

# General F-theory models with tuned $(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1))/\mathbb{Z}_6$ symmetry

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**ABSTRACT:** We construct a general form for an F-theory Weierstrass model over a general base giving a 6D or 4D supergravity theory with gauge group  $(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1))/\mathbb{Z}_6$  and generic associated matter, which includes the matter content of the standard model. The Weierstrass model is identified by unHiggsing a model with  $\text{U}(1)$  gauge symmetry and charges  $q \leq 4$  previously found by the first author. This model includes two distinct branches that were identified in earlier work, and includes as a special case the class of models recently studied by Cvetič, Halverson, Lin, Liu, and Tian, for which we demonstrate explicitly the possibility of unification through an  $\text{SU}(5)$  unHiggsing. We develop a systematic methodology for checking that a parameterized class of F-theory Weierstrass models with a given gauge group  $G$  and fixed matter content is generic (contains all allowed moduli) and confirm that this holds for the models constructed here.

**KEYWORDS:** F-Theory, Supergravity Models

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## 1 Introduction

A primary goal of string theory is to understand how the observed physics of the standard model of particle physics can arise in a UV complete quantum theory of gravity. Over the years, many different approaches have been taken to realizing standard model-like physics in the context of string compactifications, and recent work [1–3] suggests that the number of such possible realizations may be very large. One of the features of F-theory [4–6] is that it gives a good global picture of an enormous nonperturbative class of string compactifications, so that one can begin to gain some insight into what structures are typical and which require extensive fine tuning.

There are a number of different ways in which the gauge group of the standard model could in principle arise in F-theory. Some of these are reviewed in section 2.4. In this paper, we address the most straightforward approach, in which the standard model gauge group is simply directly tuned in the Weierstrass model describing the axiodilaton in the IIB/F-theory framework. Such constructions of theories with the standard model gauge group have been considered in [7–10]; recently, Cvetič, Halverson, Lin, Liu, and Tian (CHLLT) [3], building on a toric construction identified in [11] and aspects of global group structure studied in [12–14], considered one class of such models that can be realized over any weak Fano base, giving a large number of possible standard model-like constructions in F-theory. In the paper [15], two of the authors of this paper described what seems to be the most generic class of tuned F-theory models that give the gauge group  $(\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1))/\mathbb{Z}_6$ ; these models include the CHLLT models as a particular subclass. As reviewed further in section 2.2, we focus on constructions with the group  $(\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1))/\mathbb{Z}_6$ , since there is a well-defined sense in which the generic matter content of models with the gauge group  $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$  does not match the observed matter in the standard model [16]. A limitation of [15], however, is that the description given there of the generic  $(\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1))/\mathbb{Z}_6$  model was somewhat indirect. In this paper, we make this class of models more concrete by giving an explicit general form of the Weierstrass model that covers much of the range of models that were identified through more indirect means in [15]. This explicit Weierstrass realization also reveals some cases where there is an obstruction to the F-theory realization despite 6D anomaly cancellation, as we discuss in more detail in section 8.

The structure of this paper is as follows: in section 2 we review various aspects of previous work, including a rigorous definition of the notion of generic used here, and introduce the notion of a generic parameterized F-theory model with fixed gauge group  $G$ . In section 3 we give the explicit construction of the Weierstrass model for the generic F-theory model with gauge group  $(\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1))/\mathbb{Z}_6$ . In the subsequent two sections, sections 4 and 5, we examine in more detail two distinct subclasses of these models and their unHiggsings, connecting to the analysis of [15]. In section 6, we describe the matter content of these models in more detail. In section 7, we introduce a simple numerical technique to confirm that our model is generic in the sense that it captures all dimensions of moduli space for corresponding 6D supergravity theories. In section 8, we analyze the generic  $(\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1))/\mathbb{Z}_6$  F-theory models over the base  $\mathbb{P}^2$ . In section 9, we

show that our model can be realized as a specialization of the generic Morrison-Park U(1) model [17]. In section 10, we make some observations regarding the range of geometries that support these constructions, and some concluding remarks are made in section 11.

## 2 Some background

We begin with a brief review of generic matter in F-theory, some discussion of the different ways in which the standard model may be realizable in F-theory, and a review of the results of [15].

### 2.1 Weierstrass models and gauge groups in F-theory

For a general introduction to F-theory, see [18]. An F-theory compactification is associated with a Weierstrass model

$$y^2 = x^3 + fx + g, \tag{2.1}$$

where  $f, g$  are functions on a base manifold  $B$  (more technically,  $f, g$  are sections of the line bundles  $\mathcal{O}(-4K_B), \mathcal{O}(-6K_B)$ , with  $K_B$  the canonical class of  $B$ ). F-theory can be thought of as defining a nonperturbative compactification of type IIB string theory; when  $B$  is a complex surface this gives a 6D supergravity theory, and when  $B$  is a complex threefold this gives a 4D theory. Note that, in general,  $B$  is not a Calabi-Yau manifold, but rather has a positive (i.e., effective) anticanonical class  $-K_B$ . The Weierstrass model over  $B$  defines an elliptically fibered Calabi-Yau manifold over  $B$ , where the axiodilaton of type IIB theory corresponds to the elliptic curve parameter  $\tau$  defined by the Weierstrass model over each point in the base.

Nonabelian gauge group factors are associated with codimension-one loci in the base where the elliptic fiber degenerates. Using the Kodaira classification, the gauge algebra can be identified by the orders of vanishing of  $f, g$ , and the discriminant locus  $\Delta = 4f^3 + 27g^2$ . In general, there are two ways in which such a Kodaira singularity can arise. Over simple bases like  $\mathbb{P}^2$  or  $\mathbb{P}^3$  (or any other weak Fano base), the generic Weierstrass model has no codimension-one singularities and the gauge group is trivial. Over such bases, a gauge group can be “tuned” by restricting the form of the Weierstrass model to ensure a certain type of Kodaira singularity. Over bases that are not weak Fano, for example the 2D Hirzebruch surfaces  $\mathbb{F}_m$  with  $m \geq 3$ , the anticanonical class  $-K_B$  generally contains rigid components over which there are “non-Higgsable” gauge group factors [19, 20].

The gauge group can have additional U(1) factors when the elliptic fibration admits extra rational sections. According to the Mordell-Weil theorem, the sections of an elliptic fibration form the finitely generated group  $\mathbb{Z}^r \oplus \mathcal{G}$  under elliptic curve addition, where  $\mathcal{G}$  is some finite group [21]. In the most basic situation, where the only rational section of the elliptic fibration is the zero section,  $\mathcal{G}$  is trivial, and  $r$ , an integer known as the Mordell-Weil rank, is 0. But we can have non-trivial  $\mathcal{G}$  and  $r$  when there are sections other than the zero section. The finite part  $\mathcal{G}$  is generated by torsional sections, which have finite order under elliptic curve addition. The  $\mathbb{Z}^r$  subgroup, meanwhile, is generated by  $r$  sections of infinite order. When an elliptic fibration describing an F-theory model

has a non-trivial Mordell-Weil rank  $r$ , the resulting gauge group includes a  $U(1)^r$  gauge factor [6]. In other words, extra rational sections (of infinite order) signal the presence of additional  $U(1)$  factors in the gauge group. Importantly, the resulting  $U(1)$  factors are not associated with a codimension-one locus in the base with  $U(1)$  factors; they are in some sense a global feature of the model. In this work, all of the  $U(1)$  gauge factors arise due to the presence of additional rational sections.

On a practical level, rational sections occur when there are solutions for  $x$  and  $y$  in the Weierstrass equation that are, at least informally, rational functions of the base coordinates. It is often easier to work with the global Weierstrass form

$$y^2 = x^3 + fxz^4 + gz^6, \tag{2.2}$$

where  $[x : y : z]$  are the homogeneous coordinates of  $\mathbb{P}^{2,3,1}$ . Sections are then described by expressions  $[\hat{x} : \hat{y} : \hat{z}]$  written in terms of the base coordinates that solve the global Weierstrass equation. We can always recover the more typical Weierstrass form above from the global form by setting  $z$  to 1, which may seem to make the global Weierstrass form superfluous. However, the global form offers a few advantages. The zero section can be more transparently written as  $[1 : 1 : 0]$  when using the global Weierstrass form. Moreover, we can use the  $[x : y : z] \rightarrow [\lambda^2 x : \lambda^3 y : \lambda z]$  rescaling to remove denominators from the  $\hat{x}$  and  $\hat{y}$  components of the rational sections. As a result, the sections can be described in a more convenient fashion, and we therefore make use of the global Weierstrass form at various points in this work.

## 2.2 Generic matter

The appearance of (geometrically) non-Higgsable gauge factors in F-theory models over bases that are not weak Fano has the physical consequence that, in many branches of F-theory, there are gauge groups that are “generic” in the sense that they are present everywhere in that branch of the geometric moduli space. For six-dimensional theories, there is a direct correspondence between the geometric and physical moduli spaces, so that a branch of the theory with a non-Higgsable gauge group corresponds to a component of the moduli space where the gauge group arises everywhere. In four dimensions, the story is complicated by the presence of fluxes and the superpotential.

There is also a notion of genericity associated with certain matter representations that can be made rigorous in six dimensions, and which carries over naturally to F-theory compactifications in four dimensions. As described in [16], in a 6D supergravity theory with a fixed (tuned) gauge group  $G$  and fixed (and relatively small) anomaly coefficients, the set of generic matter representations corresponds to those matter fields that are found on the branch of the moduli space with largest dimension. Note that this definition of generic is well-defined in 6D supergravity without reference to F-theory or any other UV completion. Note also that while the matter representations are generic for a chosen gauge group  $G$ , in general the gauge group  $G$  itself will not be a generic feature on that branch of the moduli space, i.e., it can be broken through the Higgs mechanism by giving expectation values to some of the matter fields.

As simple examples, for a U(1) gauge theory in 6D supergravity the generic charged matter content contains only matter with charges  $q = \pm 1, \pm 2$ , and for an SU( $N$ ) gauge theory the generic matter content consists of the fundamental, adjoint, and two-index antisymmetric representations (with the last of these only included when  $N > 3$ ). On the other hand, U(1) charges  $q = \pm 3$  or greater, or the two-index symmetric representation of SU( $N$ ) for  $N > 2$ , for example, are non-generic (“exotic”) matter representations in this sense. Note that for algebras like  $\mathfrak{su}(N) \oplus \mathfrak{u}(1)$ , the generic matter depends on the global structure of the gauge group. For example, when the structure is  $(\text{SU}(N) \times \text{U}(1))/\mathbb{Z}_N$ , the U(1) charge is naturally measured in units of  $1/N$  and for the simplest embedding the jointly charged generic matter representations are  $\mathbf{N}_{1/N}$  and  $\mathbf{N}_{1/N \pm 1}$  (see for example [12–14, 16] for discussion of such issues).

This notion of generic matter for a given  $G$  matches well with both anomaly cancellation in 6D and with the framework of F-theory. For the simplest groups  $G$ , in particular those without many U(1) factors, the number of generic matter representations matches with the number of anomaly cancellation conditions, so the generic matter spectrum is essentially uniquely determined by the gauge group and anomaly coefficients. This relation becomes more complicated particularly as the number of U(1) factors increases, where there are different combinations of charges compatible with the generic matter definition. Also, for the simplest groups it turns out that the generic matter types are precisely those realized by the simplest and least singular F-theory constructions, such as for example those described in [22]. In this way, and also by considering the dimension of the underlying geometric moduli space, the notion of generic matter naturally generalizes to 4D F-theory models.

### 2.3 Generic parameterized F-theory models with fixed $G$

Given the notion of generic matter, a natural question in F-theory is whether a construction can be found for the most general model with a given gauge group  $G$  and the associated generic matter content. Such a construction would be realized by a class of Weierstrass models parameterized by various sections of certain line bundles over the base, which realize the desired gauge group.<sup>1</sup> We refer to a parameterized class of Weierstrass models as “generic  $G$  F-theory models” when the following conditions are satisfied: first, the models in the class should realize F-theory constructions with gauge group  $G$  and the associated generic matter; second, the model should be general in the sense that it realizes the full connected moduli space of such models over any given base. In section 7, we explore the second of these conditions for some known models as well as the new class constructed in this paper, and provide a simple algorithm for checking if a known parameterized Weierstrass model satisfies the second condition. As an example, the Morrison-Park U(1) model [17] is confirmed to be the generic U(1) F-theory model in the sense described here.

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<sup>1</sup>Note that the parameters in these models, associated with sections of certain line bundles, can be understood as divisors on the F-theory base manifold, which in general represent multiple independent complex degrees of freedom associated with the number of linearly independent sections of the line bundle. In a local or toric representation we can understand these complex degrees of freedom in terms of the coefficients of independent monomials in a local coordinate representation of the section.

|                   |                                  |                                  |
|-------------------|----------------------------------|----------------------------------|
|                   | GUT                              | $SU(3) \times SU(2) \times U(1)$ |
| Tuned $G$         | Tuned GUT (e.g., $SU(5)$ )       | Directly tuned $G_{SM}$          |
| Non-Higgsable $G$ | Non-Higgsable GUT (e.g., $E_6$ ) | Non-Higgsable $G_{SM}$           |

**Table 1.** Different approaches to realizing the standard model in F-theory can involve combining either tuning or non-Higgsable structure with either unification (GUTs) or non-unified standard model gauge group structure.

As observed in [16], the matter content of the standard model is far from generic for the gauge group  $SU(3) \times SU(2) \times U(1)$ . In particular, generic matter for this gauge group contains no fields that are charged under all three gauge group factors. On the other hand, for the gauge group  $(SU(3) \times SU(2) \times U(1))/\mathbb{Z}_6$ , the standard model fields are contained in the set of generic matter representations. Thus, in looking for tuned F-theory constructions of the standard model, we look for the generic F-theory model with gauge group  $(SU(3) \times SU(2) \times U(1))/\mathbb{Z}_6$ . Constructing an explicit Weierstrass model for this class of generic F-theory models is the principal goal of this paper. A more general question is how to construct generic  $G$  F-theory models for more general gauge groups with a product of nonabelian and abelian  $U(1)$  factors.

## 2.4 Approaches to the standard model in F-theory

To put this work in context, it may be helpful to briefly summarize some of the different ways that the standard model may be realized in F-theory. A table of possibilities is shown in table 1. In this table, the rows correspond to whether the gauge group is realized through tuning or non-Higgsable structure (genericity), and the columns correspond to whether or not there is a (geometric) unified group broken by fluxes. We make some brief comments on each of these possible scenarios.

**Tuned GUT scenarios.** This is the first approach that was explored to F-theory realizations of the standard model, and the one that has been studied by far the most in the literature. The general idea is to start with a Weierstrass model with a tuned unification group such as  $SU(5)$  and to use fluxes to break the  $SU(5)$  down to the standard model. This approach was initiated in [23–26]; for an overview of work in this direction see [18, 27–29].

**SM with non-Higgsable group scenarios.** In this approach, one could use non-Higgsable (geometrically generic) structure for at least the nonabelian  $SU(3) \times SU(2)$  part of the standard model gauge group. This approach was explored in [30], and is possible since the group factors  $SU(3) \times SU(2)$  can appear as non-Higgsable factors with jointly charged matter in 4D (but not 6D) F-theory models. One could also try to incorporate a non-Higgsable  $U(1)$  factor, like those identified in [31, 32], but this requires very specific base structure and is difficult to make compatible with the necessary nonabelian gauge factors in a geometrically generic fashion.

**Non-Higgsable GUT scenarios.** In this class of scenarios, one starts with a non-Higgsable unification group such as  $E_6$ ,  $E_7$ , or  $E_8$  and then carries out flux breaking to



the standard model. Since the vast majority of allowed threefold bases give rise to non-Higgsable  $E_6$ ,  $E_7$ , or  $E_8$  factors [33–35], this approach seems feasible over the widest range of bases and does not involve fine tuning of the Weierstrass model. Indeed, a naive estimate of flux vacua suggests that the set of flux vacua may be dominated by a certain elliptic Calabi-Yau fourfold geometry [36], in which the only possible approach to realizing the standard model seems to be through a non-Higgsable  $E_8$ . While it seems difficult to realize the standard model Yukawa couplings in this geometry due to general arguments given in [25], it may be possible to get around this by realizing the standard model matter through SCFT sectors arising as  $E_8$  conformal matter [37].

**Directly tuned scenarios.** This is the approach we consider in this paper, in which the full gauge group is tuned in the Weierstrass model. As discussed in the introduction, models of this type were previously considered in [3, 7–10, 15]. While these models can largely arise from deformations (which always can be understood in terms of Higgsing processes in the 6D context) from larger groups, the breaking in these cases of any possible GUT depends upon deformations of the Weierstrass model.

In addition to these basic different types of constructions associated with different kinds of Weierstrass models, there are other possibilities. For example, part of the gauge group may come from D3-branes in the type II context and not from D7-branes.<sup>2</sup> It is an interesting question to consider the relative features of these different constructions, both in terms of matching phenomenologically observed aspects of the standard model and in terms of relative frequency in the broad F-theory landscape. While a naive application of standard flux counting arguments [38–41] might suggest that most compactifications are associated with the geometry analyzed in [36], a more complete analysis including geometric factors for the density of flux vacua may give an additional exponential weighting to bases giving Calabi-Yau fourfolds with smaller  $h^{3,1}$  [42], which would suggest that typical 4D F-theory vacua may be associated with Calabi-Yau fourfolds that have more room for tuning gauge factors beyond the non-Higgsable structure. A simple counting of the number of Weierstrass parameters that must be tuned to realize  $SU(5)$  GUTs over simple bases suggests that these vacua involve much fine tuning and may be statistically disfavored [43]; similar arguments apply to tuned standard model gauge group constructions like those considered here though the tuning is less extreme. On the other hand, it may be that over many bases, and in the presence of fluxes, the number of parameters that needs to be tuned may become much smaller; in fact, fluxes may force the geometry to loci where certain groups are more prevalent. Thus, there is no obvious argument that any of these four possible scenarios is completely ruled out or necessarily overwhelmingly dominant in the F-theory landscape of  $\mathcal{N} = 1$  4D string vacua. A pragmatic approach at this time is to consider all the possibilities, exploring each to the extent possible, while simultaneously trying to understand better the statistical distribution of the different types of associated vacua in the landscape and the role of fluxes and the superpotential in pushing the geometry to certain loci or breaking non-Higgsable GUT groups to structures like the standard model.

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<sup>2</sup>Thanks to Yinan Wang for discussions on this point.



## 2.5 Review of previous work on generic $(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1))/\mathbb{Z}_6$ models

We now summarize the results of [15] and review notation introduced there.

We first consider generic  $(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1))/\mathbb{Z}_6$  models in 6D supergravity. As discussed in [16], there are ten generic charged matter fields and ten nontrivial 6D anomaly cancellation conditions constraining charged matter for the gauge group  $(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1))/\mathbb{Z}_6$ . Thus, the anomaly cancellation conditions can generally be solved exactly to give the multiplicities of generic matter representations, given a fixed choice of anomaly coefficients  $a, b_3, b_2, \tilde{b}$ , associated respectively with gravity and the gauge factors  $\text{SU}(3)$ ,  $\text{SU}(2)$ , and  $\text{U}(1)$ . It is convenient to define the quantities  $\beta, X, Y$  given by

$$\begin{aligned} \tilde{b} &= \frac{4}{3}b_3 + \frac{3}{2}b_2 + 2\beta, \\ X &= -8a - 4b_3 - 3b_2 - 2\beta, \\ Y &= a + b_3 + b_2 + \beta. \end{aligned} \tag{2.3}$$

Solving the anomaly cancellation conditions yields the matter multiplicities given in table 2.

We then proceed to classify the anomaly-consistent models using intuition gained in the case of no tensor multiplets ( $T = 0$ ), for which the anomaly coefficients  $a, b_3, b_2, \tilde{b}$  are simply integers (in general, they are vectors in an  $\text{SO}(1, T)$  lattice). In order to have nontrivial nonabelian gauge factors with properly-signed kinetic terms, we must have  $b_3, b_2 > 0$ . Given that we require the spectra to have non-negative multiplicities, this immediately implies that  $X, Y \geq 0$ . We then consider two non-disjoint classes of solutions, which together cover all 6D  $T = 0$  models:

**Class (A)** models with  $\beta \geq 0$ ,

**Class (B)** models with  $Y = 0$  (for which we may have  $\beta < 0$ ).

Although these two classes were defined using intuition gained in the case of 6D  $T = 0$ , we can generalize them to arbitrary numbers of tensor multiplets, although for  $T > 0$  it is not guaranteed that every anomaly-consistent solution falls into one or both of these classes.

As discussed in [15], these two classes of models appear to have good constructions in F-theory that generalize naturally to 4D supergravity theories. Specifically, the  $\beta \geq 0$  (Class (A)) models have spectra consistent with a Higgsing deformation of an  $\text{SU}(4) \times \text{SU}(3) \times \text{SU}(2)$  model with respective gauge anomaly coefficients  $B_4 = b_3, B_3 = b_2, B_2 = \beta$ , and we would thus expect that many of these models could be constructed in F-theory by starting with a Tate-type tuning [22, 44] of  $\text{SU}(4) \times \text{SU}(3) \times \text{SU}(2)$  (see section 8.2) and carrying out the deformation in the Weierstrass model. As we discuss in section 8, however, such a Tate construction is not possible when  $b_3, b_2, \beta$  are too large, even when anomaly cancellation naively suggests that such a model should exist. The  $Y = 0$  models correspond to (a slight generalization of) the F-theory models with toric fiber  $F_{11}$  discussed in [11]; the multiplicities given there are matched to those given in table 2 by the identification  $b_3 = \mathcal{S}_9, b_2 = \mathcal{S}_7 - \mathcal{S}_9 - K_B$ . Additionally, most of the models in the  $Y = 0$  class (Class (B)) admit an unHiggsing to a Pati-Salam  $(\text{SU}(4) \times \text{SU}(2) \times \text{SU}(2))/\mathbb{Z}_2$  model with respective gauge anomaly coefficients  $B_4 = b_3, B_2 = b_2, B'_2 = -4a - 2b_3 - b_2$ , and we would expect

| Generic Matter                    | Multiplicity                                 | MSSM Multiplet   |
|-----------------------------------|--|--|
| $(\mathbf{3}, \mathbf{2})_{1/6}$  | $b_3 \cdot b_2$                              | $\mathbf{Q}$   |
| $(\mathbf{3}, \mathbf{1})_{-4/3}$ | $b_3 \cdot Y$                                |  |
| $(\mathbf{3}, \mathbf{1})_{-1/3}$ | $b_3 \cdot X$                                | $\overline{\mathbf{D}}^c$                                      |
| $(\mathbf{3}, \mathbf{1})_{2/3}$  | $b_3 \cdot (\beta - 2a)$                     | $\overline{\mathbf{U}}^c$                                      |
| $(\mathbf{1}, \mathbf{2})_{1/2}$  | $b_2 \cdot (X + \beta - a)$                  | $\overline{\mathbf{L}}, \mathbf{H}_u, \overline{\mathbf{H}}_d$ |
| $(\mathbf{1}, \mathbf{2})_{3/2}$  | $b_2 \cdot Y$                                |  |
| $(\mathbf{1}, \mathbf{1})_1$      | $(b_3 + b_2 + 2\beta) \cdot X - a \cdot b_2$ | $\mathbf{E}^c$   |
| $(\mathbf{1}, \mathbf{1})_2$      | $\beta \cdot Y$                              |  |
| $(\mathbf{8}, \mathbf{1})_0$      | $1 + b_3 \cdot (b_3 + a)/2$                  |  |
| $(\mathbf{1}, \mathbf{3})_0$      | $1 + b_2 \cdot (b_2 + a)/2$                  |  |

**Table 2.** Generic matter representations (not including conjugates) charged under the gauge group  $(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1))/\mathbb{Z}_6$ , along with multiplicities for the generic matter solution of the 6D anomaly equations. This includes all the charged MSSM multiplets. The parameters  $\beta, X, Y$  are defined in eq. (2.3).

that these models can be realized in F-theory by a Tate tuning of this gauge group followed by a deformation of the Weierstrass model. The  $Y = 0$  model with  $b_3 = b_2 = -a$  has not only a toric fiber  $F_{11}$  description and a Pati-Salam description, but also has a spectrum compatible with an unHiggsing to  $\text{SU}(5)$  with gauge anomaly coefficient  $B_5 = -a$ .

Thus, we expect that we can construct many, and perhaps all, of the models in the above two classes in F-theory. The models of Class (A) are parameterized by three divisor classes  $b_3, b_2, \beta$ , which can be varied independently subject to the constraints that the multiplicities in table 2 are non-negative, while the models of class (B) have the additional constraint  $Y = 0$  and thus are parameterized by only the two independent divisor classes  $b_3, b_2$ . Even in cases where the Tate tuning of the unHiggsed  $\text{SU}(4) \times \text{SU}(3) \times \text{SU}(2)$  or Pati-Salam model is possible, however, this procedure is difficult to make explicit since it is hard to identify the Higgsing deformations of the given nonabelian Weierstrass model. In the current paper, we give an explicit Weierstrass model that realizes these models in all cases where the Tate tunings of the enhanced gauge groups discussed above yield the desired gauge algebra; as we discuss in section 8, there are cases where there is accidental enhancement of the gauge algebra beyond  $\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ .

### 3 Obtaining the model

In this and the following two sections, we give a direct, explicit description of the desired  $(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1))/\mathbb{Z}_6$  Weierstrass models that is relevant for both 6D and 4D constructions.

| Parameter | $b_3, b_2, \beta$ Basis        | $b_3, b_2, Y$ Basis       |
|-----------|--------------------------------|---------------------------|
| $b_1$     | $b_3$                          | $b_3$                     |
| $d_0$     | $b_2$                          | $b_2$                     |
| $s_1$     | $\beta$                        | $-K_B + Y - b_3 - b_2$    |
| $s_2$     | $K_B + b_3 + b_2 + \beta$      | $Y$                       |
| $d_1$     | $-3K_B - 2b_3 - b_2 - \beta$   | $-2K_B - Y - b_3$         |
| $d_2$     | $-6K_B - 4b_3 - 3b_2 - 2\beta$ | $-4K_B - 2Y - 2b_3 - b_2$ |
| $s_5$     | $-2K_B - b_3 - b_2$            | $-2K_B - b_3 - b_2$       |
| $s_6$     | $-K_B$                         | $-K_B$                    |
| $s_8$     | $-4K_B - 2b_3 - 2b_2 - \beta$  | $-3K_B - Y - b_3 - b_2$   |

**Table 3.** Homology classes for the divisors parameterizing the  $(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1))/\mathbb{Z}_6$  Weierstrass model in eq. (3.1), in two bases.

### 3.1 Weierstrass model

We start with a Weierstrass model, first described in [45], that supports a  $\text{U}(1)$  gauge group and matter with charges  $\pm 1$  through  $\pm 4$ . This Weierstrass model is rather lengthy, so we will not write it down here. The model can be enhanced to support a larger gauge group by setting the divisor parameters  $a_1$  and  $s_3$  to zero, leading to an  $f$  and  $g$  of the form

$$\begin{aligned}
 f &= -\frac{1}{48} [s_6^2 - 4b_1(d_0s_5 + d_1s_2)]^2 \\
 &\quad + \frac{1}{2} b_1 d_0 [2b_1(d_0s_1s_8 + d_1s_2s_5 + d_2s_2^2) - s_6(s_2s_8 + b_1d_1s_1)] , \\
 g &= \frac{1}{864} [s_6^2 - 4b_1(d_0s_5 + d_1s_2)]^3 + \frac{1}{4} b_1^2 d_0^2 (s_2s_8 - b_1d_1s_1)^2 - b_1^3 d_0^2 d_2 (s_2^2s_5 - s_2s_1s_6 + b_1d_0s_1^2) \\
 &\quad - \frac{1}{24} b_1 d_0 [s_6^2 - 4b_1(d_0s_5 + d_1s_2)] [2b_1(d_0s_1s_8 + d_1s_2s_5 + d_2s_2^2) - s_6(s_2s_8 + b_1d_1s_1)] .
 \end{aligned} \tag{3.1}$$

The homology classes for the parameters are given in the first column of table 3.

The resulting discriminant is proportional to  $b_1^3 d_0^2$ , indicating the presence of  $I_2$  singularities along  $\{d_0 = 0\}$  and  $I_3$  singularities along  $\{b_1 = 0\}$ . The  $I_2$  singularities signal that the model has an  $\text{SU}(2)$  gauge symmetry tuned on  $\{d_0 = 0\}$ . One can additionally verify that the split condition is satisfied for the  $I_3$  singularities and that the model admits an  $\text{SU}(3)$  gauge symmetry tuned on  $\{b_1 = 0\}$ . But the model also has an extra non-torsional section with components given by

$$\begin{aligned}
 \hat{x} &= \left( b_1 d_0 s_1 - \frac{1}{2} s_2 s_6 \right)^2 - \frac{1}{6} s_2^2 (s_6^2 + 2b_1 d_1 s_2 - 4b_1 d_0 s_5) , \\
 \hat{y} &= - \left( b_1 d_0 s_1 - \frac{1}{2} s_2 s_6 \right)^3 + \frac{1}{4} s_2^2 \left( b_1 d_0 s_1 - \frac{1}{2} s_2 s_6 \right) (s_6^2 + 2b_1 d_1 s_2 - 4b_1 d_0 s_5) \\
 &\quad + \frac{1}{4} s_2^4 b_1 (d_1 s_6 - 2d_0 s_8) , \\
 \hat{z} &= s_2 .
 \end{aligned} \tag{3.2}$$

As a result, we have an additional U(1) gauge factor, and the total gauge algebra is that of  $\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ , where the global structure of the group may have a quotient by a discrete group in the  $\mathbb{Z}_6$  center.

It is helpful to calculate the height  $\tilde{b}$  of the generating section (3.2), as it captures physical information about the U(1) factor. In general, the height is given by the formula [17, 46]

$$\tilde{b} = -2K_B + 2\pi(\mathcal{S} \cdot \mathcal{Z}) - (\mathcal{R}_\kappa^{-1})_{IJ} (\mathcal{S} \cdot \alpha_{\kappa,I}) (\mathcal{S} \cdot \alpha_{\kappa,J}) b_\kappa, \quad (3.3)$$

where  $\mathcal{S}$  is the homology class of the generating section,  $\mathcal{Z}$  is the homology class of the zero section, and  $\alpha_{\kappa,I}$  is the  $I$ th exceptional divisor associated with the  $\kappa$ th nonabelian gauge factor. The map  $\pi$  is the projection onto the base. Finally,  $b_\kappa$  is the divisor in the base supporting the  $\kappa$ th nonabelian gauge factor, while  $\mathcal{R}_\kappa$  is the normalized root matrix for the gauge factor. For the case at hand, the generating section meets the zero section whenever  $s_2$  is zero, so  $\pi(\mathcal{S} \cdot \mathcal{Z})$  is equal to  $[s_2] = Y$ . Meanwhile, the  $\hat{y}$  component of the generating section is proportional to both  $b_1$  and  $d_0$ , so for both the SU(3) and SU(2) gauge factors,  $(\mathcal{S} \cdot \alpha_{\kappa,I})$  is nonzero for at least one of the exceptional divisors. For the SU(2) factor, there is only one exceptional divisor, and  $\mathcal{R}_\kappa^{-1}$  is essentially the constant  $\frac{1}{2}$ . For the SU(3) factor, the generating section will hit one of the two exceptional divisors, which without loss of generality we can take to be  $I = 1$ . Additionally,  $(\mathcal{R}_\kappa^{-1})_{11}$  is  $\frac{2}{3}$ . Combining all of this information together, we have

$$\tilde{b} = -2K_B + 2Y - \frac{1}{2}b_2 - \frac{2}{3}b_3 = \frac{4}{3}b_3 + \frac{3}{2}b_2 + 2\beta. \quad (3.4)$$

Note that this expression matches the first relation in eq. (2.3).

We assert that eq. (3.1) is in fact the desired generic F-theory model with gauge group  $(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1))/\mathbb{Z}_6$ . In the succeeding sections we perform a variety of computations that support this hypothesis. In particular, we show that a generic  $(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1))/\mathbb{Z}_6$  model can be Higgsed back to the  $q = 3, 4$  model that was the starting point of this construction, we show that the model (3.1) exhibits the two classes of constructions identified in [15] and can be unHiggsed to the associated parent nonabelian groups in each case, we identify the matter loci of eq. (3.1) as appropriate for the generic  $(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1))/\mathbb{Z}_6$  model, and finally show that the dimensionality of the parameterized Weierstrass model (3.1) matches with that expected for 6D models, at least in the cases with no tensor multiplets where the Weierstrass model gives the desired gauge group. Taken together these analyses demonstrate definitively that the model (3.1) is indeed the generic Weierstrass model for F-theory constructions with gauge group  $(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1))/\mathbb{Z}_6$ . The correspondence between the parameters in the Weierstrass model (3.1) and the parameters reviewed in section 2.5 is given in table 3.

### 3.2 Higgsing to U(1) with $q = 4$

As the Weierstrass model (3.1) was found via an unHiggsing of the charge-4 U(1) Weierstrass model, it is useful to consider the corresponding Higgsing from the field theory point of view. Specifically, we consider Higgsings of the gauge group  $(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1))/\mathbb{Z}_6$

induced by giving nonzero VEVs to weights in the associated generic matter representations. Up to Weyl reflection, there are 35 distinct embeddings of  $U(1)$  into  $(SU(3) \times SU(2) \times U(1))/\mathbb{Z}_6$  that can be reached by such a Higgsing. By comparing the matter multiplicities in table 4 with those in table 7 of [45], we find that the relevant Higgsing leaves unbroken the  $U(1)$  generated by

$$\mu = \frac{4}{3}\lambda_3^{(1)} + \frac{8}{3}\lambda_3^{(2)} + \frac{3}{2}\lambda_2 + \lambda_1, \tag{3.5}$$

where  $\lambda_3^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ ,  $\lambda_3^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$  are the  $SU(3)$  Cartan generators and  $\lambda_2, \lambda_1$

are respectively the  $SU(2), U(1)$  Cartan generators. There are 24 different choices of three generic weights that can be given nonzero VEVs to yield this  $U(1)$  embedding after Higgsing. By considering intermediate Higgsings that can be seen explicitly in the Weierstrass model (3.1), we determine that, up to Weyl reflection, the relevant Higgsing in our case gives nonzero VEVs to the weights  $(1, 0, 0, -\frac{4}{3}), (-1, 1, 0, -\frac{4}{3}), (0, 0, -1, \frac{3}{2})$ , respectively in the representations  $(\mathbf{3}, \mathbf{1})_{-4/3}, (\mathbf{3}, \mathbf{1})_{-4/3}, (\mathbf{1}, \mathbf{2})_{3/2}$ , at least when there are enough fields to satisfy the D-term constraints.

As mentioned above, because three weights must be given VEVs to achieve this Higgsing, at the group theoretic level there are intermediate gauge algebras that can be reached by giving VEVs to a subset of these weights. Indeed, these intermediate subalgebras can be realized as intermediate unHiggsings of the  $q = 4$   $U(1)$  Weierstrass model. Specifically, the intermediate algebras  $\mathfrak{su}(3) \oplus \mathfrak{u}(1)$ ,  $\mathfrak{su}(2)' \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ ,  $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$ , and  $\mathfrak{su}(2)' \oplus \mathfrak{u}(1)$  can be reached, where the prime is used to indicate an  $\mathfrak{su}(2)$  subalgebra of the  $\mathfrak{su}(3)$  factor. For example, taking  $a_1 \rightarrow a'_1 b_1$  enhances the  $\mathfrak{u}(1)$  algebra to  $\mathfrak{su}(2)' \oplus \mathfrak{u}(1)$ , and subsequently taking  $a'_1 \rightarrow a''_1 b_1, s_3 \rightarrow s'_3 b_1$  further enhances this to  $\mathfrak{su}(3) \oplus \mathfrak{u}(1)$ . Similar enhancements can be made to explicitly realize all of the above intermediate Higgsings of  $(SU(3) \times SU(2) \times U(1))/\mathbb{Z}_6$  in the Weierstrass model (3.1).

## 4 Class (A) models

As discussed in section 2.5, there is an important subclass of  $(SU(3) \times SU(2) \times U(1))/\mathbb{Z}_6$  models, the Class (A) models, for which the divisor class  $\beta$  is effective. This presents the question of whether the Weierstrass model (3.1) can realize these models. Because there is no prior Weierstrass realization of the Class (A) models, this question has added significance. There are some easily seen features of the Weierstrass model that suggest that it can, in fact, support the Class (A) models. For instance, our Weierstrass model has three independent divisor classes when all parameters are non-vanishing, just as seen in the analysis of [15]. And the requirement that  $\beta$  is effective has a concrete interpretation in the Weierstrass model, as it implies that we can take the parameter  $s_1$  in eq. (3.1) to be nonzero. Thus, Class (A) is in some sense the more general of the two classes.

A more thorough way of addressing this question is to examine how we can further unHiggs the Weierstrass model (3.1). The Class (A) models are expected from [15] to admit an unHiggsing to  $SU(4) \times SU(3) \times SU(2)$ . In fact, the Weierstrass model generally allows

for such an unHiggsing. If we set  $s_2 \rightarrow 0$ , the discriminant then becomes proportional to  $b_1^4 d_0^3 s_1^2$ , indicating that we have  $I_4$  fibers along  $\{b_1 = 0\}$ ,  $I_3$  fibers along  $\{d_0 = 0\}$ , and  $I_2$  fibers along  $\{s_1 = 0\}$ . The split condition is also satisfied for the  $\{b_1 = 0\}$  and  $\{d_0 = 0\}$  singular loci. Therefore, the enhanced gauge group is  $SU(4) \times SU(3) \times SU(2)$ , with the gauge group factors tuned on the exact divisor classes predicted in [15]. The divisors  $\{b_1 = 0\}$ ,  $\{d_0 = 0\}$ , and  $\{s_1 = 0\}$  do not have any singular structure, so one would expect this model to have a generic matter spectrum. An explicit matter analysis shows this to be the case, and the spectrum agrees exactly with the expectations in [15] for the unHiggsed model. Since we see an enhancement to  $SU(4) \times SU(3) \times SU(2)$ , our Weierstrass model should realize the Class (A) spectra. We will see further confirmation of this fact when we explicitly determine the matter spectrum of the  $(SU(3) \times SU(2) \times U(1))/\mathbb{Z}_6$  model in section 6.

Before going on, it is worth understanding the enhancement to  $SU(4) \times SU(3) \times SU(2)$  in more detail. From field theoretic arguments, we know that  $SU(4) \times SU(3) \times SU(2)$  can be Higgsed down to  $(SU(3) \times SU(2) \times U(1))/\mathbb{Z}_6$  by giving VEVs to matter in the  $(\mathbf{4}, \mathbf{1}, \bar{\mathbf{2}})$  and  $(\mathbf{1}, \mathbf{3}, \bar{\mathbf{2}})$  representations (as long as there is enough matter in these representations). Depending on which bifundamental we give a VEV to first, we would have an intermediate gauge algebra of either  $\mathfrak{su}(3) \oplus \mathfrak{su}(3) \oplus \mathfrak{u}(1)$  or  $\mathfrak{su}(4) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ . These intermediate stages should be visible when unHiggsing the Weierstrass model. In fact, setting  $s_2$  to zero, as done above, should be viewed as a combination of two tunings. If we first let  $s_2 \rightarrow d_0 s'_2$ , the gauge algebra enhances to  $\mathfrak{su}(3) \oplus \mathfrak{su}(3) \oplus \mathfrak{u}(1)$ . Subsequently setting  $s'_2 \rightarrow 0$  gives us the full  $SU(4) \times SU(3) \times SU(2)$  gauge group. Alternatively, we could first enhance the gauge algebra to  $\mathfrak{su}(4) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$  by letting  $s_2 \rightarrow b_1 s''_2$  and then set  $s''_2 \rightarrow 0$ . Either way, the enhancement to  $SU(4) \times SU(3) \times SU(2)$  can generally be described as a two step process, as expected from field theory considerations.<sup>3</sup>

## 5 Class (B) models

### 5.1 Specialization to Class (B)

The Weierstrass model described by eq. (3.1) can also realize the Class (B) models. Recall that  $Y = [s_2]$  is trivial in these models (i.e., is in the zero class in cohomology), while  $\beta = [s_1]$  is allowed to be ineffective. Since  $s_1$  appears in the Weierstrass model, it naively seems that the construction is invalid when  $[s_1]$  is ineffective. But there is a way of salvaging the model in these situations: if  $[s_1]$  is ineffective, one can simply set  $s_1$  to 0 in the Weierstrass model. Of course, setting a parameter to 0 can introduce problems such as codimension-two (4,6) singularities, which are generally associated with superconformal sectors in the theory (see, e.g., [47] for a review), or an exactly vanishing discriminant. For this situation, setting  $s_1$  to zero does not introduce these more serious issues, but the discriminant does become proportional to  $s_2^2$ . This would suggest that we have an extra, undesired gauge group unless  $[s_2] = Y$  is trivial. Indeed, the analysis of [15] states that  $Y$

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<sup>3</sup>Note that in some cases, such as in 6D models where there is only a single hypermultiplet in each of the three bifundamental representations, the Higgsing involves all three bifundamentals and occurs in a single step.

must be trivial for the Class (B) models. The Weierstrass model directly reflects this fact and provides an explanation for the trivial  $Y$ : unless  $Y$  is trivial, the gauge group of the model is enhanced to something larger than the standard model gauge group.

There is another way of seeing that  $\beta = [s_1]$  can be ineffective as long as  $Y = [s_2]$  is trivial. When  $Y$  is trivial,  $s_2$  is essentially a nonzero constant, so we can freely divide by  $s_2$  without issue. With this in mind, we can redefine the parameters  $s_5$ ,  $s_6$ , and  $s_8$  as

$$s_5 = s'_5 + \frac{s_1}{s_2} (s_2 s'_6 + b_1 d_0 s_1), \quad s_6 = s'_6 + \frac{2}{s_2} b_1 d_0 s_1, \quad s_8 = s'_8 + \frac{1}{s_2} d_1 b_1 s_1. \quad (5.1)$$

Note that, as long as  $[s_2] = Y$  is trivial, these are simple redefinitions involving shifts in the parameters  $s_5$ ,  $s_6$ ,  $s_8$ . But these redefinitions remove all the terms in the Weierstrass model containing  $s_1$ . In essence, the  $s_1$  terms have been absorbed into the other parameters. And since  $s_1$  no longer appears in the Weierstrass model, clearly  $s_1$  can be ineffective.

However, it is known that the Class (B) spectra can also be realized by the  $F_{11}$  model described in [11]. This would suggest that the Weierstrass model above should match the  $F_{11}$  model, at least when  $Y$  is trivial. Indeed, we can redefine the parameters above in terms of the parameters in the  $F_{11}$  model:<sup>4</sup>

$$\begin{aligned} d_2 &= \frac{1}{s_2^2} (S_1 - S_{11} S_2 + S_{11}^2 S_3), & d_1 &= \frac{1}{s_2} (S_2 - S_3 S_{11}), & d_0 &= S_3 \\ s'_8 &= \frac{S_5}{s_2}, & s'_6 &= S_6, & b_1 &= S_9. \end{aligned} \quad (5.2)$$

Here,  $S_{11}$  is a new parameter not in the original  $F_{11}$  model, defined as

$$S_{11} = s'_5. \quad (5.3)$$

After these redefinitions, the  $f$  for the Weierstrass model defined here (when  $Y$  is trivial) is exactly the same as the  $f$  for the  $F_{11}$  model. The  $g$ 's, meanwhile, agree up to a term proportional to the new parameter  $S_{11}$ :

$$g = g_{F_{11}} - S_3^2 S_9^3 S_{11} (S_1 - S_{11} S_2 + S_{11}^2 S_3). \quad (5.4)$$

This would suggest that the Weierstrass model presented here gives a slight generalization of the  $F_{11}$  model, with exact agreement when  $S_{11} \rightarrow 0$ .

The presence of this extra term should not be too surprising. If one were to tune an  $SU(3) \times SU(2)$  gauge group, a term in  $g$  proportional to  $b_1^3 d_0^2$  (or  $S_3^3 S_3^2$ ) would only contribute terms in  $\Delta$  at order  $b_1^3 d_0^2$  or higher. Therefore, this extra term would not affect the  $SU(3) \times SU(2)$  tuning. Of course, it would have to take some special form to allow for the extra section generating the  $U(1)$ , and we indeed see additional structure in the extra term above. Nevertheless, it is natural to include this extra term in  $g$ , even though it does not appear in the  $F_{11}$  model.

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<sup>4</sup>We use the symbol  $S_i$  to refer to the parameter  $s_i$  in the  $F_{11}$  model.



## 5.2 Pati-Salam enhancement

Some of the Class (B) spectra should admit unHiggsings to Pati-Salam models [15], and as first found in [11], setting the parameter  $S_5$  in the  $F_{11}$  model to 0 leads to an F-theoretic Pati-Salam model. These two results suggest that our Weierstrass model should similarly admit an enhancement to a Pati-Salam model. Indeed, the tuning

$$s'_8 \rightarrow \frac{1}{s_2} s'_5 s'_6, \tag{5.5}$$

makes the discriminant proportional to

$$b_1^4 d_0^2 \left( d_2 s_2^2 - d_1 s'_5 s_2 + d_0 s_5'^2 \right)^2. \tag{5.6}$$

The corresponding gauge algebra of the enhanced theory is then  $\mathfrak{su}(4) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ . The  $\mathfrak{su}(4)$  is tuned on a divisor of class  $b_3$ , and the two  $\mathfrak{su}(2)$ s are tuned on divisors of classes  $b_2$  and  $-4a - 2b_3 - b_2$ . (Recall that  $Y = [s_2]$  is trivial for the Class (B) models we are currently considering.) This exactly matches the expectations from [15]. Moreover, the U(1) generating section (3.2) becomes

$$[\hat{x} : \hat{y} : \hat{z}] = \left[ \frac{1}{12} \left[ s_6'^2 - 4b_1 (d_1 s_2 - 2d_0 s_5') \right] : 0 : 1 \right]. \tag{5.7}$$

Since the  $\hat{y}$  component vanishes, the U(1) generating section has now become a torsional section of order 2.<sup>5</sup> In fact, this new torsional section is essentially the same as that identified in [11], with the only differences coming from terms proportional to  $S_{11}$ . If one performs an analysis similar to that in appendix B of [11], one can conclude that the gauge group is  $(\text{SU}(4) \times \text{SU}(2) \times \text{SU}(2))/\mathbb{Z}_2$ , exactly as expected.

## 5.3 Enhancement to SU(5)

The field theory analysis in [15] identified a second type of enhancement that should be possible for some of the Class (B) models: when  $Y = 0$  and  $b_3 = b_2 = -K_B$ , the  $(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1))/\mathbb{Z}_6$  gauge group has a spectrum suggesting that the gauge group can be enhanced to an SU(5) group tuned on a divisor of class  $-K_B$ . This unHiggsing is essentially the inverse process of the Higgsing in the Georgi-Glashow GUT model [49], in which matter in the adjoint (24) representation of SU(5) obtains a VEV. Even though the Class (B) models can be realized by the previous F-theory constructions in [11], no previous work has, to the best of our knowledge, explicitly demonstrated that the unHiggsing to SU(5) can be seen in a Weierstrass model. As we show below, the Weierstrass construction in eq. (3.1) explicitly admits an enhancement to SU(5), although the exact tunings required are somewhat complicated and cannot be seen in the toric picture.

When  $Y = 0$  and  $b_3 = b_2 = -K_B$ , the parameter  $s_1$  is ineffective, and one can remove  $s_1$  from the Weierstrass model through the redefinitions in eq. (5.1). To simplify the discussion, we also set the parameter  $s_2$ , which has a trivial divisor class, to 1.<sup>6</sup> With

<sup>5</sup>This phenomenon is similar to those observed in [48].

<sup>6</sup>Alternatively, one can remove  $s_2$  from the Weierstrass model by letting  $d_1 \rightarrow s_2^{-1} d_1$ ,  $s'_8 \rightarrow s_2^{-1} s'_8$ , and  $d_2 \rightarrow s_2^{-2} d_2$ . We are allowed to rescale parameters by inverse powers of  $s_2$  because  $[s_2]$  is trivial.

these simplifications, the Weierstrass model is given by

$$\begin{aligned}
 f &= -\frac{1}{48} \left[ s_6'^2 - 4b_1 (d_0 s_5' + d_1) \right]^2 + \frac{1}{2} b_1 d_0 \left[ 2b_1 (d_1 s_5' + d_2) - s_6' s_8' \right], \\
 g &= \frac{1}{864} \left[ s_6'^2 - 4b_1 (d_0 s_5' + d_1) \right]^3 + \frac{1}{4} b_1^2 d_0^2 \left( s_8'^2 - 4b_1 d_2 s_5' \right) \\
 &\quad - \frac{1}{24} b_1 d_0 \left[ s_6'^2 - 4b_1 (d_0 s_5' + d_1) \right] \left[ 2b_1 (d_1 s_5' + d_2) - s_6' s_8' \right],
 \end{aligned}
 \tag{5.8}$$

while the generating section for the U(1) gauge factor is

$$[\hat{x} : \hat{y} : \hat{z}] = \left[ \frac{1}{12} \left[ s_6'^2 - 4b_1 (d_1 - 2d_0 s_5') \right] : \frac{1}{2} b_1 d_0 (s_5' s_6' - s_8') : 1 \right].
 \tag{5.9}$$

The parameters  $s_6'$ ,  $s_8'$ ,  $b_1$ ,  $d_0$ ,  $d_1$ , and  $d_2$  are all sections of  $\mathcal{O}(-K_B)$ , while the parameter  $s_5'$  has a trivial divisor class.

One might expect that the gauge group enhances to SU(5) when the SU(2) locus coincides with the SU(3) locus. Operationally, one would tune  $d_0$  to be  $b_1$ , making the discriminant proportional to  $b_1^5$ . If this unHiggsing truly is the inverse process of the adjoint Higgsing, we should see the U(1) gauge factor “merge” with the other nonabelian factors. At a practical level, we would expect the  $\hat{z}$  component of the generating section to vanish such that the generating section coalesces with the zero section. But if we naively set  $d_0 \rightarrow b_1$  in the above expressions, the generating section remains distinct from the zero section. Tuning  $d_0 \rightarrow b_1$  therefore represents a different enhancement. Since the generating section remains after setting  $d_0 \rightarrow b_1$ , the enhanced gauge algebra is  $\mathfrak{su}(5) \oplus \mathfrak{u}(1)$ , not  $\mathfrak{su}(5)$ .<sup>7</sup> This enhancement cannot be the unHiggsing that we want, as we should not see any extra U(1) factors after the tuning.

The correct SU(5) enhancement procedure still involves tuning the SU(3) and SU(2) loci to coincide, but the exact tuning is more subtle. It is easiest to state the procedure before describing the underlying logic. First, we let

$$d_0 = b_1 - \epsilon \tilde{d}_0,
 \tag{5.10}$$

where  $[\epsilon] = 0$  and  $[\tilde{d}_0] = -K_B$ . This redefinition does not, by itself, restrict the structure of  $d_0$ . Since  $[\epsilon] = 0$ , we have simply performed a shift and rescaling of  $d_0$ . And we can always undo this by letting  $\tilde{d}_0 = \epsilon^{-1}(-d_0 + b_1)$ . Next, we redefine the other parameters in terms of  $\epsilon$ :

$$s_5' = \frac{1}{\epsilon^2} + \tilde{s}_5,
 \tag{5.11}$$

$$s_6' = -\tilde{d}_0 + \epsilon \tilde{s}_6,
 \tag{5.12}$$

$$s_8' = -\frac{2}{\epsilon^3} b_1 - \frac{2}{\epsilon} b_1 \tilde{s}_5 + \tilde{d}_1 + \epsilon \tilde{s}_8,
 \tag{5.13}$$

$$d_1 = -\frac{1}{\epsilon^2} \left( b_1 - \epsilon \tilde{d}_0 \right) - \tilde{s}_6 + \epsilon \tilde{d}_1,
 \tag{5.14}$$

$$d_2 = \frac{1}{\epsilon^4} b_1 + \frac{1}{\epsilon^2} \tilde{s}_5 b_1 - \frac{1}{\epsilon} \tilde{d}_1 - \tilde{s}_8 + \epsilon^2 \tilde{d}_2.
 \tag{5.15}$$

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<sup>7</sup>The section cannot be a  $\mathbb{Z}_5$  torsional section in an F-theory model without codimension-two (4, 6) loci, as can be seen from the general form of a Weierstrass model with such a section [50, 51].

The Weierstrass model is now described by

$$f = -\frac{1}{48} \left\{ (\tilde{d}_0 - \epsilon \tilde{s}_6)^2 - 4b_1 \left[ \tilde{s}_5(b_1 - \epsilon \tilde{d}_0) - (\tilde{s}_6 - \epsilon \tilde{d}_1) \right] \right\}^2 - \frac{1}{2} b_1 (b_1 - \epsilon \tilde{d}_0) \left[ 2b_1 \left( \tilde{s}_8 - \epsilon \tilde{d}_1 \tilde{s}_5 - \epsilon^2 \tilde{d}_2 \right) - (\tilde{d}_0 - \epsilon \tilde{s}_6)(\tilde{d}_1 + \epsilon \tilde{s}_8) \right] \quad (5.16)$$

and

$$g = \frac{1}{864} \left\{ (\tilde{d}_0 - \epsilon \tilde{s}_6)^2 - 4b_1 \left[ \tilde{s}_5(b_1 - \epsilon \tilde{d}_0) - (\tilde{s}_6 - \epsilon \tilde{d}_1) \right] \right\}^3 + \frac{1}{4} b_1^2 (b_1 - \epsilon \tilde{d}_0)^2 \left[ (\tilde{d}_1 + \epsilon \tilde{s}_8)^2 - 4b_1 \left( \tilde{d}_2 + \epsilon^2 \tilde{d}_2 \tilde{s}_5 \right) \right] + \frac{1}{24} b_1 (b_1 - \epsilon \tilde{d}_0) \left\{ (\tilde{d}_0 - \epsilon \tilde{s}_6)^2 - 4b_1 \left[ \tilde{s}_5(b_1 - \epsilon \tilde{d}_0) - (\tilde{s}_6 - \epsilon \tilde{d}_1) \right] \right\} \times \left[ 2b_1 \left( \tilde{s}_8 - \epsilon \tilde{d}_1 \tilde{s}_5 - \epsilon^2 \tilde{d}_2 \right) - (\tilde{d}_0 - \epsilon \tilde{s}_6)(\tilde{d}_1 + \epsilon \tilde{s}_8) \right]. \quad (5.17)$$

Miraculously,  $f$  and  $g$  do not contain any terms proportional to inverse powers of  $\epsilon$ , even though the redefinitions above include inverse powers of  $\epsilon$ . The generating section components, on the other hand, do have inverse powers of  $\epsilon$  after the redefinitions, but we can remove these inverse powers by rescaling the section components. In the end, the generating section is given by  $[\hat{x} : \hat{y} : \hat{z}]$ , with

$$\begin{aligned} \hat{x} &= \frac{1}{12} \epsilon^2 (\tilde{d}_0 - \epsilon \tilde{s}_6)^2 + b_1^2 + \frac{2}{3} \epsilon^2 \tilde{s}_5 b_1^2 - \frac{1}{3} \epsilon b_1 \left[ 3\tilde{d}_0 - \epsilon \tilde{s}_6 + \epsilon^2 (\tilde{d}_1 + 2\tilde{d}_0 \tilde{s}_5) \right], \\ \hat{y} &= -\frac{1}{2} b_1 (b_1 - \epsilon \tilde{d}_0) \left[ \epsilon^3 \tilde{d}_1 + \epsilon^4 \tilde{s}_8 - (1 + \epsilon^2 \tilde{s}_5) (2b_1 - \epsilon \tilde{d}_0 + \epsilon^2 \tilde{s}_6) \right], \\ \hat{z} &= \epsilon. \end{aligned} \quad (5.18)$$

Since  $f$  and  $g$  contain no inverse powers of  $\epsilon$ , we can now safely send  $\epsilon \rightarrow 0$ . We find that  $f$  and  $g$  do not vanish in this limit, but the discriminant becomes proportional to  $b_1^5$ . One can also verify that  $f$  and  $g$  satisfy the split condition, indicating that we have a split  $I_5$  singularity along  $\{b_1 = 0\}$ . Thus, we have an  $SU(5)$  tuned on a divisor of class  $-K_B$ , as expected. However, the  $\hat{z}$  component of the section now vanishes in the  $\epsilon \rightarrow 0$  limit, and the generating section coalesces with the zero section. There are no extra  $U(1)$  factors, and the enhanced gauge group is  $SU(5)$ . The  $\epsilon \rightarrow 0$  tuning is therefore the desired unHiggsing process.

The tuning procedure described above admittedly seems ad-hoc, but there is an underlying logic behind at least some of the steps. In particular, the need for the new parameter  $\epsilon$  can be understood from the  $SU(5) \rightarrow (SU(3) \times SU(2) \times U(1))/\mathbb{Z}_6$  branching rules:<sup>8</sup>

$$\begin{aligned} \mathbf{24} &\rightarrow (\mathbf{8}, \mathbf{1})_0 + (\mathbf{1}, \mathbf{3})_0 + (\mathbf{3}, \mathbf{2})_{-5/6} + (\bar{\mathbf{3}}, \mathbf{2})_{5/6} + (\mathbf{1}, \mathbf{1})_0, \\ \mathbf{10} &\rightarrow (\mathbf{3}, \mathbf{2})_{1/6} + (\bar{\mathbf{3}}, \mathbf{1})_{-2/3} + (\mathbf{1}, \mathbf{1})_1, \\ \mathbf{5} &\rightarrow (\mathbf{3}, \mathbf{1})_{-1/3} + (\mathbf{1}, \mathbf{2})_{1/2}. \end{aligned} \quad (5.19)$$

<sup>8</sup>Note that the representations listed in the branching rules below may differ by conjugation from those listed in table 4.

The branching rules involve two types of bifundamentals: those with charge  $q = \pm \frac{5}{6}$ , and those with charge  $q = \frac{1}{6}$ . Of course, the matter spectrum for the situation at hand would not actually contain any  $(\mathbf{3}, \mathbf{2})_{-5/6}$  matter, since, in the actual Higgsing process, all the  $(\mathbf{3}, \mathbf{2})_{-5/6}$  matter is eaten by the broken gauge bosons.<sup>9</sup> Nevertheless, if we want to see the SU(5) unHiggsing process explicitly, the Weierstrass model should contain a “would-be”  $(\mathbf{3}, \mathbf{2})_{-5/6}$  locus. Since bifundamentals come from intersections between gauge divisors, a construction that demonstrates an SU(5) unHiggsing should present two ways of having the SU(3) and SU(2) loci coincide. This motivates the specific form of  $d_0$  in eq. (5.10), as we can send  $d_0 \rightarrow b_1$  by letting  $\tilde{d}_0 \rightarrow 0$  or by letting  $\epsilon \rightarrow 0$ . Here,  $\{b_1 = \epsilon = 0\}$  is the would-be  $(\mathbf{3}, \mathbf{2})_{-5/6}$  locus, since  $[\epsilon] = 0$ . We also need to ensure that as  $\epsilon \rightarrow 0$ , the generating section coalesces with the zero section. This implies that we need to redefine terms so that the  $\hat{z}$  component of the section becomes proportional to  $\epsilon$ . In the end, we need SU(3) and SU(2) gauge factors tuned on divisors taking the forms described above, and we need a specific form of the  $\hat{z}$  component of the section. One can use these requirements to find the necessary redefinitions of the parameters, leading to eqs. (5.11) to (5.15).

Finally, we note that even though the enhancement was described for the Weierstrass construction presented here, a similar SU(5) unHiggsing should be possible in the  $F_{11}$  construction of [11]. As mentioned previously, there is a map between these two constructions when  $Y = 0$ . It is therefore possible to use this dictionary to find the necessary tunings in the  $F_{11}$  construction, although we do not go through the details of this analysis here.

## 6 Codimension-two singularities and matter

There are two sources of charged matter in the F-theory model described by eq. (3.1). First, the model can have matter in the adjoint representation of SU(3) or SU(2) coming from strings that propagate freely along the gauge divisors. In 6D F-theory models, where the gauge divisors are complex curves in the base, adjoint matter occurs when either the SU(3) divisor  $\{b_1 = 0\}$  or the SU(2) divisor  $\{d_0 = 0\}$  has a genus  $g$  greater than zero.<sup>10</sup> The number of adjoint hypermultiplets is given by the geometric genus of the corresponding gauge divisors. Specifically, there are  $1 + b_3 \cdot (b_3 + K_B)/2$  hypermultiplets of  $(\mathbf{8}, \mathbf{1})_0$  matter and  $1 + b_2 \cdot (b_2 + K_B)/2$  hypermultiplets of  $(\mathbf{1}, \mathbf{3})_0$  matter. These adjoint hypermultiplets are uncharged under the U(1) and are charged only under one of the nonabelian gauge group factors.

Matter can also be supported along codimension-two loci in the base where the elliptic curve singularity type enhances. The codimension-two loci supporting matter in the  $(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1))/\mathbb{Z}_6$  model are summarized in table 4, but let us describe the process of finding these loci and the associated matter representations in more detail.

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<sup>9</sup>This occurs because the SU(5) is tuned on a divisor of class  $-K_B$ , implying that, at least in a 6D model, the SU(5) matter spectrum contains only one hypermultiplet of  $\mathbf{24}$  matter. If we considered a model with more than one SU(5) adjoint hypermultiplet, the resulting spectrum would contain  $(\mathbf{3}, \mathbf{2})_{-5/6}$  hypermultiplets.

<sup>10</sup>We are assuming here that the divisors are smooth. When the divisors have singularities, there may be adjoint matter localized at the singular loci, or the singularities may support matter in more exotic representations [52–54].

| Locus   | Multiplicity                                   | Supported Matter                  |
|---|--|-----------------------------------|
| $\{b_1 = d_0 = 0\}$                                 | $b_3 \cdot b_2$                                | $(\mathbf{3}, \mathbf{2})_{1/6}$  |
| $\{b_1 = s_2 = 0\}$                                 | $b_3 \cdot Y$                                  | $(\mathbf{3}, \mathbf{1})_{-4/3}$ |
| $\{b_1 = d_2 s_6^2 - d_1 s_6 s_8 + d_0 s_8^2 = 0\}$ | $b_3 \cdot X$                                  | $(\mathbf{3}, \mathbf{1})_{-1/3}$ |
| $\{b_1 = s_8 s_2^2 - s_5 s_2 s_6 + s_1 s_6^2 = 0\}$ | $b_3 \cdot (\beta - 2K_B)$                     | $(\mathbf{3}, \mathbf{1})_{2/3}$  |
| $\{d_0 = \Delta_{(a)} = 0\}$                        | $b_2 \cdot (X + \beta - K_B)$                  | $(\mathbf{1}, \mathbf{2})_{1/2}$  |
| $\{d_0 = s_2 = 0\}$                                 | $b_2 \cdot Y$                                  | $(\mathbf{1}, \mathbf{2})_{3/2}$  |
| $V_{q=1}$   | $(b_3 + b_2 + 2\beta) \cdot X - K_B \cdot b_2$ | $(\mathbf{1}, \mathbf{1})_1$      |
| $\{s_2 = s_1 = 0\}$                                 | $\beta \cdot Y$                                | $(\mathbf{1}, \mathbf{1})_2$      |

**Table 4.** Codimension-two loci of the  $(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1))/\mathbb{Z}_6$  model and the associated charged matter. The multiplicities are for a 6D model constructed using the  $(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1))/\mathbb{Z}_6$  Weierstrass model (3.1). Note that  $\Delta_{(a)}$  is defined in eq. (6.3), while  $V_{q=1}$  is defined in eq. (6.6).

### 6.1 Determining the matter loci

First, let us focus on the codimension-two loci along the  $\text{SU}(3)$  divisor  $\{b_1 = 0\}$ . Any matter supported at such loci should be charged under the  $\text{SU}(3)$  gauge group. The discriminant takes the form

$$\Delta = b_1^3 d_0^2 \left[ \frac{1}{16} s_2 s_6^3 (s_8 s_2^2 - s_5 s_2 s_6 + s_1 s_6^2) (d_2 s_6^2 - d_1 s_6 s_8 + d_0 s_8^2) + \mathcal{O}(b_1) \right]. \quad (6.1)$$

From this expression, we see five different codimension-two loci along  $\{b_1 = 0\}$  where the singularity type enhances. At  $\{b_1 = d_0 = 0\}$ , the intersection locus of the  $\text{SU}(3)$  and  $\text{SU}(2)$  divisors, the singularity type enhances from  $I_3$  to  $I_6$ . As we will see shortly, this locus supports bifundamental matter. At  $\{b_1 = s_6 = 0\}$ , meanwhile, the singularity type changes from  $I_3$  to  $IV$ . Such loci do not contribute charged matter, since both describe  $\text{SU}(3)$  type singularities, so we can ignore  $\{b_1 = s_6 = 0\}$  for the purposes of the charged matter analysis. We are left with three loci where the singularity type enhances from  $I_3$  to  $I_4$ :  $\{b_1 = s_2 = 0\}$ ,  $\{b_1 = s_8 s_2^2 - s_5 s_2 s_6 + s_1 s_6^2 = 0\}$ , and  $\{b_1 = d_2 s_6^2 - d_1 s_6 s_8 + d_0 s_8^2 = 0\}$ . These loci support matter in the fundamental representation of  $\text{SU}(3)$  with different  $\text{U}(1)$  charges.

Let us now turn to the codimension-two loci along  $\{d_0 = 0\}$ , which should support matter charged under the  $\text{SU}(2)$  gauge factor. The discriminant can be written as

$$\Delta = b_1^3 d_0^2 \left[ -\frac{1}{16} s_2 (s_6^2 - 4b_1 d_1 s_2)^2 \Delta_{(a)} + \mathcal{O}(d_0) \right], \quad (6.2)$$

where

$$\begin{aligned} \Delta_{(a)} = & b_1^2 d_1^3 s_1^2 + b_1 [d_2^2 s_2^3 + d_1 d_2 (3s_1 s_6 - 2s_2 s_5) s_2 + d_1^2 (s_2 s_5^2 - s_1 s_6 s_5 - 2s_1 s_2 s_8)] \\ & - (s_8 s_2^2 - s_5 s_6 s_2 + s_1 s_6^2) (d_2 s_6 - d_1 s_8). \end{aligned} \quad (6.3)$$

The locus  $\{b_1 = d_0\}$ , where the singularity type enhances from  $I_2$  to  $I_6$ , was mentioned previously. At the locus  $\{d_0 = s_6^2 - 4b_1 d_1 s_2 = 0\}$ , the singularity type enhances to type  $III$ .

This locus, just like the  $\{b_1 = s_6 = 0\}$  locus discussed before, does not contribute charged matter since both singularity types are associated with  $SU(2)$  groups, so we will not discuss it further. We are therefore left with two loci where the singularity type enhances from  $I_2$  to  $I_3$ :  $\{d_0 = s_2 = 0\}$  and  $\{d_0 = \Delta_{(a)} = 0\}$ . These loci support matter in the fundamental representation of  $SU(2)$ .

Finally, we need to determine the codimension-two matter loci not along  $\{b_1 = 0\}$  or  $\{d_0 = 0\}$ . Such loci, which support matter charged only under the  $U(1)$  gauge group, are contained in

$$\hat{y} = 3\hat{x}^2 + f\hat{z}^4 = 0, \tag{6.4}$$

where  $[\hat{x} : \hat{y} : \hat{z}]$  are the section components of the generating section. Of course, we want to focus on the subloci not involving  $\{b_1 = 0\}$  or  $\{d_0 = 0\}$ . After some analysis, one finds that the appropriate sublocus containing the desired matter loci is

$$V = \left\{ \begin{array}{l} -b_1 d_1 s_1 s_2^3 + 2b_1 d_0 s_1 s_5 s_2^2 - 3b_1 d_0 s_1^2 s_6 s_2 + 2b_1^2 d_0^2 s_1^3 + s_8 s_2^4 - s_5 s_6 s_2^3 + s_1 s_6^2 s_2^2 \\ = -b_1 d_0 s_1^2 s_2 (d_1 s_2^2 - 2d_0 s_5 s_2 + 2d_0 s_1 s_6) + b_1^2 d_0^3 s_1^4 + d_2 s_2^6 \\ -s_2^2 (s_2 s_5 - s_1 s_6) (d_1 s_2^2 - d_0 s_5 s_2 + d_0 s_1 s_6) = 0 \end{array} \right\}. \tag{6.5}$$

There are two important subloci of  $V$ . The first is the sublocus  $\{s_1 = s_2 = 0\}$ , along which the  $[\hat{x} : \hat{y} : \hat{z}]$  components vanish. We will later show that this sublocus supports matter in the  $(\mathbf{1}, \mathbf{1})_2$  representation. The other important sublocus is essentially  $V$  with all of the previous matter loci removed. In particular,  $V$  contains the loci  $\{s_1 = s_2 = 0\}$ ,  $\{b_1 = s_2 = 0\}$ , and  $\{d_0 = s_2 = 0\}$ , so the locus we are interested in is

$$V_{q=1} = V \setminus (\{s_1 = s_2 = 0\} \cup \{b_1 = s_2 = 0\} \cup \{d_0 = s_2 = 0\}). \tag{6.6}$$

As might be anticipated by its name,  $V_{q=1}$  supports matter in the  $(\mathbf{1}, \mathbf{1})_1$  representation. Determining the multiplicity for  $V_{q=1}$  is somewhat complicated. Based on  $V$ , one might naively expect that the multiplicity for  $V$  is  $(2b_3 + 2b_2 + 3\beta) \cdot (2b_3 + 3b_2 + 4\beta)$ . But we must account for the contributions from the undesired loci  $\{s_1 = s_2 = 0\}$ ,  $\{b_1 = s_2 = 0\}$ , and  $\{d_0 = s_2 = 0\}$ . As part of this analysis, we need to determine how many copies of the undesired loci are contained within  $V$ . If we write  $V$  as  $\{v_a = v_b = 0\}$ , this information can be determined from the resultant of  $v_a$  and  $v_b$  with respect to  $s_2$  [55]. One finds that

$$\text{Res}_{s_2}(v_a, v_b) \propto b_1^8 d_0^9 s_1^{16}, \tag{6.7}$$

indicating that  $V$  contains eight copies of  $\{b_1 = s_2 = 0\}$ , nine copies of  $\{d_0 = s_2 = 0\}$ , and sixteen copies of  $\{s_1 = s_2 = 0\}$ . Therefore, the multiplicity for  $V_{q=1}$  is

$$\begin{aligned} & (2b_3 + 2b_2 + 3\beta) \cdot (2b_3 + 3b_2 + 4\beta) - (8b_3 + 9b_2 + 16\beta) \cdot Y \\ & = (b_3 + b_2 + 2\beta) \cdot (-8K_B - 4b_3 - 3b_2 - 2\beta) - K_B \cdot b_2. \end{aligned} \tag{6.8}$$

## 6.2 Determining the matter representations

Now that we have identified the important codimension-two loci, we can investigate the types of charged matter supported at these loci. While a proper analysis would require resolving singularities in the Calabi-Yau manifold, we can also determine the supported representations more quickly using heuristic arguments. In particular, the Katz-Vafa [56] method tells us about the nonabelian representations of the supported matter. And we can use our knowledge of the unHiggsing patterns and the relations to previously found models to determine the U(1) charges of the matter.

First, recall that we can exactly recover the  $(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1))/\mathbb{Z}_6$  model of [11] by setting  $s_1$  and  $s'_5$  to zero and  $s_2$  to a constant. Therefore, the matter loci in these two models should match after the appropriate variable specializations have been made. In this situation, the only relevant codimension-two loci are  $\{b_1 = d_0 = 0\}$ ,  $\{b_1 = s_8 s_2^2 - s_5 s_2 s_6 + s_1 s_6^2 = 0\}$ ,  $\{b_1 = d_2 s_6^2 - d_1 s_6 s_8 + d_0 s_8^2 = 0\}$ ,  $\{d_0 = \Delta_{(a)} = 0\}$ , and  $V_{q=1}$ . (The other loci disappear once  $s_2$  is set to a nonzero constant.) We can match each of these loci to loci in [11] and determine the matter representations:

- The locus  $\{b_1 = d_0 = 0\}$  is the intersection between the two gauge divisors and should therefore support bifundamental matter. In [11], the corresponding locus supports  $(\mathbf{3}, \mathbf{2})_{1/6}$  matter,<sup>11</sup> so  $\{b_1 = d_0 = 0\}$  should also support  $(\mathbf{3}, \mathbf{2})_{1/6}$  matter.
- At the locus  $\{b_1 = s_8 s_2^2 - s_5 s_2 s_6 + s_1 s_6^2 = 0\}$ , the singularity type enhances from  $I_3$  to  $I_4$ . This locus therefore supports matter in the fundamental representation of SU(3). After setting  $s_1$  to zero and  $s_2$  to a constant, this locus corresponds to the locus  $\{S_9 = S_5 = 0\}$  in [11], which supports  $(\mathbf{3}, \mathbf{1})_{2/3}$  matter. The locus  $\{b_1 = s_8 s_2^2 - s_5 s_2 s_6 + s_1 s_6^2 = 0\}$  therefore also supports  $(\mathbf{3}, \mathbf{1})_{2/3}$  matter.
- At the locus  $\{b_1 = d_2 s_6^2 - d_1 s_6 s_8 + d_0 s_8^2 = 0\}$ , the singularity type enhances from  $I_3$  to  $I_4$ . This locus therefore supports matter in the fundamental representation of SU(3). After setting  $s_1$  and  $s'_5$  to zero and  $s_2$  to a constant, this locus corresponds to the locus  $\{S_9 = S_3 S_5^2 - S_2 S_5 S_6 + S_1 S_6^2 = 0\}$  in [11], which supports  $(\mathbf{3}, \mathbf{1})_{-1/3}$  matter. The locus  $\{b_1 = d_2 s_6^2 - d_1 s_6 s_8 + d_0 s_8^2 = 0\}$  therefore supports  $(\mathbf{3}, \mathbf{1})_{-1/3}$  matter.
- At the locus  $\{d_0 = \Delta_{(a)} = 0\}$ , the singularity type enhances from  $I_2$  to  $I_3$ , indicating that this locus supports matter in the fundamental representation of SU(2). After setting  $s_1$  and  $s'_5$  to zero and  $s_2$  to a constant, this locus corresponds to the locus  $\{S_3 = S_2 S_5^2 - S_6 S_5 S_1 + S_9 S_1^2 = 0\}$  in [11], which supports  $(\mathbf{1}, \mathbf{2})_{1/2}$  matter. The locus  $\{d_0 = \Delta_{(a)} = 0\}$  therefore supports  $(\mathbf{1}, \mathbf{2})_{1/2}$  matter.
- At the locus  $V_{q=1}$ , the singularity type enhances to  $I_2$ . This locus therefore supports matter charged under only the U(1) gauge factor. After setting  $s_1$  and  $s'_5$  to zero and  $s_2$  to a constant, this locus corresponds to the locus  $\{S_1 = S_5 = 0\}$  in [11], which supports  $(\mathbf{1}, \mathbf{1})_1$  matter. The locus  $V_{q=1}$  therefore supports  $(\mathbf{1}, \mathbf{1})_1$  matter.

<sup>11</sup>In [11], the charge is listed as  $-1/6$ , and the signs of the charges listed are in general the negatives of those listed in [15]. Since the overall sign of the charges can be flipped freely, we flip the signs of the charges to match [15].



There are three remaining codimension-two loci that do not have counterparts in the model from [11]:  $\{s_1 = s_2 = 0\}$ ,  $\{b_1 = s_2 = 0\}$ , and  $\{d_0 = s_2 = 0\}$ . But there are alternative ways of determining the matter representations of these loci without performing a resolution. Because  $\{s_1 = s_2 = 0\}$  does not involve either of the gauge divisors, the matter supported here can only be charged under the U(1) gauge factor. In fact, the  $[\hat{x} : \hat{y} : \hat{z}]$  section components vanish to orders  $(2, 3, 1)$  at  $\{s_1 = s_2 = 0\}$ . In models with just a U(1) gauge group, the section components vanish to these orders at loci supporting charge  $q = 2$  matter [17, 45]. One would therefore expect that  $\{s_1 = s_2 = 0\}$  supports  $(\mathbf{1}, \mathbf{1})_2$  matter. Note that we have used the fact that the Shioda map gives us a standard unit for the charge: matter with U(1) charge  $q = 1$  that is not charged under the non-abelian gauge factors occurs at codimension two  $I_2$  loci where the generating section intersects the extra component transversely.

This leaves us with the loci  $\{b_1 = s_2 = 0\}$  and  $\{d_0 = s_2 = 0\}$ . At  $\{b_1 = s_2 = 0\}$ , the singularity type enhances from  $I_3$  to  $I_4$ , so the matter supported here must be in the fundamental representation of SU(3). At  $\{d_0 = s_2 = 0\}$ , meanwhile, the singularity type enhances from  $I_2$  to  $I_3$ , so the matter supported here should be in the fundamental representation of SU(2). However, we still need to determine the U(1) charges of these two types of matter. Fortunately, we know that our model can be obtained by Higgsing a model with an  $SU(4) \times SU(3) \times SU(2)$  gauge symmetry, at least when  $s_1$  is effective. Specifically, we give VEVs to bifundamental matter in the  $(\mathbf{4}, \mathbf{1}, \bar{\mathbf{2}})$  and  $(\mathbf{1}, \mathbf{3}, \bar{\mathbf{2}})$  representations. If one works out how the  $SU(4) \times SU(3) \times SU(2)$  representations branch to  $(SU(3) \times SU(2) \times U(1))/\mathbb{Z}_6$  representations, one recovers all of the representations mentioned so far, but one additionally finds the representations  $(\mathbf{3}, \mathbf{1})_{-4/3}$  and  $(\mathbf{1}, \mathbf{2})_{3/2}$ . These must be the representations associated with the two remaining loci,  $\{b_1 = s_2 = 0\}$  and  $\{d_0 = s_2 = 0\}$ . Thus,  $\{b_1 = s_2 = 0\}$  should support  $(\mathbf{3}, \mathbf{1})_{-4/3}$  matter, while  $\{d_0 = s_2 = 0\}$  should support  $(\mathbf{1}, \mathbf{2})_{3/2}$ .

This gives us the results summarized in table 4. The spectrum satisfies the 6D anomaly cancellation conditions with the appropriate anomaly coefficients, which gives us some confidence that we have obtained the correct matter spectrum. Moreover, the spectrum agrees exactly with that described [15], further supporting the assertion that the Weierstrass construction in eq. (3.1) is the expected  $(SU(3) \times SU(2) \times U(1))/\mathbb{Z}_6$  model.

As discussed to some extent already in [15], the matter spectrum distinguishes the Class (A) and (B) models. The class (B) models, with  $Y = 0$  have precisely the matter representations expected for a supersymmetric extension of the standard model, both in 6D and in 4D. While the Class (A) models are more general (parameterized by three divisor classes rather than two), additional matter is potentially supported in these models. In supersymmetric 6D models there are additional matter fields, specifically the fields with charges  $(\mathbf{1}, \mathbf{2})_{3/2}$ ,  $(\mathbf{3}, \mathbf{1})_{-4/3}$ , and  $(\mathbf{1}, \mathbf{1})_2$ . In 4D, there is also the possibility of massless chiral matter in these representations. However, since chiral matter is produced by fluxes, there are also consistent supersymmetric 4D F-theory models with only the usual MSSM fields in the massless spectrum. Understanding how much tuning is needed to avoid light exotics of these representations, and how the spectrum is affected by supersymmetry breaking, are interesting phenomenological questions left for further work.

## 7 Full dimensionality of model

Given a parameterized Weierstrass model for F-theory constructions with a given gauge group  $G$  and associated generic matter, of the type constructed in section 3, we would like to know if this is in fact the most general such Weierstrass model, or if we are missing some parameters that could be included in a more complete model. There are several ways of testing this, depending upon the situation.

In the simplest cases, with a single nonabelian gauge group factor associated with fixed orders of vanishing of the Weierstrass coefficients  $f, g$  through the Kodaira classification (e.g.  $E_6$ ), it is straightforward to check directly that the Weierstrass model is generic subject to those conditions. This is slightly more subtle when the gauge group requires an order of vanishing of the discriminant  $\Delta$  that is larger than that required by the orders of vanishing of  $f, g$ , but a direct analysis in these cases is relatively straightforward for a single gauge group factor. For example, in [53] the general form of Weierstrass models for  $SU(N)$  to be tuned (over a smooth curve) is determined by explicitly checking the Kodaira conditions for  $f, g, \Delta$  in an order-by-order expansion in the parameter  $\sigma$  associated with the  $SU(N)$  locus. Up to  $SU(5)$  this gives Weierstrass models associated with the standard Tate tuning procedure (above  $SU(5)$  there are multiple distinct branches of parameterized Weierstrass models, including non-generic matter in the 3-index antisymmetric representation of  $SU(6)$  through  $SU(8)$ , see also [57, 58]).

When the gauge group has abelian  $U(1)$  or multiple  $SU(N)$  factors, this question becomes more complicated. In particular, we do not know of a simple algebraic condition on the components of  $f, g, \Delta$  associated with the existence of a nontrivial section associated with nonzero Mordell-Weil rank, as is needed for a  $U(1)$  factor. As the simplest example of this, consider the Morrison-Park model [17] for a theory with  $U(1)$  gauge group and generic matter charges  $q = 1, 2$

$$y^2 = x^3 + \left( c_1 c_3 - b^2 c_0 - \frac{1}{3} c_2^2 \right) x + \left( c_0 c_3^2 - \frac{1}{3} c_1 c_2 c_3 + \frac{2}{27} c_2^3 - \frac{2}{3} b^2 c_0 c_2 + \frac{1}{4} b^2 c_1^2 \right). \quad (7.1)$$

Here,  $c_j$  is a section of a line bundle in the class  $-2K_B - (j-2)(K_B + L)$  where  $K_B$  is the canonical class of the base and the line bundle  $L$  parameterizes the spectra ( $b$  is a section of the line bundle  $-2K_B - L$ ). In this case, the number of distinct complex coefficients needed to choose the sections  $c_i, b$  is in general significantly larger than the number of moduli in the expected moduli space; i.e., the parameterization is redundant. In this kind of situation it is a bit more subtle to check that the model indeed spans a space of the proper dimensionality. This question can be answered most easily for such parameterized models in the context of six-dimensional theories, where the dimension of the space of Weierstrass models must match the number of uncharged hypermultiplets, which is in turn fixed by anomaly cancellation. For the Morrison-Park model, one approach to counting the number of moduli was given in [32]. Here we use a somewhat more direct method that works for any such parameterized Weierstrass model including those constructed in section 3.

The basic idea is fairly straightforward. If we write the set of complex coefficients of the component monomials in  $f, g$  as  $W_k \in \mathbb{C}$ , and the set of coefficients of the component

monomials in the parameters  $c_i, b$  of the Morrison-Park model as  $v_j \in \mathbb{C}$ , then the dimension of the moduli space around a fixed chosen background configuration is given by subtracting the dimension of the set of automorphism symmetries from the rank ( $\text{rk } J = \dim \text{Im } J$ ) of the Jacobian matrix

$$J_{kj} = \partial W_k / \partial v_j. \tag{7.2}$$

Since in the Morrison-Park model  $f$  and  $g$  are algebraic functions of the  $c_i, b$ , each of the elements of the Jacobian matrix is simply a polynomial in the  $v_j$ . Because the dimensions of the matrix are quite large, however, it is computationally difficult to check the rank algebraically. We proceed therefore numerically. To avoid precision issues, we simply choose  $\hat{v}_j$  to be random integers in a given range  $\hat{v}_j \in \{1, 2, \dots, N\}$ , and then the rank of the Jacobian matrix

$$\text{rk } J|_{v=\hat{v}} \tag{7.3}$$

in the vicinity of the specific model with  $v_j = \hat{v}_j$  is straightforward to compute using Mathematica or another computational tool. If the range  $N$  of allowed integers is sufficiently large, there will be many relatively prime factors in the distinct  $\hat{v}_j$  and the chances of a coincidental decrease in rank becomes very small. To check that this does not occur we have tested various cases with the  $\hat{v}_j$  varying in ranges from 1 to  $N = 10$  and from 1 to  $N = 1000$  and in all cases we get the same answer, so empirically the range does not need to be too large to get a correct measure of the rank. As a specific example, if we take the Morrison-Park model in the case where the base is  $\mathbb{P}^2$ , so  $-K_B = 3H$  where  $H$  is the generating class (a line in  $\mathbb{P}^2$ ), and we pick  $L = 2H$ , then there are 115 variables  $v_j$ , and we expect the dimension of the moduli space to be 106 by anomaly cancellation. (This is the case in which the dimension of the moduli space is smallest). Computing the rank using the above algorithm gives 114. There is a redundancy in 8 degrees of freedom because of possible reparameterization of the homogeneous coordinates on  $\mathbb{P}^2$  (up to an overall scale), so this gives the available moduli. We have checked this computation for the other choices of  $L$  on  $\mathbb{P}^2$  and find exact agreement in all cases, confirming that the Morrison-Park model is in fact the most generic form of the Weierstrass model with  $U(1)$  gauge group and generic charges  $q = 1, 2$

We have used this Jacobian rank method to confirm in various specific cases that the class of models defined in eq. (3.1) indeed gives the generic  $(SU(3) \times SU(2) \times U(1))/\mathbb{Z}_6$  model, in the sense that the dimensionality of the moduli space determined by the rank of the Jacobian around a random model with fixed classes and base matches the number of moduli expected in a corresponding 6D theory (after accounting for automorphism symmetries of the base). We have checked this in models of Class (A) and of Class (B). In the majority of cases, this computation verifies that the Weierstrass model (3.1) captures the full dimension of the moduli space. In particular, for the “SU(5)” type B model on the base  $\mathbb{P}^2$ , the parameters  $b_2$  and  $b_3$  are both in the class  $-K_B = 3H$  (i.e., cubics in homogeneous coordinates on  $\mathbb{P}^2$ ). In this case, 6D anomaly cancellation indicates that the number of expected massless neutral hypermultiplets is 49, and the rank of the Jacobian is 57, so we have agreement after subtracting the 8 automorphism symmetries. However, there are cases where there is a mismatch in the number of moduli calculated with this method and

the number expected from 6D anomaly cancellation. In section 8, we discuss examples of such mismatches for models on the base  $\mathbb{P}^2$ .

## 8 Dimensionality of models on $\mathbb{P}^2$

In this section, we carry out the Jacobian rank analysis described in section 7 to count the number of moduli for all 6D models described by eq. (3.1) over the base  $\mathbb{P}^2$ . As discussed in [15], there are 98 solutions to the 6D anomaly cancellation conditions when  $T = 0$ , which corresponds to F-theory on  $\mathbb{P}^2$ . These solutions are enumerated by choices of anomaly coefficients  $b_3, b_2, \beta$ , in this case integers, which correspond in the F-theory picture to choices of the homology classes of the corresponding parameters in the Weierstrass model (3.1), as described in table 3.

For 44 of the 98 models over the base  $\mathbb{P}^2$ , the Jacobian rank analysis gives a different moduli count than is expected from 6D anomaly cancellation. There are two distinct cases: when the Jacobian rank method provides an overcount or an undercount compared with 6D anomaly cancellation. In the following subsections, we discuss the reason for these mismatches in both cases.

The upshot is that there are cases where there are valid anomaly cancellation solutions, but because the anomaly coefficients are too large, the F-theory construction discussed here develops enhanced symmetries and no longer describes an  $(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1))/\mathbb{Z}_6$  model,

### 8.1 When $d_2$ becomes ineffective

The first situation, when the Jacobian rank method provides an overcount from the naive expectation, occurs when the gauge group enhances beyond  $(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1))/\mathbb{Z}_6$  due to certain parameters in the Weierstrass model becoming ineffective. Here, we will discuss the case where  $d_2$  is ineffective but  $d_1$  remains effective. This does not cover all cases where there is gauge group enhancement, but the analysis for other cases is similar, and so is omitted here.

If  $d_2$  is ineffective, we should consider the Weierstrass model (3.1), but with  $d_2$  set to 0:

$$\begin{aligned} f &= -\frac{1}{48} [s_6^2 - 4b_1(d_0s_5 + d_1s_2)]^2 + \frac{1}{2}b_1d_0 [2b_1(d_0s_1s_8 + d_1s_2s_5) - s_6(s_2s_8 + b_1d_1s_1)], \\ g &= \frac{1}{864} [s_6^2 - 4b_1(d_0s_5 + d_1s_2)]^3 + \frac{1}{4}b_1^2d_0^2(s_2s_8 - b_1d_1s_1)^2 \\ &\quad - \frac{1}{24}b_1d_0 [s_6^2 - 4b_1(d_0s_5 + d_1s_2)] [2b_1(d_0s_1s_8 + d_1s_2s_5) - s_6(s_2s_8 + b_1d_1s_1)]. \end{aligned} \tag{8.1}$$

In fact, the model gains an extra generating section when  $d_2 \rightarrow 0$ , indicating that the gauge algebra is enhanced to  $\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1)$ . The Mordell-Weil group is generated by the sections  $Q$  and  $T$ , which have the following  $[\hat{x} : \hat{y} : \hat{z}]$  components:

$$\begin{aligned} Q &: \left[ \frac{1}{12} [s_6^2 - 4b_1(d_1s_2 + d_0s_5)] : \frac{1}{2}b_1d_0(b_1d_1s_1 - s_2s_8) : 1 \right], \\ T &: \left[ \frac{1}{12} [s_6^2 + 4b_1(2d_1s_2 - d_0s_5)] : -\frac{1}{2}b_1(b_1d_0d_1s_1 + d_0s_2s_8 - d_1s_2s_6) : 1 \right]. \end{aligned} \tag{8.2}$$

| Matter  | Locus   | Multiplicity   |
|---|---|--|
| $(\mathbf{3}, \mathbf{2})_{\frac{1}{6}, -\frac{1}{3}}$  | $\{b_1 = d_0 = 0\}$                                 | $b_3 \cdot b_2$  |
| $(\mathbf{3}, \mathbf{1})_{-\frac{1}{3}, -\frac{1}{3}}$ | $\{b_1 = s_1 s_6^2 - s_2 s_5 s_6 + s_8 s_2^2 = 0\}$ | $b_3 \cdot (-b_3 - b_2 - 3K_B + Y)$  |
| $(\mathbf{3}, \mathbf{1})_{\frac{2}{3}, -\frac{1}{3}}$  | $\{b_1 = s_8 = 0\}$                                 | $b_3 \cdot (-b_3 - b_2 - 3K_B - Y)$  |
| $(\mathbf{3}, \mathbf{1})_{-\frac{1}{3}, \frac{2}{3}}$  | $\{b_1 = d_1 s_6 - d_0 s_8 = 0\}$                   | $b_3 \cdot (-b_3 - 3K_B - Y)$  |
| $(\mathbf{3}, \mathbf{1})_{\frac{2}{3}, \frac{2}{3}}$   | $\{b_1 = s_2 = 0\}$                                 | $b_3 \cdot Y$  |
| $(\mathbf{1}, \mathbf{2})_{-\frac{1}{2}, 0}$            | $\{d_0 = \Delta'_a = 0\}$                           | $b_2 \cdot (-2b_2 - 2b_3 - 6K_B)$  |
| $(\mathbf{1}, \mathbf{2})_{-\frac{1}{2}, 1}$            | $\{d_0 = d_1 = 0\}$                                 | $b_2 \cdot (-b_3 - 2K_B - Y)$  |
| $(\mathbf{1}, \mathbf{2})_{\frac{1}{2}, 1}$             | $\{d_0 = s_2 = 0\}$                                 | $b_2 \cdot Y$  |
| $(\mathbf{1}, \mathbf{1})_{-1, -1}$                     | $\{s_1 = s_2 = 0\}$                                 | $Y \cdot (-K_B - b_2 - b_3 + Y)$   |
| $(\mathbf{1}, \mathbf{1})_{1, -1}$                      | $\{d_1 = s_8 = 0\}$                                 | $(b_3 + 2K_B + Y) \cdot (b_2 + b_3 + 3K_B + Y)$                            |
| $(\mathbf{1}, \mathbf{1})_{1, 0}$                       | $I_{Q, q=1}$  | $b_2 \cdot (b_2 + 2b_3 + 5K_B) + (b_3 + 2K_B - 2Y) \cdot (b_3 + 3K_B + Y)$ |
| $(\mathbf{1}, \mathbf{1})_{0, 1}$                       | $I_{T, q=1}$  | $(b_3 + 2K_B) \cdot (b_2 + b_3 + 3K_B) - (b_3 + 4K_B + 2Y) \cdot Y$        |
| $(\mathbf{8}, \mathbf{1})_{0, 0}$                       | $\{b_1 = 0\}$                                       | $1 + \frac{1}{2} b_3 \cdot (b_3 + K_B)$                                    |
| $(\mathbf{1}, \mathbf{3})_{0, 0}$                       | $\{d_0 = 0\}$                                       | $1 + \frac{1}{2} b_2 \cdot (b_2 + K_B)$                                    |

**Table 5.** Matter spectrum for the model when  $d_2 \rightarrow 0$ , with multiplicities for a 6D model. The quantities  $\Delta'_a$ ,  $I_{Q, q=1}$ , and  $I_{T, q=1}$  are defined in the main text.

The section of the original model (with the standard model gauge group) is equal to  $-(Q + T)$ , where  $+$  represents elliptic curve addition.

The charged matter spectrum of the model is given in table 5. The quantity  $\Delta'_a$  in the table is

$$\Delta'_a = b_1 d_1 [b_1 d_1 s_1^2 + s_5 (s_2 s_5 - s_1 s_6)] + s_8 [s_6 (s_1 s_6 - s_2 s_5) - 2b_1 d_1 s_1 s_2] + s_2^2 s_8^2. \quad (8.3)$$

Meanwhile,  $I_{Q, q=1}$  and  $I_{T, q=1}$  are the loci supporting  $(\mathbf{1}, \mathbf{1})_{1, 0}$  and  $(\mathbf{1}, \mathbf{1})_{0, 1}$  matter, respectively. Let us first focus on the  $I_{Q, q=1}$  locus. This is naively given by  $\{\hat{y}_Q = 3\hat{x}_Q^2 + f\hat{z}_Q^4 = 0\}$ , but we must remove contributions from the other loci. First, we note that  $\hat{y}_Q$  and  $3\hat{x}_Q^2 + f\hat{z}_Q^4$  are both proportional to  $b_1 d_0$ . Loci where either  $b_1 = 0$  or  $d_0 = 0$  support matter charged under the nonabelian factors and should therefore be excluded from the locus. We can therefore drop these factors from  $\hat{y}_Q$  and  $3\hat{x}_Q^2 + f\hat{z}_Q^4$  when determining  $I_{Q, q=1}$ . A resultant analysis [55] reveals that the remaining locus contains one copy of  $\{b_1 = s_2 = 0\}$ , one copy of  $\{b_1 = s_8 = 0\}$ , one copy of  $\{s_1 = s_2 = 0\}$ , and one copy of  $\{d_1 = s_8 = 0\}$ . Removing the contributions of these loci leads to the multiplicity listed in the table. The multiplicity of the  $I_{T, q=1}$  locus is calculated in a similar way. The resulting matter spectrum satisfies the 6D gauge and mixed anomaly constraints with

$$\tilde{b}_{QQ} = -2K_B - \frac{2}{3} b_3 - \frac{1}{2} d_0, \quad \tilde{b}_{TT} = -2K_B - \frac{2}{3} b_3, \quad \tilde{b}_{QT} = K_B + \frac{1}{3} b_3 + Y. \quad (8.4)$$

Before turning to moduli counting, let us note an interesting fact about the spectrum. In particular, suppose that we wanted to Higgs this  $\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1)$  algebra down

to our original  $\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$  model. We would need to give VEVs to matter in the  $(\mathbf{1}, \mathbf{1})_{1,-1}$  representation, and we would need at least two hypermultiplets of this matter to satisfy the D-term constraints. From the table, this matter is localized at  $\{d_1 = s_8 = 0\}$  and has multiplicity

$$[d_1] \cdot [s_8] = (b_3 + 2K_B + Y) \cdot (b_2 + b_3 + 3K_B + Y) = \frac{1}{4} ([d_2] + [d_0]) \cdot ([d_2] - [d_0] - 2K_B). \quad (8.5)$$

Now assume that  $d_2$  is ineffective but that  $d_1$  and  $s_8$  are effective. Then, both  $[d_2] + [d_0]$  and  $[d_2] - [d_0] - 2K_B$  must be effective, implying that

$$K_B \leq [d_2] < 0, \quad -[d_2] \leq [d_0] \leq -2K_B + [d_2]. \quad (8.6)$$

As our base is  $\mathbb{P}^2$ , there are three possible choices for  $[d_2]$ :  $-3H$ ,  $-2H$ , and  $-H$ , where  $H$  is the hyperplane class. For each of these choices, we can determine the possible values of  $[d_0]$  and calculate  $[d_1] \cdot [s_8]$ . Miraculously, whenever  $[d_2]$  is ineffective,  $[d_1] \cdot [s_8]$  is never greater than 1. In other words, if  $[d_2]$  is ineffective, there are never enough  $(\mathbf{1}, \mathbf{1})_{1,-1}$  hypermultiplets to Higgs the  $\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1)$  algebra down to the original  $\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ , at least for a  $\mathbb{P}^2$  base. This fact suggests a possible explanation for the fact that the gauge algebra of the Weierstrass model enhances when  $[d_2]$  is ineffective: we are seeing a phenomenon similar to that observed for non-Higgsable clusters (NHCs). Just as in the model at hand, models with NHCs do not have enough charged hypermultiplets to break the NHC while satisfying D-term constraints. At the level of the Weierstrass model, NHCs occur when certain parameters (the coefficients in a power series expansion of  $f$  and  $g$  around the relevant divisor) become ineffective, thereby forcing the Weierstrass model to obtain gauge singularities. This is exactly the behavior observed here for  $\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ . Of course, an NHC cannot be Higgsed at all, whereas there should be ways of Higgsing  $\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1)$  down to alternative gauge groups when  $[d_2]$  is ineffective. Nevertheless, the automatic enhancement of  $\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$  to  $\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1)$  seems to have a similar origin as the enhancement seen for NHCs.

### 8.1.1 Moduli counting

Table 6 shows the moduli counts for the models on  $\mathbb{P}^2$  with  $d_2$  as the only ineffective parameter. For most of the models, the calculated number of moduli agrees with the expectations from the gravitational anomaly cancellation condition. However, there are two classes of models where there is a mismatch; these classes of models are separated off in the table by horizontal lines:

- When all three of  $\beta = [s_1]$ ,  $[s_5]$ , and  $[s_8]$  are trivial, the calculated number of moduli is one more than the expectation from anomaly cancellation. This seems problematic, since the Weierstrass model should have at most as many moduli as that expected from anomaly cancellation.
- When  $s_1$  is ineffective, the calculated number of moduli is smaller than the expectation from anomaly cancellation. In all of these cases,  $[s_5]$  is also ineffective, and  $[s_8]$  is trivial.

We will see below that we can fully account for the mismatches in both of these cases.

| $b_3$ | $b_2$ | $\beta$ | $Y$ | Expected Moduli | Computed Moduli | Expected Moduli for SM |
|-------|-------|---------|-----|-----------------|-----------------|------------------------|
| 1     | 1     | 6       | 5   | 73              | 73              | 72                     |
| 1     | 2     | 5       | 5   | 63              | 63              | 62                     |
| 1     | 3     | 4       | 5   | 56              | 56              | 55                     |
| 1     | 4     | 2       | 4   | 45              | 45              | 44                     |
| 2     | 1     | 4       | 4   | 54              | 54              | 53                     |
| 2     | 2     | 3       | 4   | 46              | 46              | 45                     |
| 2     | 3     | 1       | 3   | 37              | 37              | 37                     |
| 2     | 3     | 2       | 4   | 41              | 41              | 40                     |
| 3     | 1     | 2       | 3   | 41              | 41              | 40                     |
| 3     | 2     | 1       | 3   | 35              | 35              | 34                     |
| 4     | 1     | 0       | 2   | 34              | 34              | 33                     |
| 1     | 5     | 0       | 3   | 41              | 42              | 40                     |
| 2     | 4     | 0       | 3   | 35              | 36              | 34                     |
| 3     | 3     | 0       | 3   | 32              | 33              | 31                     |
| 4     | 5     | -6      | 0   | 44              | 43              | 43                     |
| 5     | 4     | -6      | 0   | 41              | 40              | 40                     |
| 6     | 3     | -6      | 0   | 41              | 40              | 40                     |

**Table 6.** Moduli counting for models with ineffective  $[d_2]$ . The column labeled “Expected Moduli” gives the number of moduli for the  $\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1)$  model that are expected from 6D anomaly cancellation, given the global gauge group structure consistent with the spectrum in table 5. The column labeled “Computed Moduli” gives the number of moduli calculated from the rank of the Jacobian matrix (7.2). The column labeled “Expected Moduli for SM” gives the number of moduli that would be expected for the  $\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$  model based on 6D anomaly cancellation.

**When  $[s_1]$ ,  $[s_5]$ , and  $[s_8]$  are trivial.** The first type of mismatch, occurring in models where  $[s_1]$ ,  $[s_5]$ , and  $[s_8]$  are trivial, can be explained by the presence of extra U(1) gauge factors. This phenomenon can most easily be seen when the elliptic fiber is written as a cubic in a  $\mathbb{P}^2$  ambient space:<sup>12</sup>

$$b_1 v w (d_0 v + d_1 w) + u v (s_2 u + s_6 w) + u (s_1 u^2 + s_5 u w + s_8 w^2) = 0. \tag{8.7}$$

Here,  $[u : v : w]$  are the homogeneous coordinates of the  $\mathbb{P}^2$  ambient space. Note that this equation is satisfied when  $v = s_1 u^2 + s_5 u w + s_8 w^2 = 0$ . For arbitrary  $s_1$ ,  $s_5$ , and  $s_8$ , the expression  $s_1 u^2 + s_5 u w + s_8 w^2$  does not factor. But if the expression happens to factor, we can read off two new sections of the fibration, which we refer to as  $A$  and  $B$ . Specifically, if we let  $s_1 = \alpha_1 \alpha_2$ ,  $s_5 = \alpha_1 \beta_2 + \alpha_2 \beta_1$ , and  $s_8 = \beta_1 \beta_2$ , the two new sections are given by

$$A: [u : v : w] = [-\beta_1 : 0 : \alpha_1], \quad B: [u : v : w] = [-\beta_2 : 0 : \alpha_2]. \tag{8.8}$$

<sup>12</sup>Note that the equation below is written in a form that explicitly assumes  $d_2$  has been set to 0, since we are currently interested in situations where  $[d_2]$  is ineffective. One can describe situations where  $d_2$  is nonzero by adding an extra  $b_1 d_2 w^3$  term to the left-hand side.



In Weierstrass form, these new sections are given by  $[x_A : y_A : \beta_1]$  and  $[x_B : y_B : \beta_2]$ , with

$$\begin{aligned}
 x_A &= \alpha_1^2 b_1^2 d_1^2 - \frac{1}{3} b_1 \beta_1 [\beta_1 d_0 (\alpha_2 \beta_1 - 2\alpha_1 \beta_2) + d_1 (3\alpha_1 s_6 - 2\beta_1 s_2)] + \frac{1}{12} \beta_1^2 s_6^2, \\
 y_A &= \frac{1}{2} b_1 \left\{ 2\alpha_1^3 b_1^2 d_1^3 + \alpha_1 b_1 \beta_1 d_1 [\beta_1 d_0 (2\alpha_1 \beta_2 - \alpha_2 \beta_1) + d_1 (2\beta_1 s_2 - 3\alpha_1 s_6)] \right. \\
 &\quad \left. + \beta_1^2 (\beta_1 \beta_2 d_0 - d_1 s_6) (\beta_1 s_2 - \alpha_1 s_6) \right\}, \\
 x_B &= \alpha_2^2 b_1^2 d_1^2 - \frac{1}{3} b_1 \beta_2 [\beta_2 d_0 (\alpha_1 \beta_2 - 2\alpha_2 \beta_1) + d_1 (3\alpha_2 s_6 - 2\beta_2 s_2)] + \frac{1}{12} \beta_2^2 s_6^2, \\
 y_B &= \frac{1}{2} b_1 \left\{ 2\alpha_2^3 b_1^2 d_1^3 + \alpha_2 b_1 \beta_2 d_1 [\beta_2 d_0 (2\alpha_2 \beta_1 - \alpha_1 \beta_2) + d_1 (2\beta_2 s_2 - 3\alpha_2 s_6)] \right. \\
 &\quad \left. + \beta_2^2 (\beta_1 \beta_2 d_0 - d_1 s_6) (\beta_2 s_2 - \alpha_2 s_6) \right\}.
 \end{aligned} \tag{8.9}$$

When  $[s_1]$ ,  $[s_5]$ , and  $[s_8]$  are trivial,  $s_1 u^2 + s_5 u w + s_8 w^2 = 0$  will automatically factor, and one must take these new sections into account. If one considers these new sections together with the sections  $Q$  and  $T$  from before, one finds that the Mordell-Weil rank has increased from two to three.<sup>13</sup> In turn, the gauge algebra should now be  $\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1)$ .

The matter spectrum of this new model is described in table 7. The number of moduli expected from the gravitational anomaly condition, meanwhile, is listed in table 8, along with the results of the Jacobian calculation. We now see that the expected number of moduli matches the number obtained from the Jacobian calculation. Therefore, the Weierstrass model has all of the expected moduli when  $[d_2]$  is ineffective and  $[s_1]$ ,  $[s_5]$ , and  $[s_8]$  are trivial.

**When  $[s_1]$  and  $[s_5]$  are ineffective.** In order to understand the models with the second type of mismatch, let us consider what happens when we set  $s_1$  and  $s_5$  to zero (along with  $d_2$ ). The Weierstrass model is now given by

$$\begin{aligned}
 f &= -\frac{1}{48} [s_6^2 - 4b_1 d_1 s_2]^2 - \frac{1}{2} b_1 d_0 s_6 s_2 s_8, \\
 g &= \frac{1}{864} [s_6^2 - 4b_1 d_1 s_2]^3 + \frac{1}{24} b_1 d_0 s_6 s_2 s_8 [s_6^2 - 4b_1 d_1 s_2] + \frac{1}{4} b_1^2 d_0^2 s_2^2 s_8^2.
 \end{aligned} \tag{8.10}$$

and the discriminant is proportional to  $b_1^3 d_0^2 s_2^3 s_8^2$ . However, since  $[s_2]$  and  $[s_8]$  are trivial for the models under consideration, the nonabelian part of the gauge algebra is simply  $\mathfrak{su}(3) \oplus \mathfrak{su}(2)$ . To find the abelian part of the gauge algebra, we must calculate the Mordell-Weil rank of these models. Surprisingly, the sections  $Q$  and  $T$ , which are independent when only  $d_2$  is set to 0, become related when  $s_1$  and  $s_5$  are also set to 0. Specifically,  $T$  becomes equal to  $-2Q$ , and  $Q$  becomes equal to the section originally given for the  $(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1))/\mathbb{Z}_6$  model. Therefore, when  $d_2$ ,  $s_1$ , and  $s_5$  are all ineffective, the gauge group is still  $(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1))/\mathbb{Z}_6$ . The matter spectrum, too, is essentially unchanged, but many of the original  $(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1))/\mathbb{Z}_6$  representations happen to have multiplicity 0 because  $[s_2]$  and  $[s_8]$  are trivial. The number of moduli calculated by our Jacobian procedure should therefore equal that expected from the gravitational anomaly condition for the  $(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1))/\mathbb{Z}_6$  model, which is exactly what is observed.

<sup>13</sup>One can show that  $T$  can be written as a combination of  $Q$ ,  $A$ , and  $B$  under elliptic curve addition, so these four sections are not fully independent.

| Matter   | Multiplicity                                    |
|--|---|
| $(\mathbf{3}, \mathbf{2})_{\frac{1}{6}, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}}$ | $-b_3 \cdot (b_3 + 2K_B)$                       |
| $(\mathbf{3}, \mathbf{1})_{\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}}$  | $-b_3 \cdot K_B$                                |
| $(\mathbf{3}, \mathbf{1})_{-\frac{1}{3}, -\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}}$ | $-b_3 \cdot K_B$                                |
| $(\mathbf{3}, \mathbf{1})_{-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{2}{3}}$ | $-b_3 \cdot K_B$                                |
| $(\mathbf{3}, \mathbf{1})_{-\frac{1}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}}$   | $-b_3 \cdot (b_3 + 2K_B)$                       |
| $(\mathbf{1}, \mathbf{2})_{\frac{1}{2}, 1, 0, 0}$                                  | $K_B \cdot (b_3 + 2K_B)$                        |
| $(\mathbf{1}, \mathbf{2})_{-\frac{1}{2}, 0, 1, 0}$                                 | $K_B \cdot (b_3 + 2K_B)$                        |
| $(\mathbf{1}, \mathbf{2})_{-\frac{1}{2}, 0, 0, 1}$                                 | $K_B \cdot (b_3 + 2K_B)$                        |
| $(\mathbf{1}, \mathbf{2})_{-\frac{1}{2}, 1, 1, 1}$                                 | $(b_3 + K_B) \cdot (b_3 + 2K_B)$                |
| $(\mathbf{1}, \mathbf{1})_{1, 0, -1, -1}$  | $K_B \cdot (b_3 + 2K_B)$                        |
| $(\mathbf{1}, \mathbf{1})_{0, 1, 1, 0}$  | $K_B \cdot (b_3 + 2K_B)$                        |
| $(\mathbf{1}, \mathbf{1})_{0, 1, 0, 1}$  | $K_B \cdot (b_3 + 2K_B)$                        |
| $(\mathbf{8}, \mathbf{1})_{0, 0, 0, 0}$  | $1 + \frac{1}{2}b_3 \cdot (b_3 + K_B)$          |
| $(\mathbf{1}, \mathbf{3})_{0, 0, 0, 0}$  | $1 + \frac{1}{2}(b_3 + K_B) \cdot (b_3 + 2K_B)$ |

**Table 7.** Matter spectrum for the model when  $[d_2]$  is ineffective and  $[s_1]$ ,  $[s_5]$ , and  $[s_8]$  are trivial, with multiplicities for a 6D model.

| $b_3$ | $b_2$ | $\beta$ | $Y$ | Expected Moduli | Computed Moduli | Expected Moduli for SM |
|-------|-------|---------|-----|-----------------|-----------------|------------------------|
| 1     | 5     | 0       | 3   | 42              | 42              | 40                     |
| 2     | 4     | 0       | 3   | 36              | 36              | 34                     |
| 3     | 3     | 0       | 3   | 33              | 33              | 31                     |

**Table 8.** Moduli counting for models with ineffective  $[d_2]$  and trivial  $[s_1]$ ,  $[s_5]$ , and  $[s_8]$ . The column labeled “Expected Moduli” gives the number of moduli for the  $\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1)$  model that are expected from 6D anomaly cancellation, given the global gauge group structure consistent with the spectrum in table 7. The column labeled “Computed Moduli” gives the number of moduli calculated from the rank of the Jacobian matrix (7.2). The column labeled “Expected Moduli for SM” gives the number of moduli that would be expected for the  $(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1))/\mathbb{Z}_6$  model based on 6D anomaly cancellation.

## 8.2 When the discriminant vanishes identically

The cases where the Jacobian rank method gives an undercount of the moduli compared with the expectation from 6D anomaly cancellation are exactly the cases where the discriminant vanishes identically for the given choice of parameters  $b_3, b_2, \beta$ . These cases are listed

| $b_3$ | $b_2$ | $\beta$ | $Y$ | Expected Moduli | Computed Moduli |
|-------|-------|---------|-----|-----------------|-----------------|
| 1     | 2     | 7       | 7   | 90              | 18              |
| 1     | 3     | 5       | 6   | 67              | 18              |
| 1     | 4     | 4       | 6   | 64              | 18              |
| 2     | 2     | 5       | 6   | 67              | 18              |
| 2     | 3     | 3       | 5   | 49              | 18              |
| 2     | 4     | 2       | 5   | 48              | 19              |
| 3     | 2     | 3       | 5   | 50              | 18              |
| 3     | 3     | 1       | 4   | 37              | 19              |
| 3     | 4     | 0       | 4   | 38              | 10              |
| 4     | 2     | 1       | 4   | 39              | 19              |

**Table 9.** Expected number of moduli from 6D anomaly cancellation for the gauge algebra  $\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$  along with the number of moduli counted for the Weierstrass model (3.1) using the Jacobian rank method (subtracting 8 to account for the automorphisms of the base  $\mathbb{P}^2$ ), for all  $\mathbb{P}^2$  models where the Jacobian rank method provides an undercount. These are precisely the cases where the discriminant vanishes identically. Both  $\beta$  and  $Y$  are given for convenience, though either is sufficient to determine the other, given  $b_3, b_2$ .

in table 9, and all lie in Class (A). In such a situation, eq. (3.1) is not a valid Weierstrass model. This can be thought of as a more severe version of the gauge algebra enhancement observed in the previous subsection.

It is worthwhile to discuss what happens in these cases when we try to construct a corresponding  $SU(4) \times SU(3) \times SU(2)$  Tate model. All of the models in table 9 are in Class (A), and so would appear to have an unHiggsing to a Tate  $SU(4) \times SU(3) \times SU(2)$  model, as discussed in section 2.5. Additionally, all of these models satisfy the Tate bound

$$4b_3 + 3b_2 + 2\beta \leq -8K = 24, \tag{8.11}$$

and so we would naively expect that this choice of parameters should yield a good Tate model, yet this clashes with the observation that the discriminant vanishes identically in the Higgsed  $(SU(3) \times SU(2) \times U(1))/\mathbb{Z}_6$  model. In these cases, the discriminant in fact identically vanishes for the Tate model as well.

As an example, let us try to carry out a Tate tuning of  $SU(4) \times SU(3) \times SU(2)$  over  $\mathbb{P}^2$  with the gauge factors supported on divisors  $u, v, w$  with homology classes  $[u] = H, [v] = 2H, [w] = 7H$ , where  $H$  is the hyperplane class in  $\mathbb{P}^2$ . This corresponds to the first line of table 9. Following the Tate algorithm, we can write the model in long Weierstrass form as

$$y^2 + a_1xyz + a_3yz^3 = x^3 + a_2x^2z^2 + a_4xz^4 + a_6z^6, \tag{8.12}$$

with

$$a_2 = a'_2uv, \quad a_3 = a'_3u^2vw, \quad a_4 = a'_4u^2v^2w, \quad a_6 = a'_6u^4v^3w^2. \tag{8.13}$$

From the homology classes of  $u, v, w$  and the fact that  $[a_j] = -jK_{\mathbb{P}^2} = 3jH$ , we find that

$$[a'_2] = 3H, \quad [a'_3] = -2H, \quad [a'_4] = -H, \quad [a'_6] = -6H. \quad (8.14)$$

The parameters in the Weierstrass model must be effective to be non-vanishing, and so we must have  $a_3 \rightarrow 0, a_4 \rightarrow 0, a_6 \rightarrow 0$ . However, the remaining parameters  $a_1, a_2$  have order of vanishing 0 along the divisor  $w$  that supports the  $SU(2)$ . As the  $SU(2)$  factor is supposed to be nontrivial, the only way that this can be the case is if the discriminant vanishes identically. Indeed, an explicit check shows that this is the case.

### 8.3 Swampland questions

As seen in the previous two subsections, there are cases where there is a valid low-energy  $(SU(3) \times SU(2) \times U(1))/\mathbb{Z}_6$  supergravity model that solves the anomaly cancellation conditions, but the corresponding F-theory construction given here exhibits enhancement to a larger gauge group or to a singularity structure so severe as to no longer be valid. The former case is similar to an enhancement of  $SU(2) \rightarrow SU(2)^2/\mathbb{Z}_2$  that was observed in [51] when the anomaly coefficient takes the almost-maximal values  $b = 10, 11$  in  $T = 0$  models, while the latter case can be viewed as a more extreme version of this phenomenon. In general, it seems that certain anomaly free models in 6D supergravity produce additional gauge factors or enhancement under known F-theory constructions when the anomaly coefficients are too large; it would be nice to understand this better. In particular, as the present construction does not actually realize the desired 6D supergravity  $(SU(3) \times SU(2) \times U(1))/\mathbb{Z}_6$  model in these cases, these models are in the “apparent swampland” of theories that violate no known quantum consistency constraints and yet have no known UV completion as a quantum gravity theory [59]. It remains to be seen whether these models have some other UV completion through F-theory or another approach to string compactification.

## 9 Matching to Morrison-Park form

The approach used in section 3 to construct the generic  $(SU(3) \times SU(2) \times U(1))/\mathbb{Z}_6$  model we have studied in this paper relies on the previous construction in [45] of a Weierstrass model for theories with  $U(1)$  gauge group and  $q = 3, 4$  matter, which happens to unHiggs to the desired nonabelian model of interest. One might wish for a more direct argument for this construction, or to construct other models with gauge groups of the form  $G = (G_{\text{NA}} \times U(1))/\Gamma$ , where  $G_{\text{NA}}$  is a simply-connected nonabelian group and  $\Gamma$  is a discrete subgroup, where no convenient form for a Higgsed version of the theory is already known. In such cases one might imagine carrying out the construction by starting with the abelian  $U(1)$  part of the gauge group and then combining this with the desired nonabelian Kodaira singularity structure through a method analogous to that used in [53]. We leave a systematic effort towards such constructions for future work, but it is perhaps useful in this regard to confirm that the class of models given by eq. (3.1) is indeed a specialization of the Morrison-Park model, as may be expected from the presence of the abelian factor and the generic nature of the matter representations in the theory.

Specifically, we can find a map between the parameters  $b$ ,  $c_0$ ,  $c_1$ ,  $c_2$  and  $c_3$  of the Morrison-Park form [17] and expressions in the  $(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1))/\mathbb{Z}_6$  Weierstrass model. Based on the forms of  $f$  and  $g$  reproduced in eq. (7.1), one can identify the following expressions for the parameters in the Morrison-Park form in terms of the parameters appearing in the  $(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1))/\mathbb{Z}_6$  model (3.1):

$$\begin{aligned}
 b &= s_2, \\
 c_3 &= b_1 d_0 s_1 - \frac{1}{2} s_2 s_6, \\
 c_2 &= \frac{1}{4} s_6^2 - b_1 d_0 s_5 + \frac{1}{2} b_1 d_1 s_2, \\
 c_1 &= b_1 \left( d_0 s_8 - \frac{1}{2} d_1 s_6 \right), \\
 c_0 &= \frac{1}{4} b_1^2 (d_1^2 - 4d_0 d_2).
 \end{aligned}
 \tag{9.1}$$

This confirms that our  $(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1))/\mathbb{Z}_6$  Weierstrass model is in fact a specialization of the Morrison-Park form.

## 10 Range of geometries for construction

In this section we make some simple observations regarding the range of possible F-theory geometries in which this construction is relevant. Elliptic Calabi-Yau threefolds and fourfolds are characterized by the two- or three-dimensional complex base manifold supporting the elliptic fibration. For 6D F-theory models (base surfaces), the set of possible bases is fairly well understood, and at least for Calabi-Yau threefolds with large  $h^{2,1}$  a reasonably representative sample is given by the range of 61,539 allowed toric bases [60, 61]. For 4D models (base threefolds), the set of toric bases alone seems to be of order  $10^{3000}$  [33–35]. It is natural to ask how many of these bases can support the Weierstrass models given here for  $(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1))/\mathbb{Z}_6$  gauge group and generic matter.

As was observed in [3] for the “SU(5)” special case of the construction given by the Class (B) model with  $b_3 = b_2 = -K_B$ , the constructions given here of both Class (A) and Class (B) can be carried out in a straightforward fashion on a weak Fano base (i.e., one without non-Higgsable structure). This will give a generic model with gauge group  $(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1))/\mathbb{Z}_6$ , as long as the divisors satisfy the necessary bounds, subject to the additional caveat that, as observed in section 8, when the divisors are too large the gauge group may be further enhanced or the discriminant may vanish identically. Since only a very small fraction of allowed bases are weak Fano, however, and almost all bases contain some effective divisors supporting non-Higgsable gauge groups, it is of interest to inquire to what extent the constructions described here can be realized on bases that are not weak Fano. The primary conclusion of the limited analysis we describe here is that both Class (A) and Class (B) models can be constructed on at least some bases that are not weak Fano without introducing additional exotic matter or extra gauge group factors coupled directly to the standard model gauge group  $(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1))/\mathbb{Z}_6$ .

For bases that are not weak Fano, there are in general divisors supporting non-Higgsable gauge groups (this is always true for 6D theories and generally true at the

geometric level for 4D theories). To tune the generic  $(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1))/\mathbb{Z}_6$  Weierstrass model we have found here on such a base, the divisors supporting the nonabelian  $\text{SU}(3)$  and  $\text{SU}(2)$  factors of the standard model must not contain or intersect any divisors supporting non-Higgsable gauge groups; otherwise, there can either be exotic matter charged both under the  $(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1))/\mathbb{Z}_6$  group and the non-Higgsable gauge factors, or the model can develop singularities at codimension one or two associated with vanishing orders of  $(4, 6)$  in  $f, g$ , which go beyond the Kodaira classification or would appear to involve a superconformal theory. As the Hodge number  $h^{1,1}$  of the threefold or fourfold increases, the non-Higgsable cluster structure becomes increasingly dense and it is harder to find divisors on which the gauge group  $(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1))/\mathbb{Z}_6$  can be tuned. Here we focus on those cases that are not weak Fano but where tuning of a decoupled standard model-like sector may still be possible. For clarity we focus attention on 6D models in this analysis, though similar results hold for 4D models.

We begin by considering models of Class (A). In this case, it is possible to choose divisors  $[b_1] = b_3, [d_0] = b_2, [s_1] = \beta$  on some bases that are not weak Fano so that all three of these divisors are disjoint from all divisors supporting a non-Higgsable gauge group. As long as these divisors are small enough, this allows a Tate tuning of the unHiggsed  $\text{SU}(4) \times \text{SU}(3) \times \text{SU}(2)$  model with no exotic matter charged under both this gauge group and the non-Higgsable factors, so that the associated  $(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1))/\mathbb{Z}_6$  model has the same feature; the non-Higgsable gauge groups in such a situation correspond to “hidden sectors” that communicate only gravitationally with the standard model-like part of the theory. As a concrete example, let us choose the base for a 6D model to be the Hirzebruch surface  $\mathbb{F}_3$ . This base contains a curve  $S$  of self-intersection  $S \cdot S = -3$  that supports a non-Higgsable  $\text{SU}(3)$  gauge group. The anticanonical class of the base is  $-K_B = 2S + 5F$ , where  $F$  is a  $\mathbb{P}^1$  fiber satisfying  $F \cdot F = 0, F \cdot S = 1$ . The curve  $\tilde{S} = S + 3F$  satisfies  $\tilde{S} \cdot \tilde{S} = 3, \tilde{S} \cdot S = 0$ . If we choose  $[b_3] = [b_2] = [\beta] = \tilde{S}$ , it is straightforward to check that all the parameters in the model are effective, so we get a construction of a model with the gauge group  $(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1))/\mathbb{Z}_6$  over a base that is not weak Fano, in such a way that there is no interaction with the non-Higgsable gauge groups.

In fact, to maintain the separation of the  $(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1))/\mathbb{Z}_6$  part of the theory from the non-Higgsable factors, it is generally necessary to ensure that  $b_3$  and  $b_2$  are disjoint from the non-Higgsable clusters, but  $\beta$  does not need to be fixed to be in a class that is a multiple of  $\tilde{S}$  to have a good construction. The presence of a  $\text{U}(1)$  factor can be compatible in certain cases with a non-Higgsable gauge factor without additional jointly charged matter even if the  $\text{U}(1)$  is related through unHiggsing to an  $\text{SU}(2)$  factor that intersects the divisor supporting the non-Higgsable factor; one such construction is possible when  $[b_3] = [b_2] = \tilde{S}$  and  $[\beta] = S + 2F$ , for example. We give an explicit example of this kind of situation below for a Class (B) model, where the only decoupled  $(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1))/\mathbb{Z}_6$  models involve this kind of structure; the analysis of Class (A) models with  $[\beta]$  not an integer multiple of  $\tilde{S}$  proceeds similarly.<sup>14</sup> We thus see that there are models realizing the gauge group

<sup>14</sup>Note that in these situations the anomaly coefficient  $\tilde{b}$  of the associated  $\text{U}(1)$  factor, however, must still be orthogonal to the divisors supporting non-Higgsable gauge factors for the spectrum to satisfy anomaly cancellation; this anomaly coefficient is shifted from the expected value on a weak Fano base by a (possibly fractional) multiple of the divisors in the non-Higgsable clusters.

$(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1))/\mathbb{Z}_6$  without exotic matter even on the Hirzebruch surface  $\mathbb{F}_3$ , and that there can be multiple models even for a fixed choice of  $[b_3], [b_2]$  that are disjoint from all non-Higgsable clusters. A similar result holds for other bases with non-Higgsable clusters, although as mentioned above the room for such tunings becomes increasingly constricted as  $h^{1,1}(X)$  increases. Thus, on a general base we expect that the Class (A) models are parameterized by two divisors that are disjoint from all non-Higgsable clusters and one divisor that satisfies weaker constraints. In general, the condition that a divisor is disjoint from all non-Higgsable clusters becomes increasingly stringent as  $h^{1,1}$  increases; we leave a systematic analysis of the conditions under which such divisors can be used to construct models with decoupled  $(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1))/\mathbb{Z}_6$  gauge group for further investigation.

Now let us consider the models of Class (B). In this situation, things are somewhat different. In particular, note that the divisor classes  $b_3, b_2, \eta = -4K_B - 2b_3 - b_2$  that support the factors of the unHiggsed Pati-Salam group  $(\text{SU}(4) \times \text{SU}(2) \times \text{SU}(2))/\mathbb{Z}_2$  satisfy  $2b_3 + b_2 + \eta = -4K_B$ . Every surface that is not weak Fano, however, contains some curve  $C$  that has self-intersection  $C \cdot C \leq -3$  and supports a non-Higgsable gauge group. The presence of the non-Higgsable gauge factor can be associated [19] with the fact that  $-K_B \cdot C < 0$ , so that  $-K_B$  is reducible and contains  $C$  as a component. From this we see that for a Class (B) model, one of the divisor classes  $b_3, b_2, \eta$  must satisfy  $D \cdot C < 0$  and hence must contain  $C$  as a component; furthermore, after removing this component (with possible multiplicity), the remainder of the sum of divisors  $2b_3 + b_2 + \eta - nC$  must have positive intersection with  $C$  (since  $(K_B + C) \cdot C = -2$  for any effective  $C$  in the base of self-intersection  $C \cdot C < -2$ ). If the self-intersection of  $C$  is sufficiently negative, this can lead to a singularity that goes beyond the Kodaira classification. For example, if  $C \cdot C = -12$ , then we cannot have  $b_3 \cdot C > 0$  or there would be a codimension two (4, 6) point. Even if the model is allowed, the resulting construction ends up having two properties: first, the gauge group on the non-Higgsable cluster is increased through the component of  $C$  lying in  $b_3, b_2$ . Second, there is exotic matter in the Pati-Salam enhanced model that is charged under the non-Higgsable gauge group and the  $(\text{SU}(4) \times \text{SU}(2) \times \text{SU}(2))/\mathbb{Z}_2$  factor(s) associated with the divisor(s) that intersect  $C$ . Despite this, as long as  $b_3$  and  $b_2$  are disjoint from all divisors supporting non-Higgsable factors, it can be possible to tune the Weierstrass model for  $(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1))/\mathbb{Z}_6$  without exotic matter or codimension two (4, 6) points.

As an example of tuning a Class (B) model over a base with non-Higgsable gauge factors, let us consider again the case of the base  $\mathbb{F}_3$ . We then have  $2b_3 + b_2 + \eta = -4K_B$ . We cannot choose  $b_3, b_2, \eta$  to all be divisors that do not intersect  $S$ , since  $-K_B \cdot S = -1$ . We could for example try to choose  $b_3 = b_2 = \eta = -K_B$  (the “SU(5)” case), but then we see that each of these divisors contains a reducible component of  $S$ . On the one hand, this means that the discriminant now vanishes to higher order on  $S$ , increasing the non-Higgsable gauge group factor on this divisor to a larger group. On the other hand, we have  $(b_3 - S) \cdot S = 2$ , so that the gauge factor SU(3) lies on a divisor intersecting the non-Higgsable gauge factor, so there is matter jointly charged under the SU(3) factor and the (enhanced) non-Higgsable gauge factor. A similar story holds for the SU(2) factor. So this corresponds to a situation where it is possible to tune the gauge group  $(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1))/\mathbb{Z}_6$  with generic matter, but there are exotic charged matter fields coupled under this gauge group and an enhanced non-Higgsable gauge factor.



There are nonetheless a small set of Class (B) models, however, that can be tuned on the base  $\mathbb{F}_3$  to get a gauge algebra  $\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1) \oplus \mathfrak{su}(3)$  with no matter jointly charged under the  $\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$  and  $\mathfrak{su}(3)$ . If we choose the divisors  $b_3 = n_3 \tilde{S}, b_2 = n_2 \tilde{S}$ , with  $n_3, n_2 \in \mathbb{Z}$ , the  $\mathfrak{su}(3)$  and  $\mathfrak{su}(2)$  factors of  $\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$  will be disjoint from the  $-3$  curve  $S$  supporting the non-Higgsable  $SU(3)$  factor. The Class (B) condition  $Y = 0$  then imposes the condition  $\eta = -4K_B - (2n_3 + n_2)\tilde{S}$ , which is effective when  $2n_3 + n_2 \leq 6$ . Let us consider the simplest case, where  $n_3 = n_2 = 1$ , so  $\eta = 5S + 11F$ . We can then use table 3 to determine the classes of the Weierstrass parameters:  $[b_1] = [d_0] = \tilde{S}; [s_1] = -F$ , so  $s_1 \rightarrow 0; [s_2] = 0$  as usual for Class (B) models;  $[d_1] = 3S + 7F$ , so  $[d_1] \cdot S = -2$  and  $[d_1]$  therefore contains a factor of  $S$  as a component; similarly,  $[d_2]$  contains two factors of  $S$  and  $s_5, s_6, s_8$  each contain a factor of  $S$ . From this we can read off the order of vanishing of  $(f, g)$  in the Weierstrass model (3.1) on the locus  $S$  to be  $(2, 2)$ , associated with the non-Higgsable  $SU(3)$  factor, which we see is not enhanced. Because there is no matter charged under this gauge factor, indeed this gives a construction of a Class (B) model on the base  $\mathbb{F}_3$  with gauge group  $\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1) \oplus \mathfrak{su}(3)$  with no exotic matter.<sup>15</sup> Similar constructions seem to work for  $n_3 = 1, n_2 \leq 4$  and for  $n_3 = 2, n_2 \leq 2$ , so there is no further constraint on  $b_3, b_2$ . More generally, we expect that on any base with non-Higgsable clusters the Class (B) models without exotic extra matter charged under the  $(SU(3) \times SU(2) \times U(1))/\mathbb{Z}_6$  group will be parameterized by two divisors  $(b_3, b_2)$  that are constrained to be disjoint from all non-Higgsable clusters. The parameters  $b_3, b_2$  may be subject to further constraints that we have not tried to analyze generally here associated with the intersection of  $\eta$  with curves supporting the non-Higgsable clusters, although there are no such further constraints in the  $\mathbb{F}_3$  case we have analyzed explicitly here.

We thus see that it is possible to construct generic Weierstrass models with the tuned  $(SU(3) \times SU(2) \times U(1))/\mathbb{Z}_6$  gauge group using eq. (3.1) even over base surfaces that are not weak Fano. Such models are quite constrained, particularly for Class (B) models, and are not possible for the “SU(5)” case, but may give a broad class of constructions of tuned standard model-like theories over bases with additional non-Higgsable gauge factors that could play the role of hidden sector dark matter. We leave a more comprehensive analysis of this possibility to future work.

## 11 Conclusions

In this paper we have given an explicit Weierstrass formulation for a general F-theory model with gauge group  $(SU(3) \times SU(2) \times U(1))/\mathbb{Z}_6$  and generic matter content. Matching with the results of [15], where these models were analyzed through less direct means, we find that this Weierstrass model naturally describes two distinct subclasses of models, one that is parameterized by three independent divisors in the base and one parameterized by two independent divisors. The first class of models gives a larger range of possibilities for tuning the standard model gauge group over a fixed base. The second class of models,

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<sup>15</sup>We have not gone into the details of the global structure of the gauge group here; as mentioned above, the anomaly coefficient of the  $U(1)$  factor must be shifted to be proportional to  $\tilde{S}$  to satisfy anomaly cancellation, suggesting  $\mathbb{Z}_3$  torsion involving the non-Higgsable  $SU(3)$ .

on the other hand, automatically restricts to only the matter representations realized in the standard model. The construction presented here is valid in the context of 4D  $\mathcal{N} = 1$  supergravity theories as well as for 6D  $\mathcal{N} = (1, 0)$  supergravity. We believe that the models described here give the most general way of constructing F-theory models with the standard model gauge group and generic associated matter in a way that arises from a tuning of the geometry and does not come from a unified gauge group broken by fluxes or incorporate supersymmetrically non-Higgsable gauge group components. These constructions present a broad generalization of the class of models recently considered in [3], and provide an interesting playground for considering a broad class of standard model realizations in the context of supersymmetric F-theory compactifications.

There are a number of obvious directions in which this work could be extended. It is natural to try to analyze more detailed aspects of the phenomenology of these models, starting with the fluxes and chiral matter content, for which purpose the explicit Weierstrass formulation given here should be a useful tool. The most general class of models produced by this construction can naturally produce some specific types of massive exotic matter, in particular particles with  $U(1)$  charge  $q = 2$  that are uncharged under the  $SU(3)$  and  $SU(2)$  factors,  $SU(2)$  doublets with  $U(1)_Y$  charge  $3/2$ , and right-handed quarks with  $U(1)_Y$  charge  $-4/3$ ; these and the  $SU(4) \times SU(3) \times SU(2)$  unHiggsing provide some potentially interesting new phenomenological features. It would also be interesting to explore in more detail what subset of the large range of twofold and threefold bases that support elliptic Calabi-Yau threefolds and fourfolds are compatible with this construction, and the relative frequency of such constructions in the context of 4D flux vacua. We have also identified some apparent “swampland” models in 6D where theories compatible with anomaly cancellation conditions do not arise through the general construction presented here; it would be interesting to try to identify further quantum consistency constraints ruling out these models or find alternative string constructions. On a more theoretical axis, this model represents an explicit Weierstrass realization of a rather complicated gauge group structure with nonabelian and abelian factors and a discrete quotient. At present there is no general framework available for constructing Weierstrass models with arbitrary such gauge groups; we found the model in this paper by a somewhat serendipitous happenstance. The existence of this model suggests that there may be a more general approach that could give insight into such constructions.

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