SOME RESULTS ON NONLINEAR OPTIMAL CONTROL

by

Jinghao Zhu

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APPROVED:

David L. Russell

David L. Russell, Chairman

Robert C. Rogers

James E. Thomson

Robert F. Olin

Robert Wheeler

Robert A. McCoy

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Committee Chairman: David L. Russell

Department of Mathematics

(Abstract)

This thesis consists of two individual mathematical papers which have been developed in the course of the author's thesis work. They deal with certain aspects of optimal control of systems in which the system equations are nonlinear, the cost integrand is non- quadratic, or both.

The first paper deals an extension from the linear-quadratic case to systems as just described of the so called Newton-Kleinman method. Here we carry out this extension theoretically and prove that the associated sequence of stabilizing feedback controls converges uniformly to the optimal control.

In the second chapter of this work we generalize the existence and uniqueness theory for the nonlinear-nonquadratic optimal control problem from the critical point and periodic cases studied earlier by Lukes and Zhang, respectively, to the case where the invariant target set is a compact submanifold of the state space.

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Chapter 1

Introduction

We consider certain optimal control problems in which for a given dynamic system

$$\dot{x}(t) = F(x(t), u(t)), \quad x(0) = x_0 \in R^n, \tag{1}$$

it is required to minimize the cost functional (function of trajectories)

$$V(x,u) = \int_0^\infty G(x(t), u(t))dt. \tag{2}$$

By analogy with the method of Lagrange multipliers, one may approach the minimization of $V(x,u)$ subject to the system (1) by formally stating the unconstrained problem of finding critical points of

$$V(x,u) + \langle p, \dot{x} - F(x,u) \rangle \tag{3}$$

seen as a functional in some infinite dimensional space. Here p denotes an element of the dual space (itself a function) that plays the role of a multiplier or dual state. Suitably modified and made rigorous, this approach leads to the necessary condition–Pontryagin's Maximum Principle (PMP) [3] and is related to the Hamiltonian formulation of classical mechanics.

When $F(x,u), G(x,u)$ take the linear-quadratic form, i.e. in (1)

$$F(x,u) = Ax + Bu \tag{4}$$

with A, B being a stabilizable pair and

$$G(x,u) = (x^*, u^*) \begin{pmatrix} W & R \\ R^* & U \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \tag{5}$$

with $\begin{pmatrix} W & R \\ R^* & U \end{pmatrix} > 0,\ U > 0$, D. Lukes proved in [2] the existence and uniqueness of the optimal control (1), (2). In this linear-quadratic case, the PMP necessary condition is also a sufficient condition. He obtained the explicit formula for the optimal control

$$u = -U^{-1}(R^* + B^*Q)x \tag{6}$$

in feedback form with Q being the positive definite solution of the Kalman-Lyapunov matrix equation.

Later, in [1], D.Lukes further considered extending the method to nonlinear-nonquadratic optimal control problem near a critical point. He kept the linear part unchanged in (1) but added to it a higher order part $f(x, u)$. Also the integrand in the cost (2) maintains its quadratic part while a higher order part $g(x, u)$ is added. He proved the optimal control near the origin for this extended problem takes a feedback form as well but having a higher order term:

$$u = -U^{-1}(R^* + B^*Q)x + h(x). \tag{7}$$

We see that the linear part of (7) just coincides with (6). Lukes method is based on power series solution of the Hamilton- Jacobi equation.

At the same time, there is a useful approach to the optimal control (1),(2),(6) called the Kleinmann-Newton method (cf.[6],[4]). It obtains the positive definite matrix Q in (6) as the limit of a sequence of positive matrices Q_j, i.e.

$$Q = \lim_{j\to\infty} Q_j, \tag{8}$$

with Q_j being the solution of the matrix equation:

$$(A + BK_j)^*Q_j + Q_j(A + BK_j) + (I, K_j^*) \begin{pmatrix} W & R \\ R^* & U \end{pmatrix} \begin{pmatrix} I \\ K_j \end{pmatrix} = 0, \tag{9}$$

where

$$K_j = -U^{-1}(R^* + B^*Q_{j-1}). \tag{10}$$

The most recent significant work is due to D.Russell and X.Zhang(cf.[5]). They considered the critical point being replaced by an invariant periodic flow of the uncontrolled vector field.

In chapter 2 of this work, we try to work with nonlinear-nonquadratic optimal control without periodic conditions. We consider there the real analytic nonlinear vector field $F(x,u)$ and a compact analytic manifold M detemined by

$$\{x : x \in R^n, \varphi(x) = 0\}, \tag{11}$$

which is invariant for the uncontrolled vector field $F(x,0)$ and exponentially orbitally stable for the corresponding system, by which we mean there is a twice continuously differentiable feedback $k_0(x)$ such that for given x_0,

$$d(x_{k_0}(t,x_0), M) \leq m_0 e^{-\beta_0 t}\|\varphi(x_0)\|, \ \ t \geq 0, \tag{12}$$

where $x_{k_0}(t,x_0)$ is the solution of

$$\dot{x} = F(x, k_0(x)), \ \ x(0) = x_0,$$

and $k_0(x) \equiv 0, \ \ x \in M$.

Clearly, the condition (12) coincides with that in the case of nonlinear- nonquadratic optimal control problem near a critical point.

Our optimal control problem can be stated explicitly as follows:

$$\min_{u \in U}\{\int_0^\infty G(x,u)dt\} \tag{13}$$

where $x(t)$ is subject to

$$\dot{x} = F(x,u), \ \ x(0) = x_0,$$

with x_0 near M,where $G(x,u)$ is real analytic and $G(x,u) > 0, (x,u) \notin M \times \{0\}; G(x,0) = 0, x \in M$.

The main result is as follows.

Theorem 1. The nonlinear optimal control problem described above has a unique solution expressed by a feedback function $K(x)$ which vanishes on the compact manifold M concerned and is continuously differentiable in a neighborhood of M. The synthesized system

$$\dot{x} = F(x, K(x))$$

obtained with the use of this control has M as a uniformly orbitally stable invariant set.

On the other hand, in Chapter 1 of this work we prove that the Kleinmann-Newton method is still valid for the nonlinear-nonquadratic optimal control problem near a critical point. Indeed, we can write the Kleinman-Newton process as follows: Given any stabilizing feedback control $u = K_0(x) = Dx + h_0(x)$ with $A + BD$ being stable and $h_0(0) = 0, \frac{\partial h_0(0)}{\partial x} = 0$. Then for $j = 1, 2, ...$ by the process

$$V_{K_{j-1}}(x) = \int_0^\infty G(\hat{x}(t,x), K_{j-1}(\hat{x}(t,x))dt, \tag{14}$$

with $\dot{\hat{x}} = F(\hat{x}, K_{j-1}(\hat{x})), \hat{x}(0) = x$, and

$$\frac{\partial V_{K_{j-1}}(x)}{\partial x}\frac{\partial F(x, K_j(x))}{\partial u} + \frac{\partial G(x, K_j(x))}{\partial u} = 0, \tag{15}$$

where $F(x,u) = Ax + Bu + f(x,u)$, $G(x,u) = (x^*, u^*)\begin{pmatrix} W & R \\ R^* & U \end{pmatrix}\begin{pmatrix} x \\ u \end{pmatrix} + g(x,u)$, we can in turn get the stabilizing feedback control $K_j(x), j = 1, 2, ...$

We have the following result.

Theorem 2. For the nonlinear-nonquadratic optimal control problem near the critical point, given any stabilizing feedback control $u = K_0(x)$, the sequence beginning from $K_0(x)$ of feedback controls $K_j(x), j = 1, 2, ...$ created by the process (14),(15) will be uniformly convergent in $\{x : ||x|| < \alpha\}$ for some positive constant α.

References

1. Lukes, D.L.: *Optimal regulation of dynamical systems*, SIAM J.Control 7 (1969), pp.75-100

2. Lukes, D.L.: *Stabilizability and optimal control*, Funkcialaj Ekvacioj, 11(1968), pp.39-50

3. Pontryagin, L.S., V.G.Boltyanskii, R.V.Gamkrelidze, and E.F.Mischenko,: *The Mathematical Theory of Optimal Processes*, Wiley, New York, 1962.

4. Russell, D.L.: *Mathematics of Finite Dimensional Control System*, Marcel Dekker, Inc., New York, 1979

5. Russell, D.L.and Zhang,X.: *Extending linear-quadratic optimal control laws to nonlinear systems and/or nonquadratic cost criteria*, 1993

6. Russell, D.L.: *On the positive square root of the fourth derivative operator*, Quart.Appl. Math. Volume XLVI, Number 4 December 1988, pp.751-773

7. Kleinman, D.L.: *On an iterative technique for Riccati equation computations*, IEEE Trans.Auto. Control, AC-13(1968), pp.114-115.

Chapter 2

Extending the Kleinman-Newton Approach to Optimal Control of Nonlinear/Nonquadratic System

In [1] D.L.Lukes proved the existence and uniqueness of the optimal control, in nonlinear feedback form, for the nonlinear-nonquadratic problem near a critical point of the system under study. Here we prove that the feedback control sequence created by application of the Kleinman-Newton approach[4], beginning from a globally stabilizing feedback control, converges to the optimal control described in Lukes paper.

2.1 Definition of the Optimal Control Problem

We are going to deal with the optimal control of certain autonomous systems of nonlinear differential equations

$$\dot{x} = F(x, u). \tag{2.1}$$

Throughout the paper we assume $F(x, u)$ is real analytic in some neighborhood of the origin in R^{n+m}. In particular, then, we may write

$$F(x, u) = Ax + Bu + f(x, u) \tag{2.2}$$

where A and B are $n \times n$ and $n \times m$ matrices, (A, B) is a stabilizable pair [1] and the higher order part, $f(x, u)$, satisfies

$$f(0,0) = 0, \quad \frac{\partial f(0,0)}{\partial(x, u)} = 0.$$

We assume the cost functional for the optimal control problem to have the form

$$\int_0^\infty G(x, u)dt, \tag{2.3}$$

further assuming $G(x,u)$ is also real analytic near the origin and

$$G(x,u) = (x^*, u^*) \begin{pmatrix} W, & R \\ R^*, & U \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} + g(x,u), \tag{2.4}$$

with

$$\begin{pmatrix} W, & R \\ R^*, & U \end{pmatrix} > 0, \quad U > 0, \tag{2.5}$$

and

$$g(0,0) = 0, \quad \frac{\partial g(0,0)}{\partial(x,u)} = 0, \quad \frac{\partial^2 g(0,0)}{\partial(x,u)^2} = 0.$$

The optimal control problem is

$$min_{u \in U_\infty} \int_0^\infty G(x(t), u(t))dt$$

subject to the constraints that x and u should together satisfy

$$\dot{x} = F(x,u), \quad x(0) = x_0$$

for an initial state x_0 near the origin in R^n.

In [1],D.L.Lukes proved the existence and uniqueness of the optimal control, in closed loop form, for this problem. Here we show this optimal control can be realized as the uniform limit of a sequence of stabilizing controls generated by the so called Kleinman-Newton process (see [2],pp.284-289) beginning from any globally stabilizing feedback control.

2.2 The Kleinman-Newton Method

We know from [2] for the linear-quadratic problem, i.e. for the case in which

$$F(x,u) = Ax + Bu, \tag{2.6}$$

$$G(x,u) = (x^*, u^*) \begin{pmatrix} W, & R \\ R^*, & U \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}, \tag{2.7}$$

that any given feedback control

$$K_0(x) = Dx \tag{2.8}$$

such that $A+BD$ is stabilty matrix, will give rise to a sequence of positive definite matrices $Q_0, Q_1, ..., Q_j, ...$ through solution of a sequence of Liapounov matrix equations([2],pp. 284) with an associated sequence of linear feedback controls related to the Q_j by

$$K_j(x) = -U^{-1}(R^* + B^*Q_{j-1})x. \;\; j = 1, 2, ... \tag{2.9}$$

This process is called the Kleinman-Newton method [2].

We see from [2] that there is a positive definite symmetric matrix Q such that

$$Q = \lim_{j \to \infty} Q_j, \tag{2.10}$$

such that the matrix differences $Q_j - Q$ satisfy a condition of quadratic convergence:

$$\|Q_j - Q\| \leq M\|Q_{j-1} - Q\|^2, \tag{2.11}$$

where M is a positive constant. We also see from that work that the linear feedback control generated by

$$\hat{u} = -U^{-1}(R^* + B^*Q)x \tag{2.12}$$

is the unique linear feedback control providing the solution for the linear quadratic control problem corresponding to the linear system (2.6) and the quadratic cost integrand (2.7). If we define for $G(x,u)$ in (2.7)

$$V_j(x) = \int_0^\infty G(x, K_j(x))dt,$$

we know from [2] that $V_j(x) = x^* Q_j x$ and we see that $K_{j+1}(x)$ in (2.9) can be obtained by the following equation

$$\frac{\partial V_j(x)}{\partial x} B + 2(K_{j+1}(x))^* U + 2x^* R = 0,$$

i.e.

$$\frac{\partial V_j(x)}{\partial x} \frac{\partial (Ax + BK_{j+1}(x))}{\partial u} + \frac{\partial G(x, K_{j+1}(x))}{\partial u} = 0.$$

Returning now to the nonlinear-nonquadratic problem which is our topic here, for an arbitrary globally stabilizing feedback relation $u = K(x)$ we define the associated cost function

$$V_K(x) = \int_0^\infty (\hat{x}^*, K^*(\hat{x})) \begin{pmatrix} W, & R \\ R^*, & U \end{pmatrix} \begin{pmatrix} \hat{x}, \\ K(\hat{x}) \end{pmatrix} + g(\hat{x}, K(\hat{x}))dt \tag{2.13}$$

where $\hat{x}$ is the corresponding state trajectory satisfying

$$\dot{\hat{x}} = F(\hat{x}, K(\hat{x})), \quad \hat{x}(0) = x. \tag{2.14}$$

Following the Hamilton-Jacobi-Bellman paradigm, it is then possible to define another feedback $\tilde{K}(x)$ as the (unique under appropriate hypotheses) solution of

$$\frac{\partial V_K}{\partial x} \frac{\partial F(x, \tilde{K}(x))}{\partial u} + \frac{\partial G(x, \tilde{K}(x))}{\partial u} = 0,$$

and, from that $\tilde{K}(x)$ to construct another cost functional $V_{\tilde{K}}$ as in (2.13), replacing $K(x)$ by $\tilde{K}(x)$. The Kleinman-Newton process consists in constructing, via this pattern, a sequence of stabilizing controls

$$K_j(x), j = 1, 2, ...,$$

via

$$\frac{\partial V_{K_{j-1}}}{\partial x} \frac{\partial F(x, K_j(x))}{\partial u} + \frac{\partial G(x, K_j(x))}{\partial u} = 0. \tag{2.15}$$

Our purpose in this paper is to establish the viablity of this approach for approximating the optimal control function $\hat{K}(x)$ as studied by Lukes in [1].

Theorem 2.1. For the nonlinear-nonquadratic problem described above, given any globally stabilizing feedback control

$$u = K_0(x), \quad K_0(x) = Dx + h_0(x)$$

with $A + BD$ being stable and

$$h_0(0) = 0, \quad \frac{\partial h_0(0)}{\partial x} = 0,$$

the sequence beginning with $K_0(x)$) of feedback control functions $K_j(x), j = 1, 2, ...$ created by the process

$$V_{K_{j-1}}(x) = \int_0^\infty G(\hat{x}(t,x), K_{j-1}(\hat{x}(t,x))dt,$$
$$\dot{\hat{x}} = F(\hat{x}, K_{j-1}(\hat{x})), \hat{x}(0) = x,$$
$$\frac{\partial V_{K_{j-1}}}{\partial x}\frac{\partial F(x, K_j(x))}{\partial u} + \frac{\partial G(x, K_j(x))}{\partial u} = 0 \tag{2.16}$$

converges uniformly near the origin, i.e. there exists a positive real number α such that $\{K_j(x)\}$ uniformly converges in $\{x \mid \|x\| < \alpha\}$.

2.3 Background

We summarize the results from [1] (see pp.79-82 in [1]) in the following three lemmas.

Lemma 3.1. If $\{Q_j\}$ is described in section 2 above,and $\{K_j(x)\}$ is as in Theorem 2.1, we have

$$V_{K_j}(x) = x^* Q_j x + H_j(x), (j = 1, 2, ...), \tag{2.17}$$

$$K_j(x) = \frac{-1}{2}U^{-1}(2R^*x + B^*(\frac{\partial V_{K_{j-1}}}{\partial x})^*) + h(x, \frac{\partial V_{K_{j-1}}}{\partial x})$$
$$= -U^{-1}(R^* + B^*Q_{j-1})x + h_j(x), \tag{2.18}$$

where

$$H_j(0) = 0, \frac{\partial H_j(0)}{\partial x} = 0,$$

$$\frac{\partial^2 H_j(0)}{\partial x^2} = 0;$$

$$h(0,0) = 0, \frac{\partial h(0,0)}{\partial(x,u)} = 0;$$

$$h_j(0) = 0, \frac{\partial h_j(0)}{\partial x} = 0.$$

Lemma 3.2. For each $K_j(x)$ as described in lemma 3.1, we have

$$\frac{\partial V_{Kj}}{\partial x} F(x, K_j(x)) + G(x, K_j(x)) = 0. \tag{2.19}$$

Lemma 3.3. For $j = 1, 2, ...,$ let

$$-U^{-1}(R^* + B^* Q_{j-1})$$

be denoted by

$$\hat{Q}_j.$$

Then we have

$$\frac{\partial(x^* Q_j x)}{\partial x}(Ax + B\hat{Q}_j x) + (x^*, (\hat{Q}_j x)^*) \begin{pmatrix} W, & R \\ R^*, & U \end{pmatrix} \begin{pmatrix} x, \\ \hat{Q}_j x \end{pmatrix} = 0. \tag{2.20}$$

2.4 Adaptation of the Hamilton-Jacobi Equation

Let $K(x)$ be one of the feedback functions in the control sequence determined by the Kleinman-Newton method (cf.(2.15)). From Lemma 3.1-3.3,we see that

$$\frac{\partial H_K(x)}{\partial x} F(x, K(x)) + 2(Q_K x)^*(f(x, K(x)) + Bh_K(x)) + g_1(x, Q_K x, h_K(x)) = 0, \tag{2.21}$$

where

$$g_1(x, Q_K x, h_K(x)) = (x^*, K^*(x)) \begin{pmatrix} W, & R \\ R^*, & U \end{pmatrix} \begin{pmatrix} x \\ K(x) \end{pmatrix}$$

$$-(x^*, (\hat{Q}_K x)^*) \begin{pmatrix} W, & R \\ R^*, & U \end{pmatrix} \begin{pmatrix} x, \\ \hat{Q}_K x \end{pmatrix}$$
$$+ g(x, K(x)), \tag{2.22}$$

where

$$K(x) = \hat{Q}_K x + h_K(x).$$

Let $\frac{\partial H_K(x)}{\partial x}$ be denoted by $P_K(x)$. Taking the second derivatives in (2.21), we obtain

$$F^* \frac{\partial P_K^*}{\partial x} + P_K \frac{\partial F}{\partial x} + \frac{\partial g_2}{\partial x} = 0, \tag{2.23}$$

where

$$g_2(x) = 2(Q_K x)^*(f(x, K(x)) + Bh_K(x)) + g_1(x, Q_K x, h_K(x)). \tag{2.24}$$

Since

$$g_2(0) = 0, \frac{\partial g_2(0)}{\partial x} = 0, \frac{\partial^2 g_2(0)}{\partial x^2} = 0, \tag{2.25}$$

we can show for each $K(x)$ that there is a unique $P_K(x)$ satisfying (2.23) with

$$P_K(0) = 0, \quad \frac{\partial P_K^*(0)}{\partial x} = 0. \tag{2.26}$$

Indeed, we propose to adopt the method in [1](pp.89-90) to establish the unique solution P_k of (2.23) ,(2.26) as follows.

For each solution $\hat{x}(s, x)$ of

$$\dot{\hat{x}} = F(\hat{x}, K(\hat{x})) = (A + B\hat{Q}_K)\hat{x} + Bh_K(\hat{x}) + f(\hat{x}, K(\hat{x})), \tag{2.27}$$
$$\hat{x}(0) = x,$$

where $||x||$ is sufficiently small, we see, that in association with the differential system corresponding to (2.27),(2.23) the following relationship holds

$$\begin{pmatrix} \frac{d\hat{x}(t)}{dt} \\ \frac{dP_K^*(\hat{x}(t))}{dt} \end{pmatrix} = \begin{pmatrix} (A + B\hat{Q}_K)\hat{x}(t) \\ -(A + B\hat{Q}_K)^* P_K^*(\hat{x}(t)) \end{pmatrix} + r(t, P_K^*(\hat{x}(t)), \tag{2.28}$$

with

$$r(t,p) = \begin{pmatrix} Bh_K(\hat{x}(t)) + f(\hat{x}(t), K(\hat{x}(t))) \\ (-B\frac{\partial h_K(\hat{x}(t))}{\partial x})^* p - (\frac{\partial f(\hat{x}(t), K(\hat{x}(t)))}{\partial x})^* p - \frac{\partial g_2^*(\hat{x}(t))}{\partial x} \end{pmatrix}. \tag{2.29}$$

Let

$$U_1(t) = \begin{pmatrix} e^{t(A+B\hat{Q}_K)}, & 0 \\ 0, & 0 \end{pmatrix}$$

$$U_2(t) = \begin{pmatrix} 0, & 0 \\ 0, & e^{-t(A+B\hat{Q}_K)^*} \end{pmatrix}.$$

These satisfy

$$||U_1(t)|| \leq \beta e^{-(\mu+\sigma)t}, (t \geq 0),$$

$$||U_2(t)|| \leq \beta e^{\sigma t}, (t \leq 0),$$

for some positive constants β, μ, σ.

Since $||\hat{x}(t)||$ decays exponentially, and $\frac{\partial h_K(0)}{\partial x} = 0$, $\frac{\partial f(0,0)}{\partial(x,u)} = 0$, we see, for given ϵ such that $2\epsilon\beta/\sigma < \frac{1}{2}$, that there is a positive real number δ such that when $||x|| < \delta$, we have

$$||r(t,p) - r(t,q)|| < \epsilon||p - q|| \tag{2.30}$$

uniformly for $t \in [0, \infty)$.

Now consider the integral equation

$$\theta(t,x) = U_1(t)x + \int_0^t U_1(t-s)r(s,\theta(s,x))ds - \int_t^\infty U_2(t-s)r(s,\theta(s,x))ds. \tag{2.31}$$

Using successive approximations to solve the integral equation above, starting with the initial approximation $\theta_0(t,x) = 0$, we readily obtain

$$||\theta_{(l+1)}(t,x) - \theta_{(l)}(t,x)|| \leq \frac{\beta||x||}{2^l} e^{-\mu t},$$

leading to the solution θ of the integral equation.

Then, for every x, we can write

$$P_K^*(x) = -\int_0^\infty U_2(-s)r(s,\theta(s,x))ds. \qquad (2.32)$$

Next we are going to construct another partial differential system. We define

$$\tilde{F}(x,z,u) = F(x, -U^{-1}(R^*x + B^*Qz) + u) \qquad (2.33)$$

$$\tilde{f}(x,z,u) = f(x, -U^{-1}(R^*x + B^*Qz) + u), \qquad (2.34)$$

$$\tilde{g}(x,z,u) = g(x, -U^{-1}(R^*x + B^*Qz) + u), \qquad (2.35)$$

and

$$\hat{g}(x,z,u) = 2(Qz)^*[f(x, -U^{-1}(R^*x + B^*Qz) + u) + Bu] +$$

$$+(x^*, (-U^{-1}(R^*x + B^*Qz) + u)^*) \begin{pmatrix} W, & R \\ R^*, & U \end{pmatrix} \begin{pmatrix} x, \\ -U^{-1}(R^*x + B^*Qz) + u \end{pmatrix}$$

$$-(x^*, (-U^{-1}(R^*x + B^*Qz))^*) \begin{pmatrix} W, & R \\ R^*, & U \end{pmatrix} \begin{pmatrix} x, \\ -U^{-1}(R^*x + B^*Qz) \end{pmatrix} +$$

$$g(x, -U^{-1}(R^*x + B^*Qz) + u)$$

$$= u^*Uu + 2(Qz)^*\tilde{f}(x,z,u) + \tilde{g}(x,z,u). \qquad (2.36)$$

Now we pose the following partial differential system

$$\tilde{F}^*(x,z,u)\frac{\partial P^*(x,z,u)}{\partial(x,z,u)} + P(x,z,u)\frac{\partial \tilde{F}(x,z,u)}{\partial(x,z,u)} + \frac{\partial \hat{g}(x,z,u)}{\partial(x,z,u)} = 0. \qquad (2.37)$$

Let us explain the roles played by z and u above. The variable z repesents $Q^{-1}Q_K x$ for each $K(x)$ and u represents $h_K(x)$ for each $K(x)$. Through this partial differential equation system a common framework is established for all $K(x)$.

We claim that there is a solution of (2.37),$P(x, z, u)$, satisfying

$$\lim_{||x|| \to 0} P(x, Q^{-1}Q_j x, h_j(x)) = 0, \tag{2.38}$$

$$\lim_{||x|| \to 0} \frac{\partial P^*(x, Q^{-1}Q_j x, h_j(x))}{\partial(x, z, u)} = 0, \tag{2.39}$$

for all j. In fact, let us consider

$$P(x, z, u)\tilde{F}(x, z, u) + \int_{(0,0,0)}^{(x,z,u)} R_1 dx + R_2 dz + R_3 du = 0, \tag{2.40}$$

where

$$P(x, z, u) = (P_1, ..., P_n),$$

$$\tilde{F}(x, z, u) = \begin{pmatrix} \tilde{F}_1 \\ \vdots \\ \tilde{F}_n \end{pmatrix}$$

$$(R_1, R_2, R_3) = \frac{\partial \hat{g}(x, z, u)}{\partial(x, z, u)},$$

and

$$(dx)^* = (dx_1, ..., dx_n), (dz)^* = (dz_1, ..., dz_n), (du)^* = (du_1, ..., du_m).$$

We define for $j = 1, 2, ..., n$

$$P_j(x, z, u) = \frac{-\tilde{F}_j(x, z, u)}{\sum_{j=1}^n \tilde{F}_j^2(x, z, u)} \int_{(0,0,0)}^{(x,z,u)} R_1 dx + R_2 dz + R_3 du. \tag{2.41}$$

Then if we further define

$$P(x, z, u) = (P_1, ..., P_n),$$

we see, from lemma 5.1 in next section, that $P(x, z, u)$ will satisfy (2.37) and (2.38),(2.39) and for all j, $P(x, Q^{-1}Q_j x, h_j(x))$ is analytic near $x = 0$.

Then we see this $P(x,z,u)$ is a solution of (2.37) with

$$\lim_{\|x\|\to 0} P(x, Q^{-1}Q_j x, h_j(x)) = 0, \tag{2.42}$$

$$\lim_{\|x\|\to 0} \frac{\partial P^*(x, Q^{-1}Q_j x, h_j(x))}{\partial(x,z,u)} = 0, \tag{2.43}$$

for all j. Now if we let

$$z = Q^{-1}Q_K x,$$

$$u = h_K(x),$$

we see, from (2.42),(2.43) that

$$P(x, Q^{-1}Q_K x, h_K(x)) \mid_{x=0} = 0, \tag{2.44}$$

and

$$\frac{\partial P^*(x, Q^{-1}Q_K x, h_K(x))}{\partial x} \mid_{x=0} = 0 \tag{2.45}$$

wherein $\frac{\partial h_K(0)}{\partial x} = 0$.

Also it is easy to see when $z = Q^{-1}Q_K x, u = h_K(x)$,that $P(x,z,u)$ satisfies (2.37) which, in turn,implies that $P(x, Q^{-1}Q_K(x), h_K(x))$ satisfies (2.23). But the uniqueness of the solution of (2.23), with (2.26), gives

$$P(x, Q^{-1}Q_K x, h_K(x)) = P_K(x). \tag{2.46}$$

noting by Lemma 5.1 below as well that they are real analytic in a neighborhood of the origin dependent only on F, G, K_0.

2.5 A Bounding Lemma

We begin by stating

Lemma 5.1 Following the notation of Lemma 3.1, there exist positive numbers ϵ and α such that when $\|x\| < \epsilon$, we have

$$\|h_j(x)\| \leq \alpha \|x\| \tag{2.47}$$

uniformly for all $j = 1, 2, ...$

Proof: From lemma 3.1, and using (2.18), we have the basic relationship between $h_j(x)$ and $V_{j-1}(x)$,

$$h_j(x) = \frac{-1}{2}U^{-1}B^*(\frac{\partial H_{j-1}(x)}{\partial x})^* + h(x, \frac{\partial V_{j-1}}{\partial x}). \tag{2.48}$$

Since $\hat{Q}_{K_j} \longrightarrow \hat{Q}, (j \to \infty)$, by the stability analysis (see [3],pp.314-315), we see that there are positive numbers $\delta_1 < 1, \delta_2 < 1$ such that when $||x|| \le \delta_1$,and $||h(x)|| \le \delta_2||x||$, for all j, the solution $\hat{x}_j(t)$ of

$$\dot{\hat{x}} = (A + B\hat{Q}_j)\hat{x} + Bh(\hat{x}) + f(\hat{x}, h(\hat{x})),$$

$$\hat{x}(0) = x$$

will decay exponentially.

But this implies, when x satisfies $0 < ||x|| \le \delta_1$ and $||h(x)|| \le \delta_2||x||$, that for all j, then

$$F(x, \hat{Q}_j x + h(x)) \ne 0,$$

noting that $A + B\hat{Q}_j = A + B\hat{Q} + B(\hat{Q}_j - \hat{Q})$. Indeed, by reducing δ_2 if necessary, we can choose δ_1, δ_2 so small that

$$||F(x, \hat{Q}_j x + h(x))|| \ge c\sigma||x||, \tag{2.49}$$

where $\sigma(> 0)$ is the minimal singular value of $A + B\hat{Q}$ with c being an absolutely positive constant.

Now by (2.41), for each $i, 1 \le i \le n$, for all $j, j = 1, 2, ...$, and any $h(x)$ satisfying $||h(x)|| \le \delta_2||x||$(whenever $||x|| \le \delta_1$), we have

$$P_i(x, Q^{-1}Q_j x, h(x)) = \frac{-\tilde{F}_i(x, Q^{-1}Q_j x, h(x))}{\sum_{k=1}^n \tilde{F}_k^2(x, Q^{-1}Q_j x, h(x))} \int_{(0,0,0)}^{(x,Q^{-1}Q_j x,h(x))} R_1 dx + R_2 dz + R_3 du \tag{2.50}$$

well defined and real analytic when $||x|| < \delta_1$. We define

$$N = \bigcup_{j,h} \{(x, z, u) \mid z = Q^{-1}Q_j x, u = h(x); h(0) = 0, \frac{\partial h(0)}{\partial x} = 0; ||x|| < \delta_1, ||h(x)|| < \delta_2||x||\} \tag{2.51}$$

to be a union of open submanifolds in R^{2n+m}.

By (2.49), (2.50) and (2.36), with elementary calculus, noting that $||h(x)|| < \delta_2||x||$, we see that there is a $\nu > 0$ such that when $||x|| < \nu$ and $(x, z, u) \in N$

$$||P(x, z, u)|| \leq C_0||x||, \tag{2.52}$$

$$||\frac{\partial P(x, z, u)}{\partial(x, z)}|| \leq C_1(||x|| + \delta_2^2), \quad ||\frac{\partial P(x, z, u)}{\partial u}|| \leq C_2(||x|| + \delta_2), \tag{2.53}$$

where C_0, C_1, C_2 are absolutely positive constants and here we require

$$[C_2 + C_1(1 + \max_j ||Q^{-1}Q_j||)]\delta_2 < \frac{1}{8},$$

$$||U^{-1}B^*||[C_2 + C_1(1 + \max_j ||Q^{-1}Q_j||)]\delta_2 < \frac{1}{4},$$

by reducing δ_2 a little bit if necessary(noting C_1, C_2 only depending on F, G and not depending on δ_2).

Before going to the major part of the proof, another preparatory is needed as follows. Given j and $h(x)$ as above, for fixed $x_0(||x_0|| < \delta_1), z_0 = Q^{-1}Q_jx_0, u_0 = h(x_0)$, and for each $u, u = tu_0, 0 < t < 1$, we need construct a $\tilde{h}(x)$, such that $\tilde{h}(0) = 0, \frac{\partial \hat{h}(0)}{\partial x} = 0$, and $||\tilde{h}(x)|| < \delta_2||x||, \tilde{h}(x_0) = u$. To fulfill this task, we simply choose

$$\tilde{h}(x) = th(x),$$

since it is easy to see that

$$\tilde{h}(0) = th(0) = 0, \frac{\partial \tilde{h}(0)}{\partial x} = t\frac{\partial h(0)}{\partial x} = 0,$$

$$||\tilde{h}(x)|| = t||h(x)|| < t\delta_2||x|| < \delta_2||x||,$$

(whenever $||x|| < \delta_1$) and

$$\tilde{h}(x_0) = th(x_0) = tu_0 = u.$$

Therefore when $(x, z, u) \in \{\|x\| < \nu\} \cap N$, we have by (2.53)

$$\|P(x, z, u)\| \leq C_3(\|x\|^2 + \delta_2^2\|x\|) + C_2(\|x\| + \delta_2)\|u\|. \tag{2.54}$$

where $C_3 = C_1(1 + \max_j \|Q^{-1}Q_j\|)$. In fact, assuming $z = Q^{-1}Q_j x$, $u = h(x)$, we see, by using the construction of $\tilde{h}(x)$ noted above, that

$$\|P(x, z, u)\|$$

$$= \|P(x, z, u) - P(0, 0, 0)\|$$

$$\leq \|P(x, z, u) - P(x, z, 0)\| + \|P(x, z, 0) - P(0, 0, 0)\|$$

$$\leq (\sup_{N, \|x\| < \nu} \|\frac{\partial P(x, z, u)}{\partial u}\|)\|u\| + (\sup_{N, \|x\| < \nu} \|\frac{\partial P(x, z, 0)}{\partial (x, z)}\|)(\|x\| + \|z\|). \tag{2.55}$$

Now let us begin the main step of the proof which is an induction process. We require, by reducing ν if necessary,

$$\max_{(\|x\| \leq \nu, \|w\| \leq \nu)} \|\frac{\partial h(x, w)}{\partial (x, w)}\|(2 + 2\max_j\{\|Q_j\|\}) < \frac{1}{8}\delta_2.$$

Now we begin, starting from $h_0(x)$. For the positive δ_2 given above ,we can choose a positive $\delta_3, 0 < \delta_3 < \min\{\delta_1, \nu\}$, such that when $\|x\| \leq \delta_3$,

$$\|h_0(x)\| \leq \delta_2\|x\|.$$

We can also require δ_3 to satify

$$\delta_3\delta_1 < \delta_2, 3\delta_2\delta_3 < \nu; \tag{2.56}$$

$$\max_j\{\|Q^{-1}Q_j\|\}\delta_3 < \nu, \;\; (2\max_j\{\|Q_j\|\} + 1)\delta_3 < \nu, \tag{2.57}$$

and

$$\frac{1}{2}\|U^{-1}B^*\|((C_2 + C_3)\delta_3 + (C_2 + C_3)\delta_2^2) < \frac{1}{4}\delta_2, \tag{2.58}$$

$$(C_2 + C_3)\delta_3 + (C_2 + C_3)\delta_2^2 < 1, \tag{2.59}$$

noting the remark below (2.53).

We will establish, by mathematical induction on $j = 1, 2, ...$, that when $||x|| \leq \delta_3$, we have $||h_j(x)|| \leq \delta_2 ||x||$.

We now suppose, as an induction assumption, that when $||x|| \leq \delta_3$, we have $||h_j(x)|| \leq \delta_2 ||x||$. Then by (2.53),(2.54), we have when $||x|| \leq \delta_3$,

$$\begin{aligned} &||P(x, Q^{-1}Q_j x, h_j(x))|| \\ &\leq C_3(||x||^2 + \delta_2^2 ||x||) + C_2(||x|| + \delta_2)||h_j(x)|| \\ &\leq (C_3 + C_2)\delta_3 ||x|| + (C_3 + C_2)\delta_2^2 ||x|| \\ &= [(C_2 + C_3)\delta_3 + (C_2 + C_3)\delta_2^2]||x||. \end{aligned} \tag{2.60}$$

We denote by σ_1 the quantity $(C_2 + C_3)\delta_3 + (C_2 + C_3)(\delta_2)^2$. Then let us estimate $||h_{j+1}(x)||$, when $||x|| \leq \delta_3$. By (2.51) , (2.46),(2.44),(2.45) we deduce when $||x|| \leq \delta_3$ that

$$\frac{\partial H_j(x)}{\partial x} = P(x, Q^{-1}Q_j x, h_j(x)).$$

noting that the functions so obtained are real analytic. Then from (2.60) we have

$$\begin{aligned} &||\frac{\partial H_j(x)}{\partial x}|| \\ &\leq \sigma_1 ||x||. \end{aligned} \tag{2.61}$$

Thus by (2.48),(2.56)-(2.59) we have

$$\begin{aligned} &||h_{j+1}(x)|| \leq \frac{1}{2}||U^{-1}B^*||\sigma_1||x|| \\ &+(\max_{(||x||\leq\nu, ||w||\leq\nu)} ||\frac{\partial h(x,w)}{\partial x}||)||x|| + (\max_{(||x||\leq\nu, ||w||\leq\nu)} ||\frac{\partial h(x,w)}{\partial w}||)(2||Q_j|| ||x|| + \sigma_1 ||x||) \\ &\leq \frac{1}{2}\delta_2||x|| + \frac{1}{2}\delta_2||x|| = \delta_2||x||. \end{aligned} \tag{2.62}$$

Thus we obtain, when $||x|| \leq \delta_3$, the inequality $||h_{j+1}(x)|| \leq \delta_2 ||x||$ and we see that the Lemma has been proved by mathematical induction .

2.6 A Contraction Process

Here we complete the proof of Theorem 1.2.1.. In what follows we consider $||x||$ to be sufficiently small; for example, $||x|| < \delta_3$, where δ_3 is determined by Lemma 5.1 and that quantity may be reduced several times below. Let $K(x), K'(x)$ be two feedback controls in the sequence $\{K_j(x)\}$ created in the process (2.16).

We define

$$\Delta(K,K')(x) = ||\frac{\partial H_K(x)}{\partial x} - \frac{\partial H_{K'}(x)}{\partial x}|| + ||Q_K - Q|| + ||Q_{K'} - Q||. \tag{2.63}$$

Noting that

$$P_K(x) = \frac{\partial H_K(x)}{\partial x}, P_{K'}(x) = \frac{\partial H_{K'}(x)}{\partial x},$$

we see that

$$\Delta(K,K')(x) = ||P_K(x) - P_{K'}(x)|| + ||Q_K - Q|| + ||Q_{K'} - Q||. \tag{2.64}$$

For integer $p, q, p > q > 0$, by (2.46) (2.63) we have

$$\Delta(K_{p+1}, K_{q+1})(x) = ||P(x, Q^{-1}Q_{p+1}x, h_{p+1}(x)) - P(x, Q^{-1}Q_{q+1}x, h_{q+1}(x))||$$

$$+ ||Q_{p+1} - Q|| + ||Q_{q+1} - Q||. \tag{2.65}$$

But, also using Lemma 5.1, especially (2.53),and (2.11), we have

$$||P(x, Q^{-1}Q_{p+1}x, h_{p+1}(x)) - P(x, Q^{-1}Q_{q+1}x, h_{q+1}(x)||$$

$$\leq ||P(x, Q^{-1}Q_{p+1}x, h_{p+1}(x)) - P(x, Q^{-1}Q_{q+1}x, h_{p+1}(x))||+$$

$$+||P(x, Q^{-1}Q_{q+1}x, h_{p+1}(x)) - P(x, Q^{-1}Q_{q+1}x, h_{q+1}(x))||$$

$$\leq \sigma_1(x)||Q_{p+1} - Q_{q+1}|| + \sigma_2||h_{p+1}(x) - h_{q+1}(x)||$$

$$= \sigma_1(x)||Q_{p+1} - Q_{q+1}|| + \sigma_2||(K_{p+1}(x) - \hat{Q}_p x) - (K_{q+1}(x) - \hat{Q}_q x)||$$

$$\leq (\sigma_1(x))(||Q_{p+1} - Q_{q+1}||) + \sigma_2(||K_{p+1}(x) - K_{q+1}(x)||) + \sigma_3(x)||Q_p - Q_q||$$

$$\leq (\sigma_1(x))(\|Q_{p+1}-Q\|+\|Q_{q+1}-Q\|)+\frac{1}{2}\sigma_2\|BU^{-1}\|\|\frac{\partial V_{K_p}}{\partial x}-\frac{\partial V_{K_q}}{\partial x}\|+$$

$$+\sigma_4(x)\|\frac{\partial V_{K_p}}{\partial x}-\frac{\partial V_{K_q}}{\partial x}\|+\sigma_3(x)\|Q_p-Q_q\|$$

$$\leq M(\sigma_1(x))(\|Q_p-Q\|^2+\|Q_q-Q\|^2)+\sigma_5(x)(\|Q_p-Q\|+\|Q_q-Q\|)$$

$$+\frac{1}{2}\sigma_2\|BU^{-1}\|\|\frac{\partial H_{K_p}}{\partial x}-\frac{\partial H_{K_q}}{\partial x}\|++\sigma_4(x)\|\frac{\partial H_{K_p}}{\partial x}-\frac{\partial H_{K_q}}{\partial x}\|$$

$$\leq \sigma_6(\|Q_p-Q\|+\|Q_q-Q\|)+\sigma_7\|\frac{\partial H_{K_P}}{\partial x}-\frac{\partial H_{K_q}}{\partial x}\| \tag{2.66}$$

where

$$\sigma_2 = C_2\delta_3 + C_2\delta_2,$$

and noting the remark below (2.53),

$$\sigma_j(x) > 0,$$

$$\sigma_j(x) \longrightarrow 0, (\|x\| \to 0), j = 1,3,4,5,$$

Further, when $\|x\| < \delta_3, 0 < \delta_6 < 1, 0 < \delta_7 < 1$.

Now by (2.65),(2.11), and using (2.66), we have(for some positive $\sigma_8 < 1$)

$$\Delta(K_{p+1}, K_{q+1})(x)$$

$$\leq \sigma_8(\|Q_p-Q\|+\|Q_q-Q\|)+\sigma_7\|\frac{\partial H_{K_p}}{\partial x}-\frac{\partial H_{K_q}}{\partial x}\|$$

$$\leq \sigma^*(\|\frac{\partial H_{K_p}}{\partial x}-\frac{\partial H_{K_q}}{\partial x}\|+\|Q_p-Q\|+\|Q_q-Q\|)$$

$$= \sigma^*\Delta(K_p, K_q)(x), \tag{2.67}$$

with

$$\sigma^* = max\{\sigma_7, \sigma_8\} < 1.$$

Now we can show that the sequence $\{\frac{\partial H_{K_j}}{\partial x}\}$ converges uniformly near the origin as follows. First we show, when $n \to \infty$, that for all $p > 0$,

$$\Delta(K_{n+p}, K_n)(x) \longrightarrow 0 \tag{2.68}$$

uniformly near the origin. Since by (2.67)

$$\begin{aligned} &\Delta(K_{n+1}, K_n)(x) \\ &\leq \sigma^* \Delta(K_n, K_{n-1})(x) \\ &\leq ... \leq \sigma^{*n} \Delta(K_1, K_0)(x), \end{aligned}$$

we see for $\forall p > 0$ that

$$\begin{aligned} &\Delta(K_{n+p}, K_n)(x) \\ &= \|\frac{\partial H_{K_{n+p}}}{\partial x} - \frac{\partial H_{K_n}}{\partial x}\| + \|Q_{n+p} - Q\| + \|Q_n - Q\| \\ &< \sum_{j=1}^{p} \|\frac{\partial H_{K_{n+j}}}{\partial x} - \frac{\partial H_{K_{n+j-1}}}{\partial x}\| + \|Q_{n+p} - Q\| + \\ &+2(\|Q_{n+p-1} - Q\| + ... + \|Q_{n+1} - Q\|) + \|Q_n - Q\| \\ &= \sum_{j=1}^{p} (\|\frac{\partial H_{K_{n+j}}}{\partial x} - \frac{\partial H_{K_{n+j-1}}}{\partial x}\| + \\ &+\|Q_{n+j} - Q\| + \|Q_{n+j-1} - Q\|) \\ &= \sum_{j=1}^{p} \Delta(K_{n+j}, K_{n+j-1})(x) \\ &\leq \sum_{j=1}^{\infty} \sigma^{*n+j-1} \Delta(K_1, K_0)(x) \\ &= (\frac{\sigma^{*n}}{1 - \sigma^*}) \Delta(K_1, K_0)(x) \longrightarrow 0. \end{aligned}$$

We have therefore shown (2.68) to be true.

Then since

$$\|Q_{n+p} - Q\| \longrightarrow 0,$$

$$\|Q_n - Q\| \longrightarrow 0,$$

we deduce from (2.63) that

$$\|\frac{\partial H_{K_{n+p}}}{\partial x} - \frac{\partial H_{K_n}}{\partial x}\|$$

$$\leq \Delta(K_{n+p}, K_n)(x) + \|Q_{n+p} - Q\| + \|Q_n - Q\| \longrightarrow 0, (n \to \infty, \forall p > 0).$$

We have now proved $\{\frac{\partial H_{K_j}}{\partial x}\}$ converges uniformly near the origin, as does $\{\frac{\partial V_{K_j}}{\partial x}\}$ by (2.17),(2.12). Consequently,$\{K_j(x)\}$ converges uniformly near the origin by virtue of (2.18).

2.7 Further Properties

Finally, we present a theorem which shows that the $K_j(x), V_j(x) \equiv V_{K_j}(x)$ generated by the extended Kleinman-Newton process described above continue to enjoy some of the properties established in [4] for the corresponding linear-quadratic case.

Theorem 7.1. Let $\mathcal{D}$ be a closed set, whose open interior contains the origin $x = 0$, in which all of the $K_j(x)$, $V_j(x)$ are defined. Further suppose that $\mathcal{D}$ is invariant under each of the systems (cf. (2.15))

$$\dot{x} = F(x, K_j(x)), \ j = 1, 2, 3, ... \tag{2.69}$$

generated by this process, each of the cost functions $V_j(x)$ is positive in $\mathcal{D}$ except for $V_j(0) = 0$ and has the property, for each positive number v, that the subset of $\mathcal{D}$ described by $V_j(x) \leq v$ is compact. Then i) If $G(x, u) > 0, \ x \neq 0$ for all u, each system (2.69) has the origin as a globally asymptotically stable critical point in $\mathcal{D}$; ii) The cost functions $V_j(x)$ are monotonically decreasing in $\mathcal{D}$ as j increases, i.e., for $x \in \mathcal{D}$,

$$V_{j+1}(x) \leq V_j(x), \ j = 1, 2, 3, ... \tag{2.70}$$

Proof: From the definition of the $V_j(x)$ we know that

$$\frac{\partial V_j(x)}{\partial x} F(x, K_j(x)) + G(x, K_j(x)) \equiv 0, \ x \in \mathcal{D}. \tag{2.71}$$

Since the "next" feedback function $K_{j+1}(x)$ is chosen so that $u_{j+1}(x) = K_{j+1}(x)$ minimizes the Hamiltonian function $\frac{\partial V_j(x)}{\partial x} F(x,u) + G(x,u)$, it follows that

$$\frac{\partial V_j(x)}{\partial x} F(x, K_{j+1}(x)) + G(x, K_{j+1}(x)) \leq 0, \;\; x \in \mathcal{D}$$

and therefore

$$\frac{\partial V_j(x)}{\partial x} F(x, K_{j+1}(x)) \leq -G(x, K_{j+1}), \;\; x \in \mathcal{D}$$

Our assumption under i) shows that $-G(x, K_{j+1}(x) < 0,, \;\; x \in \mathcal{D}, \;\; x \neq 0$. Since $V_j(x)$ satisfies the standard assumptions for a Lyapounov function in $\mathcal{D}$ (see, e.g., [3]) relative to the system (2.69), the conclusion of i) follows.

To show ii), suppose x is a point in $\mathcal{D}$ and let $\xi(t)$ be the solution of $\dot{\xi} = F(\xi, K_j(\xi))$ with $\xi(0) = x$. By the result in i) we conclude that $\xi(t) \in \mathcal{D}, \;\; t \geq 0$ and $\lim_{t\to\infty} \xi(t) = 0$. Then for $\tau \geq 0$ let $\zeta(t,\tau)$ satisfy $\zeta(0,\tau) = x$ and, with the dot denoting differentiation with respect to t,

$$\dot{\zeta}(t,\tau) = F(\zeta(t,\tau), K_{j+1}(\zeta(t,\tau))), \;\; 0 \leq t \leq \tau;$$

$$\dot{\zeta}(t,\tau) = F(\zeta(t,\tau), K_j(\zeta(t,\tau))), \;\; 0 \leq t; \;\; \tau \leq t < \infty.$$

Since $\zeta(t,\tau) \in \mathcal{D}$ the result of i) applies again to show that $\zeta(t,\tau) \in \mathcal{D}, \;\; t \geq 0$ and $\lim_{t\to\infty} \zeta(t,\tau) = 0$.

Let us define the cost corresponding to the composite trajectory $\zeta(t,\tau)$ by

$$V_\tau(x) = \int_0^\tau G(\zeta(t,\tau), K_{j+1}(\zeta(t,\tau)))dt + \int_\tau^\infty G(\zeta(t,\tau), K_j(\zeta(t,\tau))dt$$

$$= \int_0^\tau G(\zeta(t,\tau), K_{j+1}(\zeta(t,\tau)))dt + V_j(\zeta(\tau,\tau)).$$

We observe that $V_0(x) = V_j(x)$ while $lim_{\tau\to\infty} V_\tau(x) = V_{j+1}(x)$. Thus, to show that $V_{j+1}(x) \leq V_j(x)$ it is sufficient to show that $V_\tau(x)$ is monotone decreasing with increasing τ. To this end we differentiate $V_\tau(x)$ with respect to τ (the justification of which is routine) with the result

$$\frac{dV_\tau(x)}{d\tau} = G(\zeta(\tau,\tau), K_{j+1}(\zeta(\tau,\tau))) + \frac{\partial V_j}{\partial x}(\zeta(\tau,\tau))F(\zeta(\tau,\tau), K_{j+1}(\zeta(\tau,\tau)))$$

$$-G(\zeta(\tau,\tau), K_j(\zeta(\tau,\tau))) + \frac{\partial V_j}{\partial x}(\zeta(\tau,\tau))F(\zeta(\tau,\tau), K_j(\zeta(\tau,\tau)).$$

The last line here is zero by virtue of (2.71) so we have

$$\frac{dV_\tau(x)}{d\tau} = G(\zeta(\tau,\tau), K_{j+1}(\zeta(\tau,\tau))) + \frac{\partial V_j}{\partial x}(\zeta(\tau,\tau))F(\zeta(\tau,\tau), K_{j+1}(\zeta(\tau,\tau)))$$

Since the right hand side here vanishes when K_{j+1} is replaced by K_j and is minimized over u by the choice $u = K_{j+1}(\zeta(\tau,\tau))$, we conclude that $\frac{d}{d\tau}V_\tau(x) \leq 0$. Applying the mean value theorem we conclude that $V_\tau(x) \leq V_0(x) = V_j(x)$ for $\tau \geq 0$. Then, since $V_{j+1}(x) = lim_{\tau\to\infty} V_\tau(x)$ we have the result ii) and the proof of the theorem is complete.

It seems likely that one can show that the convergence of $V_j(x)$ and its gradient to the optimal $\hat{V}(x)$ and its gradient, respectively, are quadratic as $j \to \infty$, thus yielding a corresponding result for the convergence of the feedback functions $K_j(x)$. It is not immediately clear, however, how this result may be obtained at the present time.

References

1. Lukes, D.L.: *Optimal regulation of nonlinear dynamical systems*, SIAM J.Control 7 (1969),pp.75-100

2. Russell, D.L.: *Mathmatics of Finite Dimensional Control System*, Marcel Dekker,Inc.,New York,1979

3. Coddington, E. A. *Theory of Ordinary Differential Equations*, McGraw-Hill Book Company, New York, Toronto, London, 1955

4. Kleinman, D.L.: *On an iterative technique for Riccati equation computations*, IEEE Trans. Auto. Control, AC-13(1968), pp.114-115.

Chapter 3

A Generalization of Nonlinear-Nonquadratic Optimal Control to Invariant Compact Manifolds

In 1969, [1] D. L. Lukes proved the existence and uniqueness of the optimal control for the nonlinear-nonquadratic (NLNQ) problem for an autonomous analytic control system $\dot{x} = F(x, u)$ near a critical point $\hat{x}$ for the uncontrolled system, $\dot{x} = F(x, 0)$, i.e., a point where $F(\hat{x}, 0) = 0$. Later, in 1993, [3] D. L. Russell and X. Zhang extended the theory to cover the case wherein the critical point is replaced by a periodic solution of the uncontrolled system. Here we adapt this theory to solve the NLNQ problem near a general invariant compact manifold for the system.

3.1 An Optimal Control Problem

We begin by considering a control system in R^{n+m}:

$$\dot{x} = F(x, u), x \in R^n, u \in R^m. \tag{3.1}$$

We will suppose $F(x, u)$ to be real analytic with respect to both x and u. Further, we will suppose there is an analytic, compact m-dimensional manifold, $1 \leq m < n$, in R^n, which we denote by M, such that M is invariant under (3.1) when $u = 0$, M being given by $n - m$ equations

$$\varphi_i(x) = 0, i = 1, 2, ..., n - m, \tag{3.2}$$

which we abbreviate by

$$\varphi(x) = 0; \varphi : R^n \to R^{n-m}. \tag{3.3}$$

We will assume that $\varphi(x)$ is real analytic in the region of interest and that the Jacobian matrix $\frac{\partial \varphi}{\partial x}$ is everywhere of rank $n - m$ on the manifold M.

We suppose there is a twice continuously differentiable feedback function

$$u = k_0(x) \tag{3.4}$$

defined in the region of interest in R^n such that

$$k_0(x) \equiv 0, x \in M, \tag{3.5}$$

and such that the closed loop system

$$\dot{x} = F(x, k_0(x)) \tag{3.6}$$

has the property of exponential M-orbital stability: i.e., given x_0 near M, the solution $x_{k_0}(t, x_0)$ of the closed loop system with $x_{k_0}(0, x_0) = x_0$ satisfies, with d denoting the usual Euclidean distance in R^n,

$$d(x_{k_0}(t, x_0), M) \leq m_0 e^{-\beta_0 t} \|\varphi(x_0)\|, \ t \geq 0, \tag{3.7}$$

for some fixed positive constants m_0, β_0. In this sense we say that the system (3.1) is stabilizable near the compact manifold M.

From (3.7), consequently for $s : 0 < s < t$, we have

$$d(x_{k_0}(t, x_0), M) \leq m_0 e^{-\beta_0 (t-s)} \|\varphi(x_{k_0}(s))\|.$$

In fact, if let $T = t - s$, $x_1(T, x_{k_0}(s)) = x_{k_0}(T + s, x_0)$, we see that $x_1(T)$ is the solution of

$$\dot{x_1}(T) = F(x_{k_0}(T + s), k_0(x_{k_0}(T + s))) = F(x_1(T), k_0(x_1(T))),$$

with

$$x_1(0) = x_{k_0}(s).$$

Thus, by (3.7), we have

$$d(x_1(T, x_{k_0}(s)), M) \leq m_0 e^{-\beta_0 T} \|\varphi(x_{k_0}(s))\|,$$

i.e.

$$d(x_{k_0}(t,x_0),M) \le m_0 e^{-\beta_0(t-s)}\|\varphi(x_{k_0}(s))\|, t > s > 0.$$

Let $G(x,u)$ be a scalar function which is also real analytic in a sufficiently large region which includes M. We will suppose that

$$G(x,u) \ge 0, \tag{3.8}$$

for all x and u of interest and that

$$G(x,u) > 0, (x,u) \notin M \times \{0\},$$

$$G(x,0) \equiv 0, x \in M. \tag{3.9}$$

Our basic control objective is to minimize the cost

$$C(x_0,u) = \int_0^\infty G(x,u)dt \tag{3.10}$$

associated with the control function $u(t)$ and the response $x(t,u,x_0)$ to that control via (3.1) initiating at the point x_0 near M. Specifically, we propose to prove the following result.

Theorem 1.1 The optimal control problem for the system (3.1), stabilizable as indicated near a compact manifold M, with cost integrand described by (3.8),(3.9), has a unique solution expressed by a feedback function $K(x)$ which is identically equal to zero on M and is continuously differentiable in a neighborhood of M. Moreover, the synthesized system

$$\dot{x} = F(x,K(x)) \tag{3.11}$$

obtained with the use of this control has M as a uniformly orbitally stable invariant set.

3.2 The Variational System

We now focus on the twice continuously differentiable nonlinear feedback function

$$u = k_0(x), \;\; k_0(x) \equiv 0, x \in M, \tag{3.12}$$

defined in the region of interest in R^n such that the closed loop system obtained by replacing u with $k_0(x)$ in (3.1)), i.e.,

$$\dot{x} = F_0(x) \equiv F(x, k_0(x)) \tag{3.13}$$

having M as an invariant set, has the further property of exponential orbital stability. Since the function $k_0(x)$ is continuously differentiable, the condition $k_0(x) \equiv 0, x \in M$ and the compactness of M suffice to establish that

$$||k_0(x)|| \leq B_0 d(x, M), \tag{3.14}$$

in some neighborhood of M. Letting the control synthesized by (3.12) be denoted by $u_0(t, x)$ and writing

$$x_{k_0}(t, x_0) = x(t, u_0, x_0), \tag{3.15}$$

we see that the cost

$$C(x_0, u_0) = \int_0^\infty G(x(t, u_0, x_0), u_0(t, x_0))dt \tag{3.16}$$

is finite for all initial states x_0 in some neighborhood of M.

First of all, we describe the system (3.1) in local coordinates and the corresponding variational system based on M. Let $\hat{x}$ be a point on M. Since $\frac{\partial \varphi}{\partial x}(\hat{x})$ has rank $n - m$, we can find linearly independent vectors

$$\psi_1(\hat{x}), \psi_2(\hat{x}), ..., \psi_m(\hat{x})$$

such that

$$\frac{\partial \varphi}{\partial x}(\hat{x})\psi_k(\hat{x}) = 0, k = 1, 2, ..., m. \tag{3.17}$$

Now let Ψ be the $n \times m$ matrix whose columns are the vectors $\psi_k(\hat{x}), k = 1, 2, ..., m$. Then the matrix

$$\Phi(x) = \begin{pmatrix} \frac{\partial \varphi}{\partial x}(x) \\ \Psi^* \end{pmatrix}, \tag{3.18}$$

where $*$ denotes the transpose of a vector or matrix, is clearly such that $\Phi(\hat{x})$ is nonsingular. We define local coordinates near $\hat{x}$ by

$$\xi = \varphi(x), \xi \in R^{n-m}; \eta = \Psi^*(x - \hat{x}). \tag{3.19}$$

Clearly the Jacobian matrix of the map $x \longrightarrow (\xi, \eta)$ at $x = \hat{x}$ is $\Phi(\hat{x})$ and is therefore nonsingular. From the continuity we can find a neighborhood

$$n(\hat{x}, \rho(\hat{x})) = \{x \in R^n \mid \|x - \hat{x}\| < \rho(\hat{x})\}, \rho(\hat{x}) > 0, \tag{3.20}$$

such that the Jacobian matrix $\Phi(x)$ of the map $x \longrightarrow (\xi, \eta)$ at each point x in $n(\hat{x}, \rho(\hat{x}))$ remains nonsingular and such that we have a smooth inverse relationship

$$x = x(\xi, \eta) \tag{3.21}$$

in each such neighborhood. Since we can do this for each point $\hat{x}$ in M and since M is compact, there are finitely many points $\hat{x}_j, j = 1, 2, ..., J$, such that the neighborhoods

$$n_j \equiv n(\hat{x}_j, \rho(\hat{x}_j)), \ j = 1, 2, ..., J,$$

cover M. This defines a local coordinate system on M such that, in each such coordinate system, M is given by $\xi = 0$. Of course, the coordinate relationships (3.19) differ in each neighborhood n_j, but we will use the notation ξ, η for simplicity, resorting to ξ_j, η_j only when it is necessary to distinguish between these coordinates in different n_j. Similar conventions will be followed in connection with $\varphi, \Phi, \psi, \Psi$ etc.

The transformed control system is

$$\dot{\xi} = \frac{\partial \varphi}{\partial x}(x(\xi, \eta))F(x(\xi, \eta), u) \equiv f(\xi, \eta, u), \tag{3.22}$$

$$\dot{\eta} = \Psi^* F((x(\xi, \eta), u) \equiv g(\xi, \eta, u). \tag{3.23}$$

Again we should note that the functions f and g are actually different in each neighborhood $n_j, \ j = 1, 2, ..., J$. Since M is invariant under the flow associated with $F(x, 0)$ we have

$$f(0, \eta, 0) \equiv 0. \tag{3.24}$$

In terms of ξ and η the control synthesis (3.12) now becomes

$$u = k_0(x(\xi,\eta)) \equiv K_0(\xi,\eta) \tag{3.25}$$

and since $k_0(x) \equiv 0$ for $x \in M$ we have $K_0(0,\eta) \equiv 0$. Since solutions $\xi(t), \eta(t)$ of (3.22),(3.23) with $u(t) \equiv 0$ which initiate at points on M are such that $\xi(t) \equiv 0$ we may unambiguously refer to such solutions solely in terms of $\eta(t)$, where

$$\dot{\eta} = g(0,\eta,0). \tag{3.26}$$

Solutions of this m-dimensional system of differential equations describe the uncontrolled flow on the manifold M. We can think of such solutions as initiating at points $\eta_0 \in M$, provided we realize that it is also necessary to specify the neighborhood n_j in which the point $x(0,\eta_0)$ lies in order for η_0 to be unambiguous.

The variational system of (3.22) based on M consists of all linear systems of the form (note that (3.24) implies $\frac{\partial f}{\partial \eta}(0,\eta,0) \equiv 0$),

$$\dot{\delta\xi} = \frac{\partial f}{\partial \xi}(0,\eta,0)\delta\xi + \frac{\partial f}{\partial u}(0,\eta,0)\delta u, \tag{3.27}$$

where $\eta(t)$ is a solution of (3.26). It should be noted that this is, in effect,a set of linear systems parametrized by solutions $\eta(t)$ of (3.26). We can write this system in the form

$$\dot{\delta\xi} = A(\eta)\delta\xi + B(\eta)\delta u \tag{3.28}$$

with

$$A(\eta) = \frac{\partial f}{\partial \xi}(0,\eta,0),$$

$$B(\eta) = \frac{\partial f}{\partial u}(0,\eta,0).$$

Noting that $K_0(0,\eta) \equiv 0$ and $\frac{\partial K_0(0,\eta)}{\partial \eta} \equiv 0$, the corresponding variational form of (3.25) may be seen to be

$$\delta u(t) = \frac{\partial K_0}{\partial \xi}(0,\eta)\delta\xi(t) \equiv K_0(\eta)\delta\xi(t), \tag{3.29}$$

so that the closed loop variational system based on M takes the form

$$\dot{\delta\xi} = (A(\eta) + B(\eta)K_0(\eta))\delta\xi. \tag{3.30}$$

Lemma 2.1 Suppose that (3.7) is true, then there is a positive number μ such that each solution $\delta\xi(t)$ of

$$\dot{\delta\xi} = \{A(\eta(t)) + B(\eta(t))K_0(\eta(t))\}\delta\xi \equiv \hat{A}(\eta)\delta\xi, \tag{3.31}$$

$$\delta\xi(0) = \xi_0,$$

with $||\xi_0||$ small, satisfies

$$||\delta\xi(t)|| \leq \mu e^{-\gamma_0 t}||\xi_0||, \tag{3.32}$$

for all $t \geq t^*(t^*$ being some positive real number) and uniformly for all solutions $\eta(t)$ of (3.26),where $\gamma_0 = \frac{\beta_0}{2}$.

Proof: We redefine

$$n_j = n(\hat{x}_j, \frac{\rho(\hat{x}_j)}{4}), j = 1, ..., J. \tag{3.33}$$

and

$$4n_j = n(\hat{x}_j, \rho(\hat{x}_j)). \tag{3.34}$$

so that

$$M \subset \cup_{j=1}^{J} n_j. \tag{3.35}$$

From (3.7), also noting (3.3), there exists for all x_0, $||\varphi(x_0)||$ sufficiently small ($||\varphi(x_0)|| < \alpha$, for some absolutely positive α), a $t^* > 0$ such that

$$x_{k_0}(t, x_0) \subset \cup_{j=1}^{J} n_j, \tag{3.36}$$

whenever $t \geq t^*$.

Now given any solution $\eta(t)$ of (3.26), i.e.,

$$\dot{\eta} = g(0, \eta, 0),$$

$$\eta(0) = \eta_0,$$

let $\delta\xi$ be an arbitrary solution of (3.31), i.e.,

$$\dot{\delta\xi} = [A(\eta(t)) + B(\eta(t))K_0(\eta(t))]\delta\xi,$$

$$\delta\xi(0) = \xi_0.$$

with $||\xi_0|| < \alpha$. Moreover, noting that $f(\xi, \eta, u)$ is real analytic, we have a solution (ξ, τ) of

$$\dot{\xi} = f(\xi, \tau, K_0) - f(0, \eta, 0) = [A(\eta) + B(\eta)K_0(\eta)]\xi + \phi(t, \xi(0)),$$

$$\dot{\tau} = g(\xi, \tau, K_0(\xi, \tau))),$$

$$\xi(0) = \xi_0, \tau(0) = \eta_0,$$

where $||\phi(t, \xi(0))|| = \mathcal{O}\{||\xi(t)||^2\}$ because $\phi(t, \xi_0)$ consists of the higher order (≥ 2) terms of ξ with coefficients in η. Because of the relationship between the original system and its local form under the locally one-to-one correspondance

$$x = x(\xi, \tau),$$

and also noting that $f(0, \eta, 0) \equiv 0$, this finally determines a solution $x(t)$ of

$$\dot{x} = F(x, k_0(x)), \quad x(0) = x_0 = (\xi_0, \eta_0), \tag{3.37}$$

with $||\varphi(x_0)|| = ||\xi_0|| < \alpha$.

Therefore $x(t) \subset \cup_{j=1}^{J} n_j, t \geq t^*$. For each $t \geq t^*$ we have a j, $1 \leq j \leq J$, such that $x(t) \in n_j$. We claim that the projection of this $x(t)$ onto M denoted by $Px(t)$ will lie in $4n_j \cap M$. In fact we have, for all $y \in n_j \cap M$,

$$\|x - Px\| \leq \|x - y\| \leq \|x - \hat{x}_j\| + \|\hat{x}_j - y\| \leq \frac{\rho(\hat{x}_j)}{4} + \frac{\rho(\hat{x}_j)}{4} = \frac{\rho(\hat{x}_j)}{2} \tag{3.38}$$

and

$$\|x - \hat{x}_j\| < \frac{\rho(\hat{x}_j)}{4}. \tag{3.39}$$

But (3.38),(3.39) give

$$\|\hat{x}_j - Px\| \leq \|\hat{x}_j - x\| + \|x - Px\| < \frac{\rho(\hat{x}_j)}{4} + \frac{\rho(\hat{x}_j)}{2} < \rho(\hat{x}_j). \tag{3.40}$$

By the one-one correspondance

$$x \longleftrightarrow (\xi, \eta)$$

in $4n_j$, we see from (3.19) that, in local coordinates near $\hat{x}_j$,

$$\varphi(x(t)) = \xi(t), \varphi(Px(t)) = 0. \tag{3.41}$$

Therefore when $t > t^*$ we have

$$\begin{aligned} \|\xi(t)\| &= \|\varphi(x(t)) - \varphi(Px(t))\| \\ &\leq Sup_{x \in 4n_j} \|\frac{\partial \varphi}{\partial x}\| \|x(t) - Px(t)\| \\ &\leq Max_{1 \leq j \leq J} \Big\{ Sup_{x \in 4n_j} \Big\{ \|\frac{\partial \varphi}{\partial x}\| d(x(t), M) \Big\} \Big\}. \\ &\leq \mu_0 m_0 e^{-\beta_0 t} \|\varphi(x_0)\|, \end{aligned} \tag{3.42}$$

where we denote $Max_{1 \leq j \leq J} \Big\{ Sup_{x \in 4n_j} \Big\{ \|\frac{\partial \varphi}{\partial x}\| \Big\} \Big\}$ by μ_0 and use (3.7). Then, when $t \geq t^*$, we obtain

$$\|\xi(t)\| \leq \mu_0 m_0 e^{-\beta_0 t} \|\xi_0\|. \tag{3.43}$$

Here we can assume $\mu_0 m_0 e^{-\beta_0 t^*} < 1$ by properly choosing t^*.

Let $\Phi(t)$ stand for the fundamental matrix, reducing to the identity when $t = 0$, of

$$\dot{\delta\xi} = [A(\eta) + B(\eta)K_0(\eta)]\delta\xi. \tag{3.44}$$

We have

$$\xi(t) = \xi(t, \xi_0) = \Phi(t)\Phi^{-1}(0)\xi_0 + \Phi(t)\int_0^t \Phi^{-1}(s)\phi(s, \xi_0)ds. \tag{3.45}$$

Taking the derivative with respsct to ξ_0, we see that

$$\frac{\partial \xi(t, \xi_0)}{\partial \xi_0} = \Phi(t)\Phi^{-1}(0) + \Phi(t)\int_0^t \frac{\partial}{\partial \xi_0}(\Phi^{-1}(s)\phi(s, \xi_0))ds. \tag{3.46}$$

From (3.43), as in [1], pp.95, we have, for $||\xi_0||$ small,

$$||\frac{\partial \xi(t, \xi_0)}{\partial \xi_0}|| \leq \mu e^{-\beta_0 t}, \tag{3.47}$$

where $\mu = 2\mu_0 m_0$. For fixed t in (3.46), let $\xi_0 \to 0$. Then we see that

$$||\Phi(t)\Phi^{-1}(0)|| \leq \mu e^{-\frac{\beta_0}{2}t}. \tag{3.48}$$

Similarly, for each $s < t$, we have

$$||\Phi(t)\Phi^{-1}(s)|| \leq \mu e^{-\frac{\beta_0}{2}(t-s)}$$

Clearly,(3.48) is true for all $t > t^*$. Then since

$$\delta\xi(t) = \Phi(t)\Phi^{-1}(0)\xi_0,$$

we have

$$||\delta\xi(t)|| \leq \mu e^{-\frac{\beta_0}{2}t}||\xi_0||, \ \ t > t^*.$$

3.3 The Representation of Cost Functionals

Let $\eta(t)$ be a solution of (3.26)and suppose that, in the local coordinates, the conclusion of Lemma 2.1 is true for $K_0(\eta)$. Since M is compact, for each solution $\delta\xi(t)$ of (3.31), we have

$$\int_0^\infty \delta\xi^*(t)\tilde{W}_{K_0}(\eta)\delta\xi(t)dt < \infty, \tag{3.49}$$

where

$$\tilde{W}_{K_0}(\eta) = \begin{pmatrix} I^*, & K_0^*(\eta) \end{pmatrix} \begin{pmatrix} W(\eta), & R(\eta) \\ R^*(\eta), & U(\eta) \end{pmatrix} \begin{pmatrix} I \\ K_0(\eta) \end{pmatrix}$$

$$\begin{pmatrix} W(\eta) & R(\eta) \\ R^*(\eta) & U(\eta) \end{pmatrix} > 0, \;\; U(\eta) > 0,$$

noting that, in the local coordinates,

$$G(x,u) = \Gamma(\xi,\eta,u) = (\xi^*, u^*) \begin{pmatrix} W(\eta), & R(\eta) \\ R^*(\eta), & U(\eta) \end{pmatrix} \begin{pmatrix} \xi \\ u \end{pmatrix} + \gamma(\xi,\eta,u),$$

because

$$\Gamma(0,\eta,0) \equiv 0, \;\; \Gamma(\xi,\eta,u) > 0, \;\; (\xi,u) \neq (0,0),$$

wherein

$$\gamma(\xi,\eta,u) = o\{\|\xi\|^2 + \|u\|^2\}, \|\xi\|, \|u\| \longrightarrow 0, 0.$$

Lemma 2.2 There exists a continuously differentiable matrix function $Q(\eta)$ on M such that for every solution $\delta\xi(t)$ of (3.31) (with $\delta\xi(0) = \xi_0$, $\|\xi_0\|$ small)

$$\int_0^\infty \delta\xi^*(t)\tilde{W}_{K_0}(\eta)\delta\xi(t)dt = \xi_0^* Q(\eta_0)\xi_0.$$

Furthermore, $Q(\eta)$ is positive definite. Moreover, for the solution of

$$\dot{\xi} = [A(\eta) + B(\eta)K_0(\eta)]\xi + \phi(t,\xi_0), \;\; \dot{\eta} = g(0,\eta,0), \;\; \xi(0) = \xi_0, \;\; \eta(0) = \eta_0,$$

we have

$$J(\xi_0,\eta_0) = \int_0^\infty \Gamma(\xi,\eta,K_0(\xi,\eta))dt = \xi_0^* Q(\eta_0)\xi_0 + j(\xi_0,\eta_0),$$

where $j(\xi_0, \eta_0)$ consists of higher order(≥ 3) terms in ξ_0 with coefficients in η_0. Further, $Q(\eta)$ satisfies

$$\frac{\partial Q}{\partial \eta} g(0, \eta, 0) + \hat{A}^*_{K_0}(\eta) Q(\eta) + Q(\eta) \hat{A}_{K_0}(\eta) + \tilde{W}(\eta) = 0,$$

with $\hat{A}_{K_0}(\eta) = A(\eta) + B(\eta) K_0(\eta)$.

Proof: (1). We choose

$$\epsilon = \frac{1}{4}\alpha,$$

with α being chosen as in the proof of Lemma 2.1. In R^{n-m}, let

$$e^*_{jj} = (0, ..., 0, \epsilon, 0, ..., 0),$$

and for $i \neq j$,

$$e^*_{ij} = (0, ..., 0, \epsilon, 0, ..., 0, \epsilon, 0, ..., 0).$$

Correspondingly, let $\delta\xi_{ij}(t)$ be the solution of

$$\dot{\delta\xi} = \hat{A}_{K_0}(\eta(t)) \delta\xi. \ \dot{\eta} = g(0, \eta, 0) \tag{3.50}$$

$$\delta\xi(0) = e_{ij}, \ \ \eta(0) = \eta_0.$$

(2) Given $T > 0$, on the interval $[0, T]$ the equation

$$\dot{Q} + Q \hat{A}_{K_0}(\eta(t)) + \hat{A}^*_{K_0}(\eta(t)) Q + \tilde{W}_{K_0}(\eta(t)) = 0,$$

$$Q(T) = I(identity) \tag{3.51}$$

has a unique solution which is denoted by $Q^T(t)$. Since $\tilde{W}_{K_0}(\eta)$ is symmetric, we see $Q^T(t)$ is symmetric.

(3) From Lemma 2.1 we see that any solution of (3.50) obeys

$$\delta\xi(t) \longrightarrow 0, (t \to \infty),$$

By (3.49) above we have

$$\int_0^\infty \delta\xi^* \tilde{W}_{K_0}(\eta)\delta\xi dt$$
$$= \lim_{T\to\infty}\int_0^T \delta\xi^* \tilde{W}_{K_0}(\eta)\delta\xi dt$$
$$= \lim_{T\to\infty}\int_0^T \delta\xi^*(-\dot{Q}^T - Q^T\hat{A}_{K_0} - \hat{A}^*_{K_0}Q^T)\delta\xi dt$$
$$= \lim_{T\to\infty}\int_0^T -\frac{d}{dt}(\delta\xi^* Q^T(t)\delta\xi)dt$$
$$= \lim_{T\to\infty} \delta\xi^*(0)Q^T(0)\delta\xi(0) - \delta\xi^*(T)I\delta\xi(T).$$
$$= \lim_{T\to\infty} \delta\xi^*(0)Q^T(0)\delta\xi(0). \tag{3.52}$$

(4) We write $Q^T(0)$, displaying its entries, as follows

$$\begin{pmatrix} q^T_{11}, & \ldots, & q^T_{1n-m} \\ . & \ldots, & . \\ q^T_{n-m1}, & \ldots, & q^T_{n-mn-m} \end{pmatrix}. \tag{3.53}$$

Noting that

$$q^T_{jj} = \frac{1}{\epsilon^2}e^*_{jj}Q^T(0)e_{jj},$$

and, when $i \neq j$,

$$q^T_{ij} = \frac{1}{2}[\frac{1}{\epsilon^2}e^*_{ij}Q^T(0)e_{ij} - q^T_{ii} - q^T_{jj}],$$

we see from (3.52) that for every pair of i, j,

$$q^T_{ij} \longrightarrow q_{ij}, \;\; T \longrightarrow \infty.$$

We define

$$Q(\eta_0) = \begin{pmatrix} q_{11}, & \ldots, & q_{1n-m} \\ \ldots\ldots & & \\ q_{n-m1}, & \ldots, & q_{n-mn-m} \end{pmatrix}. \tag{3.54}$$

(5) Thus we have

$$Q^T(0) \to Q(\eta_0). \tag{3.55}$$

Since for each T, $Q^T(0)$ is symmetric, $Q(\eta_0)$ is symmetric as well. From (3.52), we see that

$$\int_0^\infty \delta\xi^*(t)\tilde{W}_{K_0}(\eta)\delta\xi(t)dt = \xi_0^* Q(\eta_0)\xi_0 \tag{3.56}$$

for arbitrarily small $\delta\xi_0$ and $\delta\xi(t)$ satisfying

$$\dot{\delta\xi} = \hat{A}_{K_0}(\eta)\delta\xi,$$

$$\delta\xi(0) = \xi_0.$$

From (3.56), when $\tilde{W}_{K_0}(\eta) \geq 0$, we see that for arbitrary ξ_0

$$\xi_0^* Q(\eta)\xi_0 \geq 0,$$

and we deduce that $Q(\eta_0)$ is nonnegative. Clearly, if $\tilde{W}_{K_0}(\eta) > 0$, then $Q(\eta_0)$ is positive definite.

(6) From (3.56) we have, for every pair i, j,

$$q_{ij}(\eta_0) = \frac{1}{\epsilon^2} e_{ij}^* Q(\eta_0) e_{ij}$$

$$= \frac{1}{\epsilon^2} \int_0^\infty \delta\xi_{ij}^*(t)\tilde{W}_{K_0}(\eta(t,\eta_0))\delta\xi_{ij}(t)dt.$$

Using the fact that the solution $(\delta\xi_{ij}(t), \eta(t,\eta_0))$ is of class C^1 with respect to η_0 near M and $\tilde{W}_{K_0}(\eta)$ is of class C^1 with respect to η near M, we then see that $q_{ij}(\eta_0)$ is of class C^1 with respect to η_0 near M and so, consequently, is $Q(\eta_0)$.

(7) First we have, by (3.16),

$$\int_0^\infty \Gamma(\xi, \eta, K_0(\xi, \eta))dt < \infty.$$

Let

$$\Delta(t,\xi_0)=\xi(t)-\delta\xi(t),\quad \Delta_1(t,\xi_0)=K_0(\xi,\eta)-K_0(\eta)\xi.$$

Clearly, for each t, $\Delta(t,\xi_0),\Delta_1(t,\xi_0)$ consists of all the terms in ξ_0 of order greater than or equal to two. Thus we see that

$$\int_0^\infty \Gamma(\xi,\eta,K_0(\xi,\eta))dt$$

$$=\int_0^\infty (\delta\xi^*,(K_0(\eta)\delta\xi)^*)\begin{pmatrix} W(\eta), & R(\eta) \\ R^*(\eta), & U(\eta)\end{pmatrix}\begin{pmatrix}\delta\xi \\ K_0(\eta)\delta\xi\end{pmatrix}dt+$$
$$+\int_0^\infty \psi(\Delta(t,\xi_0),\eta,\Delta_1(t,\xi_0))dt,$$

where

$$\int_0^\infty \psi(\Delta(t,\xi_0),\eta,\Delta_1(t,\xi_0))dt<\infty,$$

which can be expanded as a series in ξ_0 involving terms of order greater than or equal to three with coefficients depending only on η_0. Then by (3.56) it is easy to deduce(cf. [1], pp. 79-81) that

$$\int_0^\infty \Gamma(\xi,\eta,K_0(\xi,\eta))dt=\xi_0^*Q(\eta_0)\xi_0+j(\xi_0,\eta_0),$$

with $j(\xi,\eta)$ being differentiable and

$$j(\xi,\eta)=O\{\|\xi\|^3\}.$$

(8) Now we derive a first order hyperbolic partial differential equation which has the solution $Q(\eta)$. For given $t>0$, we have

$$\delta\xi^*(t)Q(\eta(t))\delta\xi(t)$$
$$=\int_0^\infty \delta\xi^*(s+t)\tilde{W}_{K_0}(\eta(s+t))\delta\xi(s+t)ds$$

$$= \int_t^{\infty} \delta\xi^*(s)\tilde{W}_{K_0}(\eta(s))\delta\xi(s)ds. \tag{3.57}$$

We write $Q(\eta)$, showing its column vectors:

$$Q(\eta) = (P_1(\eta), ..., P_{n-m}(\eta)).$$

Differentiating both sides of (3.57) with respect to t gives

$$\delta\xi^*(t)\{((\frac{\partial P_1}{\partial\eta})g, ..., (\frac{\partial P_{n-m}}{\partial\eta})g) + \hat{A}^*_{K_0}(\eta)(P_1, ..., P_{n-m}) +$$

$$+ (P_1, ..., P_{n-m})\hat{A}_{K_0}(\eta) + \tilde{W}_{K_0}(\eta)\}\delta\xi(t) = 0. \tag{3.58}$$

Since, for given $t > 0$, the matrix

$$((\frac{\partial P_1}{\partial\eta})g, ..., (\frac{\partial P_{n-m}}{\partial\eta})g) + \hat{A}^*_{K_0}(P_1, ..., P_{n-m}) + (P_1, ..., P_{n-m})\hat{A}_{K_0} + \tilde{W}_{K_0}(\eta)$$

is fixed but $\delta\xi(t)$ can be arbitrary, we see from (3.58) that for all $\eta(t)$

$$((\frac{\partial P_1}{\partial\eta})g, ..., (\frac{\partial P_{n-m}}{\partial\eta})g) + \hat{A}^*_{K_0}(\eta)(P_1, ..., P_{n-m}) +$$

$$+ (P_1, ..., P_{n-m})\hat{A}_{K_0}(\eta) + \tilde{W}_{K_0}(\eta) = 0. \tag{3.59}$$

Clearly (3.59) is a system of first order hyperbolic equations. We write it in the form

$$\frac{\partial Q}{\partial\eta}g(0, \eta, 0) + \hat{A}^*_{K_0}(\eta)Q(\eta) + Q(\eta)\hat{A}_{K_0}(\eta) + \tilde{W}_{K_0}(\eta) = 0. \tag{3.60}$$

3.4 On Kalman-Riccati Matrix Differential Equations

We begin with the following crucial lemma.

Lemma 2.3 There exists a positive definite matrix function $Q(\eta)$ which is continuously differentiable on M such that for an arbitrary solution $\eta(t)$ of (3.26): $\dot{\eta} = g(0, \eta, 0), \eta(0) = \eta_0$, we have

(i) $Q(\eta(t))$ satisfies the Kalman-Riccati differential equation

$$\frac{dQ(\eta(t))}{dt}+A^*(\eta)Q(\eta)+Q(\eta)A(\eta)+W(\eta)-(R(\eta)+Q(\eta)B(\eta))U^{-1}(\eta)(R^*(\eta)+B^*(\eta)Q(\eta))=0$$

for $t \in [0,\infty)$.

(ii) For all ξ_0, $||\xi_0||$ being small, the solution of

$$\dot{\xi} = [A(\eta) + B(\eta)(-U^{-1}(R^*(\eta) + B^*(\eta)Q(\eta))]\xi(t) \equiv \hat{A}_{\hat{Q}}(\eta)\xi(t), \tag{3.61}$$

$$\xi(0) = \xi_0, \tag{3.62}$$

satisfies

$$||\xi(t)|| < \mu e^{-\gamma_0 t}||\xi_0||, \tag{3.63}$$

where μ, γ_0 only depending on the manifold M and the constant α being chosen in the proof of Lemma 2.1.

Proof: Based on the solution $\eta(t)$, (3.60) becomes a Kalman-Riccati matrix ordinary differential equation

$$\frac{dQ(\eta(t))}{dt} + (A(\eta(t)) + B(\eta(t))K_0(\eta(t)))^*Q(\eta(t))+$$
$$+ Q(\eta(t))(A(\eta(t)) + B(\eta(t))K_0(\eta(t))) + \tilde{W}_{K_0}(\eta(t)) = 0, \tag{3.64}$$

on $t \in [0,\infty)$.

In the following we denote $Q(\eta)$ by $Q_{K_0}(\eta)$ to specify that $Q(\eta)$ is obtained corresponding to $K_0(\eta)$ in Lemma 2.2. We define as in [2]

$$c(t,\xi) = \xi^* Q_{K_0}(\eta(t))\xi.$$

We similarly define

$$H(t,\xi,u) = \frac{\partial c(t,\xi)}{\partial t} + \frac{\partial c(t,\xi)}{\partial \xi}[A(\eta)\xi + B(\eta)u]+$$

$$+ (\xi^*, u^*) \begin{pmatrix} W(\eta), & R(\eta) \\ R^*(\eta), & U(\eta) \end{pmatrix} \begin{pmatrix} \xi \\ u \end{pmatrix}. \tag{3.65}$$

From (3.64), we see that $H(t, \xi, K_0(\eta(t))\xi) = 0$ and by elementary calculation we see that

$$\frac{\partial H(t, \xi, u)}{\partial u} = 0, \tag{3.66}$$

if and only if $u = \hat{K}(\eta(t))\xi = -U^{-1}(\eta)[R^*(\eta) + B^*(\eta)Q_{K_0}(\eta)]\xi$. But the Hessian here is

$$2U(\eta(t)) > 0.$$

Therefore,

$$H(t, \xi, \hat{K}(\eta)\xi) \leq 0. \tag{3.67}$$

But this implies, for all $T > 0$, that

$$\hat{\xi}^*(T)Q_{K_0}(\eta(T))\hat{\xi}(T) - \xi_0^* Q_{K_0}(\eta_0)\xi_0 \leq -\int_0^T \hat{\xi}^*(t)\tilde{W}_{\hat{K}}(\eta(t))\hat{\xi}(t)dt, \tag{3.68}$$

where $\hat{\xi}(t)$ is the solution of

$$\dot{\hat{\xi}} = [A(\eta(t)) + B(\eta(t))\hat{K}(\eta(t))]\hat{\xi}, \ \hat{\xi}(0) = \xi_0, \ \ \|\xi_0\| < \alpha.$$

Noting that $Q(\eta) \geq 0$ and letting $T \to \infty$ we obtain

$$\xi_0^* Q_{K_0}(\eta_0)\xi_0 \geq \int_0^\infty \hat{\xi}^*(t)\tilde{W}_{\hat{K}}(\eta(t))\hat{\xi}(t)dt. \tag{3.69}$$

From Lemma 2.2 we further obtain

$$\xi_0^* Q_{K_0}(\eta_0)\xi_0 \geq \xi_0^* Q_{\hat{K}}(\eta_0)\xi_0, \tag{3.70}$$

where $Q_{\hat{K}}(\eta)$ corresponds to $\hat{K}(\eta)$ by the method described in Lemma 2.2.

If we denote $\hat{K}(\eta)$ by K_1 and carry out the same process as outlined above to obtain K_2 from K_1, we in turn obtain

$$K_0, K_1, K_2, ..., K_n, ...$$

and the corresponding bounded matrix function sequence (since M is compact, we see that $Q_{K_0}(\eta)$ is bounded)

$$Q_{K_0}(\eta), Q_{K_1}(\eta), ..., Q_{K_n}(\eta), ..., \tag{3.71}$$

with

$$Q_{K_0}(\eta) \geq Q_{K_1}(\eta) \geq ... \geq Q_{K_n}(\eta) \geq ... > 0 \tag{3.72}$$

and

$$\frac{dQ_{K_j}(\eta(t))}{dt} + A^*(\eta(t))Q_{K_j}(\eta(t)) + Q_{K_j}(\eta(t))A(\eta(t)) + W(\eta(t))$$
$$-(R(\eta(t))+Q_{K_{j-1}}(\eta(t))B(\eta(t)))U^{-1}(\eta(t))(R^*(\eta(t))+B^*(\eta(t))+B^*(\eta(t))Q_{K_{j-1}}(\eta(t))) = 0 \tag{3.73}$$

for $t \in [0, \infty)$.

By (3.72), we denote $\lim_{j\to\infty} Q_{K_j}(\eta)$ by $Q(\eta)$. In order to show, for each solution $\eta(t)$ of (3.26), that $Q(\eta(t))$ satisfies the Kalman-Riccati differential equation, we only need to show, on each closed interval $[a,b]$, that $\{Q_{K_j}(\eta(t))\}$ is uniformly convergent . We deal this proof in detail as follows.

Since $Q_{K_j}(\eta) \longrightarrow Q(\eta)$ on M, we see, for each $t \in [a,b]$, that $Q_{K_j}(\eta(t)) \longrightarrow Q(\eta(t))$. First we show the equicontinuity of each entry function of $Q_{K_j}(\eta(t))$ on $[a,b]$. In fact, for every entry $q(\eta(t))$ of $Q_{K_j}(\eta(t))$ we have (by (3.73) and the relationship between $Q_{K_j}(\eta)$ and its entry)

$$|q(\eta(t_1)) - q(\eta(t_2))| < \left\{\max_{[a,b]} |\frac{dq(\eta(t))}{dt}|\right\}|t_1 - t_2|$$
$$< \left\{3 \max_{[a,b]} ||\frac{dQ_{K_j}(\eta(t))}{dt}||\right\}|t_1 - t_2| < \beta|t_1 - t_2|$$

where (noting that M is compact and (3.72))

$$\beta = \max_{M,j} ||A^*(\eta)Q_{K_j}(\eta)+Q_{K_j}(\eta)A(\eta)+W(\eta)-(R(\eta)+Q_{K_j}(\eta)B(\eta))U^{-1}(\eta)(R^*(\eta)+B^*(\eta)Q_{K_j}(\eta))||$$

and where t_1, t_2 are between a and b. This shows the equicontinuity of each entry function on $[a,b]$. Then we will use the following theorem from D. Russell's book (cf. [4]):

Theorem. If the sequence of functions $f_j(t)$ converges pointwise to the function $f(t)$ on the compact interval $[a,b]$, and if the $f_j(t)$ are equicontinuous, then the $f_j(t)$ converge uniformly to $f(t)$ on $[a,b]$.

Proof: Choose $\epsilon > 0$ and, from the equicontinuity property, find $\delta > 0$ such that if x and y are in $[a,b]$ and $|x-y| < \delta$, then $|f_j(x) - f_j(y)| < \frac{\epsilon}{3}$ for any j. Then let $x_k, k = 1,2,...,K$ be a finite set of points such that every $x \in [a,b]$ lies within a distance δ of one of the x_k. Since there are only finitely many points x_k we can find $J(\epsilon)$ such that if i and j are both greater than $J(\epsilon)$ we have $|f_i(x_k) - f_j(x_k)| < \frac{\epsilon}{3}$ for all k, by repeated application of the Cauchy property. Then for any $x \in [a,b]$ and for i and j both greater than $J(\epsilon)$ we have, for some x_k within a distance δ of x, $|f_i(x) - f_j(x)| < |f_i(x) - f_i(x_k)| + |f_i(x_k) - f_j(x_k)| + |f_j(x_k) - f_j(x)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$. Thus the functions $f_j(t)$ are uniformly Cauchy on the interval $[a,b]$ and therefore converge uniformly to their limit $f(t)$.

Now we apply the above theorem first to the diagonal elements of the $Q_{K_j}(\eta(t))$ and then to the off-diagonal elements. Thus $\{Q_{K_j}(\eta(t))\}$ is uniformly convergent on $[a,b]$. Letting $j \to \infty$ in (3.73), we see that $Q(\eta(t))$ satisfies the Kalman-Riccati differential equation on $t \in [a,b]$. Because $[a,b]$ is an arbitrary closed interval, we see that $Q(\eta(t))$ satisfies the Kalman-Riccati differential equation on $[0,\infty)$.

On the other hand, by (3.57),(3.73) we see that (3.63) is true. But we would like to deal with this matter in detail in order to obtain some results to be used in later sections. In the following, we use C, μ, μ_1, μ_2, etc., to stand for appropriate constants.

Just as in the proof of Lemma 2.2 we have, for the solution of (3.61) and for all $t > 0$,

$$\xi^*(t)Q(\eta(t))\xi(t) = \int_t^\infty \xi^*(s)\tilde{W}_{\hat{Q}}(\eta(s))\xi(s)ds,$$

where

$$\tilde{W}_{\hat{Q}}(\eta) = (I, -[R(\eta)+Q(\eta)B(\eta)]U^{-1}(\eta)) \begin{pmatrix} W(\eta) & R(\eta) \\ R^*(\eta) & U(\eta) \end{pmatrix} \begin{pmatrix} I \\ -U^{-1}(\eta)[R^*(\eta) + B^*(\eta)Q(\eta)] \end{pmatrix}$$

Noting that $Q(\eta)$ and $\tilde{W}_{\hat{Q}}(\eta)$ all have finite upper and positive lower bounds, we have

$$\|\xi(t)\|^2 \leq C \int_t^\infty \|\xi(s)\|^2 ds.$$

But this implies

$$\|\xi(t)\| \leq \mu e^{-\gamma_0 t},$$

and the fundamental matrix $\Phi(t)$ of (3.61) satisfies

$$\|\Phi(t)\| \leq \mu e^{-\gamma_0 t}.$$

for $t > 0$. Consequently, we have

$$\xi(t) = \Phi(t)\Phi^{-1}(0)\xi_0,$$

and

$$\|\xi(t, \xi_0)\| \leq \mu_1 e^{-\gamma_0 t}\|\xi_0\|.$$

Thus we conclude that (3.63) is true.

Further, for $0 < s < t$, we will show that

$$\|\Phi(t)\Phi^{-1}(s)\| \leq \mu_2 e^{-\gamma_0(t-s)}.$$

In fact, if we let $\xi_s = \xi(s)$, we have

$$\xi(t, \xi_s) = \Phi(t)\Phi^{-1}(s)\xi_s. \tag{$*$}$$

Letting $\xi_1(T) = \xi(T + s)$, with $T = t - s$, we see that $\xi_1(T)$ is the solution of

$$\dot{\xi}_1(T) = \hat{A}_{\hat{Q}}(\eta(T + s))\xi_1(T), \;\; \xi_1(0) = \xi_s(= \xi(s)).$$

So, by use of (3.63) as just proved, we have

$$\|\xi(t,\xi_s)\| = \|\xi_1(T,\xi_s)\| \le \mu_1 e^{-\gamma_0 T}\|\xi_s\| = \mu_1 e^{-\gamma_0(t-s)}\|\xi_s\|.$$

This implies that, when $\|\xi_s\|$ is small,

$$\|\frac{\partial \xi(t,\xi_s)}{\partial \xi_s}\|. \le \mu_2 e^{-\gamma_0(t-s)}$$

Taking the derivative with respect to ξ_s in (*) above we finally obtain

$$\|\Phi(t)\Phi^{-1}(s)\| \le \mu_2 e^{-\gamma_0(t-s)}.$$

3.5 Lyapounov Functions

Lemma 2.4 Let $u = k_0(x)$ be a twice continuously differentiable nonlinear feedback function for (3.1) with $k_0(x) \equiv 0$, for all $x \in M$. Suppose that, for the corresponding $K_0(\eta)$, all solutions $\delta\xi(t)$ of (3.31) satisfy the conclusion of Lemma 2.1; then there exists a continuously differentiable function $V(x)$ such that $V(x) > 0$, $x \notin M$ and for each solution $x(t)$ of $\dot{x} = F(x, k_0(x))$ with $x(0) = x_0$ sufficiently near M, $V(x(t))$ is strictly decreasing as t increases. Consequently, from the standard result in [6], we see that M is uniformly orbitally stable for

$$\dot{x} = F(x, K_0(x)).$$

Proof: We already have

$$M \subset \cup_{j=1}^{J} n_j, \tag{3.74}$$

where

$$n_j = n(\hat{x}_j, \frac{\rho(\hat{x}_j)}{4})$$

with a one to one correspondance between x and (ξ,η) in each neighborhood $4n_j$. Now we define

$$N_k = n_k - \cup_{j=1}^{k-1} n_j, k = 1, 2, ..., J. \tag{3.75}$$

We see that

$$N_k \cap N_l = \emptyset, k \ne l,$$

and

$$\cup_{k=1}^{J} N_k = \cup_{j=1}^{J} n_j.$$

Then we define $V(x)$ as follows: for every x in $\cup_{k=1}^{J} n_k$ we have a unique k such that $x \in N_k \subset n_k$. In $4n_k$ we have a one to one correspondance

$$x \longleftrightarrow (\xi, \eta).$$

Following the same argument as in the proof of Lemma 2.1, we see that $Proj_M x \in 4n_k$, and there is a unique $(0, \eta_k)$ corresponding to $Proj_M x$, noting that, on M, $\xi = 0$. Then we define

$$V(x) = \int_0^\infty \varphi^*(\hat{x})\tilde{W}_{K_0}(P\hat{x})\varphi(\hat{x})dt, \tag{3.76}$$

with $\tilde{W}_{K_0}(\tau)$ being defined before lemma 2.2 and $\hat{x}(t)$ being the solution of

$$\dot{\hat{x}}(t) = F(\hat{x}, k_0(\hat{x})), \ \hat{x}(0) = x.$$

Clearly, $V(x)$ is well defined since $\varphi(\hat{x}(t))$ decays exponentially and $\tilde{W}_{K_0}(\tau)$ is bounded on M. Also, we see that when $x \notin M$, $V(x) > 0$ since $\tilde{W}_{K_0}(\tau) > 0$, on M.

Now let $x(t)$ be a solution of $\dot{x} = F(x, k_0(x))$ with $x(0) = x_0$ near M. We show that $V(x(t))$ is decreasing with respect to $t \longrightarrow \infty$. Given $t_1 < t_2$, we show that

$$V(x(t_1)) > V(x(t_2)). \tag{3.77}$$

In fact, we have by the definition of $V(x)$ above

$$\frac{V(x(t_2)) - V(x(t_1))}{t_2 - t_1} = \frac{d}{dt}V(x(t^*))$$

$$= \frac{d}{dt}[\int_{t^*}^\infty \varphi^*(x(s))\tilde{W}_{K_0}(Px(s))\varphi(x(s))ds]$$

$$= -\varphi^*(x(t^*))\tilde{W}_{K_0}(Px(t^*))\varphi(x(t^*)) < 0, \tag{3.78}$$

with some t^* staying between t_1, t_2. Thus we have proved that $V(x(t))$ is strictly decreasing.

3.6 A Nonlinear Hamiltonian System

Here we will complete the proof of Theorem 1.1. The optimization problem is that of finding

$$min_u\{\int_0^\infty \Gamma(\xi,\tau,u)dt\}$$

such that $\xi(t),\tau(t)$ satisfy (cf.(3.22),(3.23))

$$\dot{\xi} = f(\xi,\tau,u),$$

$$\dot{\tau} = g(\xi,\tau,u),$$

$$\xi(0) = \xi_0, \tau(0) = \eta_0,$$

where (ξ_0,τ_0) is the local coordinate of x_0 near M.

Let $\eta(t)$ be the solution of (3.26):

$$\dot{\eta} = g(0,\eta,0),$$

$$\eta(0) = \eta_0,$$

where $(0,\eta_0)$ is the local coordinate of $Proj_M x_0$. Since

$$\Gamma(0,\eta,0) \equiv 0,$$

$$f(0,\eta,0) \equiv 0,$$

we have in analytic case

$$\int_0^\infty (\Gamma(\xi,\tau,u) - \Gamma(0,\eta,0))dt$$

$$= \int_0^\infty (\xi^*,u^*) \begin{pmatrix} W(\eta), & R(\eta) \\ R^*(\eta), & U(\eta) \end{pmatrix} \begin{pmatrix} \xi \\ u \end{pmatrix} + \gamma(\xi,\eta,u)dt,$$

where $\|\gamma(\xi,\eta,u)\| = \mathcal{O}\{\|\xi\|^3 + \|u\|^3\}$, and

$$f(\xi,\tau,u) = f(\xi,\tau,u) - f(0,\eta,0) = A(\eta)\xi + B(\eta)u + f_1(\xi,\eta,u), \tag{3.79}$$

wherein we recall that $\|f_1(\xi,\eta,u)\| = \mathcal{O}\{\|\xi\|^2 + \|u\|^2\}$.

We see then that in the analytic case the only dependence is on the solution $\eta(t)$ of $\dot{\eta} = g(0,\eta,0)$, $\eta(0) = \eta_0$ and there is no explicit dependence on $\tau(t)$. Thus the original optimization problem is equivalent to finding

$$min_u\{\int_0^\infty (\xi^*, u^*) \begin{pmatrix} W(\eta), & R(\eta) \\ R^*(\eta), & U(\eta) \end{pmatrix} \begin{pmatrix} \xi \\ u \end{pmatrix} + \gamma(\xi,\eta,u)dt\}$$

wherein $\xi(t), \eta(t)$ satisfy

$$\dot{\xi} = A(\eta)\xi + B(\eta)u + f_1(\xi,\eta,u),$$

$$\dot{\eta} = g(0,\eta,0),$$

$$\xi(0) = \xi_0, \quad \eta(0) = \eta_0,$$

for ξ_0 near zero.

The adjoint system is

$$\dot{\lambda_\xi} = -A^*(\eta)\lambda_\xi - 2W(\eta)\xi - 2R(\eta)u + f_2(\xi,\eta,u), \tag{3.80}$$

noting that $\dot{\eta} = g(0,\eta,0)$ is uncontrolled. From the Pontryagin necessary conditions [5], we see that the optimal control should have the form

$$\hat{u} = -\frac{1}{2}U^{-1}(\eta)(2R^*(\eta)\xi + B^*(\eta)\lambda_\xi) + h(\xi,\eta,\lambda_\xi). \tag{3.81}$$

Carrying out the change of variable with $Q(\eta(t))$ as in Lemma 2.3:

$$\lambda_\xi(t) = \mu_\xi(t) + 2Q(\eta(t))\xi(t), \tag{3.82}$$

we obtain the systems

$$\dot{\xi} = \hat{A}_{\hat{Q}}(\eta)\xi + a(\eta)\mu_\xi + \tilde{f}_1(\xi, \eta, \mu_\xi), \tag{3.83}$$

$$\dot{\mu}_\xi = -\hat{A}^*_{\hat{Q}}(\eta)\mu_\xi - \tilde{f}_2(\xi, \eta, \mu_\xi). \tag{3.84}$$

where $a(\eta) = -\frac{1}{2}B(\eta)U^{-1}(\eta)B^*(\eta)$.

Lemma 2.5 For the family of nonlinear analytic systems (3.83) (3.84) there exists a real $m(n-m)$-dimensional analytic invariant manifold S in which the origin of the $\xi \times \mu_\xi$ space is asymptotically stable for (3.83),(3.84), uniformly for all $\eta(t)$.

Proof: In (3.83),(3.84) we write

$$r(\eta, \xi, \mu_\xi) = \begin{pmatrix} a(\eta)\mu_\xi + \tilde{f}_1(\xi, \eta, \mu_\xi) \\ -\tilde{f}_2(\xi, \eta, \mu_\xi) \end{pmatrix}. \tag{3.85}$$

Let $\Phi(t)$ denote the fundamental matrix of

$$\dot{\xi}(t) = \hat{A}_{\hat{Q}}(\eta(t))\xi(t).$$

We write

$$U_1(t,s) = \begin{pmatrix} \Phi(t)\Phi^{-1}(s), & 0 \\ 0, & 0 \end{pmatrix},$$

and

$$U_2(t,s) = \begin{pmatrix} 0, & 0 \\ 0, & \Phi^{*-1}(t)\Phi^*(s) \end{pmatrix}.$$

We consider the integral equation

$$\theta(t, a, \eta_0) = U_1(t, 0)a + \int_0^t U_1(t,s)r(\eta(s,\eta_0), \theta(s, a, \eta_0))ds$$

$$-\int_t^{\infty} U_2(t,s) r(\eta(s,\eta_0), \theta(s,a,\eta_0)) ds \tag{3.86}$$

with a being a constant $2(n-m)$-dimensional vector.

From the proof of Lemma 2.3 we see that, when $s < t$,

$$\|U_1(t,s)\| = \|\Phi(t)\Phi^{-1}(s)\| \le \mu e^{-\gamma_0(t-s)}. \tag{3.87}$$

This implies that in (3.86)

$$\int_0^t U_1(t,s) ds$$

converges as $t \to \infty$. At the same time we have, when $t < s$,

$$\|U_2(t,s)\| = \|\Phi^{-1*}(t)\Phi^*(s)\| \le \mu e^{-\gamma_0(s-t)}. \tag{3.88}$$

But this implies that in (3.86)

$$\int_t^{\infty} U_2(t,s) ds$$

converges. Since

$$\tilde{f}_1(0,\eta,0) = 0, \tilde{f}_2(0,\eta,0) = 0,$$

$$\frac{\partial \tilde{f}_1(0,\eta,0)}{\partial(\xi,\mu_\xi)} = 0, \frac{\partial \tilde{f}_2(0,\eta,0)}{\partial(\xi,\mu_\xi)} = 0,$$

and since we can assume $\|a(\eta)\|$ is specified as small as needed (since M is compact and a transformation like

$$\nu_\xi = C\mu_\xi$$

can be used with some positive constant C for this purpose), we see that for fixed $\epsilon < \frac{1}{2}$, there is a positive δ such that

$$\|r(\eta,\xi_1,\mu_{\xi 1}) - r(\eta,\xi_2,\mu_{\xi 2})\| \le \epsilon \|(\xi_1,\mu_{\xi 1})^* - (\xi_2,\mu_{\xi 2})^*\|, \tag{3.89}$$

holds for $\|(\xi_1,\mu_{\xi 1})^*\| \le \delta$,$\|(\xi_2,\mu_{\xi 2})^*\|| \le \delta$ and uniformly for $\eta \in M$, provided we already assume

$$\|a(\eta)\| < \epsilon. \tag{3.90}$$

Using successive approximations to solve (3.86) with initial approximation $\theta_0(t, a, \eta_0) = 0$, we readily obtain

$$||\theta_{l+1}(t, a, \eta_0) - \theta_l(t, a, \eta_0)|| \leq \frac{C||a||}{2^l} e^{-\gamma_0 t}, \tag{3.91}$$

for some positive constant C, which leads to the existence of a solution θ of (3.86) satisfying

$$||\theta(t, a, \eta_0)|| \leq 2C||a||e^{-\gamma_0 t}, \tag{3.92}$$

uniformly for $\eta_0 \in M$.

That θ is a solution of (3.86) is immediate for $||a||$ small and for all $\eta_0 \in M$, since by the estimate of $||\Phi^{-1*}(t)\Phi^*(s)||$ the integral in (3.86) converges. Moreover, we see that θ is analytic in ξ_0 for fixed t. From (3.86), it follows that the first $n - m$ components of $\theta(0, a, \eta_0)$ are $a_j, j = 1, 2, ..., n - m$ and the latter components are given by

$$\theta_j(0, a, \eta_0) = \{-\int_0^\infty U_2(0, s) r(\eta(s, \eta_0), \theta(s, a, \eta_0)) ds\}_j, \tag{3.93}$$

$$j = n - m + 1, ..., 2(n - m).$$

We define the function q_* by

$$q_{*j}(a_1, ..., a_{n-m}, \eta_0) = \{\int_0^\infty U_2(0, s) r(\eta(s, \eta_0), \theta(s, a, \eta_0)) ds\}_{n-m+j} \tag{3.94}$$

for $j = 1, 2, ..., n - m$, and the initial value $(\eta, \xi, q)^* = \theta(0, a, \eta)$ satisfy the equation

$$q - q_*(\xi, \eta) = 0, \tag{3.95}$$

in the $\xi \times \eta \times q$- space, thereby defining a $m(n - m)$-dimensional manifold S satisfying $S \cap R^m = M$.

Just as in [1],pp.90-91, we see that S is the desired manifold for Lemma 2.5. We can show that the trajectories of (3.83),(3.84) intersecting S do not leave S. Let $\theta_0 \times \eta_0 \in S$

and $||\theta_0||$ be small. Then the equation for the manifold, which is

$$\theta_0 - \{ \begin{pmatrix} I, & 0 \\ 0, & 0 \end{pmatrix} \theta_0 - \int_0^\infty U_2(0,s) r(\eta(s,\eta_0), \theta(s,\theta_0,\eta_0)) ds \} = 0, \tag{3.96}$$

is satisfied. Let the trajectory through θ_0 at $t = 0$, $\theta(t,\theta_0,\eta_0) = \begin{pmatrix} \xi \\ \mu_\xi \end{pmatrix} (t,\theta_0,\eta_0)$ satisfy the differential equation (3.83),(3.84). Then we have (from (3.86), by changing s to $\tilde{s} + t$)

$$\theta(t,\theta_0,\eta_0) - \{ \begin{pmatrix} I, & 0 \\ 0, & 0 \end{pmatrix} \theta(t,\theta_0,\eta_0) - \int_0^\infty U_2(0,s) r(\eta(s,\eta_0), \theta(s+t,\theta_0,\eta_0)) ds \} = 0.$$

Noting the formula

$$\theta(s+t,\theta_0,\eta_0) = \theta(s,\theta(t,\theta_0,\eta_0),\eta_0),$$

which follows from the uniqueness of the solution to (3.83),(3.84), we obtain

$$\theta(t,\theta_0,\eta_0) - \{ \begin{pmatrix} I, & 0 \\ 0, & 0 \end{pmatrix} \theta(t,\theta_0,\eta_0) - \int_0^\infty U_2(0,s) r(\eta(s,\eta_0), \theta(s,\theta(t,\theta_0,\eta_0),\eta_0)) ds \} = 0. \tag{3.97}$$

But this implies that for the trajectory through θ_0 for $||\theta_0||$ small, $\theta(t,\theta_0) \times M$ does not leave S. The proof of Lemma 2.5 is thus complete.

Now from Lemma 2.5, we see that for the initial value (ξ_0, η_0) with ξ_0 small, (3.83),(3.84) imply that

$$u(\xi,\eta) = -U^{-1}(\eta)(R^*(\eta) + B_1^*(\eta)Q(\eta))\xi + (\frac{-1}{2} B_1 U^{-1} B_1^*) q_*(\xi,\eta) + h(\xi,\eta,q_*(\xi,\eta)) \tag{3.98}$$

is the desired stabilizing control. Just as in [1] ,pp.91-92 and pp.83-84 we see that this is the unique optimal control. The only difference here is that we deal with

$$H(\xi,\eta,u) = \frac{\partial J(\xi,\eta,\hat{u})}{\partial \eta} g(0,\eta,0) + \frac{J(\xi,\eta,\hat{u})}{\partial \xi} f(\xi,\eta,u) + \Gamma(\xi,\eta,u). \tag{3.99}$$

By Lemma 2.2, we see that

$$H(\xi,\eta,\hat{u}) \equiv 0,$$

near $\xi = 0$. Also we have

$$\frac{\partial H(\xi,\eta,\hat{u})}{\partial u} \equiv 0,$$

near $\xi = 0$ and whenever

$$\frac{\partial J(\xi,\eta,\hat{u})}{\partial \xi} f_u(\xi,\eta,\hat{u}(\xi,\eta)) + \Gamma_u(\xi,\eta,\hat{u}(\xi,\eta)) = 0 \tag{3.100}$$

holds for all ξ near zero. But by lemma 2.5, using the same argument as in [1], pp.91-92, we see that this is true since $g(0,\eta,0)$ is uncontrolled. Moreover, from (3.92) we see that the optimal trajectory $\xi(t)$ satisfies

$$||\xi(t)|| \leq C||\xi_0||e^{-\gamma_0 t}, \tag{3.101}$$

for some positive constant C. Then, returning to x- coordinate by (3.19), we obtain the desired optimal control

$$u = K(x)$$

as described in Theorem 1.1. Furthermore, from (3.101), using Lemma 2.4, we see that

$$\dot{x} = F(x, K(x))$$

has M as an uniformly orbitally stable invariant set.

References

1. Lukes, D.L.: *Optimal regulation of nonlinear dynamical systems*, SIAM J. Control 7 (1969), pp.75-100

2. Lukes, D.L.: *Stabilizability and optimal control, Funkcialaj Ekvacioj*, 11 (1968) pp.39-50

3. Russell, D.L. and Zhang, X.: *Extending linear-quadratic optimal control laws to non-linear systems and/or nonquadratic cost criteria*, 1993

4. Russell, D.L.: *Mathmatics of Finite Dimensional Control System*, Marcel Dekker, Inc., New York, 1979

5. Pontryagin, L.S.,V.G. Boltyanskii, R.V. Gamkrelidze and E.F.Mishchenko: *The Mathematical Theory of Optimal Processes* (Trans.by K.N.Trirogoff,Ed.by L.W.Neustadt), Interscience Pub., John Wiley Sons, Inc., New York and London, 1962

6. Cronin, J. *Differential Equation: Introduction and Qualitative Theory*, Vol.54, Pure and Applied Mathematics, Marcel Dekker, Inc, New York, 1980

7. Reid, T.William, *Ordinary Differential Equations*, John Wiley Sons Inc., 1971

8. Coddington, Earl A : *Theory of Ordinary Differential Equations*, McGraw-Hill book Company, 1955

Vita of Jinghao Zhu

Jinghao Zhu was born in Shanghai, China on November 11, 1949. He was a undergraduate student in mathematics at East China Normal University from 1978 to 1985 and got the Master degree in mathematics in 1985. He taught Complex Analysis in Tongji University, Shanghai, from 1985 to 1992. He came to America in 1992 and received the Ph.D degree in Mathematics from Virginia Polytechnic Institute and State University in 1996.

Jinghao Zhu