$\overline{X}$ Control Charts in the Presence of Correlation

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(ABSTRACT)

In traditional quality control charts, fixed sampling interval (FSI) schemes are used where the time between samples has fixed intervals. More efficient methods called variable sampling interval (VSI) schemes have been developed where one takes the next observation sooner than usual if there is an indication that the process is operating off the target value.

Another traditional assumption behind most statistical process control charts is that the sequential observations are independent. However, there are many situations where the sequential observations should not to be treated as independent. Rather, a time series model, in particular the first order autoregressive (AR (1)) model, is appropriate. A Markov chain representation is used to study the properties of the FSI and VSI Shewhart $\overline{X}$ control charts.

First, the results show that if the process variance is properly estimated and if traditional control limits are used in the FSI control charts, then the detection time is shorter when the consecutive observations are negatively correlated than when they are positively correlated. If they are positively correlated, then the false alarm rate decreases as the correlation between consecutive observations increases. On the other hand, the detection time increases as the correlation increases.

In VSI control charts with traditional control limits, if the process mean is on or near the target, then the average time to signal (ATS) and average number of samples to signal (ANSS) tend to decrease as the correlation increases until the correlation becomes rather moderate. Then, for more highly correlated data, the ATS and ANSS tend to increase as the correlation increases.

Next, the results show that, even under the AR (1) process, the VSI chart is more efficient than the FSI chart in terms of ATS. In contrast, the VSI chart is less efficient than the FSI chart

in terms of ANSS. The efficiency (inefficiency) of ATS (ANSS) tends to decrease (increase) as the correlation between the consecutive observations becomes stronger.

Steady state ATS (ATS^*) and steady state ANSS ($ANSS^*$) under the AR (1) process show the same trend as the 'regular' ATS and 'regular' ANSS except when the deviation is very large. If the deviation is very large, then the VSI control chart does not seem to be more efficient than the FSI control chart in terms of steady state ATS.

If we have an AR (2) process, then for any given value of ϕ_2 a FSI control chart has a shorter detection time when ϕ_1 is negative than when ϕ_1 is positive. In a FSI control chart, the effect of positive ϕ_2 in addition to positive ϕ_1 is that the false alarm rate decreases even further and the detection time is even longer.

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Chapter I

INTRODUCTION

Control charts are used to monitor a process to detect changes that may produce deterioration in the quality of the output of the process. Samples are taken from a process, and some statistics which are computed from the samples are plotted on a control chart in time order. When monitoring a process in which a quality variable is normal, a sample mean can be used to keep track of the process mean and a sample variance can be used to monitor a process variance. In all control charts, there are two essential parts: a target value and control limits. Control limits are set up so that the fluctuations within the control limits could be readily explained by "chance causes." On the other hand, if a point falls outside the control limits, then an "assignable cause" of variation in the process has happened to change the process.

After each sample mean or sample variance is plotted on a control chart, the decision is made whether to signal that the process is potentially out of control and bring the process back into control or to take another sample at the next sampling period. Usually, the sampling intervals for control charts are fixed; that is, a fixed sampling interval (FSI) scheme is used for most control charts. But if there is an indication that the process is operating off the target value, then we may want to take the next observation sooner than usual because we want the detection time to be short when the process is actually out of control. This is called a variable sampling interval (VSI) scheme. The proposed scheme is to let the time to the next sample depend on what is observed in the current sample.

The usual assumption behind most statistical process control methods is that there are multiple observations, say 4 or 5, at each sampling point. However, there are many situations where the sample size is only 1: the production rate might be too slow to conveniently allow sample sizes greater than 1; it might be inconvenient to obtain more than one observation per sample; or it might also be that each observation itself is a mean of 4 or 5 observations.

Another traditional assumption in quality control charts is that the sequential observations are identically and independently distributed. But in practice, it is common to have correlation in the data; autocorrelations and other systematic time series effects are often substantial. In particular, serial correlation is a common feature in environmental, biological, and chemical data. We will consider the situation where the consecutive observations are correlated according to an autoregressive process, and investigate the properties of FSI and VSI Shewhart $\overline{X}$ control charts. In particular, we will concentrate on the situation where the sequential observations are from the first order autoregressive (AR(1)) process.

Our objective is to determine the robustness to correlation and to investigate how to design control charts in the presence of correlation.

Chapter II

LITERATURE REVIEW

2.1 Fixed Sampling Interval Control Charts

In this section, we review the development of the most widely used control chart, called the Shewhart control chart (1931), and discuss its advantages and disadvantages. The Shewhart $\overline{X}$ control chart is maintained by taking multiple observations, usually 4 or 5, at each sampling point, and plotting the sample mean on the control chart. The chart consists of the center line which is a target value μ_0, and control limits which are usually set at $\mu_0 \pm 3$ standard deviations. If a sample mean falls outside the control limits, then it is taken as an indication that the process mean is no longer equal to the target value μ_0, and a corrective action has to be taken. The $\overline{X}$ control chart is easy to use and good at detecting large shifts from the target value. However, since it uses the information only in the last sample and ignores all the information in the previous samples, it is relatively insensitive to small and moderate shifts. Some modifications of the chart have been proposed in order to correct this disadvantage. Page (1955) added warning lines within the control limits. His additional rule is that if r out of the last N sample means fall between the warning lines and control limits, then the process is deemed out of control. Another modification of the chart is adding supplementary runs rules which were proposed by Weiler (1953) and Moore (1958). They suggested giving an out of control signal if k of the last m sample means fall within a specified interval. Page (1955) showed that the $\overline{X}$ control chart with warning lines is more efficient than the standard Shewhart $\overline{X}$ chart in detecting small shifts in the process mean. Page (1962) also showed that the control charts with warning lines and those with runs rules perform similarly for small shifts in the process mean. However, for large shifts, the control charts with warning lines are more efficient than those with runs rules.

An alternative method of plotting sequential sample means is to plot the cumulative sums (CUSUM) of the deviations of the sample means from a target value in order to use all of the information in the sequence of sample means. The CUSUM control chart was proposed first by Page (1954). Recently, Champ and Woodall (1987) obtained exact run length properties of Shewhart control charts with supplementary runs rules using Markov chains, and compared the average run lengths with those of basic Shewhart $\overline{X}$ charts and CUSUM charts. It has been shown that supplementary runs rules cause the Shewart chart to be more sensitive to small shifts in the process mean, but not as sensitive as the CUSUM chart.

Another way of plotting sequential sample means is to give less weight to the older data , thus giving more importance to the more recent data. This scheme is called an exponentially weighted moving average (EWMA) which was first introduced by Roberts (1959). Using simulation to study the properties of an EWMA control chart, he showed that the EWMA control chart is not good at detecting large shifts but is good at detecting small shifts. There have been some studies on EWMA and on its applications: Box, Jenkins and MacGreger (1974), Muth (1960), Wortham and Henrick (1972), Robinson and Ho (1978), Hunter (1986), and Crowder (1987). Very recently, Lucas and Saccucci (1990) evaluated the properties of the EWMA scheme and its enhancements such as a fast initial response, a combined Shewhart EWMA control chart, and a robust EWMA using a Markov chain, and they concluded that the EWMA control schemes have average run lengths which are similar to those of CUSUM schemes.

2.2 Variable Sampling Interval Control Charts

If a sample mean falls close to a control limit, one would like to take another sample quickly rather than wait another usual delay for the next sample so that the detection time would be short. Arnold (1970) developed a Markov process proposing a sampling procedure which uses variable sampling intervals in order to study the water quality monitoring of streams. Crigler (1973) formulated an economically optimal sampling policy using Arnold's (1970) Markov process. Expressions for the variance of the sample size and for simplifying approximations to both the expected sample size and its variance have been developed by Smeach and Jernigan (1977). The problem of determining a sampling plan with variable time intervals between samples has also been investigated by Crigler and Arnold (1979, 1986). Hui (1980) and Hui and Jensen (1980) extended the variable sampling interval plans in the multivariate case. Expected sample size and its variance when one or more rejection regions are used have been derived. Reynolds and Arnold (1989) and Reynolds (1989) investigated the theoretical aspects of VSI schemes in detail, and showed that the VSI charts perform better than the standard FSI charts. They also suggest that only two sampling intervals need to be considered in the VSI control charts. Chengalur, Arnold and Reynolds (1987) considered the VSI feature when multiple parameters are monitored. Cui and Reynolds (1988) investigated the properties of the $\overline{X}$ charts with runs rules and VSI schemes using a Markov chain, and concluded that a VSI chart with runs rules is more efficient than a FSI chart with runs rules. Reynolds, Amin, Arnold, and Nachlas (1988) investigated the properties of $\overline{X}$ charts with variable sampling intervals such as the average time to signal and the average number of samples to signal, and concluded that the VSI control chart is substantially more efficient than the FSI control chart. Reynolds, Amin, and Arnold (1990) also investigated the properties of VSI CUSUM charts. Their results show that the VSI CUSUM chart is considerably more efficient than the standard CUSUM chart.

2.3 Time Series Model

It is often difficult to have sequentially independent observations that are appropriate for standard control charts. Sometimes the observations are serially correlated. One of the early works on the effect of autocorrelated data in CUSUM charts was a simulation study by Goldsmith and Withefield (1961). They used a model which is similar to AR (1). Johnson and Bagshaw (1974) and Bagshaw and Johnson (1975) also used the AR (1) and the first order moving average (MA (1)) processes in their study on the effect of correlation on CUSUM control charts. Vasilopoulos and Stamboulis (1978) assumed that observations are from an AR (1) process. Curves of the modified auxiliary quality control features were presented showing the substantial effect of dependence on the classical quality control factors. They demonstrated that the limits of an $\overline{X}$ control chart can be off by a wide margin when the correlation structure is not taken into account. Wu (1977) proposed a new modeling approach called dynamic data system (DDS) analysis. The DDS methodology uses the data that are obtained from the system in the form of a time series in order to develop a physically meaningful stochastic difference or differential equation so that the model could be used for system analysis and control or for predicting system behavior.

Statistical modeling and fitting of time series effects were proposed by Alwan and Roberts (1988). When the data suggests any systematic nonrandom pattern reflecting "common causes", a chart of fitted values without control limits based on an autoregressive integrated moving average model is suggested to provide guidance in seeking better understanding of the process. Then a standard chart with control limits to the residual series is used to detect "special causes". There is an important statistical reference by Box and Jenkins (1976) which describes discrete data process control extensively. Sahrmann (1979) proposed statistical methods to accept or reject a process when the data shows a significant amount of cyclic variation. Beneke, Leemis, Schlegel and Foote

(1988) developed a control chart which uses periodogram ordinates in order to detect cycles in the process mean. The simulation results show that the new chart is superior for detecting cyclic variations but not for detecting shifts in the process mean. Wineck (1988) showed that the exponential weighted moving average in statistical quality control can be viewed both as a statistical time series, namely AR(1), and as a low pass filter because it tends to pass low frequency signals and to block high frequency signals.

Chapter III

$\overline{X}$ CONTROL CHARTS UNDER AR(1) PROCESS

3.1 Basic Properties of FSI $\overline{X}$ Control Charts

Suppose that a Shewhart $\overline{X}$ control chart is used to detect a shift in the process mean. Let μ_0 be the target for the process mean. Suppose that a random sample of size m, $X_i = (X_{i1}, X_{i2}, \ldots, X_{im})$, is taken at each sampling point i where each X_{ij} is identically and independently distributed according to N (μ, σ^2). Then the average of this sample, $\overline{X}_i$, is distributed according to N$(\mu, \sigma^2/m)$. If $\mu = \mu_0$, then we say that the process mean is on target. Otherwise, we say that the process mean is off target. Suppose that Z is N(0, 1). Let $Z_{\alpha/2}$ be a value such that $P(Z < Z_{\alpha/2}) = \frac{\alpha}{2}$. Then the probability that any sample mean $\overline{X}$ will fall between

$$\mu_0 - Z_{\alpha/2}\frac{\sigma}{\sqrt{m}} \quad \text{and} \quad \mu_0 + Z_{\alpha/2}\frac{\sigma}{\sqrt{m}}$$

is 1 - α if the process mean is on target. It is customary to replace $Z_{\alpha/2}$ by 3 but we may denote $Z_{\alpha/2}$ by r sometimes. The control chart signals if $\overline{X} \notin (\mu_0 - r\frac{\sigma}{\sqrt{m}}, \mu_0 + r\frac{\sigma}{\sqrt{m}})$.

For a traditional FSI control chart, the run length is defined to be the number of samples required for the chart to signal and the average run length (ARL) is its expected value. If the process is in control, the ARL should be large because, in that case, the ARL can be interpreted as a false alarm rate. With $Z_{\alpha/2} = 3$, the ARL is 370.4 which means that the false alarm rate is 1 out of 370.4. On the other hand, if the process is not in control, the ARL should be small because, in that case, the ARL is the expected run length of the control chart without a signal even though the process is out of control. Since the run length could mean both the time to signal and the number of samples to signal, which we need to consider separately when a VSI scheme is used, we introduce a couple of new terms.

The random time and the random number of samples at which a control chart signals are called time to signal (T) and number of samples to signal (N), and their expected values are called average time to signal (ATS) and average number of samples to signal (ANSS). Let d be an interval for the FSI scheme. Then, for a FSI control chart, the time required to signal is simply the product of the run length and the length of the FSI chart; $T = dN$. If we let q be the probability that a sample mean $\overline{X}$ is out of control; that is, if $q = P(\overline{X} < \mu_0 - r\frac{\sigma}{\sqrt{m}}$ or $\overline{X} > \mu_0 + r\frac{\sigma}{\sqrt{m}})$, then since N is geometrically distributed,

$$E(T) = dE(N) = \frac{d}{q}$$

$$\mathrm{Var}(T) = d^2\frac{(1-q)}{q^2}.$$

3.2 VSI Control Charts

In a VSI control chart, the sampling interval to the next sample is long if the present sample mean is close to the target value, but short if it is off the target value but still within the control limits. Thus, the choice of a sampling interval for the next sample depends on what is observed at present (see Amin (1987, p.12) for detail). When evaluating the properties of a VSI control chart, ATS and ANSS are good criteria. Let R_i be the sampling interval used before the i^{th} sample mean $\overline{X}_i$ is taken. For simplicity, we assume that the chart is started at time 0. Then $T = \sum_{i=1}^{N} R_i$. Reynolds et. al. (1988) showed that, if η different sampling intervals $d_1, d_2, \ldots, d_\eta$ are used in a VSI control chart, then

$$E(T) = E(N)E(R_i) = \frac{1}{q}\sum_{j=1}^{\eta} d_j \frac{p_j}{1-q} \qquad [3.1]$$

and

$$Var(T) = \sum_{j=1}^{\eta} \frac{d_j^2 p_j}{q(1-q)} + \frac{(1-2q)\left(\sum_{j=1}^{\eta} d_j p_j\right)^2}{q^2(1-q)^2},$$

where $E(R_i)$ is calculated assuming that there is no signal, and p_j is the probability that a sampling interval d_j is used for a sample mean $\overline{X}$.

If we allow a fixed first sampling interval, say d_0, for a fair comparison between FSI and VSI control charts, then we have $T = d_0 + \sum_{i=2}^{N} R_i$. In this case,

$$E(T) = d_0 + \frac{1}{q} \sum_{j=1}^{\eta} d_j p_j$$

and

$$Var(T) = \sum_{j=1}^{\eta} \frac{d_j^2 p_j}{q} + \frac{(1-2q)(\sum_{j=1}^{\eta} d_j p_j)^2}{q^2(1-q)^2} .$$

Reynolds (1989) and Reynolds and Arnold (1989) showed that if the consecutive observations are independent, then the VSI control charts with lower numbers of different sampling intervals are more efficient. An example of the VSI control chart with only two sampling interval lengths d_1 and d_2 is shown in Figure 1. The interval $(\mu_0 - r\frac{\sigma}{\sqrt{n}}, \mu_0 + r\frac{\sigma}{\sqrt{n}})$ within the control limits is partitioned into three regions with

$$R1 = (\mu_0 + r'\frac{\sigma}{\sqrt{m}}, \mu_0 + r\frac{\sigma}{\sqrt{m}})$$

and

$$R3 = (\mu_0 - r\frac{\sigma}{\sqrt{m}}, \mu_0 - r'\frac{\sigma}{\sqrt{m}})$$

where a short interval d_1 is used, and

$$R2 = (\mu_0 - r'\frac{\sigma}{\sqrt{m}}, \mu_0 + r'\frac{\sigma}{\sqrt{m}})$$

where a longer interval d_2 is used.

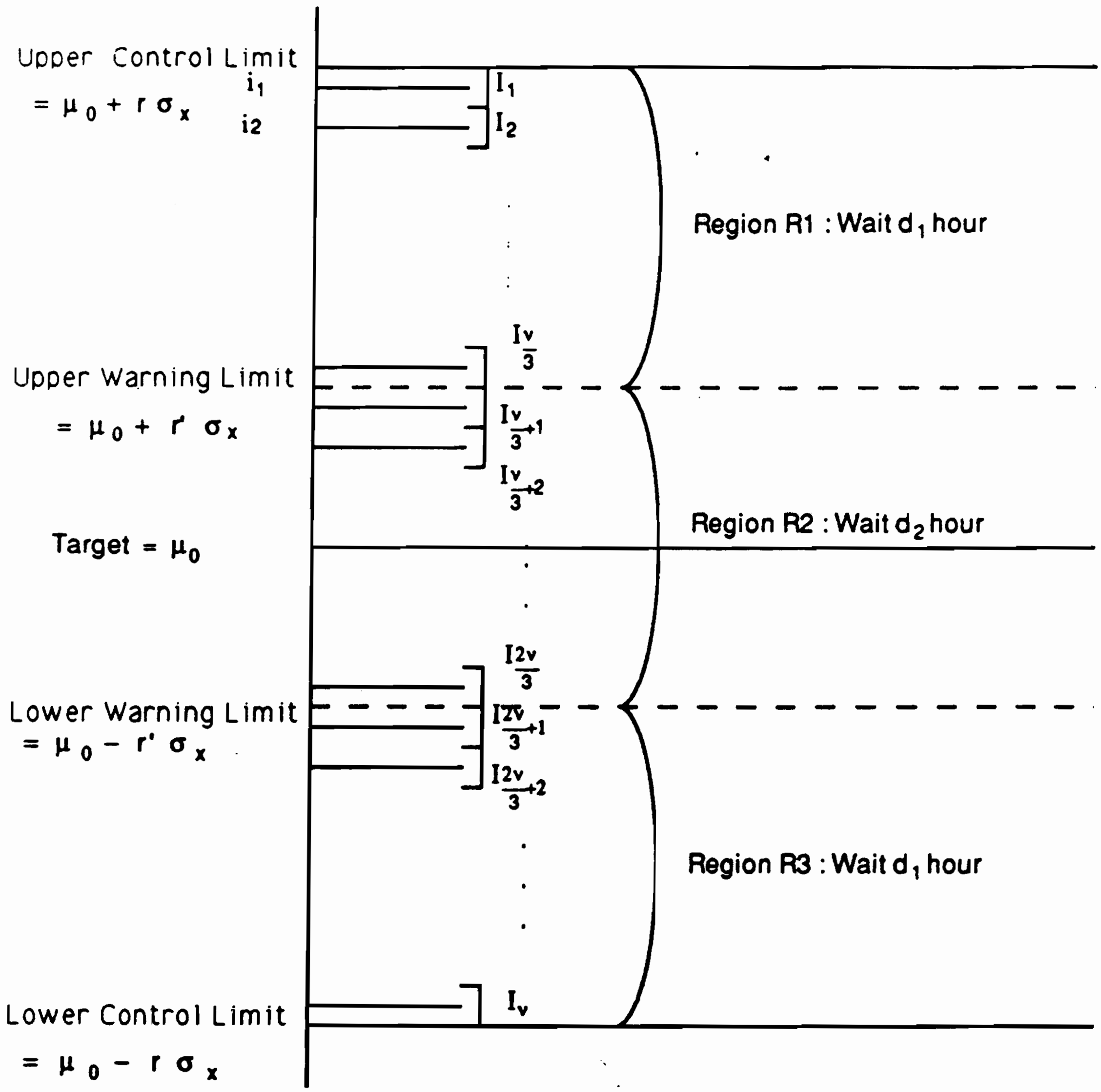

Figure 1. A Variable Sampling Interval $\overline{X}$ Control Chart with only Two Sampling Intervals

3.3 AR (1) Process

Shewhart $\overline{X}$ control charts have been used extensively for process control with the assumption that the sequential observations are independent. However, in practice, serial correlation is frequently not negligible. Therefore, it is important to know how serial correlation affects the ATS and ANSS. The most popular method to incorporate the dependence in the data is to use a time series model.

A time series is a set of observations generated sequentially in time order. We assume that the set is discrete. That is, the observations are always made at some discrete intervals. The time series is said to be deterministic if future values of a time series are exactly determined by some mathematical function. On the other hand, the time series is said to be nondeterministic or is called a statistical time series if the future values can be described only in terms of a probability distribution. We are dealing with a discrete statistical time series. A discrete time series $\{X(t)\}$ is a sequence of random variables, denoted by $X(1), X(2), \dots, X(\mathrm{n})$ possessing a joint probability distribution $F_{X(1), X(2), \dots, X(n)}(\cdot, \cdot, \dots, \cdot)$. A statistical phenomenon evolving in time according to probabilistic laws is called a stochastic process. The time series may then be thought of as one particular realization, produced by the underlying probability mechanism, of the system under study.

A stochastic process is said to be strictly stationary if the joint probability distribution associated with n observations $X(t_1), X(t_2), \dots, X(t_n)$ made at any set of times $t_1, t_2, \dots, t_n$ is the same as that associated with n observations $X(t_1 + s), X(t_2 + s), \dots, X(t_n + s)$ made at times $t_1 + s, t_2 + s, \dots, t_n + s$ for any time shift s. Hence, for a process to be strictly stationary, the joint distribution of any set of observations must not be affected by any time shift. A less restrictive requirement, called weak stationarity, is that

1. both $\mathrm{E}(\mathrm{X}(t))$ and $\mathrm{Var}(\mathrm{X}(t))$ do not depend on t and
2. $\gamma(\mathrm{s}) = \mathrm{Cov}(X(t), X(t + s))$ depends only on the time lag s and not on t.

We shall mean weak stationarity when we refer to an $X(t)$ series unless otherwise specified. Note that strict stationarity implies weak stationarity but not vice versa. But if the process $\{X(t)\}$ is Gaussian, that is, if any finite collection of random variables has a multivariate normal distribution, then weak stationarity is equivalent to strong stationarity since multivariate distributions are completely determined by their means, variances, and covariances.

If we divide $\gamma(s)$ by $\gamma(0)$, then we have an autocorrelation function (ACF); $\rho(s) = \frac{\gamma(s)}{\gamma(0)}$. The ACF is important in studying a stationary time series because it summarizes the correlated structure of $\{X(t)\}$ as a function of the lag time s. A partial correlation between X and Y adjusted for Z, which is denoted as $\rho_{X,Y|Z}$, is the correlation between adjusted variables X - β_1Z and Y - β_2Z, where β_1 and β_2 are regression coefficients of X on Z and Y on Z, respectively. Another important tool for identifying a time series model other than the ACF, according to Box and Jenkins (1976), is a partial autocorrelation function (PACF), defined by

$$\phi_{kk} = \rho_{X(t),\, X(t-k)|X(t-1),\, X(t-2),\, \ldots,\, X(t-k+1)}.$$

Without loss of generality, we assume that $\{X(t)\}$ is a time series which has already been subtracted from its process mean μ. Thus, the process mean of the time series $\{X(t)\}$ is 0. One of the stochastic models that has received a great deal of attention in quality control is the AR (1) model. In this model, the current value of the process is expressed in terms of a linear combination of the previous value of the process and a random shock at the current time. Since the AR (1) model will be used in this and following chapters, it would be appropriate to express some of the properties of the model. We assume that the observation $X'(t)$ at time t is

$$X'(t) = X(t) + \delta\sigma_X, \tag{3.2}$$

where

$$X(t) = \phi X(t-1) + a(t). \tag{3.3}$$

Note that the shift $\delta\sigma_x$ is expressed in units of the process standard deviation σ_x. The $a(t)$ may be regarded as a series of random shocks with $E(a(t)) = 0$ and $var(a(t)) = \sigma_a^2$. The parameter ϕ must satisfy the condition that $-1 < \phi < 1$ for the process to be stationary. Note that we are dealing with a stationary process in this research. The autocorrelation function of the AR (1) process is $\rho(k) = \phi\rho(k-1)$. Thus, $\rho(\mathrm{k}) = \phi^k$, $k \geq 0$. The correlation between consecutive observations $X(t)$ and $X(t-1)$ is ϕ. However, if the time between the observations is k time units apart, then the correlation between them comes down to ϕ^k. The variance of the process is $\sigma_x^2 = \frac{1}{1-\phi^2}\sigma_a^2$, which is a function of the parameter ϕ. Therefore, the process variance σ_x^2 increases as the correlation ϕ increases in absolute value.

3.4 Estimation of Process Variance or Process Standard Deviation

If the process standard deviation σ_x is known, then it can be used to set the control limits. The traditional control limits would be $\mu_0 \pm 3\sigma_x$. However, the process standard deviation σ_x is not usually known. Hence, it needs to be estimated from the n sequential observations. There are a couple of ways to estimate the process variance or standard deviation.

According to Ryan (1989), the most commonly used procedure to estimate the process standard deviation σ_x when the sample size is only 1 is to create ranges by taking differences of successive observations, and dropping the sign of the difference when it is negative. Let R_t be the range of observations $X'(t+1)$ and $X'(t)$; $R_t = |X'(t+1) - X'(t)|$. If the process mean is on target ($\delta = 0$), then the average of the moving ranges of size 2 is used to estimate σ_x by

$$\hat{\sigma}_X = \frac{\overline{R}}{d_2},$$

where d_2 is a correction factor that makes $\frac{\overline{R}}{d_2}$ an unbiased estimator of the process standard deviation σ_x if the sequential observations are independent. Note that the d_2 used in this section (Section 3.4) is a correction factor that makes $\frac{\overline{R}}{d_2}$ an unbiased estimator of σ_x for independent observations, not the longer sampling interval used in a VSI control chart. Ryan (1989) gives values of d_2 for different sample sizes. For example, $d_2 = \frac{2}{\sqrt{\pi}}$ for a moving range of size 2.

The other approach to the estimation of the process standard deviation is to use the sample standard deviation. Ryan (1989) suggests taking a sample of at least 50 observations, and calculating the sample variance $S_n^2 = \frac{1}{n-1}\sum_{t=1}^{n}(X'(t) - \overline{X'})^2$ in order to use $\frac{S_n}{c_4}$ as an estimator of the process standard deviation σ_x, where $\overline{X'}$ is the average of $X'(1), X'(2), \ldots, X'(n)$, and c_4 is a correction factor that makes $\frac{S_n}{c_4}$ an unbiased estimate of σ_x if the sequential observations are independent. Values of c_4 for different sample sizes are also given in Ryan (1989). For instance, c_4 = .9949 for n = 50. Note that c_4 is always < 1.

Both $\frac{\overline{R}}{d_2}$ and $\frac{S_n}{c4}$ are the unbiased estimators of the process standard deviation σ_x when the consecutive observations are independent ($\phi = 0$). Hence, when the data come from a common distribution, Wadsworth, Stephens, and Godfrey (1986, p. 194) propose using any one of the above two estimators. However, Cryder and Ryan (1990) showed that if the observations are independent, then the variance of $\frac{\overline{R}}{d_2}$ is at least 60 % greater than that of $\frac{S_n}{c_4}$. In other words, for independent observations, from Duncan (1986, p. 144),

$$\text{Var}(\frac{S_n}{c_4}) = \frac{1 - c_4^2}{c_4^2}\sigma_x^2$$

and, from Cryder and Ryan (1990),

$$\text{Var}(\frac{\overline{R}}{d_2}) = \frac{c_4^2(0.8264\text{n} - 1.082)}{(\text{n} - 1)(1 - c_4^2)},$$

where

$$c_4 = \sqrt{\frac{2}{\text{n} - 1}}\,\frac{\Gamma(\frac{\text{n}}{2})}{\Gamma(\frac{\text{n} - 1}{2})}.$$

Using Johnson and Kotz's (1970, p. 63) asymptotic approximation,

$$\frac{1}{c_4} = 1 + \frac{1}{4(\text{n} - 1)}$$

Cryder and Ryan (1990) showed that the ratio of the variances $\mathrm{Var}(\frac{\overline{R}}{d_2})/\mathrm{Var}(\frac{S_n}{c_4})$ is about 1.65 as $n \to \infty$. Hence, $\frac{S_n}{c_4}$ should be used for the estimator of the process standard deviation σ_x.

Cryder and Ryan (1990) also showed that if the data are correlated, then the superiority of $\frac{S_n}{c_4}$ relative to $\frac{\overline{R}}{d_2}$ is even greater. If the process mean is on target, then $X'(t+1) - X'(t)$ has a normal distribution with zero mean and variance $2(1-\rho(1))\sigma_x^2$. It can be shown that $E(\frac{\overline{R}}{d_2}) = \sqrt{1-\rho(1)}\,\sigma_x$. Since $\rho(1) = \phi$, $E(\frac{\overline{R}}{d_2}) = \sqrt{1-\phi}\,\sigma_x$. Therefore, if the consecutive observations are positively correlated, then taking the differences of the consecutive observations and using $\frac{\overline{R}}{d_2}$ as a basis for estimating the process standard deviation results in the underestimation of σ_x, because the consecutive observations tend to be close together even though the whole process may still fluctuate. However, if the consecutive observations are negatively correlated, then $\frac{\overline{R}}{d_2}$ overestimates the process standard error σ_x.

On the other hand, from the equation

$$\sum_{t=1}^{n}(X(t)-\mu)^2 = \sum_{t=1}^{n}(X(t)-\overline{X})^2 + n(\overline{X}-\mu)^2,$$

we know that

$$E(S_n^2) = \frac{n}{n-1}\sigma_x^2 - \frac{n}{n-1}\mathrm{Var}(\overline{X}).$$

According to Anderson (1971, p. 459),

$$\mathrm{Var}(\overline{X}) = \frac{\sigma_x^2}{n}[1 + 2\sum_{k=1}^{n-1}(1-\frac{k}{n})\rho(k)].$$

Hence,

$$E(S_n^2) = \sigma_x^2 - 2\frac{\sigma_x^2}{n-1}\sum_{k=1}^{n-1}(1-\frac{k}{n})\rho(k).$$

Anderson (1971, p. 459) showed that, under the assumption that $\sum_{k=1}^{\infty}\rho(k) < \infty$, $E(S_n^2)$ tends to σ_x^2 as $n \to \infty$. Note that if the observations are correlated according to an AR (1) process, then $E(S_n^2) = \sigma_x^2 - \frac{2\sigma_x^2\phi}{n}$ so that S_n^2 is biased downward for $\phi > 0$ and upward for $\phi < 0$. However, S_n^2 can still be used to estimate the process variance σ_x^2 if we have a lot of sequential observations as long as the process mean is on target or stable. Exact expressions for $E(\frac{S_n}{C_4})$ are intractable in the correlated data. However, Cryder abd Ryan (1990) used a limiting arguement to show that $E(\frac{S_n}{C_4}) \to \sigma_x$ as $n \to \infty$.

3.5 Markov Chain Representation for the AR (1) Process

Before we discuss the Markov chain approximation for the discrete AR (1) process, we introduce a continuous AR (1) process (Priestley (1981, p. 167)). Note that the parameter ϕ in equation [3.3] is the correlation between X(t-1) and X(t) when they are one unit time apart. In general, we can write the equation [3.3] as

$$X(t) = \phi_{\Delta t}X(t-\Delta t) + a(t),$$

where $\phi_{\Delta t}$ is the correlation between $X(t-\Delta t)$ and X(t). We can rewrite the above equation as

$$X(t)(1-\phi_{\Delta t}) + \phi_{\Delta t}(X(t) - X(t-\Delta t)) = a(t). \quad [3.4]$$

Then, dividing the equation [3.4] by $\phi_{\Delta t}\Delta t$, we have

$$\frac{\Delta X(t)}{\Delta t} + \frac{1-\phi_{\Delta t}}{\phi_{\Delta t}\Delta t}X(t) = \frac{a(t)}{\phi_{\Delta t}\Delta t}.$$

Let $\alpha = \frac{1 - \phi_{\Delta t}}{\phi_{\Delta t}\Delta t}$ and $\varepsilon(t) = \frac{a(t)}{\phi_{\Delta t}\Delta t}$. If we assume that α and $\varepsilon(t)$ remain finite as $\Delta t \to 0$, say $\lim_{\Delta t \to \infty} \alpha = \alpha^*$ and $\lim_{\Delta t \to \infty} \varepsilon(t) = \varepsilon^*(t)$, then the above equation becomes

$$\dot{X}(t) + \alpha^* X(t) = \varepsilon^*(t).$$

In order to study the behavior of $\phi_{\Delta t}$ and $a(t)$ as $\Delta t \to 0$, we use Landav's big Oh and little Oh notations.

Definition 3.1 $x_n = o(r_n)$ iff $\forall \varepsilon$, $\exists n_0 \ni \forall n \geq n_0 \quad \|x_n\| \leq \varepsilon \|r_n\|$.

Definition 3.2 $x_n = O(r_n)$ iff $\exists n_0$, $C > 0 \ni \forall n \geq n_0 \quad \|x_n\| \leq C\|r_n\|$.

Now,

$$\begin{aligned}
\phi_{\Delta t} &= \frac{1}{1 + \alpha\Delta t} \\
&= 1 - (\alpha\Delta t) + (\alpha\Delta t)^2 - (\alpha\Delta t)^3 + \cdots \\
&= 1 - \alpha\Delta t + o(\Delta t).
\end{aligned}$$

Hence, $\phi_{\Delta t} \to 1$ as $\Delta t \to 0$. Next,

$$\begin{aligned}
Var(X(t)) &= \sigma_a^2 \frac{1}{1 - \phi_{\Delta t}^2} \\
&\cong \sigma_a^2 \frac{1}{1 - (1 - \alpha\Delta t)^2} \\
&= \frac{\sigma_a^2}{2\alpha\Delta t} \frac{1}{(1 - \frac{\alpha\Delta t}{2})} \\
&= \frac{\sigma_a^2}{2\alpha\Delta t} (1 + (\frac{\alpha\Delta t}{2}) + (\frac{\alpha\Delta t}{2})^2 + \cdots) \\
&= \frac{\sigma_a^2}{2\alpha\Delta t} (1 + O(\Delta t)).
\end{aligned}$$

Thus, in order for Var($X(t)$) to be finite, we must have $\sigma_a^2 = O(\Delta t)$. Then

$$
\begin{aligned}
Var(\varepsilon(t)) &= \frac{\sigma_a^2}{\phi^2(\Delta t)^2} \\
&\cong \frac{\sigma_a^2}{(\Delta t)^2} \quad \text{since } \phi_{\Delta t} \rightarrow 1 \text{ as } \Delta t \rightarrow 0 \\
&= O(\frac{1}{\Delta t}).
\end{aligned}
$$

Hence, $\varepsilon(t)$ is distributed with mean 0 and standard deviation $O(\frac{1}{\sqrt{\Delta t}})$. Note that $\phi_{\Delta t} \rightarrow 1$ as $\Delta t \rightarrow 0$ and $\sigma_a^2 = O(\Delta t)$.

Now, we introduce a discrete version of the behavior of $\phi_{\Delta t}$ and σ_a^2 as $\Delta t \rightarrow 0$. Without loss of generality, let one unit time be the sampling interval for the FSI control chart; i.e., d = 1. Let FSI (1) be the FSI control chart with a one unit time sampling interval and VSI (d_1, d_2) be the VSI control chart with sampling intervals $d_1 < 1$ and $d_2 > 1$. Let $l_1 = \frac{1}{k}$ be the smallest value of d_1 to be considered where k is an integer. The values of d_1, d_2 and d to be considered are such that d_1 and d_2 are both integer multiples of $\frac{1}{k}$ and $d = \frac{k}{k} = 1$. Suppose that the observation $X'(\frac{t}{k})$ at time $\frac{t}{k}$ is

$$
X'(\frac{t}{k}) = X(\frac{t}{k}) + \delta\sigma_X, \qquad [3.5]
$$

where

$$
X(\frac{t}{k}) = \lambda X(\frac{t-1}{k}) + \varepsilon(\frac{t}{k}), t = 1, 2, \ldots, k, k+1, \ldots, 2k, 2k+1, \ldots \qquad [3.6]
$$

Here $\varepsilon(\frac{t}{k}) \sim N(0, \sigma_\varepsilon^2)$. Now let $\phi = \lambda^k$ and $a(\text{n}) = \sum_{i=1}^{k} \lambda^{k-i}\varepsilon(\frac{(\text{n}-1)k+i}{k})$. Then the process {X(t)} at times $t = \frac{k}{k}, \frac{2k}{k}, \ldots$ is an AR (1) process with parameters ϕ and σ_a^2, i.e.,

$$
X(t) = \phi X(t-1) + a(t), \qquad t = 1, 2, \ldots
$$

where $\sigma_a^2 = \frac{1-\lambda^{2k}}{1-\lambda^2}\sigma_\varepsilon^2$. Note again that $\lambda \rightarrow 1$ and $\sigma_\varepsilon^2 \rightarrow 0$ as $k \rightarrow \infty$.

Now, we can use the above properties of λ and σ_ε^2 for the construction of the Markov chain representation for both FSI and VSI control charts. Since sequential observations are not independent, the ATS cannot be explicitly expressed in the same way as in equation [3.1]. But by the Markov property of the AR (1) process, the future value $X(t+1)$ depends only on the present value $X(t)$ and on a random shock $a(t+1)$ at the future time; $X(t+1)$ does not depend on any previous observations $X(t-1), X(t-2) \ldots$. Hence, we will use a Markov chain representation in order to investigate the properties of the FSI and VSI control charts under an AR (1) process. The Markov chain approximation has been used by Brook and Evans (1972) for a one-sided CUSUM procedure to determine exact and approximate expressions for the run length distribution, its moments, and its percentage points. We investigate the properties of ATS and ANSS for both FSI and VSI $\overline{X}$ control charts by approximating the continuous state Markov chain with different size Markov chains and extrapolating to an infinite number of transient states using the method of least squares. As shown in Figure 1, we can partition the possible values of $X(t)$ within the control limits into disjoint intervals $I_1, I_2, \ldots$, and I_v. Let $i_1, i_2, \ldots$, and i_v their middle points, respectively. Let the chain have states 1, 2, ..., v, and a. Thus, the process is in state i, i = 1, 2, ..., v if the current observation is in interval I_i. Otherwise, the process is said to be in an absorbing state a. If there are only two sampling intervals, then, as shown in Figure 1, we will use three different regions R1, R2 and R3 with an equal number of intervals in each region such that

$$R1 = I_1 \cup I_2 \cup \cdots \cup I_{\frac{v}{3}}$$

$$R2 = I_{\frac{v}{3}+1} \cup I_{\frac{v}{3}+2} \cup \cdots \cup I_{\frac{2v}{3}}$$

and

$$R3 = I_{\frac{2v}{3}+1} \cup I_{\frac{2v}{3}+2} \cup \cdots \cup I_v.$$

As an illustration, consider VSI($\frac{1}{k}, \frac{2k-1}{k}$). It waits only $\frac{1}{k}$ unit time if an observation falls in region R1 or R3, and $\frac{2k-1}{k}$ unit times if it falls in region R2. Then the transition matrix is

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1v} & p_{1a} \\ p_{21} & p_{22} & \cdots & p_{2v} & p_{2a} \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ p_{v1} & p_{v2} & \cdots & p_{vv} & p_{va} \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}, \qquad [3.9]$$

where

$$p_{rj} = \begin{cases} P(X'(\frac{t+1}{k}) \in I_j \mid X'(\frac{t}{k}) = i_r) & \text{if } I_r \in \text{R1 or R3 and j} \le v \\ = P(\varepsilon(\frac{t+1}{k}) + \lambda X(\frac{t}{k}) + \delta\sigma_x \in I_j \mid X(\frac{t}{k}) + \delta\sigma_x = i_r) & \\ = P(\varepsilon(\frac{t+1}{k}) + \lambda(i_r - \delta\sigma_x) + \delta\sigma_x \in I_j) & \\ P(X'(\frac{t+(2k-1)}{k}) \in I_j \mid X'(\frac{t}{k}) = i_r) & \text{if } I_r \in R2 \text{ and j} \le v \\ = P(\varepsilon(\frac{t+(2k-1)}{k}) + \lambda\varepsilon(\frac{t+(2k-2)}{k}) + \lambda^2\varepsilon(\frac{t+(2k-3)}{k}) + \cdots & \\ \quad + \lambda^{2k-2}\varepsilon(\frac{t+1}{k}) + \lambda^{2k-1}X(\frac{t}{k}) \in I_j \mid X(\frac{t}{k}) + \delta\sigma_x = i_r) & \\ = P(\varepsilon(\frac{t+(2k-1)}{k}) + \lambda\varepsilon(\frac{t+(2k-2)}{k}) + \lambda^2\varepsilon(\frac{t+(2k-3)}{k}) + \cdots & \\ \quad + \lambda^{2k-2}\varepsilon(\frac{t+1}{k}) + \lambda^{2k-1}(i_r - \delta\sigma_x) + \delta\sigma_x \in I_j) & \\ 1 - \sum_{l=1}^{v} p_{lj} & \text{if j} = \text{a} \end{cases}$$

The matrix P can be rewritten as

$$P = \begin{bmatrix} Q & (I-Q)\underline{1} \\ \underline{0}' & 1 \end{bmatrix}, \qquad [3.10]$$

where Q is obtained by deleting the last row and last column of the matrix P, and $\underline{0}'$ is a vector of 0's. Now the fundamental matrix is $M = (I - Q)^{-1}$, where M_{ij} is the expected number of times that the process is in state j given that the process starts in state i. Hence, $\sum_{k=1}^{v} M_{ik}$ is the total expected number of times that the process is in transient states given that the process starts in state i.

3.6 Computational Results for a Random Start

Suppose that a large shift in the process mean occurs at the beginning of the process at time 0. If we apply FSI at time 0, then the first sampling interval for FSI (1) is always 1. On the other hand, if a VSI scheme is used at time 0, then the first sampling interval is almost always the short sampling interval d_1. Hence, the out of control ATS tends to be longer for FSI (1) than for VSI (d_1, d_2). In order to compare FSI and VSI control charts fairly, we assume that the sampling interval used at time 0 is always one unit time whether a FSI or VSI control chart is used; $d_0 = 1$. However, the sampling interval which is used at time 1 depends on what is observed at time 1. In FSI control charts, the sampling interval at time 1 is still one unit time. However, in VSI control charts, if $X'(1) \in$ R1 or R3, then a short sampling interval d_1 is used at time 1. Otherwise, a longer sampling interval d_2 is used at time 1.

In the sequentially independent case, the value X(1) at time 1 does not depend on the value X(0) at time 0. However, if they are dependent, it is important to know how the process starts at time 0 because it will affect the value of X(1) at time 1. In this Section 3.6, we assume that X'(0) is random with $X'(0) \sim N(\mu_0 + \delta\sigma_x, \sigma_x^2)$. Let $f_{X'(0)}(.)$ be the probability density function of $X'(0)$. Let ATS $(X'(0) = x)$ and ATS $(X'(1) = x)$ be the ATS's when $X'(0) = x$ at time 0 and when $X'(1) = x$ at time 1, respectively. Then, since X'(0) is random,

$$ATS = \int_{-\infty}^{\infty} ATS\ (X'(0) = x)\, f_{X'(0)}(x)\ dx.$$

Since we use one unit time at the beginning ($d_0 = 1$) whether a FSI or VSI control chart is used for a fair comparison

$$ATS(X'(0) = x) = 1 + \int_{\{s:s \in \text{ control limits }\}} P(X'(1) = s \mid X'(0) = x)\ ATS\ (X'(1) = s)\ ds.$$

Hence,

$$\begin{aligned} ATS &= 1 + \int_{\{s:s \in \text{ control limits }\}} \int_{-\infty}^{\infty} P(X'(1) = s \mid X'(0) = x) f_{X'(0)}(x)\ dx\ ATS\ (X'(1) = s)\ ds \\ &= 1 + \int_{\{s:s \in \text{ control limits }\}} P(X'(1) = s)\ ATS\ (X'(1) = s)\ ds. \end{aligned}$$

In the same way, if we let ANSS ($X'(1) = s$) be the ANSS when $X'(1) = s$ at time 1, then

$$ANSS = 1 + \int_{\{s:s \in \text{ control limits }\}} P(X'(1) = s)\ ANSS\ (X'(1) = s)\ ds.$$

A discrete Markov chain is used to evaluate the properties of the control charts. Let ATS_v and $ANSS_v$ be the ATS and ANSS when there are v transient states, respectively. Hence, if we let $ATS_v\ (X'(1) = i_r)$ and $ANSS_v\ (X'(1) = i_r)$ be the expected time and expected number of samples to signal when $X'(1) = i_r$ and when there are v transient states, then the ATS in the discrete case is

$$ATS_v = 1 + \sum_{r=1}^{v} P(X'(1) \in I_r)\ ATS_v\ (X'(1) = i_r)$$

and the ANSS in the discrete case is

$$ANSS_v = 1 + \sum_{r=1}^{v} P(X'(1) \in I_r)\ ANSS_v\ (X'(1) = i_r).$$

The ATS_v and $ANSS_v$ tend to stabilize as the number of transient states v increases. A greater number of transient states is needed for accurate asymptotic values of ATS and ANSS when the two sampling interval lengths d_1 and d_2 are farther apart and when ϕ is higher. In Chapter III, $ANSS_v$ and ATS_v are calculated for $v = 45, 48, \dots, 75$. Similar to the procedure of Brook and Evans (1972), our extrapolation to an infinite number of transient states is based on fitting the following formulas by least squares:

$$ATS_v = \text{asymptotic ATS} + \frac{A}{v^2} + \frac{B}{v^4} \qquad [3.11]$$

$$ANSS_v = \text{asymptotic ANSS} + \frac{C}{v^2} + \frac{D}{v^4}. \qquad [3.12]$$

The computation is done using the assumption that the random shock $a(t)$ for the unit time series is $a(t) \sim N(0, 1)$. A FORTRAN program that computes the ATS and ANSS is given in Appendix A.

3.6.1 Robustness of Two Sampling Interval Symmetric Control Charts with respect to Departure from Independence

For VSI (d_1, d_2) with $d_1 < 1$ and $d_2 > 1$, R1, R2 and R3 are chosen such that

$$d_1 p_{0,1} + d_2 p_{0,2} = q_0, \qquad [3.13]$$

where $p_{0,1} = P(X(t) \in R1 \text{ or } R3)$ and $p_{0,2} = P(X(t) \in R2)$, and $q_0 = P(X(t) \notin R1 \cup R2 \cup R3)$ when the process mean is on target. Hence, if $d_1 < 1$ and $d_2 > 1$ are symmetric around 1 as in VSI ($\frac{1}{k}, \frac{2k-1}{k}$), then R1, R2 and R3 are chosen in such a way that $P_{0,1} = P_{0,2}$, which results in $r' = 0.672$ in Figure 1.

The effect of correlation on ATS (= ANSS) for FSI (1), when the process standard deviation σ_x is known and when the usual control limits of $\mu_0 \pm 3\sigma_x$ are used, is shown in Table 1. Note that the deviation from the target value is expressed in units of σ_x, which is $\sqrt{\frac{1}{1-\phi^2}}$. Thus, the magnitude of the deviation $\delta\sigma_x$ depends on δ as well as on the parameter ϕ. If the process mean is not on target ($\delta \neq 0$), then the detection time is shorter when the correlation is negative than when it is positive. Even though the negative correlation in the consecutive observations is very rare in practical situations, the ATS values for negative correlations are added in Table 1 for completeness and for comparison between negative and positive correlations. However, from now on, we will only consider a situation where the correlation between the consecutive observations is positive. According to Table 1, whether the process mean is on or off target, the ATS and ANSS tend to increase as the correlation ϕ between the consecutive observations increases. Therefore, the false alarm rate decreases and the detection time increases as ϕ increases in FSI (1). In fact, for a high correlation of ϕ, the control limits become wide because the control limits are set at $\mu_0 \pm 3\sqrt{\frac{1}{1-\phi^2}}$. Note that for any deviation δ, the ATS increases very slowly until ϕ reaches a moderate value of about 0.4. But for a higher correlation of ϕ, the ATS becomes very large because the control limits become rapidly wider. The latter one would be the case when the sequential observations are almost the same except possibly for the measurement errors.

Next, the effect of correlation on ATS and ANSS for VSI (d_1, d_2), where d_1 and d_2 are symmetric, is shown in Tables 2 and 3, which are for VSI (0.5, 1.5) and VSI (0.1, 1.9), respectively. We know that the width of the control limits increases as ϕ increases, but the increase in the width is not noticable until ϕ reaches about 0.4. Furthermore, the process of the series $\{X(\frac{t}{k})\}$, t = 1, 2, ... k, k+1, ..., 2k, ... for VSI $(\frac{1}{k}, \frac{2k-1}{k})$ has a higher correlation than the process $\{X(\frac{t}{k})\}$, t = k, 2k, 3k, ... for FSI (1). That is, the correlation between $X(\frac{1}{k})$ and $X(\frac{2}{k})$ is higher than the correlation between $X(\frac{k}{k})$ and $X(\frac{2k}{k})$ for $k \geq 2$. Hence, according to the first columns of Tables 2 and 3, if the process mean is on target, then the ATS and ANSS decrease as ϕ increases until ϕ reaches a moderate correlation of about 0.4, and then, for higher correlations of ϕ, they increase. The same phenomenom happens when the deviations from the process mean are small or at least moderate ($\delta \leq 1$). For instance, according to the second, third and fourth columns of Tables 2 and 3, if δ = 0.25, 0.5 and 1, then the ATS and the ANSS decrease as ϕ increases until ϕ reaches 0.1, 0.2 or 0.3 depending on the magnitude of the deviation δ, and then, for a higher correlation of ϕ, they tend to increase. However, when the deviation from the process mean is large ($\delta \geq 2$), the detection time and the detection number of samples to signal increase monotonically as ϕ increases. Finally, according to Tables 1, 2 and 3, as the sampling intervals d_1 and d_2 become farther apart (FSI (1) → VSI (0.5, 1.5) → VSI (0.1, 1.9)), the ATS for any level of ϕ and δ tends to decrease.

In summary, for FSI control charts, the false alarm rate is lower, and detection time and number of samples to detection are longer when there is a correlation ($\phi > 0$) in the data than when the observations are independent ($\phi = 0$). For a VSI control chart, if the process mean is on or near target, then the ATS and ANSS are lower when there is correlation in the data than when the observations are independent, except when ϕ is very high ($\phi = 0.8$ or 0.9) depending on whether VSI (0.5, 1.5) or VSI (0.1, 1.9) is used. However, if the magnitude of the deviation is large, then the ATS and ANSS are larger when there is a correlation than when the observations are independent.

3.6.2 Number of Sampling Intervals

Reynolds (1989) and Reynolds and Arnold (1989) showed that if the consecutive observations are independent, then the detection time when the process mean is not on target becomes longer as the number of sampling intervals used in the control chart increases. They also showed that it is best to have only two sampling intervals. It is important to know how the ATS and ANSS will change as the number of sampling intervals increases for sequentially dependent observations according to AR (1). Shewhart charts with 3 and 5 different sampling intervals are evaluated numerically using the total number of transient states ν = 45, 50, ... , 75 and ν = 45, 54, ... , 81 for VSI (0.1, 1.0, 1.9) and VSI (0.1, 0.5, 1.0, 1.5, 1.9), respectively, and our extrapolation to an infinite number of transient states is based on fitting the same formula in equations [3.11] and [3.12]. The regions for each of the sampling intervals 0.1, 1.0 and 1.9 for VSI (0.1, 1.0, 1.9) and 0.1, 0.5, 1.0, 1.5 and 1.9 for VSI (0.1, 0.5, 1.0, 1.5, 1.9) are determined such that the probability that X′(t) will fall in any of the regions is the same when the process mean is on target.

Tables 4 and 5 show the ATS and ANSS for VSI (0.1, 1.0, 1.9) and VSI (0.1, 0.5, 1.0, 1.5, 1.9), respectively when the usual control limits of $\mu_0 \pm 3\sigma_x$ with a known process standard deviation σ_x are used. The first 2 rows of Tables 3, 4 and 5 again reveal the optimality of VSI control charts with only two different sampling intervals for sequentially independent observations. According to the rest of the rows, if the consecutive observations are correlated according to AR (1), then with the usual control limits of $\mu_0 \pm 3\sigma_x$ the ATS and ANSS decrease as the number of sampling intervals increases for almost all cases except when both ϕ and δ are very large ($\phi \geq 0.8$ and $\delta \geq 2$). Therefore, for sequentially dependent observations, if the process mean is on target, then a false alarm rate in terms of ATS and ANSS occurs more frequently as the number of different sampling intervals increases. On the other hand, if the process mean is not on target, then it takes less time and fewer number of samples to signal as the number of different sampling intervals increases. Note that the in control ATS's (or in control ANSS's) for VSI (0.1, 1.9), VSI (0.1, 1.0, 1.9) and VSI (0.1, 0.5, 1.0, 1.5, 1.9) when the process mean is on target ($\delta = 0$) are not the same for any particular correlation of $\phi \neq 0$. Hence, if the observations are consecutively dependent, then we cannot compare the ATS's (or ANSS's) when the process mean is off target for different numbers of sampling intervals.

We need to have the same ATS's and the same ANSS's when the process mean is on target ($\delta = 0$) so that VSI (0.1, 1.9), VSI (0.1, 1.0, 1.9) and VSI (0.1, 0.5, 1.0, 1.5, 1.9) are comparable when the process mean is off the target ($\delta \neq 0$). Note that the control and warning limits are set at $\mu_0 + r\sigma_x$ and $\mu_0 + r'\sigma_x$, respectively when we use only two sampling intervals. However, when more than two sampling intervals are used in a VSI control chart, we need additional warning limits, such as $\mu_0 + r'' \sigma_x$ for VSI (0.1, 1.0, 1.9), and $\mu_0 + r'' \sigma_x$, $\mu_0 + r''' \sigma_x$ and $\mu_0 + r'''' \sigma_x$ for VSI (0.1, 0.5, 1.0, 1.5, 1.9), respectively. For a two sampling interval control chart, we alter r and r′ so that if the process mean is on target, then the new ATS and new ANSS become close to our standard. We take 370.4 as our standard for the new ATS and new ANSS when the process mean is on target. In general, for a two sampling interval control chart both ATS and ANSS increase as r increases. However, the ATS alone increases as r′ increases. For a control chart with more than two sampling intervals, there are multiple combinations of new control and warning limits that achieve the in control ATS and in control ANSS of about 370.4. We consider the case where we alter only r and r′ for VSI (0.1, 1.0, 1.9), and r, r′ and r″ for VSI (0.1, 0.5, 1.0, 1.5, 1.9), respectively.

Table 6 shows the new in control and warning limits as well as the out of control ATS and ANSS for VSI (0.1, 1.9), VSI (0.1, 1.0, 1.9) and VSI (0.1, 0.5, 1.0, 1.5, 1.9), respectively. According to Table 6, if the correlation between the consecutive observations is weak ($\phi = 0.3$), then the ATS when the process mean is not on target ($0.25 \le \delta \le 2$) tends to decrease slightly as the number of sampling intervals increases. If the correlation is at least medium ($\phi \ge 0.6$), then the ATS when the process mean is not on target ($\delta \ge 0.5$) increase as the number of sampling intervals increases.

In summary, for some ranges of ϕ ($\phi \ge 0.6$), a two sampling interval control chart tends to be better than a control chart with more than two sampling intervals. For other ranges of ϕ ($\phi \le 0.3$), a two sampling interval control chart does not tend to be better than a control chart with more than two sampling intervals. But the difference is very small. Note also that if the new in control ATS and ANSS are closer to our standard of 370.4, say within the absolute difference of 0.1, then the interpretation of Table 6 would be a little bit different. Overall, even though we consider only one combination of r and r′ for VSI (0.1, 1.0, 1.9) and r, r′, r″, r‴ and r⁗ for VSI (0.1, 0.5, 1.0, 1.5, 1.9) that achieve the new in control ATS and ANSS of about 370.4, one is not doing better than the other with a great margin. Since a two sampling interval chart is simple to use in practice, we will only consider a control chart with only two sampling intervals.

3.6.3 Two Sampling Interval Asymmetric VSI Control Charts

So far, we have been dealing with the VSI control charts where the sampling intervals d_1 and d_2 are symmetric around 1. But now consider asymmetric control charts. In order for a VSI (0.5, 2.0) control chart to be comparable to a symmetric control chart VSI (0.5, 1.5), the probability $P_{0,1}$ in equation [3.13] of using a short sampling interval should be larger than the probability $P_{0,2}$ of using a longer sampling interval when the process mean is on target so that on the average one unit time will be used. In the case of VSI (0.5, 2.0), r′ should be 0.429.

The results in the first 2 rows of Tables 2, 7 and 8 corresponding to VSI (0.5, 1.5), VSI (0.5, 2.0) and VSI (0.5, 3.0) suggest that if the sequential observations are independent, then for a given sampling interval of d_1 it is best to choose a maximum length of d_2 as shown by Reynolds and Arnold (1989) and Reynolds (1989). This result holds for any deviations $\delta \neq 0$ considered for sequentially independent observations. However, Tables 2, 7 and 8 also suggest that if the sequential observations are correlated according to AR (1), then the ATS and ANSS when the traditional control limits are used tend to increase as the sampling intervals d_1 and d_2 become more asymmetrically apart.

Note that, for any given $\phi \neq 0$, the ATS's (or ANSS's) for VSI (0.5, 1.5), VSI (0.5, 2.0) and VSI (0.5, 3.0) when the process mean is on target ($\delta = 0$) are different. Hence, in order for VSI ((0.5, 1.5), VSI (0.5, 2.0) and VSI (0.5, 3.0) to be comparable when the process mean is off target ($\delta \neq 0$), we need to alter the control and warning limits so that new ATS and new ANSS become close to our standard 370.4 when the process mean is on target. Tables 10, 12 and 13 show the new r and r′ as well as the new ATS′s and new ANSS′s for various values of $\mu = \mu_0 \pm \delta\sigma_x$, $\delta = 0$, 0.25, 0.5, 1, 2 and 3 for VSI (0.5, 1.5), VSI (0.5, 2.0) and VSI (0.5, 3.0), respectively. According to each of those tables, unless the deviation is large ($\delta \geq 2$), the detection time and detection number of samples to signal tend to decrease as ϕ increases up to $\phi = 0.1$, and then for a higher correlation of ϕ, they tend to increase as ϕ increases. For a large shift, however, they tend to increase monotonically as ϕ increases.

Tables 10, 12 and 13 show that if the correlation is very weak ($\phi = 0.1$), then the out of control ATS (or out of control ANSS) tends to increase as the sampling intervals d_1 and d_2 become farther apart asymmetrically for small or moderate shifts ($\delta = 0.25, 0.5$ and 1). But for a large shift ($\delta \geq 2$), the out of control ATS (or out of control ANSS) tends to decreases as the sampling intervals d_1 and d_2 become farther apart asymmetrically. Next, if the correlation is at least moderate but not large ($0.4 \leq \phi \leq 0.6$), then the out of control ATS tends to decrease as the sampling intervals becomes farther apart asymmetrically. If the correlation is high ($\phi \geq 0.7$), then the out of control ATS (or out of control ANSS) does not always seem to decrease as the sampling intervals become farther apart asymmetrically when the shift is relatively small ($\delta = 0.25$ and 0.5).

In summary, for some ranges of ϕ ($\phi \cong 0.4$ or 0.5) it is better to have the two sampling intervals farther apart asymmetrically. But for some other ranges of ϕ ($\phi \cong 0.2$ or 0.8) it is not better to have the two sampling intervals farther apart asymmetrically. In the next Section 3.6.4, we will consider how symmetric control charts behave as the sampling intervals become farther apart.

3.6.4 Comparison of Symmetric Control Charts

Suppose that, for a given deviation δ from the target process mean, we want to know how fast the $\overline{X}$ control charts detect being out of control for different degrees of correlation ϕ in FSI (1). Then we need to have the same ATS's when the process mean is on target. We use 370.4 as our standard. According to Table 1, if the process mean is on target, then the ATS is bigger than 370.4 for sequentially dependent observations ($\phi \neq 0$). Hence, we decrease the control limit r until the new ATS when the process mean is on target becomes very close to 370.4. Table 9 shows the new r as well as the new out of control ATS for various values of $\mu = \mu_0 + \delta\sigma_x$, where $\delta \neq 0$. For instance, if $\phi = 0.70$, then the new upper control limit is $2.927\sigma_x$. Hence, the decrease in the upper control limit is $0.073\sigma_x$. Note that $\sigma_x = \sqrt{\frac{1}{1-\phi^2}}$, which increases as the correlation ϕ increases. According to Table 9, it takes longer to detect being out of control in the process mean as ϕ gets bigger.

Next, suppose that, for a given deviation δ from the target process mean μ_0, we want to compare the out of control ATS's (or out of control ANSS's) for different degrees of correlation in VSI (d_1, d_2). Then, we also need to have the same in control ATS's as well as the same in control ANSS's when the process mean is on target. We use 370.4 as our standard for the in control ATS and in control ANSS when the process mean is on target. Altering the control and warning limits (r and r′), we can get the new in control ATS and ANSS close to our standard. Tables 10 and 11 show the new r and r′ that achieve the new in control ATS and ANSS of about 370.4 for VSI (0.5, 1.5) and VSI (0.1, 1.9), respectively. According to each of those tables, if the deviation from the process mean is small or moderate ($\delta = 0.25, 0.5,$ or 1.0 depending on whether a VSI (0.5, 1.5) or

VSI (0.1, 1.9) is used), then the detection time and detection number of samples to signal tend to decrease as ϕ increases up to $\phi = 0.1$, but for a higher correlation of ϕ, they tend to increase as ϕ increases. However, for a large shift ($\delta \geq 2.0$), they tend to increase monotonically as ϕ increases.

Now we compare the FSI (1) with VSI (d_1, d_2) for different values of ϕ using Tables 9, 10 and 11, where altered control and warning limits are used for comparison. The ATS of VSI (d_1, d_2) is smaller than that of FSI (1), and the ATS tends to decrease as the sampling intervals d_1 and d_2 become farther apart. Relative efficiency in ATS, which is $\dfrac{ATS \text{ of } FSI\ (1)}{ATS \text{ of } VSI\ (d_1, d_2)}$, is shown in Tables 14 and 15 for VSI (0.5, 1.5) and VSI (0.1, 1.9), respectively. According to those tables, we gain maximum savings in ATS when the observations are independent and when the sampling intervals d_1 and d_2 are farther apart, and the gain becomes smaller as the correlation gets stronger.

Unfortunately, the ANSS for VSI (d_1, d_2) tends to be larger than that of FSI (1), and the ANSS tends to increase as the sampling intervals are farther apart. Tables 16 and 17 show the relative inefficiency, which is $\dfrac{ANSS \text{ of } VSI\ (d_1, d_2)}{ANSS \text{ of } FSI\ (1)}$, for VSI (0.5, 1.5) and VSI (0.1, 1.9), respectively. They show that VSI (d_1, d_2) tends to be less efficient than FSI (1) in terms of ANSS if ϕ is higher and if two sampling intervals are farther apart.

In summary, even though there is a correlation between consecutive observations, a VSI control chart is more efficient than a FSI control chart in terms of ATS. However, the VSI chart is less efficient than the FSI chart in terms of ANSS. The efficiency (inefficiency) of ATS (ANSS) tends to decrease (increase) as the correlation between the consecutive observations becomes stronger. Note also that, unlike the asymmetric control charts in Section 3.6.3, as the sampling intervals become farther apart, the detection time decreases.

3.7 Computational Results for a Fixed Start

In the previous sections, we assumed that the starting value $X'(0)$ has a probability density function $f_{X'(0)}(.)$. That is, $X'(0) \sim N(\mu_0 + \delta\sigma_x, \sigma_x^2)$. However, we may be interested in the ATS and ANSS when X′(0) is not random but fixed as $X'(0) = \mu_0 + \delta\sigma_x$. Hence, in the computation for this section, we will assume that the process starts at a fixed point $X'_0 = \mu_0 + \delta\sigma_x$ from time 0 on. If we allow a fixed first sampling interval of unit time ($d_0 = 1$) for a fair comparison between FSI and VSI control charts, then the ATS and ANSS are

$$ATS = 1 + \int_{\{s:s \in \text{ control limits }\}} P(X'(1) = s \mid X'(0) = x)\, ATS\, (X'(1) = s)\, ds$$

and

$$ANSS = 1 + \int_{\{s:s \in \text{ control limits }\}} P(X'(1) = s \mid X'(0) = x)\, ANSS\, (X'(1) = s)\, ds.$$

The ATS and ANSS in the discrete case when the process starts at x are

$$ATS_v = 1 + \sum_{r=1}^{v} P(X'(1) \in I_r | X'(0) = x)\, ATS_v\, (X'(1) = i_r)$$

and

$$ANSS_v = 1 + \sum_{r=1}^{v} P(X'(1) \in I_r | X'(0) = x)\, ANSS_v\, (X'(1) = i_r).$$

The ATS_v and $ANSS_v$ are calculated for $v = 45, 48, \dots, 75$ transient states and the extrapolation to an infinite number of transient states is based on the equations [3.11] and [3.12]. See Appendix A for a FORTRAN program.

The effect of fixed start on ATS and ANSS when the usual control limits of $\mu_0 \pm 3\sigma_x$ are used, where σ_x is assumed to be known, is shown in Table 18 for FSI (1) and VSI (0.5, 1.5) and VSI (0.1, 1.9). The ATS and ANSS values are calculated for ϕ values of 0.0, 0.3, 0.6, 0.9. Their counterparts are shown in Tables 1, 2 and 3. According to Tables 1, 2, 3 and 18, if the consecutive observations are independent, then the ATS and ANSS are the same whether the starting point is random or fixed for FSI or VSI control charts. However, if they are correlated according to AR (1), then the ATS and ANSS with a random start are different from those with a fixed start. In the case of small to relatively moderate deviations ($0.25 \le \delta \le 2$), the ATS and ANSS with a fixed start tend to be slightly larger than those with the random start, because, for instance, if the process mean is on target ($\delta = 0$) at time 0, then the fixed start assumes that the process starts at the target value at time 0 while the random start assumes that the process might start anywhere including the target value. In the case of large deviations ($\delta = 3$), however, the ATS and ANSS with a fixed start tend to be slightly smaller than those with the random start. Note that the difference tends to be bigger as the correlation ϕ becomes stronger.

3.8 Design of Control Charts

Suppose that we want to design a control chart when we have an AR (1) process with the known parameters ϕ and σ_a^2. Then we can get the process variance $\sigma_x^2 = \frac{1}{1-\phi^2}\sigma_a^2$. If we have sequentially independent observations, then the in control ATS as well as the in control ANSS are the same for both FSI and VSI control charts, as shown in the tenth row and first column of Table 1, and in the first two rows and first columns of Tables 2, 3, 4, 5, 7 and 8. In this case, we only needed to decide on the level of the in control ATS. However, if the observations are not independent, then the in control ATS as well as the in control ANSS are not the same for any FSI or VSI control charts. Therefore, we need to decide on the levels of both in control ATS and in control ANSS. In our case, they were arbitrarily set at 370.4, respectively. But they do not have to be the same. Then our objective is that for any given in control ATS and in control ANSS we need to find a combination of parameters that gives a shortest detection time and shortest detection number of samples to signal.

As mentioned in Section 3.6.2, a two sampling interval control chart tends to perform similarly to three or more sampling interval control chart. In addition to the performance, a two sampling interval control chart is easier to use in practice than a three or more sampling interval control chart. Hence, a control chart with only two sampling intervals is to be considered.

Next, Section 3.6.3 showed that asymmetric control charts did not always seem to have a shorter detection time as the sampling intervals become farther apart. On the other hand, a symmetric two sampling interval control chart tends to have a shorter detection time as the sampling intervals become farther apart. Therefore, a symmetric two sampling interval control chart is to be recommended in the design of control charts. Hence, we need to make the sampling intervals d_1 and d_2 as far apart as possible while we keep them symmetric around 1 if we are only concerned about the detection time.

Once the sampling intervals d_1 and d_2 are determined, we can alter the control and warning limits (r and r′) so that we can have new in control ATS and new in control ANSS close to our standard. As mentioned in Section 3.6.3, both ATS and ANSS tend to increase as r increases. However, the ATS alone tends to increases as r′ increases. Note that if the sampling intervals d_1 and d_2 become farther apart, then the VSI control chart tends to be less efficient in terms of ANSS. Therefore, if it turns out that it takes too many samples to signal, then we need to make the sampling intervals d_1 and d_2 less apart. As an illustration, suppose that we used VSI (0.1, 1.9) as our control chart when ϕ is 0.9 and when δ is 2. Accordign to Table 11, the average time to signal is 17.737. On the other hand, the average number of samples to signal is 56.818 which is rather large. If we cannot afford 56.818 samples, then we can make the sampling intervals d_1 and d_2 less apart, as in VSI (0.5, 1.5). In this case, we need to alter the control and warning limits (r and r′) again in order for the new in control ATS and new in control ANSS to be close to our standard. Now the new ATS is 17.85 (see Table 10) which is slightly larger than its provious counterpart 17.378. However, the new ANSS is only 27.136 which is a lot smaller than its previous counterpart 56.818. This type of procedure contiues until we get the out of control ATS as well as out of control ANSS to a certain desired level for any given correlation ϕ and deviation δ.

Table 1. ATS (= ANSS) of FSI (1) for unaltered control limits

δ / ϕ	0	0.25	0.5	1	2	3	r
-.9	831.778	545.088	281.971	84.737	9.699	1.603	3.000
-.8	555.189	388.811	202.700	58.953	7.322	1.650	3.000
-.7	462.792	335.188	177.449	50.568	6.502	1.690	3.000
-.6	419.377	309.653	166.148	46.813	6.119	1.727	3.000
-.5	396.281	296.005	160.423	44.940	5.936	1.764	3.000
-.4	383.461	288.436	157.394	43.979	5.869	1.803	3.000
-.3	376.383	284.271	155.800	43.515	5.884	1.844	3.000
-.2	372.652	282.105	155.039	43.371	5.964	1.890	3.000
-.1	370.903	281.179	154.859	43.493	6.103	1.941	3.000
.0	370.398	281.153	155.224	43.895	6.303	2.000	3.000
.1	370.903	282.019	156.263	44.638	6.572	2.069	3.000
.2	372.652	284.091	158.258	45.828	6.926	2.153	3.000
.3	376.383	288.037	161.672	47.631	7.393	2.258	3.000
.4	383.461	295.007	167.257	50.313	8.020	2.392	3.000
.5	396.281	306.985	176.294	54.347	8.893	2.574	3.000
.6	419.377	327.732	191.231	60.649	10.182	2.835	3.000
.7	462.792	365.617	217.560	71.306	12.274	3.253	3.000
.8	555.189	444.626	271.117	92.390	16.301	4.045	3.000
.9	831.778	678.247	427.224	152.999	27.704	6.259	3.000

A fixed sampling interval (d = 1) is used if an observation falls in $(-r\sigma_x, r\sigma_x)$.

Table 2. $\frac{ATS}{ANSS}$ of VSI (0.5, 1.5) for unaltered control limits

ϕ \ δ	0	0.25	0.5	1	2	3	r	r′
.0	370.398 370.398	277.563 281.153	147.637 155.224	36.685 43.895	4.209 6.303	1.520 2.000	3.000	0.672
.1	311.158 314.138	227.825 233.605	118.896 127.348	31.385 38.567	4.579 6.918	1.656 2.254	3.000	0.672
.2	275.552 281.547	205.002 212.882	110.219 119.734	30.977 38.643	4.934 7.537	1.749 2.430	3.000	0.672
.3	255.264 264.639	193.479 203.899	107.414 118.455	31.905 40.367	5.383 8.322	1.854 2.628	3.000	0.672
.4	247.148 260.689	190.720 204.469	109.167 122.417	34.035 43.691	5.977 9.362	1.982 2.873	3.000	0.672
.5	251.030 270.197	196.877 215.309	115.856 132.391	37.670 49.115	6.798 10.802	2.151 3.198	3.000	0.672
.6	269.730 297.230	214.544 240.080	129.369 151.041	43.609 57.845	8.008 12.929	2.393 3.665	3.000	0.672
.7	311.617 352.823	250.923 288.302	154.607 184.983	53.786 72.733	9.973 16.390	2.780 4.410	3.000	0.672
.8	402.231 469.803	327.668 388.041	206.114 253.608	73.886 102.124	13.766 23.079	3.516 5.832	3.000	0.672
.9	666.194 807.570	549.953 675.238	354.343 450.771	131.475 186.494	24.579 42.184	5.598 9.853	3.000	0.672

A sampling interval ($d_1 = 0.5$) is used if an observation falls in ($-r\sigma_x$, $-r'\sigma_x$) or ($r'\sigma_x$, $r\sigma_x$).

A sampling interval ($d_2 = 1.5$) is used if an observation falls in ($-r'\sigma_x$, $r'\sigma_x$).

Table 3. $\frac{ATS}{ANSS}$ of VSI (0.1, 1.9) for unaltered control limits

δ / φ	0	0.25	0.5	1	2	3	r	r′
.0	370.398 370.398	274.691 281.153	141.568 155.224	30.918 43.895	2.533 6.303	1.135 2.000	3.000	0.672
.1	164.815 217.657	123.264 170.340	66.954 104.507	19.265 42.366	3.275 11.427	1.361 3.672	3.000	0.672
.2	149.649 223.013	114.042 178.134	64.024 113.320	19.559 48.772	3.569 13.891	1.439 4.364	3.000	0.672
.3	142.251 236.935	110.392 192.672	63.944 126.306	20.549 56.802	3.919 16.728	1.523 5.125	3.000	0.672
.4	140.357 261.116	110.877 215.837	66.221 145.327	22.290 67.730	4.382 20.388	1.625 6.068	3.000	0.672
.5	144.832 301.820	116.347 253.147	71.541 174.508	25.119 83.703	5.034 25.491	1.762 7.339	3.000	0.672
.6	158.887 372.900	129.567 316.637	81.796 222.587	29.799 109.046	6.025 33.228	1.962 9.206	3.000	0.672
.7	190.701 508.204	157.466 435.603	101.636 310.657	38.148 153.987	7.694 46.336	2.290 12.276	3.000	0.672
.8	264.788 811.797	220.707 699.908	144.862 503.036	55.542 249.285	11.048 72.933	2.937 18.338	3.000	0.672
.9	496.291 1798.340	416.646 1554.513	276.971 1118.429	107.812 546.734	20.970 152.814	4.828 36.132	3.000	0.672

A sampling interval ($d_1 = 0.1$) is used if an observation falls in ($-r\sigma_x$, $-r'\sigma_x$) or ($r'\sigma_x$, $r\sigma_x$).

A sampling interval ($d_2 = 1.9$) is used if an observation falls in ($-r'\sigma_x$, $r'\sigma_x$).

Table 4. $\frac{ATS}{ANSS}$ of VSI (0.1, 1.0, 1.9) for unaltered control limits

ϕ \ δ	0	0.25	0.5	1	2	3
.0	370.398 370.398	275.154 281.153	142.513 155.224	31.700 43.895	2.660 6.303	1.146 2.000
.1	144.349 187.495	108.487 147.187	59.296 90.523	17.194 36.814	3.073 10.304	1.350 3.503
.2	130.732 188.337	100.227 151.131	56.791 96.761	17.607 42.171	3.366 12.570	1.424 4.156
.3	125.674 198.303	98.100 162.105	57.394 107.235	18.774 49.184	3.731 15.237	1.506 4.888
.4	126.440 217.683	100.405 180.961	60.538 123.209	20.739 58.885	4.217 18.695	1.609 5.804
.5	133.646 251.099	107.840 211.901	66.864 147.977	23.839 73.101	4.899 23.511	1.748 7.043
.6	150.485 309.699	123.151 264.669	78.265 188.724	28.843 95.522	5.926 30.768	1.951 8.860
.7	185.332 421.135	153.384 363.223	99.422 262.809	37.571 134.808	7.638 42.942	2.284 11.833
.8	262.479 667.623	218.984 578.835	143.967 421.818	55.329 216.548	11.030 67.335	2.935 17.663
.9	490.588 1429.444	412.628 1244.561	275.135 910.959	107.514 464.923	20.979 139.940	4.833 34.760

$r = 3.000$

$r' = 0.964$

$r'' = 0.429$

A sampling interval ($d_1 = 0.1$) is used if an observation falls in ($-r\sigma_x$, $-r'\sigma_x$) or ($r'\sigma_x$, $r\sigma_x$).

A sampling interval ($d_2 = 1.0$) is used if an observation falls in ($-r'\sigma_x$, $-r''\sigma_x$) or ($r''\sigma_x$, $r'\sigma_x$).

A sampling interval ($d_3 = 1.9$) is used if an observation falls in ($-r''\sigma_x$, $r''\sigma_x$).

Table 5. $\frac{ATS}{ANSS}$ of VSI (0.1, 0.5, 1.0, 1.5, 1.9) for unaltered control limits

ϕ \ δ	0	0.25	0.5	1	2	3
.0	370.398 370.398	275.500 281.153	143.226 155.224	32.310 43.895	2.772 6.303	1.157 2.000
.1	140.404 171.596	104.999 133.763	56.980 81.303	16.430 32.679	2.979 9.378	1.341 3.337
.2	126.312 166.641	96.574 133.205	54.548 84.820	16.933 36.989	3.288 11.428	1.415 3.955
.3	121.276 171.608	94.630 140.178	55.384 92.731	18.204 42.908	3.668 13.851	1.498 4.651
.4	122.374 185.867	97.292 154.735	58.813 105.753	20.274 51.248	4.168 16.995	1.603 5.523
.5	130.179 213.075	105.231 180.265	65.468 126.606	23.480 63.570	4.863 21.371	1.743 6.698
.6	147.855 262.370	121.193 224.816	77.243 161.282	28.594 83.016	5.903 27.947	1.949 8.418
.7	183.565 355.696	152.081 307.531	98.761 223.828	37.421 116.797	7.626 38.921	2.283 11.225
.8	260.168 555.057	217.342 482.797	143.200 354.528	55.202 185.882	11.036 60.818	2.938 16.744
.9	466.594 1112.709	395.461 978.801	267.341 731.770	106.647 390.420	21.284 127.353	4.935 33.683

$r = 3.000$
$r' = 1.275$
$r'' = 0.839$
$r''' = 0.523$
$r'''' = 0.253$

A sampling interval ($d_1 = 0.1$) is used if an observation falls in $(-r\sigma_x, -r'\sigma_x)$ or $(r'\sigma_x, r\sigma_x)$.

A sampling interval ($d_2 = 0.5$) is used if an observation falls in $(-r'\sigma_x, -r''\sigma_x)$ or $(r''\sigma_x, r'\sigma_x)$.

A sampling interval ($d_3 = 1.0$) is used if an observation falls in $(-r''\sigma_x, -r'''\sigma_x)$ or $(r'''\sigma_x, r''\sigma_x)$.

A sampling interval ($d_4 = 1.5$) is used if an observation falls in $(-r'''\sigma_x, -r''''\sigma_x)$ or $(r''''\sigma_x, r'''\sigma_x)$.

A sampling interval ($d_5 = 1.9$) is used if an observation falls in $(-r''''\sigma_x, r''''\sigma_x)$.

Table 6. Number of Sampling Intervals

$\frac{ATS}{ANSS}$ of VSI (0.1, 1.9) for unaltered control limits

ϕ \ δ	0	0.25	0.5	1	2	3	r	r′
.0	370.40	274.70	141.57	30.92	2.53	1.14	3.000	0.672
	370.40	281.15	155.22	43.90	6.30	2.00		
.3	369.50	272.64	144.62	39.71	5.92	1.88	3.263	1.129
	370.63	287.50	174.62	71.00	20.09	6.62		
.6	370.27	286.95	165.44	52.02	8.80	2.50	3.231	1.347
	370.50	302.14	199.81	93.29	31.03	10.43		
.9	370.46	315.91	216.38	88.18	17.74	4.13	2.883	1.566
	369.49	328.38	251.85	144.42	56.82	18.17		

$\frac{ATS}{ANSS}$ of VSI (0.1, 1.0, 1.9) for unaltered control limits

ϕ \ δ	0	0.25	0.5	1	2	3	r	r′	r″
.0	370.40	275.15	142.51	31.70	2.66	1.15	3.000	0.964	0.429
	370.40	281.15	155.22	43.90	6.30	2.00			
.3	371.40	271.23	141.70	38.29	5.78	1.89	3.304	1.591	0.429
	370.25	281.14	163.71	62.08	17.39	6.12			
.6	370.65	287.33	165.80	52.30	8.93	2.56	3.270	1.795	0.429
	369.42	297.07	189.82	82.40	26.44	9.38			
.9	370.32	315.88	216.48	88.29	17.78	4.14	2.887	1.934	0.429
	371.22	324.88	239.39	124.02	43.72	14.23			

$\frac{ATS}{ANSS}$ of VSI (0.1, 0.5, 1.0, 1.5, 1.9) for unaltered control limits

ϕ \ δ	0	0.25	0.5	1	2	3	r	r′	r″	r‴	r⁗
.0	370.40	275.50	143.23	32.31	2.77	1.16	3.000	1.275	0.839	0.523	0.253
	370.40	281.15	155.22	43.90	6.30	2.00					
.3	369.52	268.98	140.00	37.82	5.76	1.90	3.313	2.224	0.889	0.523	0.253
	371.09	277.93	156.73	54.77	13.82	4.96					
.6	369.57	287.22	166.41	52.75	9.04	2.59	3.231	2.693	0.989	0.523	0.253
	371.32	294.55	180.66	68.46	16.61	5.30					
.9	370.32	315.93	216.56	88.35	17.79	4.14	2.838	2.661	1.189	0.523	0.253
	370.83	320.84	228.78	105.97	27.80	7.16					

Table 7. $\frac{ATS}{ANSS}$ of VSI (0.5, 2.0) for unaltered control limits

δ / ϕ	0	0.25	0.5	1	2	3	r	r′
.0	370.398 370.398	277.214 281.153	146.933 155.224	36.125 43.895	4.130 6.303	1.514 2.000	3.000	0.429
.1	326.413 328.935	239.715 245.559	125.277 134.384	32.684 40.545	4.608 7.087	1.654 2.264	3.000	0.429
.2	297.145 302.584	220.338 228.275	117.594 127.815	32.415 40.799	4.987 7.747	1.749 2.445	3.000	0.429
.3	277.628 286.638	208.831 219.455	114.415 126.253	33.219 42.437	5.440 8.550	1.855 2.648	3.000	0.429
.4	266.617 280.236	203.942 218.198	115.049 129.246	35.110 45.581	6.027 9.599	1.983 2.896	3.000	0.429
.5	265.417 285.443	206.659 226.164	120.173 137.962	38.446 50.825	6.837 11.053	2.153 3.224	3.000	0.429
.6	278.259 308.166	220.386 248.201	131.957 155.602	44.072 59.513	8.032 13.211	2.395 3.695	3.000	0.429
.7	315.039 362.362	253.293 295.944	155.669 189.879	53.977 74.791	9.984 16.750	2.780 4.447	3.000	0.429
.8	402.780 486.860	328.053 402.169	206.290 262.913	73.919 105.811	13.768 23.643	3.516 5.886	3.000	0.429
.9	666.181 861.010	549.944 719.328	354.340 479.059	131.475 196.529	24.579 43.444	5.598 9.960	3.000	0.429

A sampling interval ($d_1 = 0.5$) is used if an observation falls in ($-r\sigma_x$, $-r'\sigma_x$) or ($r'\sigma_x$, $r\sigma_x$).

A sampling interval ($d_2 = 2.0$) is used if an observation falls in ($-r'\sigma_x$, $r'\sigma_x$).

Table 8. $\frac{ATS}{ANSS}$ of VSI (0.5, 3.0) for unaltered control limits

ϕ \ δ	0	0.25	0.5	1	2	3	r	r′
.0	370.398	277.049	146.600	35.864	4.095	1.512	3.000	0.253
	370.398	281.153	155.224	43.895	6.303	2.000		
.1	343.835	254.511	134.067	34.669	4.670	1.654	3.000	0.253
	345.509	259.911	143.311	42.926	7.235	2.270		
.2	325.539	242.210	129.363	34.996	5.093	1.751	3.000	0.253
	329.396	249.249	139.570	43.783	7.955	2.456		
.3	312.497	234.333	127.292	35.956	5.569	1.858	3.000	0.253
	319.286	243.643	138.937	45.535	8.784	2.661		
.4	303.503	230.096	127.663	37.703	6.160	1.988	3.000	0.253
	314.333	242.606	141.415	48.443	9.830	2.912		
.5	299.505	230.445	131.295	40.659	6.956	2.158	3.000	0.253
	316.141	247.662	148.253	53.157	11.255	3.241		
.6	304.861	238.863	140.444	45.711	8.123	2.400	3.000	0.253
	330.546	263.607	162.684	61.063	13.366	3.711		
.7	330.684	264.190	160.637	54.911	10.037	2.784	3.000	0.253
	372.666	302.769	192.753	75.432	16.858	4.462		
.8	407.480	331.349	207.794	74.196	13.784	3.517	3.000	0.253
	486.493	401.605	262.483	105.939	23.754	5.901		
.9	666.267	550.007	354.369	131.480	24.580	5.598	3.000	0.253
	873.356	729.666	485.961	199.257	43.839	9.994		

A sampling interval ($d_1 = 0.5$) is used if an observation falls in $(-r\sigma_x, -r'\sigma_x)$ or $(r'\sigma_x, r\sigma_x)$.

A sampling interval ($d_2 = 3.0$) is used if an observation falls in $(-r'\sigma_x, r'\sigma_x)$.

Table 9. $\frac{ATS}{ANSS}$ of FSI (1) for altered control limits

δ / φ	0	0.25	0.5	1	2	3	r
.0	370.398	281.153	155.224	43.895	6.303	2.000	3.000
.1	369.688	281.140	155.820	44.533	6.562	2.068	2.999
.2	370.222	282.330	157.366	45.616	6.905	2.150	2.998
.3	370.307	283.624	159.428	47.090	7.338	2.248	2.995
.4	370.080	285.246	162.256	49.087	7.890	2.368	2.989
.5	370.587	288.126	166.528	51.899	8.621	2.521	2.979
.6	369.876	291.074	171.958	55.675	9.596	2.718	2.960
.7	369.917	295.981	180.173	61.282	11.012	2.994	2.927
.8	370.871	303.962	193.295	70.409	13.298	3.415	2.864
.9	370.173	315.965	216.794	88.558	17.853	4.157	2.711

A fixed sampling interval (d = 1) is used if an observation falls in $(-r\sigma_x, r\sigma_x)$.

Table 10. $\frac{ATS}{ANSS}$ of VSI (0.5, 1.5) for altered control limits

δ / φ	0	0.25	0.5	1	2	3	r	r′
.0	370.398 370.398	277.563 281.153	147.637 155.224	36.685 43.895	4.209 6.303	1.520 2.000	3.000	0.672
.1	370.163 371.281	268.423 273.498	137.857 146.813	35.374 43.332	4.915 7.475	1.714 2.364	3.053	0.683
.2	371.373 370.019	271.677 275.252	141.999 150.746	37.983 46.602	5.587 8.568	1.867 2.651	3.090	0.715
.3	371.053 369.689	275.327 279.079	147.423 156.758	41.131 50.649	6.313 9.764	2.028 2.952	3.114	0.742
.4	370.077 371.281	279.058 284.859	153.483 164.384	44.708 55.434	7.129 11.127	2.204 3.285	3.125	0.762
.5	370.744 369.464	284.254 288.208	160.777 171.010	48.926 60.230	8.089 12.621	2.406 3.655	3.121	0.812
.6	370.686 369.464	289.373 293.353	168.802 179.366	53.898 66.139	9.262 14.460	2.647 4.095	3.100	0.853
.7	369.836 371.328	294.782 300.874	178.341 190.514	60.256 74.038	10.816 16.918	2.954 4.657	3.055	0.882
.8	370.989 369.475	303.605 307.145	192.587 203.522	69.958 84.658	13.207 20.449	3.397 5.438	2.972	0.939
.9	371.167 369.457	316.665 319.622	217.088 227.831	88.580 105.271	17.850 27.136	4.158 6.739	2.789	0.980

A sampling interval ($d_1 = 0.5$) is used if an observation falls in ($-r\sigma_x$, $-r'\sigma_x$) or ($r'\sigma_x$, $r\sigma_x$).

A sampling interval ($d_2 = 1.5$) is used if an observation falls in ($-r'\sigma_x$, $r'\sigma_x$).

Table 11. $\frac{ATS}{ANSS}$ of VSI (0.1, 1.9) for altered control limits

ϕ \ δ	0	0.25	0.5	1	2	3	r	r′
.0	370.398 370.398	274.691 281.153	141.568 155.224	30.918 43.895	2.533 6.303	1.135 2.000	3.000	0.672
.1	371.382 370.076	266.488 278.101	134.917 159.722	34.107 58.685	4.602 14.574	1.565 4.684	3.228	0.952
.2	369.978 371.327	269.193 283.668	139.505 167.800	36.813 65.156	5.256 17.386	1.720 5.666	3.252	1.046
.3	369.504 370.629	272.640 287.503	144.618 174.615	39.711 71.003	5.924 20.093	1.875 6.623	3.263	1.129
.4	370.279 371.259	277.185 292.403	150.635 182.267	43.071 77.440	6.691 23.132	2.047 7.691	3.265	1.204
.5	369.501 370.875	281.105 296.957	157.009 190.379	46.952 84.676	7.607 26.683	2.248 8.929	3.255	1.274
.6	370.266 370.497	286.952 302.135	165.436 199.805	52.023 93.290	8.802 31.029	2.501 10.433	3.231	1.347
.7	370.642 369.458	293.819 307.871	176.171 211.185	58.843 104.117	10.443 36.615	2.837 12.320	3.183	1.422
.8	371.160 369.886	302.932 316.572	191.281 227.292	69.071 119.507	12.958 44.504	3.320 14.833	3.091	1.497
.9	370.460 369.491	315.912 328.380	216.380 251.853	88.178 144.420	17.737 56.818	4.125 18.173	2.883	1.566

A sampling interval ($d_1 = 0.1$) is used if an observation falls in $(-r\sigma_x, -r'\sigma_x)$ or $(r'\sigma_x, r\sigma_x)$.

A sampling interval ($d_2 = 1.9$) is used if an observation falls in $(-r'\sigma_x, r'\sigma_x)$.

Table 12. $\frac{ATS}{ANSS}$ of VSI (0.5, 2.0) for altered control limits

δ / φ	0	0.25	0.5	1	2	3	r	r′
.0	370.398 370.398	277.214 281.153	146.933 155.224	36.125 43.895	4.130 6.303	1.514 2.000	3.000	0.429
.1	369.573 371.328	269.546 275.342	139.251 149.022	35.596 44.134	4.846 7.497	1.695 2.343	3.038	0.432
.2	370.946 369.446	271.519 275.312	141.810 151.149	37.659 46.792	5.460 8.520	1.834 2.607	3.066	0.452
.3	371.301 370.214	274.518 278.915	146.114 156.383	40.387 50.563	6.147 9.694	1.986 2.899	3.088	0.466
.4	371.306 370.277	278.485 283.170	151.927 162.976	43.832 55.140	6.955 11.067	2.162 3.233	3.102	0.488
.5	370.247 371.305	282.667 289.100	158.871 171.486	48.037 60.809	7.929 12.728	2.369 3.628	3.104	0.511
.6	370.492 369.458	288.507 293.427	167.665 179.988	53.333 67.164	9.156 14.723	2.622 4.102	3.090	0.561
.7	371.354 369.406	295.636 299.795	178.509 190.831	60.173 75.256	10.785 17.316	2.945 4.700	3.052	0.621
.8	371.267 369.424	303.759 307.806	192.605 205.209	69.930 86.646	13.197 21.070	3.395 5.519	2.971	0.687
.9	370.421 369.520	316.062 320.427	216.718 229.796	88.456 107.783	17.829 28.039	4.154 6.864	2.788	0.751

A sampling interval ($d_1 = 0.5$) is used if an observation falls in $(-r\sigma_x, -r'\sigma_x)$ or $(r'\sigma_x, r\sigma_x)$.

A sampling interval ($d_2 = 2.0$) is used if an observation falls in $(-r'\sigma_x, r'\sigma_x)$.

Table 13. $\frac{ATS}{ANSS}$ of VSI (0.5, 3.0) for altered control limits

δ / ϕ	0	0.25	0.5	1	2	3	r	r′
.0	370.398 370.398	277.049 281.153	146.600 155.224	35.864 43.895	4.095 6.303	1.512 2.000	3.000	0.253
.1	369.643 370.788	272.537 277.853	142.601 152.218	36.436 45.100	4.809 7.475	1.677 2.315	3.022	0.254
.2	369.638 370.684	273.013 278.533	144.019 154.190	38.130 47.492	5.363 8.411	1.799 2.548	3.038	0.258
.3	369.597 370.461	274.351 280.027	146.507 157.278	40.205 50.385	5.968 9.442	1.931 2.802	3.050	0.264
.4	370.306 369.624	277.251 282.095	150.624 161.583	42.973 54.047	6.696 10.679	2.089 3.105	3.059	0.275
.5	370.762 369.442	281.415 286.088	156.680 168.281	46.752 59.079	7.626 12.272	2.288 3.486	3.064	0.290
.6	369.648 370.961	286.123 293.061	164.737 178.410	51.857 66.123	8.854 14.397	2.546 3.980	3.059	0.308
.7	371.255 370.248	294.432 299.793	176.718 190.417	59.190 75.235	10.579 17.277	2.893 4.639	3.035	0.354
.8	370.714 369.549	303.004 308.147	191.813 206.021	69.518 87.709	13.107 21.362	3.372 5.531	2.965	0.431
.9	370.491 369.925	316.098 321.209	216.713 231.170	88.438 109.370	17.823 28.574	4.152 6.933	2.788	0.545

A sampling interval ($d_1 = 0.5$) is used if an observation falls in ($-r\sigma_x$, $-r'\sigma_x$) or ($r'\sigma_x$, $r\sigma_x$).

A sampling interval ($d_2 = 3.0$) is used if an observation falls in ($-r'\sigma_x$, $r'\sigma_x$).

Table 14. Relative Efficiency : $\dfrac{ATS \text{ of } FSI(1)}{ATS \text{ of } VSI(0.5, 1.5)}$

δ / ϕ	0	0.25	0.5	1	2	3
.0	1.000	1.013	1.051	1.197	1.498	1.316
.1	0.999	1.047	1.130	1.259	1.335	1.207
.2	0.997	1.039	1.108	1.201	1.236	1.152
.3	0.998	1.030	1.081	1.145	1.162	1.108
.4	1.000	1.022	1.057	1.098	1.107	1.074
.5	1.000	1.014	1.036	1.061	1.066	1.048
.6	0.998	1.006	1.019	1.033	1.036	1.027
.7	1.000	1.004	1.010	1.017	1.018	1.014
.8	1.000	1.001	1.004	1.006	1.007	1.005
.9	0.997	0.998	0.999	1.000	1.000	1.000

Table 15. Relative Efficiency : $\dfrac{ATS \text{ of } FSI(1)}{ATS \text{ of } VSI(0.1, 1.9)}$

δ / ϕ	0	0.25	0.5	1	2	3
.0	1.000	1.024	1.096	1.420	2.488	1.762
.1	0.995	1.055	1.155	1.306	1.426	1.321
.2	1.001	1.049	1.128	1.239	1.314	1.250
.3	1.002	1.040	1.102	1.186	1.239	1.199
.4	0.999	1.029	1.077	1.140	1.179	1.157
.5	1.003	1.025	1.061	1.105	1.133	1.121
.6	0.999	1.014	1.039	1.070	1.090	1.087
.7	0.998	1.007	1.023	1.041	1.054	1.055
.8	0.999	1.003	1.011	1.019	1.026	1.029
.9	0.999	1.000	1.002	1.004	1.007	1.008

Table 16. Relative Inefficiency : $\dfrac{ANSS \text{ of } VSI(0.5, 1.5)}{ANSS \text{ of } FSI(1)}$

ϕ \ δ	0	0.25	0.5	1	2	3
.0	1.000	1.000	1.000	1.000	1.000	1.000
.1	1.004	0.973	0.942	0.973	1.139	1.143
.2	0.999	0.975	0.958	1.022	1.241	1.233
.3	0.998	0.984	0.983	1.076	1.331	1.313
.4	1.003	0.999	1.013	1.129	1.410	1.387
.5	0.997	1.000	1.027	1.161	1.464	1.450
.6	0.999	1.008	1.043	1.188	1.507	1.507
.7	1.004	1.017	1.057	1.208	1.536	1.555
.8	0.996	1.010	1.053	1.202	1.538	1.592
.9	0.998	1.012	1.051	1.189	1.520	1.621

Table 17. Relative Inefficiency : $\dfrac{ANSS \text{ of } VSI(0.1, 1.9)}{ANSS \text{ of } FSI(1)}$

ϕ \ δ	0	0.25	0.5	1	2	3
.0	1.000	1.000	1.000	1.000	1.000	1.000
.1	1.001	0.989	1.025	1.318	2.221	2.265
.2	1.003	1.005	1.066	1.428	2.518	2.635
.3	1.001	1.014	1.095	1.508	2.738	2.946
.4	1.003	1.025	1.123	1.578	2.932	3.248
.5	1.001	1.031	1.143	1.632	3.095	3.542
.6	1.002	1.038	1.162	1.676	3.234	3.838
.7	0.999	1.040	1.172	1.699	3.325	4.115
.8	0.997	1.041	1.176	1.697	3.347	4.343
.9	0.998	1.039	1.162	1.631	3.183	4.372

Table 18. ATS and ANSS for a fixed start under AR (1) process

ATS (= ANSS) of FSI (1) for unaltered control limits

δ / ϕ	0	0.25	0.5	1	2	3	r
.0	370.398	281.153	155.224	43.895	6.303	2.000	3.000
.3	376.811	288.403	161.941	47.696	7.426	2.259	3.000
.6	421.165	329.364	192.597	61.372	10.464	2.776	3.000
.9	842.153	687.900	435.572	158.167	29.985	4.926	3.000

$\frac{ATS}{ANSS}$ of VSI (0.5, 1.5) for unaltered control limits

δ / ϕ	0	0.25	0.5	1	2	3	r	r′
.0	370.398 370.398	277.563 281.153	147.637 155.224	36.685 43.895	4.209 6.303	1.520 2.000	3.000	0.672
.3	255.627 264.989	193.775 204.190	107.612 118.665	31.922 40.421	5.393 8.363	1.850 2.627	3.000	0.672
.6	271.298 298.771	215.953 241.498	130.503 152.281	44.115 58.669	8.165 13.342	2.318 3.548	3.000	0.672
.9	676.210 817.944	559.255 685.174	362.327 460.172	136.227 194.550	26.354 46.351	4.203 7.272	3.000	0.672

$\frac{ATS}{ANSS}$ of VSI (0.1, 1.9) for unaltered control limits

δ / ϕ	0	0.25	0.5	1	2	3	r	r′
.0	370.398 370.398	274.691 281.153	141.568 155.224	30.918 43.895	2.533 6.303	1.135 2.000	3.000	0.672
.3	142.599 237.386	110.668 193.049	64.112 126.592	20.527 56.920	3.906 16.839	1.513 5.107	3.000	0.672
.6	160.386 375.303	130.890 318.916	82.807 224.809	30.136 111.202	6.060 34.572	1.870 8.688	3.000	0.672
.9	505.900 1821.359	425.558 1577.653	284.556 1143.479	112.102 577.499	22.161 170.481	3.377 23.672	3.000	0.672

Chapter IV

STEADY STATE PROPERTIES OF $\overline{X}$ CONTROL CHARTS UNDER AN AR (1) PROCESS

4.1 Steady State Properties of Shewhart $\overline{X}$ Control Charts

In Chapter 3, a shift in the process mean is assumed to have occurred at the beginning of the process. However, the process mean may start out at the target value μ_0 and then change to some other value $\mu = \mu_0 + \delta\sigma_x$, where $\delta \neq 0$, at some random time in the future. The control statistic $X'(t)$ has a distribution over its possible values in the long run. This is called a steady state distribution of the control statistic. The steady state ATS and steady state ANSS have traditionally been defined as the ATS and ANSS computed after the process has been running for some time and the shift occurs immediately after a sample is taken. However, the shift might happen at some random point in the short or long sampling interval (d_1 or d_2), or even in the middle of the regular sampling interval (d). In order to incorporate this feature, we redefine the steady state time and steady state number of samples to signal as the time and number of samples to signal from the shift to where the process signals, assuming that the shift can occur anywhere between samples. Their expected values are defined as the steady state ATS (ATS^*) and steady state ANSS ($ANSS^*$). In order to investigate the properties of ATS^* and $ANSS^*$ in $\overline{X}$ control charts, let

T^* = the steady state time to signal
N^* = the steady state number of samples to signal
U = the length of the interval in which the shift occurs
Y = the time from the process shift to the next sample
Z = the time from the first sample after the process shift to the signal

It is clear from Figure 2 that $T^* = Y + Z$. Hence, the steady state ATS is

$$E(T^*) = E(Y) + E(Z).$$

The steady state ATS and steady state ANSS are·the weighted averages of ATS's and ANSS's conditioned on the initial state the scheme was in control when the shift occured, where the weights are the steady state distribution of the control statistic. The steady state ARL differs from the conventional ARL because of the possible head start as noted in Roberts (1966). There are several ways to get steady state probabilities, and they depend on how false alarms before the actual shift in the process mean are treated.

Taylor (1968) used a simulation method in order to calculate two types of steady state ARL's. According to the first method, if there is any false alarm before the actual shift occurs, then the control statistic is reset back to its initial value μ_0 because the process mean is still on target and an observation happened to be outside the control limits. Hence, Croiser (1986) called the ARL computed using the first method "cyclical" steady state ARL. In order to construct a transition

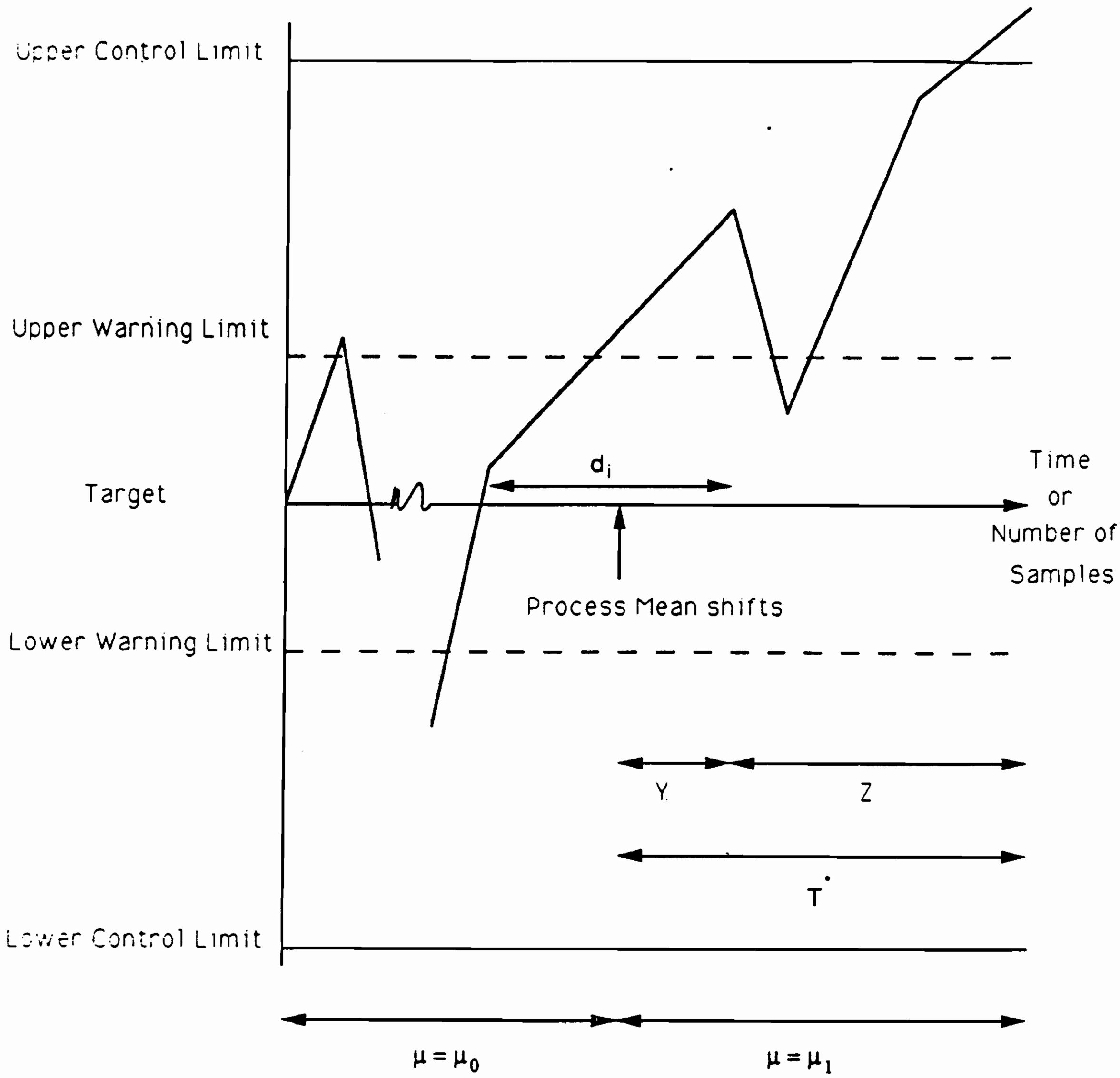

Figure 2. Steady State Time to Signal

matrix, we need to add a row corresponding to a state where we have an observation outside the control limits and reset it back to its initial value. Then the new transition matrix is

$$P^* = \begin{bmatrix} Q & (I-Q)\underline{1} \\ 0, \dots, 1, \dots, 0 & 0 \end{bmatrix}.$$

Now solve

$$\underline{p} = P^* \underline{p},$$

subject to $\underline{1}'\underline{p} = 1$, where

$$\underline{p} = [p_1, p_2, \dots, p_v, p_a].$$

Note that p_a is the probability that the process is in absorbing state in the long run. Paige, Styan and Wachter (1975) suggested an efficient algorithm for computing the stationary distribution of a Markov chain. Their recommended algorithm is $p'(I - P^* + e\,u') = u'$ where e is the $(v+1)*1$ column vector of ones and u' is any $1*(v+1)$ row vector of P^*. Then the cyclical steady state probability vector $\underline{\pi} = [\pi_1, \pi_2, \dots, \pi_v]$ is found by $\underline{\pi} = (\underline{1}'\underline{q})^{-1}\underline{q}$ where $\underline{q}$ is a vector obtained from $\underline{p}$ by deleting the entry corresponding to the absorbing state. This method of calculating the steady state probabilities reflects the way control schemes are actually used. However, since we reset the control statistic back to its initial value, we may lose the continuity of the control statistic. In particular, when the sequential observations are correlated according to AR (1), we may in fact alter the course of the original process as well as the control statistic because we are modelling the process itself with the Markov chain.

According to the second method, we get steady state probabilities assuming that the sequential observations are conditioned on not yielding a wrong signal. Therefore, the ARL which is computed using the second method is called a "conditional" steady state ARL by Croiser (1986). The conditional steady state probability vector $\underline{\pi}$ can be found by solving

$$\underline{\pi} = \tilde{Q}\underline{\pi},$$

subject to $\underline{1}'\underline{\pi} = 1$, where $\tilde{Q}$ is taken from Q by dividing Q by the sum of elements in each row. An efficient algorithm by Paige, Styan and Wachter (1975) is used to compute the steady state probabilities of a Markov chain.

Yashchin (1985) introduced a different way of doing the conditioning to get steady state probabilities. As in the conditional steady state probabilities, the false alarm is not assumed to have been triggered. Darroch and Seneta (1965) showed that the relevant steady state probabilities are obtained by the normalized left eigenvector corresponding to the maximal real eigenvalue λ_0 of the transition matrix Q.

Finally, there is still another approach to be used in the computation in this chapter to get steady state probabilities. Recall that the first 'cyclical' method resets whatever is observed outside the control limits back to its initial value. However, we might really have some observations outside the control limits even though the process mean is in control. The second and third methods assume that there are no false alarms. However, false alarms always occur in practice. In order to get more appropriate steady state probabilities, we will allow the observations to be anywhere outside the control limits and let the process go without interruption. In the computation, however, we break down the whole region into the original v discrete intervals with the control limits and $2(w+1)$ discrete intervals outside the control limits as shown in Figure 3. For convention, the sampling intervals used for those $2(w+1)$ intervals are the short sampling interval d_1. We will have 'maximum' and 'minimum' limits where those limits are so far out that there is little probability beyond them. If an observation falls anywhere in the region U_{w+1} (or V_{w+1}), then we reset it to u_{w+1} (v_{w+1}) Therefore, the altering of the process is very minimal since we do not reset it back to its initial value μ_0 as practiced in cyclical steady state ATS but rather to the value that is closest to the extreme value. Since there are $2(w+1)$ additional states, we need to have the same number of states added to the original transitional matrix Q in equation [3.10]. Then the steady state probabilities are obtained using an efficient algorithm by Paige, Styan and Wachter (1975), and will be used in this Chapter IV.

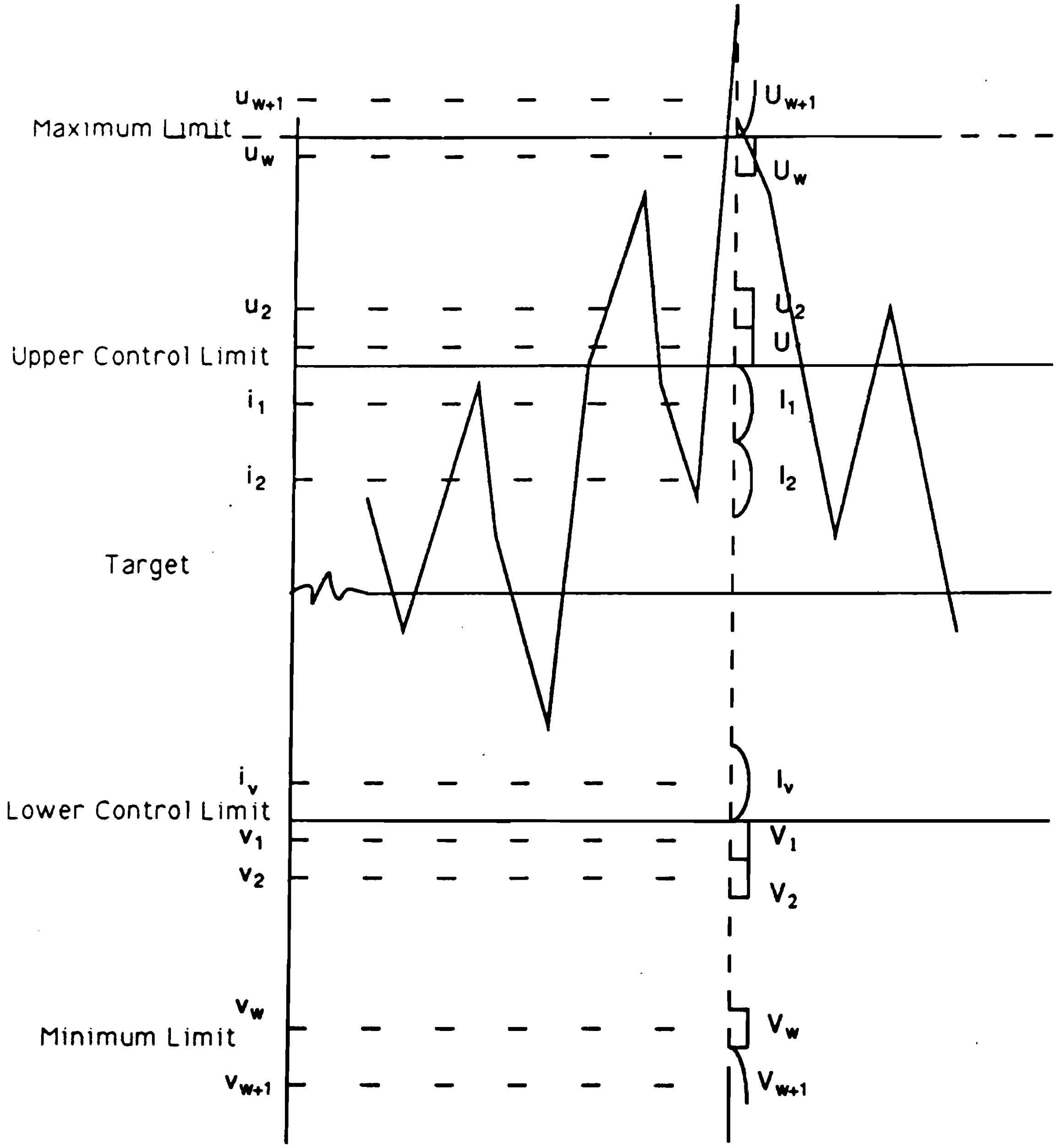

Figure 3. Steady State Probabilities

4.2 Steady State Properties of $\overline{X}$ Charts Under Sequentially Independent Observations

Suppose that η different sampling intervals $d_1, d_2, \ldots, d_\eta$ are used in a VSI control chart. Let p_{0j} be the probability that the sampling interval d_j is used when the process mean μ is on target value μ_0. On the other hand, let p_{1j} be the probability that the sampling interval d_j is used when the process mean is $\mu_1 = \mu_0 + \delta\sigma_x$, $\delta \neq 0$. The expected number of samples to signal, when the process mean is $\mu = \mu_1$, is $\frac{1}{q_1}$, where $q_1 = 1 - (p_{1,1} + p_{1,2} + \cdots + p_{1,\eta})$. Reynolds et. al. (1988) derived an expression for the steady state ATS (E(T^*)) under sequentially independent observations. In other words, letting $E(R_i(\mu_1))$ be the expected sampling interval used before the i^{th} sample is taken when the process mean is $\mu = \mu_1$, the steady state ATS is

$$E(T^*) = E(Y) + E(N-1)E(R_i(\mu_1)) \qquad [4.1]$$

$$= E(Y) + \frac{1}{q_1}\sum_{j=1}^{\eta} d_j p_{1j},$$

where E($R_i(\mu_1)$) is calculated assuming that there is no signal.

Assume that the shift in the process mean occurs in a particular sampling interval and that the position of the shift within the interval u is uniformly distributed over the interval. Then

$$f_{Y|U}(y|u) = \frac{1}{u} \qquad 0 \le y \le u$$

Since the probability of the shift falling in an interval of a particular length is proportional to the product of the length of this interval and the frequency that this interval is used when the process mean is in control,

$$E(Y) = \int_{-\infty}^{\infty} y f_Y(y) dy = \frac{\sum_{j=1}^{\eta} d_j^2 p_{oj}}{2\sum_{j=1}^{\eta} d_j p_{oj}}. \qquad [4.2]$$

Then the steady state ATS, when the process mean is $\mu = \mu_1$, is

$$E(T^*) = \frac{\sum_{j=1}^{\eta} d_j^2 p_{0j}}{2\sum_{j=1}^{\eta} d_j p_{0j}} + \frac{1}{q_1} \sum_{j=1}^{\eta} d_j p_{1j}. \qquad [4.3]$$

Reynolds et. al. (1988) also derived an expression for Var(T^*);

$$Var(T^*) = \frac{\sum_{j=1}^{\eta} d_j^3 p_{0j}}{3\sum_{j=1}^{\eta} d_j p_{0j}} - \frac{(\sum_{j=1}^{\eta} d_j^2 p_{0j})^2}{4(\sum_{j=1}^{\eta} d_j p_{0j})^2} + \frac{\sum_{j=1}^{\eta} d_j^2 p_{1j}}{q_1} + \frac{(\sum_{j=1}^{\eta} d_j p_{1j})^2}{q_1^2}. \qquad [4.4]$$

4.3 Steady State Properties of $\overline{X}$ Charts under an AR (1) Process

For sequentially dependent observations, the equation [4.1] does not hold. Hence, the expressions for E(T^*) and Var(T^*) in equations [4.3] and [4.4] are not valid for dependent data. However, let π_i, $i = 1, 2, \dots, 2(w+1)+2$, be the steady state probabilities that the process stays in one of the intervals in Figure 3 in the long run when the process mean is on target. Note that the probability of the shift falling in an interval of a particular length is proportional to the product of the length of the interval and the steady state probability of the interval when the process is in control. Then, similar to equation [4.2], the expected time from shift to the next sample is

$$E(Y) = \frac{\sum_{k=1}^{2*(w+1)+v} b_k^{*2} \pi_k}{2 \sum_{k=1}^{2*(w+1)+v} b_k^{*} \pi_k},$$

where b_k^* is defined as

$$b_k^* = \begin{cases} d_1 & \text{if } 1 \le k \le w+1+\frac{v}{3} \text{ or } w+2+\frac{2}{3}v \le k \le 2(w+1)+v \\ d_2 & \text{if } w+2 \le k \le v+w+1 \end{cases}$$

Next, we are interested in the time Z and the number of samples N^* from the first sample after the shift to the signal in Figure 2. Suppose that the shift happens during a particular sampling interval b_k^* (d_1 or d_2). Suppose also that the process is in state i , i′ and j at the time of sampling before the shift, at the time of the shift, and at the time of sampling after the shift, respectively where $1 \le i \le 2(w+1)+v$, $1 \le i' \le 2(w+1)+v$, and $w+2 \le k \le v+w+1$. In fact, the state j depends on the state i′ as well as on the distance Y. In the same way, the state i′ depends on the state i and the distance $b^* - Y$. If we let $p_{i^*j^*}(t, \mu^*) = P(\text{state } i^* \rightarrow \text{ state } j^* \text{ in t time units } \mid \mu = \mu^*)$, then the probability p_{ij}^* that the process is in state i at the time of sampling before the shift and in state j at the time of sampling after the shift is

$$p_{kj}^* = \int_0^{b_k} p_{kj}^*(y) f_Y(y)\, dy, \qquad [4.5]$$

where $p^*_{kj}(y) = \sum_{i'=1}^{2*(w+1)+v} p_{ki'}(b_k - y, \mu_0)\, p_{i',j}(y, \mu_1)$, and $f_Y(y) = \frac{1}{b^*_k}$. We assume that the shift occurs instantaneously (without any drift) at the time of the shift. Let $E(T_j(\mu_1))$ be the expected time to signal assuming that the process is in state j and the process mean is $\mu = \mu_1$. Then the expected time from the first sample after the shift to the signal is

$$E(Z) = \sum_{k=1}^{2*(w+1)+v} \pi^*_k \sum_{j=w+2}^{v+w+1} p^*_{kj} E(T_j(\mu_1)),$$

where $\pi^*_k = \pi_k b^*_k / \sum_{l=1}^{2*(w+1)+v} \pi_l b^*_l$, Thus, the steady state ATS and steady state ANSS, when the process mean is μ_1, are

$$E(T^*) = \sum_{k=1}^{2*(w+1)+v} b^*_k \pi^*_k \left[\frac{b^*_k}{2} + \sum_{j=w+2}^{v+w+1} p^*_{kj} E(T_j(\mu_1))\right]$$

$$= \frac{1}{2} \underline{\pi}^{**} \underline{b}^* + \underline{\pi}^{**} Q^* M \underline{b}$$

and

$$E(N^*) = \underline{\pi}^{**} Q^* M \underline{1},$$

where $\underline{\pi}^{**}$ is a $2(w+1)+v$ by 1 row vector whose k^{th} element is $\pi^*_k = \pi_k b^*_k / \sum_{l=1}^{2*(w+1)+v} \pi_l b^*_l$, $\underline{b}^*$ is a $2(w+1)+v$ by 1 column vector of sampling intervals for the whole region in Figure 3, Q^* is a $2(w+1)+v$ by v matrix, M $= (I-Q)^{-1}$, $\underline{b}$ is a v by 1 column vector of sampling intervals within the control limits, and $\underline{1}$ is a v by 1 column vector of 1's.

Note that if the process mean is on target ($\mu = \mu_0$), then we do not actually calculate the ATS^* and $ANSS^*$ using steady state probabilities π_k's. Instead, for comparison and evaluation of control charts, we use the regular in control ATS and ANSS in Tables 1 and 3 for FSI (1) and VSI (0.1, 1.9), respectively.

4.4 Computational Results

As mentioned in Section 4.3, the position of shift within the interval d_i is assumed to be uniformly distributed over the interval. For the computations in this chapter, however, we assume that the position of the shift within the interval d_i is uniformly distributed over a discrete number of points. Suppose that M discrete points are used over the one unit time (d = 1). Then, for FSI (1), the discrete approximation of p^{*}_{kj} in equation [4.5] becomes $\frac{1}{M}\sum_{m=1}^{M}\sum_{i'=1}^{2^{*}(w+1)+v} p_{ki'}(m, \mu_0)\, p_{i',j}(M-m, \mu_1)$. In VSI ($d_1$, d_2), if I_k is the interval where a short sampling interval is used, then Md_1 discrete points are used over the shorter sampling interval. Otherwise, Md_2 discrete points are used for the longer sampling interval. Therefore, as compared to the independent case, it takes 2M(2(w + 1) + v) more times to compute p^{*}_{kj} on the average. Since it takes too much computing time with total transient states v = 45, 48, ..., 75, the steady state ATS and steady state ANSS are calculated using only the total number of transient states v = 18, 21, ... , 33 and our extrapolation to an infinite number of transient states is based on fitting the formula in equations [3.11] and [3.12]. Also, we use a reduced array of ϕ and δ. That is, the steady state ATS and ANSS are evaluated only at ϕ = 0.0, 0.3, 0.6, and 0.9, and δ = 0.0, 0.5, 1.0, and 3.0. M and w are taken as 20 and 1, respectively.

Since total transient states of v = 18, 21, ..., 33 are used for ATS and ANSS, we may first be interested in how ATS^{*} and $ANSS^{*}$ change when total numbers of transient states v = 45, 48, ..., 74 are used. As it turned out in Table 20, unless the correlation is very large (ϕ = 0.9), the ATS^{*} and $ANSS^{*}$ with v = 45, 48, ..., 75 transient states are slightly larger than those with v = 18, 21, ..., 33 transient states. However, if ϕ = 0.9, then the ATS^{*} and $ANSS^{*}$ with v = 45, 48, ..., 75 transient states are much larger than those with v = 18, 21, ..., 33 transient states, in particular when the deviation δ is small. Therefore, we need to be careful when we compare the ATS^* (or $ANSS^*$) based on $v = 18, 21, \ldots, 33$ with the ATS (or ANSS) based on $v = 45, 48, \ldots, 75$.

4.4.1 Robustness of Control Charts with respect to Departure from Independence (Steady State Properties)

Table 19 shows the effect of correlation on ATS^* and $ANSS^*$ for FSI (1) when the usual control limits of $\mu_0 \pm 3\sigma_x$ are used, assuming that σ_x is known. First, compare the regular ATS in Table 1 with the steady state ATS (ATS^*) in Table 19 when the sequential observations are independent ($\phi = 0$). In this case, the ATS^* is about half a unit smaller than the ATS, and the $ANSS^*$ is half a unit smaller than the ATS^*. Even with dependent data ($\phi > 0$), the ATS^* in Table 19 tends to be still slightly smaller than its counterpart in Table 1 in almost all cases except when the correlation is very high ($\phi = 0.9$) and when the magnitude of the deviations is at least relatively moderate ($\delta \geq 1.0$). In general, ATS^* behaves much like ATS in FSI control charts. Hence, for a given deviation δ, the ATS^* ($= ANSS^*$) tends to increases as the correlation ϕ increases.

Next, Table 20 shows the effect of correlation on ATS^* and $ANSS^*$ for VSI (0.1, 1.9). First, the ATS^* and $ANSS^*$ in Table 20 is slightly smaller than its counterpart in Table 3 except when the correlation is relatively large ($\phi \geq 0.6$) and when the magnitude of deviation δ is very large ($\delta = 3.0$). This might be due to the fact that v = 18, 21, ..., 33 transient states are used for ATS^* and $ANSS^*$. Hence, both ATS^* and $ANSS^*$ behave similarly to the regular ATS and ANSS. Therefore, as seen in Table 3, Table 20 also shows that, if the deviations from the process mean are small or moderate ($\delta \leq 1$), then the ATS^* and $ANSS^*$ tend to decrease as ϕ increases until ϕ reaches 0.3, and then for a higher correlation of ϕ, they tend to increase. However, if the deviation is very large ($\delta \geq 3$), then the ATS^* and $ANSS^*$ tend to increase monotonically as ϕ increases.

In general, the steady state ATS and steady state ANSS behave much like the regular ATS and ANSS in FSI and VSI control charts. However, if the correlation is strong and if the shift is large, say $\phi = 0.9$ and $\delta = 3.0$, then the ATS^* and $ANSS^*$ tend to be larger than the ATS and ANSS for both FSI and VSI control charts.

4.4.2 Comparison of Symmetric Control Charts (Steady State Properties)

In order to compare the ATS^* and $ANSS^*$ for different degrees of correlation ϕ when the process mean is off target in FSI control charts, we need to alter the control limits so that the ATS^* and $ANSS^*$ when the process mean is on target may be both close to our standard of 370.4. Note that if the process mean is on target, we use the regular ATS and ANSS as our in control ATS^* and $ANSS^*$ for comparison of control charts. Hence, the altering procedure of the control limits is the same as in Table 9; the control limits in Table 21 are the same as those in Table 9. According to Table 21, it takes longer in terms of steady state ATS (ATS^*) and steady state ANSS ($ANSS^*$) to detect being out of control in the process mean if the correlation is stronger, as seen in Table 9 for regular ATS and ANSS. Note that the ATS^* and $ANSS^*$ in Table 21 are slightly smaller than the ATS (= ANSS) in Table 9 in almost all cases except when ϕ is very large ($\phi = 0.9$) possibly due to the fact that $\nu = 18, 21, \ldots, 33$ transient states are used

Next, suppose that we want to compare the ATS^*'s and $ANSS^*$'s for different degrees of correlation ϕ when the process mean is off target in VSI (0.1, 1.9) control charts. Then we need to alter the control and warning limits (r and r′) so that the ATS^* and $ANSS^*$ when the process mean is on target may be both close to our standard of 370.4. The altering procedure is the same as used in Table 11 for VSI (0.1, 1.9). Hence, the control and warning limits of Table 11 are the same as those of Table 22. Table 22 shows the altered control and warning limits, and the ATS^* and $ANSS^*$ for various values of $\delta \neq 0$. Note again that the ATS^* and $ANSS^*$ are evaluated only at $\phi = 0.0$, 0.3, 0.6, and 0.9. Therefore, Table 22 does not show whether the ATS^* decreases as the correlation ϕ increases from 0 to 0.1, as seen in Table 11. According to Table 22, the ATS^* and $ANSS^*$ tend to increase monotonically as the correlation becomes stronger.

Now, we want to compare FSI control charts with VSI control charts in terms of ATS^* and $ANSS^*$ using Tables 21 and 22. Table 23 shows the relative efficiency in ATS^*, which is $\frac{ATS^* \text{ of } FSI\ (1)}{ATS^* \text{ of } VSI\ (0.1, 1.9)}$. On the other hand, Table 24 shows the relative inefficiency in $ANSS^*$, which is $\frac{ANSS^* \text{ of } VSI\ (0.1, 1.9)}{ANSS^* \text{ of } FSI\ (1)}$. According to Tables 23 and 24, it turns out, as in Tables 15 and 17, that VSI (0.1, 1.9) tends to be more (less) efficient than FSI (1) in terms of ATS^* ($ANSS^*$), and the efficiency (inefficiency) of ATS^* ($ANSS^*$) tends to decrease (increase) as the correlation increases for almost all cases except when the deviation is large ($\delta = 3.0$). If the deviation is very large ($\delta = 3.0$), then the VSI control charts does not even seem to be more efficient than FSI control charts for dependent data ($\phi > 0$).

4.5 Computational Results Using Conditional, Cyclical and Yashchin's Steady State Probabilities

In Section 4.1, various types of steady state probabilities were introduced. Now, we want to consider how the control charts behave in terms of ATS^* and $ANSS^*$ when other types of steady state probabilities such as cyclical, conditional, or Yashchin's steady state probabilities are used. Let $\pi_1, \pi_2, \ldots, \pi_v$ be the steady state probabilities when conditional, cyclical or Yashchin's steady state probabilities were used. In this case, the steady state ATS and steady state ANSS, when the process mean is $\mu = \mu_1$, are

$$E(T^*) = \sum_{k=1}^{v} \pi_k^* \left[\frac{b_k}{2} + \sum_{j=1}^{v} p_{kj}^* E(T_j(\mu_1))\right]$$

$$= \frac{1}{2} \underline{\pi}^{*\prime} \underline{b} + \underline{\pi}^{*\prime} \underline{Q}^* M \underline{b}$$

and

$$E(N^*) = \underline{\pi}^{*'} Q^* M \underline{1}$$

where $\underline{\pi}^{*'}$ is a 1 by v row vector whose k^{th} element is $\pi_k^* = \pi_k b_k / \sum_{l=1}^{v} \pi_l b_l$, $\underline{b}$ is a v by 1 column vector of sampling intervals within the control limits, Q^* is a v by v matrix whose element p_{kj}^* is in equation [4.5], and $M = (I - Q)^{-1}$. As in the steady state probabilities in Section 4.4, M is taken as 20. The ATS^* and $ANSS^*$ are evaluated only for VSI(0.1, 1.9) and for ϕ values of 0.0, 0.3, 0.6 and 0.9 using total transient states of v = 18,21, ..., 33.

Now suppose that the process standard deviation is known and usual control limits of $\mu_0 \pm 3\sigma_x$ are used. Then, according to Table 25, regardless of the level of correlation, there is very little difference in the ATS^* and $ANSS^*$ whether conditional, cyclical or Yashchin's probabilities are used. Therefore, as seen in Table 20, if the deviation is not large, then the other types of steady state ATS and steady state ANSS tend to decrease as ϕ increases until ϕ reaches 0.3, and then for a higher correlation of ϕ, they tend to increase. However, if the magnitude of the deviation is large ($\delta = 3.0$), then the other types of steady state ATS and steady state ANSS also tend to increase monotonically as ϕ increases.

Table 19. $\frac{ATS^*}{ANSS^*}$ of FSI (1) for unaltered control limits

δ / ϕ	0	0.5	1	3	r
0.0	370.398 370.398	154.753 154.253	43.439 42.939	1.550 1.050	3.000
0.3	376.383 376.383	161.235 160.735	47.217 46.717	1.853 1.353	3.000
0.6	419.376 419.376	190.863 190.363	60.321 59.821	2.520 2.020	3.000
0.9	831.102 831.102	427.199 426.699	153.188 152.688	6.508 6.008	3.000

A fixed sampling interval (d = 1) is used if an observation falls in ($-r\sigma_x$, $r\sigma_x$).

Table 20. $\frac{ATS^*}{ANSS^*}$ of VSI (0.1, 1.9) for unaltered control limits

δ / ϕ	0	0.5	1	3	r	r′
.0	370.398 370.398	141.503 154.253	30.869 42.939	1.090 1.050	3.000	0.672
.3	142.242 236.925	63.644 125.170	20.412 55.776	1.521 4.579	3.000	0.672
.6	158.701 372.557 (158.887) (372.901)	81.379 221.498 (81.458) (221.688)	29.718 108.434	2.109 10.096	3.000	0.672
.9	441.210 1613.348 (496.291) (1798.340)	259.757 1055.845 (278.419) (1125.439)	107.530 546.279	7.046 53.308	3.000	0.672

A sampling interval ($d_1 = 0.1$) is used if an observation falls in ($-r\sigma_x$, $-r'\sigma_x$) or ($r'\sigma_x$, $r\sigma_x$).

A sampling interval ($d_2 = 1.9$) is used if an observation falls in ($-r'\sigma_x$, $r'\sigma_x$).

Note that the ATS and ANSS in the parenthesis are based on $\nu = 45, 48, \ldots, 75$.

Table 21. $\frac{ATS^*}{ANSS^*}$ of FSI (1) for altered control limits

δ / ϕ	0	0.5	1	3	r
.0	370.398 370.398	154.753 154.253	43.439 42.939	1.550 1.050	3.000
.3	370.307 370.307	158.991 158.491	46.676 46.176	1.842 1.342	2.995
.6	369.874 369.874	171.590 171.090	55.346 54.846	2.401 1.901	2.960
.9	370.087 370.087	216.931 216.431	88.749 88.249	4.310 3.810	2.711

A fixed sampling interval (d = 1) is used if an observation falls in $(-r\sigma_x, r\sigma_x)$.

Table 22. $\frac{ATS^*}{ANSS^*}$ of VSI (0.1, 1.9) for altered control limits

δ / ϕ	0	0.5	1	3	r	r′
.0	370.398 370.398	141.503 154.253	30.869 42.939	1.090 1.050	3.000	0.672
.3	369.491 370.620	144.353 173.490	39.586 69.941	1.883 5.805	3.263	1.129
.6	370.203 370.449	165.105 198.650	51.927 92.271	2.580 9.914	3.231	1.347
.9	370.505 369.505	216.568 251.467	88.642 144.539	4.582 18.846	2.882	1.566

A sampling interval ($d_1 = 0.1$) is used if an observation falls in $(-r\sigma_x, -r'\sigma_x)$ or $(r'\sigma_x, r\sigma_x)$

A sampling interval ($d_2 = 1.9$) is used if an observation falls in $(-r'\sigma_x, r'\sigma_x)$.

Table 23. **Relative Efficiency** : $\dfrac{ATS^{*} \text{ of } FSI(1)}{ATS^{*} \text{ of } VSI(0.1, 1.9)}$

δ / ϕ	0	0.5	1	3
.0	1.000	1.094	1.407	1.422
.3	1.002	1.101	1.179	0.978
.6	0.999	1.039	1.066	0.931
.9	0.999	1.002	1.001	0.941

Table 24. **Relative Inefficiency** : $\dfrac{ANSS^{*} \text{ of } VSI(0.1, 1.9)}{ANSS^{*} \text{ of } FSI(1)}$

δ / ϕ	0	0.5	1	3
.0	1.000	1.000	1.000	1.000
.3	1.001	1.095	1.515	4.326
.6	1.002	1.161	1.682	5.215
.9	0.998	1.162	1.648	4.946

Table 25. ***ATS** and *ANSS** for VSI(0.1, 1.9) using other types of steady state probabilities**

conditional steady state $\frac{ATS^*}{ANSS^*}$ for unaltered control limits

δ / φ	0	0.5	1	3	r	r′
.0	370.398 370.398	141.157 153.874	30.795 42.833	1.090 1.047	3.000	0.672
.3	142.242 236.925	63.619 125.108	20.403 55.747	1.519 4.567	3.000	0.672
.6	158.701 372.557	81.408 221.552	29.719 108.432	2.103 10.061	3.000	0.672
.9	441.210 1613.348	259.815 1056.184	107.456 546.024	7.004 53.033	3.000	0.672

cyclical steady state $\frac{ATS^*}{ANSS^*}$ for unaltered control limits

δ / φ	0	0.5	1	3	r	r′
.0	370.398 370.398	141.150 153.865	30.794 42.831	1.090 1.047	3.000	0.672
.3	142.242 236.925	63.662 125.177	20.421 55.785	1.520 4.557	3.000	0.672
.6	158.701 372.557	81.564 221.913	29.791 108.660	2.101 10.006	3.000	0.672
.9	441.210 1613.348	260.995 1060.721	108.053 549.045	6.959 52.595	3.000	0.672

Yashchin's steady state $\frac{ATS^*}{ANSS^*}$ for unaltered control limits

δ / φ	0	0.5	1	3	r	r′
.0	370.398 370.398	141.138 153.895	30.776 42.855	1.071 1.069	3.000	0.672
.3	142.242 236.925	63.365 124.559	20.308 55.434	1.517 4.463	3.000	0.672
.6	158.701 372.557	81.299 220.696	29.694 108.049	2.016 9.042	3.000	0.672
.9	441.210 1613.348	262.007 1060.405	108.633 552.535	4.985 36.330	3.000	0.672

Chapter V

$\overline{X}$ CONTROL CHARTS UNDER AN AR (p) PROCESS, $p \geq 2$

5.1 AR (p) Process, $p \geq 2$

The underlying assumption in the previous chapters was that we had an observation $X'(\frac{t}{k}) = X(\frac{t}{k}) + \delta\sigma_x$, where $X(\frac{t}{k})$ was an AR (1) process. However, it may be that $X(\frac{t}{k})$ is a second or higher order autoregressive process. That is, the observation $X'(t)$ at time t may be

$$X'(t) = X(t) + \delta\sigma_X,$$

where

$$X(t) = \phi_1 X(t-1) + \phi_2 X(t-2) + \cdots + \phi_p X(t-p) + a(t). \qquad [5.1]$$

The random shock $a(t)$ consists of uncorrelated random variables with $E(a(t)) = 0$ and Var($a(t)) = \sigma_a^2$. Some of the properties of the general autoregressive model are as follows. Rewriting equation [5.1] using a back shift operator, we have

$$\phi(B)X(t) = a(t),$$

where $\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p$. Hence, $X(t)$ at time t can now be expressed in terms of previous random shocks as

$$X(t) = \phi(B)^{-1} a(t).$$

Since we assume that $\{X(t)\}$ is a stationary process, the roots of

$$\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p = 0$$

must all be greater than 1 in absolute value. If p = 2, then the roots of $\phi(B) = 1 - \phi_1 B - \phi_2 B^2 = 0$ must be outside the unit circle. This implies that the parameters ϕ_1 and ϕ_2 satisfy

$$\phi_1 + \phi_2 < 1$$
$$\phi_2 + \phi_1 < 1$$
$$-1 < \phi_2 < 1.$$

The stationary region for the parameters ϕ_1 and ϕ_2 is shown in Figure 4.

The autocorrelation function for the general autoregressive process satisfies the p^{th} order difference equation:

$$\rho(k) = \phi_1 \rho(k-1) + \phi_2 \rho(k-2) + \cdots + \phi_p \rho(k-p) \qquad k = 1, 2, 3, \ldots \qquad [5.3]$$

which is called the Yule Walker equation. We can use equation [5.3] to determine the parameters $\phi_1, \phi_2, \ldots, \phi_p$ in terms of $\rho(0), \rho(2), \ldots, \rho(p)$ as follows :

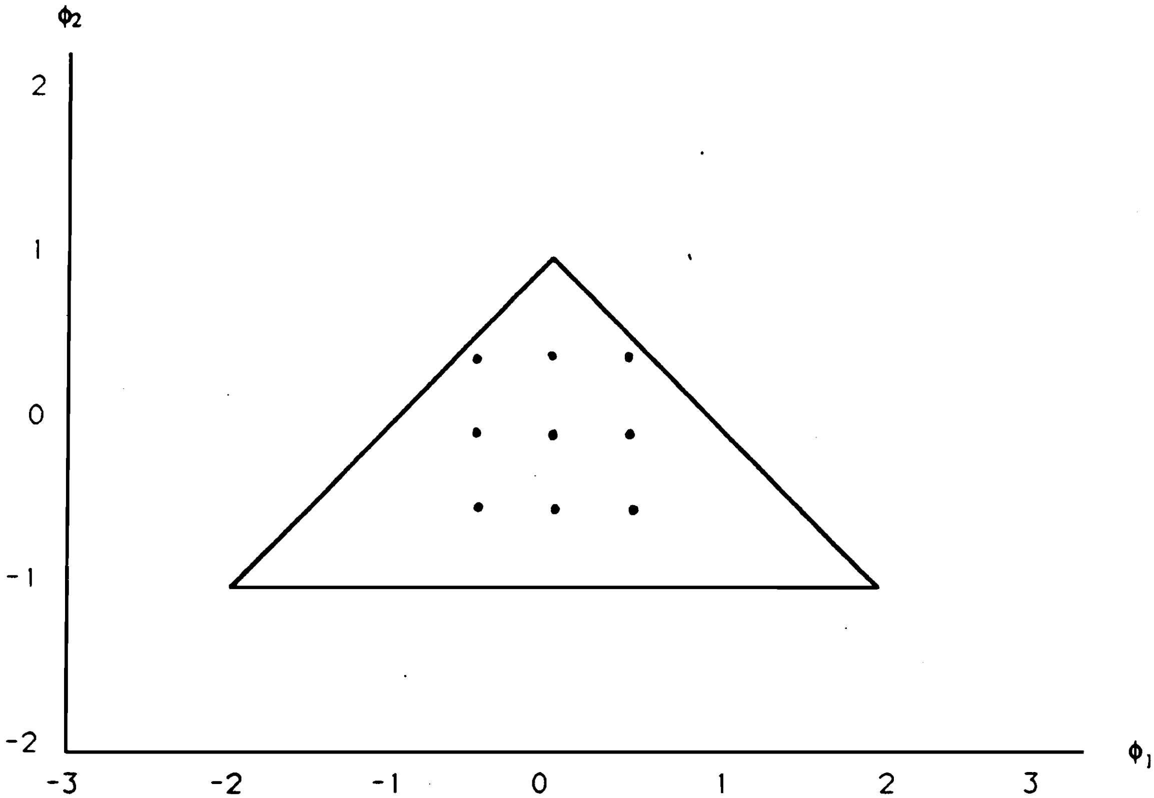

Figure 4 Stationary Region Of AR (2) Process

$$\begin{bmatrix} 1 & \rho(1) & \rho(2) & \cdots & \rho(p-1) \\ \rho(1) & 1 & \rho(1) & \cdots & \rho(p-2) \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \rho(p-1) & \rho(p-2) & \rho(p-3) & \cdots & 1 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \cdot \\ \cdot \\ \cdot \\ \phi_p \end{bmatrix} = \begin{bmatrix} \rho(1) \\ \rho(2) \\ \cdot \\ \cdot \\ \cdot \\ \rho(p) \end{bmatrix}.$$

Then the solution for $\underline{\phi}$ is

$$\underline{\phi} = P_\rho^{-1} \underline{\rho},$$

where P_ρ^{-1} is the inverse of the above p by p matrix. By the theory of difference equations, the solution to equation [5.3] has the form

$$\rho(k) = c_1 m_1^k + c_2 m_2^k + \cdots + c_p m_p^k,$$

where $m_1, m_2, \ldots, m_p$ are the roots of $m^p - \phi_1 m^{p-1} - \phi_2 m^{p-2} - \cdots - \phi_0 = 0$, and $c_1, c_2, \ldots, c_p$ are constants determined by the starting values $\rho(0), \rho(1), \ldots, \rho(p-1)$. Note that the ACF function consists of a mixture of damped exponentials and damped sine waves, which means that the correlation between any two observations gets smaller as the time between them increases. The variance of the general autoregressive process, according to Box and Jenkins (1976), is

$$\sigma_x^2 = \frac{\sigma_a^2}{1 - \rho(1)\phi_1 - \cdots - \rho(p)\phi_p}.$$

As an illustration, if $p = 2$, then the Yule Walker equation is

$$\rho(k) = \phi_1 \rho(k-1) + \phi_2 \rho(k-2) \qquad [5.4]$$

with the starting values

$$\rho(0) = 1 \text{ and } \rho(1) = \frac{\phi_1}{1-\rho_2}. \qquad [5.5]$$

The solution to equation [5.4] is

$$\rho(k) = c_1 m_1^k + c_2 m_2^k,$$

where m_1 and m_2 are the roots of $m^2 - \phi_1 m - \phi_2 = 0$, and c_1 and c_2 are determined by the equation [5.5]. The variance of the AR (2) process is

$$\sigma_x^2 = \frac{\sigma_a^2}{1-\rho(1)\phi_1 - \rho(2)\phi_2}$$
$$= \frac{1-\phi_2}{1+\phi_2}\frac{\sigma_a^2}{(1-\phi_2)^2 - \phi_1^2}.$$

Note that as ϕ_1 or (and) ϕ_2 approaches the boundary of the stationary region in Figure 4, the process variance increases.

5.2 Estimation of Process Variance or Process Standard Deviation

One may want to know how the average of size 2, $\frac{\overline{R}}{d_2}$, performs in estimating the process standard deviation σ_x under AR (p), $p \geq 2$. If the process mean is on target (i.e. $\delta = 0$), then $u_t(= X(t+1) - X(t))$ is shown to be normal with

$$E(u_t) = 0$$
$$\text{Var}(u_t) = 2(\gamma(0) - \gamma(1)),$$

where $\gamma(k)$ is $\text{Cov}(X(t), X(t+k))$ for k = 0 or 1. In an AR (p) process,

$$\gamma(k) = \phi_1\gamma(k-1) + \phi_2\gamma(k-2) + \cdots + \phi_p\gamma(k-p). \qquad k > 0$$

We can use the p-2 equations $\gamma(p-1), \gamma(p-2), \dots, \gamma(2)$ for an AR (p), $p \geq 3$, process in order to express Var(u_t) in terms of $\gamma(0)(=\sigma_x^2)$ only. For AR (3) and AR (2) processes, it turns out that

$$E(\frac{\overline{R}}{d_2}) = \sqrt{1 - \frac{\phi_1 + \phi_2\phi_3}{1 - \phi_2 - \phi_1\phi_3 - \phi_3^2}}\,\sigma_x$$

and

$$E(\frac{\overline{R}}{d_2}) = \sqrt{1 - \frac{\phi_1}{1 - \phi_2}}\,\sigma_x, \qquad [5.6]$$

respectively. Note that the result in chapter III for $E(\frac{\overline{R}}{d_2})$ for the AR (1) process is a special case of the above results. We know from chapter III that if we have positively dependent data, then $\frac{\overline{R}}{d_2}$ underestimates the process standard deviation. Now, if we have an AR (2) process, and if ϕ_2 is positive in addition to positive ϕ_1, then the underestimation is even larger.

From Section 3.4, we know that under the assumption that $\sum_{k=1}^{\infty}\rho(k) < \infty$, $E(S_n^2)$ tends to σ_x^2 as $n \to \infty$. Hence, if the process mean is on target or stable, then the $\frac{S_n}{c_4}$ should be used to estimate the process standard deviation σ_x, in particular when both ϕ_1 and ϕ_2 are positive.

5.3 FSI Control Charts when the unit time series is an AR (p) $p \geq 2$

In Chapters 3 and 4, if $\frac{1}{k}$ was the smallest value of d_1 to be considered, then we assumed that the $\frac{1}{k}$ unit time series $\{X(\frac{t}{k})\}$, t = 1, 2, ..., k ,..., 2k, 2k + 1, ..., was an AR (1) process. By the Markov property of AR (1), the unit time series $\{X(\frac{t}{k})\}$, t = k, 2k, ..., is another AR (1) process with different parameters, as shown in Section 3.5. However, suppose that the $\frac{1}{k}$ unit time series $\{X(\frac{t}{k})\}$ is an AR (p) process with $p \geq 2$. Our immediate question would be whether or not the unit time series is also an AR (p) process with $p \geq 2$ with different parameters. Since the AR (p) with $p \geq 2$ does not have a Markov property, the unit time series is not an AR (p) process with any form of parameters. However, we may be still interested in FSI and VSI control charts when $\frac{1}{k}$ unit time series $\{X(\frac{t}{k})\}$ is an AR (p) process with $p \geq 2$. Since we do not have time to do more, we will only consider the behavior of FSI control charts when the unit time series $\{X(t)\}$ is an AR (p) process with $p \geq 2$ in particular p = 2. For simplicity, we will assume that the process starts at a fixed point.

For an AR (p) process, if we want to know where the process is at time t + 1, then we need to know where the process was at time t + p-1, t + p-2, and t. At each time the process could be in a different state. Let $k_1, k_2, \ldots, k_p$ be a sequence of states the process is in at times t + p-1, t + p-2, ... , and t, where each k_r, r = 1, 2, ... p, itself is a state for an AR (1) process; k_r = 1, 2, ..., or v. In this case, the transitional probability for an AR (p) process is

$$
\begin{aligned}
& p_{k_1 k_2 \ldots k_p, j_1 j_2 \ldots j_p} \\
&= P(X'(t-p+2) = i_{j_1}, X'(t-p+3) = i_{j_2}, \ldots, X'(t+1) \in I_{j_p} \mid X'(t-p+1) = i_{k_1}, X'(t-p+2) = i_{k_2}, \ldots, X'(t) = i_{k_p}) \\
&= P(X'(t+1) \in I_{j_p} \mid X'(t-p+1) = i_{k_1}, X'(t-p+2) = i_{k_2}, \ldots, X'(t) = i_{k_p}) \text{ if } i_{j_1} = i_{k_2}, i_{j_2} = i_{k_3}, \ldots, i_{j_{p-1}} = i_{k_p} \\
&= P(a_{t+1} + \phi_1 i_{k_p} + \phi_2 i_{k_{p-1}}, \ldots, + \phi_p i_{k_1} \in I_{j_p}),
\end{aligned}
$$

where $i_{k_1}, i_{k_2}, \ldots,$ and i_{k_p} are the middle points of the intervals $I_{k_1}, I_{k_2}, \ldots,$ and I_{k_p}, and $i_{j_1}, i_{j_2}, \ldots,$ and i_{j_p} are the middle points of the intervals $I_{j_1}, I_{j_2}, \ldots,$ and I_{j_p}. Then the ATS (= ANSS) becomes

$$ATS = (I - Q)^{-1} \underline{1}, \qquad [5.7]$$

where Q is a v^p by v^p matrix whose elements are shown as in $p_{k_1 k_2 \ldots k_p, j_1 j_2 \ldots j_p}$ above, and $\underline{1}$ is a v^p by 1 column vector of 1′s. Note that we need a total of v^p transient states for an AR (p) process. As p increses, the total number of transient states increases exponentially. In addition, v has to be reasonably big for a relatively accurate asymptotic value of ATS. Hence, as an illustration, we will only use an AR (2) process. In this case, the transitional probability becomes

$$p_{k_1k_2,j_1j_2} = P(X'(t+1) \in I_{j_2} \mid X'(t-1) = i_{k_1}, X'(t) = i_{k_2}) \qquad \text{if } i_{j_1} = i_{k_2}$$
$$= P(a_{t+1} + \phi_1 i_{k_2} + \phi_2 i_{k_1} \in I_{j_2}).$$

Let the observations at times -1 and 0 be $X'(-1) = \mu_0 + \delta\sigma_x$ and $X'(0) = \mu_0 + \delta\sigma_x$, respectively. If m is such that $X'(-1) \in I_m$ and $X'_0 \in I_m$, then the $((m-1)v + m)^{th}$ element of the ATS vector in equation [5.7] is our desired ATS for a given deviation δ. Even with an AR (2) process, the total number of transient states that we need for an accurate asymptotic value of ATS increases very rapidly as v increases. Hence, the ATS is calculated only for $v = 9, 12, 15, 18$, and 21 transient states, and our extrapolation to an infinite number of transient states is based on fitting formula [3.12].

Table 26 shows the ATS (= ANSS) for some values of ϕ_1 and ϕ_2 within the stationary regions in Figure 4, and for δ = 0.0, 0.5, 1, and 3. An Appendix B gives the FORTRAN program that produces Table 26. According to Table 26, for a given value of ϕ_2, the detection time is shorter when ϕ_1 is negative than when ϕ_1 is positive, as also seen in Table 1 for an AR (1) process. Next, even though ϕ_1 is zero, ϕ_2 alone plays a similar role in determining the ATS^* and $ANSS^*$ to ϕ_1. That is, even though ϕ_1 is zero, the detection time is shorter when ϕ_2 is negative than when ϕ_2 is positive, as can seen from the second and eighth rows of Table 26. Now, we are interested in the effect of ϕ_2 in addition to the effect of ϕ_1. If $\phi_2 = 0$, then false alarm rate is less frequent and detection time is longer when ϕ_1 is positive, say $\phi_1 = 0.4 > 0$, than when $\phi_1 = 0$. If ϕ_2 is also positive, then the false alarm rate decreases even further and the detection time is even longer. Finally, the ATS increases as ϕ_1 or (and) ϕ_2 approaches the boundary of the stationary region.

Table 26. ATS(= ANSS) of FSI (1) for an AR (2) Process

ϕ_1	δ / ϕ_2	0	0.5	1	3	r
-0.40	0.40	531.065	219.015	67.650	1.886	3.00
0.00	0.40	384.947	168.048	51.552	2.239	3.00
0.40	0.40	531.142	267.862	98.703	3.564	3.00
-0.40	0.00	384.223	158.033	44.287	1.786	3.00
0.00	0.00	370.398	155.224	43.895	2.000	3.00
0.40	0.00	384.223	167.701	51.032	2.424	3.00
-0.40	-0.40	385.312	159.495	44.711	1.713	3.00
0.00	-0.40	384.947	158.646	44.563	1.862	3.00
0.40	-0.40	385.286	162.161	46.731	2.095	3.00

A fixed sampling interval (d = 1) is used if an observation falls in $(-r\sigma_x, r\sigma_x)$.

Chapter VI

SUMMARY AND TOPICS FOR FUTURE RESEARCH

6.1 Description of the Study

In this thesis, we assume that at each sampling point the sample size is one and that the sequential observations are correlated according to an autoregressive process, in particular the first order autoregressive process. The properties of the FSI and VSI control charts are investigated through Markov chain approaches. The average time to signal (ATS) and the average number of samples of signal (ANSS) are used to evaluate the performance of Shewhart $\overline{X}$ control charts. These two quantities are more appropriate criteria than the traditional average run length (ARL) , which refers to both the time to signal and the number of samples to signal. The properties of the

Shewhart $\overline{X}$ control charts are investigated under two different types of assumptions about the time of shift. The first and conventional assumption is that the process mean shifts from time zero on. In this case, the $\overline{X}$ control chart is evaluated under the assumption that the starting value for the process at time 0 is fixed and also under the assumption that it is random. Since we are comparing FSI with VSI control charts, we allow a fixed first sampling interval for both FSI and VSI control charts for a fair comparison when we assume that the process mean shifts from time zero on. The other type of assumption about the time of shift is that the shift can occur anywhere between samples in the future, not just at the beginning. In this case, the steady state ATS and steady state ANSS are redefined, and the Shewhart $\overline{X}$ charts are investigated under the assumption that the shift occurs uniformly over the sampling interval. The shift is assumed to occur instantaneously (without a gradual shift) at the time of shift for either 'regular' ATS or steady state ATS. If the unit time series is an AR (2) process, then we consider only FSI control charts. In this case, for simplicity, we assume that the starting valus is fixed.

6.2 Summary of Results

First, consider the estimation of the process variance or process standard deviation since, unlike the conventional circumstances where there are multiple observations at each sampling point and the consecutive observations are independent, we now assume that the sample size is only 1 and that the sequential observations are correlated according to an autoregressive process, in particular the first order autoregressive process. In this case, the most commonly used method is to create ranges by taking the differences between successive observations and dropping the sign of the difference when it is negative, and then use $\frac{\overline{R}}{d_2}$ as an estimator of the process standard deviation. However, it turns out that $\frac{\overline{R}}{d_2}$ underestimates the process standard deviation if the correlation between consecutive observations is positive, and overestimates the process standard deviation if the correlation is negative. On the other hand, if we have a stationary autoregressive process, then $E(S_n^2)$ approaches σ_x^2 as $n \to \infty$ under the assumption that $\sum_{k=1}^{\infty} \rho(k) < \infty$. Therefore, if the process mean is on target or stable, then $\frac{S_n}{c_4}$ is to be used as an estimator of the process standard deviation σ_x.

The following results are based on the average time to signal (ATS) and average number of samples to signal (ANSS) when we allow a random start and a fixed first sampling interval for both FSI and VSI control charts under an AR (1) process. First, if the control limits are set at the traditional "target $\pm 3*\sqrt{\text{(process variance)}}$" with known process variance, then the false alarm rate decreases and the detection time increases as the correlation between the consecutive observations increases in FSI control charts. However, for VSI control charts, if the process mean is on or near target, then the ATS and ANSS tend to decrease as the correlation increases until the correlation becomes rather moderate. Then, for more highly correlated data, the ATS and ANSS tend to increase.

The ANSS's are the same whether we have two sampling intervals or not at any given level of shift. However, if the sequential observations are correlated according to an AR (1) process, then the in control ANSS as well as the out of control ANSS are different for different numbers of sampling intervals at any given level of shift, not to mention the in control ATS. In general, as can be seen in Tables 3, 4 and 5, if the control limits are set at conventional " target $\pm 3*\sqrt{\text{(process variance)}}$" with known process variance, then the ATS and ANSS tend to decrease as the number of sampling intervals increases.

Next, if the sequential observations are independent, then the detection time for a control chart with two sampling intervals becomes shorter as the sampling intervals become farther apart asymmetrically. The in control ANSS as well as the out of control ANSS are the same for any symmetric or asymmetric control charts at the same level of shift, as can be seen from the first two rows of Tables 2, 7 and 8. However, for correlated data, the in control ANSS as well as the out of control ANSS's are different for any symmetric or asymmetric control charts at any given level of shift, not to mention the in control ATS and out of control ATS. In general, if the traditional control limits are used, then the ATS and ANSS tend to increase as the sampling intervals become more asymmetrically apart.

By altering the control and warning limits, we can set the new in control ATS and ANSS close to our desired standard so that we can see how each control chart performs when the process mean is not on target. In FSI (1), it tends to take longer to detect being out of control in the process mean as the correlation ϕ between consecutive observations increases. For a symmetric VSI control chart, if the deviation is small or moderate, then the ATS and ANSS tend to decrease as ϕ increases up to $\phi = 0.1$, but for a higher correlation of ϕ, they tend to increase as ϕ increases. However, for a large shift, they tend to increase monotonically as ϕ increases. In general, if the observations are correlated according to an AR (1) process, then the ATS is shorter for a VSI control chart than for a FSI control chart, in particular as the correlation gets weaker and as the sampling intervals become farther apart. However, the ANSS is larger for a VSI control chart than for a FSI control chart as the correlation gets stronger and as the sampling intervals become farther apart.

Even though we allow a fixed start rather than a random start, the ATS as well as the ANSS are almost the same as when we allow a random start as long as the correlation between consecutive observations is relatively small. However, if the correlation is not small, then the ATS and ANSS with a fixed start tend to be slightly larger than those with a random start when the magnitude of the deviation is small or moderate. On the other hand, if the deviation is large, then the ATS and ANSS with a random start tend to be larger than those with a fixed start.

The results based on the steady state ATS (ATS^*) and steady state ANSS ($ANSS^*$) are almost the same as the 'regular' ATS and 'regular' ANSS except when the deviation is very large. If the deviation is large, then the VSI control charts do not seem to be more efficient than the FSI control charts for dependent data. Conditional, cyclical and Yashchin's steady state ATS and ANSS agree with one another regardless of the level of correlation.

For a general autoregressive process of order p with $p \geq 2$, we only considered the case where the unit time series is an AR (p). In particular, we focused on an AR (2) process. Unlike the AR (1) process, the correlation between consecutive observations is not just ϕ_1, but $\frac{\phi_1}{1-\phi_2}$, which is a function of both ϕ_1 and ϕ_2. If we have an AR (2) process, then for any given value of ϕ_2 the detection time is shorter when ϕ_1 is negative than when ϕ_1 is positive. The ϕ_2 alone with $\phi_1 = 0$ can play a similar role to ϕ_1. The effect of positive ϕ_2 in addition to positive ϕ_1 is that the false alarm rate decreases even farther and detection time is even longer.

Designing a control chart for sequentially independent data requires deciding only on the level of the in-control ATS. However, for dependent data, we need to decide on the level of the in control ATS as well as on the level of the in-control ANSS. Even though the observations are correlated according to an AR (1) process, a two sampling interval control chart performs similarly to the control charts with three and more sampling interval control charts. In addition, a two sampling interval control chart is easier to use in practice than a control chart with more than two sampling intervals. Therefore, a control chart with only two sampling intervals is recommended in the design of control charts. Since asymmetric control charts do not always seem to have a shorter detection time as the sampling intervals become farther apart, symmetric two sampling interval control charts are recommended. Therefore, we should make the sampling intervals d_1 and d_2 as far apart as possible if we are only concerned about the time to signal, not the number of samples to signal. If d_1 and d_2 are chosen, then we need to alter the control and warning limits (r and r′) so that the in control ATS and in control ANSS become close to our standard. Tables 9, 10 and 11 can be used for design. Suppose that VSI (0.1, 1.9) (Table 11) was used in the first satge. If it turns out that it takes too many samples to signal, then the VSI (0.5, 1.5) can now be used. In this case, we need to alter the control and warning limits again in order for the new in control and new in control ANSS to be close to our standard. Now, Table 10 can be used for design. The ATS is longer than its previous counterpart and the ANSS is smaller than its previous counterpart. If the new ANSS is still too large, then we need to use a control chart with the sampling intervals close together, say VSI (0.75, 1.25). Or if we can afford to take more samples, then we can use a control chart with the sampling intervals a little bit farther apart, say VSI (0.25, 1.75).

6.3 Comments and Areas of Future Research

In the sequentially independent case, if the process mean is not on target, then the detection time is shortest if we have only two different sampling intervals, in particular when they are farther apart. However, for sequentially dependent observations, since the in control ATS's as well as the in-control ANSS's are different for different numbers of sampling intervals, we need to alter the control or (and) warning limits to get our desired levels of the new in control ATS and the new in control ANSS. However, when we consider a control chart with more than two different sampling intervals, there are multiple combinations of new control and warning limits (more than one warning limits) that can give us the desired level of in control ATS and in control ANSS. The problem here is to find a specific combination of control and warning limits that gives us the shortest detection time for that particular number of sampling intervals. More time and effort are needed for a thorough investigation of the optimal combination of parameters when more than two sampling intervals are used for dependent data.

We found that if the sampling intervals become apart symmetrically, as in FSI (1), VSI (0.5, 1.5) and VSI (0.1, 1.9), then the detection time becomes shorter in terms of ATS. However, if they become apart asymmetrically, as in VSI (0.5, 1.5), VSI (0.5, 2.0) and VSI (0.5, 3.0), then the detection time does not always get shorter. More time is needed in order to investigate the behavior of the asymmetric control charts.

We have investigated only FSI control charts when the unit time series is an AR (2) process. However, it might be that the $\frac{1}{k}$ unit time series with $k \geq 2$ is an AR (p) process with $p \geq 2$. In this case, we can use k unit times as our fixed sampling interval and $(\frac{1}{k}, \frac{2k-1}{k})$ as our variable sampling intervals. We can use a Markov chain representation or simulations in order to investigate both FSI and VSI control charts.

We have concentrated only on the autoregressive process, in particular the first autoregressive process. However, we may have a moving average process, or, in general, an autoregressive and moving average process. Our interest would be how FSI an VSI control charts perform under the general case. We may also be interested in finding the optimal combinations of control and warning limits for a certain level of shift when we have a general autoregressive and moving average process.

Since we are dealing with dependent data, we may be interested in a control chart with runs rule in an attempt to reduce the time to signal.

As the process mean goes out of control the process variance might also shift. In this case, we may be interested in a control chart which controls both the process mean and the process variance. Sometimes, the number of variables that we need to control is more than just one. The variables of interest may be correlated each other. Careful consideration has to be given to this type of multivariate control chart.

Some of the questions that we asked concerning the Shewhart $\overline{X}$ control charts can also be raised concerning other types of control charts, such as CUSUM and EWMA control charts. Finally, we have been restricted to a time series model. However, there could be situations where some other dependency model is more appropriate for the analysis.

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Appendix A
PROGRAM FOR CHAPTERS III AND IV

```
C ************************************************************************
C THIS PROGRAM CALCULATES THE ATS AND ANSS AS WELL AS THE STEADY STATE
C ATS AND ANSS IN CHAPTERS 3 AND 4 RESPECTIVELY.
C THE MACHINE THAT IS USED IN THE COMPUTATION IS IBM 3090
C THE VARIABLES WILL BE EXPLAINED IN EACH SECTION OF THE PROGRAM
      PARAMETER (MIN=45,MAX=75,INCR=3,MULTI=15,
     *            MAXUM=MAX/(INCR*MULTI),    MAXIM=MAX+2*MAXUM+2 )
      IMPLICIT REAL *8(A-H,O-Z)
      REAL RWKSP(90000)
      REAL *8IV1,IV2,IV3,IV4,IV5,DEVIAT(6)
      REAL *8XDATA((MAX-MIN)/INCR+1),YDATA((MAX-MIN)/INCR+1),
     *  YDATAN((MAX-MIN)/INCR+1),BETA(3),BETAN(3),SSPOLY(3),STAT(10)
      LOGICAL CHANGE,REGULAR,FIXEDS,YASHIN,CONDAL,CYCAL,MCYCAL
      DIMENSION B(MAX),BB(MAX)
      DIMENSION R(MAX),UR(MAX/(INCR*MULTI)+2),DR(MAX/(INCR*MULTI)+2),
     *    S(MAX+1),US(MAX/(INCR*MULTI)+1),DS(MAX/(INCR*MULTI)+1),
     *                     TOTALR(MAXIM),TOTALS(MAXIM-1),SUB(10)
      DIMENSION PINI(MAX),PO(MAX)
      DIMENSION ID(MAXIM,MAXIM),Q(MAXIM,MAXIM),AIMQ(MAX,MAX),
     *  AIMQIN(MAX,MAX),ARL(MAX),ANSS(MAX)
      DIMENSION VOLD(MAX),VINT(MAX),VNEW(MAX),ST(MAX,MAXIM),B2(MAXIM),
     *                                                        ROWS(MAX)
      DIMENSION EU(MAXIM,MAXIM),EU1ST(MAXIM),PAIGE(MAXIM,MAXIM),
     *          PAIGEIN(MAXIM,MAXIM),PAIGEST(MAXIM),pistar(maxim,maxim)
      dimension BARRAY(7),BARRAYN(7)
      DIMENSION QS(MAXIM,MAXIM),QSBYM(MAX,MAX),QSBYMHI(MAX,MAX),
     *                                                ADJ1(MAX),ADJ2(MAX)
      DIMENSION qsarl(MAXIM),qsanss(MAXIM),VAL(MAX,10,7),VALN(MAX,10,7)
      COMMON /WORKSP/ RWKSP
      DATA DEVIAT/0.0, 0.25, 0.5, 1.0, 2.0, 3.0/
      CALL IWKIN(90000)
C THE SAMPLING INTERVALS ARE TO BE DECIDED. THE SAMPLING INTERVALS TO BE
C CONSIDERED ARE IV1=1.0D0 AND IV2=1.0D0 FOR FSI(1), IV1=0.5 AND
C IV2=1.5 FOR VSI(0.5, 1.5), IV1=0.1D0 AND IV2=1.9D0 FOR VSI(0.1,1.9).
C FOR VSI(0.1, 1.0, 1.9), IV1=0.1D0 IV2=1.0D0 AND IV3=1.9D0.
```

```
C FINALLY, FOR VSI(0.1, 0.5, 1.0, 1.5, 1.9), IV1=0.1D0 IV2=0.5D0
C IV3=1.0D0 IV4=1.5D0 AND IV5=1.9D0.
      IV1=0.1D0
      IV2=1.9D0
      IV3=1.0D0
      IV4=1.5D0
      IV5=1.9D0
C IF ALTERED CONTROL LIMITS ARE DESIRED THEN CHANGE=.TRUE.
C                                       ELSE CHANGE=.FALSE.
      CHANGE=.FALSE.
C IF REGULAR ATS IS TO BE COMPUTED then regular=.true.
C IF STEADY STATE ATS IS TO BE COMPUTED THEN REGULAR=.FALSE.
      REGULAR=.FALSE.
C SUPPOSE THAT A REGULAR ATS IS DESIRED. THEN WHAT IS THE ASSUMPTION
C FOR THE STARTING POINT.
C IF A FIXED START IS DESIRED THEN FIXEDS=.TRUE.
C IF A RANDOM START IS DESIRED THEN FIXEDS=.FALSE.
      FIXEDS=.FALSE.
C IF A STEADY STATE ATS IS DESIRED THEN WHAT KIND OF STEADY STATE
C PROBABILITIES ARE USED. PICK ONE OUT OF FOUR METHODS.
C IF YASHIN METHOD IS USED THEN ONLY YASHIN=.TRUE.
C IF CONDITIONAL METHOD IS USED THEN ONLY CONDAL=.TRUE.
C IF CYCLICAL METHOD IS USED THEN ONLY CYCAL=.TRUE.
C IF MODIFIED CYCLICAL METHOD IS USED THEN ONLY MCYCAL=.TRUE.
      MCYCAL=.TRUE.
      CONDAL=.FALSE.
      CYCAL=.FALSE.
      YASHIN=.FALSE.
C WHEN ADJUSTING THE CONTROL LIMITS TO GET DESIRED IN CONTROL VALUES
C THE DISIRED IN CONTROL VALUES ARE SET AT 370.4D0.
      STAD=370.4D0
C WHEN GETTING A DOMINANT EIGEN VECTOR, TOLERANCE LIMIT AND MAXIMUM
C INTERATION ARE NEEDED.
      TOL=.000001D0
      MAXITN=500
C INITIALIZE THE STATIONARY PROBABILITIES.
      DO 121 LP1=1,MAX
        DO 121 LP2=1,MAX
121       ST(LP1,LP2)=0.0D0
C GET AN IDENTITY MATRIX
      DO 51 I4=1,maxim
         DO 52 I53=1,MAXIM
            ID(I4,I53)=0
52       CONTINUE
51    CONTINUE
      DO 53 I6=1,maxim
         ID(I6,I6)=1
53    CONTINUE
C ********************************************************************
```

```
C ********** FOR ANY PHI=IPHI*0.1D0, IPHI=0,1,..,9. *****************
C *******************************************************************
      NPHI=1
      DO 81 IPHI = 9,10,3
       PHI=IPHI*0.1D0
C WHEN DIVIDING A UNIT TIME PROCESS INTO 20 DISCRETE POINTS, THE NEW
C PROCESS HAS A CORRELATION OF PHYM WITH THE VARIANCE OF RANDOM SHOCK
C TERM SIGAM.
       IF (IPHI.EQ.0) THEN
        PHYM=0.0
        SIGAM=1.0
       ELSE
        PHYM=PHI**(0.05)
        SIGAM=SQRT((1.0-PHYM**2)/(1.0-PHYM**40))
       END IF
C GIVEN A CORRELATION OF UNIT TIME SERIES AND THE VARIANCE OF THE
C RANDOM SHOCK TERMS, NEW CORRELATION AND NEW VARIANCE OF RANDOM SHOCK
C TERMS FOR A SMALLER UNIT TIME SERIES ARE TO BE COMPUTED.
       CALL GETSIGA(PHI,PHY,SIGA,IV1)
       IF (PHI.LT.0.0D0) THEN
        PHY=IPHI*0.1D0
        SIGA=1.0D0
       END IF
C PROCESS VARIANCE IS SIGZ.
       SIGZ=SIGA*DSQRT(1.0/(1.0-PHY**2))
       RL=3.0D0
C WHEN CONSIDERING ONLY TWO DIFFERENT SAMPLING INTERVALS, B1*SIGZ IS THE
C LINE THAT SEPERATES THE SHORTER AND LONGER SAMPLING INTERVALS
       FIRST=DNORDF(RL)-DNORDF(-RL)
       B11=DNORIN((2+FIRST)/4)
       IF (IV1.EQ.0.5D0) THEN
        IF (IV2.EQ.3.0D0) THEN
          B11=DNORIN((5+FIRST)/10)
        ELSE
          IF (IV2.EQ.2.0D0) THEN
            B11=DNORIN((3+FIRST)/6)
          END IF
        END IF
       END IF
C WHEN CONSIDERING THREE DIFFERENT SAMPLING INTERVALS, B1*SIGZ AND
C C1*SIGZ ARE THE LINES THAT SEPERATE THE SHORTER, MIDDLE AND LONGER
C SAMPLING INTERVALS.
       IF ((IV1.EQ.0.1D0).AND.(IV2.EQ.1.0D0)) THEN
        B11=DNORIN((3+2.0*FIRST)/6)
        C11=DNORIN((3+1.0*FIRST)/6)
       END IF
       IF ((IV1.EQ.0.5D0).AND.(IV2.EQ.1.0D0)) THEN
        B11=DNORIN((3+2.0*FIRST)/6)
        C11=DNORIN((3+1.0*FIRST)/6)
```

```
      END IF
C IN THE SAME WAY, B1*SIGZ, C1*SIGZ, D1*SIGZ AND E1*SIGZ ARE THE LINES
C THAT SEPERATE THE FIVE DIFFERENT SAMPLING INTERVALS.
      IF ((IV1.EQ.0.1D0).and.(iv2.eq.0.5d0)) THEN
       B11=DNORIN((5+4.0*FIRST)/10)
       C11=DNORIN((5+3.0*FIRST)/10)
       D11=DNORIN((5+2.0*FIRST)/10)
       E11=DNORIN((5+1.0*FIRST)/10)
      END IF
c ********************************************************************
C ****** FOR ANY DEVIATION DEV=0, 0.25, 0.5, 1.0, 2.0 AND 3.0 *******
c ********************************************************************
      NDEV=1
      DO 25 IDEV=1,3,2
      DEV=SIGZ*DEVIAT(IDEV)
2748  CALL GETSUB(IV1,IV2,DEV,PHI,PHY,SIGA,SUB,INT1,INT2,INT3,INT4,INT5)
      UL=RL*SIGZ
      B1=B11*SIGZ
      IF ((IV1.EQ.0.1D0).AND.(IV2.EQ.1.0D0)) THEN
       C1=C11*SIGZ
      END IF
      IF ((IV1.EQ.0.1D0).and.(iv2.eq.0.5d0)) THEN
       C1=C11*SIGZ
       D1=D11*SIGZ
       E1=E11*SIGZ
      END IF
c ********************************************************************
C * FOR CERTAIN NUMBER OF TRANSIENT STATES (NR=MIN TO MAX BY INCR)   *
c ********************************************************************
      DO 811 NR=MIN,MAX,INCR
C DETERMINE THE SAMPLING INTERVALS.
      DO 812 NC1=1,NR/INCR
812     B(NC1)=IV1
      DO 813 NC1=NR/INCR+1, 2*NR/INCR
813     B(NC1)=IV2
      DO 814 NC1=2*NR/INCR+1, 3*NR/INCR
814     B(NC1)=IV1
      IF ((IV1.EQ.0.1D0).AND.(IV2.EQ.0.5D0)) THEN
       DO 821 NC1=2*NR/INCR+1, 3*NR/INCR
821      B(NC1)=IV3
       DO 822 NC1=3*NR/INCR+1, 4*NR/INCR
822      B(NC1)=IV4
       DO 823 NC1=4*NR/INCR+1, 5*NR/INCR
823      B(NC1)=IV5
       DO 824 NC1=5*NR/INCR+1, 6*NR/INCR
824      B(NC1)=IV4
       DO 825 NC1=6*NR/INCR+1, 7*NR/INCR
825      B(NC1)=IV3
       DO 826 NC1=7*NR/INCR+1, 8*NR/INCR
826      B(NC1)=IV2
```

```
        DO 827 NC1=8*NR/INCR+1, 9*NR/INCR
827       B(NC1)=IV1
       END IF
       IF (((IV1.EQ.0.5D0).AND.(IV2.EQ.1.0D0)).OR.
     *   ((IV1.EQ.0.1D0).AND.(IV2.EQ.1.0D0))) THEN
        DO 828 NC1=2*NR/INCR+1, 3*NR/INCR
828       B(NC1)=IV3
        DO 829 NC1=3*NR/INCR+1, 4*NR/INCR
829       B(NC1)=IV2
        DO 830 NC1=4*NR/INCR+1, 5*NR/INCR
830       B(NC1)=IV1
       END IF
       IUNIT=1
       DO 4812 NC1=1,NR
4812      BB(NC1)=IUNIT
C BREAK DOWN THE WHOLE REGION WITHIN THE CONTROL LIMITS INTO NR REGIONS.
       M1=NR/INCR
       M2=NR/INCR
       M3=NR/INCR
       M4=NR/INCR
       M5=NR/INCR
       M6=NR/INCR
       M7=NR/INCR
       M8=NR/INCR
       M9=NR/INCR
       DIST1=(UL-B1)/M1
       DIST2=(2*B1)/M2
       DIST3=DIST1
       IF ((IV1.EQ.0.1D0).AND.(IV2.EQ.0.5D0)) THEN
        DIST2=(B1-C1)/M2
        DIST3=(C1-D1)/M3
        DIST4=(D1-E1)/M4
        DIST5=(2.0*E1)/M5
        DIST6=DIST4
        DIST7=DIST3
        DIST8=DIST2
        DIST9=DIST1
       END IF
       IF (((IV1.EQ.0.5D0).AND.(IV2.EQ.1.0D0)).OR.
     *   ((IV1.EQ.0.1D0).AND.(IV2.EQ.1.0D0))) THEN
        DIST2=(B1-C1)/M2
        DIST3=(2.0*C1)/M3
        DIST4=DIST2
        DIST5=DIST1
       END IF
       S(1)=UL
       DO 10 I1=1,M1
         S(I1+1)=S(I1)-DIST1
         R(I1)=S(I1+1)+DIST1/2.0
10     CONTINUE
```

```
      DO 16 I1=M1+1,M1+M2
        S(I1+1)=S(I1)-DIST2
        R(I1)=S(I1+1)+DIST2/2.0
16    CONTINUE
      DO 18 I1=M1+M2+1,M1+M2+M3
        S(I1+1)=S(I1)-DIST3
        R(I1)=S(I1+1)+DIST3/2.0
18    CONTINUE
      IF ((IV1.EQ.0.1D0).AND.(IV2.EQ.0.5D0)) THEN
       DO 311 I1=M1+M2+M3+1,M1+M2+M3+M4
          S(I1+1)=S(I1)-DIST4
          R(I1)=S(I1+1)+DIST4/2.0
311    CONTINUE
       DO 312 I1=M1+M2+M3+M4+1,M1+M2+M3+M4+M5
          S(I1+1)=S(I1)-DIST5
          R(I1)=S(I1+1)+DIST5/2.0
312    CONTINUE
       DO 313 I1=M1+M2+M3+M4+M5+1,M1+M2+M3+M4+M5+M6
          S(I1+1)=S(I1)-DIST6
          R(I1)=S(I1+1)+DIST6/2.0
313    CONTINUE
       DO 314 I1=M1+M2+M3+M4+M5+M6+1,M1+M2+M3+M4+M5+M6+M7
          S(I1+1)=S(I1)-DIST7
          R(I1)=S(I1+1)+DIST7/2.0
314    CONTINUE
       DO 315 I1=M1+M2+M3+M4+M5+M6+M7+1,M1+M2+M3+M4+M5+M6+M7+M8
          S(I1+1)=S(I1)-DIST8
          R(I1)=S(I1+1)+DIST8/2.0
315    CONTINUE
       DO 316 I1=M1+M2+M3+M4+M5+M6+M7+M8+1,M1+M2+M3+M4+M5+M6+M7+M8+M9
          S(I1+1)=S(I1)-DIST9
          R(I1)=S(I1+1)+DIST9/2.0
316    CONTINUE
      END IF
      IF (((IV1.EQ.0.5D0).AND.(IV2.EQ.1.0D0)).OR.
     *  ((IV1.EQ.0.1D0).AND.(IV2.EQ.1.0D0))) THEN
       DO 317 I1=M1+M2+M3+1,M1+M2+M3+M4
          S(I1+1)=S(I1)-DIST4
          R(I1)=S(I1+1)+DIST4/2.0
317    CONTINUE
       DO 318 I1=M1+M2+m3+m4+1,M1+M2+M3+m4+m5
          S(I1+1)=S(I1)-DIST5
          R(I1)=S(I1+1)+DIST5/2.0
318    continue
      end if
C BREAK DOWN THE WHOLE REGION INSIDE AND OUTSIDE THE CONTROL LIMITS
C INTO (NR + 2*(NR/INCR*MULTI) + 2) DISCRETE REGIONS.
      IF (MCYCAL) THEN
        MAXI=NR/(INCR*MULTI)
        US(1)=S(1)
```

```
          UR(1)=R(1)
          DS(1)=S(NR+1)
          DR(1)=R(NR)
          DO 71 MZ1=1,MAXI
           US(MZ1+1)=US(MZ1)+DIST1
           UR(MZ1+1)=UR(MZ1)+DIST1
           DS(MZ1+1)=DS(MZ1)-DIST1
71         DR(MZ1+1)=DR(MZ1)-DIST1
          UR(MAXI+2)=UR(MAXI+1)+DIST1
          DR(MAXI+2)=DR(MAXI+1)+DIST1
          TOTALR(1)=UR(MAXI+2)
          TOTALS(1)=US(MAXI+1)
          DO 72 L1=1,MAXI
           TOTALR(L1+1)=TOTALR(L1)-DIST1
72         TOTALS(L1+1)=TOTALS(L1)-DIST1
          DO 73 L1=1,NR
           TOTALR(MAXI+L1+1)=R(L1)
73         TOTALS(MAXI+L1)=S(L1)
          DO 74 L1=1,MAXI+1
           TOTALR(MAXI+NR+L1+1)=
     *                           TOTALR(MAXI+NR+L1)-DIST1
           TOTALS(MAXI+NR+L1)=
     *                           TOTALS(MAXI+NR+L1-1)-DIST1
74        continue
        END IF
C NOW CALCULATE THE TRANSITION MATRIX Q.
        IF (IV1 .LT. 1.0D0) THEN
         IF ((IV1.EQ.0.1D0).AND.(IV2.EQ.0.5D0)) THEN
         do 205 jk2=1,nr
          IF ((JK2.LE.(NR/INCR)).OR.(JK2.GE.((INCR-1)*NR/INCR+1))) THEN
             DO 200 JK3 = 1,NR
             Q(JK2,JK3)=DNORDF((S(JK3)-PHY**INT1*R(JK2)-SUB(5))/SUB(6))-
     *             DNORDF((S(jk3+1)-PHY**INT1*R(jk2)-SUB(5))/SUB(6))
200          CONTINUE
          else
          IF  ( ( (JK2.GE.(NR/INCR+1)) .AND. (JK2.LE.(2*NR/INCR)) ).OR.
     * ( (JK2.GE.((INCR-2)*NR/INCR+1)).AND.(JK2.LE.((INCR-1)*NR/INCR))))
     *     THEN
             DO 201 JK3 = 1,NR
           Q(JK2,JK3)=DNORDF((S(JK3)-PHY**INT2*R(JK2)-SUB(4))/SUB(7))-
     *             DNORDF((S(JK3+1)-PHY**INT2*R(JK2)-SUB(4))/SUB(7))
201          CONTINUE
          ELSE
          IF  ( ( (JK2.GE.(2*NR/INCR+1)) .AND. (JK2.LE.(3*NR/INCR)) ).OR.
     *  ((JK2.GE.((INCR-3)*NR/INCR+1)).AND.(JK2.LE.((INCR-2)*NR/INCR))))
     *     THEN
             DO 202 JK3 = 1,NR
           Q(JK2,JK3)=DNORDF((S(JK3)-PHY**INT3*R(JK2)-SUB(3))/SUB(8))-
     *             DNORDF((S(JK3+1)-PHY**INT3*R(JK2)-SUB(3))/SUB(8))
202          CONTINUE
```

```
      ELSE
      IF  ( ( (JK2.GE.(3*NR/INCR+1)) .AND. (JK2.LE.(4*NR/INCR)) ).OR.
     * ((JK2.GE.((INCR-4)*NR/INCR+1)).AND.(JK2.LE.((INCR-3)*NR/INCR))))
     *    THEN
         DO 203 JK3 = 1,NR
       Q(JK2,JK3)=DNORDF((S(JK3)-PHY**INT4*R(JK2)-SUB(2))/SUB(9))-
     *           DNORDF((S(JK3+1)-PHY**INT4*R(JK2)-SUB(2))/SUB(9))
203      CONTINUE
      ELSE
         DO 204 JK3 = 1,NR
      Q(JK2,JK3)=DNORDF((S(JK3)-PHY**INT5*R(JK2)-SUB(1))/SUB(10))-
     *           DNORDF((S(JK3+1)-PHY**INT5*R(JK2)-SUB(1))/SUB(10))
204      CONTINUE
      END IF
      END IF
      END IF
      END IF
205   CONTINUE
     else
    IF (((IV1.EQ.0.5D0).AND.(IV2.EQ.1.0D0)).OR.
     *  ((IV1.EQ.0.1D0).AND.(IV2.EQ.1.0D0))) THEN
     DO 206 JK2=1,NR
      IF ((JK2.LE.(NR/INCR)).OR.(JK2.GE.((INCR-1)*NR/INCR+1))) THEN
         DO 207 JK3 = 1,NR
       Q(JK2,JK3)=DNORDF((S(JK3)-PHY**INT1*R(JK2)-SUB(3))/SUB(4))-
     *           DNORDF((S(JK3+1)-PHY**INT1*R(JK2)-SUB(3))/SUB(4))
207      CONTINUE
      ELSE
      IF  ( ( (JK2.GE.(NR/INCR+1)) .AND. (JK2.LE.(2*NR/INCR)) ).OR.
     * ( (JK2.GE.((INCR-2)*NR/INCR+1)).AND.(JK2.LE.((INCR-1)*NR/INCR))))
     *    THEN
         DO 208 JK3 = 1,NR
       Q(JK2,JK3)=DNORDF((S(JK3)-PHY**INT2*R(JK2)-SUB(2))/SUB(5))-
     *           DNORDF((S(jk3+1)-PHY**INT2*R(jk2)-SUB(2))/SUB(5))
208      CONTINUE
      ELSE
         DO 209 JK3 = 1,NR
       Q(JK2,JK3)=DNORDF((S(JK3)-PHY**INT3*R(JK2)-SUB(1))/SUB(6))-
     *           DNORDF((S(JK3+1)-PHY**INT3*R(JK2)-SUB(1))/SUB(6))
209      CONTINUE
      END IF
      END IF
206   CONTINUE
      ELSE
      DO 30 I2=1,NR
      IF ((I2.LE.(NR/incr)).OR.(I2.GE.((2*NR/incr)+1))) THEN
         DO 35 I3 = 1,NR
           Q(I2,I3)=DNORDF((S(I3)-PHY**INT1*R(I2)-SUB(2))/SUB(3))-
     *                  DNORDF((S(I3+1)-PHY**INT1*R(I2)-SUB(2))/SUB(3))
35       CONTINUE
```

```
         ELSE
            DO 45 I3=1,NR
              Q(I2,I3)=DNORDF((S(I3)-PHY**INT2*R(I2)-SUB(1))/SUB(4))-
     *                 DNORDF((S(I3+1)-PHY**INT2*R(I2)-SUB(1))/SUB(4))
45          CONTINUE
         END IF
30       CONTINUE
        END IF
       END IF
      ELSE
         DO 31 I8=1,NR
           DO 31 I9=1,NR
             Q(I8,I9)=DNORDF((S(I9)-SUB(1)-PHY**INT1*R(I8))/SUB(4))-
     *                  DNORDF((S(I9+1)-SUB(1)-PHY**INT1*R(I8))/SUB(4))
31          CONTINUE
      END IF
C GET I-Q MATRIX
      DO 124 I8=1,NR
        DO 124 I9=1,NR
          IF (I8.EQ.I9) THEN
            AIMQ(I8,I9)=ID(I8,I9)-Q(I8,I9)
          ELSE
            AIMQ(I8,I9)=-Q(I8,I9)
          END IF
124   CONTINUE
C GET AN INVERSE OF (I-Q)
      CALL DLINRG(NR,AIMQ,MAX,AIMQIN,MAX)
C GET INV(I-Q)*B' where B is the vector with sampling intervals.
      IPATH=1
      CALL DMURRV(NR,NR,AIMQIN,MAX,NR,B,IPATH,NR,ARL)
      CALL DMURRV(NR,NR,AIMQIN,MAX,NR,BB,IPATH,NR,ANSS)
c ******************************************************************
C ***************** GET STATIONARY PROBABILITIES ********************
c ******************************************************************
       IF (.NOT.REGULAR) THEN
       IF (IDEV .EQ. 1) THEN
C STEADY STATE PROBABILITIES WHEN YASHCHIN'S METHOD IS USED.
         IF (YASHIN) THEN
C TO GET A DOMINANT EIGENVECTOR
          ITN = 0
C      STEP 1 - DEFINE THE STARTING VECTOR VOLD AND FIND NORM(VOLD)
          RVAL = NR
          START = 1.0D0/RVAL
          DO 101 II = 1, NR
101       VOLD(II) = START
          DO 103 JJ = 1, NR
          SUMCOL = 0.0D0
          DO 102 II = 1, NR
102       SUMCOL = SUMCOL + (VOLD(II)*Q(JJ,II))
103       VINT(JJ) = SUMCOL
```

```
          SUMOLD = 0.0D0
          DO 104 II = 1, NR
104       SUMOLD = SUMOLD + (VINT(II)**2)
          ABSOLD = dsqrt(SUMOLD)
105       CONTINUE
          ITN = ITN + 1
          IF (ITN .GE. MAXITN) GO TO 120
C      STEP 2 - COMPUTE VNEW = VINT/ABSOLD
          DO 110 II = 1, NR
110       VNEW(II) = VINT(II)/ABSOLD
C      STEP 3 - COMPUTE VINT = VNEW*Q
          DO 113 JJ = 1, NR
          SUMCOL = 0.0D0
          DO 112 II = 1, NR
112       SUMCOL = SUMCOL + (VNEW(II)*Q(JJ,II))
113       VINT(JJ) = SUMCOL
C      STEP 4 - FIND NORM(VINT) = NORM(VNEW*Q)
          SUMINT = 0.0D0
          DO 114 II = 1, NR
114       SUMINT = SUMINT + (VINT(II)**2)
          ABSNEW = dsqrt(SUMINT)
C      STEP 5 - CHECK FOR CONVERGENCE
          ATOL = DABS((ABSNEW-ABSOLD)/ABSOLD)
          IF (ATOL .LT. TOL) GO TO 130
C      STEP 5 - IF DIT NOT GET CONVERGENCE FIND VNEW = VINT/NORM(VINT)
C              AND RENAME VOLD. ALSO RENAME ABSOLD
          ABSOLD = ABSNEW
          GO TO 105
120       CONTINUE
130       CONTINUE
C      NORMALIZE VNEW(I) TO OBTAIN STATIONARY DISTRIBUTION
          SUMM=0.0D0
          DO 135 II = 1, NR
135       SUMM = SUMM + VNEW(II)
          DO 136 II =1, NR
136       ST(NR,II)=VNEW(II)/SUMM
C      Determine the sampling intervals within the control limits.
          NNR=NR
          DO 139 JP=1,NNR
139       B2(JP)=B(JP)
         end if
C END OF YASHCHIN'S METHOD
C STEADY STATE PROBABILITIES WHEN CONDITIONAL OR CYCLICAL METHODS ARE
C USED
         IF (CONDAL.OR.CYCAL) THEN
          DO 701 NT0=1,NR
            ROWS(NT0)=0.0
701       CONTINUE
          DO 702 NT1=1,NR
           DO 772 NT2=1,NR
```

```
          ROWS(NT1)=ROWS(NT1)+Q(NT1,NT2)
772     CONTINUE
702    CONTINUE
        DO 149 JY=1,NR
149      B2(JY)=B(JY)
       END IF
C STEADY STATE PROBABILITIES WHEN CONDITIONAL METHOD IS USED
       IF (CONDAL) THEN
        DO 737 NT3=1,NR
         DO 719 NT4=1,NR
           Q(NT3,NT4)=Q(NT3,NT4)/ROWS(NT3)
719      CONTINUE
737     CONTINUE
        NNR=NR
       END IF
C STEADY STATE PROBABILITIES WHEN CONDITIONAL METHOD IS USED
       IF (CYCAL) THEN
        DO 773 NT5=1,NR
         Q(NR+1,NT5)=0.0D0
773      Q(NT5,NR+1)=1.0D0-ROWS(NT5)
        Q(NR+1,INT(NR/2)+1)=1.0D0
        Q(NR+1,NR+1)=0.0D0
        NNR=NR+1
       END IF
C STEADY STATE PROBABILITIES WHEN MODIFIED CYCLICAL METHOD IS USED
       IF (MCYCAL) THEN
        IF (IV1.LT.1.0D0) THEN
         DO 75 I2=1,NR+2*MAXI+2
          IF ((I2.LE.(INT(NR/INCR)+MAXI+1)).OR.
     *          (I2.GE.(2*INT(NR/INCR)+MAXI+2))) THEN
           DO 76 I3=1,NR+2*MAXI+2
            IF (I3.EQ.1) THEN
             XXX=(TOTALS(I3)-PHY**INT1*TOTALR(I2)-SUB(2))/SUB(3)
             IF (XXX.LT.(-6.0)) THEN
              Q(I2,I3)=1.0
             ELSE
              IF (XXX.GT.6) THEN
                Q(I2,I3)=0.0
              ELSE
                Q(I2,I3)=1.0D0-DNORDF(XXX)
              END IF
             END IF
            ELSE
             IF (I3.LT.(NR+2*MAXI+2)) THEN
              XX1=(TOTALS(I3-1)-PHY**INT1*TOTALR(I2)-SUB(2))/SUB(3)
              XX2=(TOTALS(I3)-PHY**INT1*TOTALR(I2)-SUB(2))/SUB(3)
              IF ((XX1.LT.(-6.0)).OR.(XX2.GT.6)) THEN
               Q(I2,I3)=0.0
              ELSE
               Q(I2,I3)=DNORDF(xx1)-dnordf(xx2)
```

```
          END IF
         ELSE
          XXX=(TOTALS(I3-1)-PHY**INT1*TOTALR(I2)-SUB(2))/SUB(3)
          IF (XXX.GT.6.0) THEN
           Q(I2,I3)=1.0
          ELSE
           IF (XXX.LT.(-6.0)) THEN
             Q(I2,I3)=0.0
           ELSE
             Q(I2,I3)=DNORDF(XXX)
           END IF
          END IF
         END IF
        END IF
76     CONTINUE
      ELSE
       DO 77 I3=1,NR+2*MAXI+2
        IF (I3.EQ.1) THEN
         XXX=(TOTALS(I3)-PHY**INT2*TOTALR(I2)-SUB(1))/SUB(4)
         IF (XXX.LT.(-6.0)) THEN
          Q(I2,I3)=1.0
         ELSE
          IF (XXX.GT.6.0) THEN
            Q(I2,I3)=0.0
          ELSE
            Q(I2,I3)=1.0D0-DNORDF(xxx)
          END IF
         END IF
        ELSE
         IF (I3.LT.(NR+2*MAXI+2)) THEN
          XX1=(TOTALS(I3-1)-PHY**INT2*TOTALR(I2)-SUB(1))/SUB(4)
          XX2=(TOTALS(I3)-PHY**INT2*TOTALR(I2)-SUB(1))/SUB(4)
          IF ((XX1.LT.(-6.0)).OR.(XX2.GT.6)) THEN
           Q(I2,I3)=0.0
          ELSE
           Q(I2,I3)=DNORDF(xx1)-dnordf(xx2)
          END IF
         ELSE
          XXX=(TOTALS(I3-1)-PHY**INT2*TOTALR(I2)-SUB(1))/SUB(4)
          IF (XXX.LT.(-6.0)) THEN
           Q(I2,I3)=0.0
          ELSE
           IF (XXX.GT.6.0) THEN
            Q(I2,I3)=1.0
           ELSE
            Q(I2,I3)=DNORDF(xxx)
           END IF
          END IF
         END IF
```

```
            END IF
77         CONTINUE
           END IF
75        CONTINUE
         ELSE
          DO 78 I8=1,NR+2*MAXI+2
          DO 78 I9=1,NR+2*MAXI+2
           IF (I9.EQ.1) THEN
            XXX=(TOTALS(I9)-PHY**INT1*TOTALR(I8)-SUB(1))/SUB(4)
            IF (XXX.LT.(-6.0)) THEN
              Q(I8,I9)=1.0
            ELSE
              IF (XXX.GT.6.0) THEN
                Q(I8,I9)=0.0
              ELSE
                Q(I8,I9)=1.0D0-DNORDF(XXX)
              END IF
            END IF
           ELSE
            IF (I9.EQ.(NR+2*MAXI+2)) THEN
             XXX=(TOTALS(I9-1)-PHY**INT1*TOTALR(I8)-SUB(1))/SUB(4)
             IF (XXX.LT.(-6.0)) THEN
               Q(I8,I9)=0.0
             ELSE
               IF (XXX.GT.6.0) THEN
                 Q(I8,I9)=1.0
               ELSE
                 Q(I8,I9)=DNORDF(XXX)
               END IF
             END IF
            ELSE
             XX1=(TOTALS(I9-1)-PHY**INT1*TOTALR(I8)-SUB(1))/SUB(4)
             XX2=(TOTALS(I9)-PHY**INT1*TOTALR(I8)-SUB(1))/SUB(4)
             IF ((XX1.LT.(-6.0)).OR.(XX2.GT.6)) THEN
               Q(I8,I9)=0
             ELSE
               Q(I8,I9)=DNORDF(xx1)-dnordf(xx2)
             END IF
            END IF
           END IF
78        CONTINUE
         END IF
         NNR=NR+2*MAXI+2
         DO 158 KF=1,NNR
          IF ((KF.LE.(INT(NR/INCR)+MAXI+1)).OR.
     *           (KF.GE.(2*INT(NR/INCR)+MAXI+2))) THEN
            B2(KF)=IV1
          ELSE
            B2(KF)=IV2
```

```
          END IF
158       CONTINUE
         END IF
C ENOF OF GETTING STEADY STATE PROBABILITIES WHEN MODIFIED CYCLICAL
C METHOD IS USED.
C FOR ANY OTHER METHOD OTHER THAN YASHCHIN'S
         IF (.NOT.YASHIN) THEN
          DO 511 NZ1=1,NNR
511       EU1ST(NZ1)=Q(1,NZ1)
          DO 512 NZ2=1,NNR
          DO 512 NZ3=1,NNR
512        EU(NZ2,NZ3)=EU1ST(NZ3)
          DO 513 NZ4=1,NNR
          DO 513 NZ5=1,NNR
513        PAIGE(NZ4,NZ5)=ID(NZ4,NZ5)-Q(NZ4,NZ5)+EU(NZ4,NZ5)
C GET AN INVERSE OF PAIGE = [I - Q + E U']
          CALL DLINRG(NNR,PAIGE,MAXIM,PAIGEIN,MAXIM)
C      GET PI' = U' INV [I - Q + E U']
          IPATH=2
          CALL DMURRV(nnr,nnr,PAIGEIN,maxim,nnr,EU1ST,IPATH,nnr,PAIGEST)
          IF (CYCAL) THEN
           DENOCYC=0.0
           DO 241 MQ=1,NR
241         DENOCYC=DENOCYC+PAIGEST(MQ)
           DO 242 MW=1,NR
242         ST(NR,MW)=PAIGEST(MW)/DENOCYC
          ELSE
           DO 79 NZ6=1,NNR
79          ST(NR,NZ6)=PAIGEST(NZ6)
          END IF
         END IF
C END OF " ANY OTHER METHODS OTHER THAN YASHCHIN'S "
C FOR ANY METHOD, GET STEADY STATE PROBABILITIES OF SHIFT HAPPENING IN
C A CERTAIN INTERVAL.
         DENO=0.0D0
         DO 774 NT6=1,NNR
774       DENO=DENO+ST(NR,NT6)*B2(NT6)
         DO 775 NT7=1,NNR
          PISTAR(NR,NT7)=ST(NR,NT7)*B2(NT7)/DENO
775      CONTINUE
       END IF
       END IF
c *********************************************************************
C ******** END OF GETTING STATIONARY PROBABILITIES *****************
c *********************************************************************
c
c *********************************************************************
C ** GET REGULAR AND STEADY STATE ATS AND ANSS WHEN OUT OF CONTROL **
c *********************************************************************
C REGULAR ATS AND ANSS ARE COMPUTED.
```

```
       IF ( (REGULAR).OR.((.NOT.REGULAR).AND.(IDEV.EQ.1)) ) THEN
        IF ((.NOT.REGULAR).AND.(IDEV.EQ.1)) THEN
         FIXEDS=.FALSE.
        END IF
C IF A FIXED START IS ASSUMED
        IF (FIXEDS) THEN
         IK=1
301      IF ( ((S(IK)/SIGZ).GE.DEVIAT(IDEV)) .AND.
     *      ((S(IK+1)/SIGZ).LT.DEVIAT(IDEV)) ) GO TO 302
           IK=IK+1
           GO TO 301
302      CONTINUE
         AVG=0.0D0
         AVGN=0.0D0
         DO 231 JB=1,NR
          PINI(JB)=PROTO1(IV1,S,MAX,IK,JB,PHY,DEV,R,SIGA)
          AVG=AVG+PINI(JB)*ARL(JB)
231       AVGN=AVGN+PINI(JB)*ANSS(JB)
         VAL(NR,NPHI,NDEV)=AVG+1.0
         VALN(NR,NPHI,NDEV)=AVGN+1.0
C IF A RANDOM START IS ASSUMED
        ELSE
         DO 770 IN=1, NR
770      PO(IN)=DNORDF((S(IN)-DEV)/SIGZ)-DNORDF((S(IN+1)-DEV)/SIGZ)
         AVG=0.0D0
         AVGN=0.0D0
         DO 771 IM=1, NR
           AVG=AVG+ARL(IM)*PO(IM)
771        AVGN=AVGN+ANSS(IM)*PO(IM)
         VAL(NR,NPHI,NDEV)=AVG+1
         VALN(NR,NPHI,NDEV)=AVGN+1
        END IF
C END OF FIXED START OR RANDOM START
       ELSE
C STEADY STATE ATS AND ANSS ARE COMPUTED WHEN OUT OF CONTROL.
        IF (IDEV.GT.1) THEN
         LENG1=(IV1+0.0001)*20
         LENG2=40-LENG1
C GET THE TRANSITION MATRIX FOR THE INTERVAL IN QUESTION OF SHIFT.
         IF (MCYCAL) THEN
          MAXI=NR/(INCR*MULTI)
          NNR=NR+2*MAXI+2
          DO 91 I2=1,nnr
          DO 91 I3=1,NR
             IF ((i2.LE.(INT(NR/INCR)+MAXI+1)).OR.
     *            (i2.GE.(2*INT(NR/INCR)+MAXI+2))) THEN
              NLEN=LENG1
             ELSE
              NLEN=LENG2
             END IF
```

```
          SUM2=0.0
          DO 580 I4=1,NLEN
           SUM1=0.0
           DO 581 I5=1,NNR
            IF (I5.EQ.1) THEN
              XXX=(TOTALS(1)-PHYM**I4*TOTALR(I2))/(SQRT(
     *              (1.0-PHYM**(2*I4))/(1.0-PHYM**2) )*SIGAM)
              IF (XXX.LT.(-6.0)) THEN
                PROIND=1.0
              ELSE
                IF (XXX.GT.6.0) THEN
                  PROIND=0.0
                ELSE
                  PROIND=1.0-DNORDF(xxx)
                END IF
              END IF
            ELSE
              IF (I5.LT.NNR) THEN
               XX1=(TOTALS(I5-1)-PHYM**I4*TOTALR(I2))/(SQRT(
     *              (1.0-PHYM**(2*I4))/(1.0-PHYM**2) )*SIGAM)
               XX2=(TOTALS(I5)-PHYM**I4*TOTALR(I2))/(SQRT(
     *              (1.0-PHYM**(2*I4))/(1.0-PHYM**2) )*SIGAM)
               IF ((XX1.LT.(-6.0)).OR.(XX2.GT.6)) THEN
                 PROIND=0.0
               ELSE
                 PROIND=DNORDF(XX1)-DNORDF(XX2)
               END IF
              ELSE
               XXX=(TOTALS(I5-1)-PHYM**I4*TOTALR(I2))/(SQRT(
     *              (1.0-PHYM**(2*I4))/(1.0-PHYM**2) )*SIGAM)
               IF (XXX.LT.(-6.0)) THEN
                 PROIND=0.0
               ELSE
                 IF (XXX.GT.6.0) THEN
                   PROIND=1.0
                 ELSE
                   PROIND=DNORDF(xxx)
                 END IF
               END IF
              END IF
            END IF
            IF (I4.LT.NLEN) THEN
             XX1=(S(I3)-PHYM**(NLEN-I4)*TOTALR(I5)-
     *       DEV)/(SQRT((1-PHYM**(2*(NLEN-I4)))/(1.0-PHYM**2))*SIGAM)
             XX2=(S(I3+1)-PHYM**(NLEN-I4)*TOTALR(I5)-
     *       DEV)/(SQRT((1-PHYM**(2*(NLEN-I4)))/(1.0-PHYM**2))*SIGAM)
             IF ((XX1.LT.(-6.0)).OR.(XX2.GT.6)) THEN
               PRODEP=0.0
             ELSE
               PRODEP=DNORDF(XX1)-DNORDF(XX2)
```

```
               END IF
              ELSE
               IF ((I5-(MAXI+1)).EQ.I3) THEN
                PRODEP=1.0
               ELSE
                PRODEP=0.0
               END IF
              END IF
581          SUM1=SUM1+PROIND*PRODEP
580         SUM2=SUM2+SUM1
91       QS(I2,I3)=SUM2/NLEN
        ELSE
         DO 531 I2=1,NR
         DO 531 I3=1,NR
            IF ((I2.LE.(INT(NR/INCR))).OR.
     *           (I2.GE.(2*INT(NR/INCR)+1))) THEN
              NLEN=LENG1
            ELSE
              NLEN=LENG2
            END IF
            SUM2=0.0
            DO 532 I4=1,NLEN
             SUM1=0.0
             DO 533 I5=1,NR
              IF (I5.EQ.1) THEN
                XXX=(S(1)-PHYM**I4*R(I2))/(SQRT(
     *                 (1.0-PHYM**(2*I4))/(1.0-PHYM**2) )*SIGAM)
                IF (XXX.LT.(-6.0)) THEN
                  PROIND=1.0
                ELSE
                  IF (XXX.GT.6.0) THEN
                    PROIND=0.0
                  ELSE
                    PROIND=1.0-DNORDF(xxx)
                  END IF
                END IF
              ELSE
               IF (I5.LT.NNR) THEN
                 XX1=(S(I5)-PHYM**I4*R(I2))/(SQRT(
     *                 (1.0-PHYM**(2*I4))/(1.0-PHYM**2) )*SIGAM)
                 XX2=(S(I5+1)-PHYM**I4*R(I2))/(SQRT(
     *                 (1.0-PHYM**(2*I4))/(1.0-PHYM**2) )*SIGAM)
                 IF ((XX1.LT.(-6.0)).OR.(XX2.GT.6)) THEN
                   PROIND=0.0
                 ELSE
                   PROIND=DNORDF(XX1)-DNORDF(XX2)
                 END IF
               ELSE
                 XXX=(S(NR+1)-PHYM**I4*R(I2))/(SQRT(
     *                 (1.0-PHYM**(2*I4))/(1.0-PHYM**2) )*SIGAM)
```

```
                  IF (XXX.LT.(-6.0)) THEN
                    PROIND=0.0
                  ELSE
                    IF (XXX.GT.6.0) THEN
                      PROIND=1.0
                    ELSE
                      PROIND=DNORDF(XXX)
                    END IF
                  END IF
                END IF
               END IF
               IF (I4.LT.NLEN) THEN
                XX1=(S(I3)-PHYM**(NLEN-I4)*R(I5)-DEV)/(SQRT(
     *            (1-PHYM**(2*(NLEN-I4) ) ) / (1.0-PHYM**2))*SIGAM)
                XX2=(S(I3+1)-PHYM**(NLEN-I4)*R(I5)-DEV)/(SQRT(
     *            (1-PHYM**(2*(NLEN-I4) ) ) / (1.0-PHYM**2) )*SIGAM)
                IF ((XX1.LT.(-6.0)).OR.(XX2.GT.6)) THEN
                  PRODEP=0.0
                ELSE
                  PRODEP=DNORDF(XX1)-DNORDF(XX2)
                END IF
               ELSE
                IF (I5.EQ.I3) THEN
                 PRODEP=1.0
                ELSE
                 PRODEP=0.0
                END IF
               END IF
533           SUM1=SUM1+PROIND*PRODEP
532          SUM2=SUM2+SUM1
531       QS(I2,I3)=SUM2/NLEN
         END IF
C END OF GET THE TRANSITION MATRIX FOR THE INTERVAL IN QUESTION OF SHIFT.
C GET THE STEADY STATE ATS AND ANSS
         IF (MCYCAL) THEN
          DO 157 KF=1,NNR
           IF ((KF.LE.(INT(NR/INCR)+MAXI+1)).OR.
     *           (KF.GE.(2*INT(NR/INCR)+MAXI+2))) THEN
            B2(KF)=IV1
           ELSE
            B2(KF)=IV2
           END IF
157       CONTINUE
          CALL DMURRV(NNR,NR,QS,MAXIM,NR,ARL,IPATH,NNR,QSARL)
          CALL DMURRV(NNR,NR,QS,MAXIM,NR,ANSS,IPATH,NNR,QSANSS)
          SUM22=0.0
          SUM23=0.0
          SUM24=0.0
          DO 674 J674=1,NNR
674        SUM22=SUM22+PISTAR(NR,J674)*B2(J674)
```

```
          DO 675 J675=1,NNR
675        SUM23=SUM23+PISTAR(NR,J675)*QSARL(J675)
          DO 676 J676=1,NNR
676        SUM24=SUM24+PISTAR(NR,J676)*QSANSS(J676)
          VAL(NR,NPHI,NDEV)=SUM22/2+SUM23
          VALN(NR,NPHI,NDEV)=SUM24
         ELSE
C GET QSBYM=QS*M
          CALL DMRRRR(NR,NR,QS,MAXIM,NR,NR,AIMQIN,MAX,NR,NR,QSBYM,MAX)
c get [I/2 + QSbyM]
          DO 776 NT8=1,NR
          DO 776 NT9=1,NR
776         QSBYMHI(NT8,NT9)=ID(NT8,NT9)/2.0 + QSBYM(NT8,NT9)
C GET [I/2 + QSBYM]*B=ADJ1
          IPATH=1
          CALL DMURRV(NR, NR, QSBYMHI,MAX,NR,B,IPATH,NR,ADJ1)
C GET QSBYM * 1 = ADJ2
          CALL DMURRV(NR, NR, QSBYM,MAX,NR,BB,IPATH,NR,ADJ2)
C GET AATS1=PISTAR*[I/2 + QSBYM]*B
C GET AANSS1=PISTAR*[QSBYM]*1
          AATS1=0.0D0
          AANSS1=0.0D0
          DO 778 N12=1,NR
           AATS1=AATS1+PISTAR(NR,N12)*ADJ1(N12)
           AANSS1=AANSS1+PISTAR(NR,N12)*ADJ2(N12)
778       CONTINUE
          VAL(NR,NPHI,NDEV)=AATS1
          VALN(NR,NPHI,NDEV)=AANSS1
         END IF
C END OF 'GET THE STEADY STATE ATS AND ANSS'
      END IF
      END IF
C     WRITE(6,*) 'PHI = ',PHI,' IDEV = ',IDEV,' NR = ',NR
C     WRITE(6,*) ' ATS AND ANSS ARE ',VAL(NR,NPHI,NDEV),
C    *                                        VALN(NR,NPHI,NDEV)
C **************************    END OF   ******************************
C ** GET REGULAR AND STEADY STATE ATS AND ANSS WHEN OUT OF CONTROL **
c *******************************************************************
811   CONTINUE
c *******************************************************************
C ******************* END OF 'NR=MIN TO MAX BY INCR' ****************
c *******************************************************************
C GET ASYMPTOTIC VALUES OF REGULAR AND STEADY STATE ATS AND ANSS USING
C ONE OF IMSL SUBROUTINE DRCURV
      DO 1235 IJ3=MIN,MAX,INCR
        AIJ3=IJ3*1.0
        ICO=(IJ3 - MIN)/incr + 1
        XDATA(ICO)=(1.0/AIJ3)*(1.0/AIJ3)
        YDATA(ICO)=VAL(IJ3,NPHI,NDEV)
```

```
        YDATAN(ICO)=VALN(IJ3,NPHI,NDEV)
1235  CONTINUE
      IF (IPHI.EQ.0) THEN
        BETA(1)=VAL(NR-INCR,NPHI,NDEV)
        BETAN(1)=VALN(NR-INCR,NPHI,NDEV)
      ELSE
        CALL DRCURV((MAX-MIN)/INCR+1,XDATA,YDATA,2,BETA,SSPOLY,STAT)
        CALL DRCURV((MAX-MIN)/INCR+1,XDATA,YDATAN,2,BETAN,SSPOLY,STAT)
      END IF
C **********************************************************************
C ****   ADJUST THE CONTROL LIMITS TO GET DESIRED ATS AND ANSS   *****
C THE YARD STICK USED IN THE COMPUTATION IS 370.4. IF THE NEW ATS AND
C ANSS ARE CLOSE TO THE YARD STICK (LESS THAN 1 IN ABSOLUTE VALUE)
C THEN THE ADJUSTING PROCEDURE STOPS.
C **********************************************************************
      DIFF=STAD-BETA(1)
      DIFFN=STAD-BETAN(1)
      IF (CHANGE) THEN
       IF (IV1 .LT. 1.0D0) THEN
        IF ((IPHI .NE. 0) .AND. (IDEV .EQ. 1)) THEN
         IF ((BETA(1) .LT. STAD) .AND. (BETAN(1) .LT. STAD)) THEN
           IF ((ABS(DIFF) .LT. 1) .AND. (ABS(DIFFN) .LT. 1)) GO TO 5003
            WRITE(6,5349) PHI, DEV, BETA(1), BETAN(1),RL,B11,C11,D11,E11
5349        FORMAT(2F6.3,2F20.5,5F7.4)
           IF ((ABS(DIFF) .LT. 5) .AND. (ABS(DIFFN) .LT. 5)) THEN
            RL=RL+.0008
           ELSE
            IF ((DIFF .GT. 100) .AND. (DIFFN .GT. 100)) THEN
               RL=RL+.1D0
            ELSE
               IF ((DIFF .GT. 50) .AND. (DIFFN .GT. 50)) THEN
                  RL=RL+.045D0
               ELSE
                  IF (DIFF .GT. 50) THEN
                    RL=RL+.031D0
                  ELSE
                    RL=RL+.024D0
                  END IF
               END IF
            END IF
           END IF
           IDENT=1
         ELSE
          IF ((BETA(1) .LT. STAD) .AND. (BETAN(1) .GT. STAD)) THEN
            IF ((ABS(DIFF) .LT. 1) .AND. (ABS(DIFFN) .LT. 1)) GO TO 5003
            WRITE(6,5345) PHI, DEV, BETA(1), BETAN(1),RL,B11,C11,D11,E11
5345        FORMAT(2F6.3,2F20.5,5F7.4)
            IF ((ABS(DIFF).LT.5).AND.(ABS(DIFFN).LT.5)) THEN
              B11=B11+.001D0
            ELSE
```

```
          IF ((ABS(DIFF).GT.100).AND.(ABS(DIFFN).LT.10)) THEN
            B11=B11+.16D0
          ELSE
            B11=B11+.07D0
          END IF
        END IF
        IDENT=2
      ELSE
       IF ((BETA(1) .GT. STAD) .AND. (BETAN(1) .LT. STAD)) THEN
        IF ((ABS(DIFF) .LT. 1) .AND. (ABS(DIFFN) .LT. 1)) GO TO 5003
        IF ((ABS(DIFF).LT.5).AND.(ABS(DIFFN).LT.5)) THEN
          B11=B11-.0007D0
        ELSE
          B11=B11-.015D0
        END IF
        IDENT=3
       ELSE
        IF ((ABS(DIFF) .LT. 1) .AND. (ABS(DIFFN) .LT. 1)) GO TO 5003
        IF ((IV1.EQ.0.1D0).AND.(IV2.EQ.0.5D0)) THEN
         IF (IDENT.EQ.2) THEN
          IF (ABS(DIFFN).GT.20) THEN
           RL=RL-0.005
           C11=C11+0.05
          END IF
         END IF
        END IF
        IF ((ABS(DIFF).GT.300).AND.(ABS(DIFFN).GT.300)) THEN
           RL=RL-.09D0
        ELSE
            IF ((ABS(DIFF) .GT. 50) .AND. (ABS(DIFFN) .GT. 50)) THEN
              RL=RL-.04D0
              IF (ABS(DIFFN).GT.100) THEN
                B11=B11+.063D0
              END IF
            ELSE
              iF ((ABS(DIFF) .GT. 5) .AND. (ABS(DIFFN) .GT. 5)) THEN
                RL=RL-.007D0
              ELSE
                RL=RL-.001D0
              END IF
            END IF
        END IF
        IDENT=4
       END IF
      END IF
     END IF
     GO TO 2748
    END IF
   ELSE
    STAD=370.9D0
```

```
         IF ((PHI .NE. 0.0D0) .AND. (DEV .EQ. 0.0D0) .AND.
     *                                (BETA(1) .GT. STAD)) THEN
           iF (ABS(DIFF).GT.300) THEN
             RL=RL-0.15D0
           ELSE
             IF (ABS(DIFF).GT.100) THEN
               RL=RL-.10D0
             ELSE
               IF (ABS(DIFF).GT.50) THEN
                 RL=RL-.012D0
               ELSE
                 RL=RL-.001D0
               END IF
             END IF
           END IF
           GO TO 2748
         END IF
        END IF
       END IF
c ***********************************************************************
c ******************** End of Adjusting procedure *********************
C ***********************************************************************
5003   BARRAY(NDEV)=BETA(1)
       BARRAYN(NDEV)=BETAN(1)
       NDEV=NDEV+1
25     CONTINUE
C ***********************************************************************
C ******       END OF 'DEV=0, 0.25, 0.5, 1.0, 2.0 AND 3.0'      ********
C ***********************************************************************
       IF (REGULAR) THEN
        IF (IV1 .LT. 1.0) THEN
          IF ((IV1.EQ.0.1D0).AND.(IV2.EQ.0.5D0)) THEN
           WRITE(6,131) (BARRAY(KC122),KC122=1,6),RL,B11,C11,D11,E11
131        FORMAT(6F10.3,3X,5F6.3)
           WRITE(6,132) (BARRAYN(KC122),KC122=1,6)
132        FORMAT(6F10.3)
          ELSE
           IF (((IV1.EQ.0.5D0).AND.(IV2.EQ.1.0D0)).OR.
     *        ((IV1.EQ.0.1D0).AND.(IV2.EQ.1.0D0))) THEN
            WRITE(6,133) (BARRAY(KC122),KC122=1,6),RL,B11,C11
133         FORMAT(6F10.3,3X,3F6.3)
            WRITE(6,134) (BARRAYN(KC122),KC122=1,6)
134         FORMAT(6F10.3)
           ELSE
            WRITE(6,596) (BARRAY(KC122),KC122=1,6),RL ,B11
596         FORMAT(6F10.3,3X,F6.3,3X,F6.3)
            WRITE(6,597) (BARRAYN(KC122),KC122=1,6)
597         FORMAT(6F10.3)
           END IF
          END IF
```

```
        ELSE
          WRITE(6,12375) (BARRAY(KC122),KC122=1,6),RL
12375     FORMAT(6F10.3,3X,F6.3)
        END IF
      ELSE
        WRITE(6,12371) (BARRAY(KC122),KC122=1,6),RL ,B11
12371   FORMAT(6F10.3,3X,F6.3,3X,F6.3)
        WRITE(6,12374) (BARRAYN(KC122),KC122=1,6)
12374   FORMAT(6F10.3)
      END IF
      NPHI=NPHI+1
81    CONTINUE
C **********************************************************************
C **********           END OF 'IPHI=1,2,...9.'           ***************
C **********************************************************************
      STOP
      END
C **********************************************************************
C ***** SUBROUTINE DETERMINING THE PARAMETERS OF A TIME SERIES ******
C ***** WITH SHORT INTERVALS.                                   ******
C **********************************************************************
      SUBROUTINE GETSIGA(PHI,PHY,SIGA,IV1)
      IMPLICIT REAL *8(A-H,O-Z)
      REAL *8IV1
      IF (IV1.EQ.0.1D0) THEN
        PHY=PHI**0.05
        SIGA=DSQRT((1.0-PHY**2)/(1.0-PHY**40))
      ELSE
        IF ((IV1 .EQ. 0.5D0) .OR. (IV1 .EQ. 0.75D0)) THEN
          PHY=PHI**0.125
          SIGA=DSQRT((1.0-PHY**2)/(1.0-PHY**16))
        ELSE
          IF (IV1 .EQ. 1.0D0) THEN
             PHY=PHI**0.5
             SIGA=DSQRT(1.0D0/(1.0D0+PHY**2))
          END IF
        END IF
      END IF
      RETURN
      END
C **********************************************************************
c ***** Subroutine used for calculating the transition matrix   ******
c **********************************************************************
      SUBROUTINE GETSUB(IV1,IV2,DEV,PHI,PHY,SIGA,SUB,INT1,INT2,INT3,
     *    INT4,INT5)
      IMPLICIT REAL *8(A-H,O-Z)
      REAL *8IV1, IV2
      DIMENSION SUB(10)
      IF (IV1 .EQ. .1D0)  THEN
         SUB(1)=DEV*(1-PHY**38)
```

```
   SUB(2)=DEV*(1-PHY**2)
   SUB(3)=DSQRT(1.0+PHY**2)*SIGA
   SUB(4)=DSQRT((1.0D0-PHY**76)/(1.0D0-PHY**2))*SIGA
   INT1=2
   INT2=38
ELSE
 IF (IV1.EQ.0.5D0) THEN
  IF (IV2.EQ.3.0D0) THEN
     SUB(1)=DEV*(1-PHY**24)
     SUB(2)=DEV*(1-PHY**4)
     SUB(3)=DSQRT((1.0-PHY**8)/(1.0-PHY**2))*SIGA
     SUB(4)=DSQRT((1-PHY**48)/(1-PHY**2))*SIGA
     INT1=4
     INT2=24
  ELSE
    IF (IV2.EQ.2.0D0) THEN
       SUB(1)=DEV*(1-PHY**16)
       SUB(2)=DEV*(1-PHY**4)
       SUB(3)=DSQRT((1.0-PHY**8)/(1.0-PHY**2))*SIGA
       SUB(4)=DSQRT((1-PHY**32)/(1-PHY**2))*SIGA
       INT1=4
       INT2=16
    ELSE
      IF (IV2.EQ.1.5D0) THEN
        SUB(1)=DEV*(1-PHY**12)
        SUB(2)=DEV*(1-PHY**4)
        SUB(3)=DSQRT((1.0-PHY**8)/(1.0-PHY**2))*SIGA
        SUB(4)=DSQRT((1-PHY**24)/(1-PHY**2))*SIGA
        INT1=4
        INT2=12
      END IF
    END IF
  END IF
 ELSE
  IF (IV1 .EQ. .75D0) THEN
   SUB(1)=DEV*(1-PHY**10)
   SUB(2)=DEV*(1-PHY**6)
   SUB(3)=DSQRT((1.0-PHY**12)/(1.0-PHY**2))*SIGA
   SUB(4)=DSQRT((1-PHY**20)/(1-PHY**2))*SIGA
   INT1=6
   INT2=10
  else
   IF (IV1 .EQ. 1.0D0) THEN
    IF (PHI.GT.0.0D0) THEN
     SUB(1)=DEV*(1-PHY**2)
     SUB(4)=DSQRT(1.0D0+PHY**2)*SIGA
     INT1=2
    ELSE
     SUB(1)=DEV*(1-PHY**1)
     SUB(4)=DSQRT(1.0D0)
```

```
            INT1=1
           END IF
          END IF
         END IF
        END IF
       END IF
       IF ((IV1.EQ.0.1D0).AND.(IV2.EQ.0.5D0)) THEN
         SUB(1)=DEV*(1-PHY**38)
         SUB(2)=DEV*(1-PHY**30)
         SUB(3)=DEV*(1-PHY**20)
         SUB(4)=DEV*(1-PHY**10)
         SUB(5)=DEV*(1-PHY**2)
         SUB(6)=DSQRT(1.0+PHY**2)*SIGA
         SUB(7)=DSQRT((1-PHY**20)/(1-PHY**2))*SIGA
         SUB(8)=DSQRT((1-PHY**40)/(1-PHY**2))*SIGA
         SUB(9)=DSQRT((1-PHY**60)/(1-PHY**2))*SIGA
         SUB(10)=DSQRT((1-PHY**76)/(1-PHY**2))*SIGA
         INT1=2
         INT2=10
         INT3=20
         INT4=30
         INT5=38
       END IF
       IF ((IV1.EQ.0.5D0).AND.(IV2.EQ.1.0D0)) THEN
         SUB(1)=DEV*(1-PHY**30)
         SUB(2)=DEV*(1-PHY**20)
         SUB(3)=DEV*(1-PHY**10)
         SUB(4)=DSQRT((1-PHY**20)/(1-PHY**2))*SIGA
         SUB(5)=DSQRT((1-PHY**40)/(1-PHY**2))*SIGA
         SUB(6)=DSQRT((1-PHY**60)/(1-PHY**2))*SIGA
         INT1=10
         INT2=40
         INT3=60
       end if
       IF ((IV1.EQ.0.1D0).AND.(IV2.EQ.1.0D0)) THEN
         SUB(1)=DEV*(1-PHY**38)
         SUB(2)=DEV*(1-PHY**20)
         SUB(3)=DEV*(1-PHY**2)
         SUB(4)=DSQRT(1.0+PHY**2)*SIGA
         SUB(5)=DSQRT((1-PHY**40)/(1-PHY**2))*SIGA
         SUB(6)=DSQRT((1-PHY**76)/(1-PHY**2))*SIGA
         INT1=2
         INT2=20
         INT3=38
       END IF
       RETURN
       END
C *******************************************************************
C * SUBROUTINE USED FOR CALCULATING THE TRANSITION MATRIX FROM TIME *
```

```
C * 0 TO TIME 1 FOR A FIXED START.                                       *
C ************************************************************************
      REAL FUNCTION PROTO1(IV1,S,MAX,IK,JB,PHY,DEV,R,SIGA)
      IMPLICIT REAL *8(A-H,O-Z)
      REAL *8IV1
      DIMENSION S(MAX+1),R(MAX)
      IF (IV1.EQ.1.0D0) THEN
         PROTO1=DNORDF( (S(JB)-PHY**2*R(IK)-DEV*(1.0-PHY**2)) /
     *  (SIGA*SQRT((1.0-PHY**4)/(1.0-PHY**2))) ) -
     *         DNORDF( (S(JB+1)-PHY**2*R(IK)-DEV*(1.0-PHY**2)) /
     *  (SIGA*SQRT((1.0-PHY**4)/(1.0-PHY**2))) )
      ELSE
       IF ((IV1.EQ.0.75D0).OR.(IV1.EQ.0.5D0)) THEN
        PROTO1=DNORDF( (S(JB)-PHY**8*R(IK)-DEV*(1.0-PHY**8)) /
     *  (SIGA*SQRT((1.0-PHY**16)/(1.0-PHY**2))) ) -
     *         DNORDF( (S(JB+1)-PHY**8*R(IK)-DEV*(1.0-PHY**8)) /
     *  (SIGA*SQRT((1.0-PHY**16)/(1.0-PHY**2))) )
       ELSE
        IF (IV1.EQ.0.1D0) THEN
         PROTO1=DNORDF( (S(JB)-PHY**20*R(IK)-DEV*(1.0-PHY**20)) /
     *    (SIGA*SQRT((1.0-PHY**40)/(1.0-PHY**2))) ) -
     *         DNORDF( (S(JB+1)-PHY**20*R(IK)-DEV*(1.0-PHY**20)) /
     *    (SIGA*SQRT((1.0-PHY**40)/(1.0-PHY**2))) )
        END IF
       END IF
      END IF
      RETURN
      END
```

Appendix B
PROGRAM FOR CHAPTER V

```
c ***********************************************************************
C THIS PROGRAM COMPUTES THE ATS AND ANSS WHEN THE UNIT TIME PROCESS IS
C AN AR (2) PROCESS BASED ON TRANSIENT STATES OF 9, 12, ..., 21
c ***********************************************************************
      PARAMETER (MIN=9,MAX=21,NDEG=2)
      IMPLICIT REAL *8(A-H,O-Z)
      REAL RWKSP(200000)
      REAL *8BETA(NDEG+1), SSPOLY(NDEG+1), STAT(10)
      REAL *8XDATA((MAX-MIN)/3+1),YDATA((MAX-MIN)/3+1)
      DIMENSION Q(MAX*MAX,MAX*MAX),R(MAX),S(MAX+1),B(MAX*MAX)
      DIMENSION AIMQIN(MAX*MAX,MAX*MAX),ROWS(MAX*MAX)
      DIMENSION ARL(MAX*MAX),ID(MAX*MAX,MAX*MAX),BARRAY(7)
      DIMENSION VAL(MAX,10,7)
      DIMENSION AIMQ(MAX*MAX,MAX*MAX),DEVIAT(4),PHI2VA(5)
      DIMENSION PHI21ST(3),PHI22ND(5),PHI23RD(7),PHI24TH(9)
      COMMON /WORKSP/ RWKSP
      DATA DEVIAT/0.0, 0.5, 1.0, 3.0/
      DATA PHI21ST/-0.4D0, 0.0D0, 0.4D0/
      DATA PHI22ND/-0.4D0, 0.0D0, 0.4D0,-0.9d0,0.9D0/
      DATA PHI23RD/-0.4D0, 0.0D0, 0.4D0,-1.3d0,-0.9d0,0.9D0,1.3D0/
      DATA PHI24TH/-.4D0,.0D0,.4D0,-1.8D0,-1.3D0,-.9D0,.9D0,1.3D0,1.8D0/
      DATA PHI2VA/0.9d0,0.4D0, 0.0D0,-0.4D0,-0.9D0/
      CALL IWKIN(200000)
      IUNIT=1
C     GET AN IDENTITY MATRIX
      DO 51 I4=1,MAX*MAX
         DO 52 I5=1,MAX*MAX
          IF (I4.EQ.I5) THEN
            ID(I4,I5)=1
          ELSE
            ID(I4,I5)=0
          END IF
52       CONTINUE
51    CONTINUE
      NPHI=1
```

```
      DO 81 NNPHI2=2,4
        PHI2=PHI2VA(NNPHI2)
        IF (PHI2.EQ.0.9D0) THEN
          PHI1=0.0D0
          IEND=1
          GO TO 591
        END IF
        CALL DETERN(PHI2,IEND)
        DO 82 KOUNT=1,IEND
          IF (PHI2.EQ.0.4D0) THEN
            PHI1=PHI21ST(KOUNT)
          END IF
          IF (PHI2.EQ.0.0D0) THEN
            PHI1=PHI22ND(KOUNT)
          END IF
          IF (PHI2.EQ.-0.4D0) THEN
            PHI1=PHI23RD(KOUNT)
          END IF
          IF (PHI2.EQ.-0.9D0) THEN
            PHI1=PHI24TH(KOUNT)
          END IF
591        CONTINUE
      SIGA=1.0
      SIGZ=SIGA*DSQRT((1-PHI2)/((1+PHI2)*((1-PHI2)**2-PHI1**2)))
      RL=3.0D0
      FIRST=DNORDF(RL)-DNORDF(-RL)
      B11=DNORIN((2+FIRST)/4)
      NDEV=1
      DO 25 IDEV=1,4,1
      DEV=SIGZ*DEVIAT(IDEV)
2748  UL=RL*SIGZ
      B1=B11*SIGZ
      DO 811 NR=MIN,MAX,3
      NTOT=NR*NR
      DO 812 NC1=1,NTOT
812      B(NC1)=IUNIT
      M1=NR/3
      M2=NR/3
      M3=NR/3
      DIST1=(UL-B1)/M1
      DIST2=(2*B1)/M2
      DIST3=DIST1
      S(1)=UL
      DO 10 I1=1,M1
         S(I1+1)=S(I1)-DIST1
         R(I1)=S(I1+1)+DIST1/2.0
10    CONTINUE
      DO 16 I1=M1+1,M1+M2
         S(I1+1)=S(I1)-DIST2
```

```
         R(I1)=S(I1+1)+DIST2/2.0
16    continue
      DO 18 I1=M1+M2+1,M1+M2+M1
         S(I1+1)=S(I1)-DIST3
         R(I1)=S(I1+1)+DIST3/2.0
18    continue
      DO 31 I0=1,M1+M2+M3
      DO 33 I1=1,M1+M2+M3
      DO 39 I2=1,M1+M2+M3
        IF (I1.EQ.I2) THEN
         DO 30 I3=1,M1+M2+M3
          Q((I0-1)*(M1+M2+M3)+I1, (I1-1)*(M1+M2+M3)+I3) =
     *       DNORDF(S(I3)-PHI1*(R(I1)-DEV)-PHI2*(R(I0)-DEV)-DEV)
     *  - DNORDF(S(I3+1)-PHI1*(R(I1)-DEV)-PHI2*(R(I0)-DEV)-DEV)
30       CONTINUE
        END IF
39       CONTINUE
33       CONTINUE
31    CONTINUE
      DO 55 IZ1=1,NTOT
         DO 60 IZ2=1,NTOT
            AIMQ(IZ1,IZ2)=ID(IZ1,IZ2)-Q(IZ1,IZ2)
60       CONTINUE
55    CONTINUE
C GET AN INVERSE OF (I-Q)
      CALL DLINRG(NTOT,AIMQ,MAX*MAX,AIMQIN,MAX*MAX)
      IPATH=1
      CALL DMURRV(NTOT,NTOT,AIMQIN,MAX*MAX,NTOT,B,IPATH,NTOT,ARL)
      IK=1
301   IF ( ((S(IK)/SIGZ).GE.DEVIAT(IDEV)) .AND.
     *        ( (S(IK+1)/SIGZ).LT.DEVIAT(IDEV)) ) GO TO 302
        IK=IK+1
        GO TO 301
302   CONTINUE
      VAL(NR,NPHI,NDEV)=ARL((IK-1)*NR+IK)
      WRITE(6,80) NR, NTOT, PHI1, PHI2, DEV/SIGZ, VAL(NR,NPHI,NDEV)
80    FORMAT(2I10,2F15.1,2F15.9)
      DO 84 JA1=1,MAX*MAX
      DO 84 JA2=1,MAX*MAX
84       Q(JA1,JA2)=0.0
811   CONTINUE
      DO 1235 IJ3=MIN,MAX,3
         AIJ3=IJ3*1.0
         ICO=(IJ3 - MIN)/3 + 1
         XDATA(ICO)=(1.0/AIJ3)*(1.0/AIJ3)
         YDATA(ICO)=VAL(IJ3,NPHI,NDEV)
1235  CONTINUE
      CALL DRCURV((MAX-MIN)/3+1,XDATA,YDATA,2,BETA,SSPOLY,STAT)
C            WRITE(6,1234) PHY, UL, DEV, BETA(1)
C 1234         FORMAT(4F10.3)
```

```
C
5003  BARRAY(NDEV)=BETA(1)
      NDEV=NDEV+1
25    CONTINUE
      WRITE(6,12375) PHI1,PHI2,(BARRAY(KC122),KC122=1,7),RL,SIGZ
12375 FORMAT(2F7.2,4F10.3,3X,F6.3,F8.3)
      NPHI=NPHI+1
82    CONTINUE
81    CONTINUE
C
      STOP
      END
c ******************************************************************
      SUBROUTINE DETERN(PHI2,IEND)
      IMPLICIT REAL *8(A-H,O-Z)
      IF (PHI2.EQ.0.4D0) THEN
        IEND =3
      ELSE
       IF (PHI2.EQ.0.0D0) THEN
        IEND=3
       ELSE
        IF (PHI2.EQ.-0.4D0) THEN
         IEND=3
        ELSE
         IF (PHI2.EQ.-0.9D0) THEN
          IEND=3
         END IF
        END IF
       END IF
      END IF
      RETURN
      END
```

Vita

Jai Wook Baik was born in Daejeon, Korea on May 21, 1958. He graduated from Chun Ang University in Seoul, Korea in 1982. He was awarded the Bachelor of Science degree in economics. In 1986, he was awarded the Master of Science in statistics from University of Wisconsin in Madison.

In 1986, He entered Virginia Polytechnic Institute and State University in the Ph.D. program in statistics. He is a member of Mu Sigma Rho honorary society in statistics and a member of the American Statistical Association. In May, 1990, he gave a talk on "Variable Sampling Interval Control Charts in the presence of correlation" at Virginia Academy of Science meeting in Fairfax, Virginia. In 1991, he presented a paper on "$\overline{X}$ Control Charts in the Presence of Correlation" at the Winter Meeting of the ASA in New Orleans, Louisiana.

Jai Wook Baik