Contributions to Experimental Design for Quality Control

by

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(ABSTRACT)

A parameter design introduced by Taguchi provides a new quality control method which can reduce cost-effectively the product variation due to various uncontrollable noise factors such as product deterioration, manufacturing imperfections, and environmental factors under which a product is actually used. This experimental design technique identifies the optimal setting of the control factors which is least sensitive to the noise factors. Taguchi's method utilizes orthogonal arrays which allow the investigation of main effects only, under the assumption that interaction effects are negligible.

In this paper new techniques are developed to investigate two-factor interactions for 2^r and 3^r factorial parameter designs. The major objective is to be able to identify influential two-factor interactions and take those into account in properly assessing the optimal setting of the control factors. For 2^r factorial parameter designs, we develop some new designs for the control factors by using a partially balanced array. These designs are characterized by a small number of runs and some balancedness property of the variance-covariance matrix of the estimates of main effects and two-factor interactions. Methods of analyzing the new designs are also developed. For 3^r factorial parameter designs, a detection procedure consisting of two stages is developed by using a sequential method in order to reduce the number of runs needed to detect influential two-factor interactions. In this paper, an extension of the parameter design to several quality characteristics is also developed by devising suitable statistics to be analyzed, depending on whether a proper loss function can be specified or not.

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Chapter I

INTRODUCTION

Since Shewhart laid the foundation of statistical methods for quality control in 1931, we have seen an increase in the application of statistical ideas to improve the quality and productivity of industrial processes and products. Most procedures of the traditional quality control methods have used control charts in which the procedures signal when process is out of control. But they cannot be used successfully to reduce the variation of the quality characteristic of a product due to uncontrollable noise factors such as production deterioration, manufacturing imperfection, and environmental factors under which a product is actually used. In quality control activities, we frequently encounter a situation in which we want to reduce such product variability without increasing manufacturing cost.

As a simple example (Taguchi and Wu, 1980, and Taguchi, 1986), consider the manufacturing process of ceramic tiles, where there are seven controllable factors: lime additive content, granularity of additive, feldspar content, agalmatolite content, type of agalmatolite, charge quantity, and waste return content. In addition there is one factor, kiln temperature that is prohibitively expensive to control. It is known that an uneven temperature distribution in the kiln causes variation in the size of tiles, which is the most important quality characteristic of the tiles. In this situation the manufacturer wants to reduce the product variation attributed to uneven distribution of temperature in the kiln without being able to control it adequately.

As this example demonstrates, there are two types of variables which affect the quality characteristic of a product. These are called control and noise factors. The control factors are those input variables which can be adjusted by the operator such as lime additive content. Noise factors are those variables which are difficult, if not impossible, to control such as kiln temperature. Those noise factors are the main sources of variation in the quality characteristic of a product.

This problem was addressed by Taguchi, and he developed an experimental design technique (***parameter design***) to reduce the product's variation cost-effectively. It does this by identifying an optimal combination of settings of given control factors, at which the quality characteristic is least sensitive to the various noise factors. In the parameter design, we combine two designs, one for control and one for noise factors, and analyze some performance statistic which Taguci termed a signal-to-noise (*SN*) ratio. Then, by using an experimental design technique we identify an optimal setting of each control factor at which the *SN* ratio is maximized with the mean value of the quality characteristics close to the target value. In order for the parameter design to be successful, a suitable *SN* ratio should be used. Taguchi developed various kinds of *SN* ratios according to the engineering structure of a product.

In constructing designs for the control and noise factors, Taguchi suggested using orthogonal arrays (especially orthogonal arrays of strength 2) with which we can investigate main effects only. The interactions are not examined in the parameter design since the number of experimental runs needed to estimate interactions in orthogonal arrays becomes unmanageably large as the number of factors increases. However, if some interaction effects among the control factors do indeed exist, then the parameter design may lead to settings which are not optimal. Moreover, Taguchi developed the parameter design technique only for a single quality characteristic, assuming that the quality of a product can be assessed by the most important quality characteristic. But frequently we want to improve several quality characteristics of a product simultaneously. Therefore, the problems we face in the parameter design are:

(i) development of the suitable *SN* ratios corresponding to the particular properties of products;

(ii) construction of designs by which we can investigate interactions with minimum cost;

(iii) extension of the parameter design to several quality characteristics.

Most research for the parameter design centered on the first problem, and several kinds of *SN* ratios for the univariate parameter design were developed by Leon, et.al.(1987), and Box (1988). Although the second and third problems are important, no research has been done for these problems. Only some transformation techniques were suggested by Taguchi and Wu (1980), and Box (1988) for the second problem.

The procedures outlined in this paper are directed toward the second and third problem. We shall develop new designs by which we can identify influential 2-factor interactions, especially for 2^t and 3^t factorial designs for the control factors. To extend the parameter design technique to several quality characteristics we shall devise some suitable *SN* ratios which take several quality characteristics into account simultaneously.

The following chapter provides relevant background information on the parameter design with an example of its use, plus details of the influence of non-negligible interaction effects. In Chapter III, we develop suitable designs for 2^t factorial designs of the control factors, through which we can detect and estimate influential 2-factor interactions under the assumption that 3-factor and higher order interactions are all negligible. Also developed in Chapter III are methods of analyzing the new designs. The procedure developed in Chapter IV is aimed to identify the influential 2-factor interactions in 3^t factorial designs for the control factors. We employ a sequential procedure to reduce the number of experimental runs needed for detecting influential 2-factor interactions. The

SN ratios for several quality characteristics are developed in Chapter V. Chapter VI contains a summary and some suggestions for further study.

Chapter II

PARAMETER DESIGN

This chapter contains a brief review of the parameter design, followed by a simple example. For the development of new designs, problems resulting from influential interaction effects are discussed as well.

II.1 Loss Function and Expected Loss

Taguchi (1986) defines quality of a product as follows:

> "I define quality as the losses a product imparts to the society from the time the product is shipped".

The essence of this definition is that the societal losses generated by a product from the time the product is shipped to the customer, determine its desirability. The smaller the losses, the more desirable the product is. There are many ways in which a product can result in losses to the society.

A typical loss on which we concentrate is failure to meet the ideal performance, that is, deviation of the quality characteristic of a product from its ideal value.

To evaluate the quality of a product we need first to identify the quality characteristic of a product, which is assumed here to be measured on a continuous scale and we denote it by Y. The ideal value of the quality characteristic is called a target value and we denote it by τ. Also we denote the continuous loss function by $\ell(y{:}\tau)$ when y is a specific value of the quality characteristic Y. The loss function $\ell(y{:}\tau)$ represents the loss in terms of a quantitative measure, for example in dollars, suffered by an arbitrary customer at an arbitrary time during the product's life span due to the deviation of the quality characteristic Y from its target value τ. Here we assume that a loss function $\ell(y{:}\tau)$ is convex and sufficiently smooth so that its second derivative exists. Then by Talyor's series expansion we can express $\ell(y{:}\tau)$ as

$$\ell(y{:}\tau) = \ell(\tau{:}\tau) + \ell'(\tau{:}\tau)(y - \tau) + \frac{\ell''(\tau{:}\tau)}{2}\,(y - \tau)^2 + \cdots \quad . \tag{2.1}$$

Since the loss function $\ell(y{:}\tau)$ has a minimum value at $y = \tau$, its first derivative $\ell'(\tau{:}\tau)$ evaluated at $y = \tau$ in (2.1) equals zero. Assuming that the loss is always zero when y is equal to its target value τ, we can then approximate the loss function as

$$\ell(y{:}\tau) = k(y - \tau)^2 \ , \tag{2.2}$$

where the unknown constant k can be determined using an economic argument (see Taguchi and Wu, 1980).

As we mentioned, there are two types of variables which affect the product performance. These are control and noise factors. Control factors are the product design variables which can be controlled by the product designer or manufacturing operator. The set of control factors is denoted here by $\underline{\Theta}$. A vector $\underline{\theta}$ of settings of the control factors in $\underline{\Theta}$ defines a product design specification and vice versa. Noise factors are those variables which cause variation in the product's performance during the life span of a product. We denote the set of noise factors by $\underline{W}$. Taguchi and Wu (1980) classified the noise factors into the following three categories.

(i) External noise factors: These noise factors include environmental conditions under which a product is actually used, such as temperature, humidity, dust, etc.

(ii) Internal noise factors: These are noise factors which cause the variation during the usage of a product. For example, deterioration of a product or components of a product can be classified into this category.

(iii) Manufacturing noise factors: These noise factors cause the variation among the product units during the manufacturing process of a product. In the example of the manufacturing process of ceramic tiles in Chapter I, the kiln temperature is a noise factor of this category. Moreover, variations among the incoming materials and supplies fall into this category.

The parameter design is used to conduct an investigation to minimize the adverse effects of various noise factors. Effects of noise factors change with different settings of the control factors. The goal of the parameter design is to evaluate the effects of the noise factors at different settings of the control factors, and identify the optimal settings at which the product is least sensitive to the noise factors.

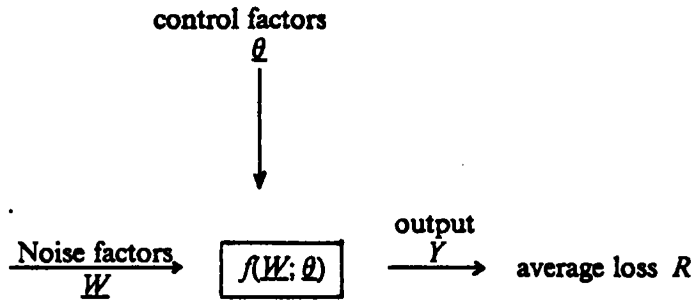
The external and internal noise factors can be investigated at the stage of research and development (see Taguchi and Wu, 1980). Even though the manufacturing noise factors can be examined in the parameter design at any stage, these noise factors are investigated primarily at the manufacturing stage. According to the noise factors included in the parameter design, we can state the purposes of parameter design as following:

(i) to minimize product sensitivity to environmental conditions;

(ii) to minimize product variability due to deterioration;

(iii) to minimize variation among the product units with the mean value of the quality characteristic close to the target value.

A simple manufacturing process is illustrated in Figure 2.1. Let $\underline{\theta} = (\theta_1, \theta_2, \dots, \theta_t)$ denote the setting of the control factors and let $\underline{W} = (W_1, W_2, \dots, W_u)$ represent the set of noise factors included in the parameter design experiment. We assume that the quality characteristic Y is a function of $\underline{\theta}$ and $\underline{W}$, that is, $Y = f(\underline{W}; \underline{\theta})$, in which f is called a transfer function. The setting of the control factors ($\underline{\theta}$) is a set of the parameters of the distribution Y, and for a given $\underline{\theta}$ the noise factors generate the distribution. The noise factors are assumed to be random variables so that the output is also a random variable.



< Figure 2.1: Block Diagram of a Manufacturing Process >

The expected loss at the setting $\underline{\theta}$ of control factors is

$$\begin{aligned} R(\underline{\theta}) &= E_{\underline{W}}(\ell(y{:}\tau)) \\ &= E_{\underline{W}}(k(y-\tau)^2) = k\text{MSE}(\underline{\theta}) \\ &= k(\sigma^2(\underline{\theta}) + (\text{bias}\,(\underline{\theta}))^2) \ , \end{aligned} \tag{2.3}$$

where we use the notation, such as MSE($\underline{\theta}$) to emphasize that MSE is a function of $\underline{\theta}$. Hereafter, however, we shall omit $\underline{\theta}$ when no ambiguity arises. This expected loss is a quantity to be minimized in the parameter design.

Note from equation (2.3) that the expected loss is proportional to MSE, and in turn the quality of a product is inversely proportional to it. Moreover, if $E(Y)$ is equal to the target value τ, the objective becomes to reduce the variance of the quality characteristic.

An example is given here (see Taguchi and Wu, 1980), which shows that the inspection method of a traditional quality control activity cannot reduce successfully the variance of the quality characteristic. Assume that the quality characteristic Y has a normal distribution with mean μ and variance σ^2. If we inspect each product and remove those whose value of the quality characteristic is outside of the tolerance interval, for example $(\mu - 3\sigma,\ \mu + 3\sigma)$, then for those having passed the inspection the density function of Y is reduced to the following truncated normal distribution

$$f(y) = \begin{cases} \frac{1}{Q}\phi(y) & \text{if } \mu - 3\sigma \leq y \leq \mu + 3\sigma \\ 0 & \text{otherwise} \end{cases}$$

,where $Q = \int_{\mu-3\sigma}^{\mu+3\sigma}\phi(y)dy = 0.9973$ and $\phi(y)$ is the normal density function with mean μ and variance σ^2.

Applying integration by parts, we obtain the variance σ_1^2 for passed products as

$$\begin{aligned} \sigma_1^2 &= E((Y-\mu)^2) \\ &= \frac{1}{Q}\int_{\mu-3\sigma}^{\mu+3\sigma}(y-\mu)^2\phi(y)dy \\ &= 0.972\sigma^2 \ . \end{aligned}$$

Consequently, the perfect inspection method can reduce the product variability only up to 2.8%. If we consider the inspection cost, then this inspection method turns out to be totally inefficient in reducing the product variability. In the next section a new statistical method (parameter design) will be described. Since the parameter design minimizes the product variability by reducing the influence of the noise factors rather than by controlling them, it is a cost-effective technique for improving the quality of a product.

II.2 Parameter Design and Its Analysis

II.2.1 Orthogonal Arrays

Before describing the analyzing technique of the parameter design, we define an orthogonal array (OA) which will be used to construct the parameter design.

< Definition 2.1 >

Let T be a $(t \times m)$ matrix with entries from a set S of s symbols. T is called an orthogonal array of strength d in m assemblies with t constraints and s levels if any $(d \times m)$ submatrix of T contains all s^d possible $(d \times 1)$ column vectors based on s symbols of S with the same frequency λ. Such an array is denoted by $OA(m,t,s,d{:}\lambda)$ in which λ is called the index of T.

OA's were introduced first into statistics by Rao(1946) under the name of hypercubes and then by Bose and Bush(1952). For given values of m and s, it is possible in general to construct OA's for a large value of t if the value of d is small. Raghavarao (1971) provided several methods for constructing OA's up to strength 3, and Seiden and Zemach(1966) investigated the construction methods of OA's of strength 4 for $s = 2$, and Kackar(1982) tabulated some important OA's.

From the above definition of OA, we can see easily that:

(i) $m = \lambda s^d$ in any $OA(m,t,s,d{:}\lambda)$;

(ii) an OA of strength d_1 is also an OA of strength $d_2 \le d_1$;

(iii) an OA remains an OA if we permute rows and/or columns of the array.

< Example 2.1 >

(i) The following is an OA(8,4,2,3:1) :

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

(ii) An OA(9,4,3,2:1) follows:

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 & 2 & 0 & 2 & 0 & 1 \\ 0 & 1 & 2 & 2 & 0 & 1 & 1 & 2 & 0 \end{bmatrix}$$

If we consider the columns of an OA as treatment combinations, then such an array is a fractional factorial design containing m (not necessarily distinct) treatment combinations from an s^t factorial design. It can be shown (Raktoe, *et.al.*, 1981) that if T is a fractional factorial design based on $OA(m,t,s,d{:}\lambda)$, then we can estimate all the main effects and the interactions up to order k, where k is the greatest integer less than $\frac{d+1}{2}$, and the estimates are mutually independent.

II.2.2 Construction of Parameter Design

The parameter design technique developed by Taguchi can be described by the following 7 steps.

< Step 1 >

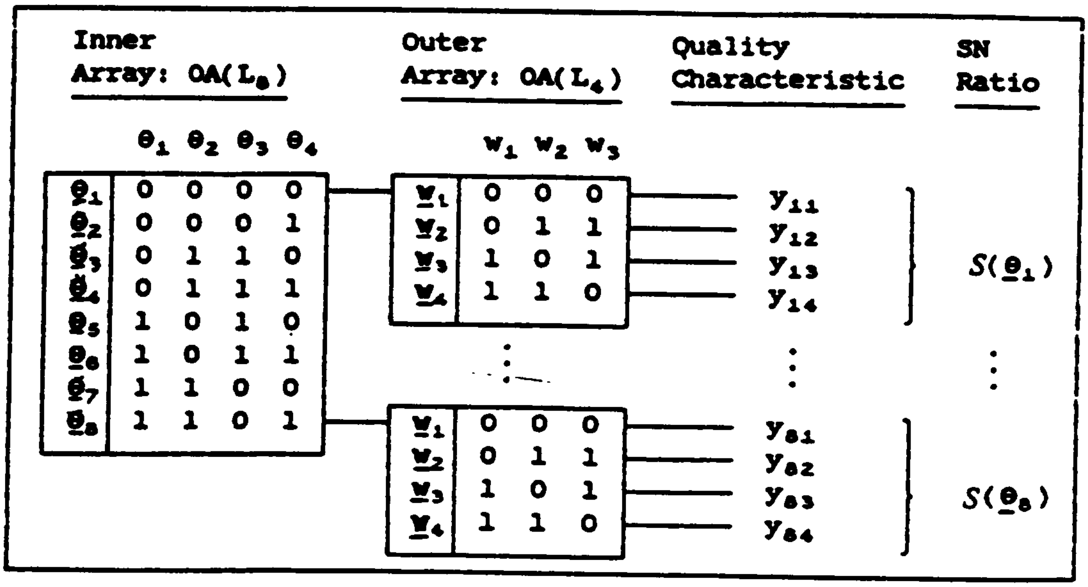
Determine the quality characteristic of a product, and identify the control and noise factors.

< Step 2 >

Determine the initial levels and the range of levels of the control factors. Also identify the range of noise factors and choose the levels to be investigated in the parameter design.

<step 3>

Using orthogonal arrays, construct the designs for the control and noise factors. Taguchi called these designs inner array and outer array,respectively. Figure 2.2 shows how we construct the parameter design. In Figure 2.2, we assign the control factors to columns of the inner array. Each row of the inner array represents the combination of levels (treatment combination, or setting) of the control factors. The columns of an outer array stand for different settings of the noise factors. A complete parameter design experiment consists of a combination of the inner array and the outer array as shown in Figure 2.2. Then, for each setting $\underline{\theta}$ of the control factors, n values of the response (quality characteristic) are generated at different settings $\underline{w}$ of the noise factors, in which n is the number of rows in the outer array for the noise factors.



< Figure 2.2: Block Diagram of Parameter Design for 4 Control Factors and 3 Noise Factors with 2 Levels >

<Step 4>

Carry out the experiment, and for each setting $\underline{\theta}$ for the control factors obtain (see section II.2.3) a signal-to-noise (*SN*) ratio by using the *n* actual response values. Note that equation (2.3) essentially says that in order to minimize the expected loss, we have to control the quality characteristic *Y* through both the mean and the variance. Therefore, in the parameter design we need to devise an objective measure which takes these two parameters into account. Taguchi called such an objective measure an *SN* ratio. The *SN* ratio is a quantity whose maximization is equivalent to minimization of the expected loss (2.3). We shall discuss the *SN* ratios in detail in section II.2.3.

<Step 5>

Perform the standard ANOVA procedure using *SN* ratios as response values and estimate the main effects for the control factors. Moreover, decompose the control factors $\underline{\Theta}$ into two groups $\underline{\Theta}_1$ and $\underline{\Theta}_2$, in which $\underline{\Theta}_2$ is a subset of the control factors which have non-significant effects on the *SN* ratio. We call the control factors in $\underline{\Theta}_2$ adjustment factors (Taguchi called them signal factors). It is sometimes argued that we do not have to test the significance of control factors in the ANOVA procedure since we must identify the optimal level of each control factor whether it has a significant effect on the *SN* ratio or not.

<Step 6>

For each control factor, find the level at which the *SN* ratio is maximized. If the mean response of the quality charateristic deviates from the target value at the combination of these identified levels of the control factors, then make the mean response close to the target value by manipulating the adjustment factors, and finally identify the optimal setting $\underline{\theta}^* = (\underline{\theta}_1^*, \underline{\theta}_2^*)$ of the control factors, at which the *SN* ratio is maximized with the mean response of the quality characteristic close to the target value. The ANOVA procedure using the mean of *n* values of quality characteristics as a response may be helpful in identifying the most efficient adjustment factor. If such a supplementary ANOVA is conducted, then in the correction procedure of the mean response we use an adjustment factor which has the most significant effect on the mean. Note here that since the adjustment factors have non-significant effects on the *SN* ratio, we can manipulate the adjustment factors without influencing the *SN* ratio.

<Step 7>

Conduct a new experiment to confirm that the identified optimal setting indeed improves the *SN* ratio. If there is no improvement in the first parameter design procedure, then iterate the procedure using the optimal setting $\underline{\theta}^*$ identified in Step 6 as the initial setting of control factors in Step2. Figure 2.3 summarizes the seven steps of the parameter design analysis in a block diagram.

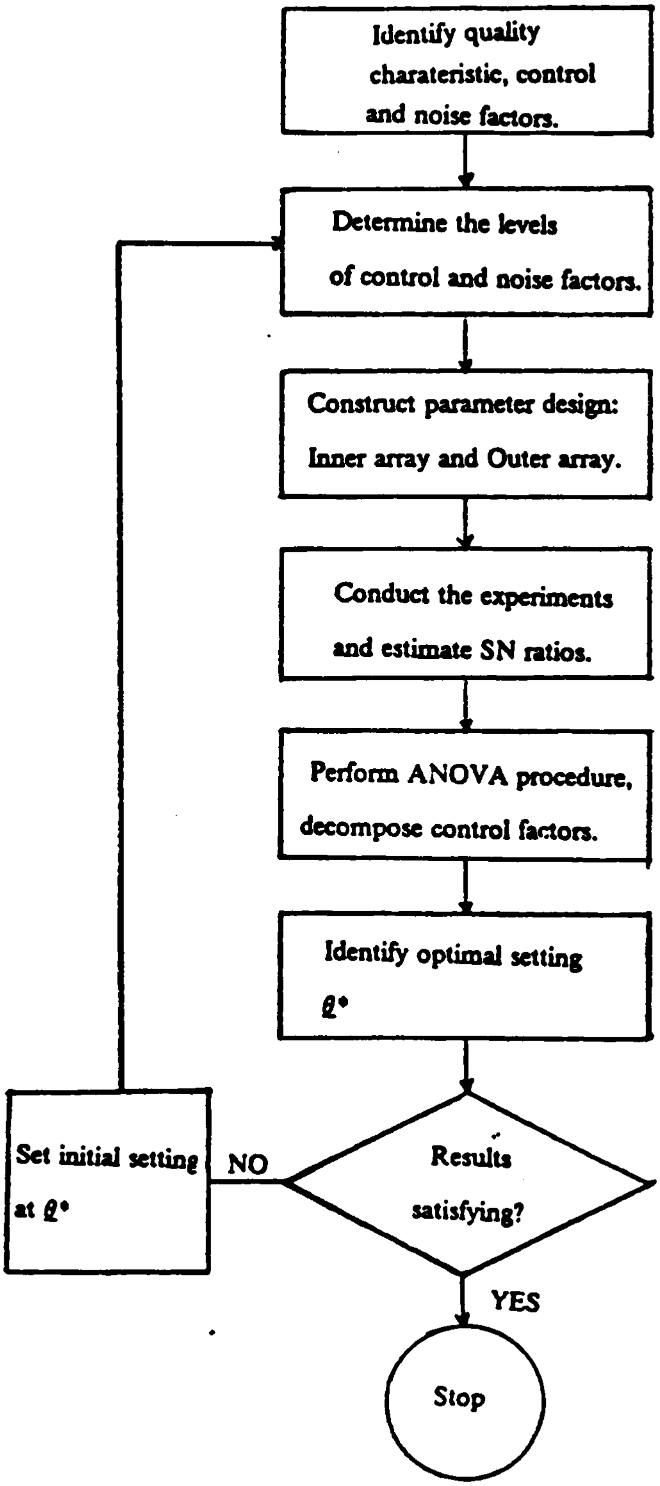
II.2.3 SN Ratios

As we mentioned before, in the parameter design we analyze *SN* ratios whose maximization is equivalent to minimization of the expected loss (2.3). The *SN* ratio estimates the effects of the noise factors on the quality characteristic. An efficient *SN* ratio takes advantage of the prior engineering knowledge about the product, the loss function, and the property of the quality characteristic. Depending on the property of the target value, Taguchi suggested the following *SN* ratios.

(1) Nominal is the Best

Here the quality characteristic is continuous and the nominal target value, say $\tau = \tau_0$, is specified and the expected loss increases as Y deviates from τ_0 in either direction. For this case, we have to control the variance and the bias simultaneously. Frequently the mean and the variance are functionally dependent so that as the mean decreases the variance also decreases and vice versa. In such cases, we cannot minimize first the variance and next reduce the bias by changing the mean to the target value. However, it may be possible to reduce the bias independently of the coefficient of variation $\frac{\sigma}{\mu}$. Therefore, we minimize first the *SN* ratio based on the coefficient variation, and then reduce the bias by manipulating the adjustment factors.

For this case, Taguchi recommends the following *SN* ratio:



< Figure 2.3: Block Diagram of Analysis of Parameter Design >

$$\eta(\underline{\theta}) = 10 \log \left(\frac{\mu^2}{\sigma^2}\right) \tag{2.4}$$

,where μ and σ^2 are expected value and variance of Y, respectively, and the base of log is 10.

As an estimate of (2.4) at the i-th setting $\underline{\theta}_i$, we use

$$S(\underline{\theta}_i) = 10 \log \left(\left(\frac{\bar{y}_i}{S_i}\right)^2\right) \quad i = 1, 2, \ldots, m \tag{2.5}$$

,where $\bar{y}_i$ and S_i^2 are the usual estimate of μ and σ^2, respectively.

(2) The Smaller the Better

Here the quality characteristic is a continuous, positive and the target value is zero. Then the expected loss (2.3) turns out to be

$$R(\underline{\theta}) = kE_{\underline{W}}(y^2) \ . \tag{2.6}$$

For this case, the SN ratio at the setting $\underline{\theta}$ is

$$\eta(\underline{\theta}) = -10 \log \left(E_{\underline{W}}(y^2)\right) \ . \tag{2.7}$$

As a moment method estimate of (2.7) at the i-th setting $\underline{\theta}_i$, we use in practice

$$S(\underline{\theta}_i) = -10 \log \left[\frac{\sum_{j=1}^{n} y_{ij}^2}{n}\right], \quad i = 1, 2, \ldots, m \tag{2.8}$$

,where m, n are the numbers of rows of the inner array and outer array, respectively.

(3) The Larger the Better

This is the case of a continuous quality characteristic which we want to be as large as possible. If we take the reciprocal $1/Y$, then we have the same situation as in the case of (2). The *SN* ratio is defined as

$$\eta(\underline{\theta}) = -10 \log_{10}\left[E_{\underline{W}}\left[\left(\frac{1}{y}\right)^2\right]\right]. \tag{2.9}$$

In practice, we use the following estimate of (2.9).

$$S(\underline{\theta}_i) = -10 \log\left[\sum_{j=1}^{n}\left(\frac{1}{y_{ij}}\right)^2/n\right], \quad i = 1, 2, \ldots, m. \tag{2.10}$$

When the target value is equal to infinity (zero), it is impossible to fit the quality characteristic exactly to the target value by manipulating adjustment factors in the parameter design. Therefore, for the cases of (2) and (3), there exist no adjustment factors in the true sense. But we can make the mean response of the quality characteristic the largest (smallest) as possible.

When the nominal value of the target value is specified, Box (1988) discussed the *SN* ratio (2.4), and showed that it was appropriate only when the standard deviation σ and the mean μ were linked through a linear function. If σ increases proportionally with μ^p, then the suitable *SN* ratio is

$$\eta(\underline{\theta}) = 10 \log\left[\left(\frac{\mu^p}{\sigma}\right)^2\right]. \tag{2.11}$$

As a special case, if σ and μ are functionally independent, *i.e.* $p = 0$, then the suitable *SN* ratio is

$$\eta(\underline{\theta}) = -10 \log(\sigma^2). \tag{2.12}$$

Therefore, information on the extent of dependency between the mean and the variance is necessary to devise the appropriate *SN* ratios. If they turn out to be dependent, Box (1988) suggested to transform the data to achieve independency. After transformation, we can then use the *SN* ratio (2.12) for the transformed data.

Leon, *et.al.* (1987), also discussed the *SN* ratio (2.4) from the point of view of the transfer function. They showed that if the transfer function is a particular multiplicative transfer model, *i.e.* the transfer function is of the form

$$y = \psi(\underline{\theta}_1,\ \underline{\theta}_2)\ \varepsilon(\underline{W},\ \underline{\theta}_1) \tag{2.13}$$

,where $E(\varepsilon(\underline{W},\ \underline{\theta}_1)) = 1$ and $E(Y) = \psi(\underline{\theta}_1,\ \underline{\theta}_2)$ is a strictly monotone function of each component of $\underline{\theta}_2$ for each $\underline{\theta}_1$, then the maximization process of the *SN* ratio (2.4) is equivalent to the minimization of expected loss (2.3).

However, for the additive transfer function

$$y = \psi(\underline{\theta}_1,\ \underline{\theta}_2)\ +\ \varepsilon(\underline{W},\ \underline{\theta}_1) \tag{2.14}$$

,where $E(\varepsilon(\underline{W},\ \underline{\theta}_1)) = 0$, they showed that (2.12) is the suitable *SN* ratio whose maximization is equivalent to the minimization of (2.3).

II.3 Example of Parameter Design Analysis

As a simple example of a parameter design, Pignatiello and Ramberg (1985) used it in the manufacturing process of leaf springs for trucks. An important quality characteristic of a leaf spring is the free height of a spring in the unloaded condition. The target value here is 8 inches.

In this production process there are 4 control factors: (A) furnace temperature, (B) heating time, (C) transfer time, and (D) hold-down time. They found that the quenching oil temperature, which was too expensive to control, caused variation in the free height of the leaf spring. Therefore, Pignatiello and Ramberg chose the quenching oil temperature (O) as a noise factor. Table 2.1 shows the two levels of each factor studied in the experiment.

They used an orthogonal array of strength 3 with 8 runs for constructing the inner array (see Table 2.2) and replicated each test run of the inner array three times at each level of the noise factor. The free height data and corresponding *SN* ratios are given in Table 2.3. Since this orthogonal array can estimate all the main effects and three 2-factor interactions under the assumption that 3-factor and higher order interactions are all negligible, they included 2-factor interactions AB, AC, and BC. Since there are no degrees of freedom for error, they pooled two small sums of squares, $SS(A)$ and $SS(AB)$, to obtain an error sum of squares.

From the ANOVA procedure based on *SN* ratios (see Table 2.4), we can see that two factors A and D are adjustment factors, and one interaction BC is significant at the $\alpha = 0.1$ level. Since estimated effects (see Table 2.5) of A and D are negative and positive, respectively, the *SN* ratio is maximized at the levels A^- and D^+. For interacting factors, we can identify the optimal combination of levels by evaluating the average *SN* ratio at four possible combinations of levels. Figure 2.4 shows that the average *SN* ratio is maximized at the combination (B^+, C^-). Therefore, the *SN* ratio is maximized at the setting $\theta = (A^-, B^+, C-, D+)$. If the mean response of the free height at this setting differs from the target value of 8 inches, then we adjust the mean response by manipulating the factor A.

< Table 2.1: Control and Noise Factors >

Letter	Factor	Levels Low(-)	High(+)
A	Furnace temperature (°F)	1840	1880
B	Heating Time (seconds)	25	23
C	Transfer Time (seconds)	12	10
D	Hold-down Time (seconds)	2	3
O	Quenching Ouil Temp. (°F)	130-150	150-170

< Table 2.2: Orthogonal Array with 8 Runs >

Factor / Test run	A	B	C	D	AB	AC	BC
1	−	−	−	−	+	+	+
2	+	−	−	+	−	−	+
3	−	+	−	+	−	+	−
4	+	+	−	−	+	−	−
5	−	−	+	+	+	−	−
6	+	−	+	−	−	+	−
7	−	+	+	−	−	−	+
8	+	+	+	+	+	+	+

< Table 2.3: Data and Summary Statistics >

Test run	O^-	O^+	$\bar{Y}$	S^2	SN Ratio $10\log(\bar{Y}/S)^2$
1	7.78 7.78 7.81	7.50 7.25 7.21	7.54	.09	28.00
2	8.15 8.18 7.88	7.88 7.88 7.44	7.90	.07	29.46
3	7.50 7.56 7.50	7.50 7.56 7.50	7.52	.001	47.70
4	7.59 7.56 7.75	7.63 7.75 7.56	7.64	.01	38.68
5	7.94 8.00 7.88	7.32 7.44 7.44	7.67	.09	28.11
6	7.69 8.09 8.06	7.56 7.69 7.62	7.79	.05	30.59
7	7.56 7.62 7.44	7.18 7.18 7.25	7.37	.04	31.55
8	7.56 7.81 7.69	7.81 7.50 7.59	7.66	.02	35.31

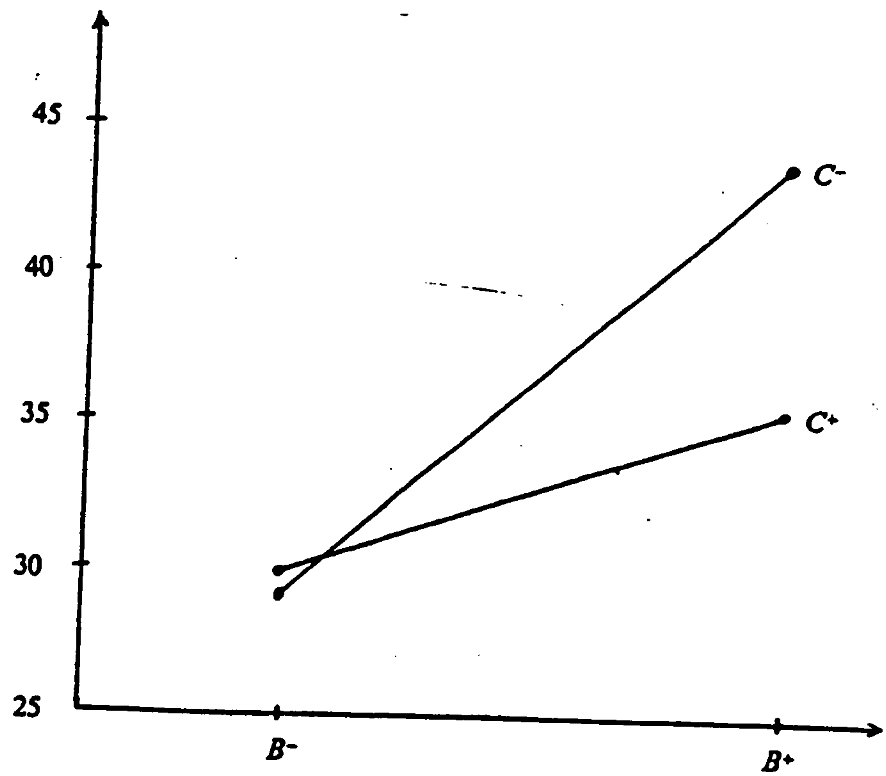
< Table 2.4: ANOVA Table for SN Ratio >

Source of Variation	SS	df	F	P
A^*	.224	1	–	–
B	171.81	1	31.80	.03
C	41.742	1	7.72	.11
D	17.293	1	3.20	.21
AB^*	10.585	1	–	–
AC	23.826	1	4.40	.17
BC	53.862	1	9.96	.09
Error	10.809	2		
Total	319.342	7		

* : $SS(A)$, $SS(AB)$ are pooled to get the error sum of squares.

< Table 2.5: Estimated Main Effects of Control Factors >

Control Factors	Estimated Effects
A	-.335
B	9.269
C	-4.569
D	2.941



< Figure 2.4: Plot of BC Interaction Effect on SN Ratio >

II.4 Influence of Non-negligible Interaction Effects

As mentioned in section II.2, Taguchi used orthogonal arrays, in particular OA's of strength 2, when constructing the arrays for the parameter design. As we mentioned in section II.2.1, OA's of strength 2 enable us in general to investigate only main effects under the assumption that all interactions are negligible. Taguchi used OA's since OA's can accomodate a maximum number of factors when all interactions are negligible (see Rao, 1946). However, in order to investigate interaction effects using OA's, the number of experimental runs needed becomes unmanageably large as the number of factors increases. For example, we need at least 64 experimental runs to examine 2-factor interaction effects for a 2^7 factorial experiment. Moreover, in the parameter design, if we have 4 rows (treatment combinations) in the outer array, then we need at least $64 \times 4 = 256$ experimental runs. In practical situations, it is not rare to encounter a situation in which we have to include a large number of control factors (say 100) in the parameter design. But the rationale of using OA's is useful only when the interaction effects are all negligible. If there exist some influential interaction effects among the control factors, then the optimal settings in the parameter design can be misleading.

In the parameter design, there are two types of factors, control and noise factors. Therefore there can be three types of interaction effects in the parameter design:

(1) interaction effects between control factors and noise factors;

(2) interaction effects among the noise factors;

(3) interaction effects among the control factors.

Now we will discuss the influence of these interaction effects in more detail.

(1) Interaction effects between the control and noise factors are the basis of the parameter design and necessary for the parameter design technique to be successful. If there exists interaction

between the control and noise factors, then the effects of the noise factors are different at different settings of the control factors. Through the parameter design analysis, we can then find the optimal setting at which the expected loss is minimized.

(2) In the parameter design, it is not important whether interaction effects among the noise factors exist or not. In the parameter design, we generate the response value Y at various settings of the noise factors for each test setting $\underline{\theta}$ of the control factors. Then we calculate SN ratios based on these responses values. Consequently, all effects among the noise factors are accumulated in calculating SN ratios.

(3) Suppose there exist some influential interactions among the control factors, then the correct linear model for the parameter design can be expressed as

$$E(\underline{S}) = X_1\underline{\beta}_1 + X_2\underline{\beta}_2 \tag{2.15}$$

,where $\underline{S}$ is a $(m \times 1)$ vector of SN ratios

$\underline{\beta}_1$ is a vector of the overall mean and main effects of the control factors

$\underline{\beta}_2$ is a vector of influential interaction effects among the control factors

X_1 and X_2 are corresponding model matrices.

Note here that in the parameter design, we include all the main effects in the model whether they are negligible or not. This is because in the parameter design our final goal is to find the optimal setting of each control factor, not to evaluate the model.

In the usual parameter design, $\underline{\beta}_2$ is assumed to be zero. Under the assumption that the error term is distributed independently and identically and this assumption, the BLUE of $\underline{\beta}_1$ is given by

$$\hat{\underline{\beta}}_1 = (X'_1X_1)^{-1}X'_1\underline{S} \ . \tag{2.16}$$

If the correct model is (2.15), then

$$E(\hat{\beta}_1) = \underline{\beta}_1 + A_1\underline{\beta}_2 \ , \tag{2.17}$$

where $A = (X'_1X_1)^{-1}X'_1X_2$ is the alias matrix.

Consequently, there is bias in estimating $\underline{\beta}_1$ with the exact amount of bias depending on X_1 and unknown X_2 , and $\underline{\beta}_2$.

However, equation (2.17) reveals that the bias will be identically zero if and only if the columns of X_1 are orthogonal to the columns of X_2, that is, $X'_1X_2 = \underline{0}$. This condition is equivalent to requiring that in the parameter design, the orthogonal array used for the inner array be one or more copies of a minimal complete factorial design. This, in general, is very inefficient and costly.

We note here that in the parameter design one of the most important considerations is to estimate the main effects of the control factors accurately. In order to get accurate estimates with small bias , the influential interactions should be included in the parameter design. Moreover, the main effect plan is liable to give misleading optimal setting of the control factors in the presence of influential interactions among the control factors for the following reasons:

(i) When there exists interaction between a significant control factor and an adjustment factor, the final setting $\underline{\theta}^*$ may not be optimal since we manipulate the adjustment factor. In other words, if the adjustment factor interact with a significant control factor, then the adjustment procedure of the mean response cannot be done without influencing the *SN* ratio (See Hunter, 1985).

(ii) The ANOVA results of main effect plan are not reliable in the presence of interactions among the control factors.

(iii) For the interacting control factors, the identified levels in the main effect plan may not be optimal since the effect due to one factor is influenced by the settings of the other factors.

Therefore, in order for the parameter design technique to be successful, the influential interaction effects among control factors should be detected and accounted for properly.

Chapter III

DETECTING INTERACTIONS IN 2-LEVEL FACTORIAL PARAMETER DESIGNS

New designs which can be used as an inner array for a 2^r factorial parameter design are developed in this chapter. These designs are intended to detect 2-factor interactions among the control factors with a minimum number of experimental runs. More specifically,

(i) the construction of new designs which allow detection and estimation of influential 2-factor interactions is developed assuming that 3-factor and higher order interactions are all negligible;

(ii) optimality criteria for the new designs are discussed;

(iii) we will discuss detection procedures for influential 2-factor interactions and introduce new detection procedures;

(iv) the analysis of the new 2^r factorial parameter design is discussed taking influential 2-factor interactions into account.

III.1 2-Level Factorial Designs

A 2^t factorial design is an experiment which involves t factors, each having two levels. In a 2^t factorial design, a treatment combination is denoted by a $(t \times 1)$ vector $\underline{x}' = (x_1, ..., x_t)$ with elements x_i from the Galois field of order 2, $GF(2)$, where 0 and 1 of $GF(2)$ represent the 2 levels of each factor. There is an one-to-one correspondence between 2^t treatment combinations and 2^t points (t-tuples over $GF(2)$) in the finite Euclidean geometry of order t,$EG(t,2)$. For a 2^t factorial experiment we express each main effect and interaction as

$$F_1^{\alpha_1} F_2^{\alpha_2} ... F_t^{\alpha_t} \tag{3.1}$$

with $\alpha_i = 0$, or 1 and $F_i^0 = 1$, $F_i^1 = F_i$ and we ignore all unity elements in this expression. For short we write (3.1) as $E^{\underline{\alpha}}$ where $\underline{\alpha}' = (\alpha_1, \alpha_2, ..., \alpha_t)$ is called a defining vector. If all $\alpha_i = 0$, then (3.1) is written as μ, the overall mean of the experiment. For example, if all $\alpha_i = 0$ except i = 1, then (3.1) reduces to F_1 which represents the main effect of factor F_1, and if all $\alpha_i = 0$ except i = 1 and 2, then (3.1) represent the interaction effect $F_1 F_2$.

For a 2^t factorial design, one degree of freedom is associated with each main effect and interaction and it is represented geometrically by a contrast of the 2 subsets generated by the pencil in the $EG(t,2)$. That is, for an effect $E^{\underline{\alpha}} = F_1^{\alpha_1} F_2^{\alpha_2} ... F_t^{\alpha_t}$, the one degree of freedom can be represented by a contrast of 2 sets of treatment combinations generated by the pencil

$$\underline{\alpha}' \underline{x} = i$$

$$\text{i.e.} \quad \alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_t x_t = i \pmod 2 \qquad i = 0 \text{ or } 1 . \tag{3.2}$$

And the effect $E^{\underline{\alpha}}$ can be expressed as

$$E^{\underline{\alpha}} = (-1)^{\Sigma \alpha_i} \cdot (E_0^{\underline{\alpha}} - E_1^{\underline{\alpha}}) , \tag{3.3}$$

where E_i^{α}, i = 0,1 are the true effects at level i of E^{α} defined by

$$E_i^{\alpha} = (\text{true mean response of treatment combinations satisfying } \underline{\alpha}'\underline{x} = i) - \mu \ .$$

Now, we can infer from (3.3) that if in $F_1^{\alpha_1}F_2^{\alpha_2}...F_t^{\alpha_t}$ the sum $\sum_{i=1}^{t}\alpha_i$ is odd, then all treatment combinations with $\sum_{i=1}^{t}\alpha_i x_i$ odd, i.e. $\underline{\alpha}'\underline{x} = 1$ (mod 2), enter positively in estimating $F_1^{\alpha_1}F_2^{\alpha_2}...F_t^{\alpha_t}$ and those with $\sum_{i=1}^{t}\alpha_i x_i$ even, i.e. $\underline{\alpha}'\underline{x} = 0$ mod(2), enter negatively. If on the other hand $\sum_{i=1}^{t}\alpha_i$ is even, then all treatment combinations satisfying $\underline{\alpha}'\underline{x} = 0$ (mod 2) enter positively, and those satisfying $\underline{\alpha}'\underline{x} = 1$ mod(2) enter negatively.

For a 2^t factorial design where all possible treatment combinations are taken, we can estimate all main effects and interactions. As the number of factors in a 2^t factorial design increases, the number of treatment combinations for a completely replicated design rapidly increases so that the experiment outgrows the resources in most cases. If the experimenter can reasonably assume that certain higher-order interactions are negligible, then information on the main effects and the lower-order interactions can be obtained by taking only a suitably chosen fraction of treatment combinations of the completely replicated factorial experiment. These fractional factorial designs are widely used in many industrial experiments.

From fractional factorial designs, we cannot estimate all main effects and interactions since the number of treatment combinations in the fractional factorial designs is less than the degrees of freedom associated with all the main effects and interactions.

< Definition 3.1 >

A fractional factorial design is said to be saturated, if the number of treatment combinations in the design is equal to the degrees of freedom associated with the effects to be estimated.

The estimability of effects can be used as a basis for classifying fractional factorial designs. Following the terminology introduced by Box and Hunter (1961) we define the "resolution" of designs.

< Definition 3.2 >

A fractional design is said to be of resolution r if all interactions up to k factors are estimable, where k is the greatest integer less than $r/2$, under the assumption that all interactions of order $r-k$ and higher are negligible.

This definition establishes the resolution of a design in terms of estimability. In particular,

1. A resolution III design is one where all main effects are estimable under the assumption that 2-factor and higher order interactions are all negligible.
2. A resolution IV design is one where all main effects are estimable in the presence of 2-factor interactions which may or may not be negligible, under the assumption that 3-factor and higher order interactions are all negligible.
3. A resolution V design is one where all main effects and 2-factor interactions are estimable under the assumption that 3-factor and higher order interactions are all negligible.

Using the usual notation, we can write the model equation for a 2^t factorial design which is either a complete factorial design or fractional factorial design,

$$E(\underline{y}) = X\underline{\beta} \ , \tag{3.4}$$

where $\underline{y}$ is a vector of responses associated with treatment combinations in the design and $\underline{\beta}$ is a vector of parameters (effects), and X is a model matrix.

We can construct the model matrix X in the following way: The first column of X is a vector with every element one corresponding to the overall mean μ and the column of X which corresponds to effect $F_1^{x_1}F_2^{x_2}\ldots F_t^{x_t}$ has elements 1 or -1, according as the treatment combination enters, respectively, positively or negatively in the estimation of the effect.

< Example 3.1 >

Consider a 2^3 fractional factorial design where we take only 4 treatment combinations

(0,0,0) (1,1,0) (1,0,1) (0,1,1).

If the parameter vector $\underline{\beta}$ includes overall mean and all main effects and interactions, then we can express

$$E\begin{bmatrix} Y_{(0,0,0)} \\ Y_{(1,1,0)} \\ Y_{(1,0,1)} \\ Y_{(0,1,1)} \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 \\ 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 \end{bmatrix} \underline{\beta}\,, \tag{3.5}$$

where $\underline{\beta}' = (\mu, F_1, F_2, F_3, F_1F_2, F_1F_3, F_2F_3, F_1F_2F_3)$ and $Y_{(x_1,x_2,x_3)}$ denotes the response corresponding to the treatment combinations (x_1, x_2, x_3). From this example we can see that each column associated with an interaction effect in the model matrix can be obtained simply by componentwise multiplication of columns corresponding to the main effects involved in the interaction.

A 2^t factorial parameter design is essentially the same as the usual 2^t factorial design as far as the structure is concerned except that in the parameter design an appropriate outer array is combined with each test setting (treatment combination of t control factors) of a 2^t factorial design and the response to be analyzed is an *SN* ratio.

For a 2^t factorial parameter design, each noise factor can have any number of levels which is not necessarily the same for each noise factor. As we discussed in the previous chapter, in the parameter design analysis we can use any design as an outer array, for example an orthogonal array, provided that the test settings of such a design can cover the possible range of all noise factors and the number of runs is small. Therefore, in this thesis we assume that an appropriate outer array is combined with each treatment combination of an inner array so that a reasonable *SN* ratio is observed. A 2^t factorial parameter design is particularly useful as a factor screening design in the early stages of the parameter design analysis, when there are likely many control factors to be investigated.

III.2 Construction of Designs for Inner Arrays

Consider a 2^t factorial parameter design in the partitioned form

$$E(\underline{S}) = X_1\underline{\beta}_1 + X_2\underline{\beta}_2 \tag{3.6}$$

with $Cov(\underline{S}) = \sigma^2 I$ where $\underline{S}$ is a vector of observed *SN* ratios

$\underline{\beta}_1$ is a vector of the overall mean and main effects of control factors

$\underline{\beta}_2$ is a vector of interaction effects among the control factors, and

X_1 and X_2 are corresponding model matrices whose elements are 1 or -1.

Then in the parameter design analysis, we want to estimate all elements of $\underline{\beta}_1$ and k non-negligible elements of $\underline{\beta}_2$ and also we want to find k, a number of non-negligible interactions. If we identify k, then for a given design T of an inner array, we have the following family D_T of competing models,

$$E(\underline{S}_T) \in D_T = \{X_1\underline{\beta}_1 + X_2{}^*\underline{\gamma}_2 : \underline{\gamma}_2 \in P_k(\underline{\beta}_2)\} \tag{3.7}$$

,where $\underline{S}_T$ is a vector of observed *SN* ratios for the inner array T

$\underline{\gamma}_2$ is $(k \times 1)$ vector of non-negligible elements of $\underline{\beta}_2$

$P_k(\underline{\beta}_2)$ is the set of all subsets of $\underline{\beta}_2$ with cardinality k

$X_2{}^*$ is a sub-matrix of X_2 corresponding to $\underline{\gamma}_2$, and

X_1, $\underline{\beta}_1$, and $\underline{\beta}_2$ are defined in (3.6).

We assume that in this thesis $\underline{\beta}_2$ contains only 2-factor interactions, that is, 3-factor and higher order interactions are assumed to be all negligible. It is a well known fact that in practical experiments the higher order interactions tend to be negligible and the most important interactions to be investigated are 2-factor interactions (Daniel, 1973). Moreover, in the parameter design analysis, if the higher order interactions are present, then the situation where all control factors have an influence on the *SN* ratio may occur easily, so that no adjustment factors exist. If there is no adjustment factor, we cannot adjust the mean response of a quality characteristic to the target value when such adjustment is necessary. Therefore, the assumption of a 2-factor interaction model has a plausible rationale in the context of parameter design analysis.

The theory and methodology for constructing designs which make it possible to draw some inferences about non-negligible interactions were first introduced in the form of "search design" by Srivastava (1975) and then extended by Srivastava and Ghosh (1976, 1977) and Srivastava and Gupta (1979), and Ghosh (1981) for 2^t factorial designs for some range of t. Most of the work about search designs assumes that the number of influential effects is known, in particular $k=1$ or 2. Moreover, it is assumed that there is no experimental error, i.e. $\sigma^2 = 0$ in (3.6). Recently, Ghosh (1987) developed a search procedure for the noisy case where $\sigma^2 > 0$.

In the parameter design analysis, the most important consideration is that the analysis should be cost-effective (Taguchi and Wu, 1980). That is, we should identify the optimal setting of the control factors with a minimum cost. In some practical situations, experimental runs (treatment combinations) may be difficult to make or expensive to apply. Moreover, in the parameter design analysis, if there are m rows (treatment combinations of control factors) in the inner array, then we need a total of $m \times n$ experimental runs, where n is the number of rows (test settings of noise factors) of the outer array. Therefore, for constructing designs which will be used as an inner array we shall develop designs with a minimum number of treatment combinations. We shall now develop designs which allow us to detect any number of non-negligible 2-factor interactions in the model (3.6). The detection procedures for the correct model in (3.7) will be developed in section III.4. Before we embark on the construction of the inner array, we shall present some definitions.

< Definition 3.3: (Srivastava, 1965) >

A fractional factorial design is called a balanced design if the variance-covariance matrix of estimates is invariant under any permutation of the factor symbols.

As an example, for a 2^t fractional factorial design of resolution V, the balancedness means that for any distinct set of 4 integers i,j,k, and l chosen from {1,2,...,t}, the quantities $Var(\hat{F_i})$, $Cov(\hat{\mu}, \hat{F_i})$, $Cov(\hat{\mu}, \hat{F_iF_j})$, $Cov(\hat{F_i}, \hat{F_j})$, $Cov(\hat{F_i}, \hat{F_iF_j})$, $Cov(\hat{F_i}, \hat{F_jF_k})$, $Cov(\hat{F_iF_j}, \hat{F_jF_k})$, and $Cov(\hat{F_iF_j}, \hat{F_kF_l})$ are independent of the subscripts i,j,k, and l, where $\hat{\mu}$, $\hat{F_i}$, $\hat{F_iF_j}$ denote the estimates of the overall mean, effect of factor F_i, and interaction effect between factor F_i and F_j, respectively. Thus we have, e.g. $Var(\hat{F_1}) = Var(\hat{F_2})$, $Cov(\hat{F_1F_2}, \hat{F_2F_3}) = Cov(\hat{F_3F_4}, \hat{F_1F_4})$, etc. but not necessarily $Cov(\hat{F_1}, \hat{F_2}) = Cov(\hat{F_1}, \hat{F_1F_2})$, $Var(\hat{F_1}) = Cov(\hat{F_1F_2})$ and $Cov(\hat{F_1F_2}, \hat{F_2F_3}) = Cov(\hat{F_1F_2}, \hat{F_3F_4})$, etc.

< Definition 3.4: (Chakravarti, 1956) >

A $(t \times m)$ array (matrix) where each column represents a treatment combination of t factors each with s levels is called a partially balanced array (PBA) of strength d if, with respect to any d of the t rows (factors), the treatment combinations $(i_1, i_2, ..., i_d)$ occurs $\lambda(i_1, i_2, ..., i_d)$ times, where $\lambda(i_1, i_2, i_3, ..., i_d)$ remains the same for all permutations of a given set $(i_1, i_2, ..., i_d)$. The set of all such numbers, as $i_1, i_2, ..., i_d$ runs over all s^t treatment combinations is called the set of index numbers of the array, or simply index set and is denoted by Λ. And we denote the array by $PBA(m,t,s,d{:}\Lambda)$.

It is customary to denote $\lambda(i_1, i_2, ..., i_d)$ in a partially balanced array with 2 symbols (levels) in which two symbols coded by 0 and 1, by λ_w where w is equal to the numbers of ones in $(i_1, i_2, ..., i_d)$. w is called the weight of $(i_1, i_2, ..., i_d)$ and the index set Λ is completely specified by $\Lambda = (\lambda_0, \lambda_1, ..., \lambda_d)$ in this case.

From this definition we can easily see that a $PBA(m,t,s,d_1{:}\Lambda_1)$ is also a $PBA(m,t,s,d_2{:}\Lambda_2)$ where $d_1 \geq d_2$, $\Lambda_1 = (\lambda_1, \lambda_2, ..., \lambda_{d_1})$ and $\Lambda_1 = (\lambda_1, \lambda_2, ..., \lambda_{d_2})$ and a partially balanced array reduces to an orthogonal array when all index numbers are equal. Therefore, it can be said that a partially balanced array is a generalized orthogonal array in a certain sense.

It can be shown (Srivastava, 1965) that a balanced 2^t fractional factorial design of resolution V is identical to a partially balanced array of strength 4 with 2 symbols, i.e. $PBA(m,t,2,4{:}\Lambda)$. Using this result and the algebraic structure of a partially balanced array, Srivastava and Chopra (1971a), and Chopra and Srivastava (1973a, 1973b, 1974, 1975) and Chopra (1975a, 1975b, 1977a, 1977b, 1979, 1983) tabulated some optimal 2^t balanced fractional factorial designs of resolution V having minimum value for the trace of the variance-covariance matrix of the estimates. But they listed the designs only for $4 \le t \le 10$ and some range of number of treatment combinations m for each t.

Now, for any number of factors t we can construct the following saturated 2^t fractional factorial designs of resolution V.

< Proposition 3.1 >

Consider a 2^t factorial experiment for $t \ge 4$, where the treatment combinations are denoted by $(x_1, x_2, ..., x_t)$ with $x_i = 0$ or 1 $(i = 1,2, ..., t)$. The following eight designs indexed by (s_1, s_2, s_3)

$$T_{(s_1,\, s_2,\, s_3)} = \{(x_1,\ x_2,\ \dots,\ x_t) \mid \sum_{i=1}^{t} x_i = s_1, \text{ or } s_2, \text{ or } s_3\}\ , \qquad (3.9)$$

$$\text{where } \begin{aligned} s_1 &= 0 \text{ or } t \\ s_2 &= 1 \text{ or } t-1 \\ s_3 &= 2 \text{ or } t-2 \end{aligned}$$

are minimal (saturated) balanced 2^t fractional factorial designs of resolution V.

< PROOF >

Since a balanced 2^t fractional factorial design of resolution V is identical with a PBA of strength 4, we shall first show that each design $T_{(s_1, s_2, s_3)}$ is a PBA of strength 4.

Consider any design $T_{(s_1, s_2, s_3)}$ written in matrix form in which each treatment combination is represented by a column. Then $T_{(s_1, s_2, s_3)}$ is a $(t \times m)$ matrix consisting of $\binom{t}{s_1}$ columns with exactly s_1 1's, $\binom{t}{s_2}$ columns with exactly s_2 1's, and $\binom{t}{s_3}$ columns with exactly s_3 1's, *i.e.* $m = 1 + t + \binom{t}{2}$. Therefore, the design $T_{(s_1, s_2, s_3)}$ is a PBA of strength t (full strength). Since a PBA of strength d_1 is also a PBA of strength $d_2 \le d_1$, we can conclude that the design $T_{(s_1, s_2, s_3)}$ is a PBA of strength 4.

Moreover, for any submatrix consisting of 4 rows of $T_{(s_1, s_2, s_3)}$, we can see that the number of columns in the submatrix that contain q 1's will be γ_q, where

$$\gamma_q = \binom{r}{q} \sum_{j=1}^{3} \binom{t-r}{s_j - q} \ .$$

Consequently, $T_{(s_1, s_2, s_3)}$ is a $PBA(m,\ t,\ 2,\ 4\colon\ \Lambda)$, where

$$\Lambda = (\lambda_0,\ \lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4),\quad \lambda_0 = \sum_{j=1}^{3} \binom{t-4}{s_j},\quad \lambda_1 = \sum_{j=1}^{3} \binom{t-4}{s_j - 1}$$

$$\lambda_2 = \sum_{j=1}^{3} \binom{t-4}{s_j - 2},\quad \lambda_3 = \sum_{j=1}^{3} \binom{t-4}{s_j - 3},\quad \text{and}\quad \lambda_4 = \sum_{j=1}^{3} \binom{t-4}{s_j - 4} \ .$$

Moreover, since the minimum number of treatment combinations for a 2^t factorial experiment of resolution V is $m = 1 + t + \binom{t}{2}$, it follows that the design $T_{(s_1, s_2, s_3)}$ is a saturated balanced design of resolution V.

Q.E.D.

To construct, for example, the design indexed by (0,1,2), we simply take one treatment combination in which all levels are 0's and t different treatment combinations in which any one level is 1, and t(t-1)/2 different treatment combinations in which any two levels are 1's. And for construction the design indexed by (t,t-1,t-2), we take one treatment combination, each level being 1 and t different treatment combinations in which any one level is 0, and t(t-1)/2 different treatment combinations in which any two levels are 0's.

< Example 3.2 >

(i) For $t = 5$, $T_{(0,1,2)}$ is given as

$$\begin{bmatrix} 0&1&0&0&0&0&1&1&1&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&1&0&0&0&1&1&1&0&0&0 \\ 0&0&0&1&0&0&0&1&0&0&1&0&0&1&1&0 \\ 0&0&0&0&1&0&0&0&1&0&0&1&0&1&0&1 \\ 0&0&0&0&0&1&0&0&0&1&0&0&1&0&1&1 \end{bmatrix}.$$

(ii) For $t = 6$, $T_{(6,1,2)}$ is given as

$$\begin{bmatrix} 1&1&0&0&0&0&0&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0 \\ 1&0&1&0&0&0&0&1&0&0&0&0&1&1&1&1&0&0&0&0&0&0 \\ 1&0&0&1&0&0&0&0&1&0&0&0&1&0&0&0&1&1&1&0&0&0 \\ 1&0&0&0&1&0&0&0&0&1&0&0&0&1&0&0&1&0&0&1&1&0 \\ 1&0&0&0&0&1&0&0&0&0&1&0&0&0&1&0&0&1&0&1&0&1 \\ 1&0&0&0&0&0&1&0&0&0&0&1&0&0&0&1&0&0&1&0&1&1 \end{bmatrix}.$$

(iii) For $t = 6$, $T_{(0,5,4)}$ is given as design of resolution V indexed by $(0, t-1, t-2)$ is

$$\begin{bmatrix} 0&0&1&1&1&1&1&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1 \\ 0&1&0&1&1&1&1&0&1&1&1&1&0&0&0&0&1&1&1&1&1&1 \\ 0&1&1&0&1&1&1&1&0&1&1&1&0&1&1&1&0&0&0&1&1&1 \\ 0&1&1&1&0&1&1&1&1&0&1&1&1&0&1&1&0&1&1&0&0&1 \\ 0&1&1&1&1&0&1&1&1&1&0&1&1&1&0&1&1&0&1&0&1&0 \\ 0&1&1&1&1&1&0&1&1&1&1&0&1&1&1&0&1&1&0&1&0&0 \end{bmatrix}.$$

This design is the dual one of $T_{(6,1,2)}$ in the sense that each treatment level 1 in $T_{(6,1,2)}$ changes to 0, and every 0 changes to 1.

These designs, as Srivastava and Chopra (1971a) said, possess the same kind of advantage over an unbalanced one, as a balanced incomplete block design does over an unbalanced or partially balanced incomplete block design. These balanced designs also lead to ease in the analysis, interpretation or usage of results since the variance-covariance matrix of the estimates has some even structure. For example, we can estimate all the main effects with the same precision and all the 2-factor interactions with the same precision. But the two variances are not necessarily equal. Therefore, these balanced fractional factorial designs of resolution V are now proposed to be used as an inner array in a 2^t factorial parameter design in which 2-factor interactions of control factors should be investigated. Next, we shall consider some optimality criteria for selecting a proper design among the eight possible balanced fractional factorial designs of resolution V.

III.3 Optimality Criteria for Proposed Designs

Under the general linear model

$$\begin{aligned} E(\underset{\sim}{y}) &= X\underset{\sim}{\beta} \\ \text{with } Var(\underset{\sim}{y}) &= \sigma^2 I, \end{aligned} \tag{3.10}$$

if the parameter vector $\underset{\sim}{\beta}$ is estimable then the best linear unbiased estimator and its variance-covariance matrix are

$$\begin{aligned} \hat{\underset{\sim}{\beta}} &= (X'X)^{-1}X'\underset{\sim}{y} \\ Var(\hat{\underset{\sim}{\beta}}) &= \sigma^2(X'X)^{-1}. \end{aligned} \tag{3.11}$$

The matrix $(X'X)$ is called the information matrix of the design and this information matrix depends on the design T. It is natural to choose a design T which produces the inverse matrix of the information matrix as small as possible in some sense. The most common measures of the size of $(X'X)^{-1}$ have generated the following three measures of a design's goodness (Kiefer, 1959).

<**Definition 3.5**>

(i) A design T is called D-optimal (determinant optimal) if the determinant of the matrix $(X'X)^{-1}$ is minimized for design T.

(ii) A design T is called an E-optimal (eigenvalue optimal) if the largest eigenvalue of $(X'X)^{-1}$ is minimized for design T.

(iii) A design T is called an A-optimal (average optimal) if the trace value of $(X'X)^{-1}$ is minimized for design T.

Choosing a design T so as to minimize $|(X'X)^{-1}|$ is equivalent to minimizing the volume of the region within which the true parameter points may lie with a certain probability, and the E-optimality criterion corresponds to a minimax criterion since this criterion minimizes the maximum possible variance of any linear function of parameters. The trace of $(X'X)^{-1}$ is proportional to the average variance of all parameters.

Srivastava and Chopra (1971b) studied the characteristic polynomial of the variance-covariance matrix V which is the inverse of information matrix for a balanced 2^t fractional factorial design of resolution V, i.e. $PBA(m,t,2,4:\Lambda)$ and obtained the following characteristic polynomial as a function of index numbers $\Lambda = (\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4)$.

$$|V - \delta I| = (c_1\delta^3 - c_2\delta^2 - c_3\delta - 1)(c_4\delta^2 - c_5\delta + 1)^p(c_6\delta - 1)^q \ , \qquad (3.12)$$

where

$$
\begin{aligned}
p &= t-1\\
q &= \frac{t(t-3)}{2}\\
c_1 &= \alpha_1^3 - \frac{t(t-1)^2}{2}\,\alpha_3^3 + (3t-5)\alpha_1^2\alpha_3 + \frac{(t-2)(t-3)}{2}\,\alpha_1^2\alpha_5\\
&\quad + \frac{(t-1)(3t-8)}{2}\,\alpha_1\alpha_3^2 + \frac{(t-1)(t-2)(t-3)}{2}\,\alpha_1\alpha_3\alpha_5 - (3t-2)\alpha_1\alpha_2^2\\
&\quad - \frac{(t-1)(t-2)^2}{2}\,\alpha_1\alpha_4^2 - 2(t-1)(t-2)\alpha_1\alpha_2\alpha_4 + 2t\alpha_2^2\alpha_3\\
&\quad - \frac{t(t-2)(t-3)}{2}\,\alpha_2^2\alpha_5 + t(t-1)(t-2)\alpha_2\alpha_3\alpha_4\\
c_2 &= 3\alpha_1^2 - (3t-2)\alpha_2^2 + \frac{(t-1)(3t-8)}{2}\,\alpha_3^2 - \frac{(t-1)(t-2)^2}{2}\,\alpha_4^2\\
&\quad + 2(3t-5)\alpha_1\alpha_3 + (t-2)(t-3)\alpha_1\alpha_5 - 2(t-1)(t-2)\alpha_2\alpha_4\\
&\quad + \frac{(t-1)(t-2)(t-3)}{2}\,\alpha_3\alpha_5\\
c_3 &= 3\alpha_1 + (3t-5)\alpha_3 + \frac{(t-2)(t-3)}{2}\,\alpha_5\\
c_4 &= (\alpha_1-\alpha_3)(\alpha_1-(t-3)\alpha_5+(t-4)\alpha_3) - (t-2)(\alpha_2-\alpha_4)^2\\
c_5 &= 2\alpha_1 + (t-5)\alpha_3 - (t-3)\alpha_5\\
c_6 &= \alpha_1 - 2\alpha_3 + \alpha_5
\end{aligned}
$$

and where the α_i's are given in terms of the λ_i's as follows:

$$
\begin{aligned}
\alpha_1 &= \lambda_0 + 4\lambda_1 + 6\lambda_2 + 4\lambda_3 + \lambda_4\\
\alpha_2 &= (\lambda_4 - \lambda_0) + 2(\lambda_3 - \lambda_1)\\
\alpha_3 &= \lambda_4 - 2\lambda_2 + \lambda_0\\
\alpha_4 &= (\lambda_4 - \lambda_0) - 2(\lambda_3 - \lambda_1)\\
\alpha_5 &= \lambda_0 - 4\lambda_1 + 6\lambda_2 - 4\lambda_3 + \lambda_4 \;.
\end{aligned}
$$

From (3.12) we can see that the variance-covariance matrix V_T for a design T will not have more than six distinct characteristic roots. Let us denote these roots by $\delta_1, \delta_2, \delta_3$ for the cubic equation

$$c_1\delta^3 - c_2\delta^2 - c_3\delta - 1 = 0 \qquad (3.13)$$

then the following relationships hold

$$\begin{aligned}\delta_1 + \delta_2 + \delta_3 &= c_2/c_1 \\ \delta_1 \cdot \delta_2 \cdot \delta_3 &= 1/c_1\end{aligned} \tag{3.14}$$

and for two roots δ_4, δ_5 for the quadratic equation

$$c_4\delta^2 - c_5\delta + 1 = 0 \tag{3.15}$$

we also have

$$\begin{aligned}\delta_4 + \delta_5 &= c_5/c_4 \\ \delta_4 \cdot \delta_5 &= 1/c_4 \ .\end{aligned} \tag{3.16}$$

Thus, we can express the three optimality criteria for the designs in (3.9) as

$$\begin{aligned}|V_T| &= (\delta_1 \cdot \delta_2 \cdot \delta_3)(\delta_4 \cdot \delta_5)^{(t-1)}(\delta_6)^{t(t-3)/2} \\ &= (1/c_1)(1/c_4)^{(t-1)}(1/c_6)^{t(t-3)/2}\end{aligned} \tag{3.17}$$

$$\begin{aligned}\text{trace}(V_T) &= \delta_1 + \delta_2 + \delta_3 + (t-1)(\delta_4 + \delta_5) + \frac{t(t-3)}{2}\delta_6 \\ &= c_2/c_1 + (t-1)\frac{c_5}{c_4} + \frac{t(t-3)}{2} \cdot \frac{1}{16}\lambda_2\end{aligned} \tag{3.18}$$

$$\text{maximum eigenvalue} = \max(\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6) \ . \tag{3.19}$$

Srivastava and Chopra (1971b) argue that since V_T has only six possible distinct eigenvalues with multiplicities $1,1,1,t-1,t-1$ and $\frac{t(t-3)}{2}$ even though the value of t is large, the maximum eigenvalue criterion may not give a good overall view of the variance-covariance matrix V_T. Neither is the determinant criterion suitable for choosing the optimal design among the eight possible proposed designs, since $|V_T|$ could be small when variances of each effect are small and (or) when the correlations between effects are large. If the correlations between two effects are large in a 2^t factorial design, it can be said that such two effects are confounded with each other in a sense (Srivastava, 1965). Moreover, the calculation of the determinants of variance-covariance matrix for the eight possible proposed designs for each t shows that the determinants become very small as t

increases. Even when $t = 4$, for example, the largest and smallest determinants are 2.32831×10^{-10} and 2.58701×10^{-11}, respectively, and the difference between these two values is negligible. Thus, the E-optimality and D-optimality criteria are not appropriate for selecting the optimal design among the 8 possible designs.

Table 3.1 gives the trace value of variance-covariance matrix V_T and an index set for each minimal balanced fractional factorial designs of resolution V for $4 \le t \le 15$. From Table 3.1 we can see the following:

1. The designs indexed by $(t,1,t-2)$ and $(0,t-1,2)$ have the smallest value for $tr(V_T)$;
2. The designs indexed by $(t,1,2)$ and $(0,t-1,t-2)$ have the same value for $tr(V_T)$;
3. The designs indexed by $(0,1,2)$ and $(t,t-1,t-2)$ have the largest value for $tr(V_T)$.

It should be noted here that an optimality criterion such as the minimum value of $tr(V_T)$ is not a real restriction in the parameter design analysis. The more important properties are that the design should be cost-effective and also balanced since unbalanced designs lead us neither to reliable analysis nor to solid interpretation of results.

In practice, some treatment combinations cost less than other treatment combinations. And for some factors the lower level is easier to apply than the higher, or vice versa. Therefore, in practical parameter design analyses, one design with a reasonable value for $tr(V_T)$ can be selected out of these eight designs by using, in addition to statistical considerations, cost considerations and engineering convenience. If the cost and other considerations are not different among the designs, then we choose the design with the smallest value for $tr(V_T)$ as an inner array of the parameter design. For example, for $t = 6$, we can choose either design indexed by $(t,1,t-1)$ or $(0,t-1,2)$ since both have the smallest trace value 1.152. For an outer array, we can use any design provided that the design has a small number of runs and the test settings of such a design can cover the possible range of all noise factors. An orthogonal array may be suitable for these purposes.

< Table 3.1 > Index Set and Trace of Variance-covariance Matrix for Minimal Balanced Fractional Factorial Designs of Resolution V, $4 \le t \le 15$.

t \ Design		1 (0,1, t-2)	2 (t, 1, t-2)	3 (t, t-1, 2)	4 (0, t-1, 2)
4	Λ tr(v)	(1,1,1,0,0) 4.375	(0,1,1,0,1) 1.486	(0,0,1,1,1) 4.375	(1,0,1,1,0) 1.486
5	Λ tr(v)	(2,1,1,1,0) 1.764	(1,1,1,1,1) 1.000	(0,1,1,1,2) 1.764	(1,1,1,1,1) 1.000
6	Λ tr(v)	(3,1,1,2,1) 1.625	(2,1,1,2,2) 1.152	(1,2,1,1,3) 1.625	(2,2,1,1,2) 1.152
7	Λ tr(v)	(4,1,1,3,3) 2.024	(3,1,1,3,4) 1.486	(3,3,1,1,4) 2.024	(4,3,1,1,3) 1.486
8	Λ tr(v)	(5,1,1,4,6) 2.719	(4,1,1,4,7) 1.942	(6,4,1,1,5) 2.719	(7,4,1,1,4) 1.942
9	Λ tr(v)	(6,1,1,5,10) 3.648	(5,1,1,5,11) 2.504	(10,5,1,1,6) 3.648	(11,5,1,1,5) 2.504
10	Λ tr(v)	(7,1,1,6,15) 4.788	(6,1,1,6,16) 3.165	(15,6,1,1,7) 4.788	(16,6,1,1,6) 3.165
11	Λ tr(v)	(8,1,1,7,21) 6.130	(7,1,1,7,22) 3.924	(21,7,1,1,8) 6.130	(22,7,1,1,7) 3.924
12	Λ tr(v)	(9,1,1,2,28) 7.667	(8,1,1,8,29) 4.778	(28,8,1,1,9) 7.667	(29,8,1,1,8) 4.778
13	Λ tr(v)	(10,1,1,9,36) 9.398	(9,1,1,9,37) 5.727	(36,9,1,1,10) 9.398	(37,9,1,1,9) 5.727
14	Λ tr(v)	(11,1,1,10,45) 11.320	(10,1,1,10,46) 6.771	(45,10,1,1,11) 11.320	(46,10,1,1,10) 6.771
15	Λ tr(v)	(12,1,1,11,55) 13.433	(11,1,1,11,56) 7.909	(55,11,1,1,12) 13.433	(56,11,1,1,11) 7.909

< Table 3.2 > Index Set and Trace of Variance-covariance Matrix for Minimal Balanced Fractional Factorial Designs of Resolution V, $4 \le t \le 15$.

t \ Design		5 (t, 1, 2)	6 (0, 1, 2)	7 (0, t-1, t-2)	8 (t, t-1, t-2)
4	Λ tr(v)	(0,1,1,0,1) 1.486	(1,1,1,0,0) 4.375	(1,0,1,1,0) 1.486	(0,0,1,1,1) 4.375
5	Λ tr(v)	(1,2,1,0,1) 2.597	(2,2,1,0,0) 10.375	(1,0,1,2,1) 2.597	(0,0,1,2,2) 10.375
6	Λ tr(v)	(3,3,1,0,1) 4.885	(4,3,1,0,0) 21.625	(1,0,1,3,3) 4.885	(0,0,1,3,4) 21.625
7	Λ tr(v)	(6,4,1,0,1) 8.649	(7,4,1,0,0) 40.375	(1,0,1,4,6) 8.649	(0,0,1,4,7) 40.375
8	Λ tr(v)	(10,5,1,0,1) 14.244	(11,5,1,0,0) 69.250	(1,0,1,5,10) 14.244	(0,0,1,5,11) 69.250
9	Λ tr(v)	(15,6,1,0,1) 22.036	(16,6,1,0,0) 111.250	(1,0,1,6,15) 22.036	(0,0,1,6,16) 111.250
10	Λ tr(v)	(21,7,1,0,1) 32.397	(22,7,1,0,0) 169.750	(1,0,1,7,21) 32.397	(0,0,1,7,22) 169.750
11	Λ tr(v)	(28,8,1,0,1) 45.698	(29,8,1,0,0) 248.500	(1,0,1,8,28) 45.698	(0,0,1,8,29) 248.500
12	Λ tr(v)	(36,9,1,0,1) 62.314	(37,9,1,0,0) 351.625	(1,0,1,9,36) 62.314	(0,0,1,9,37) 351.625
13	Λ tr(v)	(45,10,1,0,1) 82.620	(46,10,1,0,0) 483.625	(1,0,1,10,45) 82.620	(0,0,1,10,46) 483.625
14	Λ tr(v)	(55,11,1,0,1) 106.989	(56,11,1,0,0) 649.375	(1,0,1,11,55) 106.989	(0,0,1,11,56) 649.375
15	Λ tr(v)	(66,12,1,0,1) 135.797	(67,12,1,0,0) 854.125	(1,0,1,12,66) 135.787	(0,0,1,12,67) 854.125

III.4 Analysis of Proposed Designs

By the general theory of linear models for a 2^t factorial experiment(Hinkelmann,1985), we can obtain the sum of squares due to each effect E (here we omit the defining vector $\underline{\alpha}$) for the proposed designs

$$SS(E) = \frac{(\hat{E})^2}{Var(\hat{E})^2/\sigma^2} = \frac{(\hat{E})^2}{(X'X)^{-1}_{\hat{E}}} \tag{3.20}$$

,where $(X'X)^{-1}_{\hat{E}}$ denotes the element of $(X'X)^{-1}$ for the estimated effect $\hat{E}$.

It is well known that under the null hypothesis H_0: the effect E is negligible, and the normality assumption of error terms, $\frac{SS(E)}{\sigma^2}$ has the central χ^2 distribution with 1 degree of freedom. If we know σ^2 or have an independent estimate of σ^2, then we can proceed with the test procedure using the usual χ^2 statistic or F statistic, respectively. For most practical experiments, however, we don't know σ^2 nor do we have a prior estimate of σ^2, so that we have to estimate σ^2 from the experiment. But, since the proposed designs are saturated, we cannot obtain the estimate of σ^2 simply by following the usual ANOVA procedure.

For analyzing a saturated design, some statistical methods including informal methods have been developed. We shall discuss and modify some methods and propose other methods for analyzing the proposed balanced fractional factorial parameter design of resolution V.

III.4.1 Pooling Method

For a saturated design, Taguchi and Wu (1981) suggested obtaining the error sum of squares simply by pooling the sums of squares for the effects that are small. For example, each sum of squares less than 5% of the total variation may be pooled together for an error sum of squares. Following Taguchi and Wu, some authors (eg, Pignatiello and Ramberg (1985), and Quinlan (1985)) used the pooling method in some experiments of the parameter design.

However, it is a well known fact that extreme bias can be induced by this kind of pooling of sums of squares. A simulation study of Box (1988) showed that such a pooling method may lead to unreasonable conclusions. Moreover, this pooling method can be criticized to be too subjective in determining the effects to be pooled.

III.4.2 Normal Probability Plot Method

For a 2^t factorial design, or a regular fractional 2^{t-r} factorial design where we take a $(1/2)^r, r < t$ fraction of all treatment combinations, all estimated effects can be expressed as linear combinations of the data and they are mutually independent since such designs are orthogonal. Therefore, if the data are normally and independently distributed then the estimates of effects are normally distributed independently and identically with mean 0 and some variance under the assumption that there is no influential effect. For detecing influential effects, based on this fact, Daniel (1959) suggested a method of plotting the estimated effects on normal probability paper.

For the proposed designs, we have an irregular fraction of treatment combinations so that the main effects and 2-factor interactions are estimated with different variances. Moreover, estimated effects are not independent of each other. However, since the proposed designs are balanced, all 2-factor interactions are estimated with the same variance. We an also expect that the covariances between any two estimated interactions are very small, since the proposed designs are based on the partially balanced array which is a generalized orthogonal array. For designs based on an orthogonal array, any two estimated effects have covariances equal to zero.

The calculation of the variance-covariance matrix of the optimal proposed designs indexed by $(t,1,t-2)$ and $(0,t-2,2)$ in Table 3.1 reveals that

(i) for $t=4$, $Var(\hat{F_iF_j}) = 0.1389\sigma^2$, $Cov(\hat{F_iF_j}, \hat{F_jF_k}) = 0.01389\sigma^2$, $i \neq k$, and $Cov(\hat{F_iF_j}, \hat{F_kF_l}) = -0.0481\sigma^2$, $i \neq j \neq k \neq l$, so that the covariances are about $1/10$ and $1/3$, respectively of $Var(\hat{F_iF_j})$ in magnitude;

(ii) for $t=5$, the designs are orthogonal arrays so that the covariances are zero;

(iii) for $t=6$, $Var(\hat{F_iF_j}) = 0.05222\sigma^2$, $Cov(\hat{F_iF_j}, \hat{F_jF_k}) = -0.003333\sigma^2$, and $Cov(\hat{F_iF_j}, \hat{F_kF_l}) = 0.0036111\sigma^2$ so that covariances are less than $1/10$ of variance in magnitude;

(iv) for $t>6$, covariances are less than $1/10$ of the variance in magnitude, especially $Cov(\hat{F_iF_j}, \hat{F_kF_l})$ is small compared to $Cov(\hat{F_iF_j}, \hat{F_jF_k})$.

Therefore, assuming that the estimates of 2-factor interactions are approximately independent, we can develop the following plotting method to detect influential 2-factor interactions.

Arrange the absolute values of the estimates of the $\frac{t(t-1)}{2}$ interactions in ascending order, and plot the $j-th$ of these ordered values against the adjusted empirical cumulative probabilities

$$P_j = \frac{(j-0.5)}{t(t-1)/2}, \quad j = 1,2,\ldots, \frac{t(t-1)}{2} \tag{3.21}$$

on normal probability paper (or half-normal probability paper). Then the interactions that are negligible will tend to fall along a straight line on this plot, while significant interactions will be far from the line. The negligible interactions then can be combined to form an error sum of squares which is used to test the significance of the main effect of each control factor, if such a test procedure is necessary.

As Daniel (1959) pointed out, however, if there is a considerable number of non-negligible effects, then the normal-probability plot may show some irregular pattern so that drawing a true straight line on which the non-negligible effects lie may be difficult.

III.4.3 Stepwise Method

In determining the unknown number k of influential 2-factor interactions and the k influential 2-factor interaction effects themselves, we propose the following stepwise procedure:

< Step 1 >

Calculate the sum of squares due to error (SSE), S_0^2 for the main effect model,

$$E(\underline{S}_T) = X_1 \underline{\beta}_1 \ . \tag{3.22}$$

< Step 2 >

Calculate q SSE's for the models which include the main effects and only one 2-factor interaction, where $q = \frac{t(t-1)}{2}$, that is, fit the model

$$E(\underline{S}_T) = X_1 \underline{\beta}_1 + \underline{X}_{2i}\gamma_i, \quad i = 1,2,\ldots,q, \tag{3.23}$$

where γ_i is an any 2-factor interaction effect of $\underline{\beta}_2$ and $\underline{X}_{2i}$ is the corresponding model vector, and obtain the error sum of squares SSE_i. Then choose the smallest SSE, say $SSE_{r^*} \equiv S_1^2$, among these q SSE's.

< Step 3 >

Using the following test statistic

$$F_1 = \frac{(S_0^2 - S_1^2)(m-t-2)}{S_1^2} \tag{3.24}$$

$$\text{where } m = 1 + t + \frac{t(t-1)}{2},$$

test the null hypothesis that no influential 2-factor interaction exists. Note that the alternative hypothesis is that there exists only one non-negligible 2-factor interaction. It is well known (e.g. Draper and Smith, 1981) that the test statistic F_1 has a central F distribution with $(1,\ m-t-2)$ degrees of freedom under the null hypothesis and normality assumption of error terms. If we fail to reject the null hypothesis, then we conclude that there is no influential 2-factor interaction effect. If we reject the null hypothesis, then go to step 4.

< Step 4 >

Repeating Step 1-3, test the null hypothesis that γ_{r^*} is the only non-negligible interaction effect using the alternative hypothesis that there are two non-negligible 2-factor interaction effects. The model under the null hypothesis is the following one which produces the minimum SSE, S_1^2 in Step 2, say

$$E(\underline{S}_T) = X_1\underline{\beta}_1 + \underline{X}_{2r^*}\gamma_{r^*} \ . \tag{3.25}$$

And obtain the minimum SSE among the $(q-1)$ SSE's for the models obtained by adding one more 2-factor interaction effect to (3.25), that is,

$$E(\underline{S}_T) = X_1\underline{\beta}_1 + \underline{X}_{2r^*}\gamma_{r^*} + \underline{X}_{3k}\gamma_{k}, \quad k \neq j, \quad k = 1,2,\ldots,q. \tag{3.26}$$

Denote the corresponding SSE by SSE_{r^*k}, and let the minimum SSE be $SSE_{r^*k^*} \equiv S_2^2$. To test the null hypothesis, we use the following test statistic,

$$F_2 = \frac{(S_1^2 - S_2^2)(m-t-3)}{S_2^2} \tag{3.27}$$

which has a central F distribution with $(1,\ m-t-3)$ degrees of freedom under the null hypothesis and normality assumption of error terms. If we fail to reject the null hypothesis, then we conclude that there is one non-negligible 2-factor interaction effect and identify such a model that produces S_1^2 as the best one. Otherwise, continue this step until we fail to reject the null hypothesis (say, there are k non-negligible 2-factor interaction effects) and identify the best model. After identifying the best model, combine the sum of squares for the negligible interactions to obtain the error sum of squares, if it is necessary.

III.4.4 Modified Stepwise Method

If we identify the number of non-negligible interactions from the stepwise method, then we have the family D_T of competing models in (3.7). That is, there may exist more than one possible candidate for the best model in the final step of the stepwise method. To choose the best model from D_T we can modify the stepwise method, using a frequency plot method suggested by Srivastava (1975).

< Step 1 >

Determine the number of influential interaction effects, say k_1 using the stepwise method.

< Step 2 >

Calculate the p SSE's, say $S_{1:i}^2$, $i = 1,\ 2,\ \ldots,\ p$, where $S_{k_1:i}^2$ denotes the SSE for the $i-th$ model including main effects and k_1 2-factor interactions, where $p = \binom{q}{k_1}$, by fitting

$$E(\underline{S}_T) = X_1\underline{\beta}_1 + X_2^{(i)}\underline{\beta}_2^{(i)},\quad i = 1,\ 2,\ \ldots,\ p. \tag{3.28}$$

< Step 3 >

Calculate for each model

$$F_{k_1:l} = \frac{(S_0^2 - S_{k_1:l}^2)(m - t - k_1 - 1)}{k_1 \cdot S_{k_1:l}^2} \quad , l = 1,\ 2,\ \ldots,\ p. \tag{3.29}$$

Then under the null hypothesis that no influential interaction effect exists, and the normality assumption for the error term, each $F_{k_1:l}$ has the central F distribution with $(k_1{:}m - t - k_1 - 1)$ degrees of freedom. Here the alternative hypothesis is that there exist k_1 influential interactions.

< Step 4 >

Compare each $F_{k_1:l}$ with the critical value $F_{(k_1:m-t-k_1-1):\alpha}$ and retain models for which the test result is significant, that is

$$F_{k_1:l} > F_{(k_1:m-t-k_1-1):\alpha}\ . \tag{3.30}$$

These models are possible candidates for the best model. If there are u possible candidates, then we have uk_1 (not necessarily distinct) elements of $\underline{\beta}_2$.

< Step 5 >

Make a frequency distribution in which for each element of $\underline{\beta}_2$ we indicate the frequency of its occurrence among the uk_1 (not necessarily distinct) elements of $\underline{\beta}_2$. Finally, from this frequency, choose k_1 elements of $\underline{\beta}_2$ for which the frequencies are high. In case of a tie, the interactions of the model with smaller SSE are selected. Then identify these k_1 elements of $\underline{\beta}_2$ as the influential 2-factor interaction effects.

< Example 3.3 >

For a 2^6 factorial parameter design, we assume that the identified number of influential 2-factor interactions in Step 1 is four and three models are retained in Step 5, and the 2-factor interactions in the retained models are as follow;

Model I : $(F_1F_2,\ F_1F_4,\ F_2F_5,\ F_4F_6)$

Model II : $(F_1F_2,\ F_1F_4,\ F_2F_5,\ F_3F_5)$

Model III : $(F_1F_2,\ F_1F_4,\ F_3F_5,\ F_4F_5)$.

For this situation, the frequencies for interactions F_1F_2, F_1F_4, F_2F_5, F_3F_5, F_4F_5, and F_4F_6 are, respectively, 3, 3, 2, 2, 1, 1. Therefore, we identify the four interactions F_1F_2, F_1F_4, F_2F_5, and F_3F_5 as influential interactions.

III.4.5 Modified Ghosh's Procedure

Ghosh(1987) developed a detection procedure when a suitable search design is given. However, he suggested to guess the unknown number of influential effects. Therefore, his detection procedure is statistically valid only when the guessed number is correct.

After identifying the number of non-negligible interactions, say $k = k_1$ from the stepwise method, we can also modify the final step of the stepwise method following the procedure suggested by Ghosh.

Among the p possible models with main effects and k_1 2-factor interactions, denote the i-th model as

$$E(\underline{S}_T) = X_1\underline{\beta}_1 + X_2^{(i)}\underline{\beta}_2^{(i)}\ ,\quad i = 1,\ 2,\ \ldots,\ p = \binom{q}{k_1}$$

,where $X_2^{(i)}$ is an $(m \times k_1)$ submatrix of X_2 corresponding to $\underline{\beta}_2^{(i)}$ (3.31)

$\beta_2^{(i)}$ is $(k_1 \times 1)$ subvector of $\underline{\beta}_2$.

For the model (3.31), write

$$\beta_2^{(i)'} = (\beta_{21}^{(i)}, \ \ldots, \ \beta_{2j}^{(i)}, \ \ldots, \ \beta_{2k_1}^{(i)})$$
$$X_2^{(i)} = [X_{21}^{(i)}, \ \ldots, \ X_{2j}^{(i)}, \ \ldots, \ X_{2k_1}^{(i)}]$$
$$X_2^{(ij)} = \text{the matrix obtained } X_2^{(i)} \text{ by deleting the } j^{th} \text{ column}$$
$$X_{12}^{(ij)} = [X_1, \ X_2^{(ij)}]$$
$$P_{12}^{(ij)} = I - X_{12}^{(ij)}\left(X_{12}^{(ij)'} X_{12}^{(ij)}\right)^{-1} X_{12}^{(ij)'}$$
$$Z_{ij}^{(i)} = \frac{P_{12}^{(ij)} X_{2j}^{(i)}}{\sqrt{X_{2j}^{(i)'} P_{12}^{(ij)} X_{2j}^{(i)}}}$$

and

$$SSE^{(i)} = SSE \text{ for the } i-th \text{ model in (3.31)}$$
$$SSE^{(0)} = SSE \text{ for the main effects model} \ .$$

Then it can be shown (Ghosh, 1987) that under the hypothesis $H_0{:}\beta_{2j}^{(i)} = 0$ for the i-th model in (3.31) and normality assumption of error terms,

$$t^{(ij)} = \frac{Z_{ij}^{(i)'} \underline{S}_T}{\sqrt{\dfrac{SSE^{(i)}}{m-t-k_1-1}}} \qquad \begin{array}{l} i = 1, \ 2, \ \ldots, \ p \\ j = 1, \ 2, \ldots, k_1 \end{array} \tag{3.32}$$

has the central t-distribution with $(m - t - k_1 - 1)$ degrees of freedom.

Suppose that in the final step of the modified stepwise method, we have significant $F_{k_1:i}$ for $i = i_1, i_2, \ldots, i_u$. Then we have $u \bullet k_1$ (not necessarily distinct) elements of $\underline{\beta}_2$. For $v = 1,2,\ldots,q$, denote the number of i in $\{i_1, \ldots, i_u\}$ for which $t^{(iv)}$ is significant by n_v. This number n_v represents the frequency of how many times the interaction β_v appears significantly among the candidate models. We now arrange the n_v's in decreasing order and write

$$n_{(1)} \geq n_{(2)} \geq \ldots \geq n_{(q)} \ . \tag{3.33}$$

If there are more than k_1 nonzero r_i's then select k_1 interaction effects which have the largest r_i value. Otherwise, choose the interaction effects which have the nonzero value of r_i. For this case, the number of influential interactions is less than k_1. Finally obtain an error sum of squares, if necessary, by pooling the sums of squares for negligible interactions.

After identifying the influential 2-factor interactions by the detection procedures developed in this chapter , we have the following model for a 2^t factorial parameter design analysis,

$$\begin{aligned} E(\underline{S}_T) &= X_1\underline{\beta}_1 + X_2{}^*\underline{\beta}_2{}^* \\ &= X\underline{\beta} \end{aligned} \tag{3.34}$$

,where T is a balanced fractional factorial design of resolution V

$\underline{\beta}_1$ is a vector of the overall mean and main effectts

$\underline{\beta}_2{}^*$ is a vector of identified influential 2-factor interactions, and

$X = [X_1, X^*{}_2]$

$\underline{\beta}' = [\underline{\beta}'_1, \underline{\beta}^{*\prime}{}_2]$.

Since the matrix X is of full column rank, we can estimate all the effects in $\underline{\beta}$ by the usual least square procedure.

For a 2^t factorial parameter design, the estimate $\hat{E}$ is defined as in (3.3)

$$\hat{E} = \begin{cases} \hat{E}_1 - \hat{E}_0 & \text{if E is the main effect} \\ \hat{E}_0 - \hat{E}_1 & \text{if E is the 2-factor interaction} \end{cases} \tag{3.35}$$

,where we omit the defining vector $\underline{\alpha}$ in the expression of effect.

If the estimate $\hat{E}$ of the main effect is positive, then the estimated effect at level 1 (higher level) is positive since $\hat{E}_0 + \hat{E}_1 = 0$. That is, the SN ratio is maximized when we set the control factor E at the higher level. If $\hat{E}$ is negative then the estimated effect of E at level 0 (lower level) is positive so that the optimal setting of E is at the lower level. The interaction effect of factors F_i and F_j at, say the lower level of F_i and the higher level of F_j can be evaluated as the average response (SN ratio) for the treatment combinations for which the lower level and the higher level of F_i and F_j appear, respectively. (See the example in II.3.) Therefore, by evaluating main effect at two levels (high and low) and influential interaction effects at four possible combinations of levels, we can identify the optimal settings of control factors. That is, for the control factors which are not interacting with other factors, we evaluate the main effects at each level. And for the interacting control factors, we identify the best combination of levels among the four possible combinations, at which the average SN ratio is largest. Then, finally we can identify the optimal setting of control factors at which the SN ratio is maximized.

Frequently in the parameter design, especially when the nominal target value is specified, the supplementary analysis for the mean response of quality characteristics is helpful in identifying the best adjustment factors. For the analysis of the mean response, we can also use the same procedures of analysis developed in this chapter.

Chapter IV

DETECTING INTERACTIONS IN 3-LEVEL FACTORIAL PARAMETER DESIGNS

The parameter design may be more effective in identifying the optimal setting of control factors when quadratic effects of control factors exist (see Taguchi and Wu, 1980). To investigate such quadratic effects, each control factor must have three test levels. Also, for 3^t factorial parameter designs we need at least $m = 1 + 2t + 2t(t-1)$ treatment combinations to estimate main effects and 2-factor interactions. Moreover, if the outer array consists of n rows, then $n \times m$ experimental units are necessary for a 3^t parameter design. This number $n \times m$ may be too large even when the number of control factors t is moderate.

To reduce the number of experimental units necessary for detecting influential 2-factor interactions, a sequential detection procedure will be developed in this chapter. As in the 2^t factorial parameter designs, 3-factor and higher order interactions are assumed to be negligible. It is also assumed that a proper outer array of noise factors is combined with each treatment combination of the inner array and appropriate SN ratios are observed.

IV.1 3-Level Factorial Designs

IV.1.1 Definition of Effects in 3-Level Factorial Designs

In 3^t factorial designs in which each of t factors $F_1, F_2, \dots, F_t$ has 3 levels, a treatment combination can be denoted by a $(t \times 1)$ vector $\underline{x}' = (x_1, x_2, \dots, x_t)$ with elements x_i from the Galois field of order 3, $GF(3)$ in which 3 elements 0,1, and 2 of $GF(3)$ represent the 3 levels of each factor. Also, there exists an one-to-one correspondence between the 3^t treatment combinations and the 3^t points (t-tuples over $GF(3)$) in the finite Euclidean geometry of order t, $EG(t,3)$. As in 2^t factorial designs, the $\frac{(3^t-1)}{2}$ symbols for effects and interactions (each accounting for 2 d.f.) can be written as

$$E^{\underline{\alpha}} = F_1^{\alpha_1} F_2^{\alpha_2} \dots F_t^{\alpha_t} \tag{4.1}$$

with $\alpha_i = 0,1,2$ $(i = 1,2,\dots,t)$ and the convention that

(i) any letter F_i with $\alpha_i = 0$ is dropped from the expression,

(ii) the first non-zero α_i is equal to one. (This can always be achieved by multiplying each α_i by 2 mod (3)),

(iii) any letter F_i with $\alpha_i = 1$ is written as F_i,

(iv) $\underline{\alpha}' = (\alpha_1, \alpha_2, \dots \alpha_t)$ is the defining vector.

In 3^t factorial designs, each main effect is represented by one symbol in (4.1), i.e. the main effect F_i is given by $E^{\underline{\alpha}}$, in which all $\alpha_j = 0$ except $\alpha_i = 1$. The interactions, however, cannot be expressed so simply since more than one symbol (component) in (4.1) constitute each interaction. For example, the 2-factor interaction denoted by $F_i \times F_j$ between factors F_i and F_j consists of two components F_iF_j and $F_iF_j^2$. The 3-factor interaction $F_i \times F_j \times F_k$ consists of four components $F_iF_jF_k, F_iF_jF_k^2, F_iF_j^2F_k, F_iF_j^2F_k^2$. And the other higher order interactions can be defined by a similar way. Therefore, each main effect consists of one component $E^{\underline{\alpha}}$, and each 2-factor interaction consists of two components. Hereafter, we write the 2-factor interaction as $F_i \times F_j$ and we call the two interaction components F_iF_j, $F_iF_j^2$ the first interaction component and the second interaction component, respectively.

The 2 degrees of freedom for each main effect can be represented geometrically by 2 independent contrasts of the 3 sets of treatment combinations generated by the pencil in $EG(t,3)$

$$\begin{aligned} \underline{\alpha}'\underline{x} &= b \\ \Rightarrow \quad x_i &= b \\ \text{where} \quad b &= 0,1,2 \ . \end{aligned} \tag{4.2}$$

The four degrees of freedom associated with the 2-factor interaction $F_i \times F_j$ consist of each 2 degrees of freedom associated with two interaction components F_iF_j and $F_iF_j^2$. The 2 degrees of freedom corresponding to the first interaction component $E^1 = F_iF_j$ can be represented by 2 independent contrasts of the 3 sets of treatment combinations generated by the pencil in $EG(t,3)$

$$\begin{aligned} \underline{\alpha}'\underline{x} &= b \\ \Rightarrow \quad x_i + x_j &= b \\ \text{where} \quad b &= 0,1,2. \end{aligned} \tag{4.3}$$

Similarly, 2 degrees of freedom for the second interaction component $E^2 = F_iF_j^2$ can be represented by 2 independent contrasts of the 3 sets of treatment combinations generated by the pencil in $EG(t,3)$

$$\begin{aligned} \underline{\alpha}'\underline{x} &= b \\ \Rightarrow \quad x_i + 2x_j &= b \\ \text{where} \quad b &= 0,1,2. \end{aligned} \tag{4.4}$$

The two degrees of freedom associated with component E^1 can be expressed by the set of orthogonal contrasts

$$C = \begin{bmatrix} 1 & 1 \\ 0 & -2 \\ -1 & 1 \end{bmatrix} \tag{4.5}$$

,where the rows of C correspond to levels 0, 1, and 2, respectively.

That is, for the component E^{α} the linear effect of E^{α} can be expressed as

$$E_0^{\alpha} - E_2^{\alpha} \tag{4.6}$$

and the quadratic effect as

$$E_0^{\alpha} + E_2^{\alpha} - 2E_1^{\alpha} \tag{4.7}$$

,where E_i^{α}, i = 0,1,2 are the true effects at level i of E^{α} defined by

$$E_i^{\alpha} = (\text{true mean response of treatment combinations satisfying } \underline{\alpha}'\underline{x} = i) - \mu, \text{ where } \mu \text{ is the overall mean.} \tag{4.8}$$

The two parameters corresponding to 2 degrees of freedom for main effect F_i are denoted by $(F_i)^1$, $(F_i)^2$. Similarly, two parameters $(F_iF_j)^1,(F_iF_j)^2$ represent the 2 degrees of freedom associated with the first interaction component F_iF_j . And $(F_iF_j^2)^1,(F_iF_j^2)^2$ also represent the 2 degrees of freedom associated with the second interaction component $F_iF_j^2$. Therefore, we can say that $(E^{\alpha})^1,(E^{\alpha})^2$ represent the linear and quadratic effects of E^{α}, respectively.

IV.1.1 Regular Fractional Factorial Designs

A $1/3^r$ ($r < t$) regular fractional factorial design of a 3^t factorial experiment is defined as the set of 3^{t-r} treatment combinations $\underline{x}$ obtained as solutions to a set of r independent equations in t unknowns over the $GF(3)$ such that

$$A\underline{x} = \underline{b}, \tag{4.9}$$

where A is an $(r \times t)$ coefficient matrix of rank r and $\underline{b}$ is an $(r \times 1)$ vector of constants.

In the regular fraction, the components $E^{\underline{\alpha}}$ are partitioned into some number of groups in a systematic way. Such groups of components are called alias sets. The following two definitions establish that the alias sets of components are determined by the A matrix in (4.9).

< **Definition 4.1** >

Let $\underline{\alpha}' = (\alpha_1, \alpha_2, \ldots, \alpha_t)$ denote the defining vector of component $E^{\underline{\alpha}} = F_1^{\alpha_1} F_2^{\alpha_2} \ldots F_t^{\alpha_t}$. Then the component $E^{\underline{\alpha}}$ is aliased (confounded) with the overall mean μ if and only if

$$rank\begin{bmatrix} A \\ \underline{\alpha}' \end{bmatrix} = rank(A) = r. \tag{4.10}$$

< **Definition 4.2** >

Let $\underline{\alpha}'_1$ and $\underline{\alpha}'_2$ be the two defining vectors of $E^{\underline{\alpha}_1}$ and $E^{\underline{\alpha}_2}$. Assume that

$$rank \begin{bmatrix} A \\ \underline{\alpha}'_1 \end{bmatrix} = rank \begin{bmatrix} A \\ \underline{\alpha}'_2 \end{bmatrix} = r+1. \tag{4.11}$$

Then $E^{\underline{\alpha}_1}$ is aliased with $E^{\underline{\alpha}_2}$ if and only if

$$rank \begin{bmatrix} A \\ \underline{\alpha}'_1 \\ \underline{\alpha}'_2 \end{bmatrix} = r + 1. \tag{4.12}$$

The alias set S_0 containing the overall mean μ carries one degree of freedom and every another alias set S_i, $i = 1,2,..., u$ carries two degrees of freedom. Therefore, in a $1/3^r$ fractional factorial design, the components are partitioned into $1 + \frac{3^{t-r} - 1}{2} = 1 + u$ alias sets $S_0, S_1, ..., S_u$. The task of finding the alias set for a fraction is simple. By definition 4.1, the alias set S_0 will contain all components whose defining vectors are in the row space of A. By definition 4.2, to find another alias set, for example S_1 , pick some component not in S_0, and take linear combinations of its defining vector and the defining vectors of all components in S_0. To continue, pick some component neither in S_0 nor in S_1 , and take all linear combinations of its defining vector and the defining vectors of all components in S_0. Then this alias set corresponds to S_2. This process is continued until all components are accounted for.

IV.1.2 Analysis and Estimable Functions for Fractional Designs

Let T denote a design consisting of a set of m treatment combinations from a regular fractional factorial design. T is used to denote either the set of treatment combinations in the design or the $(t \times m)$ matrix having $\underline{x}_1, \underline{x}_2, ...\underline{x}_m$ as m treatment combinations. Let $T' = M = (\underline{m}_1, \underline{m}_2, ..., \underline{m}_t)$ so that M is an $(m \times t)$ matrix with each column $\underline{m}_i$ representing the levels of factor F_i for m treatment combinations in the design T. If for any two factors F_i and F_j, the first interaction component and/or the second interaction component are to be included in the model (4.13), then columns $\underline{m}_i + \underline{m}_j$ and/or $\underline{m}_i + 2\underline{m}_j$ mod(3), should be adjoined to M since the "levels" of two interaction components F_iF_j and $F_iF_j^2$ are represented by $\underline{m}_i + \underline{m}_j$ and $\underline{m}_i + 2m_j$, respectively. This procedure is followed for every pair of factors that interact and M^* represents the matrix that results after all appropriate columns corresponding to interaction components have been adjoined to M. Then, using the usual notation, we can write the model equation for a 3^{t} (fractional) factorial design, in which 3-factor and higher order interactions are assumed to be negligible,

$$E(\underline{y}) = X\underline{\beta} \tag{4.13}$$

,where the parameter vector $\underline{\beta}$ contains the overall mean μ and two parameters $(F_i)^1,(F_i)^2$ associated with each main effect F_i, and two parameters $(F_iF_j)^1,(F_iF_j)^2$, or$(F_iF_j^2)^1,(F_iF_j^2)^2$ associated with each 2-factor interaction component.

The model matrix X can be constructed in the following way:
The first column is a vector with every element one, corresponding to the overall mean μ. Since each column of M^* corresponds to two degrees of freedom, each column of M^* is transformed into two columns in the X matrix. This is done by replacing each element in M^* by the corresponding row of C in (4.5). But note here that the matrix X may not be of full column rank, which means that not all parameters in $\underline{\beta}$ are necessarily estimable, since components may be aliased with each other.

Now, we consider the estimable function(s) for each alias set. Since only one and two degrees of freedom are associated with S_0, and S_i ($i = 1,2,...,u$), respectively, we cannot estimate all components contained in each alias set. Instead, some linear functions of components in each alias set can be estimated. Consider first an estimable function for the alias set S_0. Since only one degree of freedom is associated with S_0, one estimable function for S_0 exists. Let $E^{\underline{a}_1}, E^{\underline{a}_2}, ..., E^{\underline{a}_k}$ denote the components contained in S_0, and let us denote the vector of two parameters associated with $E^{\underline{a}_l}$ by $\underline{E}_l$, that is

$$\underline{E}_l = \begin{bmatrix} (E^{\underline{a}_l})^1 \\ (E^{\underline{a}_l})^2 \end{bmatrix}. \tag{4.14}$$

Note that each component $E^{\underline{a}}$ contained in S_0 appears at the same "level" for all treatment combinations in a fractional design. Suppose that $E^{\underline{a}_1}, E^{\underline{a}_2}, \ldots E^{\underline{a}_a}$ appear at "levels" $x_1, x_2, \ldots x_a$, respectively. Then the estimable function ES_0 of the components $E^{\underline{a}_1}, E^{\underline{a}_2}, \ldots E^{\underline{a}_a}$ can be expressed as

$$ES_0 = \mu + \underline{c}'_{x_1}\underline{E}_1 + \underline{c}'_{x_2}\underline{E}_2 + \ldots + \underline{c}'_{x_a}\underline{E}_a \, , \tag{4.15}$$

where $\underline{c}'_{x_i}$ is the row of C corresponding to level x_i of $E^{\underline{a}_i}$.

Therefore, associated with the alias set S_0 is the estimable function represented by a linear combination of μ and the linear and quadratic effects of $E^{\underline{a}_i}$ aliased with μ.

Consider next the estimable functions for the alias set S_i, $i \neq 0$. Since each alias set S_i carries two degrees of freedom, we can find the two estimable functions of parameters corresponding to the components contained in each S_i. Let P denote the set of all possible permutations in $GF(3)$. Then

$$P = \{e,(012),(021),(12),(02),(01)\} \, , \tag{4.16}$$

where e is the identity permutation and (012) denotes the permutation

$$0 \rightarrow 1$$

$$1 \rightarrow 2$$

$$2 \rightarrow 0$$

and the remaining permutations are defined similarly.

Following the conventions of Mardekian (1979), define the following six matrices

$$D_e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad D_{(012)} = \begin{bmatrix} -1/2 & 3/2 \\ -1/2 & -1/2 \end{bmatrix} \quad D_{(021)} = \begin{bmatrix} -1/2 & -3/2 \\ 1/2 & -1/2 \end{bmatrix}$$

$$D_{(02)} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad D_{(01)} = \begin{bmatrix} 1/2 & 3/2 \\ 1/2 & -1/2 \end{bmatrix} \quad D_{(12)} = \begin{bmatrix} 1/2 & -3/2 \\ -1/2 & -1/2 \end{bmatrix}. \tag{4.17}$$

Suppose that the components $E^{\underline{a}_1}, E^{\underline{a}_2}, \ldots E^{\underline{a}_b}$ are contained in the alias set S_i, $i \neq 0$. Choose any component , for example $E^{\underline{a}_1}$ as a reference component. Then the "levels" of the other components are related to the levels of the reference component by some permutations (not necessarily distinct) in P, say $p_2, p_3, \ldots p_b$, respectively, since P contains all possible permutations in $GF(3)$. And it can be shown (see Appendix) that the vector ES_i of two estimable functions for S_i is

$$ES_i = D_e \, \underline{E}_1 + D_{p_2} \, \underline{E}_2 + \ldots D_{p_b} \, \underline{E}_b \ , \tag{4.18}$$

where each $\underline{E}_i$ denotes the (2×1) vector (4.14) of two parameters corresponding to the component $E^{\underline{a}_i}$ and D_{p_i} represents the matrix defined in (4.17) corresponding to the permutation p_i.

Note here that for a regular fractional design T obtained from (4.9), the alias structure is completely determined by the matrix A and the estimable functions for each alias set are determined by the constant vector $\underline{b}$.

< Example 4.1 >

Consider the $1/3^2$ fractional design T of a 3^3 factorial experiment in which 3-factor and higher order interactions are assumed to be negligible. From the following set of equations

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

we can get

$$T = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

,where each column represents a treatment combination.

The two alias sets corresponding to T are

$$S_0 = \{\mu, F_1F_2^2, F_1F_3^2, F_2F_3^2\}$$
$$S_1 = \{F_1, F_2, F_3, F_1F_2, F_1F_3, F_2F_3\}.$$

Since the components $F_1F_2^2$, $F_1F_3^2$, and $F_2F_3^2$ appear at "levels" 0, 2, and 2, respectively, for all treatment combinations in T, the estimable function for S_0 is

$$ES_0 = \mu + [1,1]\begin{bmatrix}(F_1F_2^2)^1 \\ (F_1F_2^2)^2\end{bmatrix} + [-1,1]\begin{bmatrix}(F_1F_3^2)^1 \\ (F_1F_3^2)^2\end{bmatrix} + [-1,1]\begin{bmatrix}(F_2F_3^2)^1 \\ (F_2F_3^2)^2\end{bmatrix}$$
$$= \mu + (F_1F_2^2)^1 + (F_1F_2^2)^2 - (F_1F_3^2)^1 + (F_1F_3^2)^2 - (F_2F_3^2)^1 + (F_2F_3^2)^2 .$$

If wee choose F_1 as a reference component for S_1, then the "levels" of $F_2, F_3, F_1F_2, F_1F_3, F_2F_3$ are related to the "levels" of F_1 by permutations e, (012), (12), (01), (01), respectively. Therefore,

$$ES_1 = \begin{bmatrix}(F_1)^1 \\ (F_1)^2\end{bmatrix} + \begin{bmatrix}(F_2)^1 \\ (F_2)^2\end{bmatrix} + \begin{bmatrix}-1/2 & 3/2 \\ -1/2 & -1/2\end{bmatrix}\begin{bmatrix}(F_3)^1 \\ (F_3)^2\end{bmatrix} + \begin{bmatrix}1/2 & -3/2 \\ -1/2 & -1/2\end{bmatrix}\begin{bmatrix}(F_1F_2)^1 \\ (F_1F_2)^2\end{bmatrix}$$
$$+ \begin{bmatrix}1/2 & 3/2 \\ 1/2 & -1/2\end{bmatrix}\begin{bmatrix}(F_1F_3)^1 \\ (F_1F_3)^2\end{bmatrix} + \begin{bmatrix}1/2 & 3/2 \\ 1/2 & -1/2\end{bmatrix}\begin{bmatrix}(F_2F_3)^1 \\ (F_2F_3)^2\end{bmatrix}$$

and the two estimable functions for S_1 are

$$(F_1)^1 + (F_2)^1 - \frac{1}{2}(F_3)^1 + \frac{3}{2}(F_3)^2 + \frac{1}{2}(F_1F_2)^1 - \frac{3}{2}(F_1F_2)^2 +$$
$$\frac{1}{2}(F_1F_3)^1 + \frac{3}{2}(F_1F_3)^2 + \frac{1}{2}(F_2F_3)^1 + \frac{3}{2}(F_2F_3)^2$$

and

$$(F_1)^2 + (F_2)^2 - \frac{1}{2}(F_3)^1 - \frac{1}{2}(F_3)^2 - \frac{1}{2}(F_1F_2)^1 - \frac{1}{2}(F_1F_2)^2 +$$
$$\frac{1}{2}(F_1F_3)^1 - \frac{1}{2}(F_1F_3)^2 + \frac{1}{2}(F_2F_3)^1 - \frac{1}{2}(F_2F_3)^2.$$

IV.2 Detection Procedure for Influential 2-Factor Interactions

IV.2.1 Description

Some detection procedures for influential 2-factor interactions in 3^t factorial designs were developed by Anderson and Thomas (1980), and Hussain (1986). Anderson and Thomas used a "near" minimal 3^t fractional design of resolution IV and developed detection procedure for the noiseless case, that is $\sigma^2 = 0$. Their basic assumption is that we can detect the influential 2-factor interactions using only the first interaction component. That is, if F_iF_j is negligible, then they conclude that the 2-factor interaction $F_i \times F_j$ is negligible. However, even though the first interaction component F_iF_j is negligible, the 2-factor interaction $F_i \times F_j$ is not negligible whenever the second interaction component $F_iF_j^2$ is influential. Therefore, this assumption is too strong in some experiments. Moreover, their procedure can detect at most 3 influential interactions.

Hussain (1986) developed a detection procedure from another point of view in which a sequential approach is used. He applied each factor sequentially and tested whether it interacted with other factors by taking treatment combinations sequentially from the fractional factorials. The most serious drawback of Hussain's procedure is that we perform too many tests, especially when prior information on the underlying interaction structure is not available, since we have to test all factors sequentially. In practical experiments it is not rare to encounter a situation where only a few interactions are present, although many control factors are involved. For this case, employing Hussain's method is very costly.

Moreover, both procedures include all four parameters associated with a 2-factor interaction when a 2-factor interaction is identified to be influential. However, some interaction component, for example F_iF_j may be negligible even though the interaction $F_i \times F_j$ turns out to be influential. In the parameter design analysis, one of the most important considerations is to obtain accurate estimates with small bias of main effects for control factors. We should, therefore, include as small a number of negligible interaction component as possible in the model to get accurate estimates of main effects. Anderson and Thomas, and Hussain also considered the estimation procedure of the identified influential 2-factor interactions. In the parameter design, however, we do not have to estimate the influential 2-factor interactions. This is because we can identify the optimal combination of factor levels for the interacting factors (at which the SN ratio is maximized) simply by evaluating the average SN ratio at the nine different combinations of factor levels. Therefore it is sufficient in the parameter design analysis to identify the influential 2-factor interactions only, regardless of estimability of such interactions.

The procedure developed in this chapter is designed to overcome the drawbacks of both methods. The proposed procedure consists of two stages:

(i) In Stage I, we shall use a "near" minimal resolution IV design developed by Anderson and Thomas. We split each 2-factor interaction $F_i \times F_j$ into two components F_iF_j and $F_iF_j^2$. Then we develop detection procedure for each interaction component by utilizing the same kinds of hypotheses proposed by Hussain.

If the underlying interaction structure is sufficiently simple, then we can identify all influential interaction components in this stage, and we can terminate the detection procedure. Otherwise, we go to Stage II and continue the detection procedure.

(ii) In Stage II, we shall utilize the sequential procedure developed by Hussain. Since in Stage I we can screen out the control factors which are not interacting with other factors, the number of control factors to be tested in Stage II will be small. Moreover, the interaction structure identified in Stage I and treatment combinations used in Stage I will be exploited in this stage. Therefore, the sequential procedure in Stage II will be simplified by using information obtained and treatment combinations employed in Stage I. This sequential procedure will be efficient especially when observations become available quickly.

The proposed detection procedure is, therefore, essentially a combination of extending the "pure" search procedure developed by Anderson and Thomas, and modifying the sequential method and adapting them to the parameter design analysis. As in the detection procedure for 2^t factorial parameter designs, we assume that 3-factor and higher order interactions are all negligible. We make one more assumption in this chapter, which is also used by Anderson and Thomas, and Hussain. That is, if an estimable function of components is negligible then all components involved in the estimable function will be assumed to be negligible. This assumption is reasonable since the probability of such an event is one when each component is observed from some continuous distribution. We also assume that with each treatment combination of the inner array a proper outer array is combined so that an appropriate *SN* ratio can be observed. Therefore, as in the Chapter II, the treatment combinations considered in this chapter are the settings of control factors for the inner array.

IV.2.2 Stage I: Detection Procedure with a Resolution IV Design

IV.2.2.1 Estimable Functions for a Resolution IV Design

For a given number t of control factors, consider the regular fractional factorial designs T_i, $i = 1,2,..., t$, obtained as solutions to the set of equations

$$A_i\underline{x} = \underline{0} \ , \ \ i = 1,2,..., t, \tag{4.19}$$

where $\underline{0}$ means a vector with every element 0 and A_i are $(t-2) \times t$ matrices of rank (t-2) such that

(i) the i-th column of A_i is $\underline{0}$

(ii) the second column of A_1 and the first column of A_i, $i \neq 1$ have every element equal to 2.

(iii) the remaining (t-2) columns of A_i represent the identity matrix of order (t-2) in order.

Let the design T denote the union of all treatment combinations of all fractions T_i,

$$T = \bigcup_{i=1}^{t} \{\underline{x}: \ A_i\underline{x} = \underline{0}\}. \tag{4.20}$$

Then the i-th fraction T_i has nine treatment combinations. Since 3 treatment combinations $\underline{x} = \underline{0}, \underline{1},$ and $\underline{2}$ are common to all fractions, the number of treatment combinations in T is $6t + 3$. The design T contains 6 treatment combinations more than the theoretical minimal 3^t fractional factorial design of resolution IV which has $(6t - 3)$ treatment combinations (see Margolin, 1969). The alias sets corresponding to each T_i are (see section IV.1.1)

$$\begin{aligned}
&S_{0l} = \{\mu, F_jF_k^2,\ j<k, j\neq i, k\neq l\}\\
&S_{1l} = \{F_l\}\\
&S_{2l} = \{F_j, j\neq i,\ F_jF_k, j<k, j\neq i, k\neq l\}\\
&S_{3l} = \{F_lF_j,\ j\neq l\}\\
&S_{4l} = \{F_lF_j^2,\ j\neq l\}\ .
\end{aligned} \tag{4.21}$$

We can see that each main effect F_l can be estimated from T_l so that all main effects are estimable in T. For these reasons, Anderson and Thomas (1979) called the design T a near minimal resolution IV design.

From the definition of the fraction T_l in (4.19) we can see that the nine treatment combinations in T_l are

$$\begin{matrix} \text{Factors } 1-(i-1): \\ \text{Factor } i: \\ \text{Factors } (i+1)-t: \end{matrix} \begin{bmatrix} \underline{0} & \underline{1} & \underline{2} & \underline{0} & \underline{1} & \underline{2} & \underline{0} & \underline{1} & \underline{2} \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\ \underline{0} & \underline{1} & \underline{2} & \underline{0} & \underline{1} & \underline{2} & \underline{0} & \underline{1} & \underline{2} \end{bmatrix}. \tag{4.22}$$

From (4.22) we can see that each effect $F_jF_k^2$ in the alias set S_{0l} appears at "level" 0 for all nine treatment combinations in T_l so that the estimable function for S_{0l} is

$$ES_{0l} = \mu + \sum_{\substack{j=1\\ j<k,}}^{t} \sum_{\substack{k=1\\ j,k\neq l}}^{t} (1,1) \begin{bmatrix} (F_jF_k^2)^1 \\ (F_jF_k^2)^2 \end{bmatrix} = \mu + \sum_{\substack{j=1\\ j<k,}}^{t} \sum_{\substack{k=1\\ j,k\neq l}}^{t} \left((F_jF_k^2)^1 + (F_jF_k^2)^2\right). \tag{4.23}$$

Since the main effect F_l occurs alone in S_{1l}, both parameters associated with the linear and quadratic effects of F_l are estimable from T_l. For the alias set S_{2l}, pick any main effect $F_j, j\neq i$ as a reference component. Then we can see that the levels of other main effects are related to the levels of the reference effect by a permutation e, and the levels of F_jF_k are related by a permutation (12). Therefore, we can get the vector ES_{2l} of estimable functions as

$$ES_{2l} = \sum_{\substack{j=1 \\ j \neq l}}^{t} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} (F_j)^1 \\ (F_j)^2 \end{bmatrix} + \sum_{\substack{j=1 \\ j<k}}^{t} \sum_{\substack{k=1 \\ j,k \neq l}}^{t} \begin{bmatrix} 1/2 & -3/2 \\ -1/2 & -1/2 \end{bmatrix} \begin{bmatrix} (F_jF_k)^1 \\ (F_jF_k)^2 \end{bmatrix}. \tag{4.24}$$

Similarly, we can get ES_{3l}, ES_{4l} as follow:

$$ES_{3l} = \sum_{\substack{j=1 \\ j \neq l}}^{t} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} (F_lF_j)^1 \\ (F_lF_j)^2 \end{bmatrix} \tag{4.25}$$

$$ES_{4l} = \sum_{\substack{j=1 \\ j \neq l}}^{t} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} (F_lF_j^2)^1 \\ (F_lF_j^2)^2 \end{bmatrix}. \tag{4.26}$$

Using this near minimal resolution IV design T, we now develop a detection procedure for the influential 2-factor interactions. First, we develop a detection procedure for the first interaction component, and next a procedure for detecting the second interaction component will be developed. Developing procedures separately is only for convenience of explanation. In actual experiments, we can apply both procedures simultaneously to each fraction T_l.

IV.2.2.2 Detection Procedure for the First Interaction Component

< Step 1 >

Consider the alias set S_{3l} in the fraction T_l. If the estimable function ES_{3l} in (4.25) is negligible, then all components $F_lF_j, j = 1,2,...,t, j \neq l$ involved in ES_{3l} are negligible by assumption. Therefore, by testing the null hypothesis $H_0{:}ES_{3l} = 0$ we test the hypothesis that the factor F_l does not interact with other factors.

We obtain the estimate of ES_{3l} and the sum of squares due to ES_{3l} from the fraction T_l as follows

$$\hat{ES}_{3l} = (X'_{3l}X_{3l})^{-1}X'_{3l}\underline{S}_l$$
$$SS(ES_{3l}) = \underline{S}'_l X'_{3l}(X'_{3l}X_{3l})^{-1}X'_{3l}\underline{S}_l \qquad (4.27)$$

,where X_{3l} is the (9×2) model matrix for a reference effect in the alias set S_{3l} and $\underline{S}_l$ is the (9×1) vector of observed SN ratios in T_l .

Under the null hypothesis and the normality assumption of error terms, the quantity $\frac{SS(ES_{3l})}{2\sigma^2}$ has a central χ^2 distribution with 2 degrees of freedom. Therefore, using an independent estimate of σ^2 we can test the null hypothesis. To get an independent estimate of σ^2, it may be necessary to replicate some treatment combinations. For this case, it is recommended that the three treatment combinations $\underline{x} = \underline{0}, \underline{1}, \underline{2}$ be replicated since they are common to all fractions T_l.

If the null hypothesis is not rejected for each T_l, then we can conclude that no influential first interaction component exists. For the case that the null hypothesis is rejected for only two fractions, for example T_1 and T_2, we can say that only one component, namely F_1F_2 , is influential, and we can terminate the detection procedure for the first interaction component. Otherwise, exclude the factors for which the null hypothesis is not rejected for further consideration and go to Step 2. For Step 2, we assume that the retained and excluded factors are renumbered from 1 to t_1, and from $t_1 + 1$ to t, respectively.

<Step 2>

For the fractions T_l, $l = 1,2,..., t_1$ consider the following linear combination of the first interaction components in S_{2l},

$$
\begin{aligned}
ES_{2l}{}^{*} &= ES_{2l} - \sum_{\substack{j=1\\ j\neq l}}^{t} D_e \begin{bmatrix} (F_j)^1 \\ (F_j)^2 \end{bmatrix} \\
&= \sum_{\substack{j=1\\ j<k,}}^{t_1} \sum_{\substack{k=1\\ j,k\neq l}}^{t_1} \begin{bmatrix} 1/2 & -3/2 \\ -1/2 & -1/2 \end{bmatrix} \begin{bmatrix} (F_jF_k)^1 \\ (F_jF_k)^2 \end{bmatrix} .
\end{aligned}
\tag{4.28}
$$

Since we can estimate ES_{2l} and $F_j, j = 1,2,\ldots,t, j \neq i$ from the fractions T_l and T_j, $ES_{2l}{}^{*}$ is also estimable from T. If the null hypothesis $H_0{:}ES_{2l}{}^{*} = 0$ is not rejected, then the first interaction components F_jF_k in (4.28) are all negligible by assumption. Therefore, if the null hypothesis is not rejected for some fraction T_l, then we can conclude that the influential first interaction components are $F_lF_j, j = 1,2,\ldots, t_1, j \neq i$ and we can terminate the detection procedure for the first interaction component.

The function $ES_{2l}{}^{*}$ may be estimated in several ways. One easy method is described here. We can estimate ES_{2l} from the fraction T_l, and $\hat{ES}_{2l}$ can be expressed as a linear combination of $\underline{S}$ in T

$$
\begin{aligned}
\hat{ES}_{2l} &= (X'_{2l}X_{2l})^{-1}X'_{2l}\underline{S}_l \\
&= L'_l\underline{S},
\end{aligned}
\tag{4.29}
$$

where X_{2l} is the (9×2) model matrix for a reference effect in S_{2l} and $\underline{S}_l$ is the (9×1) vector of SN ratios for T_l and $\underline{S}$ is the $((6t + 3) \times 1)$ vector of SN ratios for T, and L_l is a $((6t + 3) \times 2)$ matrix.

Also, we can estimate each $F_j, j = 1,2,\ldots,t, j \neq i$ from the fraction T_j as

$$
\begin{aligned}
\hat{F}_j &= (X'_{1j}X_{1j})X'_{1j}\underline{S}_j, \quad j = 1,2,\ldots t, j \neq i \\
&= K'_j\underline{S},
\end{aligned}
\tag{4.30}
$$

where X_{1j} is the (9×2) model matrix for F_j in the fraction T_j and $\underline{S}_j$ and $\underline{S}$ are as defined in (4.29).

Then an estimate of ES_{2l}^* can be obtained simply as the following linear combination of (4.29) and (4.30),

$$\begin{aligned} \hat{ES}_{2l}^* &= L'_l\underline{S} - \sum_{\substack{j=1 \\ j\neq l}}^{t} K'_j\underline{S} \\ &= (L'_l - \sum_{\substack{j=1 \\ j\neq l}}^{t} K'_j)\underline{S} \\ &= L'\underline{S}\,. \end{aligned} \tag{4.31}$$

Note here that (4.31) is an unbiased estimate but not necessarily the estimate with minimum variance. However, in this paper we will use this type of estimate, hereafter, whenever such an estimate is necessary.

Under the null hypothesis and the normality assumption of error terms, the quantity

$$\frac{\underline{S}'L(L'L)^{-1}L'\underline{S}}{\sigma^2} \tag{4.32}$$

has a central χ^2 distribution with 2 degrees of freedom. Therefore, using an independent estimate of σ^2 we can test the null hypothesis. If the null hypothesis is rejected for all T_i, $i = 1,2,\ldots, t_1$, then go to Step 3.

< Step 3 >

Consider a situation in which only two factors, say F_i and F_j, interact with each other but not with other factors for the first interaction component. Following the nomenclature of Hussain, these two factors are called isolated factors for the first interaction component. If two factors F_i and F_j are isolated factors for the first interaction component, then the alias sets S_{3i} and S_{3j}, respectively, from the fractions T_i and T_j contain only one first interaction component F_iF_j. Therefore, by testing the null hypothesis $H_0{:}ES_{2i} = ES_{3j}$, $i,j = 1,2,\ldots, t_1$, $i \neq j$, we can detect the isolated factors for the first interaction component.

From the fraction T_l, ES_{3l} can be estimated as

$$\hat{ES}_{3l} = (X'_{3l}X_{3l})^{-1}X'_{3l}\underline{S}_l = M'_l\underline{S}. \tag{4.33}$$

Similarly, we can estimate ES_{3j} from the fraction T_j as

$$\hat{ES}_{3j} = M'_j\underline{S}. \tag{4.34}$$

Since both ES_{3l} and ES_{3j} are estimable from the design T, the difference $ES_{3l} - ES_{3j}$ is also estimable and an esitmate is

$$\hat{ES}_{3l} - \hat{ES}_{3j} = (M'_l - M'_j)\underline{S} = M'\underline{S}. \tag{4.35}$$

Under the null hypothesis and the normality assumption of error terms, the quantity

$$\frac{\underline{S}'M(M'M)^{-1}M'\underline{S}}{\sigma^2} \tag{4.36}$$

has a central χ^2 distribution with 2 degrees of freedom. Therefore, using an independent estimate of σ^2 we can test the null hypothesis. If the null hypothesis is not rejected for some pair(s) of retained control factors, then go to the Step 4. Otherwise, go to the Stage II and continue the detection procedure by using the sequential method.

< Step 4 >

Suppose two factors F_k and F_l turn out to be isolated factors for the first interaction component. Then among the first interaction components involving either factor F_k or F_l, only the component F_kF_l is non-negligible.

Let ES_{2l}** denote the linear combination of components in S_{2l} after removing the component F_kF_l along with the main effects F_j, $j = 1,2, \ldots, t, j \neq i$,

$$ES_{2l}^{**} = ES_{2l} - \sum_{\substack{j=1 \\ j \neq l}}^{t} \begin{bmatrix} (F_j)^1 \\ (F_j)^2 \end{bmatrix} - \begin{bmatrix} 1/2 & -3/2 \\ -1/2 & -1/2 \end{bmatrix} \begin{bmatrix} (F_kF_l)^1 \\ (F_kF_l)^2 \end{bmatrix} . \tag{4.37}$$

If the null hypothesis $H_0{:}ES_{2i}^{**} = 0$ is not rejected for some fraction T_i, $i = 1,2,\ldots, t_1$, $i \neq k, l$, then we can conclude that the influential first interaction components are F_iF_j, $j = 1,2,\ldots, t_1$, $j \neq i, k, l$, and F_kF_l.

The interaction component F_kF_l can be estimated from the fractions T_k and T_l as

$$\begin{aligned} (\widehat{F_kF_l})_k &= \widehat{ES}_{3k} \\ &= (X'_{3k}X_{3k})^{-1}X'_{3k}\underline{S}_k \\ &= N'_k\underline{S} \end{aligned} \tag{4.38}$$

$$\begin{aligned} (\widehat{F_kF_l})_l &= \widehat{ES}_{3l} \\ &= (X'_{3l}X_{3l})^{-1}X'_{3l}\underline{S}_l \\ &= N'_l\underline{S}, \end{aligned} \tag{4.39}$$

where $(\widehat{F_kF_l})_i$, $i = k,l$ denote the estimates of F_kF_l from the fraction T_i.

As an estimate of F_kF_l, the average of the two estimate can be used

$$\widehat{F_kF_l} = \frac{1}{2}\{N'_k\underline{S} + N'_l\underline{S}\}. \tag{4.40}$$

Then, since ES_{2i}, $F_j, j = 1,2,\ldots t, j \neq i$ and F_kF_l are all estimable in the design T, we can also estimate (4.37) and it can be expressed as a linear combination of $\underline{S}$, say

$$\widehat{ES}_{2l}^{**} = N'\underline{S}. \tag{4.41}$$

Under the null hypothesis and the normality assumption of error terms, the quantity

$$\frac{\underline{S}'N(N'N)^{-1}N'\underline{S}}{\sigma^2} \tag{4.42}$$

also has a central χ^2 distribution with 2 degrees of freedom. Therefore, we can test the null hypothesis by using an independent estimate of σ^2.

If more than one pair of isolated factors exist for the first interaction component, then the same procedure can be used by subtracting all identified isolated factors in (4.37). Suppose that the null hypothesis is not rejected for some fraction, then terminate the detection procedure for the first interaction component, and identify the influential first interaction components. Otherwise, go to Stage II in which we continue the detection procedure.

IV.2.2.3 Detection Procedure for the Second Interaction Component

The detection procedure for the second interaction component can be described by following the same steps as for the first interaction component.

<Step 1>

Consider the alias set S_{4i} in (4.21) for the fraction T_i. If the estimable function ES_{4i} in (4.26) is negligible then all second interaction components $F_iF_j^2, j = 1,2,..., t, j \neq i$ involved in ES_{4i} are negligible by assumption. Therefore, if the null hypothesis $H_0{:}ES_{4i} = 0$ is not rejected for some fraction T_i, then we conclude that the control factor F_i does not interact with any other control factors through the second interaction component.

Since the nine observations in the fraction T_i do provide an orthogonal design for estimating $ES_{ji}, j = 0,1,2,4$, we can also devise a test statistic for the null hypothesis following the same procedure as in Step 1 for the first interaction component. As in the detection procedure for the first interaction component, if the null hypothesis is not rejected for all T_i, $i = 1,2,\ldots,t$, then terminate the detection procedure for the second interaction component and conclude that there exists no influential second interaction component. In the case in which the null hypothesis is rejected for only two fractions, say T_l and T_j, we can conclude that only one second interaction component $F_lF_j^2$ is influential. Otherwise, go to Step 2 excluding the control factors for which the null hypothesis is not rejected. In Step 2, t_2 control factors are assumed to be retained. And we also assume that retained and excluded control factor are renumbered from 1 to t_2, and from $t_2 + 1$ to t, respectively.

< Step 2 >

Consider the estimable function ES_{0l} for the alias set S_{0l} in (4.21). Since the estimate $\hat{ES}_{0l}$ is the average SN ratio for nine treatment combinations in T_l, we can also express $\hat{ES}_{0l}$ as the linear combination of SN ratios in the design T, such that

$$\hat{ES}_{0l} = \underline{R}'_l\underline{S}, \tag{4.43}$$

where $\underline{R}_l$ is a $((6t+3) \times 1)$ vector and $\underline{S}$ is defined in (4.29).

It can be shown (Anderson and Thomas, 1976) that we can estimate the overall mean μ in the design T. Therefore, since ES_{0l} and μ are estimable in T, we can also estimate $ES_{0l}{}^*$ in T as

$$\begin{aligned} ES_{0l}{}^* &= ES_{0l} - \mu \\ &= \sum_{\substack{j=1 \\ j<k}}^{t_2} \sum_{\substack{k=1 \\ j,k\neq l}}^{t_2} \left((F_jF_k^2)^1 + (F_jF_k^2)^2\right) \end{aligned} \tag{4.44}$$

and the estimate $\hat{ES}_{0l}{}^*$ can be expressed as a linear combination of $\underline{S}$,

$$\hat{ES}_{0l}{}^* = \underline{R}'\underline{S}. \tag{4.45}$$

If the null hypothesis $H_0{:}ES_{0i}^* = 0$ is not rejected for a fraction T_i, $i = 1,2,\ldots, t_2$ then all second interaction components $F_jF_k^2, j,k = 1,2,\ldots, t_2, j \neq k \neq i$, are said to be negligible. This implies that all influential second interaction components involve factor F_i. That is, $F_iF_j^2, j = 1,2,\ldots, t_2, j \neq i$ can be identified as influential. If the null hypothesis is not rejected for some fraction T_i, then terminate the detection procedure and identify the influential second interaction component. Otherwise, go to Step 3 to continue the procedure.

Under the null hypothesis and the normality assumption of error terms, the quantity $\frac{(\underline{R}'S)^2}{(\underline{R}'\underline{R})\sigma^2}$ has a central χ^2 distribution with one degree of freedom. Therefore, we can test the null hypothesis by using an independent estimate of σ^2.

<Step 3>

Consider the following null hypothesis

$$H_0{:}ES_{4i} = ES_{4j}, \quad i \neq j. \tag{4.46}$$

This null hypothesis is true if the two factors F_i and F_j are isolated factors for the second interaction component. Therefore, we can identify the isolated factors for the second interaction component by testing this null hypothesis.

For testing the null hypothesis, the same kind of test statistic can be developed as in Step 3 for the detection procedure for the first interaction component. If the null hypothesis is not rejected for some pair(s) of retained t_2 control factors then go to Step 4. Otherwise, go to Stage II in which a sequential method will be used.

<Step 4>

Suppose two factors, say F_k and F_l, are isolated factors for the second interaction component. Then we can estimate the following linear combination of the second interaction components in the same way as described in Step 4 for the first interaction component,

$$ES_{0l}^{**} = ES_{0l} - \mu - (F_jF_k^2)^1 - (F_jF_k^2)^2 \ . \tag{4.47}$$

If the null hypothesis H_0:$ES_{0l}^{**} = 0$ is not rejected for a fraction T_i, $i = 1,2,..., t_2, i \neq k,l$, then we can identify the influential second interaction components as $F_iF_j^2, j = 1,2,..., t_2, j \neq i,k,l$ and $F_kF_l^2$, and we can terminate the detection procedure for the second interaction component. Otherwise, go to Stage II in which we will continue the detection procedure.

The test statistic for the null hypothesis can be also developed following the same procedure used in Step 4 for the first interaction component. Moreover, if we identify more than one pair of isolated factors in Step 3, a similar test procedure can be developed easily by subtracting all identified isolated second interaction components from ES_{0l} .

IV. 2.3 Stage II: Detection Procedure With Sequential Designs

IV. 2.3.1 Deletion of Isolated Factors

In this stage, we will continue the detection procedure by modifying the sequential method developed by Hussain. In the sequential detection procedure developed in this section, we will not split the interaction into the two components, but consider them simultaneously. Before applying the sequential procedure, we can remove some control factors for which information obtained in Step 1 is sufficient to identify the interaction structure for those factors.

Two control factors which are isolated for both interaction components will be called completely isolated factors. If they are isolated factors for only one interaction component, then we call them partially isolated factors. Before applying the sequential procedure, we remove the completely isolated factors and the partially isolated factors which do not belong also to the interaction structure of another interaction component. These isolated factors can be removed because the interaction structure for them is completely identified. After removing such factors, we renumber the remaining factors, say from 1 to t_3.

<Example 4.2>

Consider a 3^{10} factorial parameter design in which we identify the following interaction at the end of Stage I:

For the first interaction component, we identified four non-interacting control factors F_1, F_2, F_8, F_9, two isolated control factors F_3, F_4, and we failed to identify the interaction structure for the control factors F_5, F_6, F_7, F_{10}. And for the second interaction component, we identified two pairs of isolated factors (F_3 and F_4, F_2 and F_7), and we need to continue the detection procedure for the factors F_1, F_5, F_6, F_{10}. This structure can be symbolically expressed as

First interaction component: $(F_5, F_6, F_7, F_{10}),(F_3, F_4)$

Second interaction component: $(F_1, F_5, F_6, F_{10}),(F_3, F_4),(F_2, F_7)$.

For this situation we see that the interaction structure for the completely isolated factors F_3, F_4 is identified completely. Moreover, the partially isolated factor F_2 for the second interaction component does not interact with other factors with respect to the first interaction component. Therefore, we can remove these 3 control factors and narrow the sequential procedure only to 5 factors, namely F_1, F_5, F_6, F_7, and F_{10} .

IV. 2.3.2 Hypothesis Testing in the Sequential Procedure

Consider the following $1/3^{t-1}$ fractions of 3^t factorial design obtained as the solutions to

$$B_i \underline{x} = \underline{b}, \quad i = 1,2, \dots , t \tag{4.48}$$

,where each B_i is a $(t-1)\times t$ matrix of rank (t-1) obtained from the identity matrix of order t by deleting the i-th row, and $\underline{b}$ is any $((t-1)\times 1)$ vector of constants in $GF(3)$.

In the equations (4.48), the rows of B_i represent the defining vectors $\underline{\alpha}$ of the main effects $F_j, j\neq i$ and the corresponding elements of $\underline{b}$ represent a level of each such main effect in the treatment combinations obtained as solutions to the equations.

Since there are 3 treatment combinations for the fraction (4.48), the 2 alias sets are

$$\begin{aligned} Z_{0i} &= \{\mu, F_j, F_jF_k, F_jF_k^2, \quad j,k=1,2,\ldots t, j,k\neq i\} \\ Z_{1i} &= \{F_i, F_iF_j, F_iF_j^2, \quad j=1,2,\ldots t, j\neq i\}. \end{aligned} \tag{4.49}$$

The estimable functions for these alias sets can be expressed as

$$\begin{aligned} EZ_{0i} &= \mu + \sum_{\substack{j=1\\ j\neq i}}^{t} \underline{c}'_{1j} \begin{bmatrix} (F_j)^1 \\ (F_j)^2 \end{bmatrix} + \sum_{\substack{j=1\\ j<k}}^{t} \sum_{\substack{k=1\\ j,k\neq i}}^{t} \underline{c}'_{2jk} \begin{bmatrix} (F_jF_k)^1 \\ (F_jF_k)^2 \end{bmatrix} + \sum_{\substack{j=1\\ j<k}}^{t} \sum_{\substack{k=1\\ j,k\neq i}}^{t} \underline{c}'_{3jk} \begin{bmatrix} (F_jF_k^2)^1 \\ (F_jF_k^2)^2 \end{bmatrix} \\ EZ_{1i} &= \begin{bmatrix} (F_i)^1 \\ (F_i)^2 \end{bmatrix} + \sum_{\substack{j=1\\ j\neq i}}^{t} D_{1j} \begin{bmatrix} (F_iF_j)^1 \\ (F_iF_j)^2 \end{bmatrix} + \sum_{\substack{j=1\\ j\neq i}}^{t} D_{2j} \begin{bmatrix} (F_iF_j^2)^1 \\ (F_iF_j^2)^2 \end{bmatrix}, \end{aligned} \tag{4.50}$$

where $\underline{c}_{1j}, \underline{c}_{2jk}, \underline{c}_{3jk}$ are (2×1) vectors determined from C in (4.5), and D_{1j}, D_{2j} are (2×2) matrices determined from (4.17) and the coefficient vectors $\underline{c}_{1j}, \underline{c}_{2jk}, \underline{c}_{3jk}$ and matrices D_{1j}, D_{2j} are determined by the vector $\underline{b}$ of constants in (4.48).

For any given factors F_i and F_j, consider the hypothesis

$$H_1{:}F_iF_j = F_iF_j^2 = \underline{0}, \tag{4.51}$$

where F_iF_j, $F_iF_j^2$ denote (2×1) vectors of parameters defined in (4.14), and $\underline{0}$ represents a null vector. This hypothesis implies that the two factors F_i and F_j do not interact with each other. It can be shown (Hussain ,1986) that we can test this hypothesis with six treatment combinations obtained as solutions to the following two fractions

$$B_l \underline{x} = \underline{b}_1$$
$$B_l \underline{x} = \underline{b}_2 \tag{4.52}$$

such that $b_{1k} = b_{2k}$ for all k except $k = j$, where b_{lk}, $l = 1,2$, $k = 1,2,\ldots,t$ denote the elements of $\underline{b}_l$.

Hussain showed that under the hypothesis H_1 and normality assumption of error terms, the following sum of squares

$$SS_H = (\underline{S}_1 - \underline{S}_2)'C(C'C)^{-1}C'(\underline{S}_1 - \underline{S}_2) \ , \tag{4.53}$$

where $\underline{S}_i$, $i = 1,2$ are (3×1) vectors of SN ratios for the fractions in (4.52), and C is a (3×2) matrix defined in (4.5), has mean $4\sigma^2$ and is distributed as $4\sigma^2\chi^2$ with 2 degrees of freedom. Therefore, we can develop a test statistic for H_1 by using an independent estimate of σ^2.

Moreover, Hussain showed that for a given factor F_i, the hypothesis

$$H_2{:}F_iF_k = F_iF_k^2 = \underline{0} \ \forall \ k \text{ except } k \neq j \tag{4.54}$$

can be tested with observations from the two fractions (4.52) such that $b_{1v} \neq b_{2v}$ for all v except $v = j$. The hypothesis H_2 can be used to test the null hypothesis that the factor F_i interacts only with the factor F_j. We can also develop a test statistic for this hypothesis following the same procedure for H_1.

The similar hypothesis

$$H_3{:}F_iF_k = F_iF_k^2 = \underline{0} \ \forall \ k \text{ except } k = j,l, \text{ for a given factor } F_i \tag{4.55}$$

can be also tested by two fractions (4.52) such that $b_{1v} \neq b_{2v}$ for all v except $v = j,l$. Moreover, we can construct the same kind of hypotheses H_4, H_5,..., *etc.* ,and they can be tested by using observations from the two fractions in (4.52). For testing each hypothesis, the same kind of test statistic for H_1 can be easily devised by using the of sum of squares (4.53).

IV. 2.3.2 Strategies for Selecting Fractions

Consider the following three fractions

$$\begin{aligned} B_t\underline{x} &= \underline{b}_1 \\ B_t\underline{x} &= \underline{b}_2 \\ B_t\underline{x} &= \underline{b}_3 \\ \text{where } \ \underline{b}_1 \neq \underline{b}_2 &\neq \underline{b}_3 \ . \end{aligned} \tag{4.56}$$

Since these three fractions correspond to the three parallel spaces in the finite Euclidean geometry $EG(t,3)$, there are no common treatment combinations among the three fractions. Let us rewrite (4.56) in matrix form as

$$B_t\underline{x} = [\underline{b}_1, \underline{b}_2, \underline{b}_3] \ .$$

The total number of treatment combinations in (4.56) is nine. If we take $\underline{b}_1 = \underline{0}$, $\underline{b}_2 = \underline{1}$, and $\underline{b}_3 = \underline{2}$, then the equations

$$B_t\underline{x} = [\underline{0}, \underline{1}, \underline{2}] \tag{4.57}$$

can be reduced to (4.58) by subtracting the first row from every other row,

$$\begin{bmatrix} \underline{B}'_1 \\ A_t \end{bmatrix} \underline{x} = \begin{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \end{bmatrix} \tag{4.58}$$

,where A_t is defined in (4.19), and $\underline{B}'_1$ is the first row of B_t.

This implies that (4.57) is equivalent to (4.19) in Stage I ($A_i\underline{x} = \underline{0}$) Therefore, the fraction T_i obtained from (4.19) is exactly the same as the fraction obtained from (4.57). The following fact reveals some useful information on selecting fractions in the sequential procedure.

< Proposition 4.1 >

Consider three treatment combinations obtained as solutions to any one of the following three sets of equations ,

$$\begin{aligned} B_i\underline{x} &= \underline{0}_j \\ B_i\underline{x} &= \underline{1}_j \qquad , i \neq j, \\ B_i\underline{x} &= \underline{2}_j \end{aligned} \tag{4.59}$$

in which, for example $\underline{0}_j$ denotes a vector which comes from $\underline{0}$ by changing an element corresponding to factor F_j from 0 to $v \neq 0$ ($v \in GF(3)$), leaving the other elements unchanged, where $\underline{0}$ represents a vector with every element 0. The vectors $\underline{1}_j, \underline{2}_j$ are defined similarly.

Then this fraction has one treatment combination in common with the fraction T_j obtained from (4.19).

< proof >

Without loss of generality, assume that $i < j$. Then, three treatment combinations in the fraction $B_i\underline{x} = \underline{b}_j$ in (4.59) are

$$\begin{array}{r} \text{Factors } 1-(i-1): \\ \text{Factor } i: \\ \text{Factors } (i+1)-(j-1): \\ \text{Factor } j: \\ \text{Factors } (j+1)-t: \end{array} \begin{bmatrix} \underline{b} & \underline{b} & \underline{b} \\ 0 & 1 & 2 \\ \underline{b} & \underline{b} & \underline{b} \\ v & v & v \\ \underline{b} & \underline{b} & \underline{b} \end{bmatrix}$$

and the nine treatment combinations in the fraction T_j are

$$\begin{array}{r} \text{Factors } 1-(j-1): \\ \text{Factor } j: \\ \text{Factors } (j+1)-t: \end{array} \begin{bmatrix} \underset{\sim}{0} & \underset{\sim}{1} & \underset{\sim}{2} & \underset{\sim}{0} & \underset{\sim}{1} & \underset{\sim}{2} & \underset{\sim}{0} & \underset{\sim}{1} & \underset{\sim}{2} \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\ \underset{\sim}{0} & \underset{\sim}{1} & \underset{\sim}{2} & \underset{\sim}{0} & \underset{\sim}{1} & \underset{\sim}{2} & \underset{\sim}{0} & \underset{\sim}{1} & \underset{\sim}{2} \end{bmatrix}.$$

By comparing these two sets of treatment combinations, we can easily see that they have one treatment combination in common for any value of v and the vector $\underset{\sim}{b}$.

Q.E.D.

Since the fraction T_l in Stage I is identical with the fraction (4.57), we can test the two hypotheses H_1 and H_2 in (4.51) and (4.54) by taking one additional fraction, for example

$$B_l \underset{\sim}{x} = \underset{\sim}{1}_j(10) \tag{4.60}$$

,where $\underset{\sim}{1}_j(10)$ denotes a vector which comes from the vector $\underset{\sim}{1}$ by changing an element corresponding to factor F_j from 1 to 0.

That is, we can test H_1 and H_2 by following pairs of fractions: for H_1

$$\begin{aligned} B_l \underset{\sim}{x} &= \underset{\sim}{1} \\ B_l \underset{\sim}{x} &= \underset{\sim}{1}_j(10) \end{aligned} \tag{4.61}$$

and for H_2

$$\begin{aligned} B_l \underset{\sim}{x} &= \underset{\sim}{0} \\ B_l \underset{\sim}{x} &= \underset{\sim}{1}_j(10). \end{aligned} \tag{4.62}$$

Note that the treatment combinations corresponding to the fractions $B_l \underset{\sim}{x} = \underset{\sim}{0}$ and $B_l \underset{\sim}{x} = \underset{\sim}{1}$ are available from Stage I. Moreover, we need to take two more treatment combinations to test H_1 and H_2, since one of the three treatment combinations corresponding to the fraction (4.60) is available from Stage I by the Proposition 4.1. Then each hypothesis H_1 and H_2 can be tested by using 6 treatment combinations corresponding to fractions (4.61) and (4.62).

In order to test another hypothesis, for example H_3, we have to take three more treatment combinations from a fraction, say

$$B_k \underline{x} = \underline{1}_{jk}(10) \tag{4.63}$$

,where $\underline{1}_{jk}(10)$ denotes a vector obtained from the vector $\underline{1}$ by changing two elements corresponding to factors F_j, F_k from 1 to 0.

Then the hypothesis H_3 can be tested by using six treatment combinations obtained from

$$\begin{aligned} B_k \underline{x} &= \underline{0} \\ B_k \underline{x} &= \underline{1}_{jk}(10) \ . \end{aligned} \tag{4.64}$$

Some comments about taking treatment combinations seems to be necessary at this point. For testing the hypotheses H_1, H_2 and H_3, we take the treatment combinations from the fractions such as (4.60), (4.63). However, there are many possible ways for choosing appropriate fractions. As an example, the hypotheses H_1 and H_2 can be tested by using other pairs of fractions: for H_1 ,

$$\begin{aligned} B_k \underline{x} &= \underline{2} \\ B_k \underline{x} &= \underline{2}_j(21) \end{aligned} \tag{4.65}$$

and for H_2

$$\begin{aligned} B_k \underline{x} &= \underline{1} \\ B_k \underline{x} &= \underline{2}_j(21) \ . \end{aligned} \tag{4.66}$$

For this case, we take three treatment combinations corresponding to the following fraction, among which one treatment combination is available in Stage I,

$$B_k \underline{x} = \underline{2}_j(21) \ . \tag{4.67}$$

Therefore, in practical applications of parameter design analyses we should take proper treatment combinations from, for example (4.60) or (4.67), by taking cost or engineering considerations into account as we discussed in 2^f factorial parameter designs.

Another comment is related to the testing of hypotheses. At the end of Stage I, some interaction component, for example $F_iF_j^2$ may be identified as being negligible. Even though, in testing the hypothesis, say $H_1: F_iF_j = F_iF_j^2 = 0$, the negligible interaction components $F_iF_j^2$ is involved, the sequential testing procedure is still valid. This kind of information about the negligible interaction components may be useful in identifying influential interaction components at the final step after completing the whole detection procedure. As an illustration, suppose that in Example 4.2, we conclude that interactions exist between factors F_1 and F_5, F_5 and F_6, F_6 and F_{10}, and F_5 and F_7. Then, since we can identify the interaction components $F_1F_5, F_5F_7^2$ as negligible based on the information obtained from Stage I, the following components can be identified finally as influential interaction components:

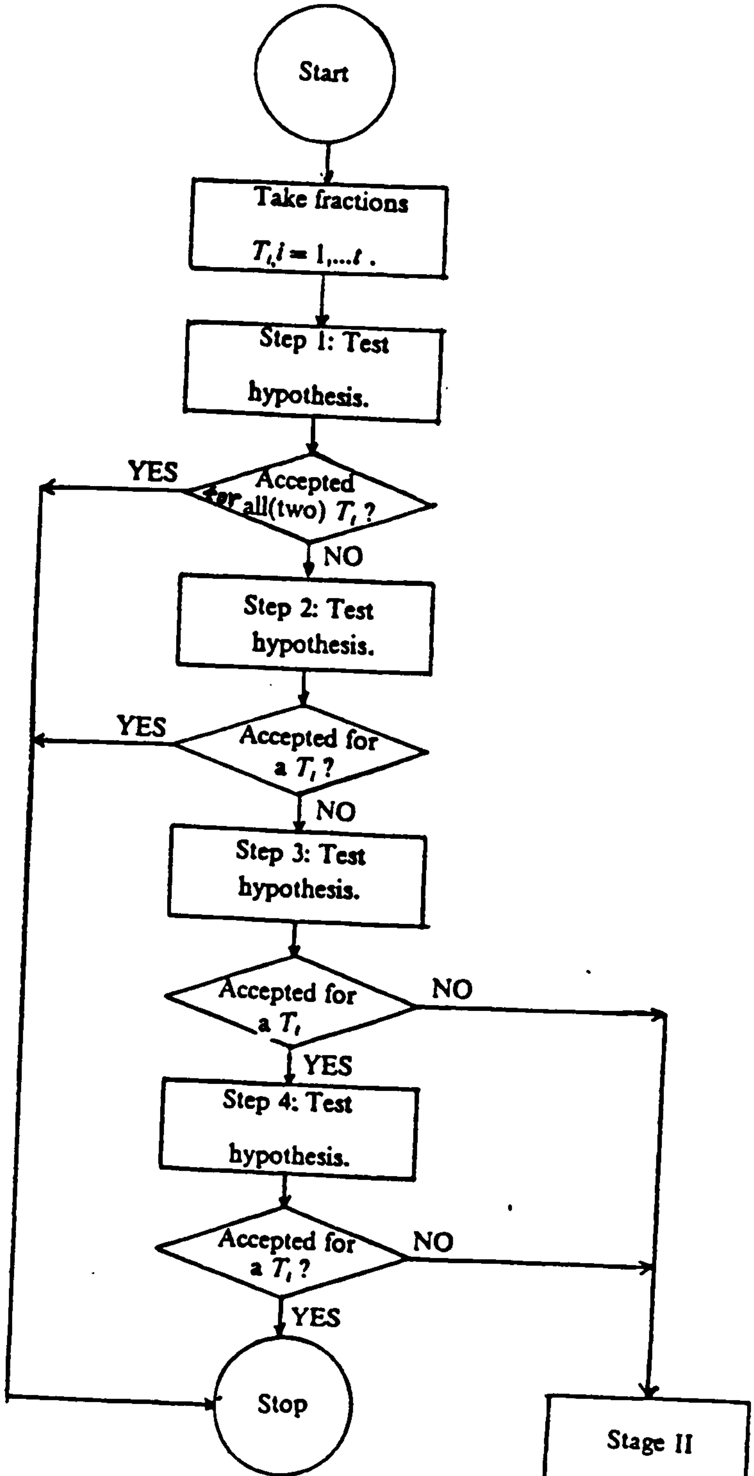
First interaction component: $F_5F_6, F_5F_7, F_6F_{10}, F_3F_4$

Second interaction component: $F_1F_5^2, F_5F_6^2, F_6F_{10}^2, F_3F_4^2, F_2F_7^2$.

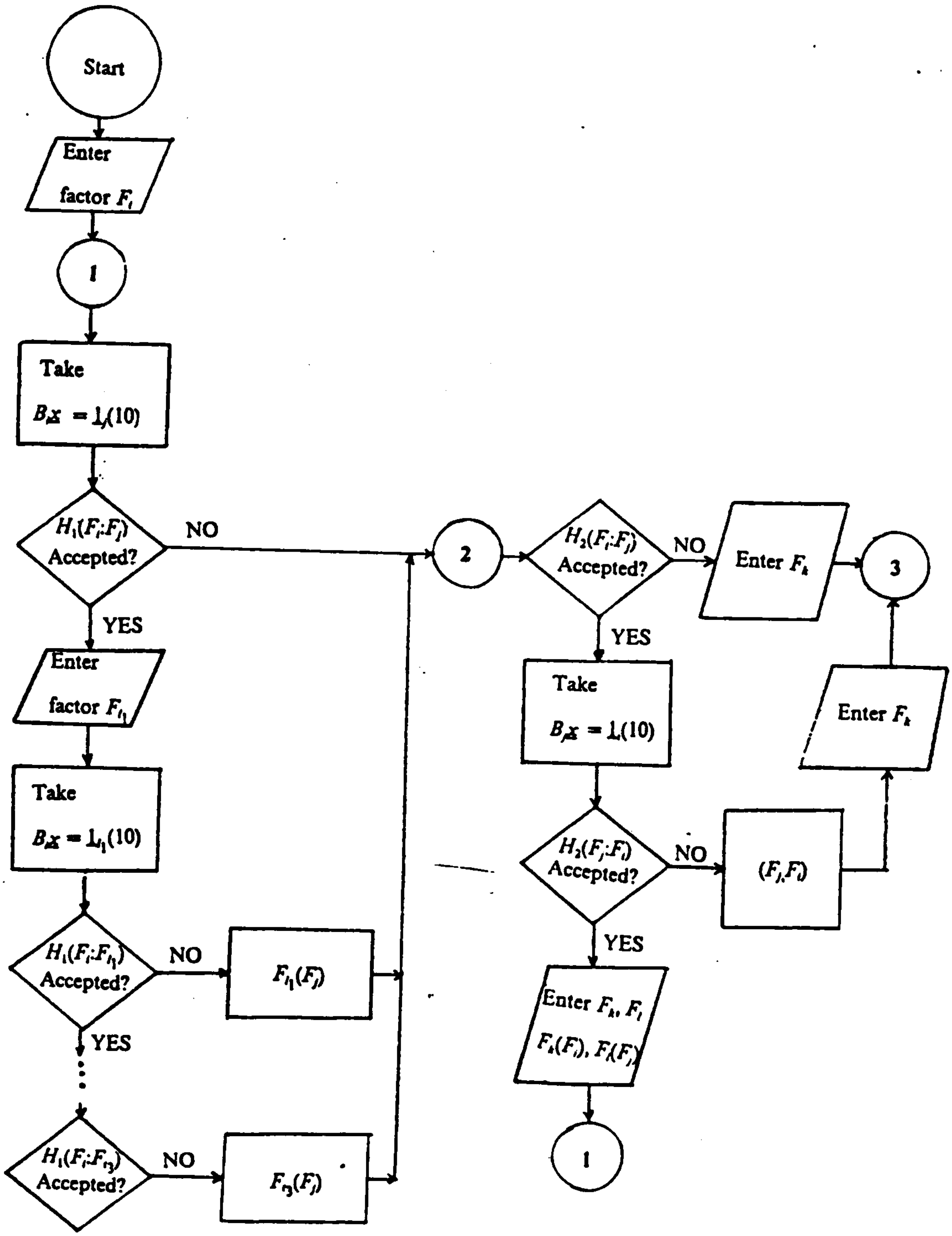
Following Hussain, the detection procedure developed in Stage I and II can be summarized in the block diagrams of Figure 4.1, Figure 4.2, and Figure 4.3. Figure 4.1 can be applied to both detection procedures for the first and second interaction component. In Figures 4.2 and 4.3, we developed the loops 1, 2, and 3 to test the hypotheses H_1, H_2 and H_3, and more loops can be developed, if necessary.

< Table 4.1: Illustration of Symbols in Figure 4.2 and 4.3 >

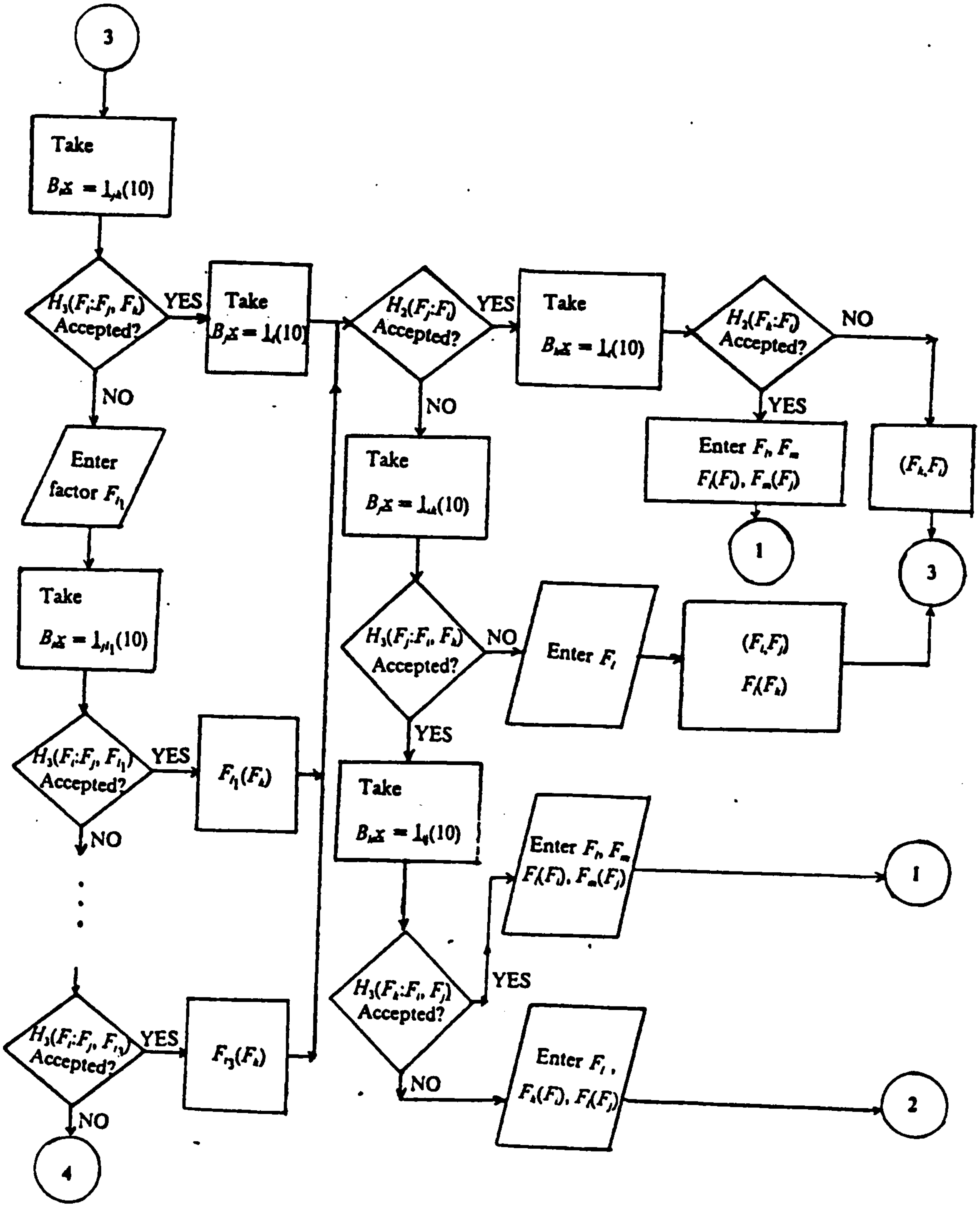
$H_1(F_i, F_j)$	: The hyphothesis H_1 in (4.51) to test that F_i, F_j do not interact each other.
$H_2(F_i : F_j)$	: The hypothesis H_2 in (4.54) to test that F_i interacts only with F_j.
$H_3(F_i : F_j, F_k)$	: The hyphothesis H_3 in (4.55) to test that F_i interacts only with F_j and F_k.
$F_j\,(F_i)$	: Replace factor F_i by factor F_j.
(F_i, F_j)	: Exchange roles of F_i and F_j.
①	: Loop 1.



< Figure 4.1: Block Diagram of Detection Procedure in Stage I >



< Figure 4.2: Block Diagram of Detection Procedure in Stage II >



< Figure 4.3: Block Diagram of Detection Procedure in Stage II >

IV. 3 Identification of Optimal Setting

Let T^* denote the design which is the union of treatment combinations taken in Stages I and II, and let $\underline{S}^*$ represent the vector of observed SN ratios for the design T^*. Then we have the final model

$$
\begin{aligned}
E(\underline{S}^*) &= X\beta \\
&= [X_1 X_2]\begin{bmatrix} \underline{\beta}_1 \\ \underline{\beta}_2 \end{bmatrix}
\end{aligned}
\tag{4.68}
$$

,where $\underline{\beta}_1$ denotes a $((2t+1)\times 1)$ vector of the overall mean μ and 2t parameters associated with t main effects, and $\underline{\beta}_2$ denotes a vector of parameters associated with the identified influential 2-factor interaction components, and X_1, X_2 are model matrices corresponding to $\underline{\beta}_1, \underline{\beta}_2$, respectively.

Since $\underline{\beta}_1$ is estimable in Stage I, it is also estimable from the final design T^*. However, $\underline{\beta}_2$ may not be estimable. To estimate β_1, write the normal equations in partitioned form

$$
\begin{bmatrix} X'_1X_1 & X'_1X_2 \\ X'_2X_1 & X'_2X_2 \end{bmatrix}\begin{bmatrix} \hat{\underline{\beta}}_1 \\ \hat{\underline{\beta}}_2 \end{bmatrix} = \begin{bmatrix} X'_1 \\ X'_2 \end{bmatrix}\underline{S}^* . \tag{4.69}
$$

Clearly, $X'X$ is not in general of full rank since not all detected interaction components are estimable. But the inverse of X'_1X_1 exists since all the main effects are estimable. Moreover, X'_1X_2 is not equal to zero since the columns of X_1 are not orthogonal to the columns of X_2. Therefore, $\hat{\underline{\beta}}_1$ is not simply $(X'_1X_1)^{-1}X'_1\underline{S}^*$. Using a generalized inverse for a partitioned matrix (see Searle (1971), and Anderson and Thomas (1979)), we obtain the solutions to (4.69) as

$$\begin{bmatrix}\hat{\underline{\beta}}_1\\ \hat{\underline{\beta}}_2\end{bmatrix} = (X'X)^{-}X'\underline{S}^*$$

$$= \begin{bmatrix}(X'_1X_1)^{-1} + (X'_1X_1)^{-1}X'_1X_2Q^{-}X'_2X_1(X'_1X_1)^{-1}, & -(X'_1X_1)^{-1}X'_1X_2Q^{-}\\ -Q^{-}X'_2X_1(X'_1X_1)^{-1} & Q^{-}\end{bmatrix}\begin{bmatrix}X'_1\underline{S}^*\\ X'_2\underline{S}^*\end{bmatrix} \tag{4.70}$$

,where $Q = X'_2X_2 - X'_2X_1(X'_1X_1)^{-1}X'_1X_2$ and A^{-} denotes a generalized inverse of A.

Then we obtain the minimum variance unbiased estimate of $\underline{\beta}_1$ as

$$\begin{aligned}\hat{\underline{\beta}}_1 &= [(X'_1X_1)^{-1} + (X'_1X_1)^{-1}X'_1X_2Q^{-}X'_2X_1(X'_1X_2)^{-1}]X'_2\underline{S}^*\\ &\quad - (X'_1X_1)^{-1}X'_1X_2Q^{-}X'_2\underline{S}^* .\end{aligned} \tag{4.71}$$

Note that $\underline{\beta}_1$ includes μ and two parameters $(E^\alpha)^1,(E^\alpha)^2$ for each main effect E^α. And the two parameters associated with main effect E^α are defined in (4.6) and (4.7) as

$$\begin{aligned}(E^\alpha)^1 &= E_0^\alpha - E_2^\alpha\\ (E^\alpha)^2 &= E_0^\alpha + E_2^\alpha - 2E_1^\alpha\end{aligned} \tag{4.72}$$

,where $E_j^\alpha, j = 0,1,2$ is defined in (4.8).

Therefore, using a relationship $E_0^\alpha + E_1^\alpha + E_2^\alpha = 0$, we can estimate E_j^α for each main effect E^α by solving the following three equations in three unknowns $\hat{E}_j^\alpha$, $j = 0, 1, 2$.

$$\begin{aligned}&\hat{E}_0^\alpha - \hat{E}_2^\alpha = (\hat{E}^\alpha)^1\\ &\hat{E}_0^\alpha + \hat{E}_2^\alpha - 2\hat{E}_1^\alpha = (\hat{E}^\alpha)^2\\ &\hat{E}_0^\alpha + \hat{E}_1^\alpha + \hat{E}_2^\alpha = 0\end{aligned} \tag{4.73}$$

Then we identify the optimal setting for each control factor as the level j for which $\hat{E}_j^\alpha$ is the largest.

For interacting two control factors, we can also identify the optimal combination of factor levels simply by evaluating the average response for the nine possible combinations of factor levels. Therefore we can identify the optimal setting of control factors in the same way as explained for 2^t factorial parameter designs.

Some control factor which has the least significant effect on the *SN* ratios and does not interact with other significant control factors can be used as an adjustment factor. If a supplementary analysis of the mean response of a quality characteristic is conducted, then we may choose a control factor which has the most significant effect on the mean as an adjustment factor among the non-significant control factors in the analysis of *SN* ratio. Then we can bring the average response of a quality characteristic close to the target value with its variance minimized by correcting adjustment factor levels.

Chapter V

SN RATIOS FOR SEVERAL QUALITY CHARACTERISTICS

In chapter II, we discussed various kinds of *SN* ratios for the parameter design analysis for one quality characteristic. The basic premise of such univariate parameter design analysis is that the quality of a product can always be assessed by the most important quality characteristic. However, the quality of a product is seldom defined by a single quality characteristic. Rather, quality is a composite of several properties which are often interrelated and are different in their relative importance in contributing to the overall quality of a product. In this chapter, we shall develop some *SN* ratios for parameter design analysis where we deal with several quality characteristics.

V.1 Loss Function and Expected Loss for Several Quality Characteristics

In the multiresponse parameter design analysis in which we are taking more than one quality characteristic into account, we have the same experimental design structure as in the univariate parameter design analysis except that we observe a vector of several quality characteristics. That is, given an inner array of control factors and outer array of noise factors for the experiment, we observe several variables (quality characteristics) for each combination of settings for inner array and outer array. Therefore, we can analyze the multiresponse parameter design following the same procedure as for the univariate parameter design except for using different types of *SN* ratios which take several quality characteristics into account simultaneously.

Let the $(g \times 1)$ vector $\underline{Y}' = (Y_1, Y_2, ..., Y_g)$ be a vector of g variables (quality characteristics) of interest. Then for a given setting $\underline{\theta}$ of control factors, the output vector $\underline{y}$ of $\underline{Y}$ is generated, through the transfer function, by the noise factors $\underline{W}$. Since the noise factors are assumed to be random, the output vector will also be a random vector which is assumed here to have some continuous distribution with mean vector $\underline{\mu}' = (\mu_1, \mu_2, ..., \mu_g)$ and $(g \times g)$ variance-covariance matrix Σ. Suppose that the vector of target values of $\underline{Y}'$ is $\underline{\tau}' = (\tau_1, \tau_2, ..., \tau_g)$ with each τ_i being a finite number. If the specific value $\underline{y}$ of $\underline{Y}$ deviates from $\underline{\tau}$, then a loss occurs. Let $l(\underline{y})$ represent the loss function (say, represented in terms of dollars) which is assumed to be convex and sufficiently smooth so that its second derivative for each variable exists.

Then, by the Taylor's series expansion of $l(\underline{y})$ at $\underline{\tau}$ we approximate $l(\underline{y})$ as

$$l(\underline{y}) \approx l(\underline{\tau}) + [(\underline{y} - \underline{\tau}) \cdot \nabla] l(\underline{y}) |_{\underline{y} = \underline{\tau}} + 1/2(\underline{y} - \underline{\tau})' \left[\frac{\partial^2 l(\underline{y})}{\partial \underline{y}' \partial \underline{y}} \right]_{\underline{y} = \underline{\tau}} (\underline{y} - \underline{\tau}) \qquad (5.1)$$

,where ∇ denotes the gradient vector.

Since the loss function has a minimum value at $\underset{\sim}{y} = \underset{\sim}{\tau}$, the second term of the right hand side in (5.1) is zero. And assuming that the loss is always zero when the values of the quality characteristics are exactly equal to their target values, we can express the loss function, apart from a constant term, as

$$l(\underset{\sim}{y}) = (\underset{\sim}{y} - \underset{\sim}{\tau})' B (\underset{\sim}{y} - \underset{\sim}{\tau}) \ , \tag{5.2}$$

where

$$B = (b_{ij}) = \left[\frac{\partial^2 l(\underset{\sim}{y})}{\partial \underset{\sim}{y}' \partial \underset{\sim}{y}} \right]_{\underset{\sim}{y} = \underset{\sim}{\tau}}$$

is the second derivative matrix of $l(\underset{\sim}{y})$ evaluated at $\underset{\sim}{y} = \underset{\sim}{\tau}$.

Then the average loss or expected loss at the setting $\underset{\sim}{\theta}$ of control factors can be approximated as

$$\begin{aligned} R(\underset{\sim}{\theta}) &= E_{\underset{\sim}{Y}}(l(\underset{\sim}{y})) \\ &= E_{\underset{\sim}{Y}}[(\underset{\sim}{y} - \underset{\sim}{\tau})' B (\underset{\sim}{y} - \underset{\sim}{\tau})] \\ &= tr(B\Sigma) + (\underset{\sim}{\mu} - \underset{\sim}{\tau})' B (\underset{\sim}{\mu} - \underset{\sim}{\tau}) \end{aligned} \tag{5.3}$$

where $tr(A)$ is the trace of the matrix A .

Note here that the values of the loss function (5.2) and the expected loss (5.3) should be invariant under change of scaling of measurement units for any variable since we express the loss function in terms of objective quantity, say in dollars. If we change scaling of measurement units for some quality characteristic, then another loss function which has the same loss as the original one can be derived simply by transforming the original loss function. Moreover, in (5.2) and (5.3), the relative importance of different quality characteristics are also taken into account in the coefficient matrix B.

From the equation (5.3) we see that the average loss $R(\underset{\sim}{\theta})$ depends on $B, \Sigma, \underset{\sim}{\mu}$, and $\underset{\sim}{\tau}$. Just as for the univariate loss function, information about independence of the variance-covariance matrix and the mean vector is useful for decomposing the average loss into the variance part and the bias part in order to develop suitable *SN* ratios. However, for the multiresponse case a general method for detecting this independence is not yet available. To devise such a method and to simplify the expression for the expected loss (5.3), consider the following transformation of the original variables $\underline{Y}$ into the canonical variables $\underline{Z}$,

$$\underline{Z} = U\underline{Y} \tag{5.4}$$

where U is $(g \times g)$ orthogonal matrix whose rows are the normalized eigenvectors of Σ.
Then we have

$$E(\underline{Z}) = U\underline{\mu} \\ = \underline{\delta}$$

$$Var(\underline{Z}) = U\Sigma U' \\ = D \tag{5.5}$$

,where D is a diagonal matrix whose diagonal elements are the eigenvalues corresponding to each row of U.

Since the value of the loss function should be invariant under any transformation, we define here the loss function $l_1(\underline{z})$ for the specific value $\underline{z}$ of the transformed variables $\underline{Z}$ as

$$l_1(\underline{z}) = l(U^{-1}\underline{z}). \tag{5.6}$$

Then we can easily see the following fact.

< Proposition 5.1 >

Consider the orthogonal transformation (5.4). Then the loss function $l_1(\underline{z})$ defined in (5.6) for the transformed variables has its minimum value at $\underline{z} = \underline{\tau}_0 = U\underline{\tau}$.

< Proof >

Since U is an orthogonal matrix, the inverse U^{-1} always exists. Since $l(\underline{y})$ its the minimum value at $\underline{y} = \underline{x}$ by assumption, we have

$$
\begin{aligned}
& l(\underline{x}) \leq l(\underline{y}) \quad \forall \underline{y} \\
\Rightarrow \; & l(U^{-1}\underline{x}_0) \leq l(U^{-1}\underline{z}) \quad \forall_{\underline{z}} \quad \text{where} \quad \underline{x}_0 = U\underline{x} \\
\Rightarrow \; & l_1(\underline{x}_0) \leq l_1(\underline{z}) \quad \forall \underline{z}
\end{aligned}
$$

which implies that $l_1(\underline{z})$ has the minimum value at $\underline{x}_0 = U\underline{x}$.

Q.E.D.

If we expand $l_1(\underline{z})$ about $\underline{z} = \underline{x}_0$ by the Taylor's series expansion, then by the proposition (5.1) we can approximate the loss function $l(\underline{y})$, apart from the constant, by

$$
\begin{aligned}
l(\underline{y}) &= l_1(\underline{z}) \\
&= l_1(\underline{x}_0) + [(\underline{z} - \underline{x}_0) \cdot \nabla] l_1(\underline{z}) |_{\underline{z} = \underline{x}_0} \\
&\quad + 1/2(\underline{z} - \underline{x}_0)' \left[\frac{\partial^2 l_1(\underline{z})}{\partial \underline{z}' \partial \underline{z}} \right]_{\underline{z} = \underline{x}_0} (\underline{z} - \underline{x}_0) + \cdots \\
&\approx (\underline{z} - \underline{x}_0)' H (\underline{z} - \underline{x}_0)
\end{aligned}
\tag{5.7}
$$

,where

$$
\begin{aligned}
H &= (h_{ij}) \\
&= \left[\frac{\partial^2 l_1(\underline{z})}{\partial \underline{z}' \partial \underline{z}} \right]_{\underline{z} = \underline{x}_0}
\end{aligned}
$$

is a $(g \times g)$ matrix of second derivatives of $l_1(\underline{z})$ at $\underline{z} = \underline{x}_0$.
Then the expected loss $R(\underline{\theta})$ at the setting $\underline{\theta}$ of control factors can be approximated by

$$
R(\underline{\theta}) = tr(HD) + (\underline{\delta} - \underline{x}_0)' H (\underline{\delta} - \underline{x}_0) \; . \tag{5.8}
$$

Since the transformed variables are mutually independent, we can investigate the independence of the variance-covariance matrix D and the mean vector $\underline{\delta}$ by using some statistical methods such as g scatter plots of paired values (δ_j, λ_j) for $j = 1,2,...,g$ or exploratory data analysis methods. That is, for each transformed variable Z_j, $j = 1,2,...,g$, calculate the sample mean values $\hat{\delta}_{ij}$ and variances $\hat{\lambda}_{ij}$, $i = 1,2,...,m$ defined by

$$\hat{\delta}_{ij} = \hat{\underline{u}}'_{ij}\hat{\underline{\mu}}_i$$
$$\hat{\lambda}_{ij} = \hat{\underline{u}}'_{ij}S_i\hat{\underline{u}}_{ij} \tag{5.9}$$

,where $\hat{\underline{\mu}}_i$ and S_i are the usual estimates of mean vector $\underline{\mu}$ and variance-covariance matrix Σ , respectively, based on n observed vectors $\underline{y}_{i1}, \underline{y}_{i2}, ..., \underline{y}_{in}$ at the i-th setting $\underline{\theta}_i$ of control factors, and $\hat{\underline{u}}'_{ij}$ is the j-th normalized eigenvector of S_i.

Then for each Z_j, based on m paired values $(\hat{\lambda}_{ij}, \delta_{ij})$, $i = 1,2,..., m$, determine the relationship between the variance λ_j and the mean δ_j . If the variance and mean vector are functionally independent for each Z_i, then we can conclude that the variance-covariance matrix D is functionally independent of the mean vector $\underline{\delta}$ for the transformed variables. Otherwise, D is said to be dependent on $\underline{\delta}$.

V.2 SN Ratios for Specified Loss Function

For some multiresponse parameter design analyses, a loss function can be provided in advance, for example from past experience. Moreover, if the approximation of a loss function up to the third term in the Taylor series expansion (5.1) is appropriate, then it may often be possible to estimate a quadratic loss function, even when a prior loss function cannot be specified. A quadratic loss function could be estimated by using economic arguments similar to those suggested by Taguchi (1986) for estimating univariate loss function. For example, if we know the actual values of the loss for a sufficiently large number of points $\underline{Y}$, we can estimate the coefficient matrix B in (5.2) by assuming the quadratic loss function. When the loss function for several quality characteristics can be specified, we can develop the different SN ratios according to whether the variance-covariance matrix D depends on the mean vector $\underline{\delta}$ or not. The maximization process of such SN ratios is then equivalent to the minimization of the approximate average loss (5.3).

V.2.1 Variance-covariance Matrix not Linked to Mean Vector

For this situation, the variance-covariance matrix D and the mean vector $\underline{\delta}$ of the transformed variables $\underline{Z}$ are functionally independent of each other. This kind of independence is the usual assumption for multivariate ANOVA procedures and such situations can occur frequently in many engineering applications. Then the bias part of average loss (5.8) can be reduced independently of the variance part $tr(HD)$. Therefore, we minimize the variance part first by analyzing the SN ratio, and then reduce the bias part by adjusting the mean responses close to target values. This adjustment can be made by correcting the levels of adjustment factors which have non-significant effects on the SN ratio. The appropriate SN ratio for this case is a measure whose maximization is equivalent to minimization of $tr(HD)$ in (5.8). Thus a proper SN ratio for this case is

$$\begin{aligned}\eta(\underline{\theta}) &= -10\log_{10}(tr(HD)) \\ &= -10\log_{10}(h_{11}\lambda_1 + h_{22}\lambda_2 + \dots + h_{gg}\lambda_g),\end{aligned} \tag{5.10}$$

where $h_{jj}, \lambda_j, j = 1,2,\dots,g$ are the diagonal elements of H and D, respectively.

The following fact enables us to estimate the SN ratio (5.10) directly from the loss functions of non-transformed variables $\underline{Y}$ or transformed variables $\underline{Z}$.

< Proposition 5.2 >

Consider the two loss function $l(\underline{y})$ and $l_1(\underline{z})$ in (5.2) and (5.6). Let $\underline{G}$ and $\underline{G}_1$ be gradient vectors of $l(\underline{y})$ and $l_1(\underline{z})$, respectively and let B and H be second derivative matrices of $l(\underline{y})$ and $l_1(\underline{z})$, respectively, then the following relationships hold.

$$\underline{G}_1 = U\underline{G}$$
$$H = UBU' \tag{5.11}$$

<Proof>

Note that

$$\underline{G}_1 = \begin{bmatrix} \frac{\partial l_1(\underline{z})}{\partial z_1} \\ \frac{\partial l_1(\underline{z})}{\partial z_2} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial l_1(\underline{z})}{\partial z_g} \end{bmatrix} = \begin{bmatrix} \frac{\partial l(\underline{y})}{\partial z_1} \\ \frac{\partial l(\underline{y})}{\partial z_2} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial l(\underline{y})}{\partial z_g} \end{bmatrix} \quad \underline{G} = \begin{bmatrix} \frac{\partial l(\underline{y})}{\partial y_1} \\ \frac{\partial l(\underline{y})}{\partial y_2} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial l(\underline{y})}{\partial y_g} \end{bmatrix}$$

and

$$\begin{aligned} \underline{y} &= U^{-1}\underline{z} \\ &= U'\underline{z} \\ &= \begin{bmatrix} \underline{y}'_1\underline{z} \\ \underline{y}'_2\underline{z} \\ \cdot \\ \cdot \\ \cdot \\ \underline{y}'_g\underline{z} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_g \end{bmatrix} \end{aligned}$$

where $\underline{y}_i$ is the i-th column of U. Then,

$$\underline{v}'_i = \left(\frac{\partial y_i}{\partial z_1}, \frac{\partial y_i}{\partial z_2}, \ldots, \frac{\partial y_i}{\partial z_g}\right), \quad i = 1,2,\ldots g.$$

And, by the chain rule

$$\begin{aligned}
\frac{\partial l_1(\underline{z})}{\partial z_i} &= \frac{\partial l(\underline{y})}{\partial z_i} \\
&= \frac{\partial l(\underline{y})}{\partial y_1}\frac{\partial y_1}{\partial z_i} + \frac{\partial l(\underline{y})}{\partial y_2}\frac{\partial y_2}{\partial z_i} + \cdots + \frac{\partial l(\underline{y})}{\partial y_g}\frac{\partial y_g}{\partial z_i} \\
&= \left(\frac{\partial y_1}{\partial z_i}, \frac{\partial y_2}{\partial z_i}, \ldots, \frac{\partial y_g}{\partial z_i}\right)\begin{bmatrix} \frac{\partial l(\underline{y})}{\partial y_1} \\ \frac{\partial l(\underline{y})}{\partial y_2} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial l(\underline{y})}{\partial y_g} \end{bmatrix} \\
&= \underline{u}'_i \underline{G}
\end{aligned}$$

,where $\underline{u}'_i$ is the i-th row of U.

Therefore,

$$\underline{G}_1 = \begin{bmatrix} \frac{\partial l_1(\underline{z})}{\partial z_1} \\ \frac{\partial l_1(\underline{z})}{\partial z_2} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial l_1(\underline{z})}{\partial z_g} \end{bmatrix} = \begin{bmatrix} \underline{u}'_1 \\ \underline{u}'_2 \\ \cdot \\ \cdot \\ \cdot \\ \underline{u}'_g \end{bmatrix} G = U\underline{G} .$$

Let h_{ij} be the (i,j) element of H. Then applying the chain rule twice, and using the fact that $\frac{\partial^2 y_i}{\partial z_j \partial z_k} = 0$ since y_i is a linear function of z_j, $i,j = 1,2,\ldots,g$, we have

$$
\begin{aligned}
h_{ij} &= \frac{\partial^2 l_1(\underline{z})}{\partial z_i \partial z_j} = \frac{\partial^2 l(\underline{y})}{\partial z_i \partial z_j} \\
&= \frac{1}{\partial z_i}\left(\frac{\partial l(\underline{y})}{\partial y_1}\cdot\frac{\partial y_1}{\partial z_j} + \frac{\partial l(\underline{y})}{\partial y_2}\cdot\frac{\partial y_2}{\partial z_j} + \cdots + \frac{\partial l(\underline{y})}{\partial y_g}\cdot\frac{\partial y_g}{\partial z_j}\right) \\
&= \frac{1}{\partial z_i}\left(\frac{\partial l(\underline{y})}{\partial y_1}\cdot\frac{\partial y_1}{\partial z_j}\right) + \frac{1}{\partial z_i}\left(\frac{\partial l(\underline{y})}{\partial y_2}\cdot\frac{\partial y_2}{\partial z_j}\right) + \cdots + \frac{1}{\partial z_i}\left(\frac{\partial l(\underline{y})}{\partial y_g}\cdot\frac{\partial y_g}{\partial z_j}\right) \\
&= \left(\frac{\partial^2 l(\underline{y})}{\partial y_1^2}\cdot\frac{\partial y_1}{\partial z_i} + \frac{\partial^2 l(\underline{y})}{\partial y_1 \partial y_2}\cdot\frac{\partial y_2}{\partial z_i} + \cdots + \frac{\partial^2 l(\underline{y})}{\partial y_1 \partial y_g}\cdot\frac{\partial y_g}{\partial z_i}\right)\cdot\frac{\partial y_1}{\partial z_j} \\
&+ \left(\frac{\partial^2 l(\underline{y})}{\partial y_2 \partial y_1}\cdot\frac{\partial y_1}{\partial z_i} + \frac{\partial^2 l(\underline{y})}{\partial y_2^2}\cdot\frac{\partial y_2}{\partial z_i} + \cdots + \frac{\partial^2 l(\underline{y})}{\partial y_2 \partial y_g}\cdot\frac{\partial y_g}{\partial z_i}\right)\cdot\frac{\partial y_2}{\partial z_j} \\
&\;\;\vdots \\
&+ \left(\frac{\partial^2 l(\underline{y})}{\partial y_g \partial y_1}\cdot\frac{\partial y_1}{\partial z_i} + \frac{\partial^2 l(\underline{y})}{\partial y_g \partial y_2}\cdot\frac{\partial y_2}{\partial z_i} + \cdots + \frac{\partial^2 l(\underline{y})}{\partial y_g^2}\cdot\frac{\partial y_g}{\partial z_i}\right)\cdot\frac{\partial y_g}{\partial z_j} \\
&= \left(\frac{\partial y_1}{\partial z_i},\ \frac{\partial y_2}{\partial z_i},\ \ldots,\ \frac{\partial y_g}{\partial z_i}\right)
\begin{bmatrix}
\frac{\partial^2 l(\underline{y})}{\partial y_1^2} & \frac{\partial^2 l(\underline{y})}{\partial y_1 \partial y_2} & \cdots & \frac{\partial^2 l(\underline{y})}{\partial y_1 \partial y_g} \\
\frac{\partial^2 l(\underline{y})}{\partial y_2 \partial y_1} & \frac{\partial^2 l(\underline{y})}{\partial y_2^2} & \cdots & \frac{\partial^2 l(\underline{y})}{\partial y_2 \partial y_g} \\
\vdots & \vdots & & \vdots \\
\frac{\partial^2 l(\underline{y})}{\partial y_g \partial y_1} & \frac{\partial^2 l(\underline{y})}{\partial y_g \partial y_2} & \cdots & \frac{\partial^2 l(\underline{y})}{\partial y_g^2}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial y_1}{\partial z_j} \\
\frac{\partial y_2}{\partial z_j} \\
\vdots \\
\frac{\partial y_g}{\partial z_j}
\end{bmatrix} \\
&= \underline{u}'_i B \underline{u}_j \; .
\end{aligned}
$$

It then follows that

$$H = (h_{ij}) = UBU' \; .$$

Q.E.D.

From this proposition, we can see analytically that the two expressions for expected loss (or loss function) in (5.3) and (5.8) are the same, that is,

$$\begin{aligned} R(\underline{\theta}) &= tr(HD) + (\underline{\delta} - \underline{\tau}_0)'H(\underline{\delta} - \underline{\tau}_0) \\ &= tr(UBU'U\Sigma U') + (\underline{\mu} - \underline{\tau})'U'UBU'U'U(\underline{\mu} - \underline{\tau}) \\ &= tr(UB\Sigma U') + (\underline{\mu} - \underline{\tau})'U'UBU'U(\underline{\mu} - \underline{\tau}) \\ &= tr(B\Sigma) + (\underline{\mu} - \underline{\tau})'B(\underline{\mu} - \underline{\tau}) \ . \end{aligned}$$

Moreover, we can easily obtain h_{jj} in the expression of the *SN* ratio (5.10) from the loss function (5.2) in terms of the original variables

$$h_{jj} = \underline{u}_j B \underline{u}'_j, \quad j = 1,2,\ldots,g \ , \tag{5.12}$$

where $\underline{u}'_j, j = 1,2,\ldots,g$ is the j-th row of the matrix U in the transformation (5.4). Therefore, in practice the *SN* ratio (5.10) at the setting $\underline{\theta}_i$ of the control factors can then be estimated as

$$S(\underline{\theta}_i) = -10\log(tr(BS_i)) \tag{5.13}$$

or

$$\begin{aligned} S(\underline{\theta}_i) &= -10\log(\hat{h}_{i11}\hat{\lambda}_{i1} + \hat{h}_{i22}\hat{\lambda}_{i2} + \ldots + \hat{h}_{igg}\hat{\lambda}_{ig}), \quad i = 1,2,\ldots,m \\ &\text{with } \hat{h}_{ijj} = \underline{u}'_{ij} B \underline{u}_{ij}, \end{aligned} \tag{5.14}$$

where $\underline{u}'_{ij}$ and $\hat{\lambda}_{ij}, j = 1,2,\ldots,g$ are the eigenvectors and corresponding eigenvalues of the sample variance-covariance matrix S_i based on n observations $\underline{y}_{i1}, \underline{y}_{i2}, \ldots, \underline{y}_{in}$ at the setting $\underline{\theta}_i$.

V.2.2 Variance-covariance Matrix Linked to Mean Vector

In this case, the variance-covariance matrix D and the mean vector $\underline{\delta}$ are functionally dependent so that the mean vector cannot be adjusted independently of the variance-covariance matrix. Therefore, we have to use the expected loss $R(\underline{\theta})$ or some monotone function of $R(\underline{\theta})$ such as $-10\log_{10}(R(\underline{\theta}))$, as an SN ratio. Thus an appropriate SN ratio for this case is

$$\begin{aligned}\eta(\underline{\theta}) &= -10\log_{10}\{tr(HD) + (\underline{\delta} - \underline{x}_0)'H(\underline{\delta} - \underline{x}_0)\} \\ &= -10\log_{10}\{tr(B\Sigma) + (\underline{\mu} - \underline{x})'B(\underline{\mu} - \underline{x})\}\,.\end{aligned} \tag{5.15}$$

Let $\underline{y}_{i1}, \underline{y}_{i2}, \ldots, \underline{y}_{in}$ be a random sample from the distribution $\underline{Y}$ at a given setting $\underline{\theta}_i$ of the control factors then the estimate of (5.15) is

$$S(\underline{\theta}_i) = -10\log\{tr(BS_i) + (\hat{\underline{\mu}}_i - \underline{x})'B(\hat{\underline{\mu}}_i - \underline{x})'\}, \qquad i = 1,2,\ldots,m \tag{5.16}$$

for a given loss function of (5.2), where S_i and $\hat{\underline{\mu}}_i$ are the sample estimates of Σ and $\underline{\mu}$, respectively at the setting $\underline{\theta}_i$.

After identifying the optimal setting at which the expected loss is minimized, we sometimes need to adjust the mean value(s) of some quality characteristic(s) to bring them close to the target value(s) when the mean value(s) deviate from the target value(s). This adjustment procedure is necessary especially when the unbiasedness property is vital for some quality characteristics. In such a situation, we recommend to use the control factors which have non-significant effects on the SN ratio as adjustment factors. However, we note that after such an adjustment process, the average loss may not be minimized at the final setting of control factors. This is a price we may have to pay in order to have some quality characteristics close to their target values.

For best results, the detection procedure for the independence of the variance-covariance matrix D and the mean vector $\underline{\delta}$, and the resulting SN ratios should be used only when all of the quality characteristics are measured in the same units. This is because the eigenvalues and corresponding eigenvectors of the sample variance-covariance matrix S are not invariant under the change of measurement scale for any quality characteristic. Otherwise, the detection procedures should be performed on the sample variance-covariance matrix S^* based on the scaled quality characteristics such as $V_j = \frac{Y_j}{\sigma_j}$ $j = 1,2,\ldots,g$, where σ_j^2 is the variance of the quality characteristic Y_j. Therefore, we summarize the procedure as follows:

<Step 1>

For each setting $\underline{\theta}_i$, $i = 1,2,\ldots,m$ of control factors, calculate the sample variance $\hat{\sigma}_j^2$ for each quality characteristic Y_j, $j = 1,2,\ldots,g$ based on n observations. Then scale each quality characteristic to $V_j = \frac{Y_j}{\hat{\sigma}_j}$ and calculate the sample-covariance matrix S_i^* and mean vector $\underline{\mu}_i^*$ for the scaled quality characteristics.

<Step 2>

Calculate the eigenvalues and corresponding eigenvectors of S_i^* and transform the scaled quality characteristics to the canonical variables by using the orthogonal transformation (5.4). Then the eigenvalues and eigenvectors are invariant under change of measurement scale.

<Step 3>

Determine the independence of the variance-covariance matrix and the mean vector for the canonical variables.

<Step 4>

Depending on the results of Step 3, determine a suitable SN ratio and estimate the SN ratio from the loss function of the scaled variables for each setting $\underline{\theta}_i$ and proceed with the appropriate parameter design analysis.

Moreover, for developing SN ratios where the loss function can be specified, we assume that all target values are finite. If the target values of some (all) quality characteristics are infinite, then we transform such quality characteristics by taking reciprocals. The target values of the transformed quality characteristics are then changed to finite numbers (actually they are zeros). Therefore, we can develop the same *SN* ratios such as (5.10) and (5.15) using the same arguments for the loss function of such transformed quality characteristics.

V.3 SN Ratios When a Loss Function Cannot Be Specified

V.3.1 Description

In the previous section, two types of *SN* ratios were developed when we can specify the loss function for several quality characteristics. However, in practice we may not be provided with the pre-specified loss function outside the experiments, nor may not it be possible to estimate a suitable loss function.

Even though a loss function cannot be specified, we can use the procedure described in section V.2 for detecting independence of the variance-covariance matrix and the mean vector. Let us assume that the variance-covariance matrix D and the mean vector $\underline{\delta}$ of the transformed variables are functionally independent, and as a result we decide to use the *SN* ratio (5.10). The coefficients h_{jj} and eigenvalues λ_j, $j = 1,2,\dots,g$ in (5.10) are all positive since both matrices H and Σ are positive definite. Therefore, the maximum value of the *SN* ratio (5.10) can be achieved when each λ_j is minimized since the h_{jj} are unknown positive constants depending on the loss function. This leads to the same kinds of problems that one encounters in developing optimal designs by minimizing all eigenvalues simultaneously. Instead we minimize the sum of the eigenvalues. In the case of *SN* ratios this leads us the following *SN* ratio

$$\eta(\underline{\theta}) = -10\log\left[\sum_{l=1}^{g}\lambda_l\right] = -10\log[tr(D)] = -10\log[tr(\Sigma)] \quad (5.17)$$

Besides (5.17), minimization of the product of eigenvalues gives us another type of SN ratio,

$$\eta(\underline{\theta}) = -10\log\left[\prod_{l=1}^{g}\lambda_l\right] = -10\log(|D|) = -10\log(|\Sigma|) \quad (5.18)$$

The SN ratio (5.17) corresponds to the minimization of the average variance for g quality characteristics and (5.18) refers to the minimization of the volume of the variance-covariance matrix Σ. However, the approximate SN ratio (5.17) is meaningful only when each quality characteristic is measured in the same units and all the quality characteristics are of the same relative importance. But, for a multiresponse parameter design analysis, variables are nearly always obtained from different scales of measurements and are not necessarily of the same relative importance. Consequently the combination of variances of several variables measured in different units as in (5.17) is physically meaningless. The same kind of difficulty also arises for the SN ratio (5.18). Moreover, the determinant of the variance-covariance matrix may also be small when some off-diagonal elements of the variance-covariance matrix are large. That is, the SN ratio (5.18) can be large especially when the correlations among the quality characteristics are large. Therefore, the SN ratios (5.17) and (5.18) are not appropriate for general usage, especially (5.18) is not suitable even when all the quality characteristics are measured in the same units and they are the same in relative importance. Moreover, when the variance-covariance matrix D and mean vector $\underline{\delta}$ of the transformed variables are functionally dependent, it is more difficult to devise the proper SN ratio since we cannot evaluate the loss function. Therefore, when a loss function cannot be specified for a multiresponse parameter design, we should develop the SN ratio from a different point of view in which a loss function is not utilized.

According to the philosophy espoused by Taguchi the primary goal of a multiresponse parameter design is to achieve minimum variance for each quality characteristic with its location close to some target value. Here the variance includes variation among the units of the product and the variation due to deterioration and the environmental conditions under which the product is actually used. In accordance with this objective, we shall develop *SN* ratios which are independent of the means of all quality characteristics and whose maximization corresponds to the minimization of some combined measure of variances of all quality characteristics. Then we can adjust the quality characteristics to their target values using the adjustment factors which have non-significant effects on the *SN* ratios if such adjustment procedure is necessary.

V.3.2 Determination of Dispersion Measure

The first step in developing the *SN* ratios is to determine the functional relationship between the variance and the mean for each quality characteristic. As we discussed in the univariate parameter design (see section II.2.3), this procedure is necessary mainly because we cannot adjust the mean response to the target value independently of the variance if the variance of a quality characteristic depends on the mean level. Therefore, we must, first of all, determine the extent of dependency of the mean and the variance for each quality characteristic. That is, for each quality characteristic Y_j we determine the value of p_j from the data

$$\sigma_j \propto (\mu_j)^{p_j} \quad j = 1,2,\ldots,g \ , \tag{5.19}$$

where σ_j and μ_j are the standard deviation and mean value of a single quality characteristic Y_j.

For each Y_j, the value of p_j in (5.19) can be determined from the m paired values $(\hat{\sigma}_{ij}, \hat{\mu}_{ij})$, $i = 1,2,\ldots,m$. If $p_j = 0$ for some Y_j, for example, then it means essentially that the variance and the mean are functionally independent and the variance is constant over the level of mean values for the quality characteristic Y_j. And $p_j = 1$ means that the standard deviation increases (decreases) proportionally with the mean.

Then, for the quality characteristic Y_j, the quantity

$$D_j = \left(\frac{\sigma_j}{(\mu_j)^{p_j}}\right)^2 \quad j = 1,2,\ldots,g \tag{5.20}$$

represents some measure of "dispersion" of y_j where "dispersion" means the measure of variation which is independent of the mean level.

After determining the relationship of the variance and the mean for each Y_j, the next step is to combine each D_j into one quantitative measure representing overall dispersion. However, as we discussed earlier, combination of several D_j obtained from different measurement units is physically meaningless and only valid when each Y_j is measured in the same units and each Y_j is of the same relative importance for evaluating the overall quality of a product. As an example, in the manufacturing process of a leaf spring in chapter II, the strength of a leaf spring may be so important that we should include this property as another quality characteristic of the leaf spring along with the property of free height in unloaded condition. The strength of the leaf spring can be measured by the pressure in psi (pounds per square inches) units at which the leaf spring is broken. And the free height is measured by in inches. If we combine directly the two measures of dispersion of these two quality characteristics, then the resulting quantity is not meaningful since the real significance of the dispersion depends on the property of each quality characteristic. That is, the measure of dispersion, say 20 for free-height is not twice as important as the measure of dispersion 40 for the strength variable. Moreover, dispersion is not invariant under the change of scale of the measurement unit.

One approach to such combining problem is to transform each dispersion measure D_j to some index number represented in a dimensionless scale. Then, after transformation combine transformed index numbers into one quantity representing the total measurement of dispersions for several quality characteristics. Harrington (1965) introduced such a transformation and combining scheme utilizing what he termed the desirability function and illustrated its use in the manufacturing process of a rubber compound for tire treads. Even though the idea of desirability function has been appealing to experimenters who want to optimize several quality characteristics, this approach has not been widely used nor has much research been done on it. We will employ the concept of a desirability function in transforming the dispersion measure of each quality characteristic and develop the appropriate *SN* ratios.

V.3.3 Transformation of Dispersion Measures

In transforming each dispersion measure D to some index variable W, there are many possible ways in determining the range of W. Following Harrington (1965), and Derringer and Suich (1980), we use a scale which ranges between 0 and 1, where $W = 0$ corresponds to an absolutely undesirable value of dispersion and $W = 1.0$ represents a completely desirable value of dispersion. And we may assign a value, say $W = 0.5$ when the value of D is acceptable but poor. This assigning scheme of W depends on the manufacturer's standpoint and the relative importance of the quality characteristic. The scale of W is a dimensionless scale to which any dispersion measure D can be transformed so that its value can be expressed in terms of goodness, say "measure of desirability for dispersion".

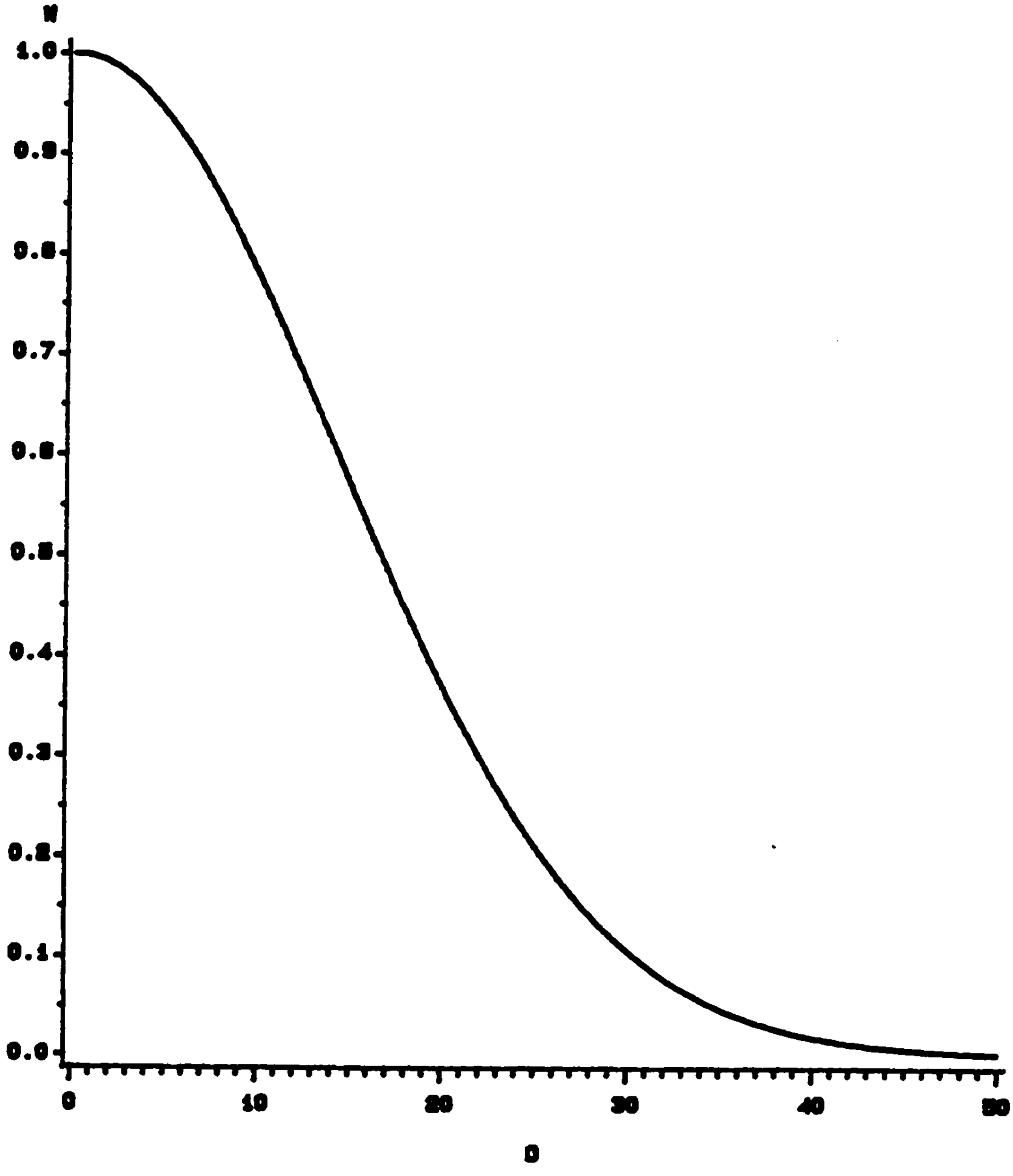
After determining the range of W, we then transform the measured value of D to the desirability scale W. If no other suitable form of transformation is specified, the following exponential transformation is convenient and adequate for the purpose of transforming measured dispersion D to the desirability scale W(see Harrington, 1965).

$$W = e^{-(cD)^2} \tag{5.21}$$

The constant c in the equation determines the slope of the curve. In Figure 5.1, the values of measured dispersion are represented along the horizontal line and the corresponding values of W are specified along the vertical line. The constant c in Figure 5.1 is 0.05. The value of c in the transformation (5.21) can be determined by selecting some values of D and corresponding values of W for the quality characteristic. Then from the paired values of (D, W) we can determine the value of c easily by the equation

$$c = -\frac{\log W}{2D} \quad . \tag{5.22}$$

Note here that the transformation (5.21) is just one example of the many possible ways in transforming the measured value D of the dispersion to the dimensionless scale W. If we can evaluate the values of W for many points of D, then we can also draw the appropriate transformation curve simply by plotting the paired values of (D, W) and connecting the points by a smooth curve.



<FIGURE 5.1>: TRANSFORMATION CURVE

One possible alternate form of equation (5.21) is

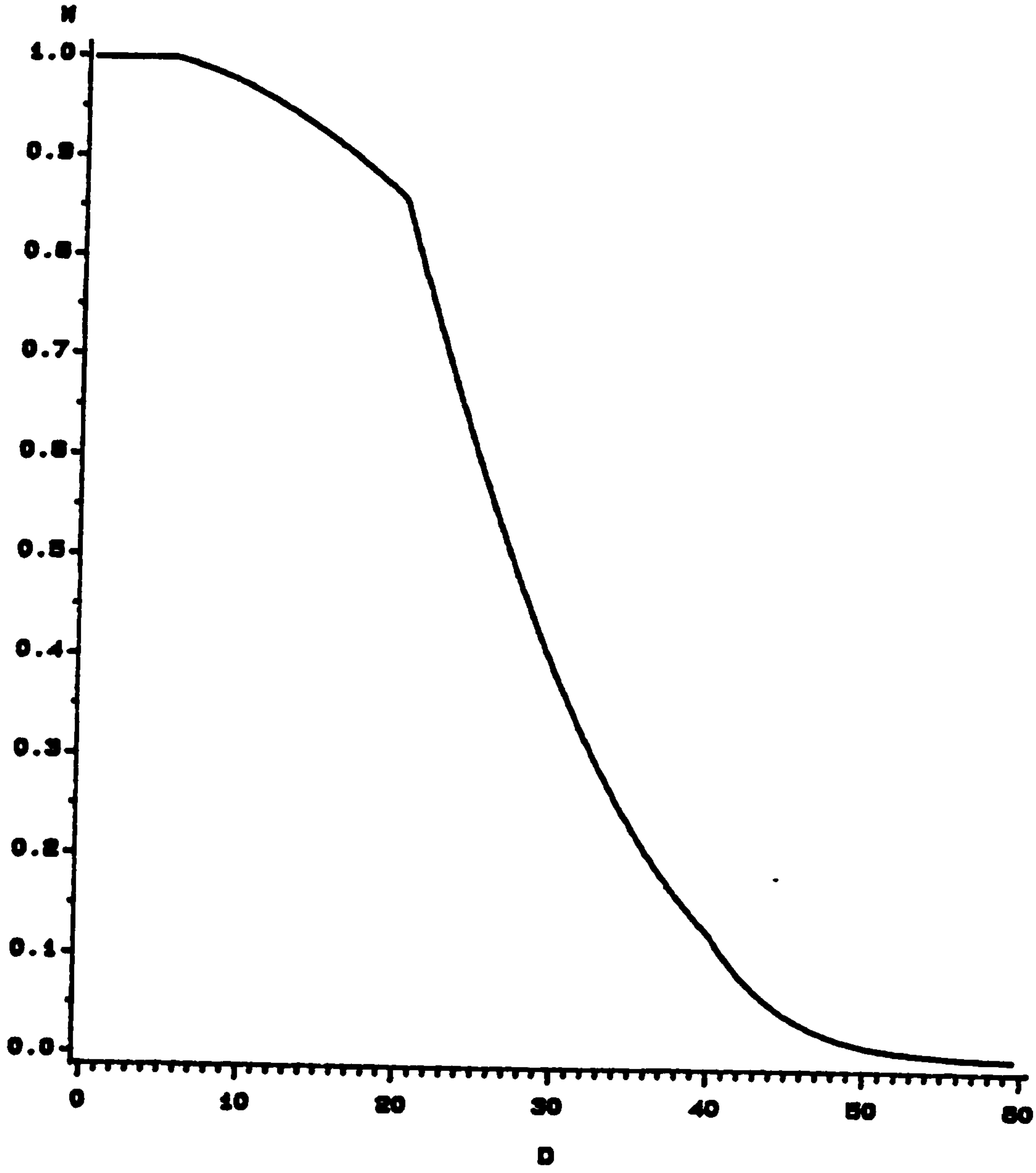
$$U = \begin{cases} a_1 \cdot e^{-(c_1 D)^2} & \text{if} \quad D \in (0, d_1) \\ a_2 \cdot e^{-(c_2 D)^2} & \text{if} \quad D \in [d_1, d_2) \\ \quad \vdots & \qquad \vdots \\ a_n \cdot e^{-(c_n D)^2} & \text{if} \quad D \in [d_{n-1}, \infty) \, . \end{cases} \tag{5.23}$$

The transformation (5.23) may be viewed as a generalization of (5.21). This kind of transformation is appropriate when we want to use different weighing schemes according to the values of D. As an example, we may want to use a transformation such that for a value of D in some interval, the corresponding value of W decreases rapidly. Figure 5.2 shows the transformation curve corresponding to the equation

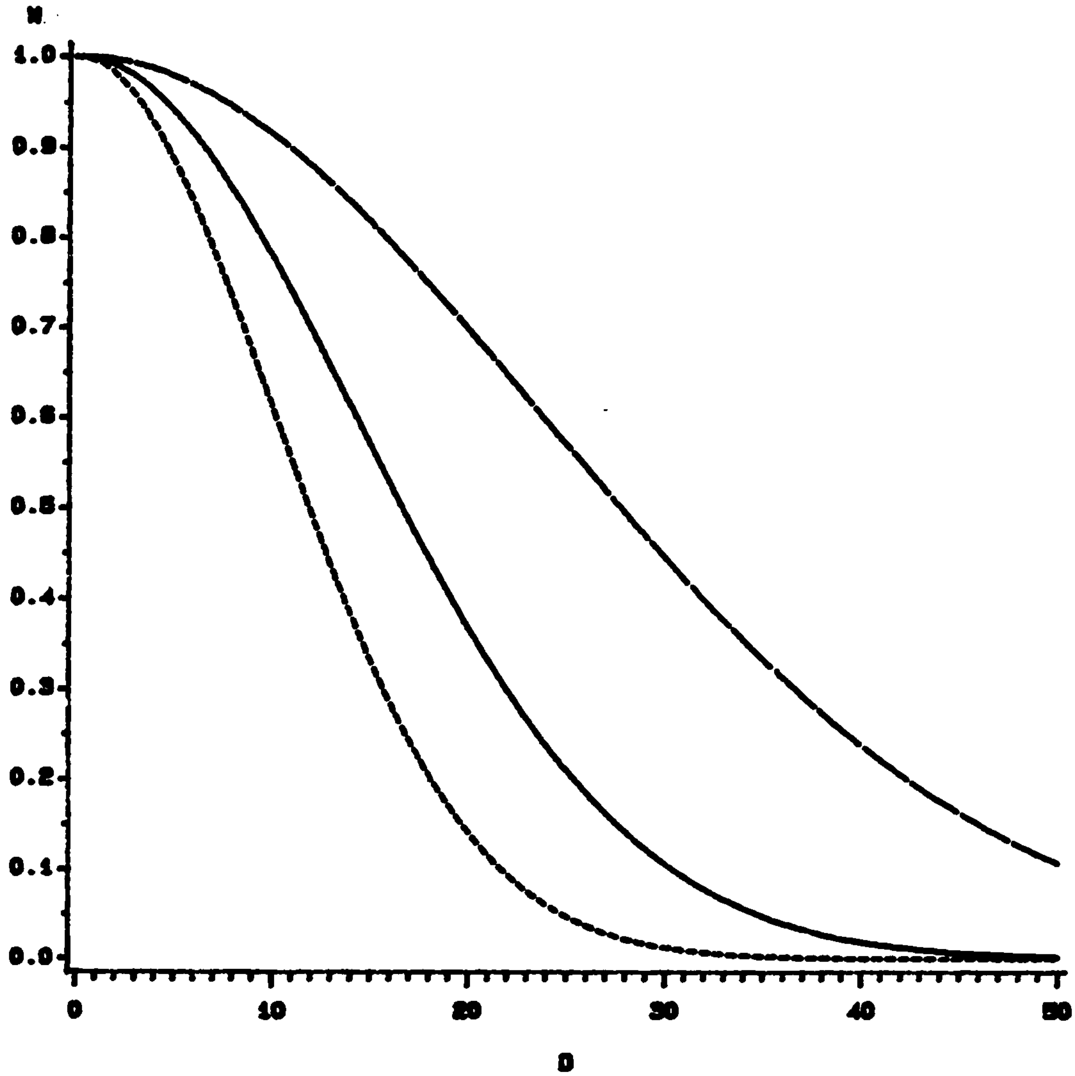
$$U = \begin{cases} 1 & \text{if} \quad 0 < D < 5 \\ 1.01 \cdot e^{-(0.02D)^2} & \text{if} \quad 5 \le D < 20 \\ .631 \cdot e^{-(0.04D)^2} & \text{if} \quad 20 \le D < 40 \\ 6.884 \cdot e^{-(0.05D)^2} & \text{if} \quad D \ge 40 \, . \end{cases} \tag{5.24}$$

For this transformation, a value of D below 5 would result in the same value 1 of W, which corresponds to the totally acceptable level. And until D reaches 20, the value of W decreases slowly. But if D is greater than 20, we transform the D with very strong weight. In other words, if D is less than 20, the corresponding desirability values are essentially the same, but the desirability decreases rapidly for the value of D above 20. Therefore it can be said that $D = 20$ is the tolerance limit of dispersion for this quality characteristic.

As we mentioned before, in any product some quality characteristics are of critical importance, and others are relatively insignificant. Therefore it is necessary that each quality characteristic be weighed according to its importance relative to the intended application. Although a number of weighing procedures are available through the exponential family transformation such as (5.21), we can accomplish this simply by adjusting the constant c in the transformation (5.21). As we noted, a small value of the constant c produces a transformation curve approaching zero rapidly. Therefore, for critical quality characteristics, we will use small values for the constant. And for less important quality characteristics relatively large values will be used. In Figure 5.3, three different curves corresponding to $c_1 = 0.03$, $c_2 = 0.05$, and $c_3 = 0.07$, respectively are presented. The Figure 5.3 reveals essentially that for more important quality characteristics, the transformation curve will be steeper and for less critical quality characteristics, the curve will be more horizontal.



<FIGURE 5.2>: TRANSFORMATION CURVE FOR DIFFERENT WEIGHING SCHEME



<FIGURE 5.3>: TRANSFORMATION CURVE FOR VARIOUS VALUES OF C

——— : W1, — — —: W2, ········: W3
W1 CORRESPONDS TO C=.05
W2 CORRESPONDS TO C=.07
W3 CORRESPONDS TO C=.09

V.3.4 SN Ratios

Having transformed the measured dispersion D for each quality characteristic to the desirability scale W at each setting $\underline{\theta}_i$ of control factors, we are now ready to combine these several values of W to develop SN ratios for multiresponse parameter design. One possible SN ratio is a simple arithmetic mean of W's,

$$\begin{aligned} SN(\underline{\theta}_i) &= \frac{W_{i1} + W_{i2} + \cdots + W_{ig}}{g} \\ &= \overline{W}_{si}, \qquad\qquad i = 1,2,\ldots,m, \end{aligned} \tag{5.25}$$

where W_{ij}, $j = 1,2,\ldots,g$ are the desirability scales of dispersion for a quality characteristic Y_j at the setting $\underline{\theta}_i$ of the control factors, and the subscript s in $\overline{W}_{si}$ denotes the simple mean.

The geometric mean used by Harrington(1965) gives us the following type of SN ratio,

$$\begin{aligned} SN(\underline{\theta}_i) &= (W_{i1} \cdot W_{i2} \cdot W_{i3} \cdot \cdots \cdot W_{ig})^{1/g} \\ &= \overline{W}_{Gi} \;. \end{aligned} \tag{5.26}$$

If one of the W's is (close to) zero, the SN ratio (5.26) as an overall measure of dispersion will also be (close to) zero. That is, if the dispersion for some quality characteristic is too large then (5.26) gives us a very poor overall measure of dispersion even though the dispersion measures for other quality characteristics may be small. Furthermore, the SN ratio (5.26) is strongly weighed by small undesirable values of W's. Therefore, this SN ratio is suitable for experiments where, if one of several quality characteristics is poor in dispersion at some setting $\underline{\theta}_i$, then such a setting should be identified as bad, regardless of other quality characteristics.

The basic rationale of the SN ratio (5.25) is different. A few small values of W_j in (5.25) can not exert much influence on the overall measure of dispersion. The SN ratio (5.25) may be suitable for a multiresponse parameter design in which poor performance in dispersion for some quality characteristic is not fatal for the overall measure. Therefore, the decision of which SN ratio should be used can be determined by the property of a product and the engineering conditions.

Since the SN ratios (5.25) and (5.26) are doubly bounded by zero and one, it may be useful to transform (5.25) and (5.26) by some monotone increasing function to another type of SN ratios whose ranges are whole real numbers. Thus

$$SN(\underline{\theta}_l) = 10\log\frac{\overline{W}_{sl}}{1-\overline{W}_{sl}} \tag{5.27}$$

$$SN(\underline{\theta}_l) = 10\log\frac{\overline{W}_{Gl}}{1-\overline{W}_{Gl}} \tag{5.28}$$

can also be used as SN ratios.

As we noted, for each quality characteristic we transform the measured value of dispersion regardless of the property of the target value. Therefore, in contrast to the univariate SN ratio suggested by Taguchi, we also separate the variance and the bias in developing the SN ratios for a quality characteristic whose target value is zero or infinite. This separation may give us a more efficient SN ratio for reducing variances, especially when the bias part is significantly larger than the variance. This is because, if we include the bias part in the SN ratio as in the univariate case, then the resulting SN ratio can be dominated by the large portion of the bias part.

If any SN ratio developed in this section is influenced by only a subset $\underline{\Theta}_1$ of the control factors $\underline{\Theta} = (\underline{\Theta}_1, \underline{\Theta}_2)$, then we can expect that the individual desirability variable W_j is also influenced by only $\underline{\Theta}_1$ since the SN ratio is a function of the W_j's. Consequently, each dispersion measure D_j also depends only on a subset $\underline{\Theta}_1$ since W_j is also a function of D_j. Equivalently, D_j is independent of μ_j since given $\underline{\Theta}_1$, μ_j depends only on $\underline{\Theta}_2$. Moreover, D_j is independent of μ_k, $k \neq j$ since D_j and μ_k depend on $\underline{\Theta}_1$ and $\underline{\Theta}_2$, respectively, and $\underline{\Theta}_1$ and $\underline{\Theta}_2$ are disjoint. That is, we can adjust the mean response μ_k to target value τ_k independently of all D_j's by manipulating levels of control factors in

Θ_2 Therefore, only the subset Θ_1 of control factors $\Theta = (\Theta_1, \Theta_2)$ will have dispersion effects and the remaining control factors in Θ_2 can be used as adjustment factors in correcting the mean response of each quality characteristic.

Therefore,using a suitable *SN* ratio developed in this section, we can proceed with the multiresponse parameter design analysis with a designed experiment such as balanced fractional factorial parameter design of resolution *V* developed in Chapter III, or a sequential parameter design developed in Chapter IV. Then with adjustment factors detected in the ANOVA procedure, we can find the optimal setting by fitting mean responses close to target values by correcting the adjustment factors, if such adjustment procedure is necessary for some quality characteristics, while fixing the subset Θ_1 of control factors at the identified levels in the ANOVA procedure.

Even though a specific loss function can be provided in a multiresponse parameter design, the approximation of the loss function up to the third term in (5.1) may turn out to be crude so that the higher-order terms are necessary. And the estimation of a quadratic loss function is valid only when the remaining terms in the Taylor expansion are negligible. However, some authors doubt the adequacy of the quadratic loss function (see Nair and Pregibon (1988)). For such cases, the *SN* ratios such as (5.10) and (5.15) based on the approximation of the loss function are not appropriate for the procedure through which we want to minimize the loss function. Instead of these, the *SN* ratios (5.27) and (5.28) developed by not utilizing loss functions can be used as alternatives. Therefore, the *SN* ratios developed in this section are widely applicable and can be thought of as generalizations suitable for multiresponse parameter design analyses.

Chapter VI

SUMMARY AND AREAS FOR FURTHER RESEARCH

VI.1 Summary of Results

Parameter designs have been shown to be cost-effective in many industrial applications to reduce the product's variation due to various noise factors. To improve the performance of parameter designs, suitable *SN* ratios were suggested by some authors. However, the parameter design technique still has some serious problems in constructing the designs, since typically only main effects of the control factors are investigated.

The goals of this dissertation have been to develop new parameter designs, especially the inner array for the control factors, through which we can detect influential two-factor interactions. For developing new designs, we restrict the scope of our research to 2^t and 3^t factorial designs for the control factors, and assume that three-factor and higher order interactions are all negligible.

For 2^t factorial parameter designs (Chapter III), we develop eight saturated balanced fractional factorial designs of resolution V for any number t of the control factors by using a partially balanced array. Those designs have the following desirable properties:

(i) They are easy to construct for any number of control factors;

(ii) They have a minimum number of treatment combinations for investigating two-factor interactions,;

(iii) They are easy to analyze and the results are easy to interpret and to use since the designs have some "balanced" structure with respect to the variance-covariance matrix of the estimates of main effects and two-factor interactions;

(iv) We can evaluate easily various optimality criteria for the designs.

Among the three optimality criteria, determinant , trace, maximum eigenvalue criterion, we argue that only the trace criterion is suitable for evaluating our designs. Therefore, we tabulate the trace of the variance-covariance matrix for the designs, and we see that two designs among the eight possible designs are optimal for the trace criterion. In order to analyze our saturated balanced designs two stepwise methods , a normal probability plot method , and a modification of Ghosh's method are being developed.

For 3^t factorial parameter designs (Chapter IV), a sequential detection procedure is developed to reduce the number of treatment combinations needed to detect influential two-factor interactions. The sequential procedure consists of two stages. In the first stage, using a "near" minimal resolution IV design we develop a series of hypotheses for detecting influential interactions. In contrast to other detection procedures in the literature, we partition two-factor interactions into the two components in the first stage, and develop a detection procedure for each component. By splitting the two-factor interactions we can obtain more accurate estimates, *i.e.* with small bias, of the main effects of the control factors. The second stage will be used when the underlying interaction structure of the control factors is too complicated to be identified in the first stage. In the second stage, a series of sequential hypotheses is developed to continue the detection procedure by taking treatment combinations sequentially. The sequential procedure in the second stage can be simplified by making use of the information obtained and the treatment combinations taken in the first stage. After identifying influential two-factor interactions, a method of analyzing 3^t factorial parameter designs in which identified influential two-factor interactions have been accounted for, is developed.

In chapter V, to extend the parameter design to several quality characteristics, we devise some suitable *SN* ratios according to whether a proper loss function for several quality characteristics can be specified or not. When a loss function can be specified, we develop two different types of *SN* ratios depending on whether the variance-covariance matrix is linked to the variance-covariance matrix or not. Moreover, a detection procedure for dependence between the mean vector and the variance-covariance matrix is developed by using an orthogonal transformation. When a proper loss function cannot be specified, we develop other kinds of *SN* ratios by utilizing the concept of a desirability function. These *SN* ratios are quite general and can be used even when a loss function can be specified, especially when the mean vector is linked to the variance-covariance matrix, or when validity of the specified loss function is doubted.

VI.2 Areas for Future Research

The development of detecting methods for influential interactions considered in this paper are restricted to 2^t and 3^t factorial designs for the control factors. Future research may be directed towards developing detection procedures for more general factorial designs, such as p^t factorial designs for the control factors, where p is a prime number, and especially for non-symmetrical factorial parameter designs, such as 2^s3^v factorial designs for the control factors.

To evaluate various analyzing methods for saturated balanced fractional factorial designs of resolution V developed in Chapter III, some simulation studies may be helpful.

Moreover, for developing detection procedures for influential interactions, we assume that three-factor and higher order interactions are all negligible. However, such an assumption may be inadequate for some experiments when higher order interactions exist. Therefore, development of more general parameter designs deserves special attention in particular with respect to detection of higher order influential interactions.

In Chapter IV, we consider simple estimates, for example (4.31), for developing test statistics. As we mentioned in section IV.2.2.2 , these estimates are unbiased estimates but not necessarily the estimates with minimum variance. Therefore, in order to develop more powerful test statistics, one needs to obtain more efficient estimates with small variance.

Another interesting area is related to a multiresponse parameter design. In Chapter V, we extend the parameter design to a mutltiresponse parameter design by devising some suitable *SN* ratios. But another extension to the multiresponse case may be directed towards hierarchical multiresponse designs in which some important quality characteristics are observed more frequently than other less important quality characteristics, and where different parameter designs are used for different quality characteristics.

APPENDIX

< Derivation of the equation (4.18) >

Let us assume that two component E_1, E_2 are contained in the alias set S_i, $i \neq 0$, and the "levels" of E_1 are determined by the "levels" of E_2 by a permutation (02), *i.e.*

$$0 \rightarrow 2$$

$$1 \rightarrow 1$$

$$2 \rightarrow 0$$

, where the first and second rows represent the "levels " of the components E_1 and E_2, respectively.

Then, by the above relationship we can see that in the submatrix X_1 of four columns of the model matrix X in (4.13) , corresponding to four parameters $(E_1)^1$, $(E_1)^2$, $(E_2)^1$, $(E_2)^2$ associated with the two components, at most three distinct rows can appear, and we can transform the submodel for the two components E_1, E_2 as

$$X_1\underline{\beta}_1 = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -2 & 0 & -2 \\ -1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} (E_1)^1 \\ (E_1)^2 \\ (E_2)^1 \\ (E_2)^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -2 \\ -1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} (E_1)^1 \\ (E_1)^2 \\ -(E_2)^1 \\ (E_2)^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} (E_1)^1 - (E_2)^1 \\ (E_1)^2 + (E_2)^2 \end{bmatrix}.$$

This implies that if two components E_1, E_2 are contained in the alias set S_i, $i \neq 0$, and the "levels" of the two components are related by the permutation (02), then instead of estimating all individual parameters associated with the components, we can estimate the following two linear combinations of the parameters,

$$\begin{bmatrix} (E_1)^1 \\ (E_1)^2 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} (E_2)^1 \\ (E_2)^2 \end{bmatrix}.$$

By the same arguments, we can generally show that if two components E_1, E_2 are contained in the same alias set S_i, $i \neq 0$, and their "levels" are related by the permutation p_i, then the vector of two estimable functions of parameters associated with the two components can be expressed as

$$D_e\underline{E}_1 + D_{p_i}\underline{E}_2,$$

where D_{p_i}, and $\underline{E}_i$ are defined in (4.14) and (4.17), respectively.

Moreover, suppose that the components $E_1, E_2, \dots, E_b$ are contained in the same alias set S_i, $i \neq 0$, and we choose a component, say E_1, as a reference effect. If the "levels" of the other components are related to the "levels" of the reference component by permutations, say $p_2, p_3, \dots, p_b$, respectively, then by the same arguments it follows that the vector ES_i of two estimable functions of parameters associated with the components is

$$ES_i = D_e E_1 + D_{p_2} E_2 + \cdots + D_{p_b} E_b \ .$$

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