Chapter 4

Displacement Dependent Pressure

In the finite element analysis, there are several types of loads which can be introduced: concentrated loads, body forces, surface forces etc. In the case of surfaces force acting normal to the surface during the deformation, these forces are called *pressures*. Pressures are follower forces and can be conservative or noncoservative. A conservative load is a load which is independent of the deformation of the body and therefore can be derived from a potential function. The nonconservative loads may not depend just on the local deformation on the body but of the deformation of the entire body, and, in general, there is no potential function from which these forces can be derived [21][57]. A clasical example of the nonconservative forces are the aerodynamic forces which depend on the fluid flow and the deformation of the entire structure. In this section we present another situation in which an applied pressure depends on the displacements of the entire body.

4.1 System classification

According to Ziegler [57] and Leipholz [37], a nonconservative load is a load which depends on spatial coordinate, displacements, displacement derivatives, velocities, and time and which

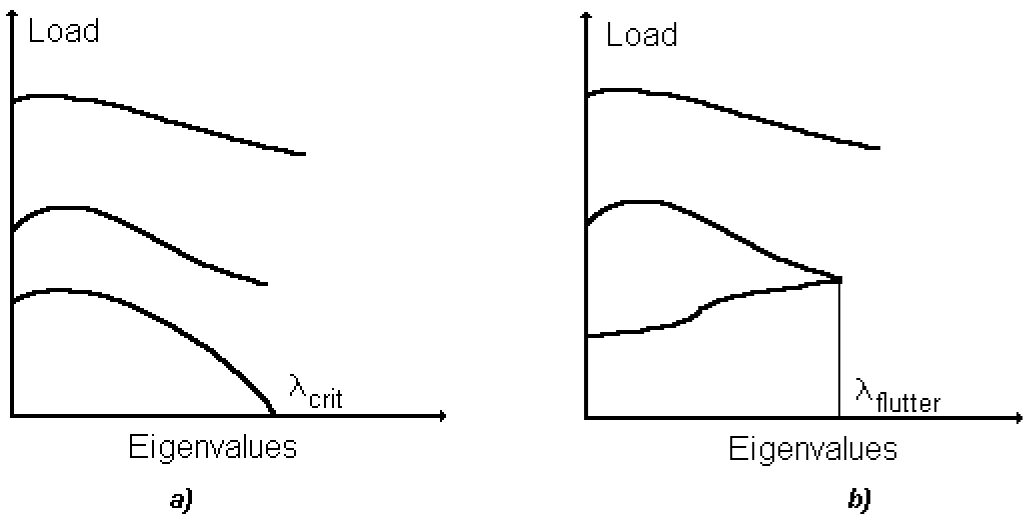

Figure 4.1: Variation of eigenvalues: *a)* divergence-type; *b)* flutter-type

can not be derived from a potential. By their nature the nonconservative forces introduce and(or) extract energy from a system (*nonconservative systems*) [57]. The effect of the extraction of energy from a system is not so dramatic as the effect of the introduction of energy in the system. The last case leads to a stability problem. Therefore many of the studies regarding nonconservative loads have been dedicated to the effect of this type of loads on the stability boundaries of a system [9][15][37][57]. Regarding the form of instability in general we distingush two forms

- *Static instability.* This type of instability, sometimes called *buckling* or *divergence*, occurs at zero eigenvalues as shown in Figure 4.1*a*. This point corresponds also to the critical point of Figure 3.2. At this point the tangent stiffness matrix becomes semi-positive definite.

- *Dynamic instability.* This type of instability is characterized by oscillations with increasing amplitudes: *flutter type.* In this case the system loses stability when two consecutive eigenvalues coalesce as shown in Figure 4.1*b*.

Based on these two types of instability Argyris [9] and Leipholz [37] give the following clasification.

1. *Purely conservative systems* (or *conservative systems of the first kind*). The forces acting on such systems are of conservative-type, i.e., they can be derived from a potential. These systems are conservative in the classical sense: i.e., with respect to energy, since their energy is conserved. Loss of instability occurs only in the form of static instability.

2. *Divergence type nonconservative systems* or *pseudo-nonconservative systems* or *conservative systems of the second kind*. The external forces acting on the system are of nonconservative-type. These systems are nonconservative in the classical sense, i.e., with respect to energy, since their energy is not conserved. However, depending on the boundary conditions, a specific functional other than the energy is conserved for such systems. These systems belong to a class of nonconservative systems, which do behave mechanically like a conservative one. Therefore they still exhibit a static type of instability.

3. *Flutter-type nonconservative systems* or *purely nonconservative systems*. Systems subjected to a nonconservative type of load exhibiting an unsymmetric tangent stiffness matrix. Loss of instability can only take place in the form of dynamic instability (flutter).

4. *Hybrid systems*. This class of systems is characterized by the presence of nonconservative-type of forces, but the systems can display either a divergence type of instability (conservative type of the second kind) or flutter instability (purely nonconservative systems). If the smallest critical load is of divergence type, the system is called *pseudo-divergence-type system*, and if the minimum load corresponds to flutter, then the system is called *pseudo-flutter-type system*.

A similar clasification of the systems based on the type of loads and reactions in a system was given by Ziegler [57]. According to Ziegler, if the loads are gyroscopic (the mechanical work done by these loads is allways zero) or noncirculatory (the loads can be derived from a potential function) and the reactions are nonworking (the mechanical work done by these reactions is zero), then the system is conservative. On the other hand, if a system contains at least one noncoservative force such as a dissipative reaction or a circulatory or instationary load, then the system is called nonconservative [57].

The introduction of nonconservative forces in the finite element analysis has been studied by Argyris and Symeonidis in [9]-[11] for the general case. The particular case of a load pressure depending on the displacements, was analyzed by Schweizerhof and Ramm [46] and Hibbit [29]. Also in the same paper they present the case of body forces which depend on displacements (centrifugal forces). A general conclusion of those analyses is that the presence of nonconservative-type loads leads to two types of correction

- A load correction.
- A tangent stiffness matrix correction. The stiffness matrix correction is in general (with some exceptions) a nonsymetric matrix and leads to a nonsymmetric tangent stiffness matrix.

We further refer to these two corrections as *load/stiffness corrections*. Schweizerhof and Ramm introduce a distinction between *space attached loads* and *body attached loads*. The space attached loads are the loads which depend on the coordinates in the deformed configuration, while the body attached loads depend only on the coordinates of the initial configuration. Their analysis shows that the *body attached loads* always lead to a nonsymmetric correction of the tangent stiffness matrix. The *space attached loads* under some particular boundary conditions lead to a symmetric correction of the tangent stiffness matrix. In order to derive the load/stiffness correction due to the presence of nonconservative loads for the finite element analysis, we will use the principle of virtual work, which remains valid for nonconservative systems [21]. We restrict the derivation to the displacement dependent pressures since our problem requires only this type of load. In the next section we follow the derivation presented by Schweizerhof and Ramm [46], and we introduce a new correction which characterizes our problem.

4.2 Load stiffness correction

Assuming that on the surface Σ of a body there is an applied pressure which depends on the displacements, the supplementary term in the principle of virtual work (3.8) is

$$D_{\boldsymbol{\chi}}\boldsymbol{\mathcal{L}}^c \cdot \boldsymbol{v} = \int_{\,^{t+\Delta t}S} {}^{t+\Delta t}t_i \upsilon_i \, d\, {}^{t+\Delta t}\Sigma, \qquad u_i = {}^{t+\Delta t}U_i - {}^{t}U_i, \qquad i = 1, 2, 3. \tag{4.1}$$

The tractions ${}^{t+\Delta t}t_i$ can be written as

$${}^{t+\Delta t}t_i = {}^{t+\Delta t}p \; {}^{t+\Delta t}n_i,$$

where ${}^{t+\Delta t}p$ is the value of the applied pressure at the time $t + \Delta t$ and ${}^{t+\Delta t}n_i$ represent the normal to the surface ${}^{t+\Delta t}\Sigma$. Referring to Figure 4.2

$${}^{t+\Delta t}\boldsymbol{x}(\xi,\eta) = {}^{t}\boldsymbol{x}(\xi,\eta) + \boldsymbol{u}(\xi,\eta), \qquad {}^{t+\Delta t}n_i \, d\, {}^{t+\Delta t}S = \epsilon_{ijk} \frac{\partial\, {}^{t+\Delta t}x_j}{\partial \xi} \frac{\partial\, {}^{t+\Delta t}x_k}{\partial \eta} \, d\xi d\eta, \tag{4.2}$$

where ϵ_{ijk} is the third order antisymetric permutation tensor. Supposing that the surface Σ is discretized in elements then (4.1) is written

$$D_{\boldsymbol{\chi}}\boldsymbol{\mathcal{L}}^c \cdot \boldsymbol{v} = \sum_e D_{\boldsymbol{\chi}}\boldsymbol{\mathcal{L}}_e^c \cdot \boldsymbol{v}.$$

Introducing (4.2) in (4.1) and considering only an element e, we obtain

$$D_{\boldsymbol{\chi}}\boldsymbol{\mathcal{L}}_e^c \cdot \boldsymbol{v} = \epsilon_{ijk} \int_\xi \int_\eta {}^{t+\Delta t}p \frac{\partial\, {}^{t+\Delta t}x_j}{\partial \xi} \frac{\partial\, {}^{t+\Delta t}x_k}{\partial \eta} \upsilon_i \, d\xi d\eta. \tag{4.3}$$

$$D_{\boldsymbol{\chi}}\boldsymbol{\mathcal{L}}_e^c \cdot \boldsymbol{v} = \int_\xi \int_\eta {}^{t+\Delta t}p \left(\frac{\partial\, {}^{t}x_j}{\partial \xi} + \frac{\partial u_j}{\partial \xi} \right) \left(\frac{\partial\, {}^{t}x_k}{\partial \xi} + \frac{\partial u_k}{\partial \xi} \right) \upsilon_i \, d\xi d\eta.$$

$$D_{\boldsymbol{\chi}}\boldsymbol{\mathcal{L}}_e^c \cdot \boldsymbol{v} = \epsilon_{ijk} \int_\xi \int_\eta {}^{t+\Delta t}p \frac{\partial\, {}^{t}x_j}{\partial \xi} \frac{\partial\, {}^{t}x_k}{\partial \eta} \upsilon_i \, d\xi \, d\eta + \epsilon_{ijk} \int_\xi \int_\eta {}^{t+\Delta t}p \frac{\partial u_j}{\partial \xi} \frac{\partial\, {}^{t}x_k}{\partial \eta} \upsilon_i \, d\xi d\eta +$$

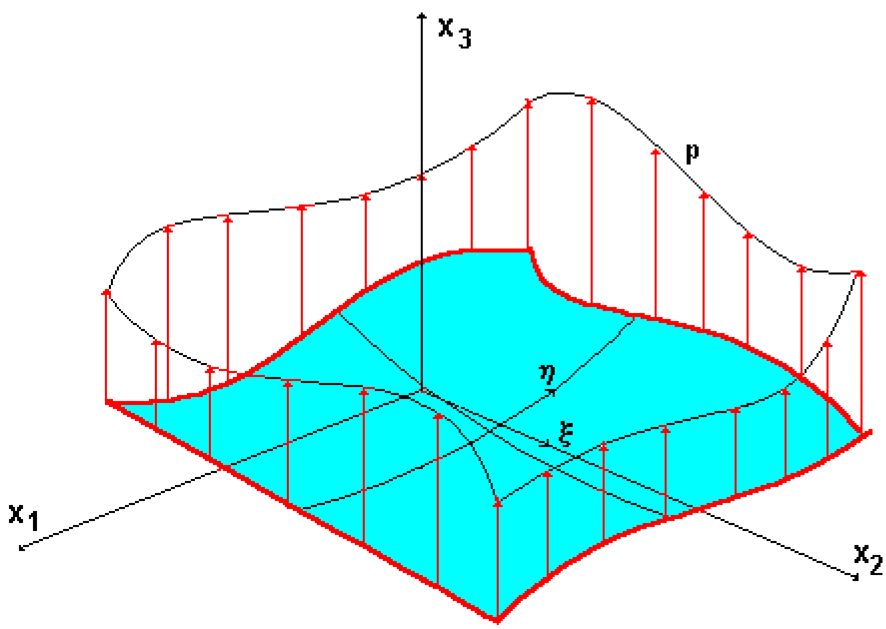

Figure 4.2: An element with a pressure load.

$$+\epsilon_{ijk}\int_{\xi}\int_{\eta}{}^{t+\Delta t}p\frac{\partial\ {}^{t}x_j}{\partial\xi}\frac{\partial u_k}{\partial\eta}\upsilon_i\,d\xi\,d\eta+\epsilon_{ijk}\int_{\xi}\int_{\eta}{}^{t+\Delta t}p\frac{\partial u_j}{\partial\xi}\frac{\partial u_k}{\partial\eta}\upsilon_i\,d\xi d\eta. \tag{4.4}$$

The last term in the equation (4.4) can be neglected because it represents a superior term of order $O(\varepsilon)$. We obtain

$$D_{\boldsymbol{\chi}}\boldsymbol{\mathcal{L}}_e^c\cdot\boldsymbol{\upsilon}=\int_{\xi}\int_{\eta}{}^{t+\Delta t}p\left(\frac{\partial\ {}^{t}x_j}{\partial\xi}+\frac{\partial u_j}{\partial\xi}\right)\left(\frac{\partial\ {}^{t}x_k}{\partial\xi}+\frac{\partial u_k}{\partial\xi}\right)\upsilon_i\,d\xi d\eta.$$

$$D_{\boldsymbol{\chi}}\boldsymbol{\mathcal{L}}_e^c\cdot\boldsymbol{\upsilon}=\epsilon_{ijk}\int_{\xi}\int_{\eta}{}^{t+\Delta t}p\frac{\partial\ {}^{t}x_j}{\partial\xi}\frac{\partial\ {}^{t}x_k}{\partial\eta}\upsilon_i\,d\xi\,d\eta+\epsilon_{ijk}\int_{\xi}\int_{\eta}{}^{t+\Delta t}p\frac{\partial u_j}{\partial\xi}\frac{\partial\ {}^{t}x_k}{\partial\eta}\upsilon_i\,d\xi d\eta+$$

$$+\epsilon_{ijk}\int_{\xi}\int_{\eta}{}^{t+\Delta t}p\frac{\partial\ {}^{t}x_j}{\partial\xi}\frac{\partial u_k}{\partial\eta}\upsilon_i\,d\xi\,d\eta.$$

At this point the pressure ${}^{t+\Delta t}p$ can be divided into a *space attached load* and a *body attached load*. For our problem at $t=0$, there is no distribution of pressure depending on the displacements. This pressure appears only when the diaphragm starts to move and, consequently, there is a change in the volume of gas V. Therefore, in our case the pressure is a *space attached load*. Expanding ${}^{t+\Delta t}p$ in a Taylor series

$$ {}^{t+\Delta t}p = {}^{t}p + \frac{d\,{}^{t}p}{dV}\Delta V + \ldots, \qquad \Delta V = {}^{t+\Delta t}V - {}^{t}V, \tag{4.5} $$

where ΔV represents the change in the gas volume enclosed by the diaphragm as shown in Figure 2.1. Introducing (4.5) in (4.3) we obtain

$$ D_{\boldsymbol{\chi}}\boldsymbol{\mathcal{L}}_e^c \cdot \boldsymbol{\upsilon} = \epsilon_{ijk}\int_\xi\int_\eta {}^{t}p\frac{\partial\,{}^{t}x_j}{\partial\xi}\frac{\partial\,{}^{t}x_k}{\partial\eta}\upsilon_i\,d\xi\,d\eta + \epsilon_{ijk}\int_\xi\int_\eta {}^{t}p\left(\frac{\partial u_j}{\partial\xi}\frac{\partial\,{}^{t}x_k}{\partial\eta} + \frac{\partial\,{}^{t}x_j}{\partial\xi}\frac{\partial u_k}{\partial\eta}\right)\upsilon_i\,d\xi d\eta + $$

$$ +\frac{d\,{}^{t}p}{dV}\Delta V\epsilon_{ijk}\int_\xi\int_\eta \frac{\partial\,{}^{t}x_j}{\partial\xi}\frac{\partial\,{}^{t}x_k}{\partial\eta}\upsilon_i\,d\xi\,d\eta + $$

$$ +\frac{d\,{}^{t}p}{dV}\Delta V\epsilon_{ijk}\int_\xi\int_\eta \left(\frac{\partial u_j}{\partial\xi}\frac{\partial\,{}^{t}x_k}{\partial\eta} + \frac{\partial\,{}^{t}x_j}{\partial\xi}\frac{\partial u_k}{\partial\eta}\right)\upsilon_i\,d\xi d\eta. \tag{4.6} $$

The first term in the equation (4.6) represents the *load correction* and can be written

$$ \epsilon_{ijk}\int_\xi\int_\eta {}^{t}p\frac{\partial\,{}^{t}x_j}{\partial\xi}\frac{\partial\,{}^{t}x_k}{\partial\eta}\upsilon_i\,d\xi\,d\eta = \int_{\Sigma_e} p\boldsymbol{\upsilon}^T\boldsymbol{n}\,dS $$

The second term can be integrated by parts as follows

$$ \epsilon_{ijk}\int_\xi\int_\eta {}^{t}p\left(\frac{\partial u_j}{\partial\xi}\frac{\partial\,{}^{t}x_k}{\partial\eta} + \frac{\partial\,{}^{t}x_j}{\partial\xi}\frac{\partial u_k}{\partial\eta}\right)\upsilon_i\,d\xi d\eta = $$

$$ = \frac{1}{2}\epsilon_{ijk}\int_\xi\int_\eta {}^{t}p\left(\frac{\partial u_j}{\partial\xi} + \frac{\partial u_j}{\partial\xi}\right)\frac{\partial\,{}^{t}x_k}{\partial\eta}\upsilon_i\,d\xi d\eta + \frac{1}{2}\epsilon_{ijk}\int_\xi\int_\eta {}^{t}p\frac{\partial\,{}^{t}x_j}{\partial\xi}\left(\frac{\partial u_k}{\partial\eta} + \frac{\partial u_k}{\partial\eta}\right)\upsilon_i\,d\xi d\eta = $$

$$ = \frac{1}{2}\epsilon_{ijk}\int_\xi\int_\eta {}^{t}p\frac{\partial\,{}^{t}x_k}{\partial\eta}\left(\upsilon_i\frac{\partial u_j}{\partial\xi} + u_i\frac{\partial\upsilon_j}{\partial\xi}\right)d\xi d\eta + \frac{1}{2}\epsilon_{ijk}\int_{b_\eta} {}^{t}p\frac{\partial\,{}^{t}x_k}{\partial\eta}u_j\upsilon_i\,d\eta + $$

$$ +\frac{1}{2}\epsilon_{ijk}\int_\xi\int_\eta {}^{t}p\frac{\partial\,{}^{t}x_j}{\partial\xi}\left(\upsilon_i\frac{\partial u_k}{\partial\eta} + u_i\frac{\partial\upsilon_k}{\partial\eta}\right)d\xi d\eta + \frac{1}{2}\epsilon_{ijk}\int_{b_\xi} {}^{t}p\frac{\partial\,{}^{t}x_k}{\partial\eta}u_j\upsilon_i\,d\xi = $$

$$= \int_{\Sigma_e} \boldsymbol{v}^T \boldsymbol{\mathcal{K}}^I \boldsymbol{u}\, d\Sigma + \int_{\Gamma_e} \boldsymbol{v}^T \left(\boldsymbol{\mathcal{K}}^{III} + \boldsymbol{\mathcal{K}}^{IV} \right) \boldsymbol{u}\, d\Gamma,$$

where $\boldsymbol{\mathcal{K}}^I$, $\boldsymbol{\mathcal{K}}^{III}$, $\boldsymbol{\mathcal{K}}^{IV}$ are matrix operators defined below

$$\boldsymbol{\mathcal{K}}^I = \frac{1}{2}\, {}^t p \left(\boldsymbol{\mathcal{D}}_\xi - \boldsymbol{\mathcal{D}}_\eta \right),$$

where

$$\boldsymbol{\mathcal{D}}_\xi = \begin{bmatrix} 0 & x_{3,\eta} \left({}_\xi\partial - \partial_\xi \right) & x_{2,\eta} \left(\partial_\xi - {}_\xi\partial \right) \\ x_{3,\eta} \left(\partial_\xi - {}_\xi\partial \right) & 0 & x_{1,\eta} \left({}_\xi\partial - \partial_\xi \right) \\ x_{2,\eta} \left({}_\xi\partial - \partial_\xi \right) & x_{1,\eta} \left(\partial_\xi - {}_\xi\partial \right) & 0 \end{bmatrix},$$

$$\boldsymbol{\mathcal{D}}_\eta = \begin{bmatrix} 0 & x_{3,\xi} \left({}_\eta\partial - \partial_\eta \right) & x_{2,\xi} \left(\partial_\eta - {}_\eta\partial \right) \\ x_{3,\xi} \left(\partial_\eta - {}_\eta\partial \right) & 0 & x_{1,\xi} \left({}_\eta\partial - \partial_\eta \right) \\ x_{2,\xi} \left({}_\eta\partial - \partial_\xi \right) & x_{1,\eta} \left(\partial_\eta - {}_\eta\partial \right) & 0 \end{bmatrix}.$$

Similarly

$$\boldsymbol{\mathcal{D}}^{III} = \frac{1}{2}\, {}^t p \begin{bmatrix} 0 & -\, {}^t x_{3,\eta} & {}^t x_{2,\eta} \\ {}^t x_{3,\eta} & 0 & -\, {}^t x_{1,\eta} \\ -\, {}^t x_{2,\eta} & {}^t x_{1,\eta} & 0 \end{bmatrix},$$

$$\boldsymbol{\mathcal{D}}^{IV} = \frac{1}{2}\, {}^t p \begin{bmatrix} 0 & {}^t x_{3,\xi} & -\, {}^t x_{2,\xi} \\ -\, {}^t x_{3,\xi} & 0 & {}^t x_{1,\xi} \\ {}^t x_{2,\xi} & -\, {}^t x_{1,\xi} & 0 \end{bmatrix}.$$

The operator $\boldsymbol{\mathcal{K}}^I$ is a symmetric operator because an element on the position ij is the same as the element on the position ji. The operators $\boldsymbol{\mathcal{K}}^{III}$ and $\boldsymbol{\mathcal{K}}^{IV}$ are instead skew-symmetric operators [46]. The integral expression in the third term of (4.6) can be written

$$\epsilon_{ijk} \int_\xi \int_\eta \frac{\partial\, {}^t x_j}{\partial \xi} \frac{\partial\, {}^t x_k}{\partial \eta} \upsilon_i \, d\xi \, d\eta = \int_{\Sigma_e} \boldsymbol{\upsilon}^T \boldsymbol{n} \, d\Sigma.$$

The variation of the gas volume ΔV can be written as

$$\Delta V = \sum_{j=1}^{N_e} \Delta V_j, \tag{4.7}$$

where ΔV_j represents the contribution to the gas volume change due to an element j. To calculate ΔV_j we use the following formula. Suppose that V is the volume enclosed by a surface S. Then

$$V = \int_V dV = \frac{1}{3} \int_V \nabla \boldsymbol{x} \, dV = \frac{1}{3} \int_S \boldsymbol{x} \cdot \boldsymbol{n} \, dS. \tag{4.8}$$

When an element e of the diaphragm is displaced, it yields to a change in the gas volume. Using (4.8) the contribution of an element e to the change in the gas volume at the time t and $t + \Delta t$ is

$${}^t V_e = \frac{1}{3} \int_{\Sigma_e} {}^t x_i n_i \, d\Sigma = \frac{1}{3} \epsilon_{ijk} \int_\xi \int_\eta {}^t x_i \frac{\partial\, {}^t x_j}{\partial \xi} \frac{\partial\, {}^t x_k}{\partial \eta} \, d\xi d\eta.$$

$${}^{t+\Delta t} V_e = \frac{1}{3} \int_{{}^{t+\Delta t} S_e} {}^{t+\Delta t} x_i \; {}^{t+\Delta t} n_i \, d \; {}^{t+\Delta t} S = \frac{1}{3} \epsilon_{ijk} \int_\xi \int_\eta {}^{t+\Delta t} x_i \frac{\partial\, {}^{t+\Delta t} x_j}{\partial \xi} \frac{\partial\, {}^{t+\Delta t} x_k}{\partial \eta} \, d\xi d\eta =$$

$$= \frac{1}{3} \epsilon_{ijk} \int_\xi \int_\eta ({}^t x_i + u_i) \left(\frac{\partial\, {}^t x_j}{\partial \xi} + \frac{\partial u_j}{\partial \xi} \right) \left(\frac{\partial\, {}^t x_k}{\partial \eta} \frac{\partial u_k}{\partial \eta} \right) \, d\xi d\eta =$$

$$= \frac{1}{3} \epsilon_{ijk} \int_\xi \int_\eta {}^t x_i \frac{\partial\, {}^t x_j}{\partial \xi} \frac{\partial\, {}^t x_k}{\partial \eta} \, d\xi \, d\eta + \frac{1}{3} \epsilon_{ijk} \int_\xi \int_\eta {}^t x_i \frac{\partial u_j}{\partial \xi} \frac{\partial\, {}^t x_k}{\partial \eta} \, d\xi d\eta +$$

$$+ \frac{1}{3} \epsilon_{ijk} \int_\xi \int_\eta {}^t x_i \frac{\partial\, {}^t x_j}{\partial \xi} \frac{\partial u_k}{\partial \eta} \, d\xi \, d\eta + \frac{1}{3} \epsilon_{ijk} \int_\xi \int_\eta {}^t x_i \frac{\partial u_j}{\partial \xi} \frac{\partial u_k}{\partial \eta} \, d\xi d\eta +$$

$$+ \frac{1}{3} \epsilon_{ijk} \int_\xi \int_\eta u_i \frac{\partial\, {}^t x_j}{\partial \xi} \frac{\partial\, {}^t x_k}{\partial \eta} \, d\xi \, d\eta + \frac{1}{3} \epsilon_{ijk} \int_\xi \int_\eta u_i \frac{\partial u_j}{\partial \xi} \frac{\partial\, {}^t x_k}{\partial \eta} \, d\xi d\eta +$$

$$+\frac{1}{3}\epsilon_{ijk}\int_\xi\int_\eta u_i\frac{\partial\ {}^tx_j}{\partial\xi}\frac{\partial u_k}{\partial\eta}\,d\xi\,d\eta+\frac{1}{3}\epsilon_{ijk}\int_\xi\int_\eta u_i\frac{\partial u_j}{\partial\xi}\frac{\partial u_k}{\partial\eta}\,d\xi d\eta.$$

$${}^{t+\Delta t}V_e=\ {}^tV_e+\frac{1}{3}\epsilon_{ijk}\int_\xi\int_\eta\ {}^tx_i\frac{\partial u_j}{\partial\xi}\frac{\partial\ {}^tx_k}{\partial\eta}\,d\xi d\eta+$$

$$+\frac{1}{3}\epsilon_{ijk}\int_\xi\int_\eta\ {}^tx_i\frac{\partial\ {}^tx_j}{\partial\xi}\frac{\partial u_k}{\partial\eta}\,d\xi d\eta+\frac{1}{3}\epsilon_{ijk}\int_\xi\int_\eta u_i\frac{\partial\ {}^tx_j}{\partial\xi}\frac{\partial\ {}^tx_k}{\partial\eta}\,d\xi\,d\eta.$$

$$\epsilon_{ijk}\int_\xi\int_\eta\ {}^tx_i\frac{\partial u_j}{\partial\xi}\frac{\partial\ {}^tx_k}{\partial\eta}\,d\xi d\eta=\epsilon_{ijk}\int_{b_\xi}\ {}^tx_iu_j\frac{\partial\ {}^tx_k}{\partial\eta}d\eta-\epsilon_{ijk}\int_\xi\int_\eta u_j\frac{\partial\ {}^tx_i}{\partial\xi}\frac{\partial\ {}^tx_k}{\partial\eta}\,d\xi\,d\eta=$$

$$=\epsilon_{ijk}\int_\xi\int_\eta u_i\frac{\partial\ {}^tx_j}{\partial\xi}\frac{\partial\ {}^tx_k}{\partial\eta}\,d\xi\,d\eta.$$

$$\epsilon_{ijk}\int_\xi\int_\eta\ {}^tx_i\frac{\partial\ {}^tx_j}{\partial\xi}\frac{\partial u_k}{\partial\eta}\,d\xi d\eta=\epsilon_{ijk}\int_{b_\xi}\ {}^tx_iu_k\frac{\partial\ {}^tx_j}{\partial\xi}d\xi-\epsilon_{ijk}\int_\xi\int_\eta u_k\frac{\partial\ {}^tx_i}{\partial\xi}\frac{\partial\ {}^tx_j}{\partial\eta}\,d\xi\,d\eta=$$

$$=\epsilon_{ijk}\int_\xi\int_\eta u_i\frac{\partial\ {}^tx_j}{\partial\xi}\frac{\partial\ {}^tx_k}{\partial\eta}\,d\xi\,d\eta.$$

It follows that

$$\Delta V_e=\epsilon_{ijk}\int_\xi\int_\eta u_i\frac{\partial\ {}^tx_j}{\partial\xi}\frac{\partial\ {}^tx_k}{\partial\eta}\,d\xi\,d\eta.$$

$$\Delta V_e=\int_\xi\int_\eta\boldsymbol{u}\cdot\left(\frac{\partial\boldsymbol{x}}{\partial\xi}\times\frac{\partial\boldsymbol{x}}{\partial\eta}\right)\,d\xi\,d\eta=\int_{\Sigma_e}\boldsymbol{n}^T\boldsymbol{u}\,d\Sigma. \tag{4.9}$$

Therefore the third expression in (4.6) can be written

$$\frac{d\ {}^tp}{dV}\Delta V\,\epsilon_{ijk}\int_\xi\int_\eta\frac{\partial\ {}^tx_j}{\partial\xi}\frac{\partial\ {}^tx_k}{\partial\eta}\upsilon_i\,d\xi\,d\eta=$$

$$= \frac{dp}{dV}\left(\int_{\Sigma_e} \boldsymbol{v}^T \boldsymbol{n}\, d\Sigma\right)\left(\sum_q \int_{\Sigma_q} \boldsymbol{n}^T \boldsymbol{u}\, d\Sigma\right) = \frac{dp}{dV}\sum_q \boldsymbol{v}_e^T \boldsymbol{\mathcal{K}}_{eq}^{II} \boldsymbol{u}_q, \tag{4.10}$$

where $\boldsymbol{\mathcal{K}}_{eq}^{II}$ is the following integral operator defined as

$$\boldsymbol{v}_e \boldsymbol{\mathcal{K}}_{eq}^{II} \boldsymbol{u} = \left(\int_{\Sigma_e} \boldsymbol{v}^T \boldsymbol{n}\, d\Sigma\right)\left(\int_{\Sigma_q} \boldsymbol{n}^T \boldsymbol{u}\, d\Sigma\right).$$

It can be seen that $\boldsymbol{\mathcal{K}}_{eq}^{II} = \boldsymbol{\mathcal{K}}_{qe}^{II}$ and therefore $\boldsymbol{\mathcal{K}}_{eq}^{II}$ is a symmetric operator. The fourth term in (4.6) contains higher order terms in $\boldsymbol{u}$ and therefore it will be neglected. Using (4.2) and (4.10), equation (4.6) can be written in the following form

$$D_{\boldsymbol{\chi}} \boldsymbol{\mathcal{L}}_e^c \cdot \boldsymbol{v} = \int_{\Sigma_e} p \boldsymbol{v}^T \boldsymbol{n}\, d\Sigma + \int_{\Sigma_e} \boldsymbol{v}^T \boldsymbol{\mathcal{K}}^I \boldsymbol{u}\, d\Sigma + \int_{\Gamma_e} \boldsymbol{v}^T \left(\boldsymbol{\mathcal{K}}^{III} + \boldsymbol{\mathcal{K}}^{IV}\right) \boldsymbol{u}\, d\Gamma + \sum_q \boldsymbol{v}_e^T \boldsymbol{\mathcal{K}}_{eq}^{II} \boldsymbol{u}_q. \tag{4.11}$$

Equation (4.11) defines the *load-stiffness* correction due to the presence of a pressure depending on the displacements. In addition to the terms proposed by Schweizerhof and Ramm [46], the expression (4.11) includes also (the last term) a new stiffness correction, derived for our problem.

4.3 Finite element implementation

Using (3.34), the relation (4.11) can be written

$$D_{\boldsymbol{\chi}} \boldsymbol{\mathcal{L}}_e^c \cdot \boldsymbol{v} = \boldsymbol{v}_e^T \mathbf{F}_e^c + \boldsymbol{v}_e^T \mathbf{K}_e^I \mathbf{u}_e + \boldsymbol{v}_e^T \left(\mathbf{K}_e^{III} + \mathbf{K}_e^{IV}\right) \mathbf{u}_e + \sum_q \boldsymbol{v}_e^T \mathbf{K}_{eq}^{II} \mathbf{u}_q, \tag{4.12}$$

where

$$\mathbf{F}_e^c = \int_{\Sigma_e} \boldsymbol{\Phi} \boldsymbol{p}\, d\Sigma, \tag{4.13}$$

$$\mathbf{K}_e^{III} = \int_{\Sigma_e} \boldsymbol{\Phi} \boldsymbol{\mathcal{K}}^{III} \boldsymbol{\Phi}^T\, d\Sigma, \qquad \mathbf{K}_e^{IV} = \int_{\Sigma_e} \boldsymbol{\Phi} \boldsymbol{\mathcal{K}}^{IV} \boldsymbol{\Phi}^T\, d\Sigma, \tag{4.14}$$

$$\mathbf{K}_e^{I} = \int_{\Sigma_e} \boldsymbol{\Phi} \boldsymbol{\mathcal{K}}^{I} \boldsymbol{\Phi}^T\, d\Sigma, \qquad \mathbf{K}_{eq}^{II} = \boldsymbol{\Phi} \boldsymbol{\mathcal{K}}_{eq}^{II} \boldsymbol{\Phi}^T. \tag{4.15}$$

The correction in the load vector is introduced by equation (4.13) and the stifness correction by equations (4.14) and (4.15). Because $\boldsymbol{\mathcal{K}}^{I,II}$ are symmetric operators the matrices (4.15) are symmetric matrices. Similarly, because $\boldsymbol{\mathcal{K}}_{eq}^{III,IV}$ are anti-symmetric operators, the corespond-ing matrices (4.14) are anti-symmetric matrices. These corrections are added to the tangent stifness matrix and the load vector (3.41). As a consequence the tangent stifness matrix may become non-symmetric. Based on the fact that the symmetry in the stiffness matrix is a consequence of the conservativeness of a system Schweizerhof and Ramm categorized the systems as

- Conservative systems. The systems for which the tangent stiffness matrix is symmetric.
- Nonconservative systems. The systems for which the tangent stiffness matrix is non-symmetric.

From equation (4.11) the symmetry in the tangent stiffness matrix is broken only if the integrals over the boundary Γ_e are not zero. The symmetry is still kept if one of the following situations occur

- Load magnitude is zero, that is

$$p|_{\Gamma_e} = 0, \qquad \text{for all } \Gamma_e.$$

- Displacements are prescribed on $\Gamma = \cup\Gamma_e$ (surface completely supported).

$$\boldsymbol{u} = \overline{\boldsymbol{u}} \quad \Rightarrow \quad \boldsymbol{v} = 0.$$

- The surface S is sufficiently supported and properly oriented. This condition can be ilustrated by writing the integrals in (4.11) on all Γ_e in the form

$$D_{\boldsymbol{\chi}}\boldsymbol{\mathcal{L}}^c_{III,IV} \cdot \boldsymbol{v} = \frac{1}{2}\int_\Gamma p\,(\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}}) \cdot \boldsymbol{\tau}\, d\Gamma = 0, \tag{4.16}$$

where $\boldsymbol{\tau}$ is that tangent vector to the boundary Γ. The condition (4.16) imposes that the body be constrained normal to the deformed surface such that the work done by the pressure forces is zero. For our problem, the displacements prescribed on the boundary Γ are zero and therefore the boundary terms vanish and the corrected tangent matrix remains symmetric. According to Schweizerhof's classification, the system is conservative. Our observations in section 1.3 indicate that the system is nonconservative. To clarify this apparent contradiction, let us write the balance of energy in the incremental form

$$\mathbf{u}^T\mathbf{F} - \mathbf{u}^T\mathbf{F}^c = \mathbf{u}^T\mathbf{K}_T\,\mathbf{u} + \mathbf{u}^T\left(\mathbf{K}^I + \mathbf{K}^{II}\right)\mathbf{u}, \tag{4.17}$$

where $\mathbf{F}$ defines the applied load vector, $\mathbf{F}^c$ the correction load vector, and the matrices $\mathbf{K}_T$, $\mathbf{K}^{I,II}$ are the tangent stiffness matrix and the correction stiffness matrix for the entire body respectively. From equation (4.17) it can be seen that from the energy introduced into the system (diaphragm) only a part is found in the deformation energy of the diaphragm. The remaining part affects the stiffness of the diaphragm. Therefore we can not say that the energy is conserved in the clasical sense. That is to say: the work done by the external forces is equal to the deformation energy. However if we consider a *fictious* system having the stiffness $\mathbf{K} \rightarrow \mathbf{K} + \mathbf{K}^{I+II}$, then we can say that the new system is conservative. Using Leipholz/Argyris clasification, the sytem is *pseudo-nonconservative* or conservative of the second kind. It appears that the classification given by Schweirzhoff is a mathematical classification rather than a physical one. The classification given by Argyris and Leipholz seems to be closer to the physics of the problem. Introducing the boundary conditions (2.42) the equation (4.12) becomes

$$D_{\boldsymbol{\chi}}\boldsymbol{\mathcal{L}}_e^c \cdot \boldsymbol{v} = \boldsymbol{v}_e^T\mathbf{F}_e^c + \boldsymbol{v}_e^T\mathbf{K}^I\mathbf{u}_e + \sum_q \boldsymbol{v}_e^T\mathbf{K}_{eq}^{II}\mathbf{u}_q, \tag{4.18}$$

and this represents the final form for our problem. The matrix $\mathbf{K}_e^I$ for an element e can be written in the form

$$\mathbf{K}_e^I = \int_{\Sigma_e} \boldsymbol{\Phi}\boldsymbol{\mathcal{K}}^I\boldsymbol{\Phi}^T\,d\Sigma = \frac{1}{2}\int_{\Sigma_e} {}^t p\boldsymbol{\Phi}\left(\boldsymbol{\mathcal{D}}_\xi - \boldsymbol{\mathcal{D}}_\eta\right)\boldsymbol{\Phi}^T\,d\Sigma.$$

For the 8-nodes brick element the surface Σ_e has 4 nodes and the structure of the matrix $\mathbf{K}_e^I$ is the following

$$\mathbf{K}_e^I = \begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} & \mathbf{K}_{13} & \mathbf{K}_{14} \\ \mathbf{K}_{21} & \mathbf{K}_{22} & \mathbf{K}_{23} & \mathbf{K}_{24} \\ \mathbf{K}_{31} & \mathbf{K}_{32} & \mathbf{K}_{33} & \mathbf{K}_{34} \\ \mathbf{K}_{41} & \mathbf{K}_{42} & \mathbf{K}_{43} & \mathbf{K}_{44} \end{bmatrix}, \tag{4.19}$$

where $\mathbf{K}_{ij}$ are 3×3 matrices defined as

$$\mathbf{K}_{ij} = \frac{1}{2}\int_{-1}^{+1}\int_{-1}^{+1} {}^t p \phi_i \left(\mathcal{D}_\xi - \mathcal{D}_\eta\right) \phi_j J \, d\xi d\eta = \begin{bmatrix} 0 & k_{12} & k_{13} \\ k_{21} & 0 & k_{23} \\ k_{31} & k_{32} & 0 \end{bmatrix}, \tag{4.20}$$

and

$$k_{12} = \frac{1}{2}\int_{-1}^{+1}\int_{-1}^{+1} {}^t p \left[x_{3,\eta}\left(\phi_{i,\xi}\phi_j - \phi_i\phi_{j,\xi}\right) - x_{3,\xi}\left(\phi_{i,\eta}\phi_j - \phi_i\phi_{j,\eta}\right)\right] J \, d\xi d\eta,$$

$$k_{13} = \frac{1}{2}\int_{-1}^{+1}\int_{-1}^{+1} {}^t p \left[x_{2,\eta}\left(\phi_i\phi_{j,\xi} - \phi_{i,\xi}\phi_j\right) - x_{2,\xi}\left(\phi_i\phi_{j,\eta} - \phi_{i,\eta}\phi_j\right)\right] J \, d\xi d\eta,$$

$$k_{21} = \frac{1}{2}\int_{-1}^{+1}\int_{-1}^{+1} {}^t p \left[x_{3,\eta}\left(\phi_i\phi_{j,\xi} - \phi_{i,\xi}\phi_j\right) - x_{3,\xi}\left(\phi_i\phi_{j,\eta} - \phi_{i,\eta}\phi_j\right)\right] J \, d\xi d\eta,$$

$$k_{23} = \frac{1}{2}\int_{-1}^{+1}\int_{-1}^{+1} {}^t p \left[x_{1,\eta}\left(\phi_{i,\xi}\phi_j - \phi_i\phi_{j,\xi}\right) - x_{1,\xi}\left(\phi_{i,\eta}\phi_j - \phi_i\phi_{j,\eta}\right)\right] J \, d\xi d\eta,$$

$$k_{31} = \frac{1}{2}\int_{-1}^{+1}\int_{-1}^{+1} {}^t p \left[x_{2,\eta}\left(\phi_{i,\xi}\phi_j - \phi_i\phi_{j,\xi}\right) - x_{2,\xi}\left(\phi_{i,\eta}\phi_j - \phi_i\phi_{j,\eta}\right)\right] J \, d\xi d\eta,$$

$$k_{32} = \frac{1}{2}\int_{-1}^{+1}\int_{-1}^{+1} {}^t p \left[x_{1,\eta}\left(\phi_i\phi_{j,\xi} - \phi_{i,\xi}\phi_j\right) - x_{1,\xi}\left(\phi_i\phi_{j,\eta} - \phi_{i,\eta}\phi_j\right)\right] J \, d\xi d\eta.$$

The jacobian J is defined by the equation (3.55). The matrix $\mathbf{K}_{eq}^{II}$ is written

$$\mathbf{K}_{eq}^{II} = \frac{dp}{dV}\left(\int_{\Sigma_e} \mathbf{\Phi}\boldsymbol{n}\, dS\right)\left(\int_{\Sigma_q} \mathbf{\Phi}\boldsymbol{n}\, dS\right)^T,$$

and has the following structure

$$\mathbf{K}_{eq}^{II} = \frac{dp}{dV} \begin{bmatrix} \mathbf{b}_e^1 \left(\mathbf{b}_q^1\right)^T & \mathbf{b}_e^1 \left(\mathbf{b}_q^2\right)^T & \mathbf{b}_e^1 \left(\mathbf{b}_q^3\right)^T & \mathbf{b}_e^1 \left(\mathbf{b}_q^4\right)^T \\ \mathbf{b}_e^2 \left(\mathbf{b}_q^1\right)^T & \mathbf{b}_e^2 \left(\mathbf{b}_q^2\right)^T & \mathbf{b}_e^2 \left(\mathbf{b}_q^3\right)^T & \mathbf{b}_e^2 \left(\mathbf{b}_q^4\right)^T \\ \mathbf{b}_e^3 \left(\mathbf{b}_q^1\right)^T & \mathbf{b}_e^3 \left(\mathbf{b}_q^2\right)^T & \mathbf{b}_e^3 \left(\mathbf{b}_q^3\right)^T & \mathbf{b}_e^3 \left(\mathbf{b}_q^4\right)^T \\ \mathbf{b}_e^4 \left(\mathbf{b}_q^1\right)^T & \mathbf{b}_e^4 \left(\mathbf{b}_q^2\right)^T & \mathbf{b}_e^4 \left(\mathbf{b}_q^3\right)^T & \mathbf{b}_e^4 \left(\mathbf{b}_q^4\right)^T \end{bmatrix},$$

where the indices e and q refer to the element e and respectively element q. A vector $\mathbf{b}_e^i$ is defined by

$$\mathbf{b}_e^i = \int_{-1}^{+1} \int_{-1}^{+1} \phi_i \boldsymbol{n}_e J \, d\xi d\eta. \tag{4.21}$$

It can be seen that the matrix $\left(\mathbf{K}_{eq}^{II}\right)^T$ is a symmetric matrix. Also the product $\mathbf{b}_e^i \left(\mathbf{b}_q^j\right)^T$ defines a 3×3 symmetric matrix. The vector $\mathbf{b}_e^i$ is a vector of 3 elements defined as follows

$$\mathbf{b}_e^i = \int_{-1}^{+1} \int_{-1}^{+1} \phi_i \boldsymbol{n}_e J \, d\xi d\eta = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix},$$

and

$$b_1 = \int_{-1}^{+1} \int_{-1}^{+1} \phi_i n_1 J \, d\xi d\eta,$$

$$b_2 = \int_{-1}^{+1} \int_{-1}^{+1} \phi_i n_2 J \, d\xi d\eta,$$

$$b_3 = \int_{-1}^{+1} \int_{-1}^{+1} \phi_i n_3 J \, d\xi d\eta.$$

The matrices $\mathbf{K}_e^I$ and $\mathbf{K}_e^{II}$ are added to the element tangent stiffness matrix $\mathbf{K}_T^e$ (3.41). Analysing these correction matrices at the element level it can be seen that $\mathbf{K}_e^I$ affects only the element stiffness matrix $\mathbf{K}_T^e$, while $\mathbf{K}_e^{II}$ affects *all* elements. This leads to a fully populated stiffness matrix. Working with a fully populated stiffness matrix in the finite element analysis is time consuming and may lead to an inefficient use of the method. Whenever such corrections of the stiffness matrix occur, we have to put in the balance the gain in the accuracy of the results versus the time and storage necessary for such matrices. Another aspect that needs to be considered is related to the changes in the basic numerical procedures, for example for solving a linear system or an eigenvalue problem.

Reffering to our problem we may want to introduce the stiffness corrections, but at the same time we would like to keep the same structure of the stiffness matrix, no modifications of the basic numerical algorithms and the same storage. This can be obtained if in the sum (4.18) we keep only the term $\mathbf{K}^{II}_{ee}$. That is to say that the variation in the volume ΔV in (4.7) can be approximated at each element by $\Delta V \approx \Delta V_e$. In other words, the applied pressure will depend not of the displacements of the entire body but just of the local displacements. Hence equation (4.18) is modified

$$D_{\boldsymbol{\chi}} \boldsymbol{\mathcal{L}}^c_e \cdot \boldsymbol{v} = \boldsymbol{v}^T_e \mathbf{F}^c_e + \boldsymbol{v}^T_e \mathbf{K}^I \mathbf{u}_e + \boldsymbol{v}^T_e \mathbf{K}^{II}_{ee} \mathbf{u}_e, \tag{4.22}$$

where

$$\mathbf{K}^{II}_{ee} = \mathbf{K}^{II}_e = \frac{dp}{dV} \left(\int_{\Sigma_e} \boldsymbol{\Phi} \boldsymbol{n} \, d\Sigma \right) \left(\int_{\Sigma_e} \boldsymbol{\Phi} \boldsymbol{n} \, d\Sigma \right)^T . \tag{4.23}$$

We will use (4.23) in the numerical results in Chapter 6. The stiffness correction introduced by $\mathbf{K}^{II}$ does not have to be confused with a stiffness correction in the problems involving an elastic foundation. In the elastic foundation case, if we admit that elastic foundation reaction is $\mathbf{k}_f \boldsymbol{u}_n$ where $\mathbf{k}_f$ is the elastic foundation stiffness matrix and $\boldsymbol{u}_n$ is the vector of the normal displacements then the element stiffness correction is

$$\mathbf{K}_f = \int_{\Sigma_e} \boldsymbol{\Phi} \mathbf{k} \boldsymbol{\Phi}^T \boldsymbol{n}^T \, d\Sigma. \tag{4.24}$$

Let suppose that $\mathbf{k}$ is such that has only diagonal terms equal with dp/dV and $\boldsymbol{n} = 1$. The equations (4.23) and (4.24) become

$$\mathbf{K}^{II}_e = \frac{dp}{dV} \left(\int_{\Sigma_e} \boldsymbol{\Phi} \, d\Sigma \right) \left(\int_{\Sigma_e} \boldsymbol{\Phi} \, d\Sigma \right)^T , \qquad \mathbf{K}_f = \frac{dp}{dV} \int_{\Sigma_e} \boldsymbol{\Phi} \boldsymbol{\Phi}^T \, d\Sigma. \tag{4.25}$$

Equation (4.25) shows clear the difference between (4.23) and (4.24). In one case we have *a product of integrals* and in the other case *an integral of a product*.