ANALYSIS OF A BENT CONSTRUCTED FROM A STRAIN-HARDENING MATERIAL

by

Dean Mook

Thesis submitted to the Graduate Faculty of the
Virginia Polytechnic Institute
in candidacy for the degree of

MASTER OF SCIENCE
in
ENGINEERING MECHANICS

December, 1960

Blacksburg, Virginia

I. TABLE OF CONTENTS

II. LIST OF FIGURES

III. INTRODUCTION

All of the classical approaches to the analysis of structures are based upon the elastic properties of the material from which the structures are made. Recently a method has been developed, which is gaining popularity, that is based upon both the elastic and the plastic properties of the material. This recent development, however, assumes that the material exhibits an ideally plastic stress-strain curve. The purpose of this thesis is to present a method for analysing a structure when part of some of the members have been loaded beyond the elastic range and when the material from which the structure is made exhibits a bilinear strain-hardening stress-strain law. This latter method should predict more accurately the collapse loads for structures made from materials such as hard steel and aluminum than those collapse loads predicted by assuming these materials to be ideally plastic. It should be noted here, however, that the collapse load found in this thesis is based upon a stress criterion. The problem is essentially a non-linear elasticity problem.

IV. REVIEW OF THE LITERATURE

The expression for the internal moment was derived in the manner presented in "Theory of Flow and Fracture of Solids" by A. Nadai (1). This development is based upon the following three assumptions that are used throughout this thesis:

1) Plane sections perpendicular to the neutral plane before bending remain plane and perpendicular to the neutral plane after bending.

2) The material has a bilinear strain-hardening stress-strain law that is the same for tension and compression.

3) All members to be considered are rectangular in cross-section.

The development of the expression for the complementary energy follows along the approach presented in "The Analysis of Structures" by N. J. Hoff (2). In the development that follows the shear energy is neglected.

A review of the literature revealed a scarcity of papers written on this topic.

V. DEVELOPMENT OF THE EXPRESSION FOR THE COMPLEMENTARY ENERGY

For this development it will be assumed that the bent is loaded with a finite number of concentrated loads, either moments or forces, and that the weight of the structure produces small effects when compared with the applied loads. Then the moment will vary linearly along the length of the members of the bent. Using the assumed stress-strain law shown in Figure 1, the stress distribution across the cross-section where there is some plastic action will be as shown in Figure 2. Where there is no plastic action, the stress will vary linearly from the neutral axis.

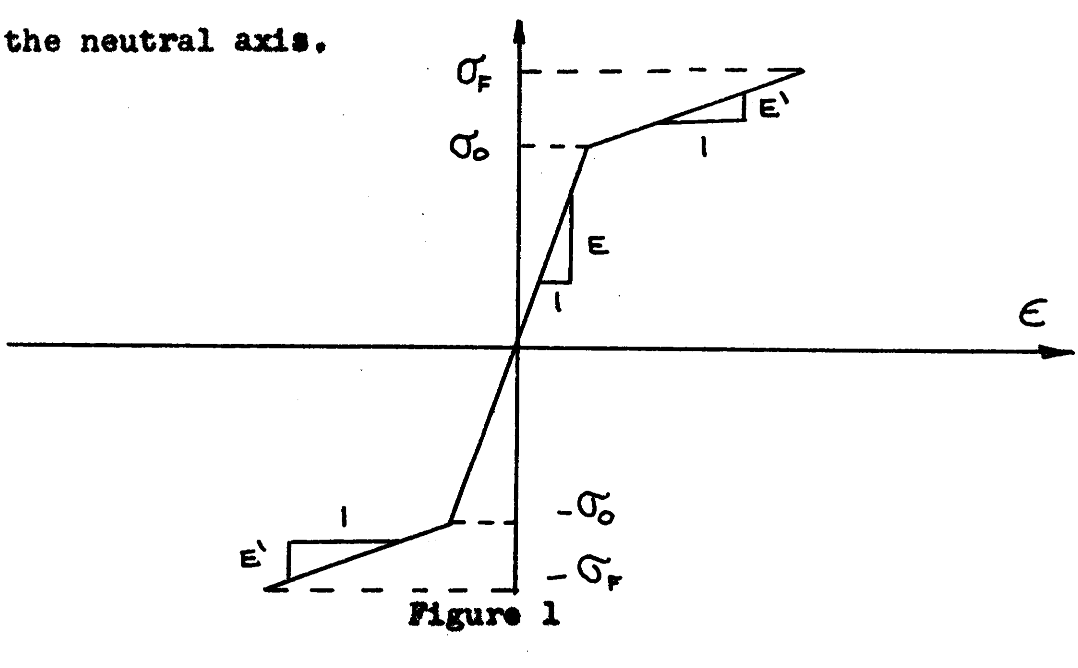

Figure 1

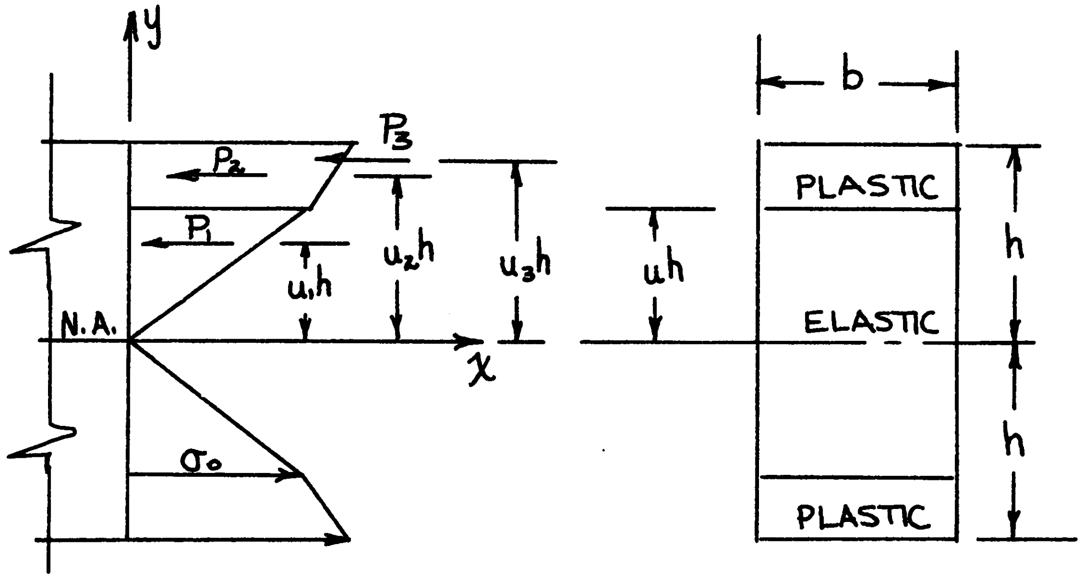

Figure 2

A typical length showing both plastic and elastic regions is shown below in Figure 3.

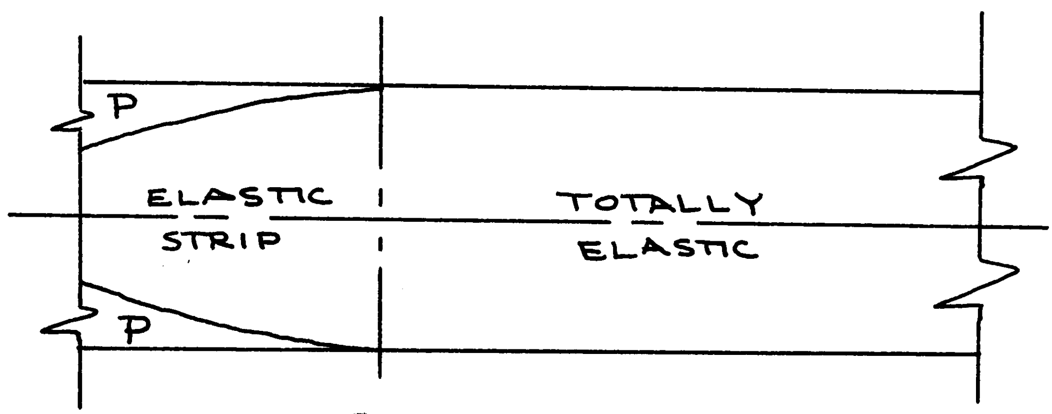

Figure 3

Note that the boundary between the elastic and plastic regions is not linear although the moment expression is linear.

First an expression for the internal moment in these regions where there is some plastic action (refer to Figure 3) will be developed.

Referring to Figure 2, the internal moment is given by:

$$M = 2h\ (P_1 u_1 + P_2 u_2 + P_3 u_3) \qquad \text{Eq. V-1}$$

where $P_1 = \frac{1}{2}\ \sigma_o buh$, $u_1 = \frac{2}{3}\ u$, $P_2 = \sigma_o bh\ (1 - u)$, and $u_2 = \frac{1}{2}\ (1 + u)$.

$P_3 = \frac{1}{2}\ (\sigma_h - \sigma_o)\ bh\ (1 - u)$. $\sigma_h - \sigma_o = E'\ (\epsilon_h - \epsilon_o)$, and from the assumption that plane sections remain plane $\frac{\epsilon_h}{h} = \frac{\epsilon_o}{uh}$ or $\epsilon_h - \epsilon_o = \frac{\epsilon_o}{u}\ (1 - u) = \frac{\sigma_o}{Eu}\ (1 - u)$.
Let $k = \frac{E'}{E}$. Then $P_3 = \frac{1}{2}\ \frac{\sigma_o kbh}{u}\ (1 - u)^2$ and

$u_3 = \frac{1}{3}\ (2 + u)$.

Putting these substitutions into the moment equation yields

$$M = \frac{\sigma_o bh^2}{3}\ [(1 - k)(3 - u^2) + \frac{2k}{u}]\ . \qquad \text{Eq. V-1a}$$

u is a function of x. The above moment equation defines the relationship between u and x.

Next the complementary energy expressions will be developed. In the totally elastic regions of the structure (refer to Figure 3) the complementary and strain energies are the same and are given by the familiar expression:

$$U'_E = \int_{L_E} \frac{M^2 dx}{2EI} \qquad \text{Eq. V-2}$$

In the elastic-plastic regions (again refer to Figure 3) there are two main parts; the elastic center strip and the two plastic outer strips. For convenience an expression for the complementary energy of each of these two main parts will be found separately.

Consider first the elastic center strip.

$$dU'_{PE} = \frac{1}{2}\sigma\epsilon\, dv \qquad \text{Eq. V-3}$$

Since this strip is elastic, it follows that $\sigma = \frac{\sigma_o y}{uh}$ and $\epsilon = \frac{\epsilon_o y}{uh} = \frac{\sigma_o y}{Euh}$.

Then $$dU'_{PE} = \frac{\sigma_o^2 y^2}{2Eu^2h^2}\, bdydx \qquad \text{Eq. V-3a}$$

or $$U'_{PE} = 2\int_{L_p}\int_{o}^{uh} \frac{\sigma_o^2 b}{2E}\,\frac{y^2}{u^2h^2}\,dydx$$ Eq. V-4

Upon integrating once:

$$U'_{PE} = \frac{\sigma_o^2 bh}{3E}\int_{L_p} udx$$ Eq. V-4a

Next consider the two plastic strips (refer to Figure 1).

$$dU'_{PP} = [\tfrac{1}{2}\sigma_o\epsilon_o + (\sigma - \sigma_o)\epsilon_o$$

$$+ \tfrac{1}{2}(\sigma - \sigma_o)(\epsilon - \epsilon_o)]\,dv$$ Eq. V-5

Making use of some expressions used in the development of the expression for the internal moment and the expression for U'_{PE}, the expression for dU'_{PP} becomes:

$$dU'_{PP} = [\frac{\sigma_o^2}{2E} + \frac{\sigma_o^2 k(y - uh)}{uhE} + \frac{\sigma_o^2 k(y - uh)^2}{2u^2h^2E}]\,bdydx$$

Eq. V-5a

$$U'_{PP} = 2\int_{L_p}\int_{uh}^{h} \frac{\sigma_o^2}{2E}\,[1 + \frac{2k(y - uh)}{uh} + \frac{k(y - uh)^2}{u^2h^2}]\,bdydx$$

Eq. V-6

Upon integrating once:

$$U'_{PP} = \frac{\sigma_o^2 bh}{3E} \int_{L_p} [3 - 3u + \frac{3k(1-u)^2}{u} + \frac{k(1-u)^3}{u^2}]\, dx \qquad \text{Eq. V-6a}$$

The total complementary energy for the elastic-plastic region, U'_P, is given by:

$$U'_P = U'_{PE} + U'_{PP} \qquad \text{Eq. V-7}$$

Therefore,

$$U'_P = \frac{\sigma_o^2 bh}{2E} \int_{L_p} [3 - 2u + \frac{3k(1-u)^2}{u} + \frac{k(1-u)^3}{u^2}]\, dx \qquad \text{Eq. V-8}$$

Recalling now that the moment equation defined the relationship between u and x, and that the load was restricted so that the moment equation was a linear function of x, it follows that:

$$M(x) = M + Cx = \frac{\sigma_o bh^2}{3} [(1 - k)(3 - u^2) + \frac{2k}{u}] \qquad \text{Eq. V-1a}$$

where C is some constant equal to the shear at the point where M is the moment, and x is the length from this point.

Upon differentiating, the moment equation yields:

$$dx = \frac{-2\,\sigma_o b h^2}{3C}\,[(1 - k)u + \frac{k}{u^2}]\,du \qquad \text{Eq. V-8}$$

Substituting this expression into the equation for U'_P, gives the following expression for U'_P:

$$U'_P = \frac{-2\,\sigma_o^3 b^2 h^3}{9CE}\int_{u_1}^{u_2} [3(1 - k)^2 u - 2(1 - k)^2 u^2 + \frac{k(1 - k)}{u} + \frac{3k(1 - k)}{u^2} + \frac{k}{u^4}]\,du \qquad \text{Eq. V-7a}$$

With the two expressions developed above, U'_E and U'_P, it is now possible to calculate the complementary energy for the entire structure.

VI. ILLUSTRATIVE EXAMPLES

The use of the expressions developed in V will now be demonstrated in the following examples.

First consider the bent shown below in Figure 4. The problem here is to find the force P that will cause the bent to collapse.

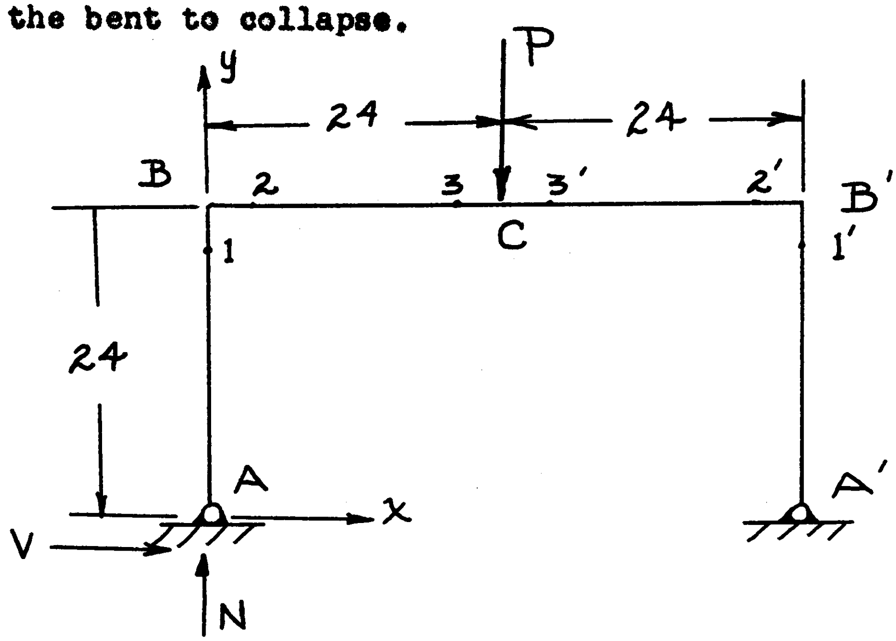

Figure 4

Refer back to Figure 1. Let $\sigma_o = 40{,}000$ psi, $\sigma_p = 60{,}000$ psi, $E = 10 \times 10^6$ psi, and $E' = 5 \times 10^5$ psi.

Refer back to Figure 2. Let $b = h = 1$ inch.

With the above information the moment at yield and the moment at failure will be determined.

Consider first the moment at which the extreme fiber reaches 40,000 psi. This moment is called the yield moment. Since the structure is completely elastic, the stress is given by:

$$\sigma_o = \frac{Mc}{I} = 40,000 = \frac{M_o\,(1)}{\frac{1}{12}\,(1)(2)^3}$$

or $\qquad M_o = 26,700$ pound-inches.

Next consider the moment at which the extreme fiber reaches 60,000 psi. This is called the collapse or failure moment, M_F.

The following calculations are necessary before proceeding with the development of M_F:

$$\epsilon_o = \frac{\sigma_o}{E} = \frac{4 \times 10^4}{10 \times 10^6} = 0.004 \text{ inches per inch.}$$

$$E'(\epsilon_h - \epsilon_o) = \sigma_h - \sigma_o = 5 \times 10^5\,(\epsilon_h - \epsilon_o) = 20,000$$

or $\qquad \epsilon_h = 0.044$ inches per inch.

Then from the assumption that plane sections remain plane $\dfrac{\epsilon_h}{\epsilon_o} = \dfrac{1}{u} = \dfrac{0.044}{0.004} = 11$

or $\qquad u = 0.0909.$

Then from the moment expression for the elastic-plastic region:

$$M_P = \frac{\sigma_o bh^2}{3}\left[(1 - k)(3 - u^2) + \frac{2k}{u}\right] \text{ from Eq. V-1a}$$

where:

$$k = \frac{E'}{E} = \frac{5 \times 10^5}{10 \times 10^6} = 0.05.$$

$$M_P = 13,333\left[0.95\ (3 - 0.0909^2) + \frac{0.10}{0.909}\right]$$

$$= 52,500 \text{ pound-inches.}$$

In order to calculate the complementary energy the regions where some plastic action occurs must be assumed. Refer back to Figure 4. Here the elastic-plastic regions are assumed to extend from point 1 on the vertical member AB around the joint B to point 2 on the horizontal member BCB' and from point 3 on BCB' to point C. As a result of symmetry there is a similar arrangement for the right side of the bent. Only the left side of the bent is to be considered in the following analysis. The validity of the assumed elastic-plastic regions can be checked by examining the final moments. The moment at all cross-sections in the elastic-plastic regions is greater than 26,700 pounds per inch, and the moment at all cross-sections outside the elastic-plastic regions is less than 26,700 pounds per inch.

For the bent to be in the state assumed above, the complementary energy is as follows. The energy expressions will be developed one region at a time for clarity.

For the length of bent between $y = 0$ and $y = y_1$, $M = Vy$ the region is completely elastic; so the complementary energy is

$$U' = \int_0^{y_1} \frac{V^2 y^2 dy}{2EI} .$$

For the region from $y = y_1$ to $y = 24$ the moment is still given by $M = Vy$. This region, however, is an elastic-plastic region. The shear in this region is V. Also, as y takes on all values from y_1 to 24, u varies from 1 to u_B.

Therefore, the complementary energy is:

$$U' = -\frac{2\sigma_o^3 b^2 h^3}{9VE} \int_1^{u_B} [3(1 - k)^2 u - 2(1 - k)^2 u^2 + \frac{k(1 - k)}{u} + \frac{3k(1 - k)}{u^2} + \frac{k}{u^4}] \, du \qquad \text{Eq. V-7a}$$

or introducing the notation

$$r(u_B) = \int_{u_B}^{1} [3(1 - k)^2 u - 2(1 - k)^2 u^2 + \frac{k(1 - k)}{u} + \frac{3k(1 - k)}{u^2} + \frac{k}{u^4}] \, du$$

U' becomes $U' = \frac{2\sigma_o^3 b^2 h^3}{9VE} r(u_B)$.

Consider now the region from $x = 0$ at point B to $x = x_2$ at point 2. For the horizontal member the moment is given by $M = 24\,V - \frac{1}{2} Px$. The slope of the moment equation is $-\frac{1}{2} P$, and as x varies from 0 to x_2, u takes on all values from u_B to 1. Therefore, U' for this region is, using the notation introduced above,

$$U' = \frac{4\sigma_o^3 b^2 h^3}{9PE} r(u_B).$$

For the totally elastic region between points 2 and 3

$$U' = \int_{x_2}^{x_3} \frac{(24 - \frac{1}{2} Px)^2}{2EI} dx \qquad \text{Eq. V-2}$$

Consider now the elastic-plastic region between point 3 and point C on the horizontal member BCB'. The moment has changed signs; so that

$$M = 24\ V - \frac{1}{2}\ Px = - \frac{\sigma_o bh^2}{3}\ [(1 - k)(3 - u^2) + \frac{2k}{u}]$$

which upon differentiation yields

$$dx = + \frac{4\sigma_o bh^2}{3P}\ [(1 - k)u + \frac{k}{u^2}]\ du$$

and since u varies from 1 at $x = x_3$ to u_C at $x = 24$, the complementary energy for this region is

$$U' = \frac{4\sigma_o{}^3 b^2 h^3}{9PE}\ r(u_C)$$

The total complementary energy for the left half of the bent is the sum of the five expressions developed above, or

$$U' = \int_0^{y_1} \frac{V^2 y^2 dy}{2EI} + \int_{x_2}^{x_3} \frac{(24\ V - \frac{1}{2}\ Px)^2}{2EI}\ dx$$

$$+ \frac{2\sigma_o{}^3 b^2 h^3}{9E}\ [\frac{r(u_B)}{V} + \frac{2r(u_B)}{P} + \frac{2r(u_C)}{P}] \qquad \text{Eq. VI-1}$$

Integration yields:

$$U' = \frac{V^3y^3}{6EIV} + \frac{(24\ V - \frac{1}{2}\ x_2P)^3}{3EIP} + \frac{(24\ V - \frac{1}{2}\ x_3P)^3}{3EIP}$$

$$+ \frac{2\sigma_o^3b^2h^3}{9E}\left[\frac{r(u_B)}{V} + \frac{2r(u_B)}{P} + \frac{2r(u_C)}{P}\right] \quad \text{Eq. VI-1a}$$

At the points where the elastic regions end and the elastic-plastic regions begin, the moment must equal the yield moment, M_o. The next three equations follow:

$$Vy_1 = + M_o \quad \text{Eq. VI-2}$$

$$24\ V - \frac{1}{2}\ Px_2 = + M_o \quad \text{Eq. VI-3}$$

$$24\ V - \frac{1}{2}\ Px_3 = - M_o \quad \text{Eq. VI-4}$$

In addition the maximum moment at the point where P is applied must equal the failure moment, M_F. Therefore:

$$24\ V - 12\ P = - M_F = - \frac{\sigma_o bh^2}{3}\left[(1 - k)(3 - u_C^2) + \frac{2k}{u_C}\right] \quad \text{Eq. VI-5}$$

If the last four relationships are substituted into Eq. VI-1, the following equation is obtained:

$$U' = \frac{M_o^3}{6EIV} + \frac{2\ M_o^3}{3EI(2V + \frac{M_F}{12})} + \frac{2\sigma_o^3 b^2 h^3}{9E}\left[\frac{r(u_B)}{V} + \frac{2r(u_B) + 2r(u_C)}{2V + \frac{M_F}{12}}\right] \qquad \text{Eq. VI-1b}$$

The problem now is to determine the value of V that will make U' a minimum. This value of V must satisfy the following expression:

$$\frac{\partial U'}{\partial V} = 0 = -\frac{M_o^3}{IV^2} - \frac{8M_o^3}{I(2V + \frac{M_F}{12})^2} + \frac{4\sigma_o^3 b^2 h^3}{3}\left[-\frac{r(u_B)}{V^2} + \frac{1}{V}\frac{\partial r(u_B)}{\partial V} + \frac{2}{2V + \frac{M_F}{12}}\frac{\partial r(u_B)}{\partial V} - \frac{4r(u_B) + 4r(u_C)}{(2V + \frac{M_F}{12})^2}\right] \qquad \text{Eq. VI-6}$$

In addition to Eq. VI-6 there is the following relationship between V and u_B

$$24\ V = \frac{\sigma_o bh^2}{3}\left[(1 - k)(3 - u_B^2) + \frac{2k}{u_B}\right] \qquad \text{Eq. VI-7}$$

Upon differentiation Eq. VI-7 becomes:

$$24 = -\frac{2\sigma_o bh^2}{3}\left[(1-k)u_B + \frac{k}{u_B^2}\right]\frac{\partial u_B}{\partial V}$$

or $$\frac{\partial u_B}{\partial V} = \frac{(3)(24)}{2\sigma_o bh^2}\left[(1-k)(u_B) + \frac{k}{u_B^2}\right]^{-1}$$

Recalling that

$$r(u_B) = \int_{u_B}^{1} [3(1-k)^2 u - 2(1-k)^2 u^2$$

$$+ \frac{k(1-k)}{u} + \frac{3k(1-k)}{u^2} + \frac{k}{u^4}]\, du$$

it follows upon differentiating that

$$\frac{\partial r(u_B)}{\partial u_B} = -[3(1-k)^2 u_B - 2(1-k)^2 u_B^2$$

$$+ \frac{k(1-k)}{u_B} + \frac{3k(1-k)}{u_B^2} + \frac{k}{u_B^4}]$$

Since $\frac{\partial r(u_B)}{\partial V}$ could be written as

$$\frac{\partial r(u_B)}{\partial V} = \frac{\partial r(u_B)}{\partial u_B}\frac{\partial u_B}{\partial V},$$

$$\frac{\partial r(u_B)}{\partial V} = \frac{3(24)}{2\sigma_o bh^2}\left[3 - 2u_B + \frac{3k(1 - u_B)^2}{u_B} + \frac{k(1 - u_B)^3}{u_B{}^2}\right] \qquad \text{Eq. VI-8}$$

Now introducing the notation

$$s(u_B) = 3 - 2u_B + \frac{3k(1 - u_B)^2}{u_B} + \frac{k(1 - u_B)^3}{u_B{}^2}$$

and substituting into Eq. VI-6, the following expression results:

$$0 = -\frac{M_o{}^3}{IV^2} - \frac{8M_o{}^3}{I(2V + \frac{M_F}{12})^2} - \frac{4\sigma_o{}^3b^2h^3}{3}\left[\frac{r(u_B)}{V^2} + \frac{4r(u_B) + 4r(u_C)}{(2V + \frac{M_F}{12})^2}\right] + \frac{48\sigma_o{}^2bh\ s(u_B)}{V} + \frac{96\sigma_o{}^2bh\ s(u_B)}{(2V + \frac{M_F}{12})} \qquad \text{Eq. VI-6a}$$

To complete the analysis, Eq. VI-6a and Eq. VI-7 must be solved simultaneously for u_B and V after the values for M_O, M_P, I, b, h, and σ_o as determined earlier for this particular problem are substituted into the expressions. The problem could be reduced, at this point easily, to a single equation in u_B; However, the author has found that the solution can be carried out with greater speed without making this reduction.

After the substitutions Eq. VI-6a and Eq. VI-7 become:

$$0 = -\frac{(26{,}700)^3}{V^2} - \frac{8(26{,}700)^3}{(2V + 4375)^2} - 2(40{,}000)^3 \left[\frac{r(u_B)}{V^2} + \frac{4r(u_B) + 4(23.46)}{(2V + 4375)^2} \right] + \frac{48(40{,}000)^2\, s(u_B)}{V} + \frac{96(40{,}000)^2\, s(u_B)}{(2V + 4375)} \qquad \text{Eq. VI-6a}$$

and

$$24\, V = \frac{40{,}000}{3} \left[(0.95)(3 - u_B{}^2) + \frac{0.10}{u_B} \right] \qquad \text{Eq. VI-7}$$

where

$$r(u_B) = \int_{u_B}^{1} [3(0.95)^2 u - 2(0.95)^2 u + \frac{0.05(0.95)}{u} + \frac{0.15(0.95)}{u^2} + \frac{0.05}{u^4}] \, du$$

and

$$s(u_B) = 3 - 2u_B + \frac{0.15(1 - u_B)^2}{u_B} + \frac{0.05(1 - u_B)^2}{u_B^2}$$

The solution to this problem is:

$P = 7960$ pounds

$V = 1770$ pounds

$y_1 = 15.0$ inches

$x_2 = 4.05$ inches

$x_3 = 17.5$ inches

$u_B = 0.255.$

The collapse load assuming ideal plasticity and a yield stress of 40,000 psi is 6,670 pounds.

As a second example consider the bent shown below in Figure 5. Again the problem is to find the P that will cause the bent to collapse.

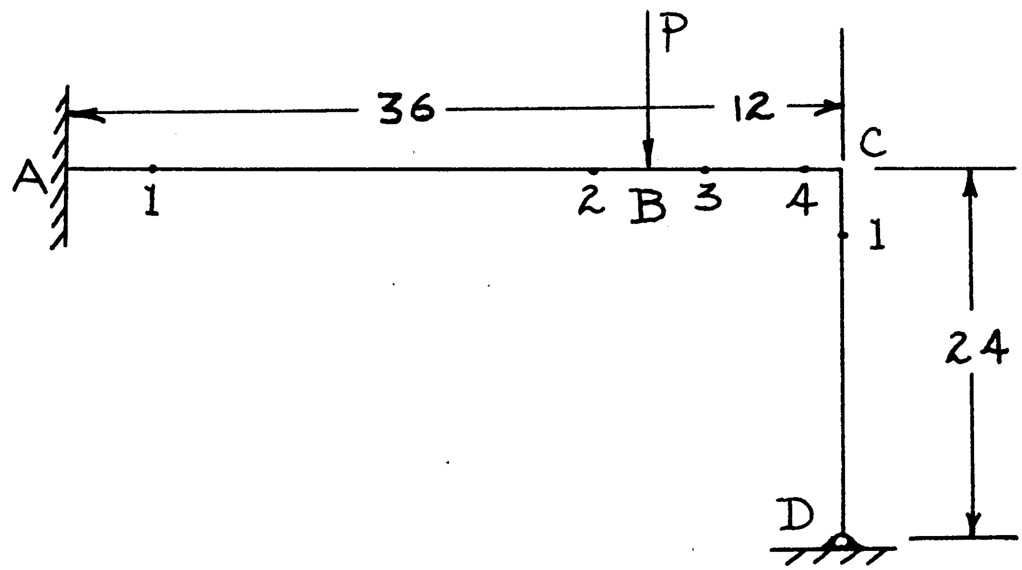

Figure 5

For this problem it will be assumed that all of the material constants and the cross-section dimensions are the same as those used in the first example.

The regions where some plastic action occurs are assumed to be between points A and 1, points 2 and 3, and points 4 and C all on member ABC. On member CD plastic action is assumed between points C and 1.

Following the same approach that was used in the first example, the expression for the complementary energy becomes:

$$U' = \frac{2\sigma_o^2}{9EV} r(u_A) + \int_{x_1}^{x_2} \frac{(Vx - M)^2 dx}{2EI} + \frac{2 \quad o^3 r(u_B)}{9EV}$$

$$+ \frac{2\sigma_o^3 r(u_B)}{9E (P - V)} + \int_{x_3}^{x_4} \frac{[36P - (P - V) x - M]^2}{2EI} dx$$

$$+ \frac{2\sigma_o^3 (u_C)}{9E (P - V)} + \frac{2\sigma_o^3 r(u_C)}{9E N}$$

$$+ \int_{y_1}^{24} \frac{(48V - 12P - M + Ng)^2}{2EI} dy \qquad \text{Eq. VI-9}$$

Upon integrating and substituting the following six relations:

$$Vx_1 - M = - 26,700 \qquad \text{Eq. VI-10}$$

$$Vx_2 - M = - 26,700 \qquad \text{Eq. VI-11}$$

$$36P - (P - V)x_3 - M = + 26,700 \qquad \text{Eq. VI-12}$$

$$36P - (P - V)x_3 - M = - 26,700 \qquad \text{Eq. VI-13}$$

$$48V + Ny_1 - 12P - M = - 26,700 \qquad \text{Eq. VI-14}$$

$$48V + 24N - 12P - M = 0 \qquad \text{Eq. VI-15}$$

$$36V - M = + M_p = 52,500 \qquad \text{Eq. VI-16}$$

into Eq. VI-9, the following expression for U' is obtained:

$$U' = \frac{2\sigma_o^3}{9EV} r(u_A) + \frac{2(26,700)^3}{6EIV} + \frac{2\sigma_o^3 r(u_B)}{9EV}$$

$$+ \frac{2\sigma_o^3 r(u_B)}{9E(2N + 4375)} + \frac{2(26,700)^3}{6EI(2N + 4375)} + \frac{2\sigma_o^3 r(u_C)}{9E(2N + 4375)}$$

$$+ \frac{2\sigma_o^3 r(u_C)}{9EN} + \frac{(26,700)^3}{6EIN} . \qquad \text{Eq. VI-9A}$$

The values of V and N that make U' a minimum are the true values for these reactions. These values can be found from $\frac{\partial U'}{\partial N} = 0$ and $\frac{\partial U'}{\partial V} = 0$.

Differentiating, Eq. VI-9a becomes:

$$\frac{\partial U'}{\partial V} = 0 = -\frac{2\sigma_o^3 r(u_A)}{9V^2} + \frac{12\sigma_o^2 s(u_A)}{V} - \frac{(26,700)^3}{2V^2}$$

$$- \frac{2\sigma_o^3 (23.46)}{9V^2} + \frac{12\sigma_o^2 s(u_C)}{9(2N + 4375)}$$

$$+ \frac{12\sigma_o^2 s(u_C)}{N} \qquad \text{Eq. VI-17}$$

And,

$$\frac{\partial U'}{\partial V} = 0 = -\frac{4\sigma_o^3\ (23.46)}{9(2N + 4375)^2} - \frac{(26,700)^3}{(2N + 4375)^2} + \frac{8\sigma_o^2\ s(u_C)}{2N + 4375}$$

$$- \frac{4\sigma_o^3\ r(u_C)}{9(2N + 4375)^2} + \frac{8\sigma_o^2\ s(u_C)}{N}$$

$$- \frac{2\sigma_o^3\ r(u_C)}{9N^2} - \frac{(26,700)^3}{4N^2} \qquad \text{Eq. VI-18}$$

In addition the following two moment expressions also exist:

$$M = \frac{\sigma_o}{3}\,[(0.95)(3 - u_A^2) + \frac{0.10}{u_A}] \qquad \text{Eq. VI-19}$$

and

$$24\ N = \frac{\sigma_o}{3}\,[0.95\ (3 - u_C^2) + \frac{0.10}{u_C}] \qquad \text{Eq. VI-20}$$

The last four equations, Eq. VI-17, 18, 19, and 20, must now be solved simultaneously.

The solution is:

V = 2650 pounds

N = 1440 pounds

u_A = 0.110

u_C = 0.625

$P = 9{,}900$ pounds

$x_1 = 6.12$ inches

$x_2 = 18.5$ inches

$x_3 = 39.5$ inches

$x_4 = 46.9$ inches

$y_1 = 18.5$ inches

$M_A = 42{,}900$ pound-inches

$Mc = 34{,}800$ pound-inches.

The collapse load for the second example assuming ideal plasticity and a yield stress of 40,000 psi is 8,890 pounds.

It should be noted at this point that the material constants used above in the two examples are typical of some aluminums. In the case of hard steels these constants change such that the difference in the two collapse loads becomes greater.

VII. DISCUSSION

From the two previous examples it can be seen that the problem reduces to solving a set of non-linear, simultaneous equations. The complexity of this set of equations increases to a very large extent with the introduction of a new unknown; so that the more general problem of three unequal sized members with the supports fixed would require solving a very complex set of equations.

In the solution of the equations in this thesis the author found it expedient to plot curves of $\frac{3M}{\sigma_o bh}$ versus u, r(u) versus u, and s(u) versus u, and for different trials to read the values from these curves. The solution to the second problem, however, was still very slow and tedious. Apparently the trouble lies in the fact that the slope of the s(u) and r(u) versus u curves is very high for small values of u. Small changes in u at this end of the graph bring about large changes in r(u) and s(u).

Another idea of approach was considered. This second approach consists of transforming the cross-section in the elastic-plastic regions into an "equivalent elastic cross-section" by reducing the width of the cross-section for values of y greater than uh while at the same time increasing the stress so that the resultant force acting on elemental strip has the same magnitude and distance from the neutral axis, and the stress varies linearly with the distance from the neutral axis. Then the stress can be predicted by the ordinary flexure formula, $\sigma = \frac{My}{I}$.

A brief description of the procedure for transforming the cross-section follows. It was shown previously for values of y greater than uh that $\epsilon - \epsilon_o = \frac{\sigma_o (y - uh)}{E\ uh}$.

The stress at this same distance y is

$\sigma = \sigma_o + E(\epsilon - \epsilon_o) = \sigma_o + \frac{\sigma_o k (y - uh)}{uh}$. If the stress were to vary linearly from the neutral axis,

$$\sigma' = \frac{\sigma_o y}{uh}.$$

Therefore, if the true stress, σ, were multiplied by the ratio $\frac{\sigma'}{\sigma}$ and the width of the cross-section by $\frac{\sigma}{\sigma'}$, the conditions mentioned in the previous paragraph for the transformed cross-section would be satisfied. The thickness for values of y greater than uh becomes

$$t = b\,\frac{\sigma_o + \frac{\sigma_o k}{uh}(y - uh)}{\sigma_o \frac{y}{uh}}$$

or

$$t = \frac{b\,[uh + k\,(y - uh)]}{y}$$

The moment of inertia for the transformed cross-section becomes

$$I = \frac{b}{12}\,(2\,uh)^3 + 2\int_{uh}^{h} \frac{b\,[uh + k\,(y - uh)]}{y}\,y^2 dy$$

Upon integrating and collecting terms,

$$I = \frac{bh^3}{3}\,u\,\left[(1 - k)(3 - u^2) + \frac{2k}{u}\right]$$

Then from the flexure formula,

$$\sigma_o = \frac{M\,uh}{I} \quad \text{or} \quad M = \frac{\sigma_o I}{uh} = \frac{\sigma_o bh^2}{3}\left[(1 - k)(3 - u^2) + \frac{2k}{u}\right]$$

which checks with Eq. V-1.

The term $\frac{M}{I}$ which frequently appears could be replaced by $\frac{\sigma_o}{uh}$; however, little if any is to be gained by this substitution. The problem has been reduced to an equivalent elastic bent with a variable moment of inertia. This problem could be handled by some technique such as the column-analogy; however, the author, after a brief attempt at a solution using the column-analogy approach, is of the opinion that the method used in parts V and VI is a superior approach.

In conclusion, it can be said that the method of approach (whether by the minimum complementary energy or column analogy) is very straightforward, and the final set of equations can be developed very quickly. However, the final simultaneous solution presents a very difficult part of obtaining the final answers.

VIII. ACKNOWLEDGMENT

The author is deeply indebted to Professor Daniel Frederick whose helpful criticisms and patience made this thesis possible.

IX. BIBLIOGRAPHY

1. A. Nadai, "Theory of Flow and Fracture of Solids," Vol. 1. McGraw-Hill Book Company, Inc., New York, N. Y., 1950.

2. N. J. Hoff, "Complementary Energy Analysis of the Failing Load of a Clamped Beam," Journal of Applied Mechanics, Vol. 19, No. 4, p. 563, December 1952.

3. Frank P. Cozzone, "Bending Strength in the Plastic Range," Journal of Aero. Sciences, Vol. 10, No. 5, p. 137, May 1943.

4. H. J. Greenberg, "On the Variational Principles of Plasticity," Office of Naval Research, Contract N7 onr-358, T. O. I., NR-041-032.

5. P. G. Hodge, Jr., "Interaction Curves for Shear and Bending of Plastic Beams," ASME Summer Conference, Berkeley, California, June 1957.

6. L. B. Kriz, "Ultimate Strength Criteria for Reinforced Concrete," Proc. ASCE, 85, EM3 (J. Engng. Mech. Div.), pp. 95-110, July 1959.

7. N. Perrone and P. G. Hodge, Jr., "Strain-Hardening Solutions with Kinematic Models," Proc. 3rd Nat. Congress of Applied Mechanics, pp. 641-648, June 1958.

8. N. Perrone and P. G. Hodge, Jr., "Applications of a Consistent Theory for Strain Hardening Plastic Solids," PIBAL Report No. 403, Polytechnic Institute of Brooklyn, Brooklyn, N. Y., 1958.

9. A. Phillips., "The Neutral Axis in Plastic Bending of Beams," Proc. ASCE, 85, EM4 (J. Engng. Mech. Div.), pp. 65-76, October, 1959.

10. W. Prager, "The Theory of Plasticity: A Survey of Recent Achievements," Proceedings of the Inst. of Mechanical Engineers, Vol. 169, pp. 41-57, 1955.

11. W. Prager, "A New Method of Analysing Stresses and Strains in Work Hardening Plastic Solids," Journal of Applied Mechanics, Vol. 23, Trans. ASME, Vol. 78, pp. 493-496, 1956.

12. W. J. Worley, "Inelastic Behavior of Aluminum Alloy. I - Beams with Web Cutouts," University of Illinois Engineering Experiment Station Bulletin 448, April 1958.

X. LIST OF SYMBOLS

σ is the stress

σ_o is the stress at the discontinuity

σ_F is the stress when the extreme fiber ruptures

ϵ is the strain

ϵ_o is the strain corresponding to σ_o

ϵ_h is the strain at the extreme fiber

M is the moment

M_o is the moment corresponding to a stress σ_o at the extreme fiber

M_F is the moment corresponding to a stress σ_F at the extreme fiber

E is the slope of the initial portion of the stress-strain curve

E' is the slope of the secondary portion of the stress-strain curve

h is one-half of the total depth of the cross-section

b is the width of the cross-section

x_1, x_2, etc., are the distances from the y axis to the boundaries between the totally elastic and elastic-plastic regions. x_1 is the distance to point 1, etc.

y_1, y_2, etc., are the distances from the x axis to the boundaries between the totally elastic and elastic-plastic regions. y_1 is the distance to point 1, etc.

k is the ratio $\frac{E'}{E}$

P is the applied force

V is the shear

N is the thrust

The vita has been removed from the scanned document

ABSTRACT

Two bents are analysed for the collapse load. The bents are constructed from a material exhibiting a bilinear, strain-hardening stress-strain curve. The first bent considered is symmetrical and the second is non-symmetrical. The solutions are obtained by using the principle of minimum complementary energy. In order to do the analysis expressions for the internal moment and the complementary energy are developed for the regions where some plastic action occurs. The collapse loads found by considering the strain-hardening are compared with those collapse loads found by considering the material to be ideally plastic for a material similar to aluminum.