

DISCRETE DYNAMICAL SYSTEMS IN SOLVING H - EQUATIONS

by

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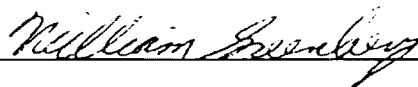
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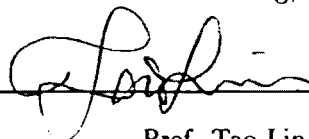
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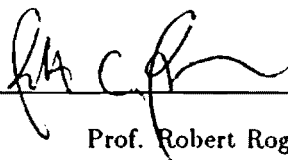
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(Abstract)

Three discrete dynamical models are used to solve the Chandrasekhar H -equation with a positive or negative characteristic function. Two of them produce series of continuous functions which converge to the solution of the H -equation. An iteration model of the n th approximation for the H -equation is discussed. This is a nonlinear n -dimensional dynamical system. We study not only the solutions of the n th approximation for the H -equation but also the mathematical structure and behavior of the orbits with respect to the parameter function, i.e. characteristic function. The dynamical system is controlled by a manifold. For $n=2$, stability of the fixed points is studied. The stable and unstable manifolds passing through the hyperbolically fixed point are obtained. Globally, the bounded orbits region is given. For parameter c in some region a periodic orbit of one dimension will cause periodic orbits in the higher dimensional system. For changing parameter c , the bifurcation points are discussed. For $c \in (-5.6049, 1]$ the system has a series of double bifurcation points. For $c \in (-8, -5.6049]$ chaos appears. For c in a window contained the chaos region, a new bifurcation phenomenon is found. For $c \leq -7$ any periodic orbits appear. For c in the chaos region the behavior of attractor is discussed. Chaos occurs in the n -dimensional dynamical system.

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For
my wife Shujing Li Chen

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Chapter 1

INTRODUCTION

The subject of radiative transfer in plane-parallel atmospheres is presented as a branch of mathematical physics with its own characteristic methods and techniques. On the physics side the novelty of the methods used consists in the employment of certain general principles of invariance which on the mathematics side leads to the study of nonlinear integral equations. For example, for a parallel beam of radiation incident on a semi-infinite plane-parallel atmosphere with diffuse reflection at the surface, the transport equation for the intensity I may be written

$$\mu \frac{dI(\tau, \mu)}{d\tau} = I(\tau, \mu) - \frac{1}{2} \int_{-1}^1 I(\tau, \mu') d\mu' - \mathfrak{S}(\tau, \mu). \quad (1.1)$$

The intensity at the surface is usually expressed in terms of a scattering function S , which itself is expressible as a product of so-called H -functions. These functions, first introduced by S. Chandrasekhar [5], satisfy nonlinear integral equations of the form

$$H(\mu) = 1 + \mu H(\mu) \int_0^1 \frac{\psi(\mu')}{\mu + \mu'} H(\mu') d\mu', \quad (1.2)$$

where $\psi(\mu)$ is a characteristic function. It is well known that this equation does not have a unique solution. However, the "physical" solution of (1.2) is subject to the constraints

$$\nu_j \int_0^1 \frac{\psi(\mu) H(\mu)}{\nu_j - \mu} d\mu = 1, \\ j = 0, \dots, \alpha,$$

where $\nu_j, j = 0, \dots, \alpha$, are zeros of the dispersion function

$$\Lambda = 1 + z \int_{-1}^{+1} \frac{\psi(s)}{s - z} ds.$$

A traditional approach to obtaining values for the H -equation is to attempt to solve (1.2) by iteration. There are two iteration models important in solving the equation in $L_1(0, 1)$ for positive $\psi(\mu)$ with

$$\int_0^1 \psi(\mu) d\mu \leq \frac{1}{2},$$

namely,

$$H_{n+1}(\mu) = 1 + \mu H_n(\mu) \int_0^1 \frac{\psi(\mu')}{\mu + \mu'} H_n(\mu') d\mu', \quad (1.3)$$

and

$$H_{n+1}(\mu) = \left(1 - \mu \int_0^1 \frac{\psi(\mu')}{\mu + \mu'} H_n(\mu') d\mu' \right)^{-1}. \quad (1.4)$$

Bittoni, Casadei, and Lorenzutta [1] have shown that when the right hand side of the H -equation is regarded as a bilinear operator from $L_1(0, 1) \times L_1(0, 1)$ to $L_1(0, 1)$ with the norm of its Fréchet derivative less than unity the bilinear operator is contractive in a ball. Thus the unique solution of H -equation can be obtained by iteration of the H -equation model (1.3). Subsequently, Bowden and Zweifel [2] showed that the solution so obtained was indeed the "physical" solution. Bowden [3] used the iteration model (1.4) to obtain a series converging to the solution of the H -equation. In Chapter 6, using both iteration models (1.3) and (1.4) we study the H -equation in $C[0, 1]$ for $\psi(\mu) \in L_1(0, 1)$, but not restricted to $\psi(\mu)$ positive. We will find a series of continuous functions which converges to the continuous solution of the H -equation.

To get the series function numerically in both iteration models (1.3) and (1.4), we need to calculate integrals. For solving the H -equation (1.2) numerically the n th approximation of the H -equation is

$$H(\mu) = 1 + \mu H(\mu) \sum_{j=1}^n \frac{a_j \psi(\mu_j)}{\mu + \mu_j} H(\mu_j) \quad (1.5)$$

S. Chandrasekhar [6] gave a solution formula with constants which are non-negative roots of an associated characteristic equation. But it is hard to solve the roots.

An iteration model of the n th approximation is given by:

$$H_{n+1}(\mu_i) = 1 + \mu_i H_n(\mu_i) \sum_{j=1}^n \frac{a_j \psi(\mu_j)}{\mu_i + \mu_j} H_n(\mu_j), \quad (1.6)$$

$$i = 1, 2, \dots, n.$$

Since there is a parameter function, $\psi(\mu)$, in the H -equation, these iteration systems above have a parameter. The system (1.6) is an n -dimension nonlinear discrete dynamical system which, in appearance, is far more complex than original the H -equation.

From the system (1.6) we have

$$\sum_{i=1}^n a_i \psi(\mu_i) H_{n+1}(\mu_i) = \sum_{i=1}^n a_i \psi(\mu_i) + \frac{1}{2} \left[\sum_{i=1}^n a_i \psi(\mu_i) H_n(\mu_i) \right]^2. \quad (1.7)$$

Let

$$X(n) = \frac{1}{2} \sum_{i=1}^n a_i \psi(\mu_i) H_n(\mu_i) \quad (1.8)$$

and

$$c = \frac{1}{2} \sum_{i=1}^n a_i \psi(\mu_i). \quad (1.9)$$

We obtain

$$X(n+1) = c + X(n)^2. \quad (1.10)$$

From (1.10) and the relation (1.8) we can see that the behavior of the one dimension system (1.8) will influence the n -dimensional system (1.6).

Iteration of continuous maps of a bounded region into itself serves as the simplest example of models for dynamical systems. These models present an interesting mathematical structure going far beyond the simple equilibrium solutions one might expect. If in addition the dynamical system depends on an experimentally controllable parameter, there is a corresponding mathematical structure telling a lot about interrelations between the behavior for different parameter values. For the system (1.6) we are interested in not only the fixed points but also the mathematical behavior for the parameter function $\psi(\mu)$. We will see in Chapter 3 that there are periodic and chaos phenomena in the dynamic system (1.6).

There are a variety of instances where chaotic phenomena can arise. In the early 1970s, it was observed that in biological populations the nonlinearities that are inherent in simple models for the regulation of plant and animal populations can lead to periodic and chaotic

dynamics. Attracting particular interest were interactions between prey and predator (including hosts and pathogens, hosts and parasitic insects, and harvested populations). Some of the complications in disentangling deterministic chaos from environmental noise were discussed. The combination of population biology with population genetics leads to an even richer assortment of nonlinear phenomena and to the suggestion that many genetic polymorphisms may vary cyclically or chaotically (rather than being steady, as usually had been assumed implicitly). Such investigations in the early 1970s led to the realization that the simplest nonlinear models for population with discrete, nonoverlapping generations (first-order difference equations with one critical point) could exhibit a surprising array of dynamical behaviors (May [7], [8]; Li and Yorke [12]; May and Oster [11]). Subsequent work showed that even richer dynamical behaviors could be generated by simple, deterministic equations for single populations with discrete but overlapping generations (higher-order difference equations), for single populations with continuous growth where regulatory effects contain time lags (time-delayed differential equations), and for two or more interacting populations. The dynamical properties of these models have been the subject of several reviews (Rogers [13]; Olsen and Degn [18]; Kloeden and Meed [14]; Lauwerier [15][16]; May [9], [10]).

The one dimensional quadratic map

$$u_{t+1} = u_t(r - qu_t). \quad (1.11)$$

is sometimes called the "logistic" difference equation. In the limit $q = 0$, it describes a population growing purely exponentially (for $r > 1$); for $q \neq 0$ quadratic nonlinearity produces a growth curve with a hump, the steepness of which is tuned by the parameter r . By writing $N = qu/\tau$, the equation may be brought into canonical form

$$N_{n+1} = rN_n(1 - N_n). \quad (1.12)$$

In many respects this simple system is typical of one dimensional dynamical systems, and has a number of properties we will encounter in our study of the H -equation. If $3 > r > 1$, the fixed point at $x^* = 1 - 1/r$ is an attractor. At $r = 3$ the system bifurcates to give

a cycle of period 2 which is stable for $1 + \sqrt{6} > r > 3$. As r increases beyond this, successive bifurcations give rise to a cascade of period doubling, producing cycles of periods $2, 4, 8, 16, \dots, 2^n$ for r in the range $3.570... > r > 3$. Beyond the point of this cascade, $4 > r > 3.570...$, there lies an apparently chaos region. In detail, the apparently chaotic region comprises infinitely many tiny windows of r -value, in which basic cycles of period k are born stable (accompanied by unstable twins), cascade down through their periodic-doubling to give stable harmonics of periods $k2^n$, and become unstable; this sequence of events recapitulates the processes seen more clearly for the basic fixed point of period 1.

The nature of the chaotic region for such "maps of the interval" is often misunderstood. The chaos is largely a mosaic of stable cycles, one giving way to another with kaleidoscopic rapidity as r increases. But for essentially all practical applications, the chaotic region has the effectively random character that superficial inspection or numerical simulations suggest.

Since 1975, sequences of bifurcations governing the route to chaos in one-dimensional discrete dynamical systems have been extensively studied [48], [37], [38], [39], [40], [41], [47]. However, models reducing to two dimensional endomorphisms are often obtained in several fields. Up to now they have been mainly studied by numerical simulations.

Among the few two dimensional discrete dynamic system which have been studied is the Hénon map [21]. This map is a quadratic with a constant jacobian. Hénon use a numerical simulation of the iteration sequence, solution for particular parameter. With a finite number of iterations of a numerical simulation, its solution was interpreted as a "strange attractor". Afterwards many papers and books were devoted to this problem ([22], [28], [29], [30], [31], [26], [32], [33], [34], [35], [36], [42], [43], [44], [22], [45], [46]).

For our system, letting $a_j \psi(\mu_j) = \frac{c\omega_j}{2}$ and $h_j = \frac{c}{4}H(\mu_j)$, we can rewrite our n -dimensional dynamic system (1.6) as

$$h_j(t+1) = \frac{c}{4} + 2\mu_j h_j(t) \sum_{i=1}^n \frac{\omega_i}{\mu_i + \mu_j} h_i(t), \quad (1.13)$$

$$i = 1, 2, \dots, n.$$

One may note that the equation looks like biological equations, where h_j is the population of the j th race,

$$2\mu_j \sum_{i=1}^n \frac{\omega_i}{\mu_i + \mu_j} h_i(t)$$

is the growth rate function for the j th race, and where the input number, $\frac{c}{4}$, is a regeneration or death rate for positive c or for negative c , respectively. So the dynamic system (1.13), which is from radiative transfer in physics, has biological meaning as well. There are no papers to study this particular system. As a mathematics problem, few papers study dynamic systems in dimensional more than two. We take three chapters to study the two dimensional system of (1.13).

In Chapter 2 we find a linear transformation to simplify the system. We find that the two dimensional system is controlled by a one dimensional system. All the stability of the fixed points is given. In Chapter 3 the invariant manifolds for hyperbolic fixed points are found. In Chapter 4, in order to find global properties of the orbits, we use methods in quantitative and stability theory of differential equations.

In the system of differential equations

$$\frac{dx}{dt} = f(x, y)$$

$$\frac{dy}{dt} = g(x, y),$$

the vector field $(f(x, y), g(x, y))$ tells us the slope and the direction of the orbits. From the field near a fixed point we will know the stability of the fixed point.

In the discrete dynamical system

$$x_{t+1} - x_t = f(x_t, y_t)$$

$$y_{t+1} - y_t = g(x_t, y_t),$$

the difference vector field $(f(x_t, y_t), g(x_t, y_t))$ tells us where the next point of the orbit is going to be.

Since the two dimensional system is governed by a one dimensional system (we call it the governing system), we use the properties of the governing one dimensional mapping to construct regions of the two dimensional plane. The properties of the two dimensional mapping are discussed. We find a boundary region which is invariant for the two dimensional system mapping. This implies that the orbits of the system in this region will be bounded. We also find that ellipses are mapped to line sections, and a region bounded by an ellipse is mapped onto a triangular region. Combining all the properties of the two dimensional mapping, we obtain a region such where all attractors lie in.

The attractors of the system depend on the parameter. Since we find a transformation such that the system is changed to new form in which the one dimensional system is equivalent to the logistic system, the problem is how to use the results of the logistic system to analyze our two dimensional problem. In Chapter 5 we prove that the existence of periodic orbits in the two dimensional system are governed by the one dimensional system, i.e. if there is an n -periodic orbit of the one dimensional system, then there is a periodic orbit with the same period for the two dimensional system. So the set of bifurcation points for the higher dimensional system includes the set of bifurcation points of the of governing system. For the governing system, we know that if there is a 3-periodic orbit then there are period orbits with any period n and chaos appears. From our result we will have that a 3-periodic orbit in the two dimensional system implies that the orbits of any period and chaos appear.

We know that many results of the logistic dynamical system are from numerical computation or computer experiment, for example, the values of bifurcation points and attractors. For our two dimensional dynamical system, computer experiments tell us that for the parameter $c \leq -7$ the system has a 3-periodic orbit. This means that for c in this region there are any periodic orbits of the two dimensional system. We also find for $c \in (-5.6049, 1]$ the system has period doubling bifurcations. When $c < -8$ for our n -dimensional system the orbits starting from a neighborhood of the origin are divergent. In the region $[-8, -5.6049]$ there are many windows. When the parameter decreases in the small region

$[-5.92849, -5.91895]$ we find a new phenomena. In this region, successive bifurcations of attractors appear. It is not period doubling but cycles of periods 6, 7, 8, 9, 10, ...

To chaos research, mathematics has become an experimental science, with the computer replacing laboratories. Chaos has become not just theory but also method, not just a canon of beliefs but also a way of doing science. Chaos has created its own technique of using computers. Chaos is making a new science [20].

Chapter 2

STABILITY OF FIXED POINTS

1. Origin of the Problem

An important nonlinear integral equation in the theory of radiative transfer is the H -equation

$$H(\mu) = 1 + \mu H(\mu) \int_0^1 \frac{\psi(\mu')}{\mu + \mu'} H(\mu') d\mu', \quad (2.1)$$

where the characteristic function $\psi(\mu)$ is an even polynomial in μ satisfying the condition

$$\int_0^1 \psi(\mu) d\mu \leq \frac{1}{2} \quad (2.2)$$

We call the solution of equation (2.1) the H -function. To find the H -function of (2.1) numerically, we consider the equation

$$H(\mu) = 1 + \mu H(\mu) \sum_{j=1}^N \frac{a_j \psi(\mu_j)}{\mu + \mu_j} H(\mu_j), \quad (2.3)$$

where $a_{\pm j} (j = 1, \dots, n)$, $a_j = -a_{-j}$ and $\mu_{\pm j} (j = 1, \dots, n)$, $\mu_j = -\mu_{-j}$ are the weights and divisions appropriate to a quadrature formula in the interval $(-1, +1)$. For equation (2.1), S. Chandrasekhar[[5]] found the solution

$$H(\mu) = \frac{1}{\mu_1 \cdots \mu_n} \frac{\prod_{j=1}^N (\mu + \mu_j)}{\prod_{i=1}^N (1 - k_i \mu)}, \quad (2.4)$$

where the k_i 's ($i = 1, \dots, N$) are the roots of the associated characteristic equation

$$1 = 2 \sum_{j=1}^N \frac{a_j \psi(\mu_j)}{1 - k^2 \mu_j^2}. \quad (2.5)$$

We can see that there are 2^N solutions of equation (2.3). Physicists are interested in the unique solution of (2.3) with k_i 's non-negative and $\mu \geq 0$. Letting

$$a_j \psi(\mu_j) = \frac{c\omega_j}{2}$$

and multiplying (2.3) by $\frac{c}{4}$, we have

$$\frac{cH(\mu)}{4} = \frac{c}{4} + 2\frac{cH(\mu)}{4}\mu \sum_{i=1}^N \frac{\omega_i}{(\mu + \mu_i)} \frac{cH(\mu_i)}{4} \quad (2.6)$$

and letting $h_i = \frac{c}{4}H(\mu_i)$ and $h(\mu) = \frac{c}{4}H(\mu)$ we have

$$h(\mu) = \frac{c}{4} + 2h(\mu)\mu \sum_{i=1}^N \frac{\omega_i}{(\mu + \mu_i)} h(\mu_i).$$

For $N = 2$ this yields

$$h_1 = \frac{c}{4} + \omega_1 h_1^2 + \frac{2\mu_1 \omega_2 h_1 h_2}{\mu_1 + \mu_2} \quad (2.7)$$

$$h_2 = \frac{c}{4} + \frac{2\mu_2 \omega_1 h_1 h_2}{\mu_1 + \mu_2} + \omega_2 h_2^2 \quad (2.8)$$

which gives

$$\omega_1 h_1 + \omega_2 h_2 = \frac{c}{4}(\omega_1 + \omega_2) + (\omega_1 h_1 + \omega_2 h_2)^2. \quad (2.9)$$

Let

$$x = \omega_1 h_1 + \omega_2 h_2$$

so that the equation (2.9) becomes

$$x = \frac{c}{4}(\omega_1 + \omega_2) + x^2.$$

From (2.7) and (2.8) we have

$$\begin{aligned} & \frac{1}{\mu_1 + \mu_2}(\mu_2 \omega_1 h_1 + \mu_1 \omega_2 h_2) \\ &= \frac{c}{4} \frac{\mu_2 \omega_1 + \mu_1 \omega_2}{\mu_1 + \mu_2} + \frac{1}{\mu_1 + \mu_2}(\mu_2 \omega_1^2 h_1^2 + \mu_1 \omega_2^2 h_2^2 + 4 \frac{\mu_1 \mu_2 \omega_1 \omega_2}{\mu_1 + \mu_2} h_1 h_2) \\ &= \frac{\mu_2 \omega_1 + \mu_1 \omega_2}{\mu_1 + \mu_2} \frac{c}{4} + \frac{1}{(\mu_1 + \mu_2)^2} [\mu_1 \mu_2 (\omega_1 h_1 + \omega_2 h_2)^2 + (\mu_2 \omega_1 h_1 + \mu_1 \omega_2 h_2)^2] \end{aligned}$$

Letting

$$y = \mu_2 \omega_1 h_1 + \mu_1 \omega_2 h_2$$

we obtain

$$x = (\omega_1 + \omega_2)\frac{c}{4} + x^2 \quad (2.10)$$

$$y = (\mu_2\omega_1 + \mu_1\omega_2)\frac{c}{4} + \frac{1}{\mu_1 + \mu_2}(\mu_1\mu_2x^2 + y^2) \quad (2.11)$$

We are interested in the iterated discrete dynamical system of (2.10) and (2.11):

$$x(n+1) = (\omega_1 + \omega_2)\frac{c}{4} + x(n)^2 \quad (2.12)$$

$$y(n+1) = (\mu_2\omega_1 + \mu_1\omega_2)\frac{c}{4} + \frac{1}{\mu_1 + \mu_2}(\mu_1\mu_2x(n)^2 + y(n)^2) \quad (2.13)$$

2. Fixed Points

The fixed points of the iterated discrete dynamical system (2.12) and (2.13) are solutions of (2.10) and (2.11) for x and y . From (2.10) we can solve

$$x_{\pm} = \frac{1}{2}[1 \pm \sqrt{1 - c(\omega_1 + \omega_2)}]$$

$$c < \frac{1}{\omega_1 + \omega_2}$$

Substituting x_{\pm} to (2.11), we have

$$y^2 - (\mu_1 + \mu_2)y + \frac{c}{4}(\mu_2\omega_1 + \mu_1\omega_2)(\mu_1 + \mu_2) + \mu_1\mu_2x_{\pm}^2 = 0$$

Then

$$\begin{aligned} y_{\pm,-} &= \frac{1}{2}\{\mu_1 + \mu_2 \\ &\quad - \sqrt{(\mu_1 + \mu_2)^2 - 4[\frac{c}{4}(\mu_2\omega_1 + \mu_1\omega_2)(\mu_1 + \mu_2) + \mu_1\mu_2x_{\pm}^2]}\} \\ &= \frac{1}{2}(\mu_1 + \mu_2)\{1 - \frac{1}{\mu_1 + \mu_2} \\ &\quad \sqrt{\mu_1^2 + \mu_2^2 - c(\mu_2^2\omega_1 + \mu_1^2\omega_2) \mp 2\mu_1\mu_2\sqrt{1 - c(\omega_1 + \omega_2)}}\} \end{aligned}$$

and

$$y_{\pm,+} = \frac{1}{2}\{\mu_1 + \mu_2$$

$$\begin{aligned}
& + \sqrt{(\mu_1 + \mu_2)^2 - 4\left[\frac{c}{4}(\mu_2\omega_1 + \mu_1\omega_2)(\mu_1 + \mu_2) + \mu_1\mu_2x_{\pm}^2\right]} \\
= & \frac{1}{2}(\mu_1 + \mu_2)\left\{1 + \frac{1}{\mu_1 + \mu_2}\right. \\
& \left. \sqrt{\mu_1^2 + \mu_2^2 - c(\mu_2^2\omega_1 + \mu_1^2\omega_2) \mp 2\mu_1\mu_2\sqrt{1 - c(\omega_1 + \omega_2)}}\right\}
\end{aligned}$$

So for $c \leq \frac{1}{\omega_1 + \omega_2}$ we have four fixed points of (2.10) and (2.11)

$$\begin{aligned}
(x_+, y_{+,+}) & \quad (x_+, y_{+,-}) \\
(x_-, y_{-,+}) & \quad (x_-, y_{-,-})
\end{aligned} \tag{2.14}$$

For $c > \frac{1}{\omega_1 + \omega_2}$, there is no real fixed point of (2.12) and (2.13).

For $N = n$ we have the n -dimensional system

$$\begin{aligned}
h_j(t+1) &= \frac{c}{4} + 2h_j(t)\mu_j \sum_{i=1}^n \frac{\omega_i}{(\mu_j + \mu_i)} h_i(t). \\
j &= 1, \dots, n
\end{aligned}$$

The formula (2.4) give us 2^n fixed points which can be constructed from the roots of the associated characteristic equation 2.5.

3. Stability of Fixed Points

For the iteration system (2.12) and (2.13) let

$$F(x) = (\omega_1 + \omega_2)\frac{c}{4} + x^2 \tag{2.15}$$

$$F(y) = (\mu_2\omega_1 + \mu_1\omega_2)\frac{c}{4} + \frac{1}{\mu_1 + \mu_2}(\mu_1\mu_2x^2 + y^2) \tag{2.16}$$

The Jacobi matrix is

$$\begin{pmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{pmatrix} = \begin{pmatrix} 2x & 0 \\ \frac{2\mu_1\mu_2}{\mu_1 + \mu_2}x & 2y \end{pmatrix} \tag{2.17}$$

The eigenvalues of the Jacobi matrix for fixed points $(x_i, y_{i,j})$, $i, j = +, -$ are

$$\lambda_i^{(1)} = 2x_i \tag{2.18}$$

$$\lambda_{i,j}^{(2)} = \frac{2}{\mu_1 + \mu_2}y_{i,j}, i, j = +, - \tag{2.19}$$

i.e.,

$$\lambda_{\pm}^{(1)} = 1 \pm \sqrt{1 - c(\omega_1 + \omega_2)}, c \leq \frac{1}{\omega_1 + \omega_2} \quad (2.20)$$

with

$$c = \frac{1}{\omega_1 + \omega_2} (\lambda_{\pm}^{(1)2} + 2\lambda_{\pm}^{(1)}) \quad (2.21)$$

and

$$\begin{aligned} \lambda_{\pm,+}^{(2)} &= 1 + \frac{1}{\mu_1 + \mu_2} \\ &\quad \sqrt{\mu_1^2 + \mu_2^2 - c(\mu_2^2\omega_1 + \mu_1^2\omega_2) \mp 2\mu_1\mu_2\sqrt{1 - c(\omega_1 + \omega_2)}, use(18)} \\ &= 1 + \frac{1}{\mu_1 + \mu_2} [\mu_1^2 + \mu_2^2 - \frac{1}{\omega_1 + \omega_2} (-\lambda_{\pm}^{(1)2} + 2\lambda_{\pm}^{(1)})(\mu_2^2\omega_1 + \mu_1^2\omega_2) \\ &\quad \mp 2\mu_1\mu_2\sqrt{1 - (-\lambda_{\pm}^{(1)2} + 2\lambda_{\pm}^{(1)})}]^{\frac{1}{2}} \\ &= 1 + \frac{1}{\mu_1 + \mu_2} \sqrt{(\mu_1 + \mu_2)^2 - \frac{1}{\omega_1 + \omega_2} (-\lambda_{\pm}^{(1)2} + 2\lambda_{\pm}^{(1)})(\mu_2^2\omega_1 + \mu_1^2\omega_2) - 2\mu_1\mu_2\lambda_{\pm}^{(1)}} \\ &= 1 + \frac{1}{\mu_1 + \mu_2} \sqrt{(\mu_1 + \mu_2)^2 - 2(\frac{\mu_2^2\omega_1 + \mu_1^2\omega_2}{\omega_1 + \omega_2} + \mu_1\mu_2)\lambda_{\pm}^{(1)} + \frac{\mu_2^2\omega_1 + \mu_1^2\omega_2}{\omega_1 + \omega_2}\lambda_{\pm}^{(1)2}} \\ &= 1 + \frac{1}{\mu_1 + \mu_2} \sqrt{(\mu_1 + \mu_2)^2 - 2(\frac{\mu_1\mu_2}{\omega_1 + \omega_2})(\omega_1\mu_2 + \omega_2\mu_1)\lambda_{\pm}^{(1)} + \frac{\mu_2^2\omega_1 + \mu_1^2\omega_2}{\omega_1 + \omega_2}\lambda_{\pm}^{(1)2}} \end{aligned}$$

Thus we have

$$\lambda_{\pm,+}^{(2)} = 1 + \sqrt{1 - 2\frac{\omega_1\mu_2 + \omega_2\mu_1}{(\omega_1 + \omega_2)(\mu_1 + \mu_2)}\lambda_{\pm}^{(1)} + \frac{\mu_2^2\omega_1 + \mu_1^2\omega_2}{(\omega_1 + \omega_2)(\mu_1 + \mu_2)^2}\lambda_{\pm}^{(1)2}} \quad (2.22)$$

and in a similar way

$$\lambda_{\pm,-}^{(2)} = 1 - \sqrt{1 - 2\frac{\omega_1\mu_2 + \omega_2\mu_1}{(\omega_1 + \omega_2)(\mu_1 + \mu_2)}\lambda_{\pm}^{(1)} + \frac{\mu_2^2\omega_1 + \mu_1^2\omega_2}{(\omega_1 + \omega_2)(\mu_1 + \mu_2)^2}\lambda_{\pm}^{(1)2}} \quad (2.23)$$

Because of the relation

$$\begin{aligned} &4\frac{(\omega_1\mu_2 + \omega_2\mu_1)^2}{(\omega_1 + \omega_2)^2(\omega_1 + \omega_2)^2} - 4\frac{\omega_1\mu_2^2 + \omega_2\mu_1^2}{(\omega_1 + \omega_2)(\mu_1 + \mu_2)^2} \\ &= \frac{4}{(\omega_1 + \omega_2)^2(\mu_1 + \mu_2)^2} [(\omega_1\mu_2 + \omega_2\mu_1)^2 - (\omega_1\mu_2^2 + \omega_2\mu_1^2)(\omega_1 + \omega_2)] \end{aligned}$$

$$\begin{aligned}
&= \frac{4}{(\omega_1 + \omega_2)^2(\mu_1 + \mu_2)^2}[-\omega_1\omega_2(\mu_1 + \mu_2)^2] \\
&= -\frac{4\omega_1\omega_2}{(\omega_1 + \omega_2)^2} < 0,
\end{aligned}$$

it is evident that the square roots of (2.22) and (2.23) are real for any $\lambda_{\pm}^{(1)}$. Also $\lambda_{\pm,\pm}^{(2)}$ are real for $c \leq \frac{1}{\omega_1 + \omega_2}$, and we can see that for $c \leq \frac{1}{\omega_1 + \omega_2}$

$$\lambda_{\pm,+}^{(2)} \geq 1 \quad (2.24)$$

$$\lambda_{\pm,-}^{(2)} \leq 1. \quad (2.25)$$

Now, we want study for what region of c

$$|\lambda_{-,\pm}^{(2)}| < 1.$$

For $\lambda_{\pm,-}^{(2)} = -1$, we have

$$1 - 2\frac{\omega_1\mu_2 + \omega_2\mu_1}{(\omega_1 + \omega_2)(\mu_1 + \mu_2)}\lambda_{\pm}^{(1)} + \frac{\mu_2^2\omega_1 + \mu_1^2\omega_2}{(\omega_1 + \omega_2)(\mu_1 + \mu_2)^2}\lambda_{\pm}^{(1)2} = 2.$$

Thus

$$\frac{\mu_2^2\omega_1 + \mu_1^2\omega_2}{(\omega_1 + \omega_2)(\mu_1 + \mu_2)^2}\lambda_{\pm}^{(1)2} - 2\frac{\omega_1\mu_2 + \omega_2\mu_1}{(\omega_1 + \omega_2)(\mu_1 + \mu_2)}\lambda_{\pm}^{(1)} - 3 = 0$$

i.e.,

$$(\mu_2^2\omega_1 + \mu_1^2\omega_2)\lambda_{\pm}^{(1)2} - 2(\omega_1\mu_2 + \omega_2\mu_1)(\mu_1 + \mu_2)\lambda_{\pm}^{(1)} - 3(\omega_1 + \omega_2)(\mu_1 + \mu_2)^2 = 0.$$

Then

$$\begin{aligned}
(\lambda_{\pm}^{(1)})_{1,2} &= \frac{1}{2(\mu_2^2\omega_1 + \mu_1^2\omega_2)}[2(\omega_1\mu_2 + \omega_2\mu_1)(\mu_1 + \mu_2) \\
&\quad \pm \sqrt{4(\omega_1\mu_2 + \omega_2\mu_1)^2(\mu_1 + \mu_2)^2 + 12(\mu_2^2\omega_1 + \mu_1^2\omega_2)(\omega_1 + \omega_2)(\mu_1 + \mu_2)^2}] \\
&= \frac{\mu_1 + \mu_2}{\mu_2^2\omega_1 + \mu_1^2\omega_2} \{ \mu_2\omega_1 + \mu_1\omega_2 \\
&\quad \pm \sqrt{\mu_2^2\omega_1^2 + \mu_1^2\omega_2^2 + 2\mu_1\mu_2\omega_1\omega_2 + 3[\mu_2^2\omega_1^2 + \mu_1^2\omega_2^2 + \omega_1\omega_2(\mu_1^2 + \mu_2^2)]} \} \\
&= \frac{\mu_1 + \mu_2}{\mu_2^2\omega_1 + \mu_1^2\omega_2} \{ \mu_2\omega_1 + \mu_1\omega_2 \\
&\quad \pm \sqrt{4(\mu_2^2\omega_1^2 + \mu_1^2\omega_2^2) + \omega_1\omega_2[3(\mu_1^2 + \mu_2^2) + 2\mu_1\mu_2]} \}
\end{aligned}$$

Let

$$\xi_1 = (\lambda_{\pm}^{(1)})_1, \xi_2 = (\lambda_{\pm}^{(1)})_2.$$

Then for

$$\xi_2 < \lambda_{\pm}^{(1)} < \xi_1 \quad (2.26)$$

the square root in (2.23) is less than 2, so

$$-1 < \lambda_{\pm,-}^{(2)} < 1. \quad (2.27)$$

For

$$\lambda_{\pm}^{(1)} < \xi_2, \text{ or } \lambda_{\pm}^{(1)} > \xi_1 \quad (2.28)$$

the square root in (2.23) is larger than 2, so

$$\lambda_{\pm,-}^{(2)} < -1. \quad (2.29)$$

Note

$$\lambda_+^{(1)} = 1 + \sqrt{1 - c(\omega_1 + \omega_2)} \geq 1 \quad (2.30)$$

$$\lambda_-^{(1)} = 1 - \sqrt{1 - c(\omega_1 + \omega_2)} \leq 1 \quad (2.31)$$

and ξ_1, ξ_2 can be written as

$$\begin{aligned} \xi_1 &= 1 + \frac{\mu_1 \mu_2 (\omega_1 + \omega_2)}{\mu_2^2 \omega_1 + \mu_1^2 \omega_2} \\ &\quad + \frac{\mu_1 + \mu_2}{\mu_2^2 \omega_1 + \mu_1^2 \omega_2} \sqrt{4(\mu_2^2 \omega_1^2 + \mu_1^2 \omega_2^2) + \omega_1 \omega_2 [3(\mu_1^2 + \mu_2^2) + 2\mu_1 \mu_2]} \\ \xi_2 &= \frac{\mu_1 + \mu_2}{\mu_2^2 \omega_1 + \mu_1^2 \omega_2} [\omega_1 \mu_2 + \omega_2 \mu_1 \\ &\quad - \sqrt{(\omega_1 \mu_2 + \omega_2 \mu_1)^2 + 3(\mu_2^2 \omega_1 + \mu_1^2 \omega_2)(\omega_1 + \omega_2)}]. \end{aligned}$$

So we can see

$$\xi_1 > 1, \quad (2.32)$$

$$\xi_2 < 0. \quad (2.33)$$

Combining of (2.30), (2.31), (2.32) and (2.33), the results (2.26)-(2.27) and (2.28)-(2.29) can be written as

Theorem 2.1 For $c \leq \frac{1}{\omega_1 + \omega_2}$

(i) if $\lambda_+^{(1)} < \xi_1$ then

$$-1 < \lambda_{+,-}^{(2)} < 1$$

(ii) if $\lambda_+^{(1)} > \xi_1$ then

$$\lambda_{+,-}^{(2)} < -1$$

(iii) if $\xi_2 < \lambda_-^{(1)}$ then

$$-1 < \lambda_{-,-}^{(2)} < 1$$

(iv) if $\xi_2 > \lambda_-^{(1)}$ then

$$\lambda_{-,-}^{(2)} < -1$$

There are relationships (2.21) between the parameters and eigenvalues of system (2.12) and (2.13). So the conditions in Theorem 2.1 above can be written in terms of the original parameters $c, \mu_1, \mu_2, \omega_1$ and ω_2 .

We have that $\lambda_+^{(1)} < \xi_1$ implies

$$\frac{1}{\omega_1 + \omega_2}(-\xi_1^2 + 2\xi_1) < c \leq \frac{1}{\omega_1 + \omega_2} \quad (2.34)$$

$\lambda_+^{(1)} > \xi_1$ implies

$$c < \frac{1}{\omega_1 + \omega_2}(-\xi_1^2 + 2\xi_1) \quad (2.35)$$

$\xi_2 < \lambda_-^{(1)}$ implies

$$\frac{1}{\omega_1 + \omega_2}(-\xi_2^2 + 2\xi_2) < c \leq \frac{1}{\omega_1 + \omega_2} \quad (2.36)$$

$\xi_2 > \lambda_-^{(1)}$ implies

$$c < \frac{1}{\omega_1 + \omega_2}(-\xi_2^2 + 2\xi_2) \quad (2.37)$$

From the expressions for ξ_1 and ξ_2 , we know that ξ_1 and ξ_2 can be written in the form:

$$\xi_1 = 1 + A + B$$

$$\xi_2 = 1 + A - B$$

with $A, B > 0$. Since the function

$$f(\xi) = \frac{1}{\omega_1 + \omega_2}(-\xi^2 + 2\xi) - \frac{1}{\omega_1 + \omega_2}[1 - (1 - \xi)^2]$$

is symmetric about $\xi = 1$, and since

$$|\xi_1 - 1| = A + B > |A - B| = |\xi_2 - 1|$$

we have

$$\frac{1}{\omega_1 + \omega_2}(-\xi_2^2 + 2\xi_2) > \frac{1}{\omega_1 + \omega_2}(-\xi_1^2 + 2\xi_1).$$

Let

$$c_i = \frac{1}{\omega_1 + \omega_2}(-\xi_i^2 + 2\xi_i), \quad (2.38)$$

$$i = 1, 2.$$

It implies

$$c_2 > c_1.$$

We know the eigenvalues at fixed points can determine the stability of the fixed point.

Now we give the result of stability for all fixed points of (2.12) and (2.13)

Theorem 2.2 (1) For $c \leq \frac{1}{\omega_1 + \omega_2}$ the fixed point $(x_+, y_{+,+})$ is a repelling point;

(2) for $c > -\frac{3}{\omega_1 + \omega_2}$ the fixed point $(x_-, y_{-,+})$ is a saddle point;

(3) for $c < -\frac{3}{\omega_1 + \omega_2}$ the fixed point $(x_-, y_{-,+})$ is a repelling point;

(4) for $c_1 < c < \frac{1}{\omega_1 + \omega_2}$ the fixed point $(x_+, y_{+,-})$ is a saddle point;

(5) for $c < c_1$ the fixed point $(x_+, y_{+,-})$ is a repelling point;

(6) if $\frac{1}{\omega_1+\omega_2} > c > -\frac{3}{\omega_1+\omega_2}$ and $\frac{1}{\omega_1+\omega_2} > c > c_2$ then the fixed point $(x_-, y_{-,-})$ is an attracting point;

(7) if $\frac{1}{\omega_1+\omega_2} > c > -\frac{3}{\omega_1+\omega_2}$ and $c < c_2$ then the fixed point $(x_-, y_{-,-})$ is a saddle point;

(8) if $c < -\frac{3}{\omega_1+\omega_2}$ and $\frac{1}{\omega_1+\omega_2} > c > c_2$ then the fixed point $(x_-, y_{-,-})$ is a saddle point;

(9) if $c < -\frac{3}{\omega_1+\omega_2}$ and $c < c_2$ then the fixed point $(x_-, y_{-,-})$ is a repelling point.

Proof: (1) For the fixed point $(x_+, y_{+,+})$, the eigenvalues are $\lambda_+^{(1)}, \lambda_{+,+}^{(2)}$. For $c < \frac{1}{\omega_1+\omega_2}$, using (2.30) and (2.24) we have

$$\lambda_+^{(1)} > 1 \quad \text{and} \quad \lambda_{+,+}^{(2)} \geq 1.$$

Therefore the fixed point $(x_+, y_{+,+})$ is a repelling point.

(2) For the fixed point $(x_-, y_{-,+})$, the eigenvalues are $\lambda_-^{(1)}, \lambda_{-,+}^{(2)}$. Using (2.24) we have $\lambda_{-,+}^{(2)} > 1$, and for $c > -\frac{3}{\omega_1+\omega_2}$, the eigenvalue $|\lambda_-^{(1)}| < 1$. Thus the fixed point $(x_-, y_{-,+})$ is a saddle point.

(3) As in the proof of (2), $\lambda_{-,+}^{(2)} > 1$, and for $c < -\frac{3}{\omega_1+\omega_2}$, the eigenvalue $\lambda_-^{(1)} < -1$, so

$$|\lambda_-^{(1)}| > 1.$$

Thus the fixed point $(x_-, y_{-,+})$ is a repelling point.

(4) Using (2.30) we know $\lambda_+^{(1)} > 1$. From the relation of $\lambda_+^{(1)} < \xi_1$ and (2.34), using (i) of Theorem 2.2 and noting the notation (2.38), we have

$$|\lambda_{+,-}^{(2)}| < 1.$$

So the fixed point $(x_+, y_{+,-})$ is a saddle point.

(5) In the same way as the proof of (4), using (ii) in Theorem 1, we have

$$\lambda_{+,-}^{(1)} < -1.$$

Then we know the eigenvalues $\lambda_+^{(1)}$ and $\lambda_{+,-}^{(2)}$ of the fixed point $(x_+, y_{+,-})$ satisfy

$$\lambda_+^{(1)} > 1 \quad \text{and} \quad |\lambda_{+,-}^{(2)}| > 1.$$

So the fixed point $(x_+, y_{+,-})$ is a repelling point.

(6) Using (2.31), for

$$\frac{1}{\omega_1 + \omega_2} > c > -\frac{3}{\omega_1 + \omega_2},$$

we have

$$-1 < \lambda_-^{(1)} < 1.$$

From the condition in (iii) of Theorem 2.2 and (2.36), we know

$$-1 < \lambda_{-,-}^{(2)} < 1.$$

So the fixed point $(x_-, y_{-,-})$ is an attracting point.

(7) The eigenvalue $\lambda_-^{(1)}$ is the same as (6) above, $|\lambda_-^{(1)}| < 1$. For $c < c_2$, using (2.37) and notation (2.38) of c_2 , we know that the condition (iv) in Theorem 2.1 is satisfied. So

$$\lambda_{-,-}^{(2)} < -1$$

i.e.,

$$|\lambda_{-,-}^{(2)}| < 1$$

Thus the fixed point $(x_-, y_{-,-})$ is a saddle point.

(8) If $-\frac{3}{\omega_1 + \omega_2} > c$, (2.31) tells us

$$\lambda_-^{(1)} < -1.$$

Using (iii) in Theorem 2.1, for $\frac{1}{\omega_1 + \omega_2} > c > c_2$, we know

$$|\lambda_{-,-}^{(2)}| < 1.$$

So the fixed point $(x_-, y_{-,-})$ is a saddle point.

(9) As in (8) above, $\lambda_-^{(1)} < -1$. And using (iv) in Theorem 1, we have

$$\lambda_{-,-}^{(2)} < -1.$$

Therefore the fixed point $(x_-, y_{-,-})$ is a repelling point. Note

$$\|\lambda_{\pm}^{(1)} - 1\| = \begin{cases} \lambda_+^{(1)} - 1 \\ -\lambda_-^{(1)} + 1 \end{cases}$$

‡

For the n dimensional system, we can not solve for the k_i 's of equation 2.5 explicitly, thus the stability of the fixed points is hard to know.

Chapter 3

QUANTITATIVE PROPERTIES OF ORBITS

In this chapter we want to discuss some quantitative properties of the iterated discrete system (2.12) and (2.13) for special values of μ_1, μ_2, ω_1 and ω_2 :

$$\begin{aligned}\omega_1 &= \frac{1}{2} \\ \omega_2 &= \frac{1}{2} \\ \mu_1 &= \frac{1}{2}\left(1 + \frac{1}{\sqrt{3}}\right) \\ \mu_2 &= \frac{1}{2}\left(1 - \frac{1}{\sqrt{3}}\right)\end{aligned}\tag{3.1}$$

The dynamical system (1.9) and (1.10) becomes

$$x_{n+1} = \frac{c}{4} + x_n^2\tag{3.2}$$

$$y_{n+1} = \frac{c}{8} + \frac{1}{6}x_n^2 + y_n^2.\tag{3.3}$$

Define the mappings

$$\begin{aligned}F(x) &= \frac{c}{4} + x^2 \\ G(x, y) &= \frac{c}{8} + \frac{1}{6}x^2 + y^2.\end{aligned}$$

Definition 3.1 *The series $\{(x_n, y_n)\}_0^\infty$ is an orbit of the system (3.2) and (3.3), if (x_n, y_n) satisfies these equations.*

To discuss the properties of the orbit $\{(x_n, y_n)\}_0^\infty$ of this system, we consider

$$\begin{aligned}\Delta x_n &= x_{n+1} - x_n = \frac{c}{4} + x_n^2 - x_n \\ \Delta y_n &= y_{n+1} - y_n = \frac{c}{8} + \frac{1}{6}x_n^2 + y_n^2 - y_n.\end{aligned}\tag{3.4}$$

For given (x_n, y_n) , we have a vector $(\Delta x_n, \Delta y_n)$ which gives the direction of the orbit at (x_n, y_n) . In order to know details about this vector field, we consider first the y -direction, by setting.

$$\frac{c}{8} + \frac{1}{6}x_n^2 + y_n^2 - y_n = 0.$$

For $c \leq 2$ we get an ellipse,

$$\frac{1}{6}x_n^2 + (y_n - \frac{1}{2})^2 = \frac{1}{8}(2 - c) \quad (3.5)$$

We can see that on the ellipse, $\Delta y_n = 0$; outside of this ellipse $\Delta y_n > 0$ and inside the ellipse $\Delta y_n < 0$.

For the x -direction in (2.4), we can see that for $c \leq 1$, when

$$\frac{1}{2}(1 - \sqrt{1 - c}) < x_n < \frac{1}{2}(1 + \sqrt{1 - c}),$$

we have $\Delta x_n < 0$, and when

$$x_n \leq \frac{1}{2}(1 - \sqrt{1 - c})$$

or

$$x_n \geq \frac{1}{2}(1 + \sqrt{1 - c}),$$

we have $\Delta x_n \geq 0$. The field of (3.1) gives us a rough idea of how to study the orbits of the system (3.2) and (3.3).

First we want to find the divergence region of the orbit $\{(x_n, y_n)\}_0^n$, i.e. if the orbit starts in this region, then the orbit will diverge.

Theorem 3.1 *Let $c \leq 1$. For some n_0 if*

$$y_{n_0} > \frac{1}{2} + \frac{1}{2\sqrt{2}}\sqrt{2 - c} \quad (3.6)$$

or

$$|x_{n_0}| > \frac{1}{2} + \frac{1}{2}\sqrt{1 - c}, \quad (3.7)$$

then the orbit $\{(x_n, y_n)\}_0^n$ is divergent.

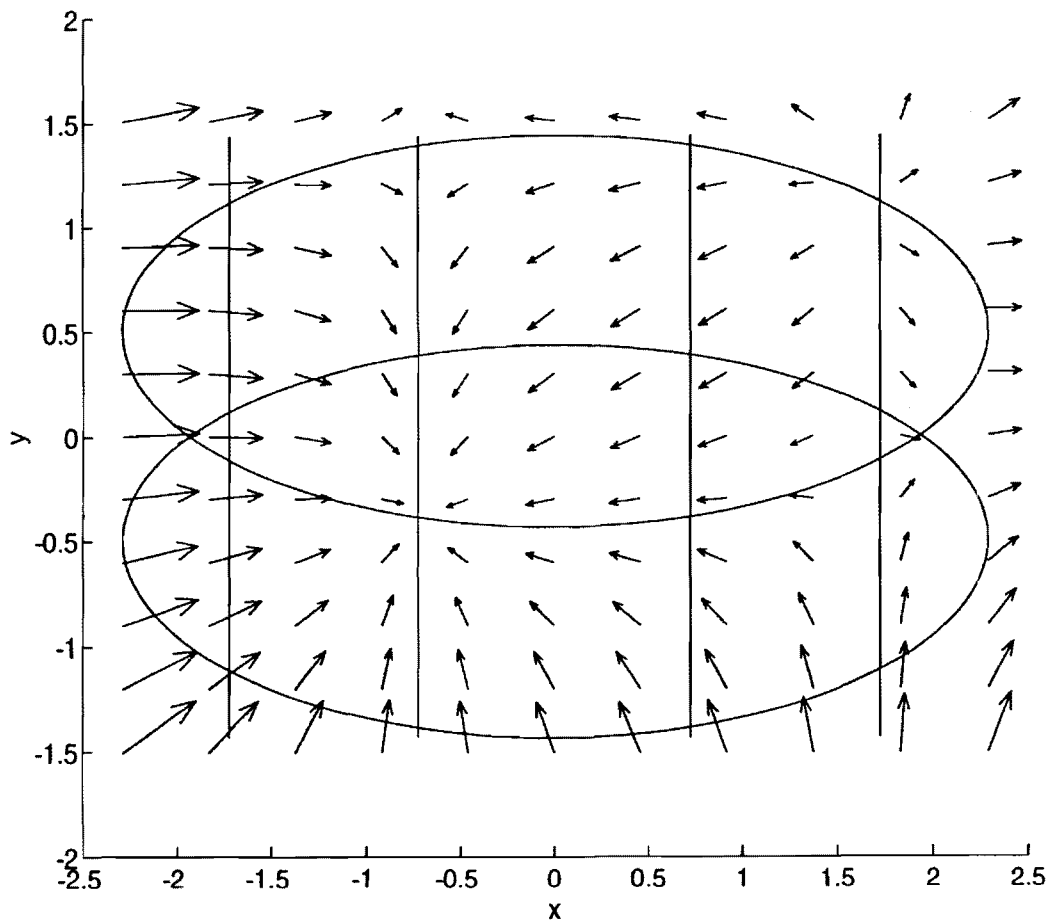


Figure 3.1: The field of vectors $(\Delta x, \Delta y)$

Proof: If

$$y_{n_0} > \frac{1}{2} + \frac{1}{2\sqrt{2}}\sqrt{2-c},$$

then (x_{n_0}, y_{n_0}) is outside of the ellipse (3.5) and we know

$$\Delta y_{n_0} > 0.$$

Thus

$$y_{n_0+1} > y_{n_0} > \frac{1}{2} + \frac{1}{2\sqrt{2}}\sqrt{2-c}.$$

Then for $n_0 < n$, we have

$$y_{n+1} > y_n > \frac{1}{2} + \frac{1}{2\sqrt{2}}\sqrt{2-c}.$$

Since from (3.4) for $n_0 < n$

$$\begin{aligned} \Delta y_n &= \frac{c}{8} + y_n^2 - y_n \\ &= \frac{c}{8} - \frac{1}{4} + (y_n - \frac{1}{2})^2 \\ &> \frac{c}{8} - \frac{1}{4} + (y_{n_0} - \frac{1}{2})^2 > 0, \end{aligned}$$

we have $\{y_n\}_0^\infty$ is divergent, so the orbit $\{x_n, y_n\}_0^\infty$ is divergent.

If $x_{n_0} > \frac{1}{2} + \frac{1}{2}\sqrt{1-c}$, then

$$\begin{aligned} \Delta x_{n_0} &= \frac{c}{4} + x_{n_0}^2 - x_{n_0} \\ &= [x_{n_0} - \frac{1}{2}(1 + \sqrt{1-c})][x_{n_0} - \frac{1}{2}(1 - \sqrt{1-c})] > 0, \end{aligned}$$

so that $x_{n_0+1} > x_{n_0}$. Then for $n > n_0$

$$x_{n+1} > x_n \geq x_{n_0+1} > x_{n_0}$$

and

$$\begin{aligned} \Delta x_n &= [x_n - \frac{1}{2}(1 + \sqrt{1-c})][x_n - \frac{1}{2}(1 - \sqrt{1-c})] \\ &> \Delta x_{n_0} > 0. \end{aligned}$$

Hence $\{x_n\}_{n_0}^\infty$ is increasing and divergent. Similarly, if

$$x_{n_0} < -\frac{1}{2}(1 + \sqrt{1-c})$$

then

$$\begin{aligned} x_{n_0+1} &= \frac{c}{4} + x_{n_0}^2 \\ &> \frac{c}{4} + \left[-\frac{1}{2}(1 + \sqrt{1-c})\right]^2 \\ &= \frac{1}{2}(1 + \sqrt{1-c}). \end{aligned}$$

Hence the orbit $\{x_n, y_n\}_{n_0}^\infty$ is divergent.

‡

Theorem 3.2 *Let $c \leq 1$. If for some n_0*

$$\frac{1}{6}x_{n_0}^2 + y_{n_0}^2 > \frac{1}{2} - \frac{c}{8} + \frac{1}{2\sqrt{2}}\sqrt{2-c}, \quad (3.8)$$

then the orbit $\{(x_n, y_n)\}_0^\infty$ is divergent.

Proof: If

$$\frac{1}{6}x_{n_0}^2 + y_{n_0}^2 > \frac{1}{2} - \frac{c}{8} + \frac{1}{2\sqrt{2}}\sqrt{2-c},$$

we have

$$\begin{aligned} y_{n_0+1} &= \frac{c}{8} + \frac{1}{6}x_{n_0}^2 + y_{n_0}^2 \\ &> \frac{1}{2} + \frac{1}{2\sqrt{2}}\sqrt{2-c}. \end{aligned}$$

Using Theorem 3.1 we have the orbit $\{x_n, y_n\}_0^\infty$ is divergent.

‡

Note that if $-c$ large enough, the orbit will diverge, even if starting from a neighborhood of the origin point.

Theorem 3.3 For $c < -8$, if for some n_0

$$x_{n_0}^2 < -\frac{c}{4} - \frac{1}{2}(1 + \sqrt{1-c}), \quad (3.9)$$

then the orbit $\{(x_n, y_n)\}_0^\infty$ is divergent.

Proof: From the condition, we have

$$\begin{aligned} x_{n_0+1} &= \frac{c}{4} + x_{n_0}^2 \\ &< \frac{c}{4} - \frac{c}{4} - \frac{1}{2}(1 + \sqrt{1-c}) \\ &= -\frac{1}{2}(1 + \sqrt{1-c}) \end{aligned}$$

Using Theorem 2.1, we have the orbit $\{(x_n, y_n)\}_0^\infty$ is divergent.

For the rest we only need to prove that for $c \leq -8$, the neighborhood of the origin (3.9) is not empty. i.e. we need prove

$$-\frac{c}{4} - \frac{1}{2}(1 + \sqrt{1-c}) \geq 0.$$

It implies that there is an x_{n_0} such that the orbit $\{x_n, y_n\}_{n_0}^\infty$ is divergent.

In fact, for $c \leq -8$, we have

$$c(c+8) \geq 0.$$

Then

$$\left(-\frac{c}{2} - 1\right)^2 = \frac{c^2}{4} + c + 1 \geq 1 - c,$$

so

$$-\frac{c}{2} - 1 \geq \sqrt{1-c}.$$

Hence

$$-\frac{c}{4} - \frac{1}{2}(1 + \sqrt{1-c}) \geq 0$$

#

Theorem 3.4 (1) For $1 \geq c \geq -8$ if

$$|x_n| \leq \left| \frac{1}{2}(1 - \sqrt{1-c}) \right|$$

then

$$-\frac{1}{2}(1 + \sqrt{1-c}) \leq x_{n+1} \leq \frac{1}{2}(1 - \sqrt{1-c})$$

(2) For $c \leq 1$, if

$$\frac{1}{2}(1 + \sqrt{1-c}) \geq |x_n| \geq \frac{1}{2}(-1 + \sqrt{1-c})$$

then

$$\frac{1}{2}(1 - \sqrt{1-c}) \leq x_{n+1} \leq \frac{1}{2}(1 + \sqrt{1-c})$$

and

$$|x_{n+1}| \leq |x_n|.$$

(3) For $1 \geq c \geq -8$, $\{x_n\}_0^\infty$ is bounded if and only if

$$|x_0| \leq \frac{1}{2}(1 + \sqrt{1-c})$$

(4) For $0 \geq c$, we have

$$\frac{1}{2}(1 + \sqrt{1-c}) \geq |x_n| > \frac{1}{2}\sqrt{2\sqrt{1-c} - 2 - c}$$

iff

$$\frac{1}{2}(1 + \sqrt{1-c}) \geq x_{n+1} > \frac{1}{2}(-1 + \sqrt{1-c});$$

and

$$\frac{1}{2}(-1 + \sqrt{1-c}) \leq |x_n| \leq \frac{1}{2}\sqrt{2\sqrt{1-c} - 2 - c}$$

iff

$$|x_{n+1}| \leq \frac{1}{2}(-1 + \sqrt{1-c}).$$

Proof: (1) For $1 \geq c \geq -8$ we have

$$-\frac{1}{2}(1 + \sqrt{1-c}) \leq \frac{c}{4}. \quad (3.10)$$

For

$$|x_n| \leq \left| \frac{1}{2}(1 - \sqrt{1-c}) \right|$$

we have

$$\begin{aligned} \frac{c}{4} &\leq x_{n+1} \\ &= \frac{c}{4} + x_n^2 \\ &\leq \frac{c}{4} + \left| \frac{1}{2}(1 - \sqrt{1-c}) \right|^2 \\ &= \frac{1}{2}(1 - \sqrt{1-c}). \end{aligned}$$

Using the inequality (3.10) we get

$$-\frac{1}{2}(1 + \sqrt{1-c}) \leq x_{n+1} \leq \frac{1}{2}(1 - \sqrt{1-c}).$$

(2) For $c \leq 1$ we know

$$\frac{1}{2}(-1 + \sqrt{1-c}) \geq 0$$

The condition

$$\frac{1}{2}(1 + \sqrt{1-c}) \geq |x_n| \geq \frac{1}{2}(-1 + \sqrt{1-c})$$

implies that for $c \leq 0$

$$\begin{aligned} \frac{c}{4} + \frac{1}{4}(1 - \sqrt{1-c})^2 &\leq x_{n+1} = \frac{c}{4} + x_n^2 \\ &\leq \frac{c}{4} + \frac{1}{4}(1 + \sqrt{1-c})^2 \end{aligned}$$

and for $0 \leq c \leq 1$

$$\begin{aligned} \frac{c}{4} &\leq x_{n+1} = \frac{c}{4} + x_n^2 \\ &\leq \frac{c}{4} + \frac{1}{4}(1 + \sqrt{1-c})^2. \end{aligned}$$

Thus

$$\frac{1}{2}(1 - \sqrt{1-c}) \leq x_{n+1} \leq \frac{1}{2}(1 + \sqrt{1-c}).$$

Now we shall prove

$$|x_{n+1}| \leq |x_n|.$$

For $x_{n+1} \geq 0$ we have

$$\begin{aligned} |x_{n+1}| - |x_n| &= |x_n|^2 + \frac{c}{4} - |x_n| \\ &= (|x_n| - \frac{1 + \sqrt{1-c}}{2})(|x_n| - \frac{1 - \sqrt{1-c}}{2}), \end{aligned}$$

so for

$$\frac{1}{2}(1 + \sqrt{1-c}) \geq |x_n| \geq \frac{1}{2}(-1 + \sqrt{1-c}),$$

then

$$|x_{n+1}| \leq |x_n|.$$

For $x_{n+1} < 0$ we have

$$\begin{aligned} |x_{n+1}| - |x_n| &= -|x_n|^2 - \frac{c}{4} - |x_n| \\ &= -(|x_n| - \frac{-1 + \sqrt{1-c}}{2})(|x_n| - \frac{-1 - \sqrt{1-c}}{2}), \end{aligned}$$

so for

$$|x_n| \geq \frac{1}{2}(-1 + \sqrt{1-c}),$$

then

$$|x_{n+1}| \leq |x_n|.$$

(3) This result follows from Theorem 2.1 and the result (1) and (2) above.

(4) It is easy to check

$$\frac{1}{2}(-1 + \sqrt{1-c}) \leq \frac{1}{2}\sqrt{2\sqrt{1-c} - 2 - c} \frac{1}{2}(1 + \sqrt{1-c}).$$

The inequality

$$\frac{1}{2}(1 + \sqrt{1-c}) \geq |x_n| > \frac{1}{2}\sqrt{2\sqrt{1-c} - 2 - c}$$

holds iff

$$\frac{1}{2}(1 + \sqrt{1-c}) - \frac{c}{4} \geq x_n^2 > \frac{1}{2}(\sqrt{1-c} - 1) - \frac{c}{4},$$

so

$$\frac{1}{2}(1 + \sqrt{1-c}) \geq x_{n+1} > \frac{1}{2}(\sqrt{1-c} - 1).$$

Otherwise the inequality

$$\frac{1}{2}(-1 + \sqrt{1-c}) \leq |x_n| \leq \frac{1}{2}\sqrt{2\sqrt{1-c} - 2 - c},$$

holds iff

$$\frac{1}{2}(1 - \sqrt{1-c}) - \frac{c}{4} \leq x_n^2 \leq \frac{1}{2}(\sqrt{1-c} - 1) - \frac{c}{4},$$

i.e.,

$$|x_{n+1}| \leq \frac{1}{2}(\sqrt{1-c} - 1).$$

‡

Theorem 3.4 give us the means to study $\{(x_n, y_n)\}_0^\infty$. The iterated form (3.3) can be written as

$$y_{n+1} = \frac{1}{4}\left(\frac{c}{2} + \frac{2}{3}x_n^2\right) + y_n^2. \quad (3.11)$$

We can treat $\frac{c}{2} + \frac{2}{3}x_n^2$ above as c in Theorem 3.4. Let the mapping of (3.2) and (3.3) be

$$R(x, y) = (F(x), G(x, y)). \quad (3.12)$$

In order to study the boundary orbit region of the system (3.2) and (3.3) we divide the domain of the mapping into the following regions (see Figure 3.2):

$$D1 = \left\{ (x, y) : \begin{array}{l} -\frac{1}{2}(1 + \sqrt{1-c}) \leq x < \frac{1}{2}(1 - \sqrt{1-c}), \\ -\frac{1}{2}[1 - \sqrt{1 - (\frac{c}{2} + \frac{2}{3}x^2)}] < y \leq \frac{1}{2}[1 + \sqrt{1 - (\frac{c}{2} + \frac{2}{3}x^2)}] \end{array} \right\}$$

$$\begin{aligned}
D2 &= \left\{ (x, y) : \begin{array}{l} |x| \leq \frac{1}{2}|1 - \sqrt{1-c}|, \\ -\frac{1}{2}[1 - \sqrt{1 - (\frac{c}{2} + \frac{2}{3}x^2)}] < y \leq \frac{1}{2}[1 + \sqrt{1 - (\frac{c}{2} + \frac{2}{3}x^2)}] \end{array} \right\}, \\
D3 &= \left\{ (x, y) : \begin{array}{l} -\frac{1}{2}(1 - \sqrt{1-c}) \leq x < \frac{1}{2}(1 + \sqrt{1-c}), \\ -\frac{1}{2}[1 - \sqrt{1 - (\frac{c}{2} + \frac{2}{3}x^2)}] < y \leq \frac{1}{2}[1 + \sqrt{1 - (\frac{c}{2} + \frac{2}{3}x^2)}] \end{array} \right\}, \\
D4 &= \left\{ (x, y) : \begin{array}{l} -\frac{1}{2}(1 + \sqrt{1-c}) \leq x < \frac{1}{2}(1 - \sqrt{1-c}), \\ |y| \leq \frac{1}{2}|1 - \sqrt{1 - (\frac{c}{2} + \frac{2}{3}x^2)}| \end{array} \right\}, \\
D5 &= \left\{ (x, y) : \begin{array}{l} |x| \leq \frac{1}{2}|1 - \sqrt{1-c}|, \\ |y| \leq \frac{1}{2}|1 - \sqrt{1 - (\frac{c}{2} + \frac{2}{3}x^2)}| \end{array} \right\}, \\
D6 &= \left\{ (x, y) : \begin{array}{l} -\frac{1}{2}(1 - \sqrt{1-c}) < x \leq \frac{1}{2}(1 + \sqrt{1-c}), \\ |y| \leq \frac{1}{2}|1 - \sqrt{1 - (\frac{c}{2} + \frac{2}{3}x^2)}| \end{array} \right\}, \\
D7 &= \left\{ (x, y) : \begin{array}{l} -\frac{1}{2}(1 + \sqrt{1-c}) \leq x < \frac{1}{2}(1 - \sqrt{1-c}), \\ -\frac{1}{2}[1 + \sqrt{1 + (\frac{c}{2} + \frac{2}{3}x^2)}] \leq y \leq \frac{1}{2}[1 - \sqrt{1 - (\frac{c}{2} + \frac{2}{3}x^2)}] \end{array} \right\}, \\
D8 &= \left\{ (x, y) : \begin{array}{l} |x| \leq \frac{1}{2}|1 - \sqrt{1-c}|, \\ -\frac{1}{2}[1 + \sqrt{1 - (\frac{c}{2} + \frac{2}{3}x^2)}] \leq y \leq \frac{1}{2}[1 - \sqrt{1 - (\frac{c}{2} + \frac{2}{3}x^2)}] \end{array} \right\}, \\
D9 &= \left\{ (x, y) : \begin{array}{l} -\frac{1}{2}(1 - \sqrt{1-c}) < x \leq \frac{1}{2}(1 + \sqrt{1-c}), \\ -\frac{1}{2}[1 + \sqrt{1 - (\frac{c}{2} + \frac{2}{3}x^2)}] \leq y \leq \frac{1}{2}[1 - \sqrt{1 - (\frac{c}{2} + \frac{2}{3}x^2)}] \end{array} \right\}.
\end{aligned}$$

Also we divide part of the image of the mapping $R(x, y)$ into the following regions (see Figure 3.3):

$$\begin{aligned}
R1 &= \left\{ (x, y) : \begin{array}{l} -\frac{1}{2}(1 + \sqrt{1-c}) \leq x \leq \frac{1}{2}(1 - \sqrt{1-c}), \\ -\frac{1}{2}[1 - \sqrt{1 - (\frac{c}{3} + \frac{2}{3}x)}] \leq y \leq \frac{1}{2}[1 + \sqrt{1 + (\frac{c}{3} + \frac{2}{3}x)}] \end{array} \right\} \\
R2 &= \left\{ (x, y) : \begin{array}{l} \frac{1}{2}(1 - \sqrt{1-c}) \leq x \leq \frac{1}{2}(1 + \sqrt{1-c}), \\ -\frac{1}{2}[1 - \sqrt{1 - (\frac{c}{3} + \frac{2}{3}x)}] \leq y \leq \frac{1}{2}[1 + \sqrt{1 + (\frac{c}{3} + \frac{2}{3}x)}] \end{array} \right\} \\
R3 &= \left\{ (x, y) : \begin{array}{l} -\frac{1}{2}(1 + \sqrt{1-c}) \leq x \leq \frac{1}{2}(1 - \sqrt{1-c}), \\ -\frac{1}{2}[1 + \sqrt{1 - (\frac{c}{3} + \frac{2}{3}x)}] \leq y \leq \frac{1}{2}[1 - \sqrt{1 + (\frac{c}{3} + \frac{2}{3}x)}] \end{array} \right\}
\end{aligned}$$

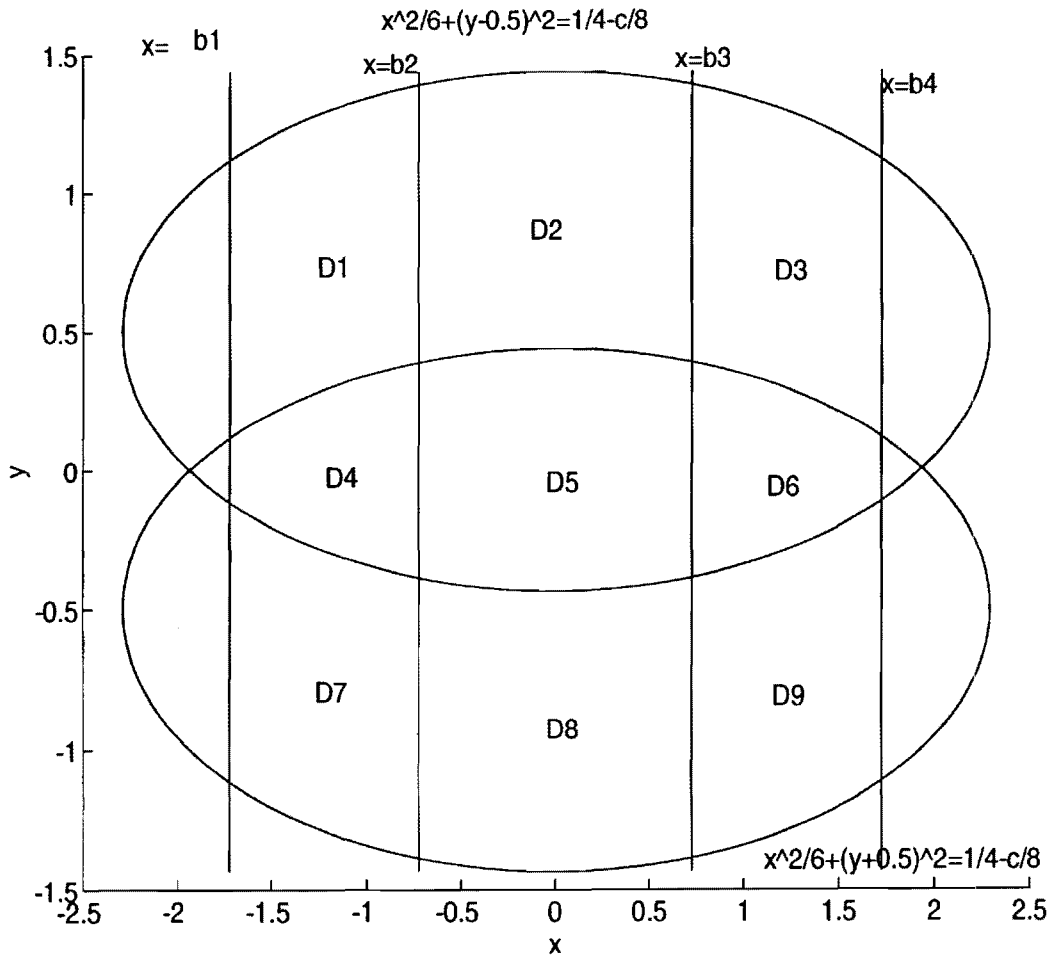


Figure 3.2:

The top ellipse is

$$\frac{1}{6}x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4} - \frac{c}{8}$$

and the bottom ellipse is

$$\frac{1}{6}x^2 + \left(y + \frac{1}{2}\right)^2 = \frac{1}{4} - \frac{c}{8}.$$

$$b1 = -\frac{1}{2}(1 + \sqrt{1 - c}), b2 = \frac{1}{2}(1 - \sqrt{1 - c}),$$

$$b3 = -b2, b4 = -b1.$$

$$R4 = \left\{ \begin{array}{l} (x, y) : \\ \frac{1}{2}(1 - \sqrt{1-c}) \leq x \leq \frac{1}{2}(1 + \sqrt{1-c}), \\ -\frac{1}{2}[1 + \sqrt{1 - (\frac{c}{3} + \frac{2}{3}x)}] \leq y \leq \frac{1}{2}[1 - \sqrt{1 + (\frac{c}{3} + \frac{2}{3}x)}] \end{array} \right\}$$

We have the following result for the mapping $R(x, y)$:

Theorem 3.5 *The function R maps Di to $R(Di)$, $i = 1, \dots, 9$ such that:*

$$R1 \supset R(D2 \cup D8), \quad (3.13)$$

$$R2 \supset R(D1 \cup D3 \cup D7 \cup D9), \quad (3.14)$$

$$R3 \supset R(D5), \quad (3.15)$$

$$R4 \supset R(D4 \cup D6). \quad (3.16)$$

Proof: (1) For $(x, y) \in D2 \cup D8$, we have

$$\begin{aligned} |x| &\leq \frac{1}{2}|1 - \sqrt{1-c}|, \\ -\frac{1}{2}[1 - \sqrt{1 - (\frac{c}{2} + \frac{2}{3}x^2)}] &\leq y \leq \frac{1}{2}[1 + \sqrt{1 - (\frac{c}{2} + \frac{2}{3}x^2)}], \end{aligned}$$

or

$$\begin{aligned} |x| &\leq \frac{1}{2}|1 - \sqrt{1-c}|, \\ -\frac{1}{2}[1 + \sqrt{1 - (\frac{c}{2} + \frac{2}{3}x^2)}] &\leq y \leq \frac{1}{2}[1 - \sqrt{1 - (\frac{c}{2} + \frac{2}{3}x^2)}]. \end{aligned}$$

Using (1) of Theorem 3.4 we know

$$-\frac{1}{2}(1 + \sqrt{1-c}) \leq F(x) \leq \frac{1}{2}(1 - \sqrt{1-c}).$$

Treating x_n and c in (2) of Theorem 3.4 as y and $\frac{c}{2} + \frac{2}{3}x^2$, respectively, we get

$$\frac{1}{2}[1 - \sqrt{1 - (\frac{c}{2} + \frac{2}{3}x^2)}] \leq G(x, y) \leq \frac{1}{2}[1 + \sqrt{1 - (\frac{c}{2} + \frac{2}{3}x^2)}].$$

By the definition of $F(x)$, we know

$$x^2 = -\frac{c}{4} + F(x).$$

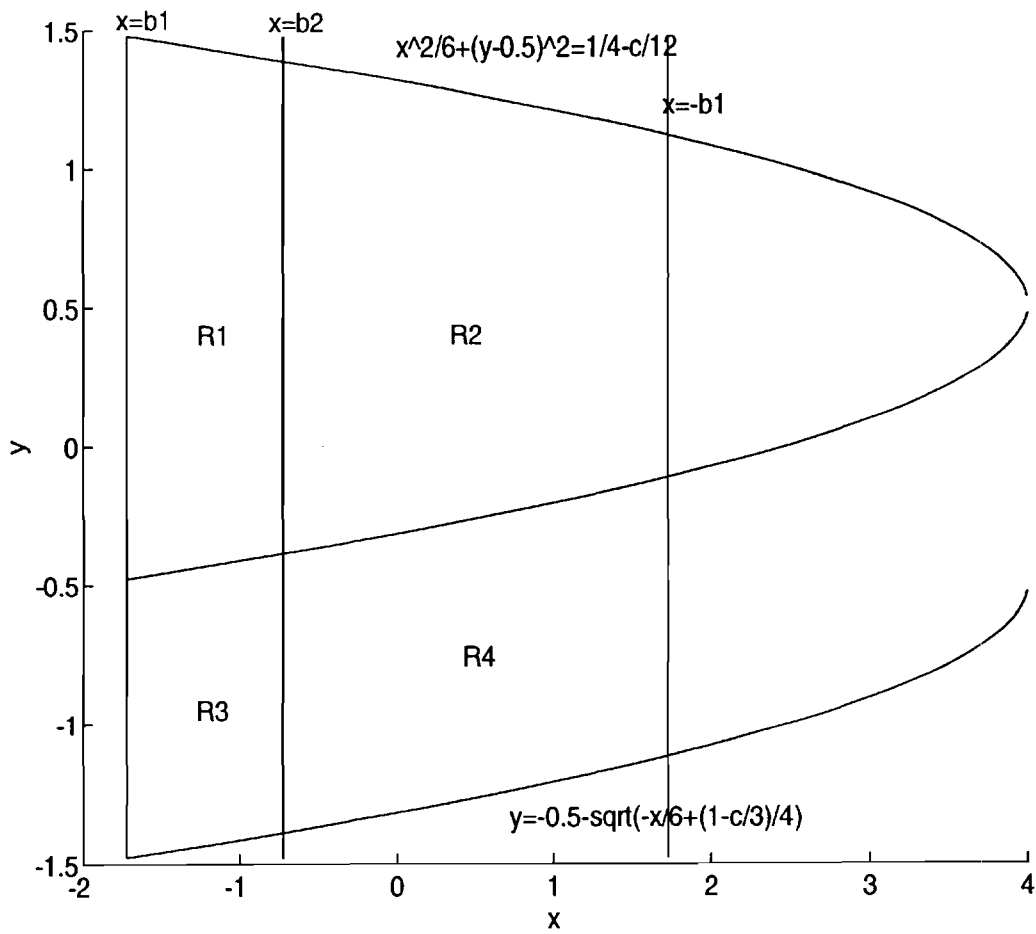


Figure 3.3:

The top parabola is

$$\frac{1}{6}x + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4} - \frac{c}{12}$$

and the bottom parabola is

$$y = -\frac{1}{2} - \sqrt{-\frac{1}{6}x + \frac{1}{4} - \frac{c}{12}}$$

$$b1 = -\frac{1}{2}(1 + \sqrt{1+c}), b2 = \frac{1}{2}(1 - \sqrt{1+c}), b4 = -b1.$$

Substituting the relation between x and $F(x)$ in the inequality of $G(x, y)$ above, we have

$$\begin{aligned} & \frac{1}{2}[1 - \sqrt{1 - (\frac{c}{2} + \frac{2}{3}x^2)}] \\ &= \frac{1}{2}[1 - \sqrt{1 - (\frac{c}{3} + \frac{2}{3}F(x))}] \\ &\leq G(x, y) \\ &\leq \frac{1}{2}[1 + \sqrt{1 - (\frac{c}{3} + \frac{2}{3}F(x))}]. \end{aligned}$$

So $R(x, y) = (F(x), G(x, y)) \in R1$, then $R2 \supset R(D2 \cup D8)$.

(2) For $(x, y) \in D1 \cup D3 \cup D7 \cup D9$, we have

$$\begin{aligned} & \frac{1}{2}|1 - \sqrt{1 - c}| \leq |x| \leq \frac{1}{2}(1 + \sqrt{1 - c}), \\ & -\frac{1}{2}[1 - \sqrt{1 - (\frac{c}{2} + \frac{2}{3}x^2)}] \leq |y| \leq \frac{1}{2}[1 + \sqrt{1 - (\frac{c}{2} + \frac{2}{3}x^2)}]. \end{aligned}$$

Using (2) of Theorem 3.4 we know

$$\frac{1}{2}(1 - \sqrt{1 - c}) \leq F(x) \leq \frac{1}{2}(1 + \sqrt{1 - c}).$$

Treating x_n and c in (2) of Theorem 3.4 as y and $\frac{c}{2} + \frac{2}{3}x^2$ respectively, we know

$$\frac{1}{2}[1 - \sqrt{1 - (\frac{c}{2} + \frac{2}{3}x^2)}] \leq G(x, y) \leq \frac{1}{2}[1 + \sqrt{1 - (\frac{c}{2} + \frac{2}{3}x^2)}].$$

In the same way in (1), changing x to $F(x)$, we have

$$\frac{1}{2}[1 - \sqrt{1 - (\frac{c}{3} + \frac{2}{3}F(x))}] \leq G(x, y) \leq \frac{1}{2}[1 + \sqrt{1 - (\frac{c}{3} + \frac{2}{3}F(x))}].$$

Thus $R(x, y) = (F(x), G(x, y)) \in R2$. Then

$$R2 \supset R(D1 \cup D3 \cup D7 \cup D9).$$

(3) If $(x, y) \in D5$, then

$$\begin{aligned} & |x| \leq \frac{1}{2}(1 - \sqrt{1 - c}), \\ & |y| \leq \frac{1}{2}|1 - \sqrt{1 - (\frac{c}{2} + \frac{2}{3}x^2)}|. \end{aligned}$$

Using (1) of Theorem 3.4 we know

$$-\frac{1}{2}(1 + \sqrt{1-c}) \leq F(x) \leq \frac{1}{2}(1 - \sqrt{1-c}).$$

And treating x_n and c in (1) of Theorem 3.4 as y and $\frac{c}{2} + \frac{2}{3}x^2$ respectively, we have

$$-\frac{1}{2}\left[1 + \sqrt{1 - \left(\frac{c}{2} + \frac{2}{3}x^2\right)}\right] \leq G(x, y) \leq \frac{1}{2}\left[1 - \sqrt{1 - \left(\frac{c}{2} + \frac{2}{3}x^2\right)}\right].$$

Then

$$-\frac{1}{2}\left[1 + \sqrt{1 - \left(\frac{c}{3} + \frac{2}{3}F(x)\right)}\right] \leq G(x, y) \leq \frac{1}{2}\left[1 - \sqrt{1 - \left(\frac{c}{3} + \frac{2}{3}F(x)\right)}\right].$$

So $R(x, y) = (F(x), G(x, y)) \in R3$, then $R3 \supset R(D5)$.

(4) For $(x, y) \in D4 \cup D6$, (x, y) satisfies

$$\frac{1}{2}|1 - \sqrt{1-c}| \leq |x| \leq \frac{1}{2}(1 + \sqrt{1-c}),$$

and

$$|y| \leq \frac{1}{2}\left|1 - \sqrt{1 - \left(\frac{c}{2} + \frac{2}{3}x^2\right)}\right|.$$

Using (1) of Theorem 3.4 we know

$$\frac{1}{2}(1 - \sqrt{1-c}) \leq F(x) \leq \frac{1}{2}(1 + \sqrt{1-c}).$$

In the same manner, we have

$$-\frac{1}{2}\left[1 + \sqrt{1 - \left(\frac{c}{3} + \frac{2}{3}F(x)\right)}\right] \leq G(x, y) \leq \frac{1}{2}\left[1 - \sqrt{1 - \left(\frac{c}{3} + \frac{2}{3}F(x)\right)}\right].$$

So $R(x, y) = (F(x), G(x, y)) \in R4$, then $R4 \supset R(D4 \cup D6)$.

‡

Let us compare the domain region of $R(x, y)$, $\cup_{i=1}^9 Di$, with its image region $\cup_{i=1}^4 Ri$.

The top boundary of $\cup_{i=1}^9 Di$ is a section of an ellipse

$$|x| \leq \frac{1}{2}(1 + \sqrt{1-c}),$$

$$y = \frac{1}{2}[1 + \sqrt{1 - (\frac{c}{2} + \frac{2}{3}x^2)}].$$

And the top boundary of $\cup_{i=1}^4 Ri$ is a section of a parabola, x in the same x -region as above and

$$y = \frac{1}{2}[1 + \sqrt{1 - (\frac{c}{3} + \frac{2}{3}x)}].$$

To compare these two boundaries, we subtract the parabola function from the ellipse function,

$$\begin{aligned} & \frac{1}{2}[1 + \sqrt{1 - (\frac{c}{2} + \frac{2}{3}x^2)}] - \frac{1}{2}[1 + \sqrt{1 - (\frac{c}{3} + \frac{2}{3}x)}] \\ &= \frac{1}{2}\sqrt{1 - (\frac{c}{2} + \frac{2}{3}x^2)} - \frac{1}{2}\sqrt{1 - (\frac{c}{3} + \frac{2}{3}x)}, \end{aligned}$$

where in the square roots

$$\begin{aligned} & \frac{c}{2} + \frac{2}{3}x^2 - (\frac{c}{3} + \frac{2}{3}x) \\ &= \frac{2}{3}(x^2 - x + \frac{c}{4}) \\ &= \frac{2}{3}(x - \frac{1 + \sqrt{1-c}}{2})(x - \frac{1 + \sqrt{1-c}}{2}). \end{aligned}$$

Then we know that for

$$\frac{1 - \sqrt{1-c}}{2} < x < \frac{1 + \sqrt{1-c}}{2},$$

the top boundary function of $\cup_{i=1}^4 Ri$ is less than the top boundary function of $\cup_{i=1}^9 Di$. For

$$-\frac{1 + \sqrt{1-c}}{2} \leq x \leq \frac{1 - \sqrt{1-c}}{2},$$

the top boundary function of $\cup_{i=1}^4 Ri$ is greater than the top boundary function of $\cup_{i=1}^9 Di$,

In the same way we have that for

$$\frac{1 - \sqrt{1-c}}{2} < x < \frac{1 + \sqrt{1-c}}{2},$$

the bottom boundary function of $\cup_{i=1}^4 Ri$ is greater than the top boundary function of $\cup_{i=1}^9 Di$. For

$$-\frac{1 + \sqrt{1-c}}{2} \leq x \leq \frac{1 - \sqrt{1-c}}{2},$$

the top boundary function of $\cup_{i=1}^4 Ri$ is less than the top boundary function of $\cup_{i=1}^9 Di$, see Figure 3.4. So, by the definition of Di and Rj , $i = 1, \dots, 9, j = 1, \dots, 4$, we have

$$D1 \cup D4 \cup D7 \subset R1 \cup R3, \quad (3.17)$$

and

$$D2 \cup D3 \cup D5 \cup D6 \cup D8 \cup D9 \supset R2 \cup R4. \quad (3.18)$$

Let

$$K1 = \left\{ (x, y) : \begin{array}{l} -\frac{1}{2}(1 + \sqrt{1-c}) \leq x \leq \frac{1}{2}(1 - \sqrt{1-c}), \\ \frac{1}{2}[1 + \sqrt{1 - (\frac{c}{2} + \frac{2}{3}x^2)}] \leq y \leq \frac{1}{2}[1 + \sqrt{1 - (\frac{c}{3} + \frac{2}{3}x)}], \end{array} \right\} \quad (3.19)$$

$$K2 = \left\{ (x, y) : \begin{array}{l} -\frac{1}{2}(1 + \sqrt{1-c}) \leq x \leq \frac{1}{2}(1 - \sqrt{1-c}), \\ -\frac{1}{2}[1 + \sqrt{1 - (\frac{c}{2} + \frac{2}{3}x^2)}] \geq y \geq \frac{1}{2}[1 + \sqrt{1 - (\frac{c}{3} + \frac{2}{3}x)}] \end{array} \right\}. \quad (3.20)$$

Then we have

$$K1 \cup K2 = R1 \cup R3 \setminus (D1 \cup D4 \cup D7),$$

Since $K1$ and $K2$ are out of the region $\cup_{i=1}^9 Di$, it is possible that part of the image $R(K1 \cup K2)$ is in the divergence region (3.8). The following theorem specifies a region in the $x - y$ plane such that all orbits $\{(x_n, y_n)\}_0^\infty$ with initial point in the region are boundary.

Theorem 3.6 *If the initial point (x_0, y_0) of an orbit $\{(x_n, y_n)\}_0^\infty$ is in the region (see Figure 3.5)*

$$BD = \left\{ (x, y) : \begin{array}{l} |x| \leq \frac{1}{2}|1 - \sqrt{1-c}|, \\ |y| \leq \sqrt{\frac{1}{2}[1 + \sqrt{1 - (\frac{c}{2} + \frac{2}{3}(\frac{c}{4} + x^2)^2)}] - (\frac{c}{8} + \frac{1}{6}x^2)}; \\ \frac{1}{2}|1 - \sqrt{1-c}| \leq |x| \leq \frac{1}{2}(1 + \sqrt{1-c}), \\ |y| \leq \frac{1}{2}[1 + \sqrt{1 - (\frac{c}{2} + \frac{2}{3}x^2)}]. \end{array} \right\} \quad (3.21)$$

then the orbit will remain in this region.

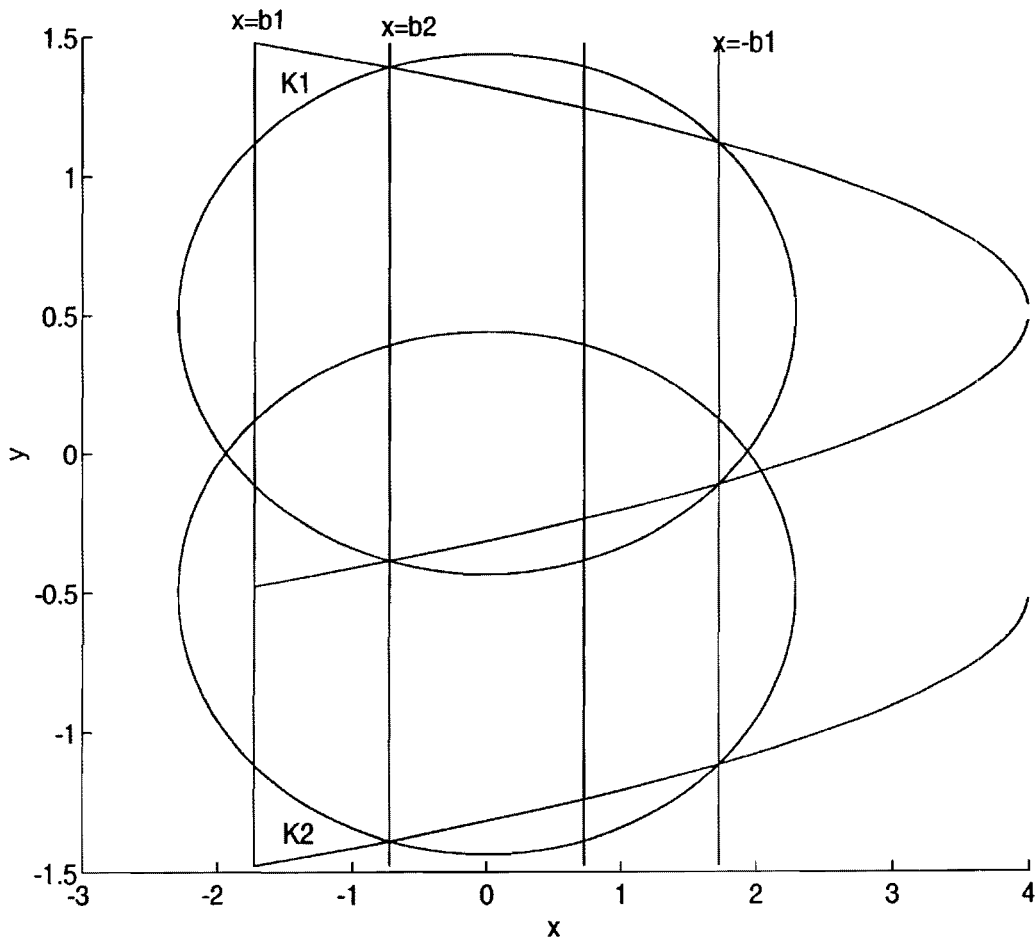


Figure 3.4:

$$b1 = -\frac{1}{2}(1 + \sqrt{1+c}),$$

$$b2 = \frac{1}{2}(1 - \sqrt{1+c}),$$

$$b3 = -b2, b4 = -b1.$$

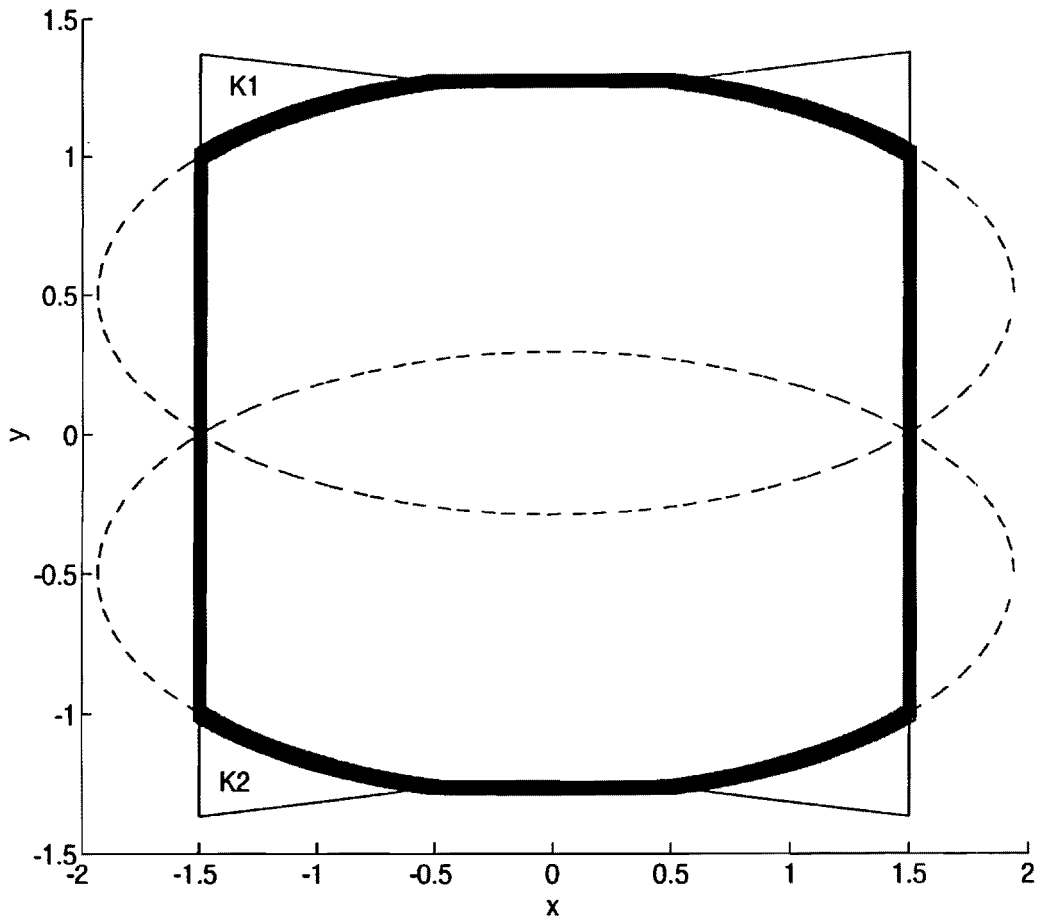


Figure 3.5: The region with the solid boundary is BD

Proof: First we need find the inverse of the mapping R on $K1$, i.e. $R^{-1}(K1)$. From (1) of Theorem 3.4 we know if $(F(x), G(x, y)) \in K1$ then

$$|x| \leq \frac{1}{2}|1 - \sqrt{1-c}|,$$

and

$$\frac{1}{2}[1 + \sqrt{1 - (\frac{c}{2} + \frac{2}{3}(\frac{c}{4} + x^2)^2)}] \leq \frac{c}{8} + \frac{1}{6}x^2 + y^2 \leq \frac{1}{2}[1 + \sqrt{1 - (\frac{c}{3} + \frac{2}{3}(\frac{c}{4} + x^2))}].$$

So y satisfies

$$\frac{1}{2}[1 + \sqrt{1 - (\frac{c}{2} + \frac{2}{3}(\frac{c}{4} + x^2)^2)}] - (\frac{c}{8} + \frac{1}{6}x^2) \leq y^2 \leq \frac{1}{2}[1 + \sqrt{1 - (\frac{c}{2} + \frac{2}{3}x^2)}] - (\frac{c}{8} + \frac{1}{6}x^2).$$

This implies

$$R^{-1}(K1) = \left\{ \begin{array}{l} (x, y) : \\ \quad |x| \leq \frac{1}{2}|1 - \sqrt{1-c}|, \\ \quad \frac{1}{2}[1 + \sqrt{1 - (\frac{c}{2} + \frac{2}{3}(\frac{c}{4} + x^2)^2)}] - (\frac{c}{8} + \frac{1}{6}x^2) \\ \quad \leq y^2 \leq \frac{1}{2}[1 + \sqrt{1 - (\frac{c}{2} + \frac{2}{3}x^2)}] - (\frac{c}{8} + \frac{1}{6}x^2) \\ \quad \quad \quad = \frac{1}{4}[1 + \sqrt{1 - (\frac{c}{2} + \frac{2}{3}x^2)}]^2 \end{array} \right\} \quad (3.22)$$

In the same way we can get the inverse of $K2$,

$$R^{-1}(K2) = \left\{ \begin{array}{l} (x, y) : \\ \quad |x| \leq \frac{1}{2}|1 - \sqrt{1-c}|, \\ \quad -\frac{1}{2}[1 + \sqrt{1 - (\frac{c}{2} + \frac{2}{3}(\frac{c}{4} + x^2)^2)}] - (\frac{c}{8} + \frac{1}{6}x^2) \\ \quad \geq y^2 \geq -\frac{1}{2}[1 + \sqrt{1 - (\frac{c}{2} + \frac{2}{3}x^2)}] - (\frac{c}{8} + \frac{1}{6}x^2) \end{array} \right\} \quad (3.23)$$

But from the y -inequality above we can see that $R^{-1}(K2)$ is empty.

From (1) of Theorem 3.5 we can see that for positive y in $R^{-1}(K1)$,

$$D2 \setminus R^{-1}(K1) = \left\{ \begin{array}{l} (x, y) : \\ \quad |x| \leq \frac{1}{2}|1 - \sqrt{1-c}|, \\ \quad -\frac{1}{2}[1 - \sqrt{1 - (\frac{c}{2} + \frac{2}{3}x^2)}] \leq y \leq \\ \quad \sqrt{\frac{1}{2}[1 + \sqrt{1 - (\frac{c}{2} + \frac{2}{3}(\frac{c}{4} + x^2)^2)}] - (\frac{c}{8} + \frac{1}{6}x^2)} \end{array} \right\} \quad (3.24)$$

and for negative y in $R^{-1}(K1)$

$$D8 \setminus R^{-1}(K1) = \left\{ \begin{array}{l} (x, y) : \\ \quad |x| \leq \frac{1}{2}|1 - \sqrt{1-c}|, \\ \quad \frac{1}{2}[1 - \sqrt{1 - (\frac{c}{2} + \frac{2}{3}x^2)}] \geq y \\ \quad \geq -\sqrt{\frac{1}{2}[1 + \sqrt{1 - (\frac{c}{2} + \frac{2}{3}(\frac{c}{4} + x^2)^2)}] + (\frac{c}{8} + \frac{1}{6}x^2)} \end{array} \right\}. \quad (3.25)$$

From Theorem 3.5 and the definition of $R^{-1}(K1)$ we have

$$R((D2 \cup D5 \cup D8) \setminus R^{-1}(K1)) \subset D1 \cup D4 \cup D7 \quad (3.26)$$

Now we want to prove

$$R(D1 \cup D4 \cup D7) \subset (D2 \cup D5 \cup D8) \setminus R^{-1}(K1) \quad (3.27)$$

In fact, we can see by definitions $D2, D5, D8$ and $R^{-1}(K1)$

$$R(D1 \cup D4 \cup D7) \subset \left\{ (x, y) : \begin{array}{l} |x| \leq \frac{1}{2}|1 - \sqrt{1-c}|, \\ |y| \leq \frac{1}{2}[1 + \sqrt{1 - (\frac{c}{2} + \frac{2}{3}x)}] \end{array} \right\},$$

and

$$(D2 \cup D5 \cup D8) \setminus R^{-1}(K1) = \left\{ (x, y) : \begin{array}{l} |x| \leq \frac{1}{2}|1 - \sqrt{1-c}|, \\ |y| \leq \sqrt{\frac{1}{2}[1 + \sqrt{1 - (\frac{c}{2} + \frac{2}{3}(\frac{c}{4} + x^2)^2)}] - (\frac{c}{8} + \frac{1}{6}x^2)} \end{array} \right\}.$$

So to prove the relation (3.27) it is sufficient to prove that

$$\frac{1}{2}[1 + \sqrt{1 - (\frac{c}{2} + \frac{2}{3}x)}] \leq \sqrt{\frac{1}{2}[1 + \sqrt{1 - (\frac{c}{2} + \frac{2}{3}(\frac{c}{4} + x^2)^2)}] - (\frac{c}{8} + \frac{1}{6}x^2)}$$

i.e.,

$$\frac{1}{2}[1 + \sqrt{1 - (\frac{c}{2} + \frac{2}{3}x)}] - \frac{1}{4}(\frac{c}{3} + \frac{2}{3}x) \leq \frac{1}{2}[1 + \sqrt{1 - (\frac{c}{2} + \frac{2}{3}(\frac{c}{4} + x^2)^2)}] - (\frac{c}{8} + \frac{1}{6}x^2)$$

To prove this inequality we note from Theorem 3.4 that for $|x| \leq -\frac{1}{2}[1 - \sqrt{1-c}]$,

$$-\frac{1}{2}[1 + \sqrt{1-c}] \leq \frac{c}{4} + x^2 \leq \frac{1}{2}[1 - \sqrt{1-c}] \leq x.$$

Then we have

$$\begin{aligned} & \frac{1}{2}\sqrt{1 - [\frac{c}{2} + \frac{2}{3}(\frac{c}{4} + x^2)^2]} - (\frac{c}{8} + \frac{1}{6}x^2) \\ & \geq \frac{1}{2}\sqrt{1 - [\frac{c}{2} + \frac{2}{3}x^2]} - \frac{c}{12} - \frac{1}{6}x \\ & = \frac{1}{2}\sqrt{1 - [\frac{c}{3} + \frac{2}{3}(\frac{c}{2} + x^2)]} - \frac{c}{12} - \frac{1}{6}x \\ & \geq \frac{1}{2}\sqrt{1 - [\frac{c}{3} + \frac{2}{3}x]} - (\frac{c}{12} + \frac{1}{6}x). \end{aligned}$$

So (3.27) is proved.

From (3.26) and (3.27) we know

$$R(D1 \cup D4 \cup D7 \cup [(D2 \cup D5 \cup D8) \setminus R^{-1}(K1)]) \subset D1 \cup D4 \cup D7 \cup [(D2 \cup D5 \cup D8) \setminus R^{-1}(K1)]. \quad (3.28)$$

Hence for

$$(x, y) \in D1 \cup D4 \cup D7 \cup [(D2 \cup D5 \cup D8) \setminus R^{-1}(K1)]$$

the image of (x, y) is in the region too,

$$F(x, y) \in D1 \cup D4 \cup D7 \cup [(D2 \cup D5 \cup D8) \setminus R^{-1}(K1)].$$

Since for any (x, y)

$$R(x, y) = R(-x, y) = R(-x, -y) = R(x, -y), \quad (3.29)$$

by the definition of the region BD we know

$$BD = D1 \cup D4 \cup D7 \cup [(D2 \cup D5 \cup D8) \setminus R^{-1}(K1)].$$

So for any $(x_0, y_0) \in BD$, $R(x_0, y_0) \in BD$. Therefore the orbit $\{(x_n, y_n)\}_0^\infty$ is in BD .

‡

From (3.29) we see that the inverse of the mapping $R(x, y)$ is four to one. It tell us that from an initial point in the region BD the orbit is contained in BD . The following lemma give us a property of the mapping $R(x, y)$.

Lemma 3.1 *R maps the elliptic region*

$$\frac{1}{6}x^2 + y^2 \leq r_0^2 \quad (3.30)$$

to the triangular region:

$$\Delta = \left\{ (x, y) : \begin{array}{l} \frac{c}{4} + 6r_0^2 \geq x \geq \frac{c}{4}. \\ \frac{c}{8} + r_0^2 \geq y \geq \frac{1}{6}x + \frac{c}{12} \end{array} \right\} \quad (3.31)$$

Proof: Let $0 \leq r \leq r_0$. R maps the ellipse Γ

$$\frac{1}{6}x_n^2 + y_n^2 = r^2$$

to a line section

$$\begin{aligned} \frac{c}{4} &\leq x_{n+1} \leq \frac{c}{4} + 6r^2 \\ y_{n+1} &= \frac{c}{8} + r^2. \end{aligned}$$

So R maps the region D_0 on the (x_n, y_n) plane to the region on the (x_{n+1}, y_{n+1}) plane with boundary

$$\begin{aligned} x_{n+1} &= \frac{c}{4} + 6r^2 \\ y_{n+1} &= \frac{c}{8} + r^2 \end{aligned}$$

with parameter r in $[0, r_0]$, i.e.,

$$\begin{aligned} \frac{c}{8} + r_0^2 &\geq y_{n+1} \geq \frac{1}{6}x_{n+1} + \frac{c}{12} \\ \frac{c}{4} + 6r_0^2 &\geq x_{n+1} \geq \frac{c}{4} \end{aligned}$$

‡

As r_0 goes to infinity, the Lemma above tells us the image of R is an unbounded triangular region Δ . Using Theorem 3.6 and the Lemma we have

Theorem 3.7 *For initial point $(x_0, y_0) \in BD$ the orbit $\{(x_n, y_n)\}_0^\infty$ is attracted by the region $BD \cap \Delta$.*

Proof: If $(x_0, y_0) \in BD$ from Theorem 3.6 we know $\{(x_n, y_n)\}_0^\infty \subset BD$. Using Lemma 3.1 we have that $\{(x_n, y_n)\}_1^\infty \subset \Delta$. Hence $(x_n, y_n) \in BD \cap \Delta$.

‡

Chapter 4

INVARIANT MANIFOLDS

In this chapter we want to study the invariant manifolds of the system (3.2) and (3.3). Let the invariant manifold passing through (u, v) be $(x(t), y(t))$, where

$$x(t) = u + \sum_{i=1}^{\infty} a_i t^i \quad (4.1)$$

$$y(t) = v + \sum_{i=1}^{\infty} b_i t^i. \quad (4.2)$$

Then there are β_1 and β_2 such that

$$\frac{c}{4} + x^2(t) = x(\beta_1 t) \quad (4.3)$$

$$\frac{c}{8} + \frac{c}{6} x^2(t) + y^2(t) = y(\beta_2 t). \quad (4.4)$$

Substituting (4.1) into (4.3) we have

$$\frac{c}{4} + \left[u + \sum_{i=1}^{\infty} a_i t^i \right]^2 = u + \sum_{i=1}^{\infty} a_i \beta_1^i t^i. \quad (4.5)$$

We compute explicitly the coefficient of t^n .

For $n = 1$, the coefficient of t satisfies

$$a_1(2u - \beta_1) = 0. \quad (4.6)$$

Let $\beta_1 = 2u$. Now let $a_1 = 1$. For $i = n > 1$, we have

$$2ua_n + \sum_{k=1}^{n-1} a_k a_{n-k} t^i = a_n \beta_1^n. \quad (4.7)$$

For $u = 0$ or $a_1 = 0$, then $a_k = 0, k = 1, 2, \dots$

Theorem 4.1 (1) If $\beta_1 = 2u = \pm 1$, then

$$a_k = 0 \quad k = 1, 2, \dots$$

and

$$x(t) = u$$

(2) If $2u \neq \pm 1$, the sequence of $a_n, n = 1, 2, \dots$, satisfies

$$a_n = \frac{1}{\beta_1^2 - 2u} \sum_{k=1}^{n-1} a_k a_{n-k}, \quad n \geq 2 .$$

Proof: (1) If $\beta_1 = 2u = 1$, then

$$a_n(2u - \beta_1^n) = \sum_{k=1}^{n-1} a_k a_{n-k}$$

$$n = 2, 3, \dots$$

Hence for $n > 1$, a_n satisfies

$$\sum_{k=1}^{n-1} a_k a_{n-k} = 0.$$

It implies that for $n = 1, \dots$, the coefficients $a_n = 0$.

If $\beta_1 = 2u = -1$, then using relation (4.7) we have

$$a_{2n} = -\frac{1}{2} \sum_{k=1}^{2n-1} a_k a_{2n-k} \tag{4.8}$$

$$n = 1, 2, \dots$$

and

$$\sum_{k=1}^{2n} a_k a_{2n+1-k} = 0, \tag{4.9}$$

$$n = 1, 2, \dots$$

For $n = 1$ from (4.8), (4.9) we have

$$a_2 = -\frac{1}{2} a_1^2 \tag{4.10}$$

and

$$\sum_{k=1}^2 a_k a_{3-k} = 2a_1 a_2 = 0. \quad (4.11)$$

Then (4.10) and (4.11) imply that

$$a_1 = a_2 = 0.$$

If $a_i = 0, i = 1, \dots, 2m$, then

$$a_{2J} = -\frac{1}{2} \sum_{k=1}^{2J-1} a_k a_{2J+1-k} = -\frac{1}{2} [2 \sum_{k=1}^{J-1} a_k a_{2J-k} + a_J^2] = -\frac{1}{2} a_J^2 = 0,$$

$$J \leq 2m.$$

i.e.,

$$a_n = 0 \text{ for } n = 2k \leq 4m \quad (4.12)$$

For $n = 2m_1$, we need to prove

$$a_{2m+1} = 0.$$

We consider $a_{3(2m+1)}$, i.e. $a_{2(3m+1)+1}$. Using (4.9) and (4.12), we have

$$\begin{aligned} 0 &= a_{3(2m+1)} \cdot 0 = \sum_{k=1}^{2(3m+1)} a_k a_{2(3m+1)+1-k} \\ &= 2 \sum_{k=1}^{2m+1} a_k a_{2(3m+1)+1-k} \\ &= 2 \left[\sum_{k=1}^{2m} a_k a_{2(3m+1)+1-k} + a_{2m+1} a_{4m+2} \right] \\ &= 2 \left[0 + a_{2m+1} a_{2(2m+1)} \right] = 2 \left[a_{2m+1} \left(-\frac{1}{2} \right) a_{2m+1}^2 \right] \\ &= -a_{2m+1}^3 \end{aligned}$$

i.e., $a_{2m+1} = 0$. Hence we have $a_k \equiv 0$ for all $k = 1, 2, \dots$

(2) It is easy to prove (2) by (4.7).

‡

If $2u \neq \pm 1, 0$, we can obtain the coefficients of the series $x(t)$ from (2) in Theorem 4.1,

$$a_n = \frac{1}{(2u)^2 - 2u} \sum_{k=1}^{n-1} a_k a_{n-k} \quad n \geq 2 \text{ and } a_1 = \text{const} \quad (4.13)$$

We want to find the coefficients of the series $y(t)$. From (4.1), (4.2) and (4.4) we have

$$\frac{c}{8} + \frac{1}{6} \left[u + \sum_{i=1}^{\infty} a_i t^i \right]^2 + \left[v + \sum_{i=1}^{\infty} b_i t^i \right]^2 = v + \sum_{i=1}^{\infty} b_i \beta_2 b_i t^i.$$

Computing the coefficient of t^n , we have

$$\frac{1}{6} 2ua_1 + 2vb_1 = b_1 \beta_2 \quad (4.14)$$

and

$$\frac{1}{6} \left[2ua_n + \sum_{i=1}^{n-1} a_i a_{n-i} \right] + \left[2vb_n + \sum_{i=1}^{n-1} b_i b_{n-i} \right] = b_n \beta_2^n$$

So

$$\begin{aligned} b_1 &= \frac{ua_1}{3(\beta_2 - 2v)} \\ b_n &= \frac{1}{\beta_2^n - 2v} \left[\frac{1}{6} \left(2ua_n + \sum_{i=1}^{n-1} a_i a_{n-i} \right) + \sum_{i=1}^{n-1} b_i b_{n-i} \right] \\ &= \frac{1}{\beta_2^n - 2v} \left\{ \frac{1}{6} \left[2ua_n + a_n(\beta_2 - 2u) \right] + \sum_{i=1}^{n-1} b_i b_{n-i} \right\} \\ &= \frac{1}{\beta_2^n - 2v} \left[\frac{1}{6} \beta_2^n a_n + \sum_{i=1}^{n-1} b_i b_{n-i} \right] \end{aligned}$$

Hence we have

$$\begin{aligned} a_n &= \frac{1}{\beta_1^n - 2u} \sum_{i=1}^{n-1} a_i a_{n-i} \\ b_n &= \frac{1}{\beta_2^n - 2v} \left[\frac{1}{6} (2u)^n a_n + \sum_{i=1}^{n-1} b_i b_{n-i} \right] \quad (4.15) \\ & \quad n \geq 2 \end{aligned}$$

where a_1 and b_1 satisfy

$$\begin{aligned} a_1(2u - \beta_1) &= 0 \\ \frac{1}{3}ua_1 + (2v - \beta_2)b_1 &= 0 \end{aligned} \quad (4.16)$$

Theorem 4.2 For the system (3.2) and (3.3), if a fixed point is hyperbolic, there are a stable invariant manifold, W^s , and a unstable invariant manifold, W^u , passing through the fixed point.

Proof: In (4.16) let $a_1 = a_1^{(u)} = 1$, $\beta_1 = \beta_2 = 2u$, Let the manifold $M(u) = (x^{(u)}(t), y^{(u)}(t))$ with parameter t be defined by

$$\begin{aligned} x^{(u)}(t) &= u + \sum_{i=1}^{\infty} a_i^{(u)} t^i \\ y^{(u)}(t) &= v + \sum_{i=1}^{\infty} b_i^{(u)} t^i \end{aligned} \quad (4.17)$$

where the coefficients of t^n , $a_n^{(u)}$, $n = 2, 3, \dots$ are given by (4.15). It satisfies

$$\frac{c}{4} + x^2(t) = x(2ut) \quad (4.18)$$

For the y component of the manifold $M(u)$, taking ,

$$b_1 = b_1^{(u)} = \frac{u}{6(u-v)},$$

the coefficients $b_n^{(u)}$, $n = 1, 3, \dots$ are given by (4.16). $y(t)$ satisfies

$$\frac{c}{8} + \frac{c}{6}x^2(t) + y^2(t) = y(2ut). \quad (4.19)$$

The equations (4.18) and (4.19) imply that if the initial point (x_0, y_0) of the orbit of the dynamical system (3.2) and (3.3) is on the manifold $M(2u)$, i.e., there is t_0 such that $(x_0, y_0) = (x^{(u)}(t_0), y^{(u)}(t_0))$, then the orbit $\{(x_n, y_n)\}_0^\infty$, starting from the point (x_0, y_0) on the manifold $M(u)$ satisfies

$$(x_n, y_n) = (x^{(u)}((2u)^n t_0), y^{(u)}((2u)^n t_0)), \quad n = 0, 1, \dots \quad (4.20)$$

We can construct another manifold, $M(v) = (x^{(v)}, y^{(v)})$, which passes through the point (u, v) ,

$$\begin{aligned} x^{(v)}(t) &= u + \sum_{i=1}^{\infty} a_i^{(v)} t^i \\ y^{(v)}(t) &= v + \sum_{i=1}^{\infty} b_i^{(v)} t^i, \end{aligned} \quad (4.21)$$

with $a_n^{(v)} \equiv 0$. Then $x^{(v)}(t) \equiv u$. Let $b_1^{(v)} = 1$ and $\beta_2 = 2v$. We can see that $a_1^{(v)}$ and $b_1^{(v)}$ satisfy the relation (4.16). Then $b_n^{(v)}, n = 2, 3, \dots$ are obtained by

$$b_n^{(v)} = \frac{1}{(2v)^n - 2v} \sum_{i=1}^{n-1} b_i^{(v)} b_{n-i}^{(v)}. \quad (4.22)$$

In the same way as for the manifold $M(u)$, we know that for any initial point of an orbit, (x_0, y_0) on the manifold $M(v)$, there is t_0 such that $(x_0, y_0) = (x^{(v)}(t_0), y^{(v)}(t_0))$. Then the orbit $\{(x_n, y_n)\}_0^\infty$ in the manifold $M(v)$ satisfies

$$(x_n, y_n) = (x^{(v)}((2v)^n t_0), y^{(v)}((2v)^n t_0)), \quad n = 0, 1, \dots \quad (4.23)$$

Due to the hyperbolicity of the fixed point (u, v) , the radius of convergence of (4.17) and (4.21) are positive [22].

For any given fixed point (u, v) , from (3.2) and (3.3) we know the eigenvalues at this fixed point are $2u$ and $2v$. Since we consider a hyperbolic fixed point, one of the eigenvalues is in $(-1, 1)$ and another is outside of $[-1, 1]$. (4.20) and (4.23) imply that one of manifolds, $M(u)$ and $M(v)$, is stable and the other is unstable, depending on which absolute value of the eigenvalue at the fixed point is less than 1 and which absolute value of the eigenvalue at the fixed point is greater than 1. In any case, we have the stable manifold W^s and unstable manifold W^u .

¶

In the same manner we can get this result for the general model (2.14) and (2.15).

If we continued, then we would find a number c such that the dynamical system (3.2) has an attracting 8-periodic orbit. In fact there is a sequence of numbers $c_1 > c_2 > \dots$ such that when $c_n > c > c_{n+1}$ the dynamical system has an attracting 2^n -periodic orbit. We already know that $c_1 = -3, c_2 = -5.0025$. But there is more. There are a series of values, c_n , which are ALL greater then the number -5.6049 . The closer c is to -5.6049 , the larger the attracting periodic orbit of (3.2). We get a series of period doubling bifurcations, each closer to -5.6049 than the one before. All of the branches combine to give an attracting 2^n -periodic orbit.

For c decreasing from 1 to -3, there are only two fixed points of (3.2). What happens for the two dimensional system (3.2) and (3.3)? The following theorem provides the answer.

Theorem 5.1 *For the two dimensional system (3.2) and (3.3), if $c \in (-3, 1]$, there are no bifurcation points. There are only four fixed points,*

$$(x_{-}^{(1)}, y_{-,-}^{(1)}), (x_{-}^{(1)}, y_{-,+}^{(1)}),$$

$$(x_{+}^{(1)}, y_{+,-}^{(1)}), (x_{+}^{(1)}, y_{+,+}^{(1)}),$$

where

$$y_{-,\pm}^{(1)} = \frac{1}{2} \left[1 - \sqrt{1 - \frac{c}{2} + \frac{2}{3}(x_{\pm}^{(1)})^2} \right], \quad (5.1)$$

$$y_{+,\pm}^{(1)} = \frac{1}{2} \left[1 + \sqrt{1 - \frac{c}{2} + \frac{2}{3}(x_{\pm}^{(1)})^2} \right] \quad (5.2)$$

and there are no more n -periodic orbit with $n > 1$. For $c \in (-8, 1]$ there is only one bifurcation point, -7.14589034 , at which one of the fixed points, $(x_{-}^{(1)}, y_{-,-}^{(1)})$, bifurcates to a 2-periodic orbit:

$$(x_{-}^{(1)}, y_{-}^{(2)}), (x_{-}^{(1)}, y_{+}^{(2)})$$

where

$$y_{\pm}^{(2)} = \frac{-1 \pm \sqrt{-\left(\frac{c}{2} + \frac{2}{3}(x_{-}^{(1)})^2\right) - 3}}{2}. \quad (5.3)$$

Proof: We know that for $c \in (-3, 1]$ there are only two fixed points $x_-^{(1)}$ and $x_+^{(1)}$ for the system (3.2). At these fixed points the system (3.3) is

$$y_{k+1} = \frac{1}{8}c + \frac{1}{6}(x_{\pm}^{(1)})^2 + y_k^2,$$

i.e.

$$y_{k+1} = \frac{1}{4}\left(\frac{c}{2} + \frac{2}{3}(x_{\pm}^{(1)})^2\right) + y_k^2. \quad (5.4)$$

We can get two fixed points for (3.4):

$$y_{-, \pm}^{(1)} = \frac{1}{2}\left[1 - \sqrt{1 - \frac{c}{2} + \frac{2}{3}(x_{\pm}^{(1)})^2}\right],$$

$$y_{+, \pm}^{(1)} = \frac{1}{2}\left[1 + \sqrt{1 - \frac{c}{2} + \frac{2}{3}(x_{\pm}^{(1)})^2}\right].$$

Thus we get four fixed points for (3.2) and (3.3)

$$(x_-^{(1)}, y_{-, -}^{(1)}), (x_-^{(1)}, y_{-, +}^{(1)}),$$

$$(x_+^{(1)}, y_{+, -}^{(1)}), (x_+^{(1)}, y_{+, +}^{(1)}).$$

Now we consider for $c \in (-3, 1]$ if these fixed points bifurcate n -periodic. In the equation (5.4) we can replace y_k with x_k and $\frac{c}{2} + \frac{2}{3}(x_{\pm}^{(1)})^2$ with c in (3.2). Then using the result of the system (3.2) we have that c satisfies the following inequality

$$\frac{c}{2} + \frac{2}{3}(x_{\pm}^{(1)})^2 < -3. \quad (5.5)$$

But we can check that this is not true. From the inequality above, by the definition of $x_{\pm}^{(1)}$ we have

$$\frac{c}{2} + \frac{2}{3}\left[\frac{1}{2}(1 \pm \sqrt{1-c}) - \frac{4}{c}\right] = \frac{c}{3} + \frac{1}{3}(1 \pm \sqrt{1-c}) < -3,$$

i.e.,

$$c + (1 \pm \sqrt{1-c}) < -9,$$

$$\pm\sqrt{1-c} < -10 - c.$$

But for $1 \geq c > -3$, $-10 - c \in (-10, -7)$ and $\pm\sqrt{1-c} \in (-2, 2)$. This is a contradiction. So for $c \in (-3, 1]$ the four fixed points can not bifurcate periodic orbits.

For $c \in (-8, 1]$, we can solve (5.5) and have $c \in (-8, -7.14589034)$. For $c > -7.14589034$ (5.5) is not true. Replacing $\frac{c}{2} + \frac{2}{3}(x_{\pm}^{(1)})^2$ and y in (5.4) with c and x in (3.2), respectively, we can get two periodic orbits of the system (5.4)

$$y_{\pm}^{(2)} = \frac{-1 \pm \sqrt{-(\frac{c}{2} + \frac{2}{3}(x_{\pm}^{(1)})^2) - 3}}{2}$$

It means $(x_{-}^{(1)}, y_{-,-}^{(1)})$ bifurcates to a 2-periodic orbit $(x_{-}^{(1)}, y_{-}^{(2)}), (x_{-}^{(1)}, y_{+}^{(2)})$ at the bifurcation point $c = -7.14589034$.

#

For $-5.0025 < c < -3$ there is an attracting two periodic orbit of the system (3.2). For the two dimensional system (3.2) and (3.3) are there any periodic orbit? We have the following result.

Theorem 5.2 *For $-5.0025 < c < -3$ there are at least two 2-periodic orbits of the two dimensional system (3.2) and (3.3).*

Proof: From Theorem 5.1, for $-5.0025 < c < -3$ there are no periodic orbits of the two dimensional system with the x component a fixed point of (3.2). So we only need to consider the x component to given a two-periodic orbit of (3.2). To find 2-periodic orbits of the two dimensional system we need to solve the equations:

$$y_2 = \frac{c}{8} + \frac{1}{6}(x_{+}^{(2)})^2 + (y_1)^2, \quad (5.6)$$

$$y_1 = \frac{c}{8} + \frac{1}{6}(x_{-}^{(2)})^2 + (y_2)^2. \quad (5.7)$$

This is equivalent to solving the equation

$$y_2 = \frac{c}{8} + \frac{1}{6}(x_{+}^{(2)})^2 + [\frac{c}{8} + \frac{1}{6}(x_{-}^{(2)})^2 + y_2]^2. \quad (5.8)$$

Let

$$G_1(y) = \frac{c}{8} + \frac{1}{6}(x_+^{(2)})^2 + y^2$$

and

$$G_2(y) = \frac{c}{8} + \frac{1}{6}(x_-^{(2)})^2 + y^2.$$

Then the problem is to solve the equation

$$y = G_1(G_2(y)). \quad (5.9)$$

For $c \leq -3$ the function G_1 at $y = 0$ is

$$\begin{aligned} G_1(0) &= \frac{c}{8} + \frac{1}{6}x_+^{(2)} \\ &= \frac{c}{8} + \frac{1}{6}\left[\frac{-1 + \sqrt{-c-3}}{2}\right]^2 \\ &= \frac{1}{12}[c-1 - \sqrt{-c-3}] \\ &< 0. \end{aligned}$$

We want to find one point such that $G_2 = 0$. Since for $c < 0$

$$\begin{aligned} &\frac{c}{8} + \frac{1}{6}x_-^{(2)} \\ &= \left[\frac{c}{8} + \frac{1}{6}\left(\frac{-1 - \sqrt{-c-3}}{2}\right)^2\right] \\ &= \frac{1}{12}[c-1 - \sqrt{-c-3}] \\ &< 0, \end{aligned}$$

we can find the point $y = \sqrt{-\frac{1}{12}[c-1 - \sqrt{-c-3}]}$ such that at this point the value of the function G_2 is zero, i.e.

$$G_2\left(\sqrt{-\frac{1}{12}[c-1 - \sqrt{-c-3}]}\right) = 0.$$

Hence, the right side function of the equation (5.9) at $y = \sqrt{-\frac{1}{12}[c-1 - \sqrt{-c-3}]}$ for $x \leq -3$ is

$$\begin{aligned} &G_1(G_2(\sqrt{-\frac{1}{12}[c-1 - \sqrt{-c-3}]})) = G_1(0) \\ &= \frac{1}{12}[c-1 - \sqrt{-c-3}] < 0. \end{aligned}$$

Hence

$$G_1(G_2(\sqrt{-\frac{1}{12}[c-1-\sqrt{-c-3}]}) - \sqrt{-\frac{1}{12}[c-1-\sqrt{-c-3}]}) < 0.$$

On the other hand it is easy to see that

$$\lim_{|y| \rightarrow \infty} G_1(G_2(y)) = \infty. \quad (5.10)$$

Then by continuity of the function $G_1(G_2(y))$ we have there are at least two solutions of (5.9). Let $y_1^{(2)}$ and $y_2^{(2)}$ be these solutions. Then we get the two 2-periodic orbits of the system (3.2) and (3.3):

$$(x_+^{(2)}, y_1^{(2)}) \quad , \quad (x_-^{(2)}, G_1(y_1^{(2)})),$$

and

$$(x_+^{(2)}, y_2^{(2)}) \quad , \quad (x_-^{(2)}, G_1(y_2^{(2)})).$$

;

In general, for given k and c if there is a k -periodic orbit of the one dimensional system (3.2), what happens for the two dimensional system? We have the following result.

Theorem 5.3 *For given c and any k , if there is a k -periodic orbit of the one dimensional system (3.2) then there are at least two k -periodic orbits of the two dimensional system (3.2) and (3.3).*

Proof: Since we have the results, Theorem 5.1 and Theorem 5.2, we only need prove this theorem for $c < -5.0025$. Let $\{x_n^{(k)}\}_{n=1}^{\infty}$ be a k -periodic orbit of (3.2), i.e.,

$$x_{n+k}^{(k)} = x_n^{(k)}. \quad (5.11)$$

Let

$$G_n(y) = \frac{c}{8} + \frac{1}{6}(x_n^{(k)})^2 + y^2.$$

To find k -periodic orbits of the two dimensional system we want to prove there are solutions of the equation

$$y = G_k \circ G_{k-1} \circ \dots \circ G_1(y). \quad (5.12)$$

Let us consider the value of the function G_n at $y = 0$,

$$G_n = \frac{c}{8} + \frac{1}{6}(x_n^{(k)})^2.$$

We know from (3) of Theorem 3.4 that

$$|x_n^k| \leq \frac{1}{2}(1 + \sqrt{1-c}).$$

So for $c < -3$ we have

$$\begin{aligned} G_k &\leq \frac{c}{8} + \frac{1}{6}\left[\frac{1}{2}(1 + \sqrt{1-c})\right]^2 \\ &= \frac{c}{8} + \frac{1}{6}\left[\frac{1}{2}(1 + \sqrt{1-c}) - \frac{c}{4}\right] \\ &= \frac{1}{12}[c + 1 + \sqrt{1-c}] \\ &= -\frac{1}{12}[(1-c) - \sqrt{1-c} - 2] \\ &= \frac{1}{12}(1 + \sqrt{1-c})(\sqrt{1-c} - 2) \\ &< 0. \end{aligned}$$

It is easy to see the mapping of the system (3.2) and (3.3) is a four to one mapping. So we can not find a unique inverse. But we can define one inverse function of $G(y)$:

$$G_n^{-1} = \sqrt{y - \left(\frac{c}{8} + \frac{1}{6}x_n^2\right)} \quad (5.13)$$

It satisfies

$$G_n \circ G_n^{-1}(y) = y.$$

Let

$$y_0 = G_1^{-1} \circ G_2^{-1} \circ \dots \circ G_k^{-1}(0).$$

Then we have

$$\begin{aligned}
& G_k \circ G_{k-1} \circ \dots \circ G_1(y_0) \\
&= G_k \circ G_{k-1} \circ \dots \circ G_1 \circ G_1^{-1} \circ G_2^{-1} \circ \dots \circ G_k^{-1}(0) \\
&= G_k \circ G_{k-1} \circ G_{k-1}^{-1}(0) \\
&= G_k(0) < 0.
\end{aligned}$$

So the function $G_k \circ G_{k-1} \circ \dots \circ G_1(y)$ is negative at y_0 . By the definition of the inverse function of G_n we have $y_0 \geq 0$. Hence

$$G_k \circ G_{k-1} \circ \dots \circ G_1(y_0) - y_0 < 0.$$

Since the function $G_k \circ G_{k-1} \circ \dots \circ G_1(y)$ is an even polynomial, we have

$$\lim_{|y| \rightarrow \infty} G_k \circ G_{k-1} \circ \dots \circ G_1(y) = \infty.$$

By continuity, there are at least two solutions of (5.12). Let these solutions be $y_1^{(k)}$ and $y_2^{(k)}$. Then for the two dimensional system, the two orbits with initial points $(x_1^{(k)}, y_1^{(k)})$ and $(x_1^{(k)}, y_2^{(k)})$, respectively, are k -periodic orbits.

‡

Let us quote a result for one dimensional system from Sarkovskit [17].

Theorem 5.4 *Let $f : R \rightarrow R$ be continuous. Suppose*

$$x_{n+1} = f(x_{n+1})$$

has a three-periodic orbit. Then it has periodic points of all other periods.

We know that for $c \leq -5.6049$ the system (3.2) has 2^n -periodic orbits. But for what c the system has n -periodic orbit with any given n ? And how about the two dimensional system? We have following result.

Theorem 5.5 (1) *The two dimensional system (3.2) and (3.3) has n -periodic orbits for any given n iff $c \leq -7$.*

(2) *For $c \in (-8, -7]$ there is an uncountable set $S \subset BD \cap \Delta$ (containing no periodic points), which satisfies the following conditions:*

for every $p, q \in S$ with $p \neq q$, we have

$$\limsup_{n \rightarrow \infty} |R^n(p) - R^n(q)| > 0 \quad (5.14)$$

and

$$\liminf_{n \rightarrow \infty} |F^n(p) - F^n(q)| = 0. \quad (5.15)$$

for every $p \in S$ and periodic point $q \in BD \cap \Delta$, we have

$$\limsup_{n \rightarrow \infty} |R^n(p) - R^n(q)| > 0, \quad (5.16)$$

where

$$|(x, y)| = \sqrt{x^2 + y^2}.$$

Proof: (1) From Theorem 5.3 we have that the two dimensional system (3.2) and (3.3) has n -periodic orbits for any given n iff there are any periodic orbits for the one dimensional system (3.2). Using Theorem 5.4 we know that the system (3.2) has any periodic orbits iff it has 3-periodic orbit. Now we want to find a 3-periodic orbits of (3.2).

For given c , to find 3-periodic orbit we need find a real solution of the equation

$$F^3(x) = x. \quad (5.17)$$

Using MAPLE on IBM 486/50 running more than 12 hours, we have that for $c > -7$ the equation above has no real solution which is one point of a three periodic orbit of the one dimension system and when $c \leq -7$ there is a real solution which is one point of a three periodic orbit. So (1) proved.

(2) Using Tien-Yien Li and J.A Yorke Theorem [12], we have that there is a set

$$S_x \subset \left[\frac{1}{2}(1 - \sqrt{1-c}), \frac{1}{2}(1 + \sqrt{1-c}) \right],$$

such that for every $v, u \in S_x$ with $v \neq u$,

$$\lim_{n \rightarrow \infty} \sup |F^n(v) - F^n(u)| > 0, \quad (5.18)$$

$$\lim_{n \rightarrow \infty} \inf |F^n(v) - F^n(u)| = 0, \quad (5.19)$$

and for every $p \in S$ and periodic point $q \in BD \cap \Delta$,

$$\lim_{n \rightarrow \infty} \sup |F^n(p) - F^n(q)| > 0. \quad (5.20)$$

Taking

$$S = \{(x, y) : x \in S_x \text{ and } (x, y) \in BD \cap \Delta\}$$

from (5.18) and (5.20) we can get (5.15) and (5.17) for S_x the x -component of S .

‡

What happens when $c \leq -5.6049$? At first the numbers x_n in the orbit of (3.2) seem to have no pattern. But this is not exactly true. Behavior is that all of the x_n values eventually fall onto some intervals. This means these intervals attract orbits.

Definition 5.1 *A set of points S is called an attraction set or an attractor for a dynamical system $x_{n+1} = f(x_n)$ iff there are a number δ and $s \in S$ with $|x_0 - s| < \delta$, such that*

$$\lim_{n \rightarrow \infty} |x_n - S| = 0.$$

In fact, the set of all attracting fixed points is an attractor. We know for $c \in [1, -5.6049)$ there is one 2^n -periodic attractor for (3.2). Hence from Theorem 5.3 the two dimensional system has at most one 2^n -periodic attractor.

For $-8 < c \leq -5.6049$ it is hard to see that the attractor is a finite or periodic orbit or not.

We use a computer to do the experiments for the two dimensional system and find the attractor changes quickly by changing c . For instance, as decreasing c passes through the

point $c = -5.89879$, the attractor changes from sections of curves to a 6-periodic orbit. It can be seen in the figures at $c = -5.89, -5.89878$ and -5.918998 .

Theorem 5.6 *c decreasing from -5.6049 there are a series bifurcation points of c for the system (3.2) and (3.3) :*

$$\begin{aligned}
 & -5.89879, -5.91895, -5.91904, -5.91911, -5.91914, -5.91923, \\
 & \quad -5.92438, -5.92844, -5.92845, -5.92849, \dots \\
 & \quad -7, -7.07403, -7.109, -7.1096, -7.1114, \dots
 \end{aligned}$$

For c in the following intervals there are periodic orbit attractors:

- $(-5.91895, -5.89879]$ attractor is 6-periodic,*
- $(-5.91904, -5.91895]$ attractor is 7-periodic,*
- $(-5.91911, -5.91904]$ attractor is 8-periodic,*
- $(-5.91914, -5.91911]$ attractor is 9-periodic,*
- $(-5.91923, -5.91914]$ attractor is 10-periodic,*
- $(-5.92438, -5.91923]$ attractor is 11-periodic,*
- $(-5.92844, -5.92438]$ attractor is 12-periodic,*
- $(-5.92845, -5.92844]$ attractor is 13-periodic,*
- $(-5.92849, -5.92845]$ attractor is 14-periodic,*
- $(-5.92849 - \delta, -5.92849]$ for small δ , attractor is 15-periodic,*
-*
- $(-7.07403, -7.00000]$ attractor is 3-periodic,*
- $(-7.10900, -7.07403]$ attractor is 6-periodic,*
- $(-7.1096, -7.10900]$ attractor is 8-periodic,*
- $(-7.1114, -7.1096]$ attractor is 11-periodic,*
- $(-7.1114 - \delta, -7.1114]$ for small δ , attractor is 12-periodic,*
-*

These attractors are not in sensitive dependence on the initial values in a neighborhood of the origin.

When decreasing c approaches -7 or -8 the attractor becomes an infinite point attractor. It can be seen in the figures at c near -7 and -8 .

Definition 5.2 *A dynamical system is said to be transitive if, when x_0 is close to some point in an attractor S , then for every point s in the attractor there is a subsequent x_{n_k} of x_n that converge to s , that is*

$$\lim_{n_k \rightarrow \infty} x_{n_k} = s.$$

By the definition of attractor and transitive we know that if a system has a k -periodic orbit attractor then the orbit is transitive. But if an attractor includes two attracting periodic orbits then the attractor is not transitive.

When $c = -8$ for different initial points which are close to each other, the orbits corresponding to the initial values are much different. For example, (See the figures for $c = -8$)
for $(x_0, y_0) = (1, 0)$ or $(1, 0.1)$ the orbit goes to 2-periodic
for $(x_0, y_0) = (0, 0)$ the orbit goes to fixed point
for $(x_0, y_0) = (1.2, 0)$ or $(0.1, 0.1)$ the orbit goes to non-periodic attractor
It shows that the orbits are sensitive dependence on the initial points.

Definition 5.3 *A dynamical system has sensitive dependence on the initial values if, whenever you take two initial values, A_0 and B_0 , which are close together, the orbits, A_n and B_n , eventually get further apart. To be more precise, there exists a number ϵ such that, whenever $0 < |A_0 - B_0| < \epsilon$, then there exists a value N such that $|A_N - B_N| > \epsilon$.*

For system (3.2), for given $c = -5.6564$ the orbit $\{A_n\}$ with initial point $A_0 = 0$ and other orbit $\{B_n\}$ with $A_0 = 0.001$ we have that $|A_{40} - B_{41}| = 0.00540085$ it means the system at this c is sensitive dependence on the initial values. The quadratic map exhibits in stunning fashion a phenomenon which is only partially understood: the chaotic behavior of orbits of a dynamical system. There are many possible definitions of chaos. Here we give one of them which is easily understood.

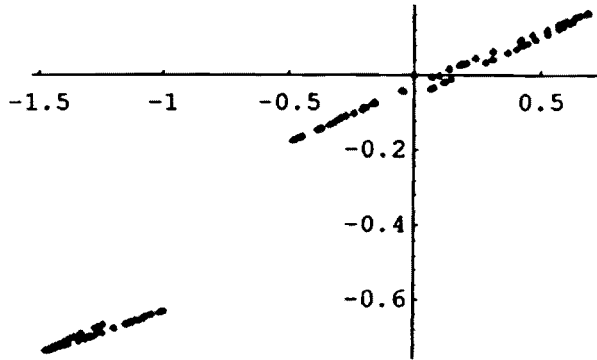
In Theorem 5.6, c changing slightly, the attractor changes. Specifically for c decreasing in the small region $[-5.91895, -5.92849]$ period of the orbit, attractor, increase one by one.

Definition 5.4 *We call the region of parameter of a dynamical system, c , is a WINDOW, if for c in this region the dynamic system have periodic attractors which are not sensitive dependence of initial values.*

There are some windows in $[-8, -5.6049]$. Beside the windows the attractors are unpredictable.

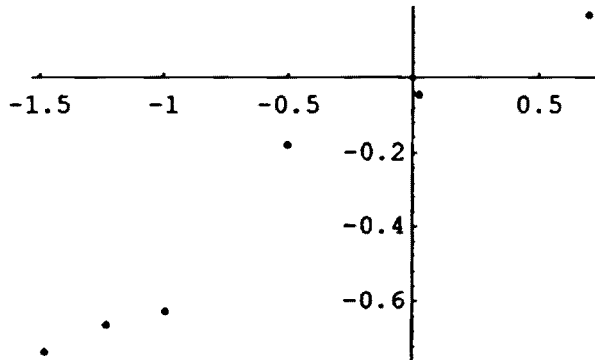
Definition 5.5 *Suppose a dynamical system (i) is transitive on its attractor S , (ii) has sensitive dependence of initial values, and (iii) has repelling periodic orbit that are 'close' to the attractor S . Then this dynamical system exhibits chaos.*

We have know that for $-8 \leq c \leq -5.6049$ the system (3.2) and (3.3) have a series of repelling 2^n -periodic for every value of n . And for $-8 \leq c \leq -7$ the system have any periodic orbit. It causes the orbit is unpredictability except windows. The dynamical system for $c \in [-8, -5.6049]$, except window, exhibits chaos.



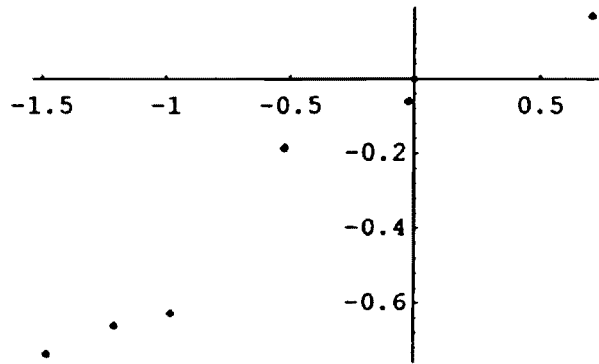
(1) $c = -5.89$

The orbit with initial point $p(0) = (0, 0)$,
 $p(3101), \dots, p(3252)$, goes to an attractor.



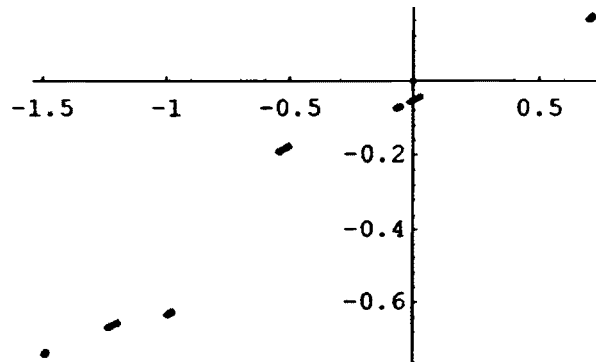
(2) $c = -5.89878$

The orbit with initial point $p(0) = (0, 0)$, $p(3101)$,
 $\dots, p(3202)$, goes to a six periodic attractor.



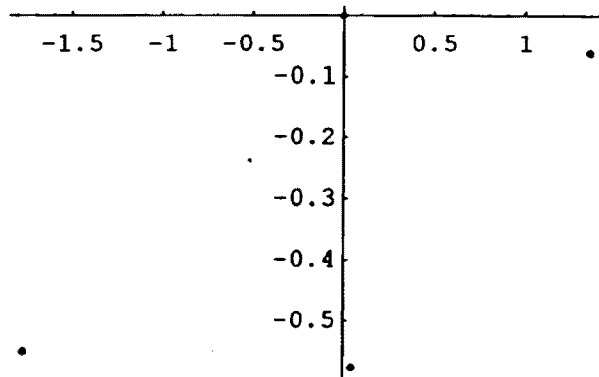
(3) $c = -5.918998$

The orbit with initial point $p(0) = (0, 0)$, $p(3101)$,
 \dots , $p(3050)$, goes to a seven periodic attractor.



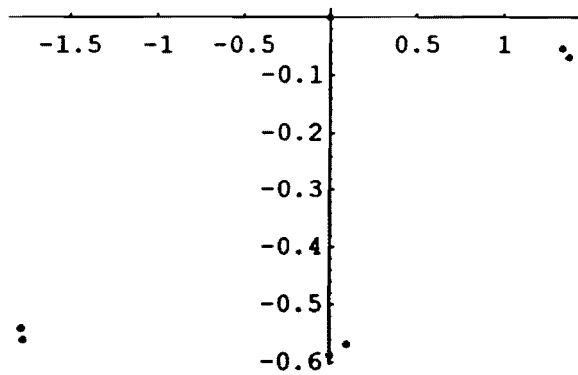
(4) $c = -5.9315$

The orbit with initial point $p(0) = (0, 0)$, $p(3101)$, \dots ,
 $p(3152)$, goes to an attractor.



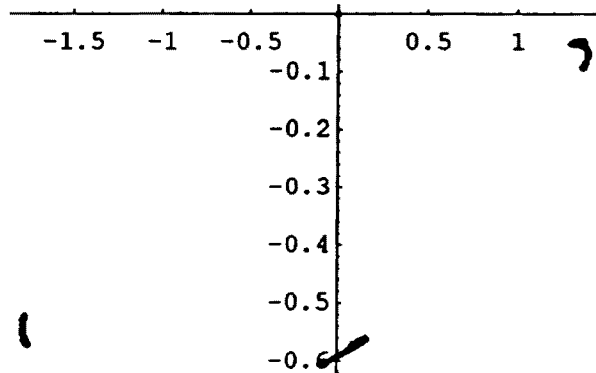
(5) $C = -7.07$

The orbit $p(n)$ with initial point $(0,0)$ and $n=2000, \dots, 3000$ goes to a three periodic attractor.



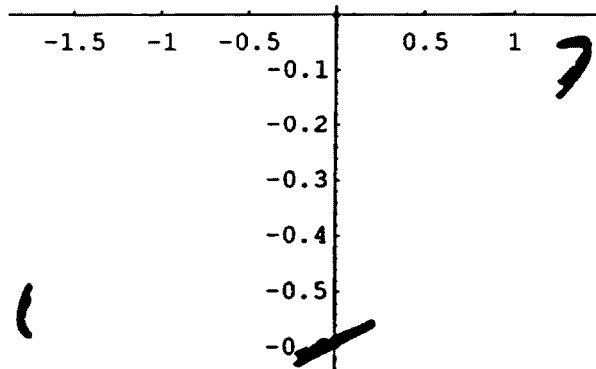
(6) $C = -7.09$

The orbit $p(n)$ with initial point $(0,0)$ and $n=2000, \dots, 3000$ goes to a six periodic attractor.



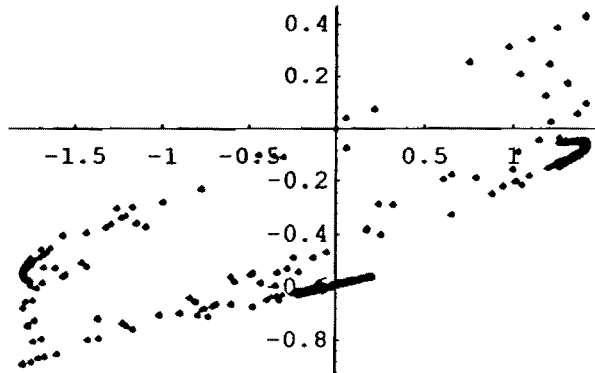
(7) $C = -7.13$

The orbit $p(n)$ with initial point $(0,0)$ and $n=2000, \dots, 4000$ goes to a three section attractor.



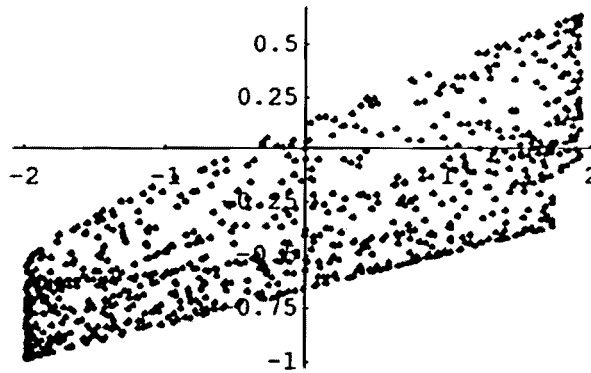
(8) $C = -7.16$

The orbit $p(n)$ with initial point $(0,0)$ and $n=2000, \dots, 4000$ goes to a three section attractor.



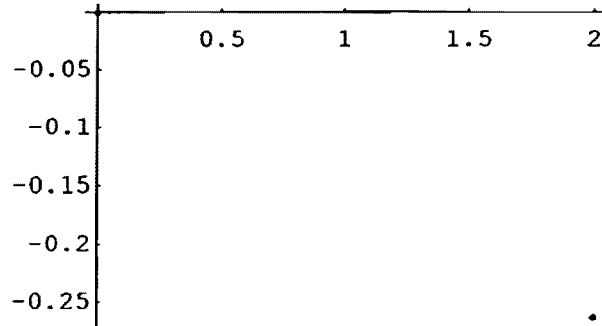
(9) $C = -7.162$

The orbit $p(n)$ with initial point $(0,0)$ and $n=2000, \dots, 4000$ goes to attractor dense on a three section area.

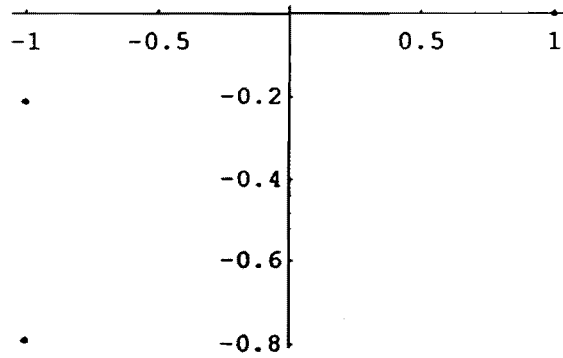


(10) $C = -7.9$

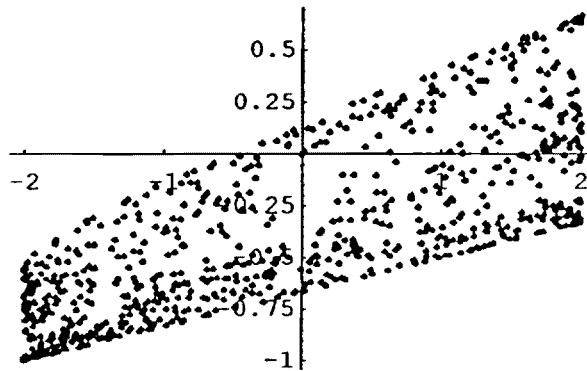
The orbit $p(n)$ with initial point $(0,0)$ and $n=2000, \dots, 4000$ goes to an attractor.



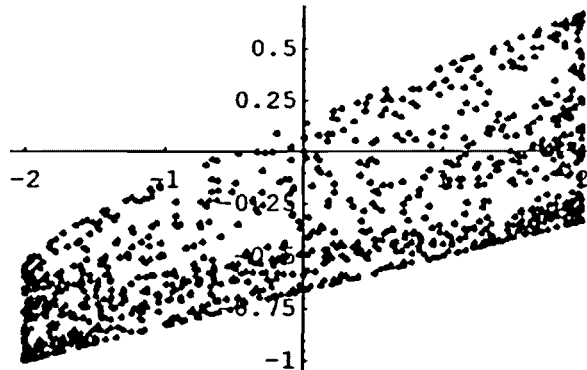
(11) $c = -8$, initial point $p(0)=(0,0)$, the orbit $p(n)$ with $n=101\dots 2000$, goes to a fixed point.



(12) $c = -8$, initial point $p(0)=(1,0)$, the orbit $p(n)$ goes with $n=101\dots 2000$ a 2-periodic orbit.



(13) $c = -8$, initial point $p(0) = (0.01, 0.01)$, the orbit $p(n)$ with $n = 101 \dots 1000$.



(14) $c = -8$, initial point $p(0) = (0.001, 0)$, the orbit $p(n)$ with $n = 101 \dots 1200$:

Chapter 6

EXISTENCE OF THE SOLUTION FOR THE H-EQUATION

In this chapter we consider the H -equation in $C[0, 1]$,

$$H(x) = 1 + H(x) \int_0^1 \frac{x}{x+t} \psi(t) H(t) dt \quad (6.1)$$

where $\psi \in L_1[0, 1]$.

From the equation (6.1), we can see its solution $H(x)$ should satisfy

$$H(0) = 1$$

and

$$H(x) \neq 0.$$

So the continuous solution of (6.1) must be positive.

Lemma 6.1 (i) *If $H(x) \in C[0, 1]$ is a solution of (6.1), then*

$$\int_0^1 \psi(t) H(t) dt = 1 - [1 - 2 \int_0^1 \psi(t) dt]^{\frac{1}{2}} \quad (6.2)$$

or

$$\int_0^1 \psi(t) H(t) dt = 1 + [1 - 2 \int_0^1 \psi(t) dt]^{\frac{1}{2}}; \quad (6.3)$$

(ii) *If $H \in C[0, 1]$ is a solution of (6.1), then*

$$\int_0^1 \psi(t) dt \leq \frac{1}{2}$$

and $H(x) \geq 1$, for $x \in [0, 1]$;

(iii) *The necessary and sufficient conditions of the positive function $H(x) \in C[0, 1]$ satisfying*

$$H(x)^{-1} = [1 - 2 \int_0^1 \psi(t) dt]^{\frac{1}{2}} + \int_0^1 \frac{t}{x+t} \psi(t) H(t) dt \quad (6.4)$$

are that $H(x)$ satisfies (6.1) and (6.2), where $\psi(x) \geq 0$ or $\psi(x) \leq 0$;

(iv) For $\psi(t) \leq 0$, there is no solution of (6.1) and (6.3) in $C[0, 1]$.

Proof: Let $H(x)$ be a solution of (6.1). Then

$$\int_0^1 \psi(x)H(x)dx = \int_0^1 \psi(x)dx + \int_0^1 \int_0^1 \frac{x\psi(x)H(x)\psi(t)H(t)}{x+t} dx dt$$

$$\int_0^1 \psi(x)H(x)dx = \int_0^1 \psi(x)dx + \int_0^1 \int_0^1 \frac{x\psi(t)H(t)\psi(x)H(x)}{x+t} dx dt.$$

So

$$\int_0^1 \psi(x)H(x)dx = \int_0^1 \psi(x)dx + \frac{1}{2}[\int_0^1 \psi(x)H(x)dx]^2.$$

This implies

$$\int_0^1 \psi(x)H(x)dx = 1 \pm \sqrt{1 - 2 \int_0^1 \psi(x)dx}.$$

Hence (i) and (ii) are proved. We will prove (iii) as follows. To prove the necessary condition, let $H(x)$ be a positive solution of (6.4). Then

$$1 = \sqrt{1 - 2 \int_0^1 \psi(t)dt} H(x) + \int_0^1 \frac{t}{x+t} \psi(t)H(t)H(x)dt.$$

Multiplying by $\psi(x)$ and integrating both sides we have

$$\int_0^1 \psi(x)dx = \sqrt{1 - 2 \int_0^1 \psi(t)dt} \int_0^1 \psi(x)H(x)dx + \frac{1}{2}[\int_0^1 \psi(t)H(t)dt]^2.$$

So

$$\begin{aligned} \int_0^1 \psi(t)H(t)dt &= -\sqrt{1 - 2 \int_0^1 \psi(t)dt} \pm \sqrt{1 - 2 \int_0^1 \psi(x)dx + 2 \int_0^1 \psi(x)dx} \\ &= \pm 1 - \sqrt{1 - 2 \int_0^1 \psi(x)dx} \end{aligned}$$

i.e.,

$$\int_0^1 \psi(t)H(t)dt = \pm 1 - \sqrt{1 - 2 \int_0^1 \psi(x)dx}. \quad (6.5)$$

Since $\psi(t) \geq 0$ or ≤ 0 , $H(t)$ satisfies (6.2). So we should take positive sign in (6.5). In fact, for $\psi(t) \leq 0$ it is easy to see, and for $\psi(t) \geq 0$, substituting (6.5) to (6.4) we have

$$\begin{aligned} H(x)^{-1} &= \sqrt{1 - 2 \int_0^1 \psi(t) dt} + \int_0^1 \frac{t}{x+t} \psi(t) H(t) dt \\ &= \sqrt{1 - 2 \int_0^1 \psi(t) dt} + \int_0^1 \psi(t) H(t) dt - \int_0^1 \frac{x}{x+t} \psi(t) H(t) dt \\ &= \pm 1 - H(x) \int_0^1 \frac{x}{x+t} \psi(t) H(t) dt. \end{aligned}$$

That $H(x)$ is positive and $H(0) = \pm 1$ implies that we should take positive sign. Substituting (6.2) in (6.4) we have

$$\begin{aligned} 1 &= H(x) - H(x) \int_0^1 \psi(t) H(t) dt + H(x) \int_0^1 \frac{t}{x+t} \psi(t) H(t) dt \\ &= H(x) - H(x) \int_0^1 \frac{x}{x+t} \psi(t) H(t) dt, \end{aligned}$$

so $H(x)$ satisfies (6.1) and the necessary condition is proved.

To prove sufficient condition, if $H(t)$ satisfies (6.1) and (6.2), from (6.2) we have

$$\begin{aligned} \int_0^1 \frac{x}{x+t} \psi(t) H(t) dt &= \int_0^1 \psi(t) H(t) dt - \int_0^1 \frac{t}{x+t} \psi(t) H(t) dt \\ &= 1 - [1 - 2 \int_0^1 \psi(t) dt]^{\frac{1}{2}} - \int_0^1 \frac{t}{x+t} \psi(t) H(t) dt \end{aligned}$$

and using (6.1) we obtain

$$\int_0^1 \frac{x}{x+t} \psi(t) H(t) dt = 1 - H(x)^{-1}.$$

So

$$1 - H(x)^{-1} = 1 - [1 - 2 \int_0^1 \psi(t) dt]^{\frac{1}{2}} - \int_0^1 \frac{t}{x+t} \psi(t) H(t) dt,$$

i.e., $H(t)$ satisfies (6.4).

To prove (iv), note that $\psi(t) \leq 0$ and $H(t) \geq 0$ imply

$$\int_0^1 \psi(t) H(t) dt \leq 0.$$

But this is a contradiction to (6.3).

Definition 6.1 T is an increasing operator on $C[0, 1]$ if for $h_1, h_2 \in C[0, 1]$ with $h_1(x) \leq h_2(x), x \in [0, 1]$, then

$$Th_1(x) \leq Th_2(x);$$

T is a decreasing operator on $C[0, 1]$ if for $h_1, h_2 \in C[0, 1]$ with $h_1(x) \leq h_2(x), x \in [0, 1]$, then

$$Th_1(x) \geq Th_2(x).$$

Let

$$\rho = \sqrt{1 - 2 \int_0^1 \psi(t) dt}$$

and

$$A_\rho = \{h \in C[0, 1] : h(x) \geq \rho, x \in [0, 1]\}.$$

For $\rho > 0$, we define operators, $S : A_\rho \rightarrow C[0, 1]$ and $T : A_\rho \rightarrow C[0, 1]$ by

$$Sh(x) = 1 + h(x) \int_0^1 \frac{x}{x+t} \psi(t) h(t) dt, h \in A_\rho$$

and

$$Th(x) = \rho + \int_0^1 \frac{t}{x+t} \psi(t) h(t)^{-1} dt, h \in A_\rho.$$

It is easy to see that for $\psi \geq 0$ S and T are continuous and S is increasing, T is decreasing. $h(x)$ is a solution of (6.1) iff $Sh = h$. From Lemma 6.1 $Th = h$ iff h^{-1} is a solution of (6.1) and (6.2). For $\psi(t) \leq 0$, T is increasing.

Let

$$C_d[0, 1] = \{f \in C[0, 1], |f| > d\}$$

with a positive constant d .

Lemma 6.2 Let $\psi \in L_1[0, 1]$.

(i) For given $d > 0$, $TC_d[0, 1]$ is equicontinuous;

(ii) For $\psi \geq 0$, or $\psi \leq 0$, if there is a solution of (6.1), let $H(\mu)$ is the solution. Then the series

$$\{S^{(n)}1\}_{n=1}^{\infty}$$

is equicontinuous where 1 represents the constant function.

Proof: (i) Let $h \in C_d[0, 1]$. Since $\psi \in L_1[0, 1]$, for any $\epsilon > 0$ there is γ small enough such that

$$\int_0^\gamma \left| \frac{\psi(t)}{h(t)} \right| dt < \frac{\epsilon}{4}.$$

For $x, y \in [0, 1]$ and $|x - y| < \delta$ with

$$\delta = \frac{d\gamma^2\epsilon}{2 \int_\gamma^1 |\psi(t)| dt},$$

we have

$$\begin{aligned} |Th(x) - Th(y)| &= \left| \int_0^1 \left[\frac{t}{x+t} - \frac{t}{y+t} \right] \psi(t) \frac{1}{h(t)} dt \right| \\ &\leq \left| \int_0^\gamma \left[\frac{t}{x+t} - \frac{t}{y+t} \right] \psi(t) \frac{1}{h(t)} dt \right| + \left| \int_\gamma^1 \left[\frac{t}{x+t} - \frac{t}{y+t} \right] \psi(t) \frac{1}{h(t)} dt \right| \\ &\leq \int_0^\gamma \left| \frac{t}{x+t} + \frac{t}{y+t} \right| |\psi(t)| \frac{1}{h(t)} dt + \int_\gamma^1 \left| \frac{t}{x+t} - \frac{t}{y+t} \right| |\psi(t)| \frac{1}{h(t)} dt \\ &\leq 2 \int_0^\gamma \left| \frac{\psi(t)}{h(t)} \right| dt + |x - y| \int_\gamma^1 \left| \frac{t}{(x+t)(y+t)} \right| |\psi(t)| \frac{1}{h(t)} dt \\ &\leq \frac{\epsilon}{4} + \delta \int_\gamma^1 \frac{1}{\gamma^2} |\psi(t)| \frac{1}{d} dt \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

So $TC_d[0, 1]$ is equicontinuous.

(ii) For $\psi \geq 0$ let

$$E = \{h \in C[0, 1] : 1 \leq h \leq H\},$$

and $R : E \mapsto C[0, 1]$

$$Rh(x) = \int_0^1 \frac{x}{x+t} \psi(t) h(t) dt, x \in [0, 1], h \in E.$$

For $x, y \in [0, 1]$

$$|Rh(x) - Rh(y)| = \left| \int_0^1 \left[\frac{x}{x+t} - \frac{y}{y+t} \right] \psi(t) h(t) dt \right|,$$

and for any $\epsilon > 0$, there is δ , $0 < \delta < 1$ such that

$$\int_0^\delta |\psi(t) H(t)| dt < \frac{\epsilon}{4},$$

$$\int_\delta^1 |\psi(t) H(t)| dt > 0.$$

So for $x, y \in [0, 1]$

$$\begin{aligned} \left| \int_0^\delta \left[\frac{x}{x+t} - \frac{y}{y+t} \right] \psi(t) H(t) dt \right| &\leq \int_0^\delta \left[\left| \frac{x}{x+t} \right| + \left| \frac{y}{y+t} \right| \right] |\psi(t) H(t)| dt \\ &\leq 2 \int_0^\delta |\psi(t)| dt \\ &\leq \frac{\epsilon}{2}. \end{aligned}$$

Then for

$$|x - y| \leq \delta_1$$

with

$$\delta_1 = \frac{\epsilon}{2} \delta^2 \left(\int_\delta^1 |\psi(t) H(t)| dt \right)^{-1},$$

we have

$$\begin{aligned} |Rh(x) - Rh(y)| &\leq \frac{\epsilon}{2} + \int_\delta^1 \left| \frac{x}{x+t} - \frac{y}{y+t} \right| |\psi(t) H(t)| dt \\ &\leq \frac{\epsilon}{2} + \int_\delta^1 \frac{t|x-y|}{(x+t)(y+t)} |\psi(t) H(t)| dt \\ &\leq \frac{\epsilon}{2} + \int_\delta^1 \frac{|x-y|}{\delta^2} |\psi(t) H(t)| dt \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

So $R(E)$ is equicontinuous.

For $h(x) \in E, x \in [0, 1]$, from

$$\begin{aligned} Rh(x) &= \int_0^1 \frac{x}{x+t} \psi(t)h(t)dt \\ &\leq \int_0^1 \psi(t)H(t)dt \\ &= 1 - \sqrt{1 - 2 \int_0^1 \psi(t)dt} \\ &\leq 1, \end{aligned}$$

we see that there is a $\beta, 0 < \beta < 1$, such that for any $h(x) \in E$ and $x \in [0, 1]$,

$$Rh(x) < \beta.$$

For $\epsilon_0 > 0$, from equicontinuity of $R(E)$, there is $\delta_0 > 0$ such that for $|x - y| < \delta_0$, and for $g \in R(E)$

$$|g(x) - g(y)| < \|H\|^{-1}(1 - \beta)\epsilon_0.$$

Since $S(1)$ is continuous, there is $\delta_1, 0 < \delta_1 < \delta_0$, such that for $|x - y| < \delta_1$,

$$|S^k(1)(x) - S^k(1)(y)| < \epsilon_0$$

for $k = 1, 2, \dots, n$. For such δ_1 and $|x - y| < \delta_1$, we have

$$|S^{n+1}1(x) - S^{n+1}1(y)| < \epsilon_0$$

In fact, by the definition of S we know $S(1), S^2(1), \dots, S^n(1) \in E$. Let $h(x) = S^n(1)(x)$, then $h(x) \in E$.

Letting $|x - y| < \delta_1$, we have

$$\begin{aligned} |Sh(x) - Sh(y)| &= |h(x)Rh(x) - h(y)Rh(y)| \\ &\leq |h(x)Rh(x) - h(x)Rh(y)| + |h(x)Rh(y) - h(y)Rh(y)| \\ &= h(x)|Rh(x) - Rh(y)| + Rh(y)|h(x) - h(y)| \\ &< \|H\| \|H\|^{-1}(1 - \beta)\epsilon_0 + \beta\epsilon_0 \\ &= \epsilon_0. \end{aligned}$$

So $|x - y| < \delta_1$ implies

$$|S^{n+1}(1)(x) - S^{n+1}(1)(y)| < \epsilon_0.$$

Hence for $\epsilon_0 > 0$, there is $\delta_1 = \delta_1(\epsilon_0)$ such that for $|x - y| < \delta_1$ and for $n = 1, 2, \dots$, we have

$$|S^n(1)(x) - S^n(1)(y)| < \epsilon_0$$

i.e., $\{S^n(1)(x)\}_{n=1}^{\infty}$ is equicontinuous.

In the same manner, for $\psi \leq 0$, let

$$E = \{h \in C[0, 1] : H \leq h \leq 1\}$$

we can prove the same result.

‡

Theorem 6.1 For $\psi(t) \geq 0 \in L_1[0, 1]$ there is a solution $H(x)$ of (6.1) and (6.2) iff

$$\int_0^1 \psi(t) dt \leq \frac{1}{2}. \quad (6.6)$$

and $\{S^n(1)\}_{n=0}^{\infty}$ converges to $H(x)$. If there is a strict inequality in (6.6), then $\{T^n(\rho)\}_{n=0}^{\infty}$ converges to $H(x)^{-1}$

with

$$|H(x)^{-1} - T^n \rho(x)| \leq |T^n \rho(x) - T^{n+1} \rho(x)|, \quad x \in [0, 1]. \quad (6.7)$$

Proof: If there is a solution of (6.1) satisfying the condition (6.2), then from Lemma 6.1, $\psi(t)$ must satisfy (6.6). So the necessary condition is proved. To prove the sufficient condition following. Suppose

$$\int_0^1 \psi(t) dt < \frac{1}{2}.$$

Then $\rho > 0$. By $\psi(t) \geq 0$, we have $\rho \leq T\rho$ and $\rho \leq T^2\rho$. Since T is decreasing, we have

$$\rho \leq T^2\rho \leq T\rho,$$

$$\rho < T^2\rho \leq T^3\rho \leq T\rho,$$

..., ...

$$\rho \leq T^2\rho \leq T^4\rho \leq T^6\rho \leq \dots, \dots \leq T^5\rho \leq T^3\rho \leq T\rho.$$

Using Lemma 6.2. we know

$$\{Th : \rho \leq h \leq T\rho\}$$

is equicontinuous. Using Ascoli-Arzerà Theorem, the function series

$$\{T^{2n}(\rho)\}_{n=1}^{\infty} = \{T(T^{2n-1}\rho)\}_{n=1}^{\infty}$$

and

$$\{T^{2n+1}(\rho)\}_{n=0}^{\infty} = \{T(T^{2n}\rho)\}_{n=0}^{\infty}$$

have subsidies which converge to u and v . Since T^{2n} is increasing, $T^{2n+1}\rho$ is decreasing and T is continuous, we have

$$\lim_{n \rightarrow \infty} T^{2n}\rho = u,$$

$$\lim_{n \rightarrow \infty} T^{2n+1}\rho = v,$$

with

$$\rho \leq u \leq v$$

and

$$Tu = v, Tv = u.$$

By

$$\min\{T^{2n}\rho\}_{n=0}^{\infty} \geq \rho > 0$$

there is a constant

$$a = \max\{k : kv \leq u\}$$

with $0 < a \leq 1$. For $a = 1$ we have $u = v$. Now, letting $a < 1$ and T_1 :

$$T_1h = Th - \rho, h \in D(T).$$

we have

$$\begin{aligned}
u &= \rho + T_1 v > \rho + T_1(a^{-1}u) = \rho + aT_1 u \\
&= (1-a)\rho + a(\rho + T_1 u) \\
&= (1-a)\rho + av \geq bv + av \\
&= (a+b)v,
\end{aligned}$$

where b is a positive constant. This is in contradiction with the definition of a . So

$$Tu = u = v,$$

and the function $H = u^{-1}$ is a solution of (6.1) and (6.2) with

$$H^{-1} = \lim_{n \rightarrow \infty} T^n \rho.$$

From

$$T^{2n} \rho \leq H^{-1} \leq T^{2n+1} \rho$$

$$n = 1, 2, \dots, \dots$$

it is easy to get (6.7).

Let us consider the case $\int_0^1 \psi dt = \frac{1}{2}$. Let $\{k_n\}_{n=1}^{\infty}$ be a series with $0 < k_i < K_{i+1}$, ($i = 1, 2, \dots, \dots$) and $k_i \rightarrow 1$ for $i \rightarrow \infty$. Since $\int_0^1 k_n \psi dt = \frac{1}{2} k_n < \frac{1}{2}$, we have that there is a positive solution $H_n \in C[0, 1]$ of

$$H(x) = 1 + H(x) \int_0^1 \frac{x}{x+t} k_n \psi dt$$

$$n = 1, 2, \dots$$

Then $H_n(x) \geq 1$ for $x \in [0, 1]$, and

$$\begin{aligned}
(H_n(x))^{-1} &= \sqrt{1 - 2 \int_0^1 k_n \psi(t) dt} + \int_0^1 \frac{t}{x+t} k_n \psi(t) H_n(t) dt \\
&\geq k_n \int_0^1 \frac{t}{x+t} \psi(t) dt \\
&\geq k_1 \int_0^1 \frac{t}{x+t} \psi(t) dt.
\end{aligned}$$

Hence there is $a > 0$ such that $(H_n)^{-1} \geq a$, for $x \in [0, 1], n = 1, 2, \dots$

Let

$$B = \{h \in C[0, 1] : a \leq h(x) \leq 1, x \in [0, 1]\}.$$

It implies $H_n^{-1} \in B, n = 1, 2, \dots$

Let $T : B \rightarrow C[0, 1]$

$$Th(x) = \int_0^1 \frac{t}{x+t} \psi(t)(h(t))^{-1} dt, h \in B$$

It is easy see $T(B)$ is bounded and equicontinuous. Letting $h_n = H_n^{-1}, n = 1, 2, \dots$ we have

$$\begin{aligned} h_n(x) &= \sqrt{1 - 2 \int_0^1 k_n \psi(t) dt} + \int_0^1 \frac{t}{x+t} k_n \psi(t)(h_n(t))^{-1} dt \\ &= \sqrt{1 - 2 \int_0^1 k_n \psi(t) dt} + k_n(Th_n)(x). \end{aligned}$$

Since $Th_n \in T(B)$, there is subsidies of $\{Th_n\}_{n=1}^\infty, \{Th_{n_j}\}_{j=1}^\infty$, such that

$$\lim_{n \rightarrow \infty} Th_{n_j} = h_0 \in C[0, 1].$$

From

$$h_{n_j}(x) = \sqrt{1 - 2 \int_0^1 k_{n_j} \psi(t) dt} + k_{n_j}(Th_{n_j})(x), \quad (6.8)$$

we have

$$\lim_{j \rightarrow \infty} h_{n_j} = h_0,$$

then

$$h_0(x) = \sqrt{1 - 2 \int_0^1 \psi(t) dt} + \int_0^1 \frac{t}{x+t} \psi(t)(h_0(t))^{-1} dt.$$

Using Lemma 6.1 we obtain that the function h_0^{-1} satisfies (6.1) and (6.2) (Since $\int_0^1 \psi(t) dt = \frac{1}{2}$, (6.2) is the same as (6.3)). So we have proved that if ψ satisfies (6.6) there is a positive continuous function satisfying (6.1) and (6.2).

Now we want to prove if (6.6) is satisfied, the solution of (6.1) and (6.2) is unique. Letting the function H be a solution of (6.1) and (6.2). we have $1 \leq H$, and $1 \leq S(1)$.

From the increasing operator S we have

$$1 \leq S(1) \leq S^2(1) \leq S^3(1) \leq \dots \leq H.$$

So $\{S^n(1)\}_{n=1}^{\infty}$ is uniform bounded. From Lemma (6.2), there is a subsidies $\{S^{n_k}(1)\}_{k=1}^{\infty}$ such that

$$\lim_{k \rightarrow \infty} S^{n_k}(1) = h \leq H.$$

Since $S^n(1) \geq S^{n+1}(1)$, we have

$$\lim_{n \rightarrow \infty} S^n(1) = h.$$

Then $S(h) = h$, i.e., h satisfies (6.2) or (6.3). Since $0 < h < H$, h satisfies (6.2), and

$$\begin{aligned} (h(x))^{-1} &= \sqrt{1 - 2 \int_0^1 \psi(t) dt} + \int_0^1 \frac{t}{x+t} \psi(t) h(t) dt \\ &\leq \sqrt{1 - 2 \int_0^1 \psi(t) dt} + \int_0^1 \frac{t}{x+t} \psi(t) H(t) dt \\ &= (H(x))^{-1}, \end{aligned}$$

i.e.,

$$h^{-1} \leq H^{-1}.$$

But from $h \leq H$ we have

$$h = H.$$

So H is a unique solution of (6.1) and (6.2) and

$$\lim_{n \rightarrow \infty} S^n(1) = H.$$

‡

Lemma 6.3 For $\psi \in L_1[0, 1]$, let $h_{n+1} = Th_n$ we have

$$\begin{aligned} \left(\frac{1}{h_{n+1}} - \frac{1}{h_n}\right)h_{n+1} &= -\frac{1}{h_n} \int_0^1 \frac{t}{x+t} \psi(t) \left(\frac{1}{h_n} - \frac{1}{h_{n-1}}\right) dt \\ n &= 0, 1, 2, \dots \end{aligned}$$

If $\psi \leq 0$, and $h_1 \geq h_0$ we have

$$\rho \geq h_{n+1} \geq h_n \geq h_0 \geq 0.$$

Proof: By the definition of T we have

$$1 = \frac{\rho}{h_n} + \frac{1}{h_n} \int_0^1 \frac{t}{x+t} \psi(t) \frac{1}{h_{n-1}(t)} dt,$$

$$1 = \frac{\rho}{h_{n+1}} + \frac{1}{h_{n+1}} \int_0^1 \frac{t}{x+t} \psi(t) \frac{1}{h_n(t)} dt.$$

Subtraction of the first equation from second above yields

$$0 = \rho \left(\frac{1}{h_{n+1}} - \frac{1}{h_n} \right) + \frac{1}{h_{n+1}} \int_0^1 \frac{t}{x+t} \psi(t) \frac{1}{h_n(t)} dt$$

$$- \frac{1}{h_n} \int_0^1 \frac{t}{x+t} \psi(t) \frac{1}{h_{n-1}(t)} dt.$$

Then

$$0 = \rho \left(\frac{1}{h_{n+1}} - \frac{1}{h_n} \right) + \left(\frac{1}{h_{n+1}} - \frac{1}{h_n} \right) \int_0^1 \frac{t}{x+t} \psi(t) \frac{1}{h_n(t)} dt$$

$$+ \frac{1}{h_n} \int_0^1 \frac{t}{x+t} \psi(t) \left(\frac{1}{h_n(t)} - \frac{1}{h_{n-1}(t)} \right) dt$$

$$= \left(\rho + \int_0^1 \frac{t}{x+t} \psi(t) \frac{1}{h_n(t)} dt \right) \left(\frac{1}{h_{n+1}} - \frac{1}{h_n} \right)$$

$$+ \frac{1}{h_n} \int_0^1 \frac{t}{x+t} \psi(t) \left(\frac{1}{h_n(t)} - \frac{1}{h_{n-1}(t)} \right) dt$$

$$= h_{n+1} \left(\frac{1}{h_{n+1}} - \frac{1}{h_n} \right)$$

$$+ \frac{1}{h_n} \int_0^1 \frac{t}{x+t} \psi(t) \left(\frac{1}{h_n(t)} - \frac{1}{h_{n-1}(t)} \right) dt.$$

So

$$h_{n+1} \left(\frac{1}{h_{n+1}} - \frac{1}{h_n} \right) = - \frac{1}{h_n} \int_0^1 \frac{t}{x+t} \psi(t) \left(\frac{1}{h_n(t)} - \frac{1}{h_{n-1}(t)} \right) dt.$$

If $h_1(t) \geq h_0 > 0$ and we suppose for $n = N$ the relation

$$\rho \geq h_N \geq h_{N-1} \geq 0$$

is true, then by the definition of h_{N+1} and $\psi \leq 0$ we obtain

$$\begin{aligned} h_N &= \rho + \int_0^1 \frac{t}{x+t} \psi(t) \frac{1}{h_{N-1}(t)} dt \\ &\leq \rho + \int_0^1 \frac{t}{x+t} \psi(t) \frac{1}{h_N(t)} dt \\ &= h_{N+1} < \rho. \end{aligned}$$

‡

From Lemma 6.3 if we can find h_0 such that $h_1 \geq h_0$, then it yields an increasing series $\{h_n\}_0^\infty$ for index n . From Lemma 6.2 we know this series converges to a continuous function $h(x)$ such that $h(x) = Th(x)$. Then from Lemma 6.1 $h(x)$ is a solution of (6.1) and (6.2). the following Theorem gives a sufficient condition of the existence of a solution.

Theorem 6.2 *For $\psi \leq 0 \in L_1[0, 1]$ and $-\frac{1}{2} < \int_0^1 \psi(t) dt$ there is an increasing series $\{h_n\}_0^\infty$ converging to a continuous solution of (6.1) and (6.2).*

Proof: From the definition of T and ρ we have

$$\begin{aligned} h_0 - h_1 &= h_0 - Th_0 \\ &= h_0 - \rho - \int_0^1 \frac{t}{x+t} \psi(t) \frac{1}{h_0(t)} dt \\ &= \frac{1}{h_0} [h_0^2 - \rho h_0 - \int_0^1 \frac{t}{x+t} \psi(t) dt] \\ &\leq \frac{1}{h_0} [h_0^2 - \rho h_0 - \int_0^1 \psi(t) dt] \\ &= \frac{1}{h_0} [h_0 - \frac{1}{2}\rho - \frac{1}{2}\sqrt{\rho^2 + 4 \int_0^1 \psi(t) dt}] [h_0 - \frac{1}{2}\rho + \frac{1}{2}\sqrt{\rho^2 + 4 \int_0^1 \psi(t) dt}] \\ &= \frac{1}{h_0} [h_0 - \frac{1}{2}\rho - \frac{1}{2}\sqrt{1 - 2 \int_0^1 \psi(t) dt + 4 \int_0^1 \psi(t) dt}] \\ &\quad [h_0 - \frac{1}{2}\rho + \frac{1}{2}\sqrt{1 - 2 \int_0^1 \psi(t) dt + 4 \int_0^1 \psi(t) dt}] \\ &= \frac{1}{h_0} [h_0 - \frac{1}{2}\rho - \frac{1}{2}\sqrt{1 + 2 \int_0^1 \psi(t) dt}] [h_0 - \frac{1}{2}\rho + \frac{1}{2}\sqrt{1 + 2 \int_0^1 \psi(t) dt}]. \end{aligned}$$

Taking positive h_0 such that

$$\frac{1}{2}\rho - \frac{1}{2}\sqrt{1 + 2 \int_0^1 \psi(t)dt} < h_0 < \frac{1}{2}\rho + \frac{1}{2}\sqrt{1 + 2 \int_0^1 \psi(t)dt}$$

we obtain

$$h_1 > h_0.$$

From Lemma 6.3 this implies that there is a series $\{h_n\}_0^\infty$ increasing and converging to a continuous solution of (6.1) and (6.2).

∴

Theorem 6.3 For $\psi(x) \leq 0 \in L_1[0, 1]$, there is a $b_0 > 0$ such that for $\gamma \geq b_0$,

$$T^2\gamma \leq \gamma \tag{6.9}$$

and

$$\frac{1}{\rho} < T\gamma \leq \gamma \tag{6.10}$$

where

$$\rho = \sqrt{1 - 2 \int_0^1 \psi(t)dt}$$

Proof: From the definition of T

$$Th(x) = \rho + \int_0^1 \frac{t}{x+t} \psi(t) \frac{1}{h(t)} dt$$

T is increasing for h . For $\rho > 0$, we have

$$\begin{aligned} \gamma - T\gamma &= \gamma - \rho - \frac{1}{\gamma} \int_0^1 \frac{t}{x+t} \psi(t) dt \\ &\geq \gamma - \rho - \frac{1}{\gamma} \int_0^1 \frac{t}{1+t} \psi(t) dt \\ &= \frac{1}{\gamma} [\gamma(\gamma - \rho) - \int_0^1 \frac{t}{1+t} \psi(t) dt] \end{aligned}$$

Let

$$b_0 = \frac{\rho}{2} + \frac{1}{2} \sqrt{\rho^2 + 4 \int_0^1 \frac{t}{1+t} \psi(t) dt}$$

if the square root is real. Otherwise let b_0 be any constant greater than $\frac{\rho}{2}$. In both cases $b_0 \geq \frac{\rho}{2}$.

For $\gamma \geq b_0$ we have

$$\gamma(\gamma - \rho) - \int_0^1 \frac{t}{1+t} \psi(t) dt \geq 0,$$

and then

$$\gamma - T\gamma \geq \frac{1}{\gamma} [\gamma(\gamma - \rho) - \int_0^1 \frac{t}{1+t} \psi(t) dt] > 0$$

i.e.,

$$\gamma > T\gamma.$$

By the definition of T we also have

$$\begin{aligned} T\gamma &= \rho + \int_0^1 \frac{t}{x+t} \psi(t) \frac{1}{\gamma} dt \\ &= \frac{1}{\rho} \left[\rho^2 + \frac{\rho}{\gamma} \int_0^1 \frac{t}{x+t} \psi(t) dt \right] \\ &\geq \frac{1}{\rho} \left[\rho^2 + \frac{\rho}{\gamma} \int_0^1 \psi(t) dt \right] \\ &= \frac{1}{\rho} \left[1 - 2 \int_0^1 \psi(t) dt + \frac{\rho}{\gamma} \int_0^1 \psi(t) dt \right] \\ &= \frac{1}{\rho} \left[1 - \left(2 - \frac{\rho}{\gamma} \right) \int_0^1 \psi(t) dt \right] \\ &> \frac{1}{\rho}. \end{aligned}$$

(Note: $\gamma > b_0 \geq \frac{\rho}{2}$)

On the other hand, the estimation

$$\begin{aligned} &\rho - \gamma + \int_0^1 \frac{t}{t+x} \psi(t) \frac{1}{T\gamma} dt \\ &< \rho - \gamma + \frac{1}{\gamma} \int_0^1 \frac{t}{t+x} \psi(t) dt \\ &= T\gamma - \gamma < 0 \text{ (using } \gamma > T\gamma \text{)} \end{aligned}$$

implies

$$\begin{aligned} T^2\gamma &= \rho + \int_0^1 \frac{t}{t+x} \psi(t) \frac{1}{T\gamma} dt \\ &= \gamma + [(\rho - \gamma) + \int_0^1 \frac{t}{t+x} \psi(t) \frac{1}{T\gamma} dt]. \end{aligned}$$

Hence

$$T^2\gamma < \gamma.$$

#

Theorem 6.4 For $\frac{1}{2} \leq \psi(x) \leq 0 \in L_1[0, 1]$, let $b_1 > 0$ such that $\rho - b_1 \geq b_0$ and let

$$\frac{b_1 - \sqrt{b_1^2 + 4}}{2} \leq \rho \leq \frac{b_1 + \sqrt{b_1^2 + 4}}{2}, \quad (6.11)$$

If

$$\rho - b_1 \leq \gamma \leq \rho + b_1, \quad (6.12)$$

then

$$\rho - b_1 \leq \frac{1}{\rho} \leq T\gamma \leq \rho + b_1. \quad (6.13)$$

Proof: Taking $b_0 \leq \gamma$ in Theorem 6.3 we have

$$\frac{1}{\rho} \leq T\gamma < \rho.$$

The condition (6.11) implies

$$\rho - b_1 \leq \frac{1}{\rho}.$$

so

$$\rho - b_1 \leq T\gamma < \gamma \leq \rho + b_1.$$

#

Theorem 6.5 For $\psi(x) \leq 0 \in L_1[0, 1]$ and $\frac{1}{2} < \int_0^1 \psi(t)dt$ there is a unique solution of (6.1) and (6.2), $H(x) \in C[0, 1]$ with

$$0 < H(x) \leq 1, \quad x \in [0, 1].$$

Proof: Taking b_0 and b_1 satisfying the conditions (6.11) and (6.12), from (6.4) we have

$$\rho - b_0 \leq \frac{1}{\rho} \leq T\gamma \leq \gamma < \rho + b_0.$$

Using the fact that T is increasing, we know

$$\frac{1}{\rho} < T\frac{1}{\rho} < T^n\gamma < T^{n-1}\gamma, \dots, \dots < \gamma.$$

So the series $\{T^n\gamma\}_{n=0}^\infty$ is uniform bounded. In the same way as the proof of Lemma 6.2 we have $T^n\gamma_{n=0}^\infty$ is equicontinuous. Hence, from the Ascoli-Arzerla Theorem and the fact that T is increasing, there is $h_0 \geq 0 \in C[0, 1]$ such that

$$h_0(x) = \lim_{n \rightarrow \infty} T^n\gamma$$

Since T is continuous we have

$$h_0(x) = \lim_{n \rightarrow \infty} T^n\gamma = \lim_{n \rightarrow \infty} (TT^{n-1}\gamma) = Th_0(x).$$

i.e., $h_0(x)$ is a positive solution of (6.4). Using (iii) of Lemma 6.1, $h_0(x)^{-1}$ is a positive solution of (6.1) and (6.2).

If there is another $h(x)$ which is a positive continuous solution of $h(x) = Th(x)$, then

$$0 \leq h(x) < \rho.$$

From the definition of T , we have

$$T\rho < \rho,$$

and using that T is increasing, we know

$$Th(x) = h(x) \leq T\rho < \rho.$$

Hence

$$h(x) < T^n \rho < T^{n-1} \rho, \dots, \dots < T \rho < \rho.$$

Then there is $h_1(x) \in C[0, 1]$ such that

$$h_1(x) = \lim_{n \rightarrow \infty} T^n \rho$$

in $C[0, 1]$. Therefore $Th_1(x) = h_1$ and $h(x) \leq h_1$. Since $h(x) = Th(x)$ and $h_0(x) = Th_0(x)$ we know

$$H(x) = \frac{1}{h(x)} \text{ and } H_0(x) = \frac{1}{h_0(x)}$$

are solutions of (6.1).

Since $h(x) \leq h_0(x)$ we have

$$H(x) \geq H_0(x). \tag{6.14}$$

But S is decreasing, so

$$SH(x) \leq SH_0(x). \tag{6.15}$$

The solutions of (6.1), $H(x)$ and H_0 , and (6.15) imply

$$H(x) = SH(x) \leq SH_0(x) = H_0(x),$$

thus

$$H(x) = H_0(x).$$

So the solution of (6.1) and (6.2) is unique. Continuity of $H(x)$ with $H(0) = 1$ implies that $H(x)$ is positive. By the equation (6.1) with $\psi(x) \leq 0$, $H(x)$ must be less than one.

‡

It is possible there are other solutions of (6.1) satisfying (6.3). Only when $\int_0^1 \psi(t) dt = \frac{1}{2}$, the solution satisfying (6.1) and (6.2) is a unique solution of (6.1). When $\int_0^1 \psi(t) dt \leq \frac{1}{2}$ we have following theorem.

Theorem 6.6 For $\psi(x) \geq 0 \in L_1[0, 1]$ and $\int_0^1 \psi(t)dt \leq \frac{1}{2}$, let $H(x)$ be a solution of (6.1) and (6.2). Then there is a solution of (6.1) and (6.3) iff

$$\int_0^1 \frac{\psi(t)}{1-t} H(t) > 1. \quad (6.16)$$

When (6.16) is satisfied, the function

$$H_1(x) = \frac{1+kx}{1-kx} H(t) \quad (6.17)$$

is a unique solution of (6.1) and (6.3), where k is a unique value satisfying

$$\int_0^1 \frac{\psi(t)}{1-kt} H(t) = 1 \quad (6.18)$$

with $0 \leq k \leq 1$.

Proof: Since $(1-kt)^{-1}$ is increasing for $k \in (0, 1)$ we have

$$\lim_{k \rightarrow 1} \int_0^1 \frac{\psi(t)}{1-kt} H(t) dt = \int_0^1 \frac{\psi(t)}{1-t} H(t) dt \quad (6.19)$$

Letting k and $\psi(x)$ satisfy (6.16) and

$$f(k) = \int_0^1 \frac{\psi(t)}{1-kt} H(t) dt,$$

using (6.2), we have

$$f(0) = \int_0^1 \psi(t) H(t) dt = 1 - \sqrt{1 - 2 \int_0^1 \psi(t) dt} < 1.$$

(6.16) and $f(k)$ increasing strictly for k imply that there is a unique $k \in (0, 1)$ such that (6.18) holds. Taking the function H_1 from (6.17), we have

$$\begin{aligned} \int_0^1 \frac{x}{x+t} \psi(t) H_1 dt &= \int_0^1 \frac{x}{x+t} \psi(t) \frac{1+kt}{1-kt} H(t) dt \\ &= \frac{1-kx}{1+kx} \int_0^1 \frac{x}{x+t} \psi(t) H(t) dt + \frac{2kx}{1+kx} \int_0^1 \frac{1}{1-kt} H(t) \psi(t) dt \\ &= \frac{1-kx}{1+kx} \int_0^1 \frac{x}{x+t} \psi(t) H(t) dt + \frac{2kx}{1+kx} \\ &= \frac{1-kx}{1+kx} \left[1 - \frac{1}{H(x)} + \frac{2kx}{1+kx} \right] \\ &= 1 - \frac{1}{H_1(x)}. \end{aligned}$$

So H_1 satisfies (6.1). Then H_1 must satisfy (6.2) or (6.3). From $H_1(x) > H(x)$ for $x \in (0, 1]$ we know H_1 satisfies (6.3).

Let $H_1 \in C[0, 1]$ and satisfying (6.1) and (6.3). From (6.3) we know H_1 satisfies

$$\int_0^1 \psi(t)H_1(t)dt > 1,$$

and

$$\int_0^1 \frac{\psi(t)}{1+t}H_1(t)dt = 1 - \frac{1}{H_1(t)} < 1.$$

There is a unique k , $0 < k < 1$, such that

$$\int_0^1 \frac{\psi(t)}{1+kt}H_1(t)dt = 1.$$

Let $H_2 \in C[0, 1]$:

$$H_2(x) = \frac{1-kx}{1+kx}H_1, \quad x \in [0, 1].$$

Then

$$\begin{aligned} \int_0^1 \frac{x}{x+t}\psi(t)H_2dt &= \int_0^1 \frac{x}{x+t}\psi(t)\frac{1-kt}{1+kt}H_1(t)dt \\ &= \frac{1+kx}{1-kx} \int_0^1 \frac{x}{x+t}\psi(t)H_1(t)dt - \left(\frac{2kx}{1+kx} \int_0^1 \frac{1}{1+kt}H_1(t)\psi(t)dt\right) \\ &= \frac{1+kx}{1-kx} \left[1 - \frac{1}{H_1(x)}\right] - \frac{2kx}{1-kx} \\ &= 1 - \frac{1+kx}{1-kx} \frac{1}{H_1(x)} \\ &= 1 - \frac{1}{H_2(x)}, \end{aligned}$$

i.e., H_2 is a solution of (6.1). So H_2 satisfies (6.2) or (6.3). From $H_2(x) < H_1(x)$, $x \in (0, 1]$, we obtain that $H_2(x)$ satisfies (6.2), so $H_2 = H$, and

$$\int_0^1 \frac{\psi(t)}{1+kt}H(t)dt = \int_0^1 \frac{\psi(t)}{1-kx} \frac{1-kt}{1+kt}H_t dt = 1.$$

Hence

$$\int_0^1 \frac{\psi(t)}{1-t}H(t)dt > 1.$$

‡

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