7-1}$$

Such a "rectangular" energy distribution function is found to have certain properties resembling those of a cold plasma while at the same time allowing a spread of velocities at each point in space just as in the case of a hot plasma. Perhaps a more illuminating representation of this particular function is its corresponding distribution function in phase space, $f(x,v)$, which is given by the equation

$$f(x,v) = \begin{cases} f_o, \text{ a constant, inside region } R \text{ in } x - v \text{ plane} \\ 0 \quad \text{outside region } R \end{cases} \tag{7-2}$$

The Vlasov equation (2.1.1a), in analogy with the Liouville equation of statistical mechanics, implies that the "density" $f(x,v,t)$ of that equation is constant in the neighborhood of any (moving) particle in the plasma. This in turn implies that if at $t = 0$ the distribution function were of the form (7-2) then its behavior would be such that only the bounding curve $V(x)$ could change with time. The value f_o must remain constant, as must the area of the region R since the total number of particles must remain constant. Thus study of the distribution function reduces to the study of the curve $V(x)$ (or $V(x,t)$). This special model of the distribution function was introduced by Depackh (ref. 20) in the study of the focusing of electron beams by magnetic fields, and was later used by Ehrman (ref. 21) for similar problems. Its use in the study of neutral plasmas was introduced by Staton and Feix (ref. 22) and the specific results of this chapter were introduced by Staton, Hohl, and Feix (ref. 23). Other workers have since adopted the model; in particular, Roberts and Berk (ref. 24) have used it for obtaining numerical results for the long time nonlinearized behavior of a plasma. At this point some interesting properties of this particular type of nonhomogeneous plasma will be developed.

A stationary electron distribution of the type described in equation (7-2) can be supported by a fixed ion distribution (smeared positive background) of uniform number density N_o between the points $-X_o$ and $+X_o$ and zero density beyond these points. The corresponding electron distribution is

$$f(x,v) = A_o\left[H(v + V(x)) - H(v - V(x))\right] \qquad (7\text{-}3)$$

where A_o is a constant and the Heaviside unit step function H is

$$H(y) = \begin{cases} 0 & y < 0 \\ 1 & y > 0 \end{cases}$$

$V(x)$ is that function of x for which $f(x,v)$ is a solution of equations (6-5). In order to have charge neutrality f_o must be such that

$$\int_{-\infty}^{\infty} dx \int_{-V(x)}^{V(x)} f(x,v)\, dv = 2X_o N_o \tag{7-4}$$

where the magnitudes of the ionic and electronic charges are taken to be equal. One may note that $f(x,v)$ is necessarily symmetric about $v = 0$ since equation (6-4b) precludes the existence of an electric current in the stationary state.

Using the above information in the Poisson equation (6-5b), multiplying by $-\frac{e}{m}$ and noting that $-\frac{e}{m}\frac{dE}{dx} = \frac{da}{dx}$, where $a(x)$ is the acceleration of an electron, one finds

$$\frac{da}{dx} = \omega_p^2 \left\{ \int_{-\infty}^{\infty} f_o\Big[H(v + V(x)) - H(v - V(x))\Big]\, dv - \Big[H\big(x + X_o\big) - H\big(x - X_o\big)\Big] \right\} \tag{7-5}$$

where $f_o = A_o/N_o$ and $\omega_p^2 = 4\pi N_o e^2/m$, the usual plasma frequency. If the assumed form of $f(x,v)$, equation (7-3), is substituted into equation (6-5a) and $-\frac{e}{m}E$ is replaced by a, there results

$$\delta(v + V(x))\left[v\frac{dV}{dx} + a\right] - \delta(v - w(x))\left[-v\frac{dV}{dx} + a\right] = 0 , \qquad (7\text{-}6)$$

where $\delta(y)$ is the Dirac delta function. Integrating this equation over a small neighborhood around $-V(x)$ and again over a small neighborhood around $+V(x)$ yields

$$V\frac{dV}{dx} - a = 0 . \qquad (7\text{-}7)$$

Differentiating this equation with respect to x and substituting equation (7-5) finally yields

$$\frac{d}{dx}\left(V\frac{dV}{dx}\right) + \omega_p^2\left[H\left(x + X_o\right) - H\left(x - X_o\right)\right] - 2\omega_p^2 f_o V(x) = 0 \qquad (7\text{-}8)$$

as the differential equation describing $V(x)$, which in turn describes the stationary electronic distribution function. It will now be demonstrated further that this $V(x)$ possesses a very interesting property with regard to its accessibility from a more general state $V(x,t)$ of the plasma.

The total energy associated with a slab dx of the plasma is the sum of the energy stored in the electric field at point x, $\frac{[E(x)]^2}{8\pi}$, and the kinetic energy of the electrons in the slab dx. This kinetic energy is

$$T(x)dx = dx \int_{-\infty}^{\infty} \frac{1}{2} mv^2 f(x,v)dv = dx\ A_o \int_{-V(x)}^{V(x)} \frac{1}{2} mv^2 dv \tag{7-9}$$

or

$$T(x)dx = \frac{A_o}{3} m \left[V(x)\right]^3 dx \tag{7-10}$$

The electric field may be found from the indefinite integral of equation (7-5) to be

$$\begin{aligned} E(x) &= 4\pi e \int_{x_1}^{x} \left\{ N_o \left[H\left(x' + X_o\right) - H\left(x' - X_o\right) \right] - 2A_o V(x') \right\} dx' \\ &= 4\pi e\ N_o \left\{ x \left[H\left(x + X_o\right) - H\left(x - X_o\right) \right] + X_o \left[H\left(x + X_o\right) + H\left(x - X_o\right) \right] \right. \\ &\quad \left. - \frac{2A_o}{N_o} W(x) \right\} \end{aligned} \tag{7-11}$$

where x_1 is the left-most extremity of the plasma and

$$W(x) = \int_{x_1}^{x} V(x')dx' \ . \tag{7-12}$$

The total energy of the plasma is then

$$U_T = \int_{x_1}^{x_2} \left\{ \frac{1}{3} mA_o \left[V(x)\right]^3 + \frac{\left[E(x)\right]^2}{8\pi} \right\} dx \qquad (7\text{-}13)$$

It will now be demonstrated that the total energy U_T of the plasma is an extremum subject to the condition that

$$\int_{x_1}^{x_2} V(x)\, dx = \frac{N}{2A_o} \qquad (7\text{-}14)$$

where N is the total number of electrons; i.e., $N = 2N_oX_o$. The problem is thus of the following type. An integral

$$I = \int_{X_1}^{X_2} g(x,y,w)\, dx\ , \qquad (7\text{-}15)$$

where $y = y(x)$, and $w = \int^x y(x)\, dx$, is to be extremized subject to the contraint

$$\int_{X_1}^{X_2} y(x)\, dx = C, \text{ a constant.} \qquad (7\text{-}16)$$

The end points X_1 and X_2 are unknown, but it is required that $y\left(X_1\right) = y\left(X_2\right) = 0$.

It is shown in the appendix that a necessary condition that I be so extremized is that $y(x)$ satisfy these equations:

$$\frac{\partial g^*}{\partial w} - \frac{d}{dx}\left(\frac{\partial g^*}{\partial y}\right) = 0 \tag{7-17a}$$

$$\frac{g^*\left(X_2, y\left(X_2\right), w\left(X_2\right)\right)}{y'\left(X_2\right)} = 0 \tag{7-17b}$$

$$\frac{g^*\left(X_1, y\left(X_1\right), w\left(X_1\right)\right)}{y'\left(X_1\right)} = 0 \tag{7-17c}$$

$$\left.\frac{\partial g^*}{\partial y}\right|_{x=X_1} = 0 \tag{7-17d}$$

$$\left.\frac{\partial g^*}{\partial y}\right|_{x=X_2} = 0 \tag{7-17e}$$

where $g^*(x,y,w) = g(x,y,w) + \lambda y(x)$ and λ is an undetermined Lagrange multiplier. Equations (7-16) and (7-17) are sufficient to determine X_1, X_2, λ, and $y(x)$.

For the problem at hand then,

$$y = V(x) \ , \quad w = W(x) \ , \tag{7-18a}$$

$$g^* = g + \lambda y = \frac{mA_o}{3} V^3 + \frac{1}{8\pi}\left[4\pi e N_o\left\{x\left[H\left(x + X_o\right) - H\left(x - X_o\right)\right]\right.\right.$$
$$\left.\left. + X_o\left[H\left(x + X_o\right) + H\left(x - X_o\right)\right]\right\} - 8\pi A_o e W\right]^2 + \lambda V \tag{7-18b}$$

Substituting this expression for g^* into equations (7-17d) and (7-17e) yields immediately, upon noting again that $V\left(x_1\right) = V\left(x_2\right) = 0$,

$$\lambda = 0 \tag{7-19}$$

The constraint equation is thus trivially satisfied, the reason being the similarity between that equation and the definition of w. For finite $V'(x_1)$ and $V'(x_2)$ equations (2.3.13b and c) require that

$$g^*\left(x_1, V(x_1), W(x_1)\right) = g^*\left(x_2, V(x_2), W(x_2)\right) = 0 \tag{7-20}$$

and one sees from equations (7-18b), (7-19), and (7-14) that these relations are indeed identically satisfied. Equation (7-17a) now reduces to

$$\frac{\partial g}{\partial w} - \frac{d}{dx}\left(\frac{\partial g}{\partial V}\right) = 0 \tag{7-21}$$

Using equation (7-18b) then immediately yields

$$V\,\frac{dV}{dx} + \frac{4\pi N_o e^2}{m}\,x\left[H\left(x + X_o\right) - H\left(x - X_o\right)\right] + X_o\left[H\left(x + X_o\right) + H\left(x - X_o\right)\right] - \frac{4\pi N_o e^2}{m}\,\frac{2A_o}{N_o}\,W(x) = 0 \tag{7-22}$$

which is the same as equation (7-7) and is a first integral of

$$\frac{d}{dx}\left(V\,\frac{dV}{dx}\right) + \omega_p{}^2\left[H\left(x + X_o\right) - H\left(x - X_o\right)\right] - 2\omega_p{}^2 f_o V(x) = 0\ . \tag{7-23}$$

Equation (7-23) is the same as equation (7-8), demonstrating that the differential equation for the stationary curve $V(x)$ is just the Euler-Lagrange equation corresponding to extremization of the total energy of the plasma system. By noting that the transformation $W \rightarrow y$ and $V = \frac{dW}{dx} \rightarrow y'$ transforms equation (7-21) into the more usual form of the Euler-Lagrange equation, one may apply the Legendre test (ref. 25) to the second variation of the integral U_T in order to gain more information about the character of the extremum.

The Legendre test states that a necessary condition that the solution $V(x)$ of equation (7-22) or (7-23) be such that the integral U_T is maximized is that the quantity $\frac{\partial^2 g^*}{\partial V^2} = 2A_o mV$ be negative definite over the interval $[x_1, x_2]$. This quantity however is seen to be positive definite over the interval so that the extremum represented by $V(x)$ cannot be a maximum, but must be either a minimum or an inflection. Numerical calculations of the energies of systems with the same number of particles and with bounding curves very close to a given $V(x)$ solution have invariably led to larger total energies than that for the given $V(x)$, so that one may with some assurance conclude that the extremum is a minimum.

In view of the minimum energy character of the stationary solutions $V(x)$, one must conclude that a plasma system prepared initially in a state

$$f(x,v,t = 0) = A_o\Big[H(v + V(x,t = 0)) - H(v + V(x,t = 0))\Big] \tag{7-24}$$

may never have access to a stationary state, since the Vlasov equation (6-4a), in analogy with the Liouville equation of statistical mechanics, dictates that $f(x,v,t)$ for such a system will always remain of the form (7-24). That is, only the shape of the boundary curve $V(x,t)$ may change with time, and even that in just such a way that its enclosed area is constant in time, since A_o and the total number of particles remain constant. Further, any such system initially prepared in the stationary state characterized by $V(x)$ must necessarily be stable to small perturbations in $V(x)$. Only for such a simplified model of an inhomogeneous plasma system is it possible to obtain such detailed information concerning the form of the distribution function as time progresses or to supply an answer concerning the accessibility of a stationary state. With the single exception of the work of Armstrong mentioned briefly above, no report has apparently appeared in the literature concerning the accessibility of any of the B-G-K modes, even though these modes are basic to the study of one-dimensional nonhomogeneous plasmas.

Solutions of equation (7-8) or (7-23) for two special cases are shown in figure 1. The upper curve corresponds to $N_o = 10^{18}$, $X_o = 20$, and $A_o = 10^{10}$. The lower curve corresponds to the special case of $N_o \to \infty$, $X_o \to 0$, such that the product $2X_oN_o = N$, the total number of ions. Thus the entire positive charge is located on a thin sheet at $x = 0$. For this case equation (7-8) reduces to

$$\frac{d}{dx}\left(V\,\frac{dV}{dx}\right) - \frac{8\pi A_o e^2}{m}\,V(x) = 0 \qquad (7\text{-}25)$$

whose solution by elementary methods is found to be

$$V(x) = \begin{cases} \left(C - \sqrt{\dfrac{4\pi A_o e^2}{m}}\,x\right)^2 & x > 0 \\ \left(C + \sqrt{\dfrac{4\pi A_o e^2}{m}}\,x\right)^2 & x < 0 \end{cases} \qquad (7\text{-}26)$$

where C must be $(3\pi N^2 e^2/4mA_o)^{1/6}$ for overall charge neutrality. The curve shown corresponds to $A_o = 96.5 \times 10^{-7}$ and $N = 5.72 \times 10^3$.

Equations governing the behavior of small perturbations to the type of distribution functions discussed here will be derived in the following chapter.

VIII. FORMULATION OF LINEARIZED PROBLEM

In this and subsequent chapters it is assumed that a stationary solution of the one-dimensional Vlasov-Maxwell equations (6-2) is given and that it is desired to examine this solution to determine how small perturbations applied to it would change with time. In particular it is desirable to know whether such perturbations are unstable and consequently grow with time, or stable, in which case they would either dampen with time or persist indefinitely at their initial level. The procedure to be followed then is to linearize the Vlasov-Maxwell equations about the given inhomogeneous solutions and then to attempt to solve these linearized equations for the perturbed quantities.

8.1 Linearized Equations

In line with this goal $f^{(o)}(x,v)$ is thus considered to be a solution of equations (6-5a) and (6-5b) with $E^{(o)}(x)$ being the electric field which is consistent with this solution; i.e.,

$$v \frac{\partial f^{(o)}}{\partial x} - \frac{e}{m} E^{(o)} \frac{\partial f^{(o)}}{\partial v} = 0 \qquad (8\text{-}1a)$$

$$\frac{dE^{(o)}}{dx} = 4\pi e \left[n_o - \int_{-\infty}^{\infty} f^{(o)}(x,v)\, dv \right] \qquad (8\text{-}1b)$$

Again the ions are regarded as a uniform, smeared, positive background corresponding to an effective number density of ions n_o each bearing a charge + e. With this solution in mind, it is now desired to solve the Vlasov-Maxwell equations (6-4) for that case in which their solution $f(x,v,t)$ can be expressed in the form $f(x,v,t) = f^{(o)}(x,v) + f^{(1)}(x,v,t)$, where the absolute value of $f^{(1)}(x,v,t)$ is small compared with that of $f^{(o)}(x,v)$ for all x and v, and for all time. Consistently with this $f(x,v,t)$, the electric field and current density are written respectively as $E(x,t) = E^{(o)}(x) + E^{(1)}(x,t)$ and $J(x,t) = J^{(o)}(x) + J^{(1)}(x,t)$. If these forms of solutions are substituted into equations (6-4) and equations (8-1) are taken into account, the resulting equations are

$$\frac{\partial f^{(1)}}{\partial t} + v\,\frac{\partial f^{(1)}}{\partial x} - \frac{e}{m}\,E^{(o)}\,\frac{\partial f^{(1)}}{\partial v} = \frac{e}{m}\,E^{(1)}\,\frac{\partial f^{(o)}}{\partial v} \tag{8-2a}$$

$$4\pi J^{(1)} + \frac{\partial E^{(1)}}{\partial t} = 0 \tag{8-2b}$$

$$\frac{\partial E^{(1)}}{\partial x} = 4\pi\rho^{(1)} \tag{8-2c}$$

$$\rho^{(1)}(x,t) = -\,e\int_{-\infty}^{\infty} f^{(1)}(x,v,t)\,dv \tag{8-2d}$$

$$J^{(1)}(x,t) = -\,e\int_{-\infty}^{\infty} vf^{(1)}(x,v,t)\,dv \tag{8-2e}$$

Reference to equation (6-4b) shows that the assumptions that $E^{(o)}$ is independent of time and that any external magnetic field present has zero curl preclude the existence of any current density $J^{(o)}(x)$

within the framework of this treatment. This restriction however still leaves ample room for physically interesting situations.

Now it is well known that equations (8-2b) and (8-2c) (or indeed equations (6-2b) and (6-2c)) are not independent since the charge continuity equation, which is embodied in the Vlasov equation (8-2a) or (6-2a), along with the generalized Ampere law, equation (6-2b), and the Faraday law, equation (6-2d), will ensure that $\nabla \cdot E = 4\pi\rho$ and $\nabla \cdot B = 0$ for all time if the latter are true at $t = 0$. In the present chapter only equations (8-2a,b,d, and e) will be used. It can be shown that for the cases to be treated here the results are identical with those obtained using the Poisson equation (8-2c) rather than equation (8-2b). The difference in the two approaches is only that the latter approach yields a differential-integral equation for the electric field rather than an integral equation as obtained in the following section.

8.2 Integral Equation Formulation

In order to derive the equation describing the electric field $E^{(1)}$, one begins by taking the Laplace transform of equation (8-2a). Thus

$$i\omega f^{(1)}(x,v;\omega) + v\,\frac{\partial f^{(1)}(x,v;\omega)}{\partial x} - \frac{e}{m}\,E^{(o)}(x)\,\frac{\partial f^{(1)}(x,v;\omega)}{\partial v}$$

$$= \frac{e}{m}\,E^{(1)}(x;\omega)\,\frac{\partial f^{(o)}(x,v)}{\partial v} + f^{(1)}(x,v,t = 0) \qquad (8\text{-}3)$$

where

$$f^{(1)}(x,v;\omega) = \int_0^\infty e^{-i\omega t} f^{(1)}(x,v,t)\, dt$$

and

$$E^{(1)}(x;\omega) = \int_0^\infty e^{-i\omega t} E^{(1)}(x,t)\, dt$$

are the Laplace transforms of the indicated quantities and $f^{(1)}(x,v,t = 0)$ is the distribution function at the initial time $t = 0$. ω is a complex number having a small negative imaginary part.

One may now note that just as the left side of equation (8-2a) represents the total derivative of $f^{(1)}(x,v,t)$ with respect to time along an unperturbed characteristic curve of equation (8-1a), so the second and third terms on the left side of (8-3) represent the total derivative of $f^{(1)}(x,v;\omega)$ along that same unperturbed characteristic curve. Thus equation (8-3) can be rewritten

$$\frac{d}{d\tau} f^{(1)}(x,v;\omega) + i\omega f^{(1)}(x,v;\omega) = \frac{e}{m} E^{(1)}(x;\omega) \frac{\partial f^{(o)}(x,v)}{\partial v} + f^{(1)}(x,v,t = 0) \tag{8-4}$$

where τ is a measure of the time required for a particle moving on the unperturbed characteristic curve, i.e., moving with the velocity

$v(x) = \pm\sqrt{\frac{2}{m}\left(U^{(o)} + e\varphi^{(o)}(x)\right)}$, where $U^{(o)}$ is the total energy of the particle and $\varphi^{(o)}(x)$ is the electric potential corresponding to $E^{(o)}(x)$, to move from a given point (say the origin) to the point x. Equation (8-4) is a first-order linear equation whose formal solution is

$$f^{(1)}(x(\tau),v(\tau);\omega) = e^{i\omega\left(\tau_o-\tau\right)} f^{(1)}\left(x\left(\tau_o\right),v\left(\tau_o\right);\omega\right)$$

$$+ \int_{\tau_o}^{\tau} e^{i\omega(\tau'-\tau)} f^{(1)}(x(\tau'),v(\tau'),\ t = 0)\ d\tau'$$

$$+ \int_{\tau_o}^{\tau} \frac{e}{m} E^{(1)}(x(\tau');\omega) e^{i\omega(\tau'-\tau)} \frac{\partial f^{(o)}}{\partial v}\ d\tau' \qquad (8\text{-}5)$$

The first term on the right is, as indicated, to be evaluated at the point $x\left(\tau_o\right)$ from which a particle initially having velocity $v\left(\tau_o\right)$ could travel along an unperturbed particle trajectory and arrive at the point x with velocity v after a time interval $\tau - \tau_o$. $x(\tau')$ and $v(\tau')$ are similarly related to each other and to τ' in the integral terms. Dobrott (ref. 26) was first to express the solution for $f^{(1)}$ in this way, using it to formulate that integral equation relating the current density to the electric field which had been previously suggested by Drummond, Gerwin, and Springer (ref. 27).

In this work the main concern lies in the intrinsic long term behavior of the plasma, rather than in transient solutions, which depend explicitly upon the initial conditions. Thus $f^{(1)}(x,v,t = 0)$ and hence the second term on the right of equation (8-5) may be set to zero. A choice for the value of τ_o which is consistent with the desired goal must now be found. Examination of the first term on the right side of equation (8-5) shows that its inverse Laplace transform is simply $f^{(1)}\left(x\left(\tau_o\right), v\left(\tau_o\right), t - \left(\tau - \tau_o\right)\right)$; i.e., the contribution of this term to $f^{(1)}(x,v,t)$ is the value of $f^{(1)}$ at the point $x\left(\tau_o\right)$, $v\left(\tau_o\right)$ in phase space at an earlier time such that a particle moving along an unperturbed characteristic curve could move from the point $x\left(\tau_o\right)$, $v\left(\tau_o\right)$ to the point x,v arriving at the time t. Since initial conditions on $f^{(1)}$ are being neglected (which corresponds to examining the solution at large t after all transients have died away) it is consistent to let $\tau_o \rightarrow -\infty$ and to drop the first term altogether. Leavens (ref. 11) has also used these arguments, and in agreement with him one now obtains

$$f^{(1)}(x,v;\omega) = \int_{-\infty}^{\tau} \frac{e}{m} E^{(1)}(x(\tau');\omega) e^{i\omega(\tau'-\tau)} \frac{\partial f^{(o)}}{\partial v} d\tau' \qquad (8\text{-}6)$$

At this point it is convenient to change from x and v as variables to x and U, where $U = \frac{1}{2} mv^2 - e\varphi^{(o)}(x)$. Thus $f^{(1)}(x,v;\omega) = F_{\pm}^{(1)}(x,U;\omega)$ and $f^{(o)}(x,v) = F^{(o)}(U)$. Equation (8-6) then becomes

$$F_{\pm}^{(1)}(x,U;\omega) = \int_{-\infty}^{\tau} eE^{(1)}(x(\tau');\omega) e^{i\omega(\tau'-\tau)} v_{\pm}(\tau') \frac{dF^{(o)}(U)}{dU} d\tau'$$

(8-7)

The upper signs correspond to particles with energy U at point x having positive velocities and the lower signs to similar particles but with negative velocities. The integration variable in this equation may now be changed from τ' to x' by noting the relation

$$\tau = \int_0^x \frac{dx'}{v_\pm(x')} = \int_0^x \frac{dx'}{\pm\sqrt{\frac{2}{m}\left(U + e\varphi^{(o)}(x)\right)}} \tag{8-8}$$

between τ and x. If attention is restricted to those functions $F^{(o)}(U)$ for which there are no particles of interest trapped in potential minima of $-e\varphi^{(o)}(x)$, then all such particles travelling to the right at point x were at the point $x' = -\infty$ at $\tau' = -\infty$ and similarly those particles travelling to the left at point x were at $x' = \infty$ at $\tau' = -\infty$. Thus upon noting that $dx' = v_\pm(\tau')d\tau'$ one gets

$$F_\pm(x,U;\omega) = \int_{\mp\infty}^{x} eE^{(1)}(x';\omega)e^{i\omega(\tau'-\tau)} \frac{dF^{(o)}(U)}{dU}\, dx' \tag{8-9}$$

where τ and τ' are related to x and x' through equation (8-8).

The current density $J^{(1)}(x;\omega)$ is related to the distribution function by

$$J^{(1)}(x;\omega) = -e\int_{-\infty}^{\infty} vf^{(1)}(x,v;\omega)\, dv$$

$$= -\frac{e}{m}\int_{-e\varphi^{(o)}(x)}^{\infty}\left[F_+^{(1)}(x,U;\omega) - F_-^{(1)}(x,U;\omega)\right] dU \tag{8-10}$$

If $F_{\pm}^{(1)}$ from equation (8-9) is used in the integral term here and the two integrations are interchanged one obtains

$$J^{(1)}(x;\omega) = -\frac{e^2}{m}\int_{-\infty}^{\infty} E(x';\omega)K(x,x';\omega)\,dx' \qquad (8\text{-}11)$$

where

$$K(x,x';\omega) = \operatorname{sgn}(x - x')\int_{-e\varphi^{(o)}(x)}^{\infty} \frac{dF^{(o)}(U)}{dU} \cdot \exp\left[-i\omega\operatorname{sgn}(x - x')\int_{x'}^{x} \frac{dx''}{\sqrt{\frac{2}{m}\quad U + e\varphi^{(o)}(x)}}\right]dU$$

(8-12)

Use has been made of equations (8-8) and (8-9) and the definition

$$\operatorname{sgn}(x) = \begin{cases} -1 & x < 0 \\ +1 & x > 0 \end{cases}$$

Equation (8-11) is the "conductivity kernel equation" of Drummond, Gerwin, and Springer (ref. 27), and represents the form of Ohm's law in the collisionless plasma. If this equation is used in the Laplace transformed version of equation (8-2b), the following integral equation for the electric field is obtained:

$$\frac{1}{i\omega} E^{(1)}(x,t = 0) = E^{(1)}(x;\omega) - \frac{4\pi e^2}{mi\omega} \int_{-\infty}^{\infty} E^{(1)}(x;\omega)K(x,x';\omega)\, dx' \tag{8-13}$$

Because of the complicated form of the kernel of this equation one may despair of drawing many broad conclusions concerning its solutions. Except for the cases of linear and quadratic functions for the $\varphi^{(o)}(x)$ it is not possible to write the kernel in terms of elementary functions. It is no surprise then to find that the bulk of work aimed at discussing this equation has been directed to such simple cases. The case $\varphi^{(o)}(x) \propto x^2$ has been treated by many authors. Harker (ref. 13), Pavkovitch (ref. 10), Watson (ref. 14), and Leavens (ref. 11) have all treated this case through integral equations like (8-13). For more general cases one may note that the complicated dependence of equation (8-13) upon the frequency ω renders that equation most suitable for treating externally driven oscillations in the plasma, just as most of the above authors have done for the quadratic case, rather than for finding the eigenfrequencies of the plasma. Accordingly, in the present work equation (8-13) will be used only for treating the case of the rectangular energy distribution function discussed in Chapter VII, with more general $\varphi^{(o)}(x)$ being reserved for the more powerful method of Chapter IX.

For examining questions of stability it suffices to examine the homogeneous counterpart of equation (8-13). Also it is helpful for the following work to change the variables again through the relation

$$\theta = \int_0^x \frac{dx'}{\sqrt{\frac{2}{m}\left(U + e\varphi^o(x')\right)}} \tag{8-14}$$

so that the equation to be treated becomes

$$E^{(1)}(\theta;\omega) - \frac{4\pi e^2}{mi\omega}\int_{-\infty}^{\infty} E^{(1)}(\theta';\omega)\left[\int_{-e\varphi^o(\theta)}^{\infty} \operatorname{sgn}(\theta - \theta') \cdot \frac{dF^o(U)}{dU}\exp\left[-i\omega \operatorname{sgn}(\theta - \theta')(\theta - \theta')\right] dU\right] v(\theta')\, d\theta' = 0 \tag{8-15}$$

where

$$v(\theta') = \sqrt{\frac{2}{m}\left(U + e\varphi^{(o)}(\theta')\right)} .$$

8.3 Application to Rectangular Energy Distribution Case

In Chapter VII the rectangular distribution function

$$F^{(o)}(U) = \begin{cases} F_o & 0 \le U \le U_o \\ 0 & U > U_o \end{cases} \tag{8-16}$$

was introduced and it was shown there that the corresponding distribution function in phase space $f^{(o)}(x,v)$ is such that the latter is described by an Euler-Lagrange equation minimizing the total system energy. In this section the equations governing small perturbations in the electric field of such a stationary state will be obtained through the integral equation of the last section.

From equation (8-16) one finds

$$\frac{dF^{(o)}(U)}{dU} = - F_o \delta\left(U - U_o\right) \tag{8-17}$$

and if this expression is substituted into equation (8-15) the result is

$$E^{(1)}(\theta;\omega) + \frac{4\pi e^2}{mi\omega} F_o \int_{-\infty}^{\infty} E^{(1)}(\theta';\omega)\mathrm{sgn}(\theta - \theta')$$

$$\cdot \exp\left[- i\omega \mathrm{sgn}(\theta - \theta')(\theta - \theta')\right] v(\theta')\, d\theta' = 0 \tag{8-18}$$

It follows that $F_o v(\theta')$ here is just $n(\theta')$, the number density of electrons, so that $4\pi e^2 F_o v(\theta')/m = 4\pi e^2 n(\theta')/m$ may be defined as the local plasma frequency $\omega_p{}^2(\theta')$. Then differentiating equation (8-18) twice with respect to θ yields

$$\frac{d^2E^{(1)}(\theta;\omega)}{d\theta^2} + \left(\omega^2 - \omega_p{}^2(\theta)\right)E^{(1)}(\theta;\omega) = 0 \tag{8-19}$$

This equation is very interesting indeed in that it is closely analogous to a second-order differential equation for the electric field obtained by Hoh (ref. 28) for a plasma in the hydrodynamic or "fluid" approximation. It is well known that this approximation holds rigorously for a cold plasma, but equation (8-19) has been obtained from the full kinetic theory, so that the rectangular distribution emerges in a sort of intermediate position between the hot and cold plasmas. That is, it possesses a spread of velocities at each point just as a hot plasma does but allows a simple equation for its electric field just like the cold plasma.

As far as solutions of equation (8-19) are concerned, it is perhaps sufficient here to note that it has the form of the time independent Schroedinger equation and that it possesses only real eigenvalues ω^2. Further, since $\omega_p^{\;2}(\theta')$ is non-negative, ω^2 is positive and the eigenfrequencies ω are therefore real, a fact which confirms the conclusion in Chapter VI that such a plasma is stable, possessing no natural modes which grow with time.

IX. THE FOURIER-HERMITE METHOD FOR THE INHOMOGENEOUS PLASMA

9.1 Introductory Remarks

As before, it is assumed now that a solution of the time independent Vlasov and Poisson equations (8-1) is given and that equations (8-2) are to be solved for the small perturbation function. It is convenient for the following work to normalize the variables in those equations according to the following scheme. First, let the smeared positive background have an effective charge density $en_o(x) = eN_o\eta(x)$ where $\eta(x)$ is of order unity. Then let $f^{(o)}(x,v) = N_oF^{(o)}(x,v)$, where $f^{(o)}$ is the unperturbed solution of equations (8-1). Introducing the thermal velocity $v_{TH} = \sqrt{\frac{kT}{m}}$, where k is the Boltzmann constant, T the temperature, and m the electron mass, along with the Debye length $\lambda_D = \sqrt{\frac{kT}{4\pi N_o e^2}}$, one may write $x = \lambda_D x'$ and $v = v_{TH}v'$ in equations (8-1) and (8-2). The time variable then scales as $t = \lambda_D/v_{TH} \cdot t' = \omega_p t'$ where ω_p is the usual plasma frequency $\sqrt{4\pi N_o e^2/m}$, and the potential energy $\varphi^{(o)}$ may be replaced with $\frac{kT}{e}\,\Phi^{(o)}$. The primes may then be dropped and the equations of interest become

$$v\,\frac{\partial F^{(o)}(x,v)}{\partial x} - E^{(o)}\,\frac{\partial F^{(o)}(x,v)}{\partial v} = 0 \tag{9-1a}$$

$$\frac{dE^{(o)}}{dx} = -\frac{d^2\Phi^{(o)}}{dx^2} = \left[\eta(x) - \int_{-\infty}^{\infty} F^{(o)}(x,v)\,dv\right] \tag{9-1b}$$

$$\frac{\partial F^{(1)}(x,v,t)}{\partial t} + v\,\frac{\partial F^{(1)}(x,v,t)}{\partial x}$$

$$- E^{(o)}(x)\,\frac{\partial F^{(1)}(x,v,t)}{\partial v} = E^{(1)}(x)\,\frac{\partial F^{(o)}(x,v)}{\partial v} \qquad (9\text{-}1c)$$

$$\frac{dE^{(1)}}{dx} = -\int_{-\infty}^{\infty} F^{(1)}(x,v,t)\,dv \qquad (9\text{-}1d)$$

Note that, as opposed to the integral equation development of Chapter VIII, it is found convenient to use the Poisson equation (8-2c) rather than the Ampere equation (8-2b) for the following work.

Berk, Rosenbluth, and Sudan (ref. 8) have given a solution of equations (9-1) under the assumption that $\varphi^{(o)}(x)$ is a monotonically increasing function having the value zero at $x = -\infty$ and ∞ at $x = +\infty$. Such a potential allows them to make use of the W.K.B. approximation, which renders the equations somewhat more tractable. Jackson and Raether (ref. 9) have given perhaps a more interesting treatment of equations (9-1). They considered conditions for which the product of the ratios of the Debye length to the wavelength of the perturbation and of the Debye length to the dimension of the plasma is small compared to unity. They argued that for those conditions the third term on the left of equation (9-1c) is negligible and may be dropped from the equations. With this assumption they treated in detail the case where $F^{(o)}(x,v) = (1 + \epsilon \cos Kx)e^{-v^2/2}$ where $\epsilon < 1$ is a parameter characterizing the strength of the inhomogeneity. The fixed background distribution $\eta(x)$, which is calculated from equation (9-16) after having chosen (arbitrarily) an unperturbed potential $\varphi^{(o)}(x)$ can be rather complicated, but the background appears only in the $E^{(o)}$ term which they were able to neglect. Jackson and Raether were thus able to reduce their problem for all $\epsilon < 1$ to the Mathieu differential equation and so to obtain both the real frequencies and the damping factors for the problem at hand. Their results closely parallel those of Weissglas (ref. 29) who also obtained the Mathieu equation from the hydrodynamic approximation for $\epsilon \ll 1$.

In this thesis an $F^{(o)}(x,v)$ similar to that of Jackson and Raether is to be treated but including the term neglected by them so that the corresponding results are valid for arbitrary wavelengths of the perturbation. In principle the results also hold for all $\epsilon < 1$, but as will be seen practical results are difficult for ϵ greater than about 0.1.

A second area to be treated in the present thesis is the solution of equations (9-1) for an $F^{(o)}(x,v)$ which is such that the system supports the two-stream instability. This instability has been discussed for the case of the homogeneous plasma by many authors (e.g., Buneman (ref. 30), Jackson (ref. 31), and Grant (ref. 16)) but the effects of spatial inhomogeneity in the unperturbed distribution function have not been examined heretofore. The effects of adding a small collision term to the right side of the Vlasov equation (6-2a) upon the growth of the two-stream instability in a homogeneous plasma have been considered by Tidman and Weiss (ref. 22), May (ref. 33), and Buti and Trehan (ref. 34). Since the methods of the present thesis also embrace the homogeneous case, a small collision term (following Grant (ref. 16) and Lenard and Bernstein (ref. 35)) is adopted here also and helps to clarify some apparent contradictions among the three papers mentioned above. This collision term, assumed to have the form

$$\left(\frac{\partial F^{(1)}(x,v,t)}{\partial t}\right)_c = B\left[\frac{\partial\, vF^{(1)}}{\partial v} + \langle v^2\rangle \frac{\partial^2 F^{(1)}}{\partial v^2}\right] \qquad (9\text{-}2)$$

where B is a small parameter characterizing an effective collision frequency and $\langle v^2\rangle$ is the mean square velocity corresponding to $F^{(o)}(x,v)$, also serves in the present work as a useful adjunct in the case of the Maxwellian velocity distribution.

9.2 The Fourier-Hermite Transformation

The basic method of attack on equations (9-1) will be through the Fourier-Hermite transformation on space and velocity space, respectively, which was introduced into the study of the Vlasov equation by Grant and Feix (ref. 35). Grant (ref. 16) used this transformation both for eigenfrequency studies in the homogeneous, linearized case and for the nonlinearized time integration of small disturbances on an initially homogeneous plasma, the latter case also being treated by Armstrong (ref. 15). In the present work interest centers upon eigenfrequency studies of inhomogeneous plasmas. To this end the distribution functions satisfying equations (9-1) are expressed as

$$F^{(o)}(x,v) = \sum_{m=-\infty}^{\infty} \sum_{r=0}^{\infty} f_{mr}^{(o)} e^{ik_m x} \left[\frac{1}{\sqrt{2\pi r!}} e^{-\frac{v^2}{2}} He_r(v) \right] \quad (9\text{-}3a)$$

and

$$F^{(1)}(x,v;\omega) = \sum_{m=-\infty}^{\infty} \sum_{r=0}^{\infty} f_{mr}^{(1)}(\omega) e^{ik_m^{(1)} x} \left[\frac{1}{\sqrt{2\pi r!}} e^{-\frac{v^2}{2}} He_r(v) \right] \quad (9\text{-}3b)$$

Here $F^{(1)}(x,v;\omega)$ refers to the Laplace transform of $F^{(1)}(x,v,t)$ and $He_r(v)$ is the Hermite polynomial of degree r satisfying the orthogonality relation

$$\int_{-\infty}^{\infty} He_m(v) He_n(v) e^{-\frac{v^2}{2}} dv = \delta_{mn} \sqrt{2\pi m!} \quad (9\text{-}4a)$$

and the recursion relations

$$He_{n+1}(v) = vHe_n(v) - nHe_{n-1}(v) \quad (9\text{-}4b)$$

$$\frac{dHe_n(v)}{dv} = nHe_{n-1}(v) \quad (9\text{-}4c)$$

At this point the Laplace transforms of equations (9-1) are taken, the initial conditions are set to zero (which is consistent with seeking the normal mode behavior), and the expansions (9-3) are substituted. Making use of equations (9-4b) and (9-4c), multiplying by $\frac{1}{L} \cdot \frac{1}{q!} e^{-ik_n x} He_q(v)$, and integrating over x from 0 to L and v from $-\infty$ to ∞ then gives finally

$$i\omega f_{nq}^{(1)} + ik_n^{(1)}\left[\sqrt{q}\, f_{n,q-1}^{(1)} + \sqrt{q+1}\, f_{n,q+1}^{(1)}\right] + \sqrt{q} \sum_m E_{n-m}^{(0)} f_{m,q-1}^{(1)}$$

$$+ \sqrt{q} \sum_m E_m^{(1)} f_{n-m,q-1}^{(o)} = 0 \qquad (9\text{-}5a)$$

and similarly,

$$ik_r E_r^{(o)} = \eta_r - f_{r,o}^{(o)} \qquad (9\text{-}5b)$$

and

$$ik_r^{(1)} E_r^{(1)} = - f_{r,o}^{(1)} \qquad (9\text{-}5c)$$

Equations (9-5) represent an infinite set of linear, homogeneous, algebraic equations for the $f_{nq}^{(1)}(\omega)$. Practically, of course, only a finite number of terms may be kept in the Fourier and Hermite series so that there will be a maximum value N_F of n, the Fourier index, and a maximum value Q of q, the Hermite index. Before proceeding it is interesting to consider that special case of equations (9-5) which corresponds to a single Fourier component for the distribution function and a vanishing $E^{(o)}$. This, the usual homogeneous, linearized case, was treated by Grant (ref. 16) using the Fourier-Hermite representation. Thus, combining equations (9-5a) and (9-5c) and dropping the (superfluous) Fourier subscript gives

$$i\omega f_q^{(1)} + ik\left[\sqrt{q}\, f_{q-1}^{(1)} + \sqrt{q+1}\, f_{q+1}^{(1)}\right] + \frac{i\sqrt{q}}{k} f_{q-1}^{(o)} f_o^{(1)} = 0 \tag{9-6}$$

which may be written in the matrix form

$$\left(M_{\alpha\beta} + i\omega\delta_{\alpha\beta}\right) f_\beta^{(1)} = 0 \tag{9-7}$$

Equation (9-6) or (9-7) is thus a standard eigenvalue problem for which the vanishing of the determinant of $M_{\alpha\beta} + i\omega\delta_{\alpha\beta}$ yields the eigenfrequencies ω. Eigenfrequencies for various values of Q were found in this way by Grant for both Maxwellian and "doubled-humped" velocity distributions.

For the inhomogeneous plasma, to be treated here, the problem is somewhat more complicated. First, the field $E^{(o)}$ must be retained, and second, many Fourier components are necessary to describe the distribution function and the electric field. The analogue of equation (9-7) is then

$$\left(M_{\alpha\beta\gamma\xi} + i\omega\delta_{\alpha\beta}\delta_{\gamma\xi}\right) f_{\beta\xi}^{(1)} = 0 \tag{9-8}$$

which is obviously not in the form of a matrix equation, so that the problem is not tractable using the standard methods of Grant.

Instead, the central idea of the present work is to use equation (9-5a) as a recursion relation to construct the higher Hermite indices from lower ones, beginning at $q = 0$ and ending at $q = Q$, the index at which the Hermite series is to be truncated. Thus, from equation (9-5a) with $q = 0$ and $q = 1$, respectively, one finds (note that for $n = 0$ the second and third terms on the left side of equation (9-5a) do not appear)

$$f_{n1}^{(1)} = \begin{cases} -\frac{\omega}{k_n^{(1)}} f_{no}^{(1)} & n \neq 0 \\ 0 & n = 0 \end{cases} \tag{9-9}$$

Making use of this expression, setting $q = 2$ in (9-5a), and putting $n = 0$ gives

$$f_{02}^{(1)} = i\sqrt{2} \sum_m \left(E_{-m}^{(o)} f_{mo}^{(1)} / k_m^{(1)} \right) - \frac{\sqrt{2}}{i\omega} \sum_m E_m^{(1)} f_{-m,1}^{(o)} \tag{9-10a}$$

Similarly for $n \neq 0$ and $q = 1$,

$$f_{n2}^{(1)} = \frac{1}{\sqrt{2}} \left(\frac{\omega^2}{k_n^2} - 1 \right) f_{no}^{(1)} + \frac{i}{\sqrt{2}\, k_n} \sum_m E_{n-m}^{(o)} f_{mo}^{(1)}$$

$$+ \frac{i}{\sqrt{2}\, k_n} \sum_m f_{n-m,q-1}^{(o)} E_m^{(1)} \tag{9-10b}$$

Equations (9-10) give the second member of the sequence of the $f_{nq}^{(1)}$ expressed in terms of the $f_{mo}^{(1)}$. Before proceeding it is convenient to express the $E^{(0)}$ and $E^{(1)}$ in terms of the corresponding distribution functions. Letting k denote the fundamental wave number of the stationary distribution $F^{(o)}$ (such that $k_n = nk$), and $k^{(1)} = \mu k$ that of the perturbation $F^{(1)}$, where μ is a parameter which will be chosen later, one finds from equations (9-5a) and (9-5b)

$$E_r^{(o)} = -\frac{i}{rk}\left[\eta_r - f_{ro}^{(o)}\right] \equiv -\frac{i}{k} G_r^{(o)} \tag{9-11}$$

and

$$E_r^{(1)} = \frac{i}{\mu k} \frac{f_{ro}^{(1)}}{r} \tag{9-12}$$

for $r \neq 0$. Since the source of the electric fields is the charge distribution in the plasma itself, $E_o^{(1)} = E_o^{(o)} = 0$.

Now suppose that N Fourier components are needed to represent the stationary electric field $E^{(o)}$ and that M components are needed for $F^{(o)}$. Then in the first summation in equations (9-10) the index m must range from $n - N$ to $n + N$. Similarly in the second summations m must range from $n - M$ to $n + M$. (Of course $n = 0$ in equation (9-10a).) Making use of these facts one may now write the equations for the general term of the above-mentioned $f_{nq}^{(1)}$ sequence by using equation (9-5a). For $n = 0$,

$$f_{or}^{(1)} = \begin{cases} -\dfrac{\sqrt{r}}{\omega k} \displaystyle\sum_{\substack{m=-N \\ m\neq 0}}^{N} G_m^{(o)} f_{m,r-1}^{(1)} + \dfrac{\sqrt{r}}{\omega\mu k} \displaystyle\sum_{m=-M}^{M} \dfrac{f_{m,r-1}^{(o)}}{m} f_{mo}^{(1)} \ ; & r \text{ even} \\ 0 \ ; & r \text{ odd} \end{cases} \tag{9-13a}$$

and for $n \neq 0$

$$f_{nr}^{(1)} = -\frac{\omega}{n\mu k\sqrt{r}} f_{n,r-1}^{(1)} - \sqrt{\frac{r-1}{r}}\, f_{n,r-2}^{(1)} + \frac{\sqrt{(r-1)/r}}{n\mu k^2} \sum_{\substack{m=n-N \\ m\neq n}}^{n+N} G_{n-m}^{(o)} f_{m,r-2}^{(1)}$$

$$- \frac{\sqrt{(r-1)/r}}{n\mu^2 k^2} \sum_{\substack{m=n-M \\ m\neq 0}}^{n+M} \frac{f_{n-m,r-2}^{(o)}}{m} f_{mo}^{(1)} \qquad (9\text{-}13b)$$

Equation (9-9) through (9-13) then define the entire sequence of the $f_{nr}^{(1)}$ expressed in terms of the $f_{mo}^{(1)}$ provided that the $G_m^{(o)}$ and $f_{mq}^{(o)}$ are given. If it is decided that the highest Hermite index needed in the representation is Q, then $f_{n,Q+1}^{(1)}$ may be set to zero and the problem has been formally reduced to a matrix equation of the form

$$A_{\alpha\beta}(Q) f_{\beta o}^{(1)} = 0 \qquad (9\text{-}14)$$

where the diagonal elements of $A_{\alpha\beta}$ can be seen to be polynomials in ω, each of degree Q. Even though equation (9-14) is not of the standard eigenvalue form, the vanishing of the determinant of $A_{\alpha\beta}$ will nevertheless yield the allowed (complex) eigenfrequencies ω. As might be expected, the actual implementation of the scheme outlined here is a computational problem of enormous proportions since typical values of Q and N_F are 160 and 10, respectively, for the range of problems treated in this thesis. Without the availability of an extremely high speed, high precision computing machine the problem would not in any sense be practical.

In order to establish some feeling for the structure of the problem and its relation to the simpler problem of equation (9-7), equation (9-14) will now be written for the case $Q = 1$ (corresponding to the retention of two terms in the Hermite series), and $N_F = N = M = 2$. The result is (for $\mu = 1$)

$$\begin{bmatrix} 0 & \frac{4}{k^2} G_1^{(o)} & \frac{2}{k^2} G_2^{(o)} \\ \frac{1}{k^2} G_1^{(o)} & \frac{\omega^2}{k^2} - 1 + G_2^{(o)} - \frac{f_{00}^{(o)}}{k^2} + \frac{f_{20}^{(o)}}{k^2} & -\frac{G_1^{(o)}}{k^2} - \frac{1}{2k^2} f_{10}^{(o)} \\ \frac{1}{2k^2} G_2^{(o)} & \frac{1}{2k^2} G_1^{(o)} - \frac{1}{2k^2} f_{10}^{(o)} & \frac{\omega^2}{4k^2} - 1 - \frac{1}{4k^2} f_{00}^{(o)} \end{bmatrix} \begin{bmatrix} f_{00}^{(1)} \\ f_{10}^{(1)} \\ f_{20}^{(1)} \end{bmatrix} = 0 \tag{9-15}$$

In order to facilitate the discussion some results to be obtained later will be anticipated here. First, if ϵ is a parameter less than unity which characterizes the degree of spatial inhomogeneity in the plasma, then $G_n^{(o)}$ and $f_{no}^{(o)}$ are proportional to $\left(\frac{\epsilon}{2}\right)^n$. Also for the cases treated in this work $f_{00}^{(o)} = 1$. An immediate consequence then is that as $\epsilon \to 0$ so that the unperturbed plasma approaches the homogeneous state, the off-diagonal elements of the matrix of (9-15) tend to zero. It may be observed also that $f_{00}^{(o)}$ in the column vector, if it were nonzero, would represent a change in the average density of the electrons from that in the unperturbed plasma. Since it is desired that the total number of particles in the system not be affected by the perturbation, then one must require $f_{00}^{(1)} \equiv 0$ and correspondingly delete the first row and column of the matrix. It is important to note that such deletion must take place only after (9-15) is obtained rather than at the outset of the recursion scheme, since the $f_{0q}^{(1)}$ are not necessarily zero for $q \neq 0$. Thus equation (9-15) must be reduced to

$$\begin{bmatrix} \frac{\omega^2}{k^2} - 1 + G_2^{(o)} - \frac{1}{k^2} + \frac{f_{20}^{(o)}}{k^2} & -\frac{G_1^{(o)}}{k^2} - \frac{1}{2k^2} f_{10}^{(o)} \\ \frac{1}{2k^2} G_1^{(o)} - \frac{1}{2k^2} f_{10}^{(o)} & \frac{\omega^2}{4k^2} - 1 - \frac{1}{4k^2} \end{bmatrix} \begin{bmatrix} f_{10}^{(1)} \\ f_{20}^{(1)} \end{bmatrix} = 0 \tag{9-16}$$

which becomes as $\epsilon \to 0$

$$\begin{bmatrix} -\omega^2 + 1 + k^2 & 0 \\ 0 & -\omega^2 + 1 + 4k^2 \end{bmatrix} \begin{bmatrix} f_{10}^{(1)} \\ f_{20}^{(1)} \end{bmatrix} = 0 \tag{9-17}$$

The vanishing of the determinant here then determines the allowed values of ω. At this point, contact is made with the work of Grant (ref. 16) for the Fourier-Hermite analysis of the homogeneous plasma. From his work it can be seen that the dispersion relation for a mode of wavelength k in that case for which two Hermite terms are retained is simply

$$\omega^2 = 1 + k^2 \tag{9-18}$$

The present method then contains the entire determinant of Grant's method in each of its diagonal elements, the first element corresponding to the dispersion relation for the fundamental k-mode and the other element to its first harmonic. In this homogeneous case the two modes obviously do not interact.

For larger values of Q and for larger numbers of Fourier components the problem is qualitatively similar to the above case. The principal differences are that the diagonal elements add more complicated ϵ-dependent terms and that the off-diagonal elements begin to depend on ω in addition to containing higher and higher powers of ϵ as their distance from the main diagonal increases. The structure of the overall determinant becomes so complicated with large Q that even the 14 digit precision of the Langley Research Center computers is unable to push ϵ much beyond 0.1. Here, just as in every other attempt reported in the literature, one finds that the inhomogeneous plasma does not easily yield its secrets.

9.3 Description of the Inhomogeneous States

Attention will now be turned to the specific details of the $F^{(o)}(x,v)$ to be treated. It is convenient to choose a case for analysis which not only yields intrinsically interesting results but which may also be readily compared with other work in the literature, particularly the above-mentioned work of Grant and of Jackson and Raether. A suitable choice for the unperturbed distribution function is

$$F^{(o)}(x,v) = A_o(\alpha U + \beta)e^{-U} \tag{9-19}$$

where $U = \frac{v^2}{2} - \Phi(x)$ and

$$\Phi^{(o)}(x) = \ln g(x) + B_o \tag{9-20}$$

where $g(x) > 0$ is the periodic function

$$g(x) = 1 + \epsilon \cos \frac{2\pi x}{L} . \tag{9-21}$$

A_o and B_o are constants and L is the length of the plasma system, which is supposed to be confined between perfectly reflecting walls at $x = 0$ and $x = L$. When $\alpha = 0$ and $\beta = 1$ the present case reduces to the unperturbed distribution function of Jackson and Raether. When $\alpha = 1$ and $\beta = 0$ the present case is a generalization of the maximally unstable two-stream case treated by Grant, and it reduces to his case as $\epsilon \to 0$.

Noting that

$$g(x) = 1 + \frac{\epsilon}{2}\left(e^{ikx} + e^{-ikx}\right) \tag{9-22}$$

where $k = \frac{2\pi}{L}$, one may write

$$E^{(o)}(x) = -\frac{d\Phi^{(o)}(x)}{dx} = -\frac{1}{g(x)}\frac{dg(x)}{dx}$$

$$= -\frac{ik\epsilon/2}{1 + \frac{\epsilon}{2}\left(e^{ikx} + e^{-ikx}\right)}\left(e^{ikx} - e^{-ikx}\right) \tag{9-23}$$

and using the binomial expansion on the term in the denominator

$$E^{(o)}(x) = -ik\frac{\epsilon}{2}\left(e^{ikx} - e^{-ikx}\right)\sum_{m=0}^{\infty}(-1)^m\left(\frac{\epsilon}{2}\right)^m\left(e^{ikx} + e^{-ikx}\right)^m \tag{9-24}$$

or

$$E^{(o)}(x) = -ik\sum_{m=0}^{\infty}\sum_{n=0}^{m}(-1)^m\left(\frac{\epsilon}{2}\right)^{m+1}\binom{m}{n}\left(e^{ik(m+1-2n)x} - e^{ik(m-1-2n)x}\right) \tag{9-25}$$

where $\binom{m}{n} = m!/n!(m-n)!$, the binomial coefficient. The Fourier coefficients of $E^{(o)}(x)$ (or equivalently the $G_m^{(o)}$ of equation (9-11) may be extracted directly from this equation after deciding upon the number of terms to be retained in the representation. The present work considers the retention of up to 20 Fourier components of $E^{(o)}$, a sufficient number to guarantee the accuracy of the representation. It can be noted that $G_m^{(0)} = -G_{-m}^{(0)}$.

The Fourier-Hermite representation of $F^{(o)}(x,v)$ also proceeds in this manner. For the $\alpha = 0$, $\beta = 1$ case one has

$$F_M^{(o)}(x,v) = A_o e^{-U} = A_o e^{B_o} e^{ln\, g(x)} e^{-\frac{v^2}{2}} \tag{9-26}$$

with the capitol M subscript merely distinguishing this Maxwellian case from the two-stream case considered later. Then

$$F_M^{(o)}(x,v) = A_o e^{B_o}\left[1 + \frac{\varepsilon}{2}\left(e^{ikx} + e^{-ikx}\right)\right] e^{-\frac{v^2}{2}} \tag{9-27}$$

B_o is chosen such that the minimum of the electron potential energy curve $- \Phi(x)$ is zero, giving $B_o = - ln(1 + \varepsilon)$ and A_o is then chosen such that $A_o e^{B_o} = 1/\sqrt{2\pi}$. One then finds $f_{00}^{(o)} = 1$ and $f_{-1,0}^{(o)} = f_{1,0}^{(o)} = \frac{\varepsilon}{2}$. This case, which goes to the pure Maxwellian velocity distribution as $\varepsilon \to 0$, is precisely the problem treated by Jackson and Raether. As will be seen later, however, the present method is capable of, and in fact is best suited for, treating wavelengths of the order of a few Debye lengths rather than the long wavelength regime of Jackson and Raether.

The distribution function

$$F_{TS}^{(o)}(x,v) = A_1(U + \alpha)e^{-U} = A_1 e^{B_o}\left[\frac{v^2}{2} + \alpha - B_o - ln\, g(x)\right] g(x) e^{-\frac{v^2}{2}} \tag{9-28}$$

goes to

$$F_{TS}^{(o)}(x,v) = A_1 v^2 e^{-\frac{v^2}{2}} \qquad (9\text{-}29)$$

for $\alpha = 0$ and $\epsilon \to 0$, which is precisely the maximally unstable two-streaming case treated by Grant, and which is qualitatively similar to the two-streaming distribution function of Buti and Trehan (ref. 34). Substituting the expressions for $g(x)$ and B_o, and putting $\alpha = 0$ yields

$$F_{TS}^{(o)} = \frac{A_1}{1+\epsilon}\left[\frac{v^2}{2} + \ln(1+\epsilon) - \ln\left[1 + \frac{\epsilon}{2}\left(e^{ikx} - e^{-ikx}\right)\right]\right] \cdot \left(1 + \frac{\epsilon}{2}\left(e^{ikx} - e^{-ikx}\right)\right) e^{-\frac{v^2}{2}} \qquad (9\text{-}30)$$

A_1 is to be chosen such that $f_{00}^{(o)} = 1$. The second logarithmic term may be expanded in its Taylor series around unity, and each resulting term may be replaced by its binomial expansion to get

$$F_{TS}^{(o)}(x,v) = \frac{A_1}{1+\epsilon}\left[\frac{v^2}{2} + \ln(1+\epsilon) - \sum_{m=0}^{\infty}\sum_{n=0}^{m}\left(\frac{\epsilon}{2}\right)^n \binom{m}{n} e^{ik(m-2n)x}\right] \cdot \left(1 + \frac{\epsilon}{2}\left(e^{ikx} + e^{-ikx}\right)\right) e^{-\frac{v^2}{2}} \qquad (9\text{-}31)$$

Again, after selecting the number of terms to be retained (terms $\sim \left(\frac{\epsilon}{2}\right)^{20}$ in the present work) the Fourier-Hermite coefficients may be extracted from this equation.

9.4 Effects of Adding a Collision Term

The Fourier-Hermite method suffers from the restriction that for a Maxwellian velocity distribution in the absence of collisions between particles the eigenfrequencies of equations (9-7) or (9-14) are all real, so that the method does not directly display the well-known phenomenon of Landau damping. Grant (ref. 16) showed, however, that the introduction of a small collision term through equation (9-2) modified the eigenfrequencies in such a way as to bring them into agreement with the Landau theory of Lenard and Bernstein (ref. 35). The adoption of collisions through this equation in the present work likewise allows the recovery of Landau damping in the inhomogeneous plasma, thus providing a method for computing the changes in Landau damping as a function of the degree of inhomogeneity of the system.

The use of the Fourier-Hermite expansion in equation (9-2) and the addition of the latter to the right side of equation (9-5a) makes it possible to add the effects of the collision term to the recursion relations (9-13). Letting the additional subscript c refer to the quantities with collisions taken into account, equations (9-13a) and (9-13b) become

$$f_{or\,c}^{(1)} = \frac{1}{i\omega + B\sqrt{(r-1)(r-2)}} \left[f_{or}^{(1)} + B(r-2) \sum_{m=-M}^{M} f_{mo}^{(o)} f_{m,r-2}^{(1)} \right] \tag{9-32a}$$

and

$$f_{nr\,c}^{(1)} = f_{nr}^{(1)} + \frac{iB}{n\mu k\sqrt{r}} \left[\sqrt{r(r-1)}\, f_{n,r-1}^{(1)} - (r-1) \sum_{\substack{m=n-M \\ m\neq n}}^{n+M} f_{n-m,0}^{(o)} f_{m,r-3}^{(1)} \right] \tag{9-32b}$$

Equation (9-9) remains unchanged. Equation (9-10a) is modified to

$$f_{02\,c}^{(1)} = \frac{1}{i\omega + B\sqrt{6}}\, f_{02}^{(1)} + 2B \sum_{m=-M}^{M} f_{m,0}^{(o)} f_{mo}^{(1)} \tag{9-33a}$$

while equation (9-10b) becomes

$$f_{n2\,c}^{(1)} = f_{n2}^{(1)} - \frac{i\omega B}{n^2 \mu^2 k^2}\, f_{no}^{(1)} \tag{9-33b}$$

With these changes the scheme outlined in section 9.2 may be carried out as before, not only for the Maxwellian case but also for the two-stream case.

Buti and Trehan have recently carried out calculations for the effect of the collision term (9-2) upon the eigenvalues of a two-stream situation consisting of two displaced Maxwellian distributions in a homogeneous plasma. They asserted the startling result that the addition of collisions increases the growth rate of the two-stream instability over its collisionless rate. Such a result contradicts the time domain analyses of Tidman and Weiss (ref. 32) and May (ref. 33), who found that collisions, whether treated by the method of equation (9-2) or by another, simpler method, always tend to suppress the growth of the instability. The present frequency domain analysis is qualitatively very similar to the treatment of Buti and Trehan, so that meaningful comparison can be made between their results and those of the present method for the homogeneous case. It is found here that the effect of collisions is to lower the growth rate of the instability, a result which favors the earlier results of Tidman and Weiss and May, and which tends to cast doubt on the Buti and Trehan results.

9.5 Details of the Solution

Some comment concerning the types of modes for $f_{mo}^{(1)}$ (and hence for $E_m^{(1)}$) which the present method supports must be given here. So far nothing has been said about the character of the eigenvectors $f_{\beta o}^{(1)}$ of equation (9-14) except that $f_{OO}^{(1)}$ must be zero. With $f_{OO}^{(1)}$ being zero it is then necessary to suppress the first row and column of the matrix $A_{\alpha\beta}$ of that equation in order that solutions other than the null solution shall exist. In the limit $\epsilon \rightarrow 0$ the reduced matrix is diagonal with the n^{th} diagonal elements corresponding to the homogeneous plasma dispersion relation for a pure Fourier mode of wavenumber $nk^{(1)} = n\mu k$. Thus any, or all, of the $f_{mo}^{(1)}$ components may have an arbitrary value in this limit. It is fruitful, however, to allow only the component $f_{10}^{(1)}$ to have a value different from zero since it is only in this way that the effects of adding a small inhomogeneity to the system can be observed to bring into play the harmonics of $f_{10}^{(1)}$ in a consistent way.

To make this clearer suppose that the m^{th} element of the column vector is unity for $\epsilon = 0$ and that all other components are zero. Then the m^{th} element of the diagonal of the reduced matrix must vanish, yielding an eigenfrequency corresponding to the homogeneous plasma mode of wavelength $m\mu k$. When ϵ is increased slightly from zero it is found that the other components $f_{ro}^{(1)}$ take on values proportional to $\left(\frac{\epsilon}{2}\right)^{|m-r|}$. While this is a perfectly legitimate solution of the equations one can see that the physical situation represented by such a solution cannot be expected to be well represented by the overall method. The reason for this is that only a finite number of Fourier components can be retained and in order that the representation be a good one it is necessary that the neglected components would have had negligibly small amplitudes had they been retained. Thus only for $m = 1$ in the above example is one certain that the amplitudes of the higher harmonics fall to zero in the appropriate way. Accordingly all of the solutions reported here are such that they approach the pure Fourier mode μk as $\epsilon \to 0$. This, of course, is no restriction on the method since μ may be chosen such that the fundamental wave number of the perturbation is any number times the fundamental wave number of the background plasma, subject only to certain boundary conditions which will now be discussed.

Since it is desired that the equations describe a plasma system with perfectly reflecting walls at $x = 0$ and $x = L$, and since there is no surface charge on these walls, then $f_{mo}^{(1)}$, which is proportional to $iE_m^{(1)}$, must be purely real in order that the electric field shall have only sine terms in its Fourier expansion. The minimum value of $k^{(1)}$ is thus $\frac{1}{2}k$, and one sees then that permissible values of μ are $1/2$, 1, $3/2$, 2, and so on. The $\mu = 1/2$ mode corresponds to the lowest mode of the Jackson and Raether analysis.

The actual computing procedure involves the following steps: (1) ϵ is set to zero and the corresponding eigenfrequency is computed for the given μ by means of solving equation (9-7) by the method of Grant (ref. 16); (2) for a very small ϵ, say .001, the recursion scheme of equation (9-13) is implemented by means of an electronic computer until the desired number of Hermite terms Q has been reached, making use of the value of the eigenfrequency found in step one; (3) step two is repeated for the same ϵ but with the frequency value changed by an amount equal to 10^{-9} of its original value; (4) using the results of steps two and three a numerical estimate of the derivative of the determinant is made and is used to predict a new value of the frequency so as to drive the determinant to zero; (5) steps two through four are repeated until the value of the frequency is unchanged in the first ten significant figures, and this value of the frequency is tentatively called the desired root; (6) with this tentative root the value of the determinant is checked for smallness and the eigenvector corresponding to the root is computed; (7) if the eigenvector is of the appropriate form, i.e., if when normalized one finds $f_{10}^{(1)} = 1$ and $f_{mo}^{(1)}$ proportional to $\left(\frac{\epsilon}{2}\right)^m$, then this value of the frequency is the desired root; (8) a new value of ϵ is chosen and steps two through seven are repeated.

A similar procedure is followed for various values of the collision parameter B in that ϵ is held fixed throughout and B is changed slightly in steps two and eight above. It is found that the changes in ϵ and/or B must be kept very small in order that the method track the movement of the eigenfrequency correctly. If a step size in ϵ or B is too large then an eigenfrequency corresponding to one of the disallowed modes discussed above begins to be tracked. This is evidenced, of course, in the nature of the eigenvector as mentioned in step seven above. It is invariably found that the correct mode can be tracked with changing ϵ and B by using a six-point extrapolation method to predict the new frequency and by keeping the step size in ϵ or B sufficiently small, but the computing time required becomes very large for ϵ greater than say 0.1. Presumably the use of double precision computations with the given computer could alleviate this problem. $\epsilon = 0.1$ however is certainly large enough to regard the computations as significant.

The computer program used to carry out the present work is quite lengthy and rather complicated. It is written in Fortran IV and is intended for use on the Control Data Corporation series 6000 computers. A copy may be obtained from the author by writing to the NASA Langley Research Center, Hampton, Virginia.

X. DISCUSSION OF RESULTS

10.1 Results for the Maxwellian Velocity Distribution

Many of the properties of the Fourier-Hermite transformation pointed out by Grant (ref. 16) in his eigenfrequency studies of the homogeneous plasma are found to carry over to the inhomogeneous case. The central property is, of course, that all the eigenfrequencies for the Maxwellian velocity distribution are real in the absence of collisions. Grant showed that the superposition of these real eigenmodes correctly represents the temporal behavior of a homogeneous plasma for as long a time as one chooses provided that a sufficiently large number of Hermite functions is retained in the analysis. Further he found that the addition of a small collision frequency removes a kind of "degeneracy" in these real eigenfrequencies in that one of these frequencies moves away from the real axis in the complex frequency plane and intercepts the position of the "Landau pole" of the Lenard and Bernstein analysis (ref. 35), while the remaining frequencies move rapidly past the Landau pole and into the very highly damped portion of the plane. (The Landau pole is the least damped solution of the dispersion relation of the usual homogeneous plasma theory (ref. 31) and its movement in the complex frequency plane as a function of the collision frequency B of equation (9-2) was investigated by Lenard and Bernstein.) For the $\epsilon = 0$ case the present method gives exact agreement with the Grant results.

Figure 2 shows the change in the real frequency as a function of B for several values of ϵ and for several values of Q, the number of Hermite polynomials retained. The curves are for $k = 1.0$ and for the lowest mode, $\mu = 1/2$. The curves for $\epsilon = 0$ are precisely the same as those of the Grant analysis and the straight line to which the $\epsilon = 0$ curves are asymptotic is precisely the plot of the movement of the Landau pole of the Lenard and Bernstein work. Thus the vertical axis intercept of the uppermost dotted straight line is the value of the real frequency part of the Landau pole of the usual homogeneous theory. For the $k = 1.0$ value, which corresponds to a wavelength of 6.28 Debye lengths for the unperturbed inhomogeneous plasma, one sees that $Q = 39$ represents too few polynomials to see the asymptotic behavior for any reasonably small B. In general, the shorter the wavelength the more polynomials are required to obtain the desired behavior.

As can be seen from the figure the $B = 0$ values of the $\epsilon = 0$ curves are markedly different. There is no easily discernible way of predicting which, of the multiplicity of real frequencies which exist for given Q and $\epsilon = 0$, will be that frequency which tends toward the Lenard-Bernstein line. Only by following all eigenfrequencies in the neighborhood as they move with B and selecting the one whose imaginary part changes least with B is it possible to select the $B = 0$ eigenfrequencies which will show the behavior of figure 2.

There is at present no analysis indicating how the Lenard-Bernstein plot is altered when an inhomogeneous plasma is considered.

Observation of the curves obtained for $\epsilon > 0$ in figure 2 strongly suggests, however, that the effect is merely a downward displacement of the straight line. This conclusion is borne out for all the calculations made in the present work, and in every case the description of the inhomogeneity is assured by retaining more and more Fourier components until the results become insensitive to the number kept. It is with strong assurance of accuracy then that one may fit a least square straight line to the straight portions of the $\epsilon > 0$ curves of figure 2, and regard the vertical intercepts of these lines as the correct values of the real frequencies corresponding to the least damped eigenfrequencies (i.e., the "inhomogeneous plasma Landau pole") for the inhomogeneous cases. A plot of these real frequencies obtained in this way versus the degree of inhomogeneity ϵ is shown in figure 3 for the unperturbed distribution function of equation (9-27). The curve is a straight line with a slope of - 0.117.

Figure 4 shows a plot of the imaginary part of the eigenfrequencies γ versus B, again for $k = 1.0$, $\mu = 1/2$ and for the same values of Q. The $\epsilon = 0$ curves again correspond to the Grant results, with the straight line portion being the Lenard-Bernstein curve for the change in the Landau damping with B. The effect of $\epsilon > 0$ is to shift the straight line upward, so that fitting a least squares straight line to the straight portions of such curves for many values of ϵ and reading the vertical intercepts allows one to construct figure 5.

Figure 5 gives the collisionless damping of the least damped eigenfrequency as a function of ϵ. That is, it gives the change in the Landau damping with ϵ for that electrostatic oscillation having a fundamental wave number $\mu k = \frac{1.0}{2} = 0.5$. This figure, along with figure 3 giving the real frequency variation for the same case, is to be considered one of the basic contributions of this work, since there has been no other work reported which has been able to predict this variation for this important wavelength range.

For long wavelength situations the present method requires relatively few Hermite polynomials for a good description of the plasma. Thus $Q = 9$ represents the plasma just as well as $Q = 159$. Figure 6 shows a plot of the real frequency part of the least damped eigenfrequency for $k = 0.1$ and $\mu = 1/2$ versus ϵ. Also shown in the figure are the results of Jackson and Raether for this same case. For $\epsilon < 0.04$ their method agrees quite closely with the present method but some deviation occurs for the larger values. This difference can be attributed to their long-wavelength approximations being slightly in error even for this relatively long wavelength of 125 Debye lengths for the electrostatic oscillation.

A measure of the error of the long wavelength approximation for the $k = 1.0$ case can be seen from figures 7 and 8. Comparing the vertical intercepts and slopes of these curves with those of figures 3 and 4 shows that the Landau damping in the homogeneous plasma is overestimated by a factor of 3.3 and the real frequency underestimated by 6.5 percent by the Jackson-Raether approximation. Also the slopes of the curves are roughly twice as large in their approximation, thereby overestimating the effect of the inhomogeneity.

10.2 Results for the Two-Stream Velocity Distribution

The eigenfrequencies for the two-stream distribution of equation (9-30) are purely imaginary and correspond to pure exponential growth with time for the range of unstable wavenumbers. For the $\epsilon = 0$ case this range is 0 to 1.0 for the wavenumber $k^{(1)}$.

Figure 9 shows some typical results for the collisionless case. The upper curve for $\epsilon = 0$ agrees with Grant's corresponding case for $Q = 159$. As noted by him the curve is in slight error at the extreme right-hand end even for Q as large as this since the constraint of infinite slope where the curve meets the axis is an artificial one imposed by the Fourier-Hermite method for all Q. The curve may be considered completely accurate below $k^{(1)} = 0.8$.

As can be seen from equation (9-30), increasing ϵ from zero has two major effects. In addition to bringing the higher harmonics into play, it also has the effect of increasing the relative size of $f_{02}^{(o)}$ compared with $f_{00}^{(o)}$. The latter effect serves to lower the maximum unstable wavenumber as well as to decrease the growth rate for all $k^{(1)}$. The dashed curve on figure 9 shows this effect alone for $\epsilon = .03$, achieved by arbitrarily setting the higher Fourier harmonics to zero. The lower solid curve is the full solution for $\epsilon = .03$ taking both effects into account. For $k^{(1)}$ less than 0.35 the solid curve lies above the dashed curve while the reverse is true for $k^{(1)}$ greater than this value. Such behavior is reasonable when one takes into account that for small $k^{(1)}$ the addition of the harmonics of $k^{(1)}$ through ϵ serves to add components which have a higher growth rate than $k^{(1)}$, so that the composite effect is to raise the growth rate of the entire inhomogeneous mode above that of the fundamental. For higher $k^{(1)}$ the harmonics tend more and more to fall outside the range where unstable growth occurs with the resultant effect that the growth rate for the inhomogeneous mode is depressed.

The addition of the collision term to the analysis depresses the growth rate across the entire unstable range although the effect is more pronounced at the higher $k^{(1)}$. This is in direct contradiction with the results of Buti and Trehan (ref. 34) who used a similar collision term in computing the eigenfrequencies of a homogeneous plasma with two contrastreaming electron beams, each with a Maxwellian velocity distribution. They assert that their calculations show that collisions produce an enhancement of the growth rate although there are obvious discrepancies in some of their plotted curves. Tidman and Weiss (ref. 32) did an earlier calculation, again using essentially the same collision term, but found that the presence of collisions tended to suppress the growth rate. May (ref. 33) repeated some of the calculations of Tidman and Weiss, using a simpler model for the collision term, and again found that the collisions opposed the growth of the instability. Figure 10 shows the results of the present method for the effect of the collision frequency B upon the growth rate of the instability for a wavenumber of 0.6 in the homogeneous case. There is seen to be a linear decrease in the growth rate for all values out to $B = 1.0$, the limit of the calculations.

The present results then tend to confirm the work of Tidman, Weiss, and May and to cast some doubt upon the Buti-Trehan results.

XI. CONCLUSIONS

The one-dimensional inhomogeneous plasma has been treated in several areas by different methods. It has been shown by means of an extremizing principle based upon the total system energy that the stationary rectangular energy distribution function is inaccessible in general from other initial states of the plasma. An integral equation method has been applied to this plasma model resulting in a Schroedinger-type differential equation for the electric field in the plasma having the frequency as an eigenvalue.

The Fourier-Hermite transformation of the appropriate Vlasov-Maxwell equations has yielded new results in the intermediate wavelength range in Maxwellian plasmas and has established this method as a powerful tool in the analysis of inhomogeneous plasmas. The effects of plasma inhomogeneities upon the growth rate of the two-stream instability in hot plasmas has been demonstrated for the first time, and the effect of adding a small collision term to the equations describing such a situation has been shown to tend toward stabilization. The latter result helps to shed light upon a disagreement in the current literature.

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XIV. APPENDIX

Derivation of Equations for Extremizing Function

This appendix applies the methods of the calculus of variations toward the extremization of integrals of the form

$$I = \int_{X_1}^{X_2} g(x,y,w)\,dx \tag{A-1}$$

subject to the condition that

$$C = \int_{X_1}^{X_2} y\,dx \tag{A-2}$$

where C is a constant. g is a known function of x, $y(x)$, and $w(x) = \int^{x} y(x)\,dx$. X_1 and X_2 are not fixed at the outset but instead it is required that $y(x)$ vanish at these end points. The goal is to establish necessary conditions which the smooth function $y(x)$ must satisfy in order that the integral I will be an extremum, i.e., a maximum, minimum, or inflection. Toward this end the one parameter families of integrals

$$I(\epsilon) = \int_{X_1(\epsilon)}^{X_2(\epsilon)} g(x,Y,W)\,dx \tag{A-3}$$

and

$$N(\epsilon) = \int_{X_1(\epsilon)}^{X_2(\epsilon)} Y \, dx \tag{A-4}$$

are constructed, where

$$Y(x, \epsilon) = y(x) + \epsilon \, \eta(x) \tag{A-5a}$$

$$W(x, \epsilon) = w(x) + \epsilon \, \xi(x) \tag{A-5b}$$

Here $y(x)$ is regarded as the (as yet unknown) actual extremizing function which passes through the (as yet unknown) actual end points x_1 and x_2, $\eta(x)$ is an arbitrary, differentiable function, ϵ is the parameter mentioned above, and $\xi(x)$ is the indefinite integral of $\eta(x)$. $Y(x, \epsilon)$ will be called a trial function and it can be seen that $Y(x, \epsilon)$ approaches $y(x)$ as ϵ approaches zero. The intersections of a given $Y(x, \epsilon)$ with the x-axis define the points X_1 and X_2. It can be seen that as $Y(x, \epsilon)$ approaches the actual $y(x)$, the points X_1 and X_2 approach the points x_1 and x_2, respectively, which are the actual end points.

By the usual arguments of the calculus of variations (ref. 25) one can now see that for given $\eta(x)$ the value of $I(\epsilon)$ in equation (A-3) will be an extremum when ϵ is set to zero. In order to take the constraint equation (A-4) into proper account the integral

$$I^*(\epsilon) = \int_{X_1(\epsilon)}^{X_2(\epsilon)} g^*(x,Y,W)\,dx \qquad \text{(A-6)}$$

is constructed, where $g^* = g + \lambda Y$ and λ is an as yet undetermined Lagrange multiplier. The condition for an extremum is then

$$\left.\frac{\partial I^*}{\partial \epsilon}\right|_{\epsilon=0} = 0 \qquad \text{(A-7)}$$

With the use of equation (A-5),

$$\frac{\partial I^*}{\partial \epsilon} = \frac{\partial X_2}{\partial \epsilon}\left[g^*\left(X_2, Y\left(X_2\right), W\left(X_2\right)\right)\right] - \frac{\partial X_1}{\partial \epsilon}\left[g^*\left(X_1, Y\left(X_1\right), W\left(X_1\right)\right)\right]$$

$$+ \int_{X_1}^{X_2}\left(\frac{\partial g^*}{\partial Y}\frac{\partial Y}{\partial \epsilon} + \frac{\partial g^*}{\partial W}\frac{\partial W}{\partial \epsilon}\right)dx \qquad \text{(A-8)}$$

Also, $\frac{\partial Y}{\partial \epsilon} = \eta$ and $\frac{\partial W}{\partial \epsilon} = \xi$, so that upon using these relations in equation (A-7) and letting $\epsilon = 0$, one obtains

$$\left.\frac{\partial I^*}{\partial \epsilon}\right|_{\epsilon=0} = \left.\frac{\partial X_2}{\partial \epsilon}\right|_{\epsilon=0}\left[g^*\left(x_2, y\left(x_2\right), w\left(x_2\right)\right)\right] - \left.\frac{\partial X_1}{\partial \epsilon}\right|_{\epsilon=0}\left[g^*\left(x_1, y\left(x_1\right), w\left(x_1\right)\right)\right]$$

$$+ \int_{x_1}^{x_2}\left(\frac{\partial g^*}{\partial y}\,\eta + \frac{\partial g^*}{\partial w}\,\xi\right)dx = 0 \qquad \text{(A-9)}$$

The end point conditions must now be examined in detail in order to evaluate $\left.\frac{\partial X_1}{\partial \epsilon}\right|_{\epsilon=0}$ and $\left.\frac{\partial X_2}{\partial \epsilon}\right|_{\epsilon=0}$.

Since $Y(X_1, \epsilon) = Y(X_2, \epsilon) = 0$ by definition, then

$$\frac{\partial Y(X_1, \epsilon)}{\partial \epsilon} = \frac{\partial Y(X_2, \epsilon)}{\partial \epsilon} = 0 \tag{A-10}$$

Thus the use of equations (A-5a) and (A-5b) gives

$$\frac{\partial w(X_1)}{\partial \epsilon} = \frac{\partial w}{\partial X_1}\frac{\partial X_1}{\partial \epsilon} = y'(X_1)\frac{\partial X_1}{\partial \epsilon} + \eta(X_1) + \epsilon\,\eta'(X_1)\frac{\partial X_1}{\partial \epsilon} = 0 \tag{A-11}$$

For $\epsilon = 0$ this becomes

$$y'(x_1)\left.\frac{\partial X_1}{\partial \epsilon}\right|_{\epsilon=0} + \eta(x_1) = 0$$

or

$$\left.\frac{\partial X_1}{\partial \epsilon}\right|_{\epsilon=0} = -\frac{\eta(x_1)}{y'(x_1)} \tag{A-12}$$

By similar reasoning,

$$\left.\frac{\partial X_2}{\partial \epsilon}\right|_{\epsilon=0} = -\frac{\eta(x_2)}{y'(x_2)} \tag{A-13}$$

Substitution of equations (A-12) and (A-13) into (A-9) and use of integration by parts in the first integral of (A-9) gives

$$- \frac{g^*\left(x_2, y\left(x_2\right), w\left(x_2\right)\right)}{y'\left(x_2\right)} \eta\left(x_2\right) + \frac{g^*\left(x_1, y\left(x_1\right), w\left(x_1\right)\right)}{y'\left(x_1\right)} \eta\left(x_1\right)$$

$$+ \xi \frac{\partial g^*}{\partial y}\Bigg|_{x_1}^{x_2} + \int_{x_1}^{x_2} \left[\left(\frac{\partial g^*}{\partial w} - \frac{d}{dx}\left(\frac{\partial g^*}{\partial y}\right)\right)\right] \xi \, dx = 0 \qquad \text{(A-14)}$$

This equation must hold for a rather wide class of functions $\eta(x)$ and $\xi(x)$ as described earlier. In particular, it must hold for those functions $\xi(x)$ which are not identically zero, but which are compatible with $\xi\left(x_1\right) = \xi\left(x_2\right) = \eta\left(x_1\right) = \eta\left(x_2\right) = 0$. For such a case one must have

$$\frac{\partial g^*}{\partial w} - \frac{d}{dx}\left(\frac{\partial g^*}{\partial y}\right) = 0 \,. \qquad \text{(A-15)}$$

Again, one may have η different from zero at one end point but zero at the other end point at the same time that ξ vanishes at both end points, and so forth. In this way one finds

$$\frac{g^*\left(x_2, y\left(x_2\right), w\left(x_2\right)\right)}{y'\left(x_2\right)} = 0 \,, \qquad \text{(A-16)}$$

$$\frac{g^*\left(x_1, y\left(x_1\right), w\left(x_1\right)\right)}{y'\left(x_1\right)} = 0 \,, \qquad \text{(A-17)}$$

$$\left.\frac{\partial g^*}{\partial y}\right|_{x=x_1} = 0 \ , \qquad \text{(A-18)}$$

and

$$\left.\frac{\partial g^*}{\partial y}\right|_{x=x_2} = 0 \ . \qquad \text{(A-19)}$$

Equation (A-15) through (A-19) along with the original constraint equation (A-2) (with x_1 and x_2 replacing X_1 and X_2) are thus the equations which determine $y(x)$, x_1, x_2, and λ.

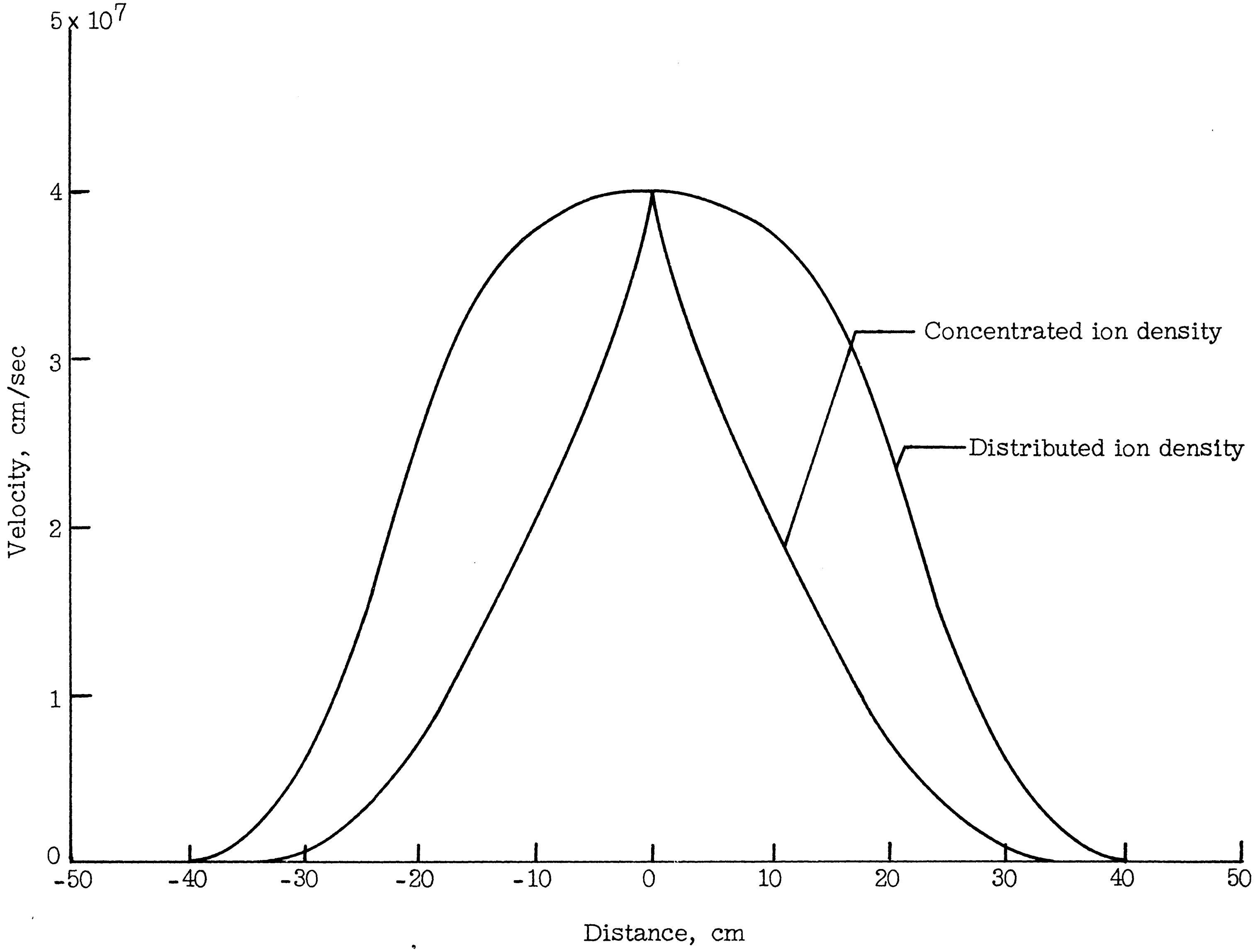

Figure 1.- V(x) curves for distributed and concentrated ion densities.

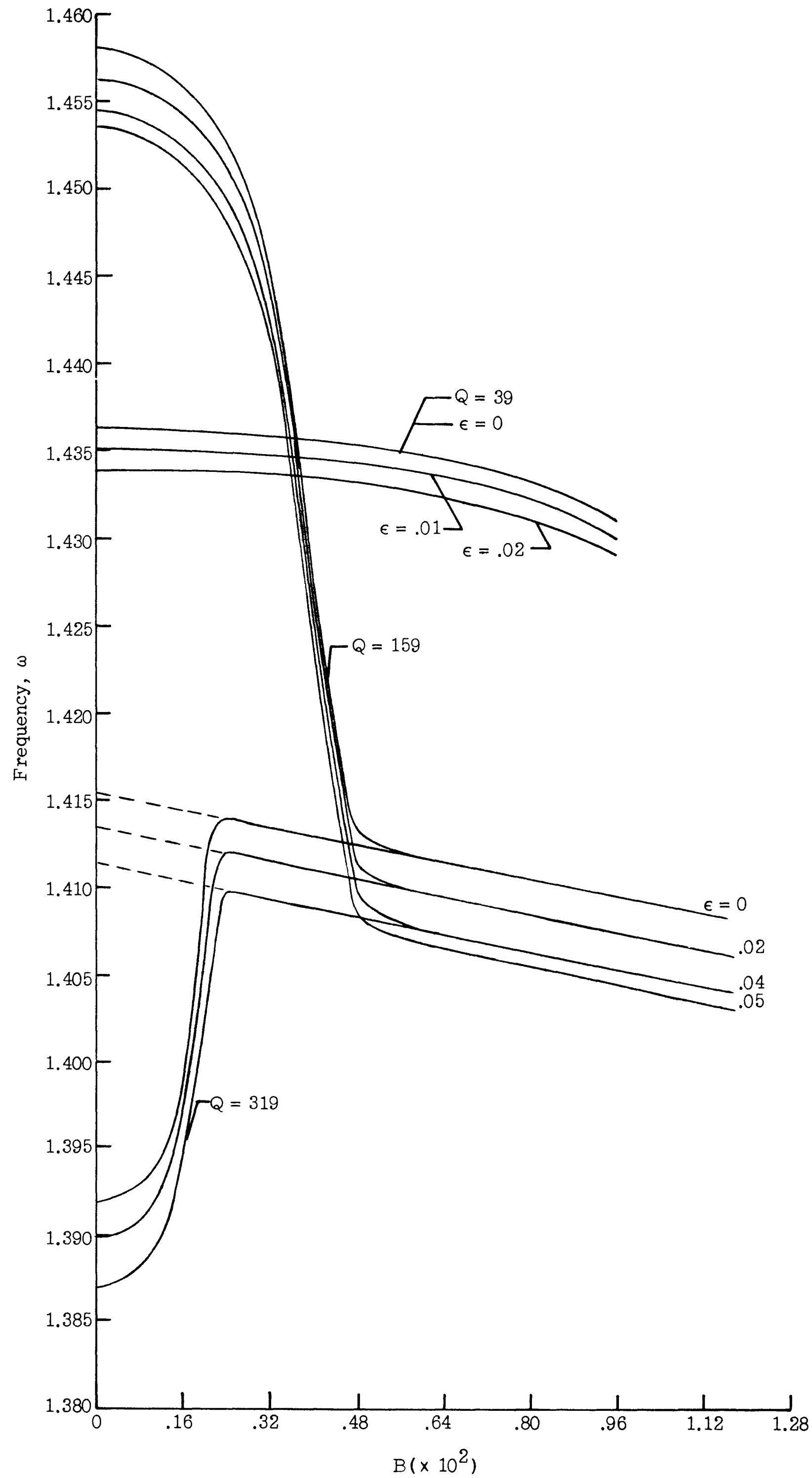

Figure 2.- Effect of collisions on real part of eigenfrequency.

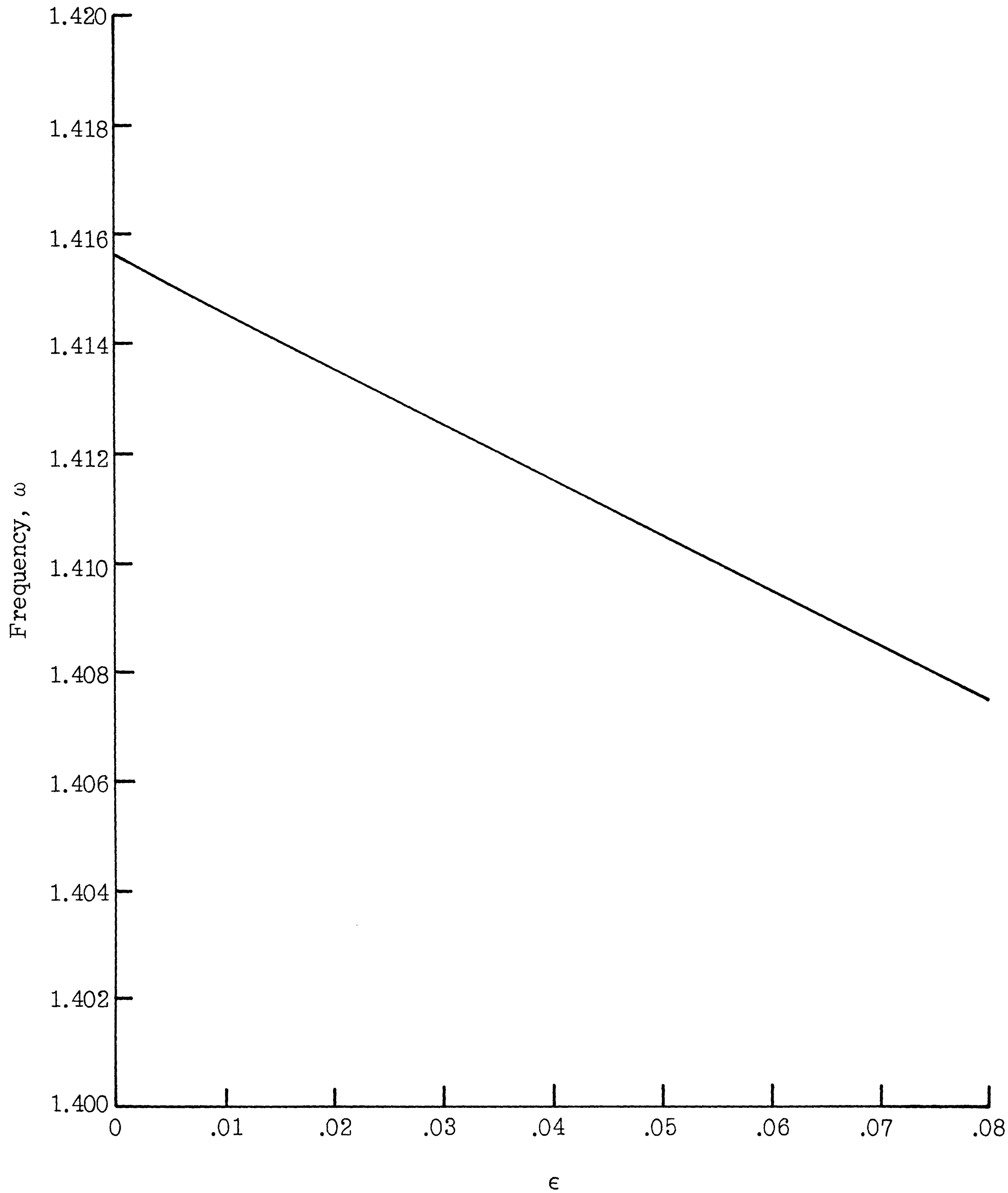

Figure 3.- Influence of plasma inhomogeneity on real part of eigenfrequency.

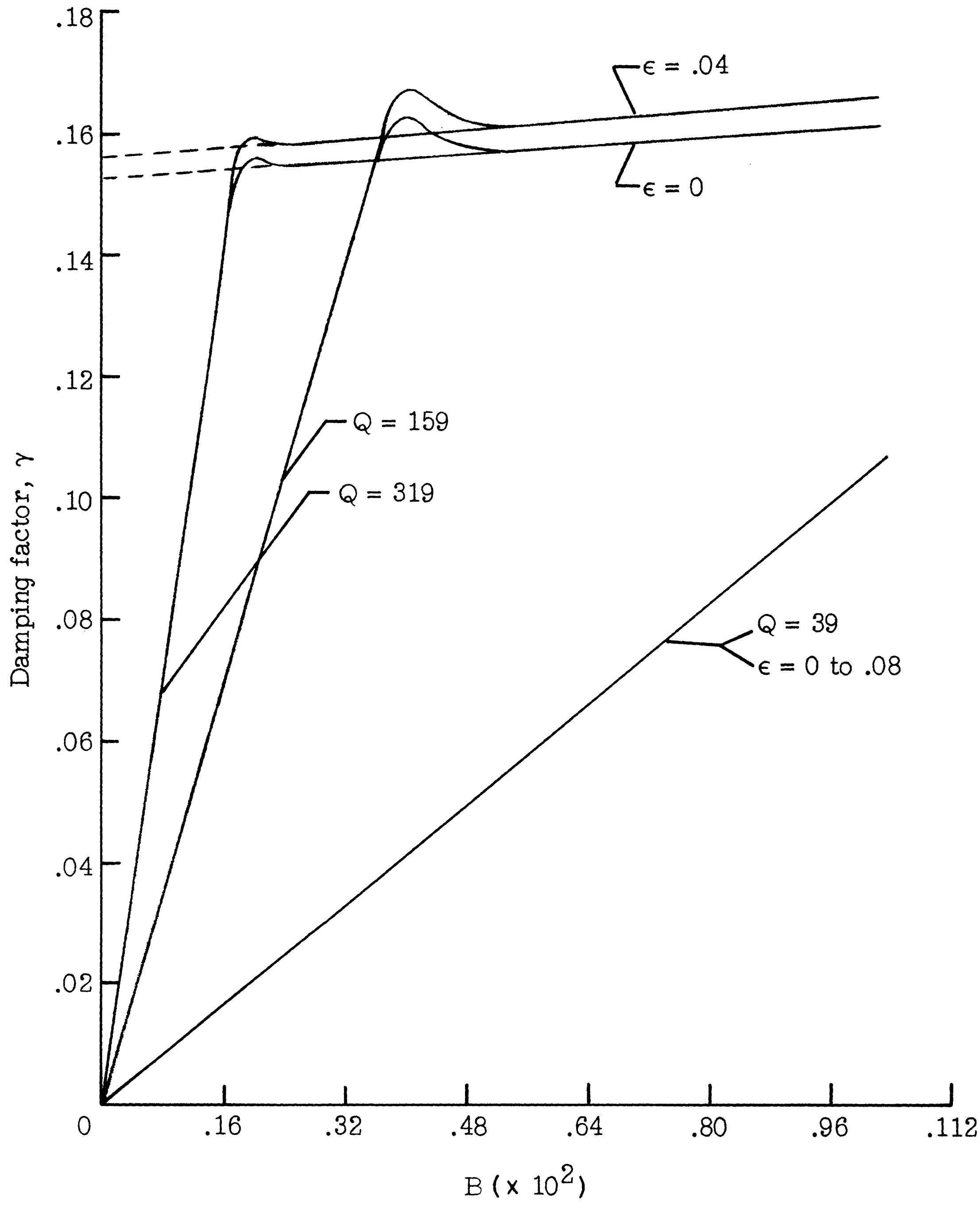

Figure 4.- Collisional damping for varying degrees of plasma inhomogeneity and different Hermite index cut-off values.

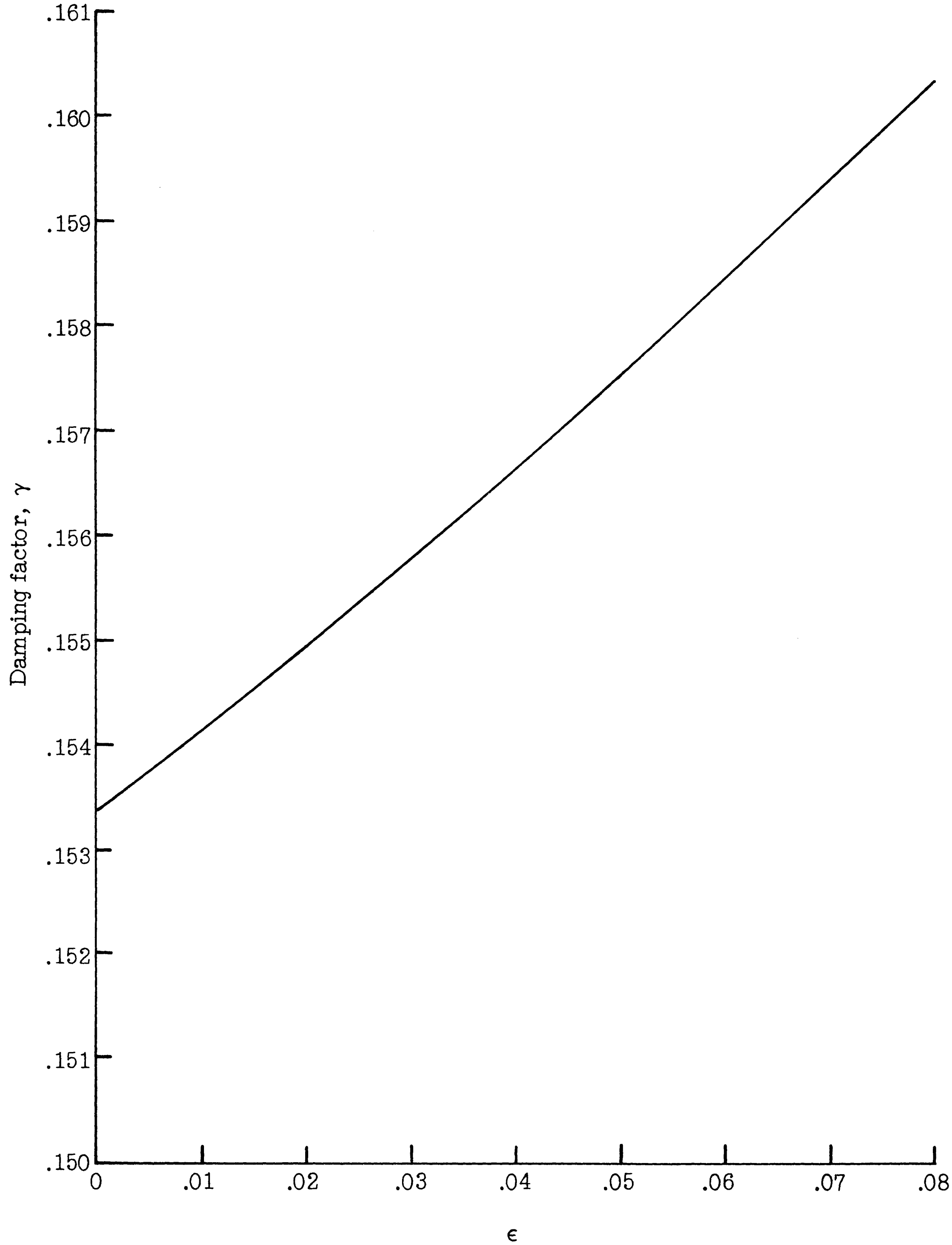

Figure 5.- Influence of plasma inhomogeneity on collisionless damping.

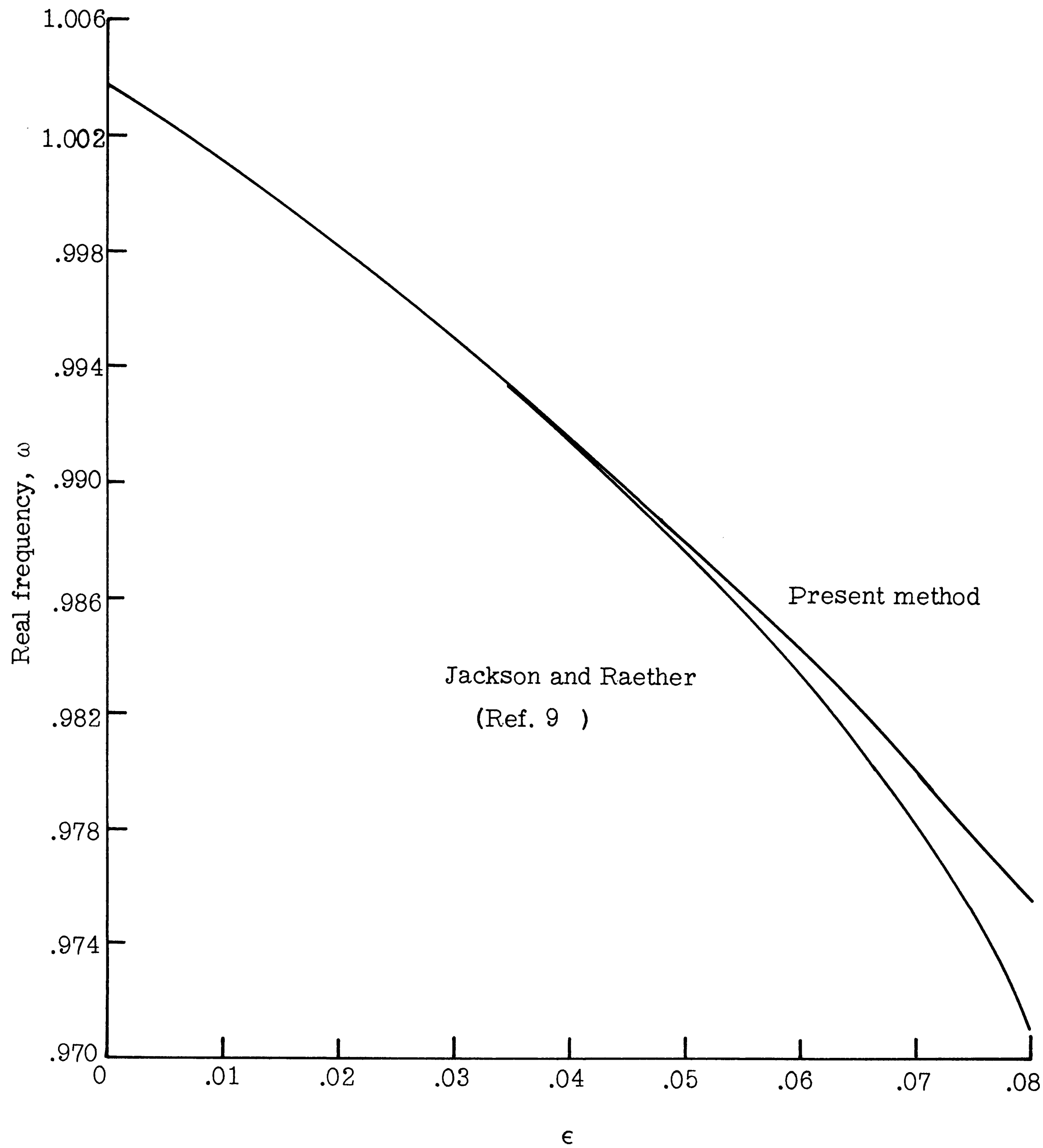

Figure 6.- Effect of plasma inhomogeneity upon real part of eigenfrequency for long wavelength case $k^{(1)} = 0.05$.

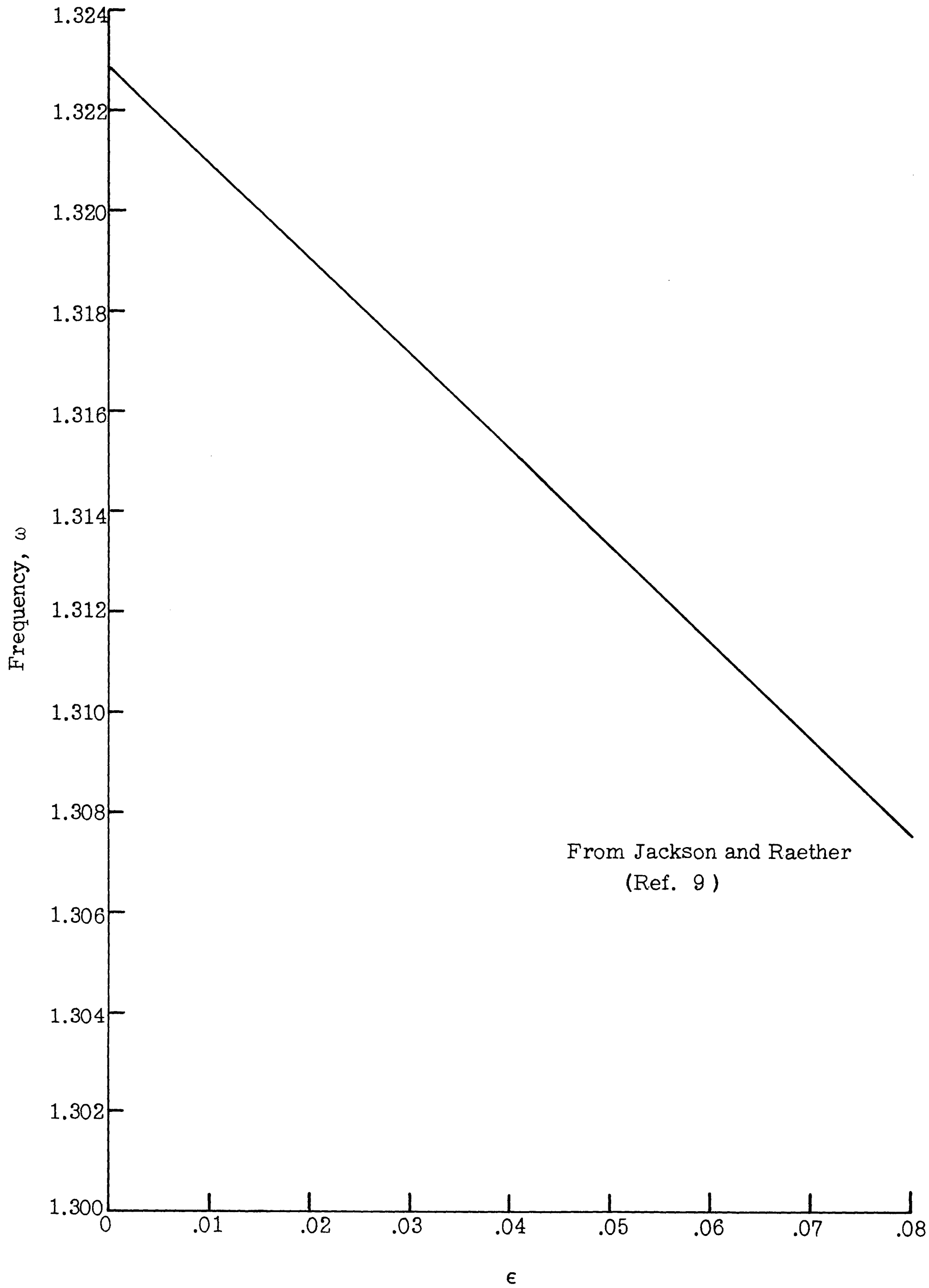

Figure 7.- Effect of plasma inhomogeneity upon real part of eigenfrequency for $k^{(1)} = 0.5$, using long wavelength approximation of reference 9.

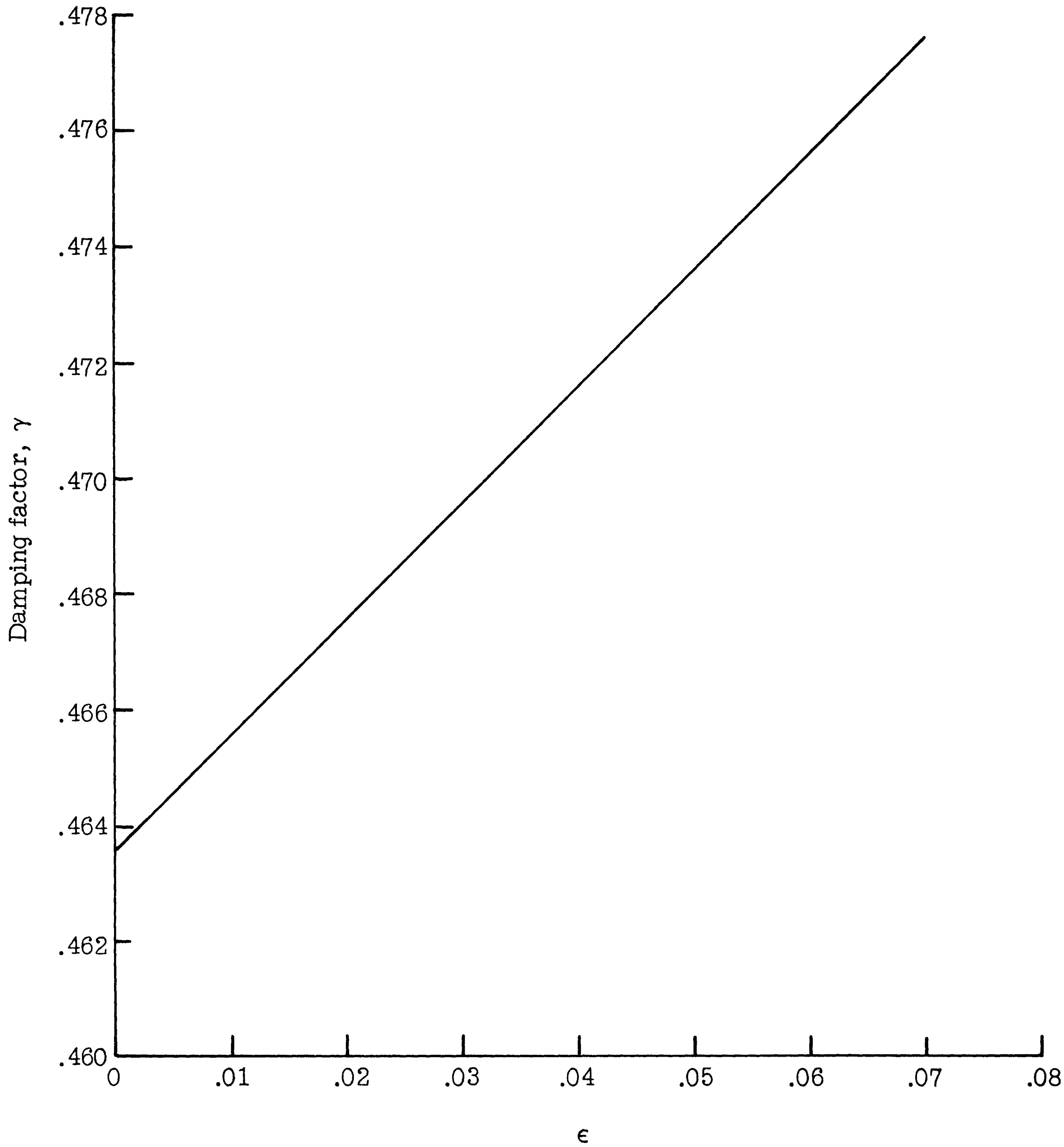

Figure 8.- Collisionless damping as a function of plasma inhomogeneity for $k^{(1)} = 0.5$, using long wavelength approximation of reference 9.

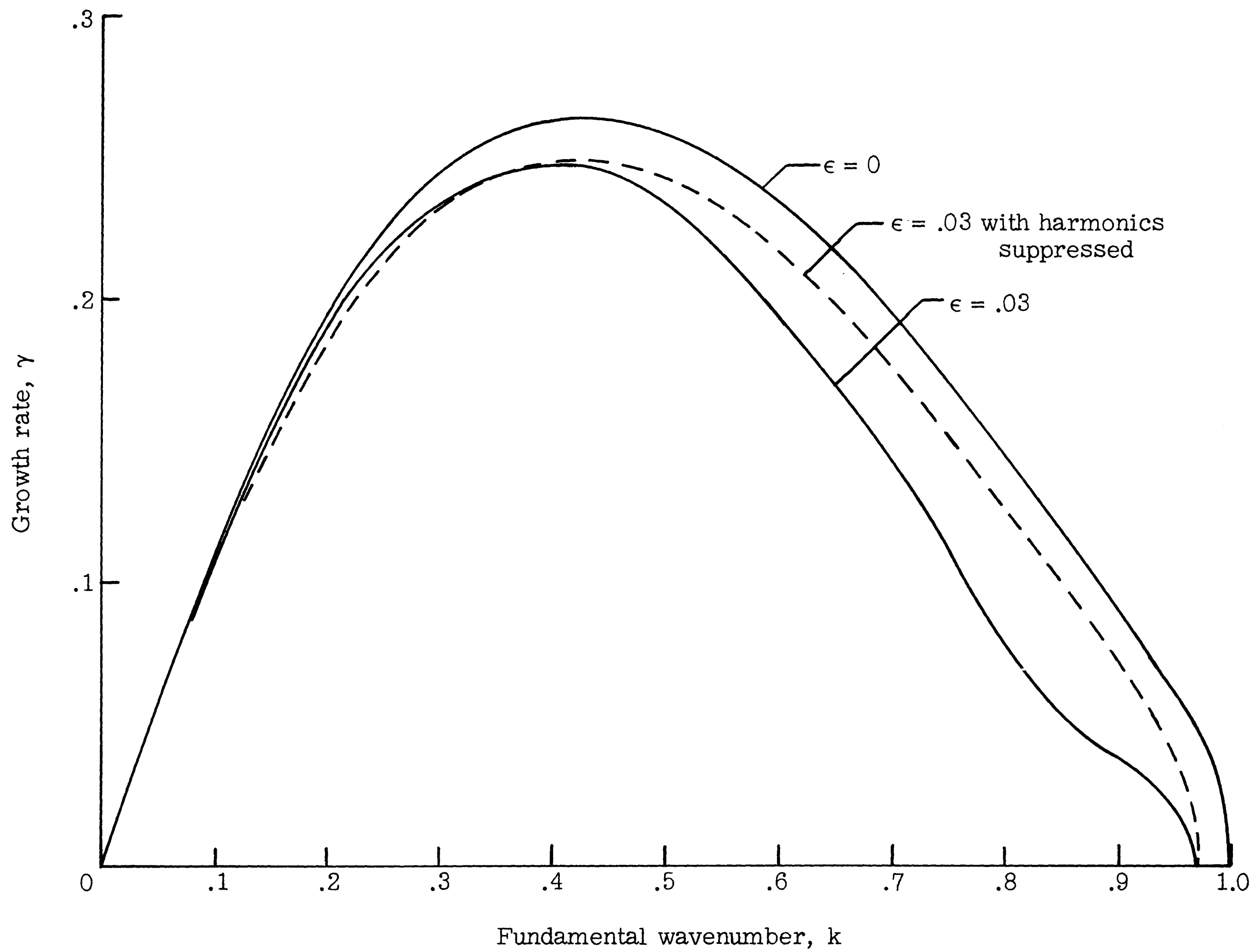

Figure 9.- Growth rate of two-stream instability versus fundamental wavenumber $k^{(1)}$ with Hermite index cut-off $Q = 159$.

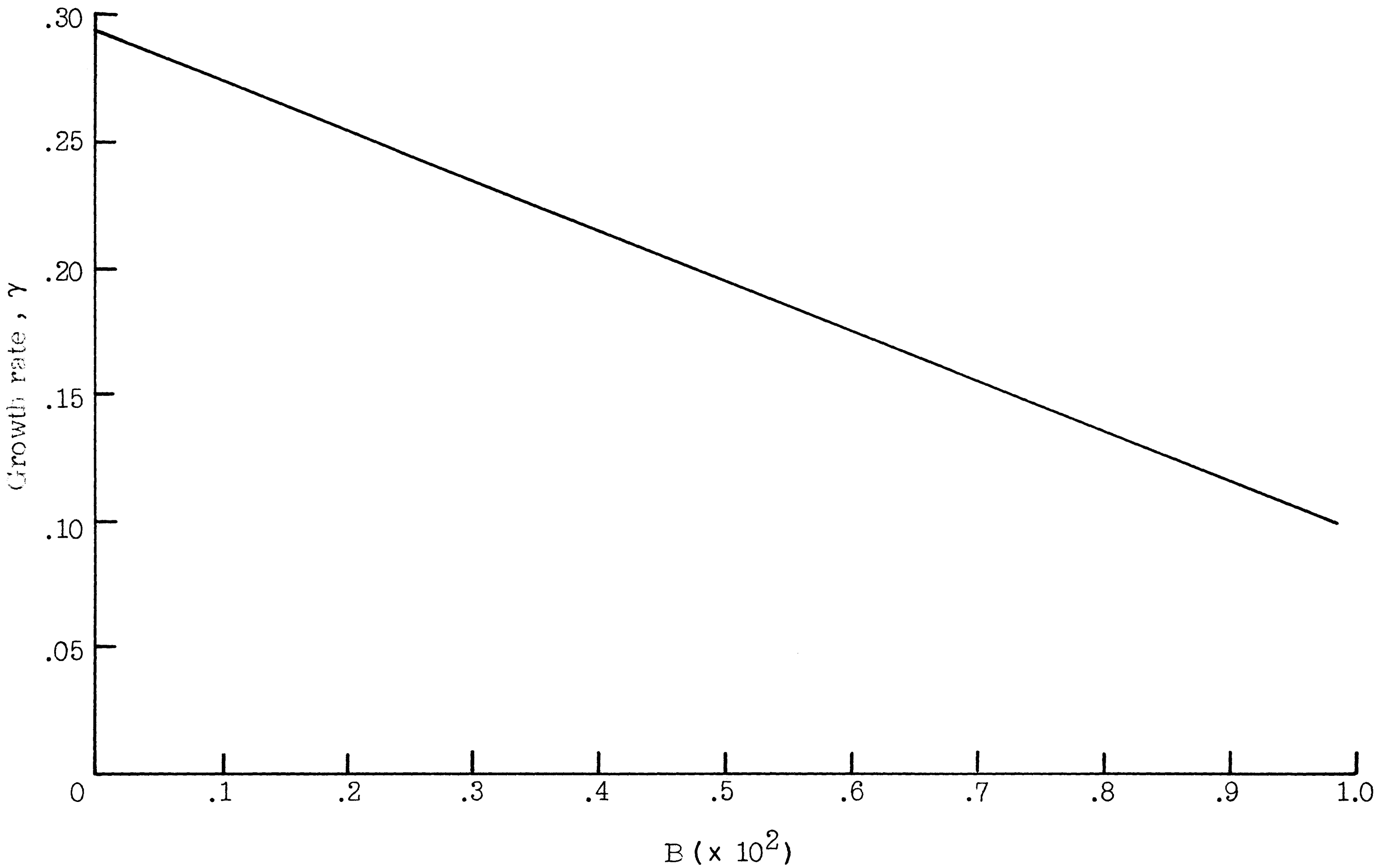

Figure 10.- Growth rate of homogeneous plasma two-stream instability in presence of collisions for $k^{(1)} = 0.6$.

ELECTROSTATIC OSCILLATIONS IN INHOMOGENEOUS PLASMAS

By

Leo Douglas Staton

ABSTRACT

The problem of longitudinal electrostatic oscillations in inhomogeneous, one-dimensional plasmas is examined in several areas. First the question is considered regarding whether the Vlasov-Maxwell equations will allow a plasma system to evolve naturally toward a preassigned stationary, inhomogeneous state such as one of the Bernstein-Greene-Kruskal modes. For the particular case in which the stationary state is charterized by an energy distribution function which is a constant for all energies up to a given maximum and zero for energies beyond, an extremization principle is developed which indicates that the total system energy is a minimum for the plasma. This stationary state then is inaccessible in general for a plasma prepared initially in a less restrictive manner. Further, such an inhomogeneous, stationary state is necessarily stable, a fact which is demonstrated directly by treating this case through an integral equation method. This integral equation is shown to degenerate into a Schroedinger-type differential equation, which is formally similar to the differential equation obtained in the hydrodynamic approximation for Maxwellian velocity distribution functions.

Electrostatic oscillations in more general cases are treated through the Fourier-Hermite transformation of the coordinate and velocity variables. A new recursion technique utilizing an electronic

computer is used to reduce the resulting algebraic equations to a homogeneous matrix equation whose column vector is proportional to the Fourier components of the electric field. The addition of a small collision term to the Vlasov equation facilitates the calculation of the Landau damping in the inhomogeneous plasma and allows the determination of the effects of collisions on the growth rate of the two-stream instability. The method is capable of treating arbitrary wavelengths, but works best for small degrees of inhomogeneity.

Results of the work show that the usual practice in the long-wavelength regime of dropping the stationary electric field term from the Vlasov equation has the effect of overestimating the Landau damping when applied to the intermediate wavelength range. The effect of inhomogeneity upon the growth rates of the two-stream instability is shown to be a relative increase in the growth rates for the smaller wavenumbers and a more pronounced relative decrease for the higher wavenumbers. Collisions are shown to depress the growth rate of this instability, in contrast to some other work reported in the literature.