Chapter 3

Fan noise

In this chapter, the analytical modeling of the fan noise is presented. As an introduction, the propagation and radiation characteristics of the modes generated by a ducted fan are reviewed in section 3.1. Then, in section 3.2, two fan noise models suitable for the TBIEM3D code are presented. For each fan model, an analytical representation of the pressure field generated by the fan in free space, i.e., the incident field, is derived.

3.1 Theory

Control of the noise radiating from a turbofan engine is an easier task if the characteristics of the noise propagating in the duct, or radiating into the far field, are well known and understood. For example, if the sound power that radiates within a particular sector of the far field has to be reduced, the mode (or modes) that predominantly radiate within that sector will be targeted for control. Then, in the instance where anti-noise is to be used to control that specific mode, knowledge of the mode pattern and propagation characteristics will help to configure a control source system that generates the same mode pattern out of phase, the intent being that control and target modes cancel each other as they propagate through the duct. Therefore, knowledge of the duct modal content and of the far field radiation directivity of the propagating modes is requisite to intelligent control of the noise radiating from a turbofan engine.

3.1.1 Generation and transmission of the duct acoustic modes

One of the principal components of fan noise in a turbofan engine is loading noise, which is caused by the aerodynamic forces acting on the fan blades. The characteristics of the pressure field generated by the fan loading can be described as follows (Tyler and Sofrin 1962, Morfey 1964): The pressure pattern associated with the rotating fan blades spins with fan shaft frequency Ω. The fan pressure field consists of a superposition of lobed patterns all turning with shaft speed Ω. The number of lobes in each pattern is an integer multiple of the number of fan blades. Thus, if there are N blades in the fan, the pressure pattern composed of N lobes is associated with the fundamental blade passage frequency $N\Omega$, while the pattern composed of 2N lobes is associated with the first harmonic $2N\Omega$ and the 3N lobes pattern is associated with the second harmonic $3N\Omega$, and so on.

The spinning pressure patterns generated by the rotating fan blades are called acoustic modes. An acoustic mode is defined by its circumferential order m, and its radial order n, written as (m,n). In a cylindrical coordinate system (r,ψ,z) (as previously described in Figure 2.2), the spinning modes forming the disturbance pressure field in a cylindrical duct of radius r_d can be expressed as

$$p_{mn}(r,\psi,z) = A_{mn}\ J_m(k_{r\psi}^{mn} r)\ \exp(i[m(\Omega t-\psi)-k_z^{mn} z])\ . \qquad (3.1)$$

A_{mn} is the complex amplitude of the (m,n) spinning mode, J_m is the m^{th} order Bessel function of the first kind, and k_z^{mn} is the axial wavenumber of the (m,n) mode. (Note that for simplicity, the tilded notation, used in chapter 2 to indicate dimensional variables, is omitted in this section.) In the case where the ducted fan is immersed in a mean flow of a propagating medium, the axial wavenumber of the (m,n) acoustic mode is given by

$$k_Z^{mn} = \frac{k}{1-M^2}\left[M \pm \sqrt{1-\left(\frac{\sqrt{1-M^2}\,k_{r\psi}^{mn}}{k}\right)^2}\,\right], \tag{3.2}$$

where the acoustic wavenumber k is given by

$$k = \frac{n_h N \ \Omega}{c} \tag{3.3}$$

and $k_{r\psi}^{mn}$ is the combined radial-circumferential wavenumber of the (m,n) mode in the (r,ψ) plane. In the absence of mean flow Eq. (3.2) reduces to

$$k_z^{mn} = \sqrt{k^2 - \left(k_{r\psi}^{mn}\right)^2}\ . \tag{3.4}$$

The wavenumbers $k_{r\psi}^{mn}$, which for simplicity will be referred in the rest of this study as the radial wavenumbers, are determined by the radial boundary conditions. Thus, for the rigid wall duct,

$$k_{r\psi}^{mn} = \frac{j'_{mn}}{r_d} \tag{3.5}$$

where j'_{mn} are the zeros of dJ_m/dr. From Eq. (3.1) it can be seen that the (m,n) mode will propagate when its axial wavenumber k_z^{mn} is real, and it will decay when k_z^{mn} has a nonzero imaginary part for hard walled boundary conditions. The frequency at which this change occurs is called the cut-off frequency. The pressure patterns and radial wavenumbers (for a duct of radius 1 m) associated with the duct acoustic modes are presented in Figure 3.1 for circumferential order m = 0 through 2, and radial order n = 0 through 2. The pressure patterns associated with the duct modes that are cut on spin with the fan shaft frequency Ω as they propagate through the duct, and radiate to the far field once they reach the duct openings.

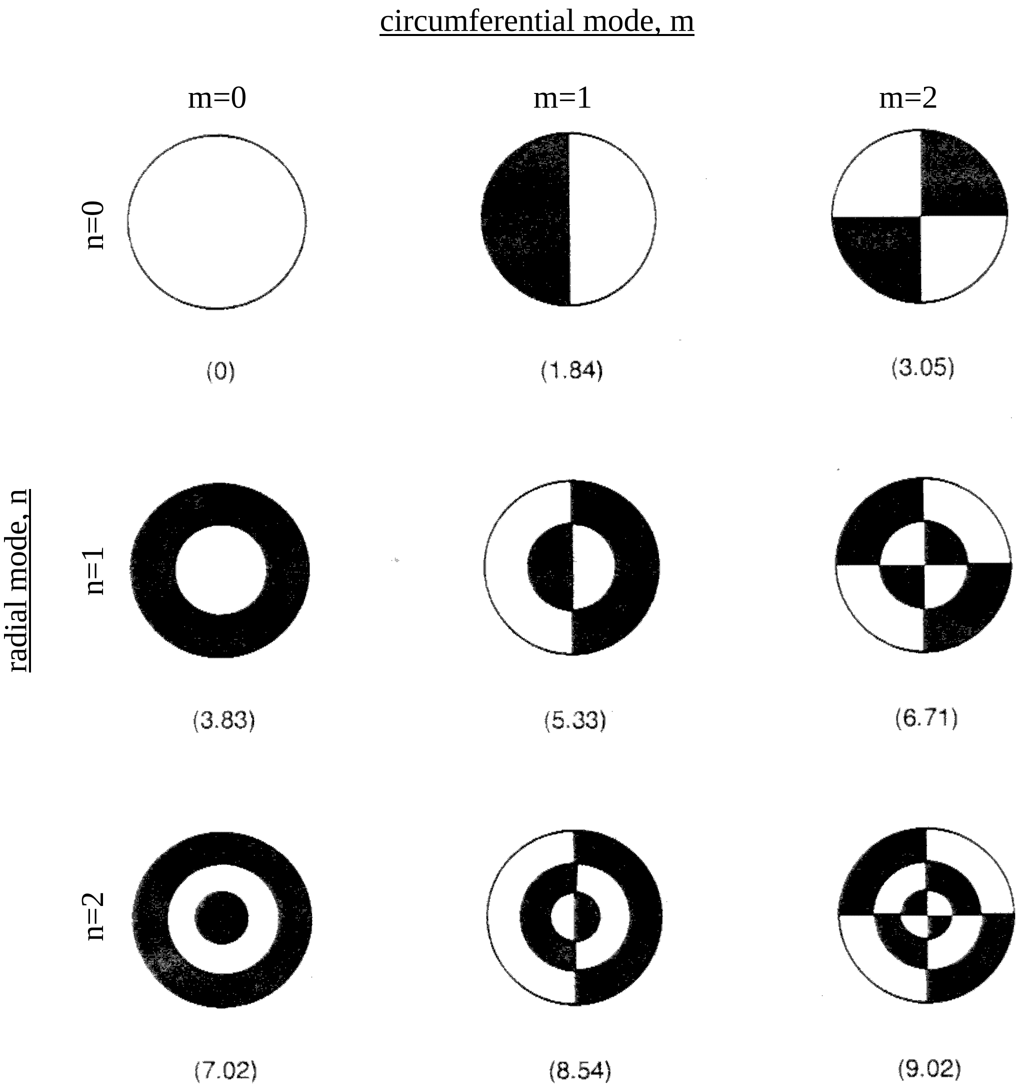

Figure 3.1: Cylindrical mode patterns with radial wavenumbers $k_{r\psi}^{mn}$ for a rigid wall duct of radius 1m. m=0,1,2 and n=0,1,2.

3.1.2 Angles of peak radiation of the duct acoustic modes

The determination of the directions of peak radiation of the duct acoustic modes has been the subject of a number of papers, most notably by Rice, Heidmann and Sofrin (Rice et al. 1979). They established the following expression for the computation of the angle θ_{peak}^{mn} for the main lobe of radiation of the (m,n) mode from the opening of a cylindrical duct

$$\cos\,\theta_{peak}^{mm} = \sqrt{1-M^2}\left[\frac{1-\frac{1}{\xi_{mn}^2}}{1-M^2\left(1-\frac{1}{\xi_{mn}^2}\right)}\right]^{\frac{1}{2}} \tag{3.6}$$

where

$$\xi_{mn} = \frac{k}{k_{r\psi}^{mn}\,\sqrt{1-M^2}} \tag{3.7}$$

is the mode cut-off ratio. The (m,n) mode is cut-off when $\xi_{mn} \le 1$ and propagates when $\xi_{mn} > 1$.

A mode propagating in a rectangular wave guide can be seen as four plane waves propagating skewed to the wave guide axis (Kinsler et al., 1982). In a similar manner, the wave fronts of the modes propagating in a cylindrical duct can be seen as a superposition of plane waves propagating at an angle with the duct axis. Based on this plane wave approximation, it was established (Rice and Heidmann, 1979) that the angle of the mode main lobe of radiation corresponds to the angle that the group velocity vector (Lighthill 1964) of the plane wave components makes with the axis of the duct. If an approximate plane wave representation of the duct modes is considered, the group velocity vector can be expressed as (Farassat 1996)

$$\vec{V}_g = c\left(\hat{n} + M\,\hat{i}_z\right), \tag{3.8}$$

where c is the speed of sound, M is the flow Mach number, $\hat{i}_z$ is a unit vector parallel to the duct axis, and $\hat{n}$ is a unit vector normal to the modal component wave front. In other words, the group velocity vector is the sum of the phase velocity vector $c\hat{n}$ and of the uniform flow velocity vector $cM\hat{i}_z$. This relationship between phase velocity, group velocity and axial flow velocity is represented graphically in Figure 3.2.

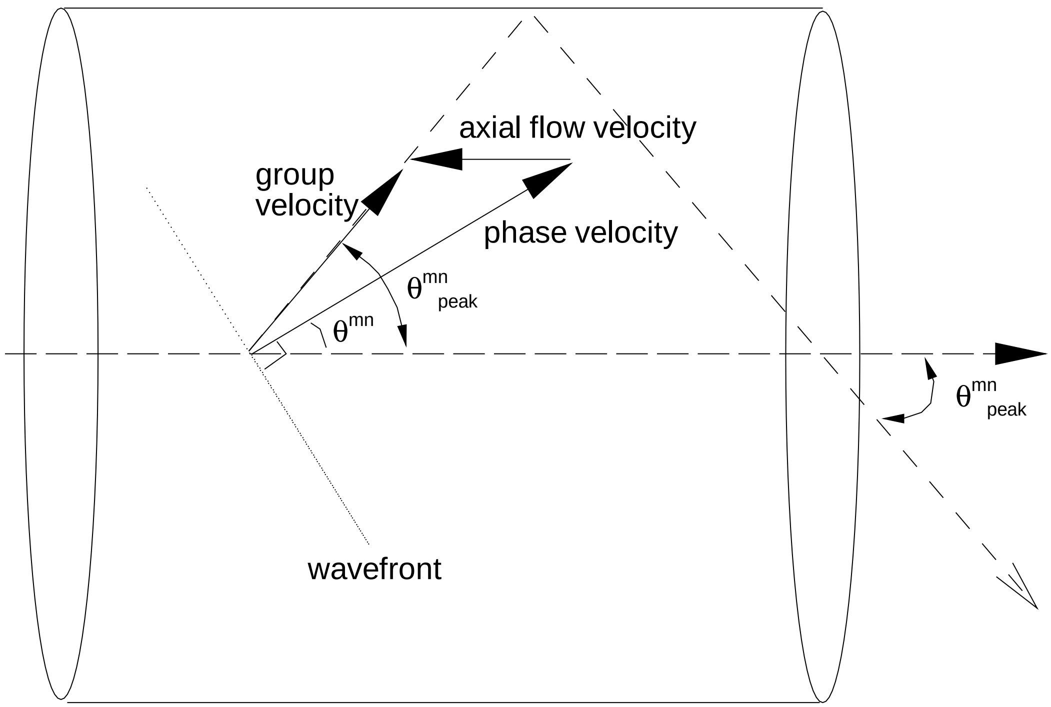

Figure 3.2: Relationship between phase, group and axial flow velocities.

From this figure, it can be seen that as the (m,n) mode gets closer to cut-off, i.e. as θ_{mn} increases, θ^{mn}_{peak} increases, and vice versa. In other words, for a given axial flow velocity, the modes that are well above their cut-on frequencies radiate toward the axis of the duct, while the modes that are closer to cut-off radiate at larger far field angles. Also, from Eq. (3.6) and Eq. (3.7), it can be noted that, independently of the direction of the flow, as the flow Mach number increases, the cut-off ratio of the (m,n) mode increases and the main lobe of radiation tilts toward the duct axis (i.e., θ^{mn}_{peak} decreases).

3.2 Analytical model of the fan noise

As it was explained in Chapter 2, the BIEM treats the prediction of sound radiation from a ducted fan as a scattering problem, in which the scattered field represents the modification of the incident free space fan noise field resulting from the presence of the duct. Since this scattering approach decouples the duct from the fan, models for the (incident) fan noise can be derived independently of the duct shape and wall impedance(s). This scattering approach allows for the use of an arbitrarily complex incident field model. However, the present work utilizes simple source models in order to avoid the extensive numerical calculations required to produce the incident field with a more realistic model (Dunn and Farassat 1992). Nevertheless, as it will be demonstrated in Chapter 5, the two fan noise models used in this work can provide a fairly good simulation of actual engine fan noise radiation.

The original fan noise model implemented into the TBIEM3D code considered a collection of spinning point dipoles to model the loading component of the fan noise. One point dipole was used to model the loading force on each blade of the fan (Morfrey 1964). The point dipoles were located toward the tip of the blade where the loading force (and hence the loading noise) is maximum. The derivations of this fan model based on spinning point sources were first done by Lan (Lan 1993). The analytical expression for the incident pressure field generated by this fan model is presented in Appendix A with some minor modifications.

In the instance where the radial distributions of the forces acting on the fan blades (in the absence of the duct) are known, the original fan model can be improved by modeling each blade as a spinning line source instead of a single spinning point source. Thus, the strength of the line source can be given a continuous radial distribution that will model that of the loading force acting on each fan blade. The loading forces (and consequently the loading noise) will therefore be modeled more accurately than with the original fan model. An analytical expression for the incident pressure field generated in free space by this fan noise model is derived in section 3.2.1. Note that the strength distribution given to each line source models a radial variation of the amplitude of the acoustic pressure along the fan blades in the absence of the duct. Therefore, by giving the line source strength a radial distribution of the form $A_{mn} J_m(k_r^{mn} r)$, which referring to Eq. (3.1) is the radial distribution of the pressure amplitude of the (m,n) duct acoustic mode, the generation of the (m,n) mode would not be accurately modeled since such a pressure distribution occurs only in the presence of the duct.

In the case where the propagation and radiation of modes of specific phases and amplitudes have to be reproduced, spinning radial arrays of point dipoles (instead of line sources) can be used to model their generation. The strength and phase of the point dipoles in each array can be adjusted according to the amplitudes and phases of the modes that need to be reproduced. This procedure and this fan noise model are described in section 3.2.2.

3.2.1 Spinning line sources

An analytical representation of the loading noise field generated by N evenly spaced spinning line sources in an unbounded space is derived in this section. This fan model is schematically represented in Figure 3.3.

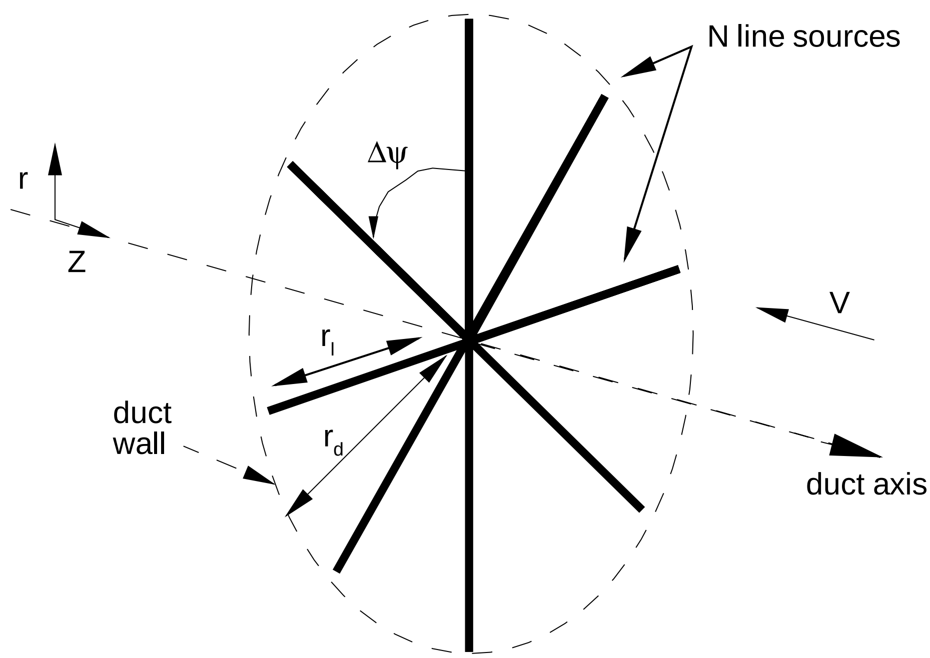

Figure 3.3: Schematic of the fan noise model based on spinning line sources.

The line sources extend from r = 0 (duct axis) to r = r_l (tip of the fan blade). The incident acoustic field generated in free space by these spinning line sources is obtained by solving the following inhomogeneous wave equation (Farassat and Myers 1993)

$$[\frac{1}{\tilde{c}^2}\frac{\partial^2}{\partial\tilde{t}^2} - \frac{1}{\tilde{r}}\frac{\partial}{\partial\tilde{r}}(\tilde{r}\frac{\partial}{\partial\tilde{r}}) - \frac{1}{\tilde{r}^2}\frac{\partial^2}{\partial\psi^2} - \frac{\partial^2}{\partial\tilde{z}^2}]\tilde{p}_i(\tilde{r},\psi,\tilde{z},\tilde{t}) =$$

$$-\tilde{\nabla}.[\tilde{\vec{F}}\sum_{m_d=-\infty}^{\infty}\tilde{h}_{m_d}(\tilde{r},\psi,\tilde{z},\tilde{t})] \tag{3.9}$$

where

$$\tilde{h}_{m_d}(\tilde{r},\psi,\tilde{z},\tilde{t}) = H(\tilde{r}_l - \tilde{r})\,\delta(\tilde{z} - \tilde{V}\tilde{t})\,\delta(\tilde{\Omega}\tilde{t} - \psi - \frac{2\pi m_d}{N}) \tag{3.10}$$

is the multi-dimensional delta function (Farassat 1994),

$$\tilde{\nabla} = \frac{\partial}{\partial\tilde{r}}\hat{i}_r + \frac{1}{\tilde{r}}\frac{\partial}{\partial\psi}\hat{i}_\psi + \frac{\partial}{\partial\tilde{z}}\hat{i}_z \tag{3.11}$$

is the divergence operator, H represents the step function and δ represents the Dirac Delta function. In addition, N is the number of blades, and $\vec{F} = F_r(r)\,\hat{i}_r + F_\psi(r)\,\hat{i}_\psi + F_z(r)\,\hat{i}_z$ is the force (in $N.m^{-1}$) applied by one blade to the fluid. The line sources are considered to be spinning with angular velocity $\tilde{\Omega}$ and translating in the axial direction (i.e., positive $\hat{i}_z$ direction) with speed V. The right hand side of Eq. (3.9) is a dipole term and models the loading component of the fan noise (Dowling 1983).

In order to simplify the explanation of the derivations and to avoid extremely lengthy equations, the solution of Eq. (3.9) is broken down into two parts. Thus, the computation of the loading noise due to the axial component of the loading force is derived first, followed by the derivation of the loading noise due to the radial and azymuthal components of the loading force.

a. Loading noise (axial force)

When considering the loading noise generated by the axial component of the force that the fan blades apply on the medium, the wave equation takes the form

$$[\frac{1}{\tilde{c}^2}\frac{\partial^2}{\partial \tilde{t}^2} - \frac{1}{\tilde{r}}\frac{\partial}{\partial \tilde{r}}(\tilde{r}\frac{\partial}{\partial \tilde{r}}) - \frac{1}{\tilde{r}^2}\frac{\partial^2}{\partial \psi^2} - \frac{\partial^2}{\partial \tilde{z}^2}]\tilde{p}_i(\tilde{r}, \psi, \tilde{z}, \tilde{t}) =$$

$$-\tilde{F}_z(\tilde{r})\, H(\tilde{r}_l - \tilde{r})\, \frac{\partial}{\partial \tilde{z}}[\delta(\tilde{z} - \tilde{V}\tilde{t})] \sum_{m_d=-\infty}^{\infty} \delta(\tilde{\Omega}\tilde{t} - \psi - \frac{2\pi m_d}{N}) . \qquad (3.12)$$

In order to compute its solution, this equation is first nondimensionalized by dividing length by $\tilde{r}_d$, mass by $\tilde{\rho}_0\, \tilde{r}_d^3$, and time by $\tilde{\Omega}^{-1}$. Hence, noting that (cf. Farassat, 1994)

$$H(a\, x) = \frac{1}{a} H(x) \qquad (3.13)$$

and

$$\delta(a\, x) = \frac{1}{a}\delta(x)\, , \qquad (3.14)$$

where a is an arbitrary number, Eq. (3.9) can be written after nondimensionalization as

$$[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) - \frac{1}{r^2}\frac{\partial^2}{\partial \psi^2} - \frac{\partial^2}{\partial z^2}]\, p_i(r, \psi, z, t) =$$

$$-F_z(r)\, H(r_l - r)\, \frac{\partial}{\partial z}[\delta(z - Vt)] \sum_{m_d=-\infty}^{\infty} \delta(t - \psi - \frac{2\pi m_d}{N})\, , \qquad (3.15)$$

where the nondimensional distribution of the axial force is defined by

$$F_z(r) = \frac{\tilde{F}_z(\tilde{r})}{\tilde{\rho}_0 \tilde{\Omega}^2\, \tilde{r}_d^3} \, . \qquad (3.16)$$

This nondimensional inhomogeneous wave equation, Eq. (3.15), is then written in a moving and stretched reference frame defined earlier by

$$Z = \frac{1}{\beta}(z - Vt) \qquad (3.17)$$

where V is the velocity of translation of the fan (i.e., of the spinning line sources), and

$$\beta = \sqrt{1 - M^2} \ . \tag{3.18}$$

Thus, in this new reference frame, Eq. (3.15) takes the following form

$$[\frac{1}{c^2}\left(\frac{\partial}{\partial t} - \frac{V}{\beta}\frac{\partial}{\partial Z}\right)^2 - \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) - \frac{1}{r^2}\frac{\partial^2}{\partial \psi^2} - \frac{1}{\beta^2}\frac{\partial^2}{\partial Z^2}]p_i(r,\psi,Z,t) =$$

$$-\frac{1}{\beta^2}F_z(r)\,H(r_l - r)\ \frac{\partial}{\partial Z}[\delta(Z)]\sum_{m_d=-\infty}^{\infty}\delta(t - \psi - \frac{2\pi m_d}{N})\ . \tag{3.19}$$

Now, it is noted that in this moving, dilated reference frame, the source term of Eq. (3.19) is periodic in (t-ψ) with period $\frac{2\pi}{N}$, and it can therefore be expanded into a Fourier series of the form

$$\sum_{m_d=-\infty}^{\infty}\delta(t - \psi - \frac{2\pi m_d}{N}) = \sum_{l=-\infty}^{\infty} c_l\, e^{-il(t-\psi)}\,, \tag{3.20}$$

where

$$c_l = \frac{N}{2\pi}\int_0^{2\pi}\sum_{m_d=-\infty}^{\infty}\delta(t - \psi - \frac{2\pi m_d}{N})\, e^{il(t-\psi)}\, d\psi$$

$$= \frac{N}{2\pi}\sum_{m_d=-\infty}^{\infty} e^{il\frac{2\pi m_d}{N}}\ . \tag{3.21}$$

Thus,

$$\sum_{m_d=-\infty}^{\infty}\delta(t - \psi - \frac{2\pi m_d}{N}) = \frac{N}{2\pi}\sum_{l=-\infty}^{\infty}\ \sum_{m_d=-\infty}^{\infty} e^{-il(t-\psi-\frac{2\pi m_d}{N})} \tag{3.22}$$

which simplifies to

$$\sum_{m_d=-\infty}^{\infty}\delta(t - \psi - \frac{2\pi m_d}{N}) = \frac{N}{2\pi}\sum_{n_h=-\infty}^{\infty} e^{i\,n_h\,N\,(t-\psi)}\ . \tag{3.23}$$

Therefore, in the moving and stretched reference frame, the inhomogeneous wave equation can be written as

$$\left[\frac{1}{c^2}\left(\frac{\partial}{\partial t}-\frac{V}{\beta}\frac{\partial}{\partial Z}\right)^2-\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r})-\frac{1}{r^2}\frac{\partial^2}{\partial\psi^2}-\frac{1}{\beta^2}\frac{\partial^2}{\partial Z^2}\right]p_i(r,\psi,Z,t)=$$

$$-\frac{N}{2\pi\beta^2}F_z(r)\,H(r_l-r)\,\frac{\partial}{\partial Z}\delta(Z)\sum_{n_h=-\infty}^{\infty}e^{i\,n_h\,N\,(t-\psi)}. \qquad (3.24)$$

The solution of the above equation is of the form

$$p_i(r,\psi,z,t)=\sum_{n_h=-\infty}^{\infty}P_i^{n_h}(r,Z)\,e^{i\,n_n N\,(t-\psi)}. \qquad (3.25)$$

Combining Eq. (3.24) and Eq. (3.25), yields

$$[\frac{1}{c^2}\left(i\,n_h N-\frac{V}{\beta}\frac{\partial}{\partial Z}\right)^2-\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r})+\frac{n_h^2 N^2}{r^2}-\frac{1}{\beta^2}\frac{\partial^2}{\partial Z^2}]\,P_i^{n_h}(r,Z)=$$

$$-\frac{N}{2\pi\beta^2}F_z(r)\,H(r_l-r)\,\frac{\partial}{\partial Z}\delta(Z)\;. \qquad (3.26)$$

Next, the wave operator (i.e., the left hand side of Eq. (3.26)) is manipulated in order to express it into a form that has a known solution. Thus, recalling that

$$\kappa=\frac{k}{\beta}, \qquad (3.27)$$

where

$$k=\tilde{k}\,\tilde{r}_d=\frac{n_h N\tilde{\Omega}}{\tilde{c}}\,\tilde{r}_d=\frac{n_h N}{c} \qquad (3.28)$$

is the nondimensional wave number, the left hand side of Eq. (3.26) can be expressed as

$$\text{L.H.S.}=\left[\frac{1}{\beta^2}\left(i\,\beta^2\kappa-M\frac{\partial}{\partial Z}\right)^2-\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r})+\frac{n_h^2 N^2}{r^2}-\frac{1}{\beta^2}\frac{\partial^2}{\partial Z^2}\right]P_i^{n_h}(r,Z)\,. \qquad (3.29)$$

Expanding the squared term and regrouping like terms yields

$$\text{L.H.S.}=\left[-\frac{1}{\beta^2}(1-M^2)\frac{\partial^2}{\partial Z^2}-\beta^2\kappa^2-2\,i\,\kappa M\frac{\partial}{\partial Z}-\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r})+\frac{n_h^2 N^2}{r^2}\right]P_i^{n_h}(r,Z)\,. \qquad (3.30)$$

Recalling that $\beta^2 = 1\text{-}M^2$, the above equation can be written as

$$\text{L.H.S.} = \left[-\left(\frac{\partial}{\partial Z} + 2\,i\,\kappa M \right) + \kappa^2 - \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) + \frac{n_h^2 N^2}{r^2} \right] P_i^{n_h}(r,Z)\,. \tag{3.31}$$

Finally, noting that

$$\left(\frac{\partial}{\partial Z} + i\,\kappa M \right) P_i^{n_h} = e^{-i\kappa M Z} \frac{\partial}{\partial Z}\left(e^{i\kappa M Z} P_i^{n_h} \right) \tag{3.32}$$

and

$$\left(\frac{\partial}{\partial Z} + i\,\kappa M \right)^2 P_i^{n_h} = e^{-i\kappa M Z} \frac{\partial^2}{\partial Z^2}\left(e^{i\kappa M Z} P_i^{n_h} \right), \tag{3.33}$$

Eq. (3.31) yields

$$\text{L.H.S.} = -e^{-i\kappa M Z} \left[\frac{\partial^2}{\partial Z} + \kappa^2 + \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) - \frac{n_h^2 N^2}{r^2} \right] e^{i\kappa M Z} P_i^{n_h}(r,Z)\,. \tag{3.34}$$

Therefore, the inhomogeneous wave equation can be written as

$$\left[\frac{\partial^2}{\partial Z} + \kappa^2 + \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) - \frac{n_h^2 N^2}{r^2} \right] e^{i\kappa M Z} P_i^{n_h}(r,Z) =$$

$$e^{i\kappa M Z} \frac{N}{2\pi\beta^2} F_z(r)\, H(r_l - r)\, \frac{\partial}{\partial Z}\delta(Z)\,. \tag{3.35}$$

Finally, defining

$$Q_i^{n_h}(r,Z) = e^{i\kappa M Z} P_i^{n_h}(r,Z)\,, \tag{3.36}$$

and noting that

$$f(x)\,\delta(x) = f(0)\,\delta(x)\,, \tag{3.37}$$

the inhomogeneous wave equation, Eq. (3.35) takes the form

$$\left[\frac{\partial^2}{\partial Z} + \kappa^2 + \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) - \frac{n_h^2 N^2}{r^2} \right] Q_i^{n_h}(r,Z) =$$

$$\frac{N}{2\pi\beta^2} F_z(r)\, H(r_l - r) \left[\frac{\partial}{\partial Z}\delta(Z) - i\,\kappa M\,\delta(Z) \right]. \tag{3.38}$$

This equation is the inhomogeneous Helmholtz equation for $Q_i(r,Z)$. The Green's function for the Helmholtz operator of this equation is known, and is defined by

$$G_{n_h}(r, r', Z - Z') = \frac{1}{2\pi}\int_0^{\pi} \cos(n_h N \psi') \frac{e^{-i\kappa R}}{R} \, d\psi' \tag{3.39}$$

where $R = \sqrt{r^2 + r'^2 - 2 r r' \cos\psi' + (Z - Z')^2}$ is the distance in cylindrical coordinates between two points M and M'.

Therefore, applying the Green's function technique, the solution of Eq. (3.38) is given by

$$Q_i^{n_h}(r, Z) = \int_{r'=0}^{\infty} \frac{N}{2\pi\beta^2} F_Z(r') \; H(r_l - r') \; r'$$

$$\times \int_{Z'=-\infty}^{\infty} G_{n_h}(r, r', Z - Z') \left[\frac{\partial}{\partial Z}\delta(Z) - i\kappa M \delta(Z)\right] dZ' \, dr' . \tag{3.40}$$

Solving the Z-integral in the above equation using integration by parts, and noting that

$$\int_{r'=0}^{\infty} g(r') H(r_l - r') \, dr' = \int_{r'=0}^{r_l} g(r') \, dr' , \tag{3.41}$$

where g(r) is an arbitrary function, yield

$$Q_i^{n_h}(r, Z) = \frac{-N}{4\pi^2\beta^2} \int_{r'=0}^{r_l} r' F_Z(r') \int_0^{\pi} \cos(n_h N \psi') \frac{i\kappa M R^2 + Z(1 + i\kappa R)}{R^3} e^{-i\kappa R} \, d\psi' dr' \tag{3.42}$$

This is the solution for the inhomogeneous Helmholtz equation, Eq. (3.38). The pressure field corresponding to the loading component of the fan noise (axial force only) radiating in an unbounded space can therefore be retrieved from the following equation

$$p_i(r, \psi, Z, t) = \sum_{n_h=-\infty}^{\infty} Q_i^{n_h}(r, Z) \, e^{-i\kappa M Z} \, e^{i n_h N (t - \psi)} \tag{3.43}$$

In order to compute the scattered field through the procedure described in chapter 2, both the Q_i term defined by Eq. (3.42) and its derivative with respect to r need to be known. Taking the derivative of Eq. (3.42) with respect to r yields

$$\frac{\partial Q_i^{n_h}}{\partial r}(r, Z) = -\frac{N}{4\pi^2\beta^2} \int_{r'=0}^{r_l} r' F_Z(r') \int_0^{\pi} [\cos(n_h N \psi')$$

$$\times \frac{-\kappa^2 M R^3 + (i\kappa M - \kappa^2 Z) R^2 + 3 i\kappa Z R + 3Z}{R^5} (r' \cos\psi' - 1) \, e^{-i\kappa R} \;] \, d\psi' dr' \tag{3.44}$$

The terms described in Eq. (3.42) and Eq. (3.44) are computed numerically using the method of Gauss-Legendre for the integration procedure (Ortega and Poole 1981). In this integration procedure, the integrand of an integral I(f) is interpolated by a polynomial of degree n of the form $\sum_{i=0}^{n} \alpha_i f(x_i)$, where f(x) is the integrand, x_i are points within the limits of the integral and α_i are the coefficients of interpolation. The coefficients α_i and points x_i are determined such that I(f) be evaluated with the maximum accuracy for the least amount of calculations. This numerical integration procedure is also known as the Gaussian quadrature method.

b. Loading noise (radial and azymuthal forces)

When calculating the loading noise generated by the radial and azymuthal forces that the fan blades apply to the medium, the inhomogeneous wave equation to be solved has the following form

$$[\frac{1}{\tilde{c}^2}\frac{\partial^2}{\partial \tilde{t}^2} - \frac{1}{\tilde{r}}\frac{\partial}{\partial \tilde{r}}(\tilde{r}\frac{\partial}{\partial \tilde{r}}) - \frac{1}{\tilde{r}^2}\frac{\partial^2}{\partial \psi^2} - \frac{\partial^2}{\partial \tilde{z}^2}]\tilde{p}_i(\tilde{r}, \psi, \tilde{z}, \tilde{t}) =$$

$$-\tilde{\nabla}.\left[\tilde{\vec{F}}_{r,\psi}\ H(\tilde{r}_l - \tilde{r})\ \delta(\tilde{z} - \tilde{V}\tilde{t}) \sum_{m_d=-\infty}^{\infty} \delta(\tilde{\Omega}\tilde{t} - \psi - \frac{2\pi m_d}{N})\right], \tag{3.45}$$

where $\tilde{\vec{F}}_{r,\psi}$ is defined as

$$\tilde{\vec{F}}_{r,\psi} = \tilde{F}_r(r)\ \hat{i}_r + \tilde{F}_\psi(r)\ \hat{i}_\psi\ . \tag{3.46}$$

Following the solution procedure applied in part (a), Eq. (3.45) is nondimensionalized, yielding

$$[\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r})-\frac{1}{r^2}\frac{\partial^2}{\partial \psi^2}-\frac{\partial^2}{\partial z^2}]\,p_i(r,\psi,z,t)=$$

$$-\nabla.\left[\vec{F}_{r,\psi}\ H(r_l-r)\ \delta(z-Vt)\sum_{m_d=-\infty}^{\infty}\delta(t-\psi-\frac{2\pi m_d}{N})\right]. \tag{3.47}$$

Before writing this inhomogeneous wave equation in the moving and stretched reference frame, where, as seen in part (a), the wave operator has a form for which a solution is known, the right hand side of Eq. (3.46) is expanded.

Thus, noting that

$$\nabla.\left[\vec{F}\ A\ B\ C\right]=A\,B\,C\,(\nabla.\vec{F})+\left[AB\ \nabla C+AC\ \nabla B+BC\ \nabla A\right].\vec{F} \tag{3.48}$$

where A and B are arbitrary functions of r, ψ and z, the right hand side of Eq. (3.47) can be rewritten as

$$\begin{aligned}R.H.S.=&-\left[H(r_l-r)\ \delta(z-Vt)\sum_{m_d=-\infty}^{\infty}\delta(t-\psi-\frac{2\pi m_d}{N})\ (\nabla.\vec{F}_{r,\psi})\right]\\&-\left[H(r_l-r)\ \delta(z-Vt)\ \nabla\left(\sum_{m_d=-\infty}^{\infty}\delta(t-\psi-\frac{2\pi m_d}{N})\right).\vec{F}_{r,\psi}\right]\\&-\left[H(r_l-r)\sum_{m_d=-\infty}^{\infty}\delta(t-\psi-\frac{2\pi m_d}{N})\ \nabla\left(\delta(z-Vt)\right).\vec{F}_{r,\psi}\right]\\&-\left[\delta(z-Vt)\sum_{m_d=-\infty}^{\infty}\delta(t-\psi-\frac{2\pi m_d}{N})\ \nabla\left(H(r_l-r)\right).\vec{F}_{r,\psi}\right].\end{aligned} \tag{3.49}$$

Now, noting that

$$\frac{\partial}{\partial\psi}(\hat{i}_r)=\hat{i}_\psi \tag{3.50}$$

and

$$\frac{\partial}{\partial\psi}(\hat{i}_\psi)=-\hat{i}_r\,, \tag{3.51}$$

$(\nabla.\vec{F}_{r,\psi})$ is computed, and it is found that

$$(\nabla.\vec{F}_{r,\psi}) = \frac{\partial F_r(r)}{\partial r} + \frac{F_r(r)}{r}. \tag{3.52}$$

Therefore, combining Eq. (3.52) with Eq. (3.49) yields

$$\begin{aligned} \text{R.H.S.} = &-\left[H(r_l - r)\, \delta(z - Vt) \sum_{m_d=-\infty}^{\infty} \delta(t - \psi - \frac{2\pi m_d}{N}) \left(\frac{F_r(r)}{r} + \frac{\partial F_r(r)}{\partial r} \right) \right] \\ &-\left[H(r_l - r)\; \delta(z - Vt) \frac{F_\psi(r)}{r} \;\; \frac{\partial}{\partial \psi} \sum_{m_d=-\infty}^{\infty} \delta(t - \psi - \frac{2\pi m_d}{N}) \right] \\ &-\left[\sum_{m_d=-\infty}^{\infty} \delta(t - \psi - \frac{2\pi m_d}{N})\, \delta(z - Vt)\;\; F_r(r)\;\; \frac{\partial}{\partial r} H(r_l - r) \right]. \end{aligned} \tag{3.53}$$

Next, Eq. (3.47) is expressed in the moving and stretched reference frame, yielding

$$\begin{aligned} &[\frac{1}{c^2}\left(\frac{\partial}{\partial t} - \frac{V}{\beta}\frac{\partial}{\partial Z}\right)^2 - \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) - \frac{1}{r^2}\frac{\partial^2}{\partial \psi^2} - \frac{1}{\beta^2}\frac{\partial^2}{\partial Z^2}]p_i(r,\psi,Z,t) = \\ &-\left[H(r_l - r) \frac{\delta(Z)}{\beta} \sum_{m_d=-\infty}^{\infty} \delta(t - \psi - \frac{2\pi m_d}{N}) \left(\frac{F_r(r)}{r} + \frac{\partial F_r(r)}{\partial r} \right) \right] \\ &-\left[H(r_l - r)\;\; \frac{\delta(Z)}{\beta} \frac{F_\psi(r)}{r} \;\; \frac{\partial}{\partial \psi} \sum_{m_d=-\infty}^{\infty} \delta(t - \psi - \frac{2\pi m_d}{N}) \right] \\ &-\left[\sum_{m_d=-\infty}^{\infty} \delta(t - \psi - \frac{2\pi m_d}{N}) \frac{\delta(Z)}{\beta}\;\; F_r(r)\;\; \frac{\partial}{\partial r} H(r_l - r) \right] \end{aligned} \tag{3.54}$$

Continuing to follow the analysis done in part (a), Eq. (3.23) is combined with Eq. (3.54), yielding

$$\begin{aligned} &[\frac{1}{c^2}\left(\frac{\partial}{\partial t} - \frac{V}{\beta}\frac{\partial}{\partial Z}\right)^2 - \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) - \frac{1}{r^2}\frac{\partial^2}{\partial \psi^2} - \frac{1}{\beta^2}\frac{\partial^2}{\partial Z^2}]p_i(r,\psi,Z,t) = \frac{-N}{2\pi\beta} \sum_{n_h=-\infty}^{\infty} e^{i n_h N (t-\psi)} \\ &\delta(Z)\left[\left(-i\, n_h N\; \frac{F_\psi(r)}{r} + \frac{\partial F_r(r)}{\partial r} + \frac{F_r(r)}{r}\right) H(r_l - r) + F_r(r) \frac{\partial}{\partial r} H(r_l - r)\right]. \end{aligned} \tag{3.55}$$

Then, since the solution p_i of the above equation can be written as a sum of spinning modes, Eq. (3.25) and Eq. (3.36) are combined with Eq. (3.55), yielding

$$[\frac{1}{c^2}\left(i\,n_h N - \frac{V}{\beta}\frac{\partial}{\partial Z}\right)^2 - \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) + \frac{n_h^2 N^2}{r^2} - \frac{1}{\beta^2}\frac{\partial^2}{\partial Z^2}]\, Q_i^{n_h}(r,Z) = \frac{-N}{2\pi\beta}e^{-iM\kappa Z}$$

$$\delta(Z)\left[\left(-i\,n_h N \ \frac{F_\psi(r)}{r} + \frac{\partial F_r(r)}{\partial r} + \frac{F_r(r)}{r}\right)H(r_l - r) + F_r(r)\frac{\partial}{\partial r}H(r_l - r)\right]. \quad (3.56)$$

Finally, repeating the manipulations done (as in part (a)) to the left hand side of Eq. (3.56), and noting that

$$e^{-i\,M\kappa Z}\,\delta(Z) = \delta(Z)\,, \quad (3.57)$$

the inhomogeneous wave equation can be written as

$$\left[\frac{\partial^2}{\partial Z^2} + \kappa^2 + \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) - \frac{n_h^2 N^2}{r^2}\right]Q_i^{n_h}(r,Z) = \frac{N}{2\pi\beta}\delta(Z)$$

$$\delta(Z)\left[\left(-i\,n_h N \ \frac{F_\psi(r)}{r} + \frac{\partial F_r(r)}{\partial r} + \frac{F_r(r)}{r}\right)H(r_l - r) + F_r(r)\frac{\partial}{\partial r}H(r_l - r)\right]. \quad (3.58)$$

The green's function corresponding to this wave operator is known and is defined in Eq. (3.39). Therefore, applying the Green's function technique, the solution of Eq. (3.58) can be expressed as

$$Q_i^{n_h}(r,Z) = \frac{N}{2\pi\beta}\int_{r'=0}^{\infty}\int_{Z'=\infty}^{\infty} G_{n_h}(r,r',Z-Z')\,\delta(Z')$$

$$\left[\left(-i\,n_h N \ \frac{F_\psi(r)}{r} + \frac{\partial F_r(r)}{\partial r} + \frac{F_r(r)}{r}\right)H(r_l - r) + F_r(r)\frac{\partial}{\partial r}H(r_l - r)\right]r'\,dr'\,dZ'. \quad (3.59)$$

Finally, using the method of integration by parts and applying the properties of the step function $H(r_l - r)$ and of the Dirac delta function , Eq. (3.59) yields

$$Q_i^{n_h}(r,Z) = \frac{-N}{(2\pi)^2\beta}\int_{r'=0}^{\infty} i\,n_h N\,(F_r(r')\int_{o}^{\pi}\cos(n_h N\psi')\frac{e^{-i\kappa R}}{R}d\psi'$$

$$+ r'\,F_r(r')\int_{o}^{\pi}\cos(n_h N\psi')\,(-1 - i\,\kappa R)(r' - r\cos\psi)\frac{e^{-i\kappa R}}{R^3}d\psi'\,dr'. \quad (3.60)$$

This solution can be expressed in terms of the incident pressure using Eq. (3.43).

The derivative with respect to r of Eq. (3.60) is given by

$$\frac{\partial Q_i^{n_h}}{\partial r}(r,Z) = \frac{-N}{2\pi\beta} \int_{r'=0}^{\infty} T_1(r') \int_{o}^{\pi} T_2(r',\psi')\, T_3(r',\psi')\, d\psi' + $$

$$T_4(r') \int_{o}^{\pi} T_2(r',\psi')\, T_5(r',\psi')\, d\psi' \quad dr' \tag{3.61}$$

where

$$T_1(r') = i\, n_h N \; F_\psi(r'), \tag{3.62.a}$$

$$T_2(r',\psi') = \cos(n_h N\psi') \frac{e^{-i\kappa R}}{R}, \tag{3.62.b}$$

$$T_3(r',\psi') = \frac{(1 + i\,\kappa R^2)(r'\cos\psi' - 1)}{R^3}, \tag{3.62.c}$$

$$T_4(r') = r'\, F_r(r'), \tag{3.62.d}$$

and

$$T_5(r',\psi') = \frac{-\kappa^2 R^3 + i\,\kappa R^2 + 2\, i\,\kappa R + 3}{R^5}. \tag{3.62.e}$$

3.2.2 Radial arrays of spinning point dipoles

Radial arrays of spinning point dipoles can be used to model fan noise in the case where the generation of modes of specific phases and amplitudes has to be reproduced. Thus, to model the generation of the (m,n) mode of complex modal amplitude A_{mn}, a fan of m blades will be modeled. Each blade is modeled by an identical radial array of N_{dip} point dipoles as described in Figure 3.4.

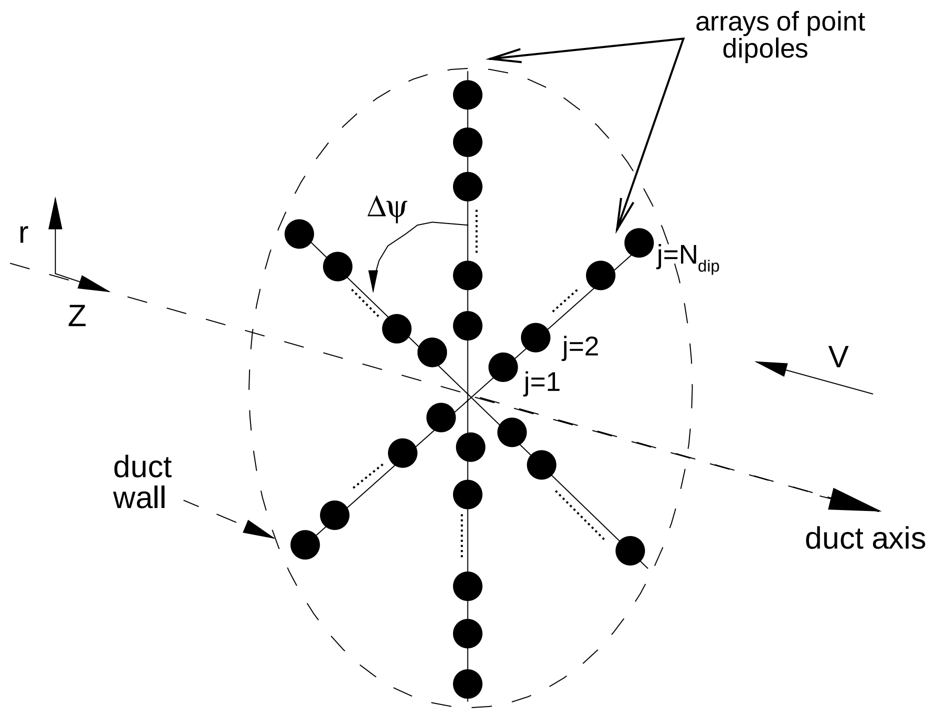

Figure 3.4: Schematic of the fan noise model based on radial arrays of point dipoles.

As indicated in section 3.1, the analytical expression for the acoustic pressure field formed by the (m,n) spinning mode propagating in the duct is

$$p_{mn}(r,\psi,z) = A_{mn}\, J_m(k_{mn} r)\, e^{i(m(\Omega t-\psi)-k_z^{mn} z)} \tag{3.63}$$

Therefore, at any fixed angular and axial location in the duct (and hence along each radial array of point dipoles), the amplitude of the acoustic pressure has the following radial distribution

$$\left\| p_{mn}(r,\psi,z) \right\| = \left\| A_{mn}\, J_m(k_{mn} r) \right\| . \tag{3.64}$$

This acoustic pressure distribution is plotted in Figure 3.5, for the (1,0) spinning mode in a rigid wall duct.

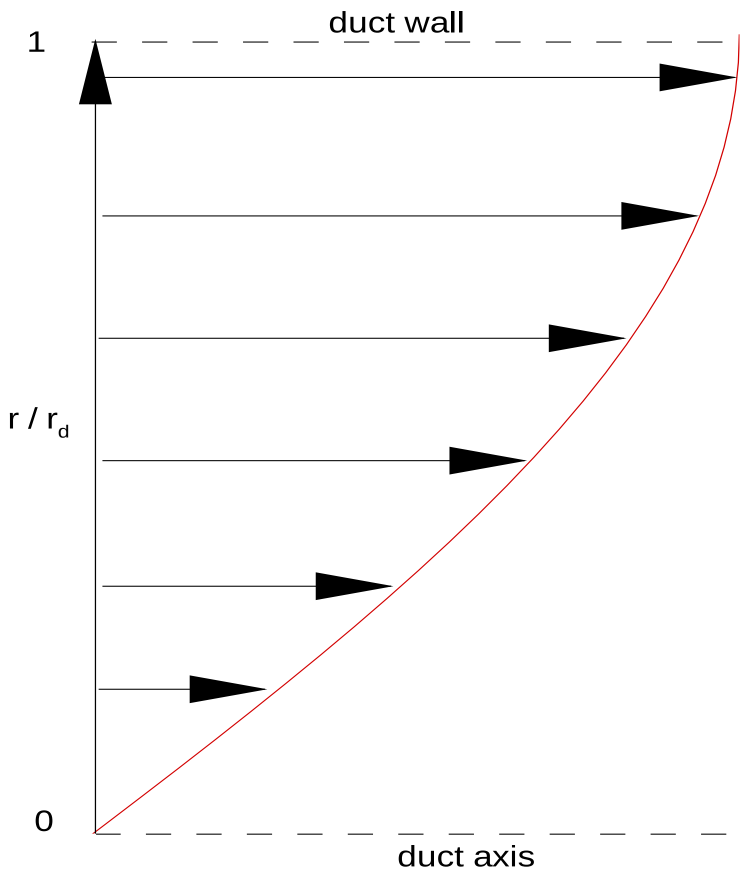

Figure 3.5: Radial distribution of the axial pressure in a rigid wall duct for the (1,0) mode.

Thus, in order to reproduce the pressure field generated in the duct by the (m,n) mode, the complex amplitude, or strength, of the j^{th} point dipole in each radial array of N_{dip} point dipoles has to be given by

$$q_{mn}(j) = A_{mn} J_m(k_{mn} r_j) \tag{3.65}$$

where r_j is the radial location of the j^{th} point dipole. If the generation of more than one mode were to be simulated, then the strength of the point dipole located at $r = r_j$ would be given by

$$q(j) = \sum_m \sum_n q_{mn}(j) = \sum_m \sum_n A_{mn} J_m(k_{mn} r_j) \tag{3.66}$$

Note that since pressure fluctuations in the axial direction of the duct are modeled, the dipoles axis are set parallel to the duct axis.

In order to model the propagation and radiation of modes of specific modal amplitudes, the pressure field generated by each of the N_{dip} point dipoles contained in a radial array has to be computed first, where the point dipoles are given a unit strength. A matrix $[P^0]$ is obtained. The elements $P^0_{l,j}$ of $[P^0]$ are the acoustic pressures at the l^{th} location of the field due to the j^{th} point dipole of unit strength. The pressure field resulting from the propagation and radiation of modes of desired amplitudes is then obtained by multiplying the matrix $[P^0]$ by the vector of dipole strengths defined by Eq. (3.66). Thus, the acoustic pressure at the l^{th} location of the field is given by

$$P_l = P^0_{l,j}\, q_j \tag{3.67}$$

Hence, in order to compute the elements of the matrix $[P^0]$, an analytical expression for the incident pressure field generated by an axial point dipole of unit strength is required (the scattered part of the field being subsequently obtained through the BIEM procedure). This expression is derived next.

When calculating the pressure field generated in free space by a spinning point dipole of unit strength whose axis is parallel to the axis of the duct (i.e., is along the $\hat{i}_z$ direction), the wave equation takes the form

$$[\frac{1}{\tilde{c}^2}\frac{\partial^2}{\partial\tilde{t}^2}-\frac{1}{\tilde{r}}\frac{\partial}{\partial\tilde{r}}(\tilde{r}\frac{\partial}{\partial\tilde{r}})-\frac{1}{\tilde{r}^2}\frac{\partial^2}{\partial\psi^2}-\frac{\partial^2}{\partial\tilde{z}^2}]\tilde{p}_i(\tilde{r},\psi,\tilde{z},\tilde{t}) =$$

$$-\frac{\delta(\tilde{r}-\tilde{r}_j)}{\tilde{r}}\ \frac{\partial}{\partial\tilde{z}}[\delta(\tilde{z}-\tilde{V}\tilde{t})]\sum_{m_d=-\infty}^{\infty}\delta(\tilde{\Omega}\tilde{t}-\psi-\frac{2\pi m_d}{N}). \qquad (3.68)$$

where $\tilde{r}_j$ is the radial location of the (j^{th}) point dipole.

The procedure used in section 3.2.1 to compute an analytical expression for the field generated by spinning line sources is also applied in this section in order to find the solution of Eq. (3.68). This procedure consists of expressing the equation to be solved (i.e., Eq. (3.68)) in a form that has a known solution.

Following the first step of the procedure, Eq. (3.68) is nondimensionalized , yielding

$$[\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r})-\frac{1}{r^2}\frac{\partial^2}{\partial\psi^2}-\frac{\partial^2}{\partial z^2}]\,p_i(r,\psi,z,t) =$$

$$-\frac{\delta(r-r_j)}{r}\ \frac{\partial}{\partial z}[\delta(z-Vt)]\sum_{m_d=-\infty}^{\infty}\delta(t-\psi-\frac{2\pi m_d}{N}), \qquad (3.69)$$

This nondimensional inhomogeneous wave equation is then written in a moving and stretched reference frame defined earlier by Eq. (3.17) and (3.18). In this new reference frame, and applying the property of the delta function described by Eq. (3.14), Eq. (3.69) takes the following form

$$[\frac{1}{c^2}\left(\frac{\partial}{\partial t}-\frac{V}{\beta}\frac{\partial}{\partial Z}\right)^2-\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r})-\frac{1}{r^2}\frac{\partial^2}{\partial\psi^2}-\frac{1}{\beta^2}\frac{\partial^2}{\partial Z^2}]p_i(r,\psi,Z,t) =$$

$$-\frac{1}{\beta^2}\ \frac{\delta(r-r_j)}{r}\ \frac{\partial}{\partial Z}[\delta(Z)]\sum_{m_d=-\infty}^{\infty}\delta(t-\psi-\frac{2\pi m_d}{N}). \qquad (3.70)$$

Replacing $\sum_{m_d=-\infty}^{\infty} \delta(t-\psi-\frac{2\pi m_d}{N})$ by its Fourier series, defined in Eq. (3.23), and noting that the solution of the resulting equation can be written as a sum of spinning modes, yields

$$[\frac{1}{c^2}\left(i\,n_h N-\frac{V}{\beta}\frac{\partial}{\partial Z}\right)^2-\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r})+\frac{n_h^2 N^2}{r^2}-\frac{1}{\beta^2}\frac{\partial^2}{\partial Z^2}]\,Q_i^{n_h}(r,Z)=$$

$$e^{i\kappa M Z}\,\frac{N}{2\pi\beta^2}\,\frac{\delta(r-r_j)}{r}\,\frac{\partial}{\partial Z}\delta(Z) \qquad (3.71)$$

where $Q_i(r,Z)$ is defined by Eq. (3.36) and Eq. (3.25). Applying the property of the delta function described in Eq. (3.37), Eq. (3.71) can be rewritten as

$$[\frac{1}{c^2}\left(i\,n_h N-\frac{V}{\beta}\frac{\partial}{\partial Z}\right)^2-\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r})+\frac{n_h^2 N^2}{r^2}-\frac{1}{\beta^2}\frac{\partial^2}{\partial Z^2}]\,Q_i^{n_h}(r,Z)=$$

$$\frac{N}{2\pi\beta^2}\,\frac{\delta(r-r_j)}{r}\left[\frac{\partial}{\partial Z}\delta(Z)-i\,\kappa M\,\delta(Z)\right] \qquad (3.72)$$

Next, the wave operator (i.e., the left hand side of Eq. (3.72)) is manipulated in order to express it into a form that has a known solution. Thus, referring to Eq. (3.29) through Eq. (3.34), Eq. (3.72) can be written as

$$\left[\frac{\partial^2}{\partial Z}+\kappa^2+\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r})-\frac{n_h^2 N^2}{r^2}\right]Q_i^{n_h}(r,Z)=$$

$$\frac{N}{2\pi\beta^2}\,\frac{\delta(r-r_j)}{r}\left[\frac{\partial}{\partial Z}\delta(Z)-i\,\kappa M\,\delta(Z)\right] \qquad (3.73)$$

This equation is the inhomogeneous Helmholtz equation for $Q_i(r,Z)$. The Green's function for the Helmholtz operator of this equation is known and was defined earlier in Eq. (3.39). Therefore, applying the Green's function technique, the solution of Eq. (3.73) is given by

$$Q_i^{n_h}(r,Z) = \int_{r'=0}^{\infty} \frac{N}{2\pi\beta^2} \quad \delta(r'-r_j)$$

$$\times \int_{Z'=-\infty}^{\infty} G_{n_h}(r,r',Z-Z') \left[\frac{\partial}{\partial Z}\delta(Z) - i\,\kappa\, M\, \delta(Z)\right] dZ'\, dr'\,. \qquad (3.74)$$

Solving the Z-integral in the above equation using integration by parts, and noting that

$$\int_{Z'=-\infty}^{\infty} f(Z')\ \delta(Z'-Z_0)\ dZ' = f(Z_0)\,, \qquad (3.75)$$

yields

$$Q_i^{n_h}(r,Z) = \int_{r'=0}^{\infty} \frac{-N}{2\pi\beta^2} \quad \delta(r'-r_j)$$

$$\times\left[\left.\frac{\partial\, G_{n_h}(r,r',Z-Z')}{\partial Z}\right|_{Z'=0} + i\,\kappa M\ G_{n_h}(r,r',Z)\right] dr'\,. \qquad (3.76)$$

Finally, noting that

$$\int_{r'=0}^{\infty} f(r')\ \delta(r'-r_j)\ dr' = f(r_j)\,, \qquad (3.77)$$

Eq. (3.76) reduces to

$$Q_i^{n_h}(r,Z) = \frac{-N}{4\pi^2\beta^2} \int_0^{\pi} \cos(n_h N\,\psi')\, \frac{i\,\kappa M R^2 + Z(1+i\,\kappa R)}{R^3}\, e^{-i\,\kappa R}\ d\psi'\,, \qquad (3.78)$$

where

$$R = \sqrt{r^2 + r_j^2 - 2\, r\, r_j \cos\psi' + Z^2}\,. \qquad (3.79)$$

This is the solution of the inhomogeneous Helmholtz equation, Eq. (3.73). The corresponding expression for the incident pressure field generated by a spinning point dipole is retrieved by using Eq. (3.43), i.e.,

$$p_i(r,\psi,Z,t) = \sum_{n_h=-\infty}^{\infty} Q_i^{n_h}(r,Z)\, e^{-i\,\kappa M Z}\ e^{i\, n_h N(t-\psi)} \qquad (3.80)$$

In order to compute the scattered field using the procedure described in Chapter 2, both the Q_i term defined by Eq. (3.78) and its derivative with respect to r need to be known. Taking the derivative of Eq. (3.78) with respect to r yields

$$\frac{\partial Q_i^{n_h}}{\partial r}(r,Z) = -\frac{N}{4\pi^2\beta^2}\int_0^\pi [\cos(n_h N\psi')$$

$$\times \frac{-\kappa^2 M R^3 + (i\kappa M - \kappa^2 Z)R^2 + 3i\kappa Z R + 3Z}{R^5}(r'\cos\psi' - 1)\, e^{-i\kappa R}\,]\, d\psi' \quad (3.81)$$

In this chapter, two models for the fan noise were described. For each of these models, analytical expressions for the computation of the free space acoustic field, i.e., incident field, were derived. These expressions serve as input to the BIEM described in Chapter 2 for the computation of the scattered part of the field due to the presence of the duct. The resulting acoustic field generated by the ducted fan is obtained by adding the incident and scattered field. These fan models will be validated in chapter 5.